The Annals of Applied Probability
2023, Vol. 33, No. 3, 1701–1731
https://doi.org/10.1214/22-AAP1808

UNIVERSALITY OF THE TIME CONSTANT FOR 2D CRITICAL FIRST-PASSAGE PERCOLATION

BY MICHAEL DAMRON[1, a], JACK HANSON[2, b] AND WAI-KIT LAM[3, c]

[1]*School of Mathematics, Georgia Institute of Technology,* [a]*mdamron6@protonmail.com*
[2]*Department of Mathematics, City University of New York, City College,* [b]*jhanson@ccny.cuny.edu*
[3]*Department of Mathematics, National Taiwan University,* [c]*waikitlam@ntu.edu.tw*

We consider first-passage percolation (FPP) on the triangular lattice with vertex weights (t_v) whose common distribution function F satisfies $F(0) = 1/2$. This is known as the critical case of FPP because large (critical) zero-weight clusters allow travel between distant points in time which is sublinear in the distance. Denoting by $T(0, \partial B(n))$ the first-passage time from 0 to $\{x : \|x\|_\infty = n\}$ we show existence of a "time constant" and find its exact value to be

$$\lim_{n\to\infty} \frac{T(0, \partial B(n))}{\log n} = \frac{I}{2\sqrt{3}\pi} \quad \text{almost surely},$$

where $I = \inf\{x > 0 : F(x) > 1/2\}$ and F is any critical distribution for t_v. This result shows that this time constant is universal and depends only on the value of I. Furthermore, we find the exact value of the limiting normalized variance, which is also only a function of I, under the optimal moment condition on F. The proof method also shows an analogous universality on other two-dimensional lattices, assuming the time constant exists.

1. Introduction.

Let $\mathbb{T}$ be the triangular lattice. We will take $\mathbb{T}$ to be embedded in $\mathbb{R}^2$ with vertex set $\mathbb{Z}^2$ and with edges between points of the form (x_1, y_1) and (x_2, y_2) with either (a) $\|(x_1, y_1) - (x_2, y_2)\|_1 = 1$ or (b) both $x_2 = x_1 + 1$ and $y_2 = y_1 - 1$.

We will consider first-passage percolation (FPP) on $\mathbb{T}$, which is defined as follows. Let $(\omega_x)_{x\in\mathbb{Z}^2}$ be a family of i.i.d. uniform random variables on $(0, 1)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Fix a distribution function F with $F(0^-) = 0$. We define $t_x = F^{-1}(\omega_x)$ (so that t_x has distribution F), where for $t \in (0, 1)$,

$$F^{-1}(t) := \inf\{y \in \mathbb{R} : F(y) \geq t\}.$$

A path is a sequence of vertices $(x_1, \dots, x_n)$ with x_i being adjacent to x_{i+1} for all $i = 1, \dots, n-1$, and a circuit is a path $(x_1, \dots, x_n)$ with $x_1 = x_n$. We will always assume that paths and circuits are self-avoiding. (A self-avoiding circuit is one such that $(x_1, \dots, x_{n-1})$ is self-avoiding.) For a path $\gamma = (x_1, \dots, x_n)$, we define its passage time by

$$T(\gamma) = \sum_{i=2}^{n} t_{x_i},$$

and for vertex sets $A, B \subseteq \mathbb{Z}^2$, we define the first-passage time from A to B by

$$T(A, B) = \inf\{T(\gamma) : \gamma \text{ is a path from a vertex in } A \text{ to a vertex in } B\}.$$

For notational simplicity, for $x \in \mathbb{Z}^2$ and $B \subseteq \mathbb{Z}^2$, we will write $T(x, B)$ for $T(\{x\}, B)$. Note that unlike in the case of edge-FPP, our definition of T is not symmetric; this will not matter in

Received January 2021; revised January 2022.
MSC2020 subject classifications. Primary 60K35; secondary 82B43.
Key words and phrases. First-passage percolation, universality, time constant.

our arguments.) In first-passage percolation, one studies the asymptotic behavior of random variables such as $T(0, \partial B(n))$, where $B(n) = \{x \in \mathbb{Z}^2 : \|x\|_\infty \le n\}$ and $\partial B(n) = \{x \in \mathbb{Z}^2 : \|x\|_\infty = n\}$.

In this paper, we will study the critical case, namely $F(0) = p_c = 1/2$, where p_c is the critical threshold for site percolation on $\mathbb{T}$. In this case, it is shown by Damron–Lam–Wang in [2] that under suitable moment assumptions on t_x, one has

$$\mathbf{E}T\big(0, \partial B(n)\big) \asymp \sum_{k=2}^{\lfloor \log n \rfloor} F^{-1}\big(p_c + 2^{-k}\big) \quad \text{and} \qquad (1.1)$$
$$\operatorname{Var} T\big(0, \partial B(n)\big) \asymp \sum_{k=2}^{\lfloor \log n \rfloor} \big[F^{-1}\big(p_c + 2^{-k}\big)\big]^2,$$

and further if $\operatorname{Var}(T(0, \partial B(n))) \to \infty$ as $n \to \infty$, then one also has a Gaussian central limit theorem for the variables $(T(0, \partial B(n)))$.[1] (Here we write $a_n \asymp b_n$ if the ratio a_n/b_n is bounded away from 0 and ∞.) Sharper asymptotics were proved by C.-L. Yao in [13, 14] (these results were further developed in [4, 15]) in the special case where t_x is Bernoulli (i.e., $t_x = 0$ with probability $1/2$ and $t_x = 1$ with probability $1/2$):

$$\frac{T(0, \partial B(n))}{\log n} \to \frac{1}{2\sqrt{3}\pi} \quad \text{a.s.}, \quad \frac{\mathbf{E}T(0, \partial B(n))}{\log n} \to \frac{1}{2\sqrt{3}\pi}, \qquad (1.2)$$
$$\frac{\operatorname{Var}(T(0, \partial B(n)))}{\log n} \to \frac{2}{3\sqrt{3}\pi} - \frac{1}{2\pi^2}$$

as $n \to \infty$. (The existence of the limit without explicit values, shown in [13], preceded [2].) By analogy with the noncritical case of FPP, we will refer to the limit on the left (of $T(0, \partial B(n))/\log n$) as the *time constant* for the model. In [13], Remark 1.3, Yao asks whether one can extend these limit theorems to general distributions.

The behaviors in (1.1) show that the limits in the limit theorems, if existent, should also depend on the behavior of F^{-1} near p_c, or equivalently the behavior of F near 0. We will show existence of these limits and, from their explicit forms, it is manifest that this is indeed the case. Let

$$I = \inf\{x > 0 : F(x) > p_c\}$$

be the infimum of the support of the law of t_x excluding 0.

THEOREM 1.1. *On the triangular lattice* $\mathbb{T}$, *we have a law of large numbers*:

$$\lim_{n\to\infty} \frac{T(0, \partial B(n))}{\log n} = \frac{I}{2\sqrt{3}\pi} \quad \textit{almost surely.} \qquad (1.3)$$

Furthermore, if $\mathbf{E}\min\{t_1, \ldots, t_6\}^2 < \infty$, *where* $t_1, \ldots, t_6$ *are i.i.d. copies of* t_v, *then*

$$\lim_{n\to\infty} \frac{\operatorname{Var}(T(0, \partial B(n)))}{\log n} = I^2 \left(\frac{2}{3\sqrt{3}\pi} - \frac{1}{2\pi^2}\right). \qquad (1.4)$$

---

[1]These results were proved for edge-FPP on the square lattice, but similar arguments give them for the current setting.

REMARK 1.

1. One can also show that if $\mathbf{E}\min\{t_1,\ldots,t_6\} < \infty$, then

$$\lim_{n\to\infty} \frac{\mathbf{E}T(0,\partial B(n))}{\log n} = \frac{I}{2\sqrt{3}\pi}. \tag{1.5}$$

This can be proved by using a similar, but simpler, method than that for (1.4). We omit the details here. Note, however, that (1.5) follows from (1.3) and (1.4) under the stronger moment assumption that is used for (1.4).

2. Our proof method extends to a large class of two-dimensional lattices (including planar lattices where RSW tools are available, like the square grid). It gives a weaker result, since the limits (1.2) are only known to hold for the Bernoulli distribution on the triangular lattice (due to the use of CLE $_6$ in their proofs). However, if any of these are shown to exist on other lattices (with possibly different limits), then our method shows that they also hold for general distributions (under suitable moment assumptions).

In the course of the proof, one would need to replace the exact values of arm exponents used in Lemma 5.3 by inequalities for these exponents on general lattices given in [8]. Precisely, we use the fact that the five-arm exponent in a half-plane or $3/4$-plane, and the six-arm exponent in the full-plane, are all strictly bigger than 2. The five-arm inequalities follow from the five-arm exponent on the full-plane, which is known to be 2 on general lattices, combined with the arguments of [8], which only require RSW methods (not conformal invariance). The six-arm inequality follows from the five-arm exponent on the full-plane combined with the BK-Reimer inequality.

3. In the standard case of FPP, where $F(0) < p_c$, there is no similar universality of the time constant. Indeed, using [12], Theorem (2.13), one can construct two bounded distributions for t_x such that they have the same infimum, but the limits $\lim_{n\to\infty} \frac{T(0,\partial B(n))}{n}$ for the different distributions are positive and distinct real numbers.

4. The moment conditions in Theorem 1.1 and equation (1.5) are optimal in the following senses. By a variant of [1], Lemma 3.1, one has for any $q > 0$, $\mathbf{E}T(0,\partial B(n))^q < \infty$ if and only if $\mathbf{E}\min\{t_1,\ldots,t_6\}^q < \infty$. Therefore if the above moment conditions fail, then either the mean or the variance of $T(0,\partial B(n))$ will be infinite. There is no need for a moment condition in (1.3) because the infinite path γ constructed in Section 4 has all but finitely many edges with weight $\le I+1$.

5. One can prove point-to-point analogues of the statements of Theorem 1.1 (replacing $T(0,\partial B(n))$ with $T(0,x)$ and $\log n$ by $\log \|x\|$) with a.s. convergence in (1.3) replaced by convergence in probability. For (1.4), one needs a slightly stronger moment condition. See a similar modification in [14].

1.0.0.1. *Question*. According to (1.1), there exist distributions such that $\mathbf{E}\,T(0,\partial B(n)) = o(\log n)$ and $\operatorname{Var}(T(0,\partial B(n))) = o(\log n)$, but both quantities diverge to infinity as $n\to\infty$. In this case, does

$$\lim_{n\to\infty} \frac{T(0,\partial B(n))}{\sum_{k=2}^{\log n} F^{-1}(p_c+2^{-k})} \text{ exist?}$$

At the time of this writing, this question appears to be open.

In this paper, the symbol C_i (where $i\in\mathbb{N}$) denotes a (possibly) large constant, and the symbol c_i ($i\ge 4$) denotes a (possibly) small constant. c_1, c_2, c_3 are reserved for Definition 3.1 and the definition of a good circuit. The symbol $\|\cdot\|$ will refer to the Euclidean norm.

1.1. *Sketch of proofs.*

1.1.1. *Sketch of* (1.3). We begin by coupling together our vertex weights with Bernoulli weights: we define the Bernoulli weights as $t_v^{\mathrm{B}} = I \cdot \mathbf{1}_{\{t_v>0\}}$. Because $t_v^{\mathrm{B}} \le t_v$, one has

$$\frac{I}{2\sqrt{3}\pi} = \lim_{n\to\infty} \frac{T^{\mathrm{B}}(0, \partial B(n))}{\log n} \le \liminf_{n\to\infty} \frac{T(0, \partial B(n))}{\log n} \quad \text{almost surely.}$$

(Here, T^{B} is the passage time using the Bernoulli weights.)

To show the other inequality, it will suffice to prove that

$$T\,\big(0, \partial B(n)\big) - T^{\mathrm{B}}\,\big(0, \partial B(n)\big) = o(\log n) \quad \text{almost surely.}$$

The idea for this proof is to use that T^{B} is equal to the maximal number of disjoint closed (i.e., with weight > 0) circuits separating 0 from $\partial B(n)$. To construct such circuits, we note in Lemma 2.3 that results of [7] allow us to find an infinite sequence of disjoint closed circuits surrounding the origin which are successively "outermost". Specifically, if $\mathcal{C}_k$ is the kth circuit in the sequence, and $\mathring{\mathcal{C}}_k$ is its interior, then $\mathcal{C}_k$ lies in $\mathring{\mathcal{C}}_{k+1}$, and any other closed circuit $\mathcal{C}$ inside $\mathring{\mathcal{C}}_{k+1}$ has $\mathring{\mathcal{C}} \subseteq \mathring{\mathcal{C}}_k$. (We make the notion "outermost" precise in Definition 2.2.)

In particular, the sequence $(\mathcal{C}_k)$ is maximal, so it is not possible to find a closed circuit lying entirely in the region strictly between two adjacent circuits in the sequence. In addition, by the outermost property, from each vertex on one circuit, there is a zero-weight path starting at an adjacent vertex and ending adjacent to the next circuit. Using this property, one can construct an infinite path by starting at 0, following any open (i.e., with weight $= 0$) path from 0 to the first circuit, using a vertex from this circuit, following any open path to the next circuit, and so on. (Here we remark that if we were to use "innermost" circuits, we would need to build paths starting on a circuit and proceeding inward to circuits in its interior. This would produce only a finite path, and then we would need to take a limit of these paths.) One can show that if $_n$ is the portion of until its first intersection with $\partial B(n)$, then

$$T^{\mathrm{B}}(\ _n) - T^{\mathrm{B}}\,\big(0, \partial B(n)\big) = o(\log n) \quad \text{almost surely.}$$

The goal then is to show that

$$T(\ _n) - T^{\mathrm{B}}(\ _n) = o(\log n) \quad \text{almost surely} \tag{1.6}$$

for a particular choice of .

To choose , we use an adaptation of the "good circuit" construction from [7]. Given an > 0, we consider a vertex v to be of "low weight" if $t_v \le I +$. (This is not the full definition; low-weight vertices are defined more precisely in Definition 3.1.) In Section 3, we show that with high probability, any open path starting at one circuit in the above sequence and ending at the next—so long as these circuits have sufficient distance from each other—can be modified to retain the same initial point, but to end adjacent to a vertex in the second circuit which has low weight. (See Figure 1.) This idea underlies the main construction (Lemma 3.2) in which we build an infinite path γ starting at 0 which passes through each circuit exactly once and contains only finitely many vertices which are not of low weight. We take $= \gamma$, so that $_n$ is an initial segment γ_n of γ. Then

$$\begin{aligned} T(\gamma_n) - T^{\mathrm{B}}(\gamma_n) &= \sum_{v: t_v>0, v\in\gamma_n} (t_v - I) \le C + \sum_{v: t_v>0, v\in\gamma_n} \\ &= C + (\ /I)\,T^{\mathrm{B}}(\gamma_n) \le C\ \log n. \end{aligned}$$

This is true for any , so this shows (1.6) and completes the sketch.

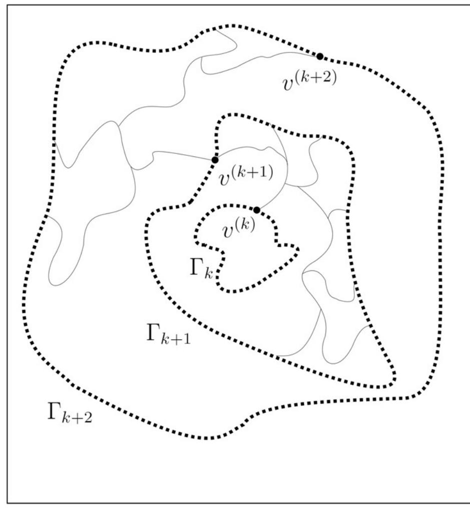

FIG. 1. *Illustration of the modification of open paths from Lemma 3.2. The circuits Γ_{k+i}, $i = 0, 1, 2$ are consecutive outermost circuits from the construction of Lemma 2.3. The light gray paths connecting vertices $v^{(i)}$ on the circuits are open. So long as the circuits have sufficient distance from each other, one can find many modified open paths which begin at the same point and end at the next circuit. With high probability, at least one such path will end at a low-weight vertex.*

1.1.2. *Sketch of* (1.4). To show universality of the variance, we represent the passage time as a sum of martingale differences Δ_k, so that

$$\operatorname{Var} T(0, \mathcal{O}_n) = \sum_{k=0}^{n} \mathbf{E}\, \Delta_k^2,$$

where $\mathcal{O}_n$ is the innermost open circuit in any annulus of the form $B(2^{m+1}) \setminus B(2^m)$ for $m \geq n$. Here, Δ_k is the martingale difference of $T(0, \mathcal{O}_n)$ using the filtration generated by weights on and inside $\mathcal{O}_k$. (The argument for the variance uses, in addition to closed circuits as in the proof of (1.3), an open circuit construction. The purpose of using open circuits is to decompose the passage time from the origin to the boundary of a box as a sum of passage times between successive open circuits.) Using a representation of Δ_k from [9], we split the difference

$$\operatorname{Var} T(0, \mathcal{O}_n) - \operatorname{Var} T^{\mathrm{B}}(0, \mathcal{O}_n) = \sum_{k=0}^{n} \mathbf{E}\left[\Delta_k^2 - \left(\Delta_k^{\mathrm{B}}\right)^2\right]$$

into a sum of three terms, each of which is bounded similarly. The term we focus on can be written (see Lemma 5.1, where the term we are discussing is called Y) as a difference of passage times between two open circuits:

$$T(\mathcal{O}_k, \check{\mathcal{O}}_k) - T^{\mathrm{B}}(\mathcal{O}_k, \check{\mathcal{O}}_k).$$

(In Lemma 5.1, this difference is called $\tilde{Y}$.) Here $\check{\mathcal{O}}_k$ is the next circuit of the form $\mathcal{O}_m$ which is not equal to $\mathcal{O}_k$. Therefore the proof reduces to showing

$$\sum_{k=0}^{n} \mathbf{E}\left[T(\mathcal{O}_k, \check{\mathcal{O}}_k) - T^{\mathrm{B}}(\mathcal{O}_k, \check{\mathcal{O}}_k)\right]^2 = o(n). \tag{1.7}$$

(Note the $o(n)$ appears instead of $o(\log n)$ because we are working on logarithmic scale.) Once this is done, then the proof is completed by bounding the difference between point-to-box passage times of the form $T(0, \partial B(2^n))$ and point-to-circuit passage times of the form

$T(0, \mathcal{O}_n)$. Although such bounds have been derived in previous works under stronger moment assumptions, the situation here is more delicate. We give this argument after (5.27).

To bound the terms of (1.7), we use the construction of low-weight paths from Section 3. To use this method, we need to show in Lemma 5.3 that with high enough probability, the circuit $\mathcal{O}_k$ is sufficiently far away from closed circuits $\mathcal{C}_j$ from the sequence in Lemma 2.3. When this occurs, one can, as in the proof of universality of the time constant, bound the difference of passage times in (1.7) using paths connecting these circuits which pass through some number of closed circuits using only low-weight vertices. Partitioning the expectation according to this "sufficiently far" event, one has a bound for the summands of (1.7) of the form

$$o(1) + \mathbf{E}(\text{maximal \# of closed circuits between } \mathcal{O}_k \text{ and } \check{\mathcal{O}}_k)^2.$$

Here, the $o(1)$ term corresponds to the bound $C/\sqrt{k}$ in (5.22), and represents the expectation on the event that $\mathcal{O}_k$ is close to a closed circuit $\mathcal{C}_j$. The second term appears in (5.25) (where is written as a k-dependent term a_k^2). After showing that this expectation is bounded by a constant, we obtain the bound $o(1) + C$ for summands in (1.7), and this completes the sketch.

2. Preliminaries.

We begin with some definitions.

DEFINITION 2.1.

1. For a circuit , we define $\mathring{}$ to be the interior of , namely the bounded connected component of $\mathbb{Z}^2 \setminus$ seen as a subgraph of $\mathbb{T}$.
2. We say that a vertex x is open if $\omega_x \leq 1/2$ and closed otherwise.
3. A path (or circuit) is open if all its vertices are open; it is closed if all its vertices are closed.

Many of our arguments in this paper will involve careful analysis of circuits, so we recapitulate many of their most important properties. We recall that $\mathring{\mathcal{C}}$ denotes the interior of the self-avoiding circuit $\mathcal{C}$.

DEFINITION 2.2. Let $A \subseteq \mathbb{Z}^2$ be a Jordan domain (we identify A with the set of vertices $\mathbb{Z}^2 \cap A$ and with the subgraph induced by these vertices), and let $B \subseteq A$ be a connected vertex set.

- An open circuit $\mathcal{C} \subseteq A \setminus B$ is said to be the outermost open circuit in $A \setminus B$ if $B \subseteq \mathring{\mathcal{C}}$, and if, for each open circuit $\mathcal{D} \subseteq A \setminus B$ with $B \subseteq \mathring{\mathcal{D}}$, we have $\mathcal{D} \subseteq (\mathcal{C} \cup \mathring{\mathcal{C}})$.
- Similarly, an open circuit $\mathcal{C} \subseteq A \setminus B$ is said to be the innermost open circuit in $A \setminus B$ if $B \subseteq \mathring{\mathcal{C}}$ and if, for each open circuit $\mathcal{D} \subseteq A \setminus B$ with $B \subseteq \mathring{\mathcal{D}}$, we have $\mathcal{C} \subseteq (\mathcal{D} \cup \mathring{\mathcal{D}})$.

Of course, we extend Definition 2.2 to the case of outermost or innermost closed circuits, replacing the word "open" with "closed" throughout. It is immediate from the definition that if it exists, the innermost or outermost open (resp. closed) circuit is unique.

If there exists an open (resp. closed) circuit in $A \setminus B$ having B in its interior, then outermost and innermost open (resp. closed) circuits also exist. This is by now well known, following from arguments along the lines of the proof of existence of extremal crossings (see [5], Lemma 1). We describe the idea in the case of outermost circuits, for definiteness. One can define a partial ordering on open circuits surrounding B in A, where $\mathcal{D}$ precedes $\mathcal{C}$ in this ordering if $\mathcal{D} \subseteq [\mathcal{C} \cup \mathring{\mathcal{C}}]$. It is easy to see that every chain in this ordering has an upper bound, so there exists a maximal element. Moreover, this maximal element is unique and succeeds all open circuits in $A \setminus B$ having B in their interior: if $\mathcal{D}$ and $\mathcal{C}$ are not comparable, they intersect, and one can loop-erase their union to produce a circuit $\mathfrak{C}$ with $[\mathcal{C} \cup \mathcal{D}] \subseteq [\mathfrak{C} \cup \mathring{\mathfrak{C}}]$. The existence of a unique maximal element for the ordering—the outermost open circuit—follows.

Another consequence of the definition which makes outermost (resp. innermost) circuits particularly useful is that it requires no reference to an exploration process, though such processes are often useful. Indeed, the definition gives a more or less explicit representation for the indicator function of the event $\{\mathcal{C}$ is the outermost open circuit in $A \setminus B\}$ (and similarly for innermost circuits). The final consequence we will use, which is of extreme importance, is that knowing the value of the outermost (resp. innermost) circuit in $A \setminus B$ tells us nothing about the status of vertices inside (resp. outside) this circuit. Formally,

$$\{\mathcal{C} \text{ is the outermost open circuit in } A \setminus B\} \text{ is independent of } \{\omega_x : x \in \mathring{\mathcal{C}}\},$$

with similar statements holding for the innermost circuit (as usual, "open" may be replaced with "closed" as well). See, for example, the statement of the analogous statement for extremal crossings which appears below Lemma 1 of [5].

We will use several (modified) lemmas from [7]. We sketch some of the proofs; they are mostly intricate RSW-type arguments that are valid for general lattices. The first provides an infinite sequence of "outermost" closed circuits $(\mathcal{C}_k)$ surrounding 0. This sequence will be maximal in the sense that $\mathcal{C}_k \subseteq \mathring{\mathcal{C}}_{k+1}$ and there is no closed circuit contained in the region between $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$.

LEMMA 2.3. *Almost surely, there exists a sequence of random disjoint circuits* $(\mathcal{C}_k)_{k \geq 1}$ *with* $0 \in \mathring{\mathcal{C}}_k$, *so that each of these circuits is closed*, $\mathcal{C}_k \subseteq \mathring{\mathcal{C}}_{k+1}$, *and* $\mathcal{C}_k$ *is the outermost circuit in* $\mathring{\mathcal{C}}_{k+1}$ *which is entirely closed: all its vertices are closed.* (*Also there is no closed circuit surrounding* 0 *in* $\mathring{\mathcal{C}}_1$.) *Moreover, there exist constants* $C_1, C_2, c_4 > 0$ *such that almost surely,* $\operatorname{diam}(\mathcal{C}_{k+1}) \leq C_1 \operatorname{diam}(\mathcal{C}_k)$ *for all large* k, *and*

$$c_4 \leq \liminf_{k \to \infty} \frac{\log(\operatorname{diam}(\mathcal{C}_k))}{k} \leq \limsup_{k \to \infty} \frac{\log(\operatorname{diam}(\mathcal{C}_k))}{k} \leq C_2. \tag{2.1}$$

PROOF. This statement is the same as that of [7], Lemma 1, except that in that paper, the circuits may be open or closed, and there is no mention of there being no circuit surrounding 0 in $\mathring{\mathcal{C}}_1$. For the reader's convenience, we include here a sketch of the proof.

Consider a large box $B(2^N)$; let $\mathfrak{C}(N)$ be the outermost open circuit in $B(2^N) \setminus \{0\}$. (There exists such a circuit with high probability for N large, by the Russo–Seymour–Welsh theorem). Either there is no closed circuit in $\mathring{\mathfrak{C}}(N) \setminus \{0\}$, or we can find an outermost closed circuit $\mathcal{D}$ in $\mathring{\mathfrak{C}}(N) \setminus \{0\}$. Continuing in this way, we can find an outermost closed circuit in $\mathring{\mathcal{D}} \setminus \{0\}$ and so on, until we find a circuit with no closed circuits in its interior.

We enumerate the circuits we found from the inside out: let $\mathcal{C}_1$ denote the innermost closed circuit found above (i.e., the final circuit that we find as we progress inward from $\mathfrak{C}(N)$), let $\mathcal{C}_2$ denote the second-innermost, and so on. We conclude the enumeration with $\mathcal{D}$. The sequence of $\mathcal{C}_i$'s so found has all the properties in the lemma, except it is finite. The final piece of the construction is to show that the sequence we find is consistent with respect to the choice of box: if we choose $M > N$ and construct a sequence $(\mathcal{C}_i')$ of circuits inside $\mathfrak{C}(M)$ via the above procedure, then $\mathcal{C}_i = \mathcal{C}_i'$ whenever both of these are defined.

Letting $\mathcal{C}_j'$ denote the outermost circuit of the $(\mathcal{C}_i')$ sequence which lies in $\mathfrak{C}(N) \cup \mathring{\mathfrak{C}}(N)$, we claim that $\mathcal{C}_j' = \mathcal{D}$. We note that $\mathcal{C}_j'$ cannot intersect $\mathfrak{C}(N)$, and so lies entirely in $\mathring{\mathfrak{C}}(N)$; similarly, $\mathcal{C}_{j+1}'$ does not intersect $\mathfrak{C}(N) \cup \mathring{\mathfrak{C}}(N)$ (if $\mathcal{C}_{j+1}'$ does not exist, one can replace it for the remainder of this paragraph by $\mathfrak{C}(M)$). By the outermost property of $\mathcal{D}$, we have

$\mathcal{C}_j \subseteq [\mathcal{D} \cup \mathring{\mathcal{D}}]$. But then $\mathcal{C}_j = \mathcal{D}$, since otherwise $\mathcal{D}$ would be larger in the partial ordering on circuits but still lie entirely within $\mathcal{C}_{j+1}$. It is now clear from the construction that $\mathcal{C}_j = \mathcal{C}_j$, and that $\mathcal{C}_i = \mathcal{C}_i$ for $i < j$.

The diameter bounds follow from standard Russo–Seymour–Welsh arguments; we briefly describe the reasoning behind the final upper bound of (2.1). Indeed, the RSW theorem gives the existence of a uniform constant $c > 0$ such that, uniformly in each $k \geq 3$, $\mathbf{P}(Q_k) \geq c$, where

$$Q_k := \begin{array}{c}\text{there exist open circuits } \mathfrak{D}, \mathfrak{D} \text{ and a closed circuit } \mathcal{C} \\ \text{in } B(2^k) \setminus B(2^{k-1}) \text{ with } \mathfrak{D} \subseteq \mathring{\mathcal{C}} \text{ and } \mathcal{D} \subseteq \mathring{\mathfrak{D}}\end{array}.$$

Moreover, Q_k is independent of $\{Q\}_{=k}$. In particular, with probability one, the density of k for which Q_k occurs is positive.

Now, when Q_k occurs, it is easy to see that some $\mathcal{C}_i$ from the sequence constructed above must lie in the region between $\mathfrak{D}$ and $\mathfrak{D}$. In particular, there is a $c > 0$ such that, almost surely, the box $B(2^n)$ contains at least cn many $\mathcal{C}_i$, for all large n. The bound follows.

The next lemma controls the number of circuits from the above sequence which intersect fixed boxes. It is a combination of [7], Lemma 6, and the inequality in its proof (second paragraph in [7], p. 23).

LEMMA 2.4. *For $c \in (0, 1)$, $j \geq 1$ and $r, s \in \mathbb{Z}$, define*

$$\tau(r, s) = \tau(r, s; c, j) = r2^{c(j+1)}, (r+3)2^{c(j+1)} \times s2^{c(j+1)}, (s+3)2^{c(j+1)}$$

and $N(j, c)$ to be the number of squares $\tau(r, s)$ which intersect $B(2^{j+1})$ and intersect two successive circuits $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$ with $\operatorname{diam}(\mathcal{C}_k) \geq 2^j$. *Then for fixed $c \in (0, 1)$, there exists $C_3 > 0$ such that*

$$\mathbf{P}\, N(j, c) > j^2 \leq \frac{C_3}{j^2}. \tag{2.2}$$

In particular, for fixed $c \in (0, 1)$, almost surely,

$$N(j, c) \leq j^2 \quad \textit{for all large } j. \tag{2.3}$$

Moreover, if $N^{(3)}(j, c)$ is the number of squares $\tau(r, s)$ which intersect $B(2^{j+1})$ and intersect three successive circuits $\mathcal{C}_k, \mathcal{C}_{k+1}, \mathcal{C}_{k+2}$ with $\operatorname{diam}(\mathcal{C}_k) \geq 2^{j-(\log j)^2}$, *then there exists $C_4 > 0$ such that*

$$\mathbf{P}\, N^{(3)}(j, c) > 0 \leq \frac{C_4}{j^2}. \tag{2.4}$$

In particular, almost surely,

$$N^{(3)}(j, c) = 0 \quad \textit{for all large } j. \tag{2.5}$$

The following deterministic lemma is a simplified version of [7], Lemma 7, which is a consequence of the pigeonhole principle. For any sets of vertices S, S, we write $d(S, S)$ for $\min\{x - y : x \in S, y \in S\}$.

LEMMA 2.5. *Let $\mathcal{C}_{k+1}$ and $\mathcal{C}_{k+2}$ be two successive circuits from the sequence of Lemma 2.3. Assume that $2^j \leq \operatorname{diam}(\mathcal{C}_{k+1}) < 2^{j+1}$. Let $v_1^{(k+1)}, v_2^{(k+1)}, \ldots, v_M^{(k+1)}$ be M arbitrary vertices of $\mathcal{C}_{k+1}$ for which*

$$v_p^{(k+1)} - v_q^{(k+1)} \geq 8 \cdot 2^{cj}, \quad \textit{for all } p, q \textit{ with } p = q. \tag{2.6}$$

If $N(j, c) \leq j^2$, then at least $M - j^2$ of the vertices $v_m^{(k+1)}$ satisfy $d(v_m^{(k+1)}, \mathcal{C}_{k+2}) > 2^{cj}$.

We will also define "good circuits," which will allow us in Section 3 to construct an infinite path starting at zero whose intersection with those circuits have low weights. Let be a circuit surrounding the origin with $2^j \leq \mathrm{diam}(\quad 2^{j+1}$. For constants $c_1, c_3 \in (0, 1)$, we consider open connected sets $\mathcal{D}$ with some or all of the following properties:

(2.7) $$\mathcal{D} \subseteq {}^{\circ} \text{ and } \mathcal{D} \text{ contains exactly one vertex adjacent to } ;$$

(2.8) $$\mathrm{diam}(\mathcal{D}) \geq \mathrm{diam}(\quad {}^{c_1};)$$

(2.9) the open cluster of $\mathcal{D}$ in $^{\circ}$ contains at least $(\mathrm{diam}(\quad {}^{c_2}))$ vertices w_m which are adjacent to and satisfy $w_p - w_q \geq (\mathrm{diam}(\quad {}^{c_1 c_3}))$ for $p = q$, moreover, there exists a vertex $z \in \mathcal{D}$ and for each of the w_m an open path from z to w_m such that only its endpoint w_m is adjacent to .

Here, the open cluster of $\mathcal{D}$ in $^{\circ}$ is the largest open connected set in $^{\circ}$ that contains $\mathcal{D}$. We say a closed circuit is (c_1, c_2, c_3)-good if any open connected set $\mathcal{D}$ with (2.7) and (2.8) also satisfies (2.9). Note that if is (c_1, c_2, c_3)-good, then for any $\hat{c}_2 \leq c_2$, it is also $(c_1, \hat{c}_2, c_3)$-good. We will use this definition with $\mathcal{D}$ equal to the open cluster of an open segment in a geodesic from a coupled Bernoulli FPP model. Property (2.9) will allow us to reroute this segment to end at a vertex w_m which is adjacent to a low-weight vertex.

LEMMA 2.6. *For any $c_1, c_3 \in (0, 1)$ with c_1 sufficiently close to* 1, *we can choose $c_2 \in (0, 1)$, depending only on c_1 and c_3, such that*

(2.10) $$\mathbf{P}\ \exists \text{ a circuit } \mathcal{C}_k \text{ with } 2^j \leq \mathrm{diam}(\mathcal{C}_k) < 2^{j+1} \text{ which is not } (c_1, c_2, c_3)\text{-good} \leq C_5 e^{-c_5 j},$$

where C_5, c_5 depend only on c_1, c_3. Moreover, given such c_i's, we can choose $c_1', c_2', c_3' \in (0, 1)$ with $c_1' c_3' > c_2$ and $c_2' = c_2$ such that (2.10) *holds with c_i replaced by c_i'*, $i = 1, 2, 3$.

PROOF. The lemma follows directly from [7], Proposition 1, and for this reason we sketch that proposition in the Appendix. Here, we describe how to apply the proposition to prove the lemma. In [7], the authors define a circuit to be good if any open connected set $\mathcal{D}$ with (2.7) and (2.8) also satisfies the following (instead of (2.9)):

> the open cluster of $\mathcal{D}$ in $^{\circ}$ contains at least $(\mathrm{diam}(\quad {}^{c_2}))$ self-avoiding paths θ_m which are adjacent to , have length $\geq c_6 \log\log(\mathrm{diam}(\quad))$ (where $c_6 > 0$ is a constant) and satisfy $d(\theta_p, \theta_q) \geq (\mathrm{diam}(\quad {}^{c_1 c_3}))$ for $p = q$, moreover, there exists a vertex $z \in \mathcal{D}$ and for each of the θ_m an open path from z to θ_m such that only its endpoint on θ_m is adjacent to .

Since the above requires more than (2.9), by [7], Proposition 1, we obtain (2.10).

3. Construction of a low-weight path. Let $(\mathcal{C}_i)$ be fixed as $(\quad {}_i)$ for some sequence of circuits with ${}_i \subseteq {}^{\circ}{}_{i+1}$. We would like to construct a self-avoiding path from 0 such that (except for finitely many vertices) it contains only open vertices or some $v^{(k)} \in {}_k$ with $v^{(k)}$ being of low weight. For the definition of low weight, let $c_2 \in (0, 1)$ (which will be taken to be the same as the c_2 in (2.9)) and recall that $I = \inf\{x > 0 : F(x) > p_c\}$.

DEFINITION 3.1. Let $\mathcal{C}$ be a circuit such that $0 \in \mathcal{C}^\circ$ with $2^j \le \operatorname{diam}(\mathcal{C}) < 2^{j+1}$.

1. If $\mathbf{P}(t_v = I) > 0$ and $I > 0$, we say that a vertex $v \in \mathcal{C}$ is of (j-)low-weight if $t_v = I$ and we define $a_j = 0$ for all j.
2. If $\mathbf{P}(t_v = I) = 0$ or $I = 0$, we fix any nonincreasing sequence (a_j) such that $\mathbf{P}(I < t_v \le I + a_j) \ge 2^{-c_2 j/2-1}$ and $a_j \to 0$ as $j \to \infty$, and we say that a vertex $v \in \mathcal{C}$ is of (j-)low-weight if $I < t_v \le I + a_j$.

For the following statement, given a configuration of open/closed vertices, let $\bar{\mathbf{P}}$ be the (regular) conditional distribution of the variables (ω_x) given this configuration.

LEMMA 3.2. *Choose c_1, c_3 and c_1', c_3' in $(0,1)$ with corresponding $c_2 = c_2'$ as dictated by Lemma 2.6, and both $c \in (c_2', c_1' c_3')$ and $\hat{c} \in (0, c_1' c_3')$. Also fix $C_6, C_7, c_7 > 0$. There exists $c_8 > 0$ such that the following holds for all sufficiently large k. For a given configuration η of open/closed vertices, suppose $2^j \le \operatorname{diam}(\mathcal{C}_{k+1}) < 2^{j+1}$ and $v^{(k)} \in \mathcal{C}_k$ satisfies*

$$d\left(v^{(k)}, \mathcal{C}_{k+1}\right) \ge 2\left(\operatorname{diam}(\mathcal{C}_{k+1})\right)^{c_1}. \tag{3.1}$$

Assume that:

- *$N(i, c) \le i^2$, $N(i, \hat{c}) \le i^2$, and $N^{(3)}(i, c_1') = 0$ for all $i \ge j$,*
- *one has*

$$\operatorname{diam}(\mathcal{C}_{i+2}) \le (i+1)^{C_6} \operatorname{diam}(\mathcal{C}_{i+1}) \quad \text{for all } i \ge k, \tag{3.2}$$

- *one has*

$$c_7 i \le \log \operatorname{diam}(\mathcal{C}_{i+2}) \le C_7 i \quad \text{for all } i \ge k, \tag{3.3}$$

- *$\mathcal{C}_{i+1}$ and $\mathcal{C}_{i+2}$ are (c_1, c_2, c_3)- and (c_1', c_2', c_3')-good for all $i \ge k$*

With $\bar{\mathbf{P}}$-probability at least $1 - e^{-c_8 j}$, (conditioned on η) we can find sequences $(v^{(i)})_{i \ge k+1}$ and $(\mathcal{D}^{(i)})_{i \ge k}$ such that for all i:

1. *for $i \ge k$, $v^{(i)} \in \mathcal{C}_i$ and $\mathcal{D}^{(i)}$ is an open path connecting a neighbor of $v^{(i)}$ with a neighbor of $v^{(i+1)}$*;
2. *for $i \ge k+1$, $v^{(i)}$ is of low weight*; *and*
3. *for $i \ge k$, $\mathcal{D}^{(i)}$ contains only one vertex adjacent to $\mathcal{C}_{i+1}$.*

PROOF. The proof is similar to the construction of double paths in [7], Section 5. We will first construct $v^{(k+1)}$ and $\mathcal{D}^{(k)}$. Because $v^{(k)} \in \mathcal{C}_k$, there exists an open path $\hat{\mathcal{D}}^{(k)}$ from a neighbor of $v^{(k)}$ to a neighbor of $\mathcal{C}_{k+1}$ that only contains one vertex adjacent to $\mathcal{C}_{k+1}$. Due to (3.1), we have

$$\operatorname{diam}\left(\hat{\mathcal{D}}^{(k)}\right) \ge 2\left(\operatorname{diam}(\mathcal{C}_{k+1})\right)^{c_1} - 4 \ge \left(\operatorname{diam}(\mathcal{C}_{k+1})\right)^{c_1}.$$

Since $\mathcal{C}_{k+1}$ is (c_1', c_2', c_3')-good, there are at least $(\operatorname{diam}(\mathcal{C}_{k+1}))^{c_2'} \ge 2^{c_2' j}$ many vertices w_m in the open cluster of $\hat{\mathcal{D}}^{(k)}$ which are adjacent to $\mathcal{C}_{k+1}$ and satisfy $\|w_p - w_q\| \ge (\operatorname{diam}(\mathcal{C}_{k+1}))^{c_1' c_3'}$ if $p \ne q$. This implies (for k large) $\|w_p - w_q\| \ge 16(\operatorname{diam}(\mathcal{C}_{k+1}))^{c}$ if $p \ne q$. So if we choose $(v_m^{(k+1)})_m$ as vertices in $\mathcal{C}_{k+1}$ adjacent to the w_m's (in some deterministic and η-measurable way), then if $p \ne q$

$$\left\|v_p^{(k+1)} - v_q^{(k+1)}\right\| \ge 8\left(\operatorname{diam}(\mathcal{C}_{k+1})\right)^{c} \ge 8 \cdot 2^{cj}.$$

Since $N(c, j) \le j^2$, by Lemma 2.5, at least $2^{c_2' j} - j^2$ of the $v_m^{(k+1)}$'s satisfy $d(v_m^{(k+1)}, \mathcal{C}_{k+2}) > 2^{cj}$. We claim that with conditional probability at least $1 - e^{-c_9 j}$, more than j^2 of the

$v_m^{(k+1)}$'s have low weight. To see why, we use the Chernoff bound: letting $X_1, \dots, X_r$ be i.i.d. Bernoulli random variables with parameter $2^{-c_2 j/2}$, and $r = 2^{c_2 j}$, then

$$\begin{aligned}\bar{\mathbf{P}}\big(\text{at most } j^2 \text{ of the } t_{v_m^{(k+1)}}\text{'s have low weight}\big) &\le \mathbf{P}\big(X_1 + \cdots + X_r \le j^2\big)\\ &= \mathbf{P}\big(\exp\big(-(X_1 + \cdots + X_r)\big) \ge e^{-j^2}\big)\\ &\le e^{j^2}\big(\mathbf{E}\exp(-X_1)\big)^r\\ &= e^{j^2}\big(\big(1-2^{-c_2 j/2}\big) + e^{-1}2^{-c_2 j/2}\big)^{2^{c_2 j}}\\ &\le e^{-c_{10}2^{c_{11} j}}.\end{aligned}$$

(Here, we have assumed that Item 2 of Definition 3.1 holds; otherwise, the proof is even easier.) This shows the claim. When it holds (i.e., more than j^2 of the vertices have low weight), we say that "the first stage is successful". Note that given η, the outcome of the first stage depends only on the weights for vertices on ${}_{k+1}$.

Assuming that the first stage is successful, we can choose $v^{(k+1)} \in \{v_m^{(k+1)}\}$ such that both conditions hold:

$$d\big(v^{(k+1)}, {}_{k+2}\big) > 2^{cj} \quad \text{and} \quad v^{(k+1)} \text{ is of low-weight.}$$

This implies by (3.2) that for large k,

$$d\big(v^{(k+1)}, {}_{k+2}\big) \ge 2\big(\operatorname{diam}({}_{k+2})\big)^{c_1}. \tag{3.4}$$

Last, we define $\mathcal{D}^{(k)}$ by modifying $\hat{\mathcal{D}}^{(k)}$ so that it begins at a neighbor of $v^{(k)}$ and ends at a neighbor of $v^{(k+1)}$ (with only one vertex adjacent to ${}_{k+1}$).

Now we construct the further paths $\mathcal{D}^{(k+1)}, \mathcal{D}^{(k+2)}, \dots$ and $v^{(k+2)}, v^{(k+3)}, \dots$. Inequality (3.4) means we can find an open path $\hat{\mathcal{D}}^{(k+1)}$ connecting a neighbor of $v^{(k+1)}$ with a neighbor of ${}_{k+2}$, with only one vertex adjacent to ${}_{k+2}$, such that

$$\operatorname{diam}\big(\hat{\mathcal{D}}^{(k+1)}\big) \ge 2\big(\operatorname{diam}({}_{k+2})\big)^{c_1} - 4 \ge \big(\operatorname{diam}({}_{k+2})\big)^{c_1}.$$

Therefore we can repeat the argument leading to (3.4) with $\hat{\mathcal{D}}^{(k+1)}$ in place of $\mathcal{D}^{(k)}$, using now that ${}_{k+2}$ is (c_1, c_2, c_3)-good (and putting $\hat{c}$ in place of c), to construct an open path $\mathcal{D}^{(k+1)}$ from a neighbor of $v^{(k+1)}$ to a neighbor of some $v^{(k+2)} \in {}_{k+2}$ of low weight that has only one vertex adjacent to ${}_{k+2}$. Again, we will only be able to do this with conditional probability $\ge 1 - \exp\big(-c_9\big(\log_2 \operatorname{diam}({}_{k+2})\big)\big)$, given both η and the outcome of the first stage. (Note that conditioning on the outcome of the first stage only gives information about the weights on ${}_{k+1}$.) If we are able to find such $\hat{\mathcal{D}}^{(k+1)}$ and $v^{(k+2)}$, then we say that "the Stage 2a is successful".

Unfortunately the argument above only gives $d(v^{(k+2)}, {}_{k+3}) \ge 2(\operatorname{diam}({}_{k+3}))^{\alpha}$ for some $\alpha < c_1$ and this is not enough to iterate the argument (the estimate will continue to deteriorate at each further iteration). We now claim that we can choose $\mathcal{D}^{(k+1)}$ and $v^{(k+2)}$ such that

$$d\big(v^{(k+2)}, {}_{k+3}\big) \ge 2\big(\operatorname{diam}({}_{k+3})\big)^{c_1}. \tag{3.5}$$

To show (3.5), we argue as follows. If it so happens that $\operatorname{diam}(\mathcal{D}^{(k+1)}) \ge (\operatorname{diam}({}_{k+2}))^{c_1}$, then we repeat the argument leading to (3.4) with $\mathcal{D}^{(k+1)}$ in place of $\mathcal{D}^{(k)}$ (and the same value of c) to produce yet another open path $\mathcal{D}^{(k+1)}$ connecting a neighbor of $v^{(k+1)}$ with a neighbor of some $v^{(k+2)} \in {}_{k+2}$ of low weight, with only one vertex adjacent to ${}_{k+2}$, but this time we will have the estimate (3.5) using $\mathcal{D}^{(k+1)}$ in place of $\mathcal{D}^{(k+1)}$. The conditional probability that we can find such $\mathcal{D}^{(k+1)}$ and $v^{(k+2)}$ is again at least $1 - \exp\big(-c_9\big(\log_2 \operatorname{diam}({}_{k+2})\big)\big)$. (If we can find such a path and vertex, we say that “the Stage 2b is successful”.) Otherwise, we must have

$$\mathrm{diam}\ \mathcal{D}^{(k+1)} < \mathrm{diam}(\ _{k+2})^{c_1},$$

and so

$$v^{(k+1)} - v^{(k+2)} \le \mathrm{diam}\ \mathcal{D}^{(k+1)} + 4 < 2\ \mathrm{diam}(\ _{k+2})^{c_1}. \tag{3.6}$$

In this case, we always declare Stage 2b to be successful. If (3.5) fails, for large k, we use (3.2) to see that

$$d\ v^{(k+2)},\ _{k+3} < 2\ \mathrm{diam}(\ _{k+2})^{c_1}.$$

This, together with (3.6), implies that each of $_{k+1}$, $_{k+2}$, $_{k+3}$ must have a point in $v^{(k+2)} + B(2(\mathrm{diam}(\ _{k+2}))^{c_1})$. Letting i be such that $2^i \le \mathrm{diam}(\ _{k+2}) < 2^{i+1}$, there exist r, s so that

$$v^{(k+2)} \in r2^{c_1(i+1)}, (r+1)2^{c_1(i+1)} \times s2^{c_1(i+1)}, (s+1)2^{c_1(i+1)} .$$

Therefore, if q is chosen such that $c_1 q \ge 4$,

$$\begin{aligned} &\tau\ \ 2^{-c_1 q}(r-2)\ ,\ 2^{-c_1 q}(s-2)\ ; c_1, i+q \\ &\supseteq\ r2^{c_1(i+1)}, (r+1)2^{c_1(i+1)} \times s2^{c_1(i+1)}, (s+1)2^{c_1(i+1)} + B\ 2 \cdot 2^{c_1(i+1)} \end{aligned}$$

would intersect $B(2^{i+1}) \subseteq B(2^{i+q+1})$, as well as $_{k+1}$, $_{k+2}$, and $_{k+3}$. Since $\mathrm{diam}(\ _{k+3}) \ge \mathrm{diam}(\ _{k+2}) \ge 2^i$ and by (3.2) and (3.3), $\mathrm{diam}(\ _{k+1}) \ge (k+1)^{-C_6}\,\mathrm{diam}(\ _{k+2}) \ge 2^{i+q-(\log(i+q))^2}$ for large i, we would have $N^{(3)}(i+q, c_1) = 0$, but this is impossible by the hypothesis. Therefore (3.5) holds.

At this point, we have constructed $\mathcal{D}^{(k)}$, $\mathcal{D}^{(k+1)}$, $v^{(k+1)}$, and $v^{(k+2)}$. The conditional probability that Stages 1, 2a, 2b are all successful is at least

$$\begin{aligned} &\bar{\mathbf{P}}(\text{Stage 2b is successful} \mid \text{Stages 2a and 1 are successful}) \\ &\quad\times \bar{\mathbf{P}}(\text{Stage 2a is successful} \mid \text{Stage 1 is successful}) \\ &\quad\times \bar{\mathbf{P}}(\text{Stage 1 is successful}) \\ &\ge\ 1 - \exp\ -c_9\ \log_2 \mathrm{diam}(\ _{k+2})\ \ ^2\ 1 - e^{-c_9 j}\ . \end{aligned}$$

Now that we have constructed $v^{(k+2)}$ and $\mathcal{D}^{(k+1)}$ such that (3.5) holds, we can now repeat the argument we just gave that derived (3.5) from (3.4), but with $v^{(k+2)}$, $v^{(k+3)}$ in place of $v^{(k+1)}$, $v^{(k+2)}$ and $\mathcal{D}^{(k+1)}$, $\mathcal{D}^{(k+2)}$ in place of $\mathcal{D}^{(k)}$, $\mathcal{D}^{(k+1)}$. From this, we reproduce the estimate (3.5) with $v^{(k+3)}$, $_{k+4}$ in place of $v^{(k+2)}$, $_{k+3}$, so long as the corresponding steps 2a and 2b (which we will label 3a and 3b) are successful. The probability that these Stages 3a and 3b are both successful, conditioned on the success of Stages 1, 2a, and 2b, is at least

$$1 - \exp\ -c_9\ \log_2 \mathrm{diam}(\ _{k+3})\ \ ^2.$$

Continuing in this way, we produce all paths $\mathcal{D}^{(i)}$ and vertices $v^{(i)}$ with conditional probability at least

$$\begin{aligned} &1 - e^{-c_9 j} \prod_{i=k}^{\infty} 1 - \exp\ -c_9\ \log_2 \mathrm{diam}(\ _{i+2})\ \ ^2 \\ &\ge\ 1 - e^{-c_9 j} \prod_{i=k}^{\infty} 1 - \exp(-c_9 c_7 i/2)^2 \\ &\ge\ 1 - e^{-c_9 j}\ \ 1 - e^{-c_{12} k} \\ &\ge 1 - e^{-c_8 j}. \end{aligned}$$

Here, we have used (3.3) to go from the first to second line, and then (3.3) again, along with the inequality diam$(\ {}_{k+1}) \geq 2^j$, to go from the third to fourth line. This completes the proof.

REMARK 2. In the statement of Lemma 3.2, the vertex $v^{(k)}$ is assumed to be on ${}_k$ and to obey the distance bound (3.1). It is straightforward to check that these conditions may be replaced by the following: there is an open path starting at a neighbor of $v^{(k)}$ of diameter $(\text{diam}(\ {}_{k+1}))^{c_1}$ that contains only one vertex adjacent to ${}_{k+1}$. (Here $v^{(k)}$ is not assumed to be on ${}_k$.) In this case, the result holds with the same conditional probability bound: at least $1 - e^{-c_8 \log_2 \text{diam}(\ {}_{k+1})}$.

4. Universality of the time constant (asymptotic form). In this section, we prove (1.3). We define $(t_x^{\mathrm{B}})_{x \in \mathbb{Z}^2}$ to be a family of Bernoulli random variables coupled to the weights (t_x): we set $t_x^{\mathrm{B}} = I \cdot \mathbf{1}_{\{\omega_x > 1/2\}}$. We also write T^{B} for the first-passage time using the weights (t_x^{B}). By [14], Proposition 3.6,

$$\lim_{n \to \infty} \frac{T^{\mathrm{B}}(0, \partial B(n))}{\log n} = \frac{I}{2\sqrt{3}\pi} \quad \text{almost surely.} \tag{4.1}$$

We now construct an infinite path using Lemma 3.2, so choose c_1, c_3 and c_1', c_3' with corresponding $c_2 = c_2'$ as dictated by Lemma 2.6. Also fix $C_6, C_7, c_7 > 0$, $c \in (c_5, c_1' c_3')$, and $\hat{c} \in (0, c_1 c_3)$. For $j, k \geq 1$, let ${}_{j,k}$ be the set of η for which there exists $v^{(k)}$ such that the hypotheses of Lemma 3.2 hold for $\eta, j, k, v^{(k)}$. (In particular, $2^j \leq \text{diam}(\mathcal{C}_{k+1}) < 2^{j+1}$.) By the lemma, the probability that there exists an infinite path γ starting at some vertex $v^{(k)}$ of $\mathcal{C}_k$ (for k fixed and large) that is open except for its intersection with each $\mathcal{C}_i$ ($i \geq k$), which consists of a low-weight vertex, is at least

$$\begin{aligned} \sum_{j=1}^{\infty} \mathbf{P}(\ {}_{j,k}) \left(1 - e^{-c_8 j}\right) &\geq \sum_{j=c_4 k/2}^{\infty} \mathbf{P}(\ {}_{j,k}) \left(1 - e^{-c_8 j}\right) \\ &\geq \left(1 - e^{-c_8 c_4 k/2}\right) \sum_{j=c_4 k/2}^{\infty} \mathbf{P}(\ {}_{j,k}). \end{aligned} \tag{4.2}$$

By Lemma 2.4, almost surely, there exists a (random) integer j such that

$$\max\left\{N(i, c), N(i, \hat{c}), \hat{N}(i, c_1)\right\} \leq i^2 \quad \text{and} \quad N^{(3)}(i, c_1') = 0 \quad \text{for all } i \geq j.$$

Also, by a minor adaptation of [7] and the above Lemma 2.3, equation (5.25), the probability that (3.2) and (3.3) hold goes to one as $k \to \infty$, so long as C_6 and C_7 are large enough and c_7 is small enough. Last, by Lemma 2.6, the probability tends to one as $k \to \infty$ that for all $i \geq k$, $\mathcal{C}_{i+1}$ and $\mathcal{C}_{i+2}$ are (c_1, c_2, c_3)- and (c_1', c_2', c_3')-good. As for condition (3.1), if it fails for our k, then we must have $N(i, c_1') > i^2$ for some $i \geq (\log_2 k) - 1$. This has probability tending to zero as $k \to \infty$. These facts, in conjunction with the fact that $(\ {}_{j,k})_j$ is a disjoint family for each k, shows that $\mathbf{P}(\bigcup_j \ {}_{j,k}) \to 1$ as $k \to \infty$. Therefore, to show that (4.2) tends to 1 as $k \to \infty$, we must show that $\sum_{j=1}^{c_4 k/2} \mathbf{P}(\ {}_{j,k}) \to 0$ as $k \to \infty$. But by disjointedness again,

$$\sum_{j=1}^{c_4 k/2} \mathbf{P}(\ {}_{j,k}) \leq \mathbf{P}\left(\text{diam}(\mathcal{C}_{k+1}) \leq 2^{c_4 k/2+1}\right) \to 0 \quad \text{as } k \to \infty$$

by (2.1). We conclude that almost surely, there exists an infinite path γ as described above, starting at a vertex $v^{(k)}$ of $\mathcal{C}_k$ for some (random) k.

Fix a configuration ω for which γ exists, and write γ_n for the segment of γ starting at $v^{(k)}$ and ending at the first intersection of γ with $\partial B(n)$. (γ_n is set to be empty if $v^{(k)} \notin B(n)$.) Then

$$
\begin{aligned}
T^{\mathrm{B}}\big(0, \partial B(n)\big) &\le T\big(0, \partial B(n)\big) \le T\big(0, v^{(k)}\big) + T(\gamma_n) \\
&\le T\big(0, v^{(k)}\big) + \sum_{\ell = 1}^{\infty} \big(I + a(\ell)\big) \mathbf{1}_{\{\mathcal{C}_\ell \cap B(n) \neq \emptyset\}},
\end{aligned} \tag{4.3}
$$

where $a(\ell) = a_j$ if $2^j \le \operatorname{diam}(\mathcal{C}_\ell) < 2^{j+1}$. The sum on the right behaves like $I\#\{\ell : \mathcal{C}_\ell \cap B(n) \neq \emptyset\}$, which we will show is similar to $T^{\mathrm{B}}(0, \partial B(n))$. Rigorously, given $\varepsilon > 0$, since $a_j \le \varepsilon$ for j large, the above is bounded by

$$
T\big(0, v^{(k)}\big) + \sum_{\ell = 1}^{L} \big(I + a(\ell)\big) \mathbf{1}_{\{\mathcal{C}_\ell \cap B(n) \neq \emptyset\}} + (I + \varepsilon)\#\big\{\ell : \mathcal{C}_\ell \cap B(n) \neq \emptyset\big\},
$$

where L is some (random) finite number independent of n. We next use that $T^{\mathrm{B}}(0, \partial B(n))$ equals I times the maximal number of disjoint closed circuits surrounding 0 in $B(n)$ (see, for instance, [14], Proposition 2.4) to bound this above by

$$
T\big(0, v^{(k)}\big) + \sum_{\ell = 1}^{L} \big(I + a(\ell)\big) \mathbf{1}_{\{\mathcal{C}_\ell \cap B(n) \neq \emptyset\}} + (I + \varepsilon)\big(T^{\mathrm{B}}\big(0, \partial B(n)\big)/I + N_m\big), \tag{4.4}
$$

where N_m is the maximal number of disjoint closed circuits surrounding 0 that intersect $B(2^{m+1}) \setminus B(2^m)$, and $m = \lceil \log_2 n \rceil$. By the RSW theorem and the BK inequality (see [3], Ch. 11), one has

$$
\mathbf{P}(N_m \ge K) \le e^{-c_{13}K} \quad \text{for some } c_{13} > 0 \text{ and all } K \ge 1,
$$

and so by the Borel–Cantelli lemma, $N_m \le \log^2 m$ for all large m, almost surely. Returning to (4.4), for large n, we obtain the bound

$$
T\big(0, v^{(k)}\big) + \sum_{\ell = 1}^{L} \big(I + a(\ell)\big) \mathbf{1}_{\{\mathcal{C}_\ell \cap B(n) \neq \emptyset\}} + (I + \varepsilon)\big(T^{\mathrm{B}}\big(0, \partial B(n)\big)/I + \log^2(\lceil \log_2 n \rceil)\big).
$$

Combining this with (4.1) and (4.3), we obtain

$$
\frac{I}{2\sqrt{3}\pi} \le \liminf_{n\to\infty} \frac{T(0, \partial B(n))}{\log n} \le \limsup_{n\to\infty} \frac{T(0, \partial B(n))}{\log n} \le \frac{I + \varepsilon}{2\sqrt{3}\pi} \quad \text{almost surely.}
$$

As ε was arbitrary, this completes the proof.

5. Universality of limiting variance. In this section, we prove (1.4) under the moment assumption $\mathbf{E} \min\{t_1, \ldots, t_6\}^2 < \infty$. We will use the martingale introduced in [9]. Define, for $k \ge 0$,

$$
A(k) = B\big(2^{k+1}\big) \setminus B\big(2^k\big),
$$

and

$$
m(k) = \inf\big\{n \ge k : A(n) \text{ contains an open circuit surrounding } 0\big\},
$$

which is a.s. finite due to the RSW theorem. We also define

$$
\mathcal{O}_k = \text{innermost open circuit surrounding 0 in } A(m(k))
$$

for $k \ge 0$ and $\mathcal{O}_{-1} = \{0\}$. Finally, for $k \ge -1$, we define

$$
\mathcal{F}_k = \sigma\text{-field generated by } \big\{\{\mathcal{O}_k = \mathcal{C}\}, \{t_{x_1} \in A_1, \ldots, t_{x_n} \in A_n\}\big\}_{i, A_i, x}
$$

for a circuit surrounding 0 outside of $B(2^k)$, $x_i \in \bar{}$, and $A_i \subseteq \mathbb{R}$ Borel. (This is the sub-σ-field of our original σ-field $\mathcal{F}$ "generated by the weights on and inside $\mathcal{O}_k$," and $\bar{}$ refers to the union of and its interior $^\circ$.) We note that $\mathcal{O}$ and $T(0, \mathcal{O})$ are measurable with respect to $\mathcal{F}_k$ for $\le k$.

Since $\mathcal{O}_{k-1} \subseteq \overline{\mathcal{O}_k}$, we have $\mathcal{F}_{k-1} \subseteq \mathcal{F}_k$, and hence

$$T(0, \mathcal{O}_n) - \mathbf{E}T(0, \mathcal{O}_n) = \sum_{k=0}^{n} \mathbf{E}\, T(0, \mathcal{O}_n) | \mathcal{F}_k - \mathbf{E}\, T(0, \mathcal{O}_n) | \mathcal{F}_{k-1}$$
$$=: \sum_{k=0}^{n} {}_k.$$

Then

$$\operatorname{Var} T(0, \mathcal{O}_n) = \sum_{k=0}^{n} \mathbf{E}\ {}_k^2.$$

Write ${}_k^{\mathrm{B}}$ for the corresponding ${}_k$ with Bernoulli weights. (Here, as in the last section, we couple (t_x) with (t_x^{B}), a family of i.i.d. Bernoulli$(1/2)$ random variables, and write T^{B} for the passage time using (t_x^{B}).) We would like to compare $\operatorname{Var}(T(0, \mathcal{O}_n))$ with $\operatorname{Var}(T^{\mathrm{B}}(0, \mathcal{O}_n))$, and so we would like to bound $\mathbf{E}\ {}_k^2 - \mathbf{E}({}_k^{\mathrm{B}})^2$.

First we need another formula for ${}_k$. Let $(, \mathcal{F}, \mathbf{P})$ be another copy of the probability space $(, \mathcal{F}, \mathbf{P})$. Let $\mathbf{E}$ denote the expectation with respect to $\mathbf{P}$ and ω denote a sample point in . Define

$$n, \omega, \omega = mm(n, \omega) +, \omega .$$

Now, we have

$$_k(\omega) = T\mathcal{O}_{k-1}(\omega), \mathcal{O}_k(\omega)(\omega) + \mathbf{E}\, T\ \mathcal{O}_k(\omega), \mathcal{O}_{(k,\omega,\omega)}\ \omega\ \ \omega$$
$$- \mathbf{E}\, T\ \mathcal{O}_{k-1}(\omega), \mathcal{O}_{(k,\omega,\omega)}\ \omega\ \ \omega .$$

The above comes from the following facts: for $0 \le k \le n$,

- $T(0, \mathcal{O}_n) = T(0, \mathcal{O}_{k-1}) + T(\mathcal{O}_{k-1}, \mathcal{O}_k) + T(\mathcal{O}_k, \mathcal{O}_n)$,
- by (conditional) independence,

$$\mathbf{E}\, T(\mathcal{O}_k, \mathcal{O}_n) | \mathcal{F}_k = \mathbf{E}\, T\ \mathcal{O}_k, \mathcal{O}_n\ \omega\ \ \omega ,$$

- $T(\mathcal{O}_k(\omega), \mathcal{O}_n(\omega))(\omega) = T(\mathcal{O}_k(\omega), \mathcal{O}_{(k,\omega,\omega)}(\omega))(\omega) + T(\mathcal{O}_{(k,\omega,\omega)}(\omega), \mathcal{O}_n(\omega))(\omega)$.

See the proofs of [9], Lemmas 1, 2, for details, and the discussion around [9], equation (2.22), for the motivation of the random variable (k, ω, ω).

By the Cauchy–Schwarz inequality,

$$\operatorname{Var} T(0, \mathcal{O}_n) - \operatorname{Var} T^{\mathrm{B}}(0, \mathcal{O}_n) \le \sum_{k=0}^{n} \mathbf{E}\ {}_k^2 - \mathbf{E}\ {}_k^{\mathrm{B}\,2}$$
$$\le \sum_{k=0}^{n} \mathbf{E}\ {}_k - {}_k^{\mathrm{B}\ 2\ 1/2}\ \mathbf{E}\ {}_k + {}_k^{\mathrm{B}\ 2\ 1/2}.$$

Now,

$$\mathbf{E}\ {}_k + {}_k^{\mathrm{B}\ 2} \le 2\mathbf{E}\ {}_k^2 + 2\mathbf{E}\ {}_k^{\mathrm{B}\ 2} \le 2\ \sqrt{\mathbf{E}\ {}_k - {}_k^{\mathrm{B}\ 2}} + \sqrt{\mathbf{E}\ {}_k^{\mathrm{B}\ 2}}^{\ 2} + 2\mathbf{E}\ {}_k^{\mathrm{B}\ 2}$$
$$\le 4\mathbf{E}\ {}_k - {}_k^{\mathrm{B}\ 2} + 6\mathbf{E}\ {}_k^{\mathrm{B}\ 2},$$

so using $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for $a = 4\mathbf{E}(\Delta_k - \Delta_k^{\mathrm{B}})^2$ and $b = 6\mathbf{E}(\Delta_k^{\mathrm{B}})^2$, we obtain

$$\begin{aligned}\left|\operatorname{Var} T(0,\mathcal{O}_n) - \operatorname{Var} T^{\mathrm{B}}(0,\mathcal{O}_n)\right| &\le 2\sum_{k=0}^{n} \mathbf{E}\left(\Delta_k - \Delta_k^{\mathrm{B}}\right)^2 \\ &\quad + \sqrt{6}\,\sup_k \sqrt{\mathbf{E}\left(\Delta_k^{\mathrm{B}}\right)^2} \sum_{k=0}^{n} \left(\mathbf{E}\left(\Delta_k - \Delta_k^{\mathrm{B}}\right)^2\right)^{1/2}.\end{aligned}$$

Due to [2], Lemma 5.5, $\sup_k \mathbf{E}(\Delta_k^{\mathrm{B}})^2 < \infty$, and so

$$\left|\operatorname{Var} T(0,\mathcal{O}_n) - \operatorname{Var} T^{\mathrm{B}}(0,\mathcal{O}_n)\right| \le C_8 \sum_{k=0}^{n} \left[\left(\mathbf{E}\left(\Delta_k - \Delta_k^{\mathrm{B}}\right)^2\right)^{1/2} + \mathbf{E}\left(\Delta_k - \Delta_k^{\mathrm{B}}\right)^2\right]. \tag{5.1}$$

To bound $\mathbf{E}(\Delta_k - \Delta_k^{\mathrm{B}})^2$, we write

$$\begin{aligned} X &= X(k) = T\big(\mathcal{O}_{k-1}(\omega),\mathcal{O}_k(\omega)\big)(\omega) - T^{\mathrm{B}}\big(\mathcal{O}_{k-1}(\omega),\mathcal{O}_k(\omega)\big)(\omega), \\ Y &= Y(k) = T\big(\mathcal{O}_k(\omega),\mathcal{O}_{(k,\omega,\omega')}[\omega']\big)(\omega) - T^{\mathrm{B}}\big(\mathcal{O}_k(\omega),\mathcal{O}_{(k,\omega,\omega')}[\omega']\big)(\omega),\end{aligned}$$

and

$$Z = Z(k) = T\big(\mathcal{O}_{k-1}(\omega),\mathcal{O}_{(k,\omega,\omega')}[\omega']\big)(\omega) - T^{\mathrm{B}}\big(\mathcal{O}_{k-1}(\omega),\mathcal{O}_{(k,\omega,\omega')}[\omega']\big)(\omega).$$

Then by the Cauchy–Schwarz and Jensen inequalities,

$$\mathbf{E}\left(\Delta_k - \Delta_k^{\mathrm{B}}\right)^2 = \mathbf{E}\left(X + \mathbf{E}'Y - \mathbf{E}'Z\right)^2 \le 9\max\left\{\mathbf{E}X^2, \mathbf{E}\mathbf{E}'Y^2, \mathbf{E}\mathbf{E}'Z^2\right\}. \tag{5.2}$$

We will only bound $\mathbf{E}\mathbf{E}'Y^2$, as bounding $\mathbf{E}\mathbf{E}'Z^2$ is similar and bounding $\mathbf{E}X^2$ is even easier.

We first give an alternate representation for $\mathbf{E}\mathbf{E}'Y^2$ which only depends on ω.

LEMMA 5.1. *One has*

$$\mathbf{E}\mathbf{E}'Y^2 = \mathbf{E}\tilde{Y}^2, \tag{5.3}$$

where $\tilde{Y} = T(\mathcal{O}_k(\omega),\mathcal{O}_{(k,\omega,\omega)}(\omega))(\omega) - T^{\mathrm{B}}(\mathcal{O}_k(\omega),\mathcal{O}_{(k,\omega,\omega)}(\omega))(\omega)$

PROOF. To show this, it suffices to show that

$$\left(T_Y, T_Y^{\mathrm{B}}\right) = \left(T_{\tilde{Y}}, T_{\tilde{Y}}^{\mathrm{B}}\right) \text{ in distribution},$$

where

$$\begin{aligned} T_Y &= T\big(\mathcal{O}_k(\omega),\mathcal{O}_{(k,\omega,\omega')}[\omega']\big)(\omega), \\ T_{\tilde{Y}} &= T\big(\mathcal{O}_k(\omega),\mathcal{O}_{(k,\omega,\omega)}(\omega)\big)(\omega)\end{aligned}$$

and T_Y^{B}, $T_{\tilde{Y}}^{\mathrm{B}}$ are defined analogously using the Bernoulli variables (t_x^{B}). To prove this, note that for any Borel set $E \subseteq \mathbb{R}^2$,

$$\begin{aligned} &\mathbf{P}\times\mathbf{P}'\left(\left(T_Y, T_Y^{\mathrm{B}}\right) \in E\right) \\ &\quad = \sum_{\ell \ge k} \sum_{\Lambda \subseteq A(\ell)} \mathbf{P}\times\mathbf{P}'\left(\left(T_Y, T_Y^{\mathrm{B}}\right) \in E, m(k,\omega) = \ell, \mathcal{Q}_k = \Lambda\right).\end{aligned}$$

The summand equals

$$\begin{aligned} &\mathbf{P}\times\mathbf{P}'\left(\left(T\left(\Lambda,\mathcal{O}_{m(\ell+1,\omega')}[\omega']\right)(\omega), T^{\mathrm{B}}\left(\Lambda,\mathcal{O}_{m(\ell+1,\omega')}[\omega']\right)(\omega)\right) \in E, m(k,\omega) = \ell, \mathcal{Q}_k = \Lambda\right) \\ &\quad = \mathbf{P}\left(\left(T\left(\Lambda,\mathcal{O}_{m(\ell+1,\omega')}[\omega']\right)(\omega), T^{\mathrm{B}}\left(\Lambda,\mathcal{O}_{m(\ell+1,\omega')}[\omega']\right)(\omega)\right) \in E\right) \\ &\qquad \times \mathbf{P}\left(m(k,\omega) = \ell, \mathcal{Q}_k = \Lambda\right) \\ &\quad = \mathbf{P}\left(\left(T\left(\Lambda,\mathcal{O}_{m(\ell+1,\omega)}(\omega)\right)(\omega), T^{\mathrm{B}}\left(\Lambda,\mathcal{O}_{m(\ell+1,\omega)}(\omega)\right)(\omega)\right) \in E\right)\mathbf{P}\left(m(k,\omega) = \ell, \mathcal{Q}_k = \Lambda\right).\end{aligned}$$

Note that the event $\{m(k,\omega) = \mathcal{O}_k = \}$ depends only on variables associated to vertices in $\overline{}$. (For more detail, see the discussion above [9], equation (1.20).) So we can regroup the probabilities by independence, and reverse the steps to obtain $\mathbf{P}((T_{\tilde{Y}}, T^{\mathrm{B}}_{\tilde{Y}}) \in E)$. This shows (5.3).

We will also need some preliminary bounds on moments of $\tilde{Y}$ and $_k$.

LEMMA 5.2. *One has* $\mathbf{E}\tilde{Y}^2 < \infty$ *for all* k. *Also, there exists* $C_9 < \infty$ *and* k_0 *such that if* $k \ge k_0$, *then* $\mathbf{E}\tilde{Y}^4 \le C_9$. *The same bounds hold for* $_k$: *one has* $\mathbf{E}\ _k^2 < \infty$ *for all* k *and* $\mathbf{E}\ _k^4 \le C_9$ *for all* $k \ge k_0$.

PROOF. The proofs for $_k$ and $\tilde{Y}$ are very similar, so we show the case of $\tilde{Y}$. The first step is to (nonoptimally) bound the pth moment of annulus passage times. Let m, n be nonnegative integers with $m \le n$. We will show that for any $p > 0$, there exist $C_{10}, C_{11} > 0$ and m_0 such that

$$\mathbf{E}T\big(B(m), \partial B(n)\big)^p \le C_{10}\left(\log\frac{n}{m}\right)^{C_{11}} \quad \text{if } n \ge m \ge m_0. \tag{5.4}$$

To do this, we take R to be a large fixed integer, and construct R disjoint sectors as follows. Define the first sector S_1 to be the open region of $\mathbb{R}^2$ whose boundary consists of the circle of radius m centered at 0, the circle of radius $2n$ centered at zero, the positive e_1-axis, and the ray started at 0 (in the first quadrant) with angle π/R with the positive e_1-axis. S_i for $i = 2, \ldots, R$ is the rotation of S_1 by the angle $\pi(i-1)/R$. (We choose the constant m_0 to prevent the sectors from being exiguous.) For $i = 1, \ldots, R$, let π_i be a path connecting the inner boundary of S_i to the outer boundary with the minimal number N_i of nonzero-weight vertices. (See Figure 2.) Then since the variables $T(\pi_i)$ are independent, for $\in (0, 1/6)$ (so that a vertex-weight t_v satisfies $\mathbf{E}t_v < \infty$),

$$\begin{aligned}
\mathbf{E}T\big(B(m), \partial B(n)\big)^p &\le \mathbf{E}\min\{T(\pi_1), \ldots, T(\pi_R)\}^p \\
&\le 1 + \int_1^\infty p y^{p-1} \max_i \mathbf{P}\big(T(\pi_i) \ge y\big)^R \, \mathrm{d}y \\
&\le 1 + \int_1^\infty p y^{p-1} \left(\frac{\max_i \mathbf{E}T(\pi_i)}{y}\right)^R \mathrm{d}y \\
&= 1 + \left(\max_i \mathbf{E}T(\pi_i)\right)^R \int_1^\infty p y^{p-1-R} \, \mathrm{d}y \\
&\le C_{12}\left(1 + \max_i \mathbf{E}T(\pi_i)\right)^R,
\end{aligned} \tag{5.5}$$

so long as $R > p/$. The inner expected value is computed by introducing $K = \log^3(n/m)$ and writing

$$\mathbf{E}T(\pi_i) = \mathbf{E}T(\pi_i)\mathbf{1}_{\{N \le K\}} + \sum_{j=1}^\infty \mathbf{E}T(\pi_i)\mathbf{1}_{\{N \in (jK,(j+1)K]\}}.$$

By conditioning on the sigma-algebra generated by the family (t^{B}_x), we can bound $\mathbf{E}T(\pi_i)\mathbf{1}_{\{N \in (a,b]\}} \le 2b\mathbf{E}t\ \mathbf{P}(N_i \ge a)$ for any integers a, b with $0 \le a \le b$. Here, $\mathbf{E}t$ is the th moment of a vertex weight. Applying this in the above, we obtain

$$\mathbf{E}T(\pi_i) \le 2K\mathbf{E}t\left(1 + \sum_{j=1}^\infty (j+1)\mathbf{P}(N_i \ge jK)\right). \tag{5.6}$$

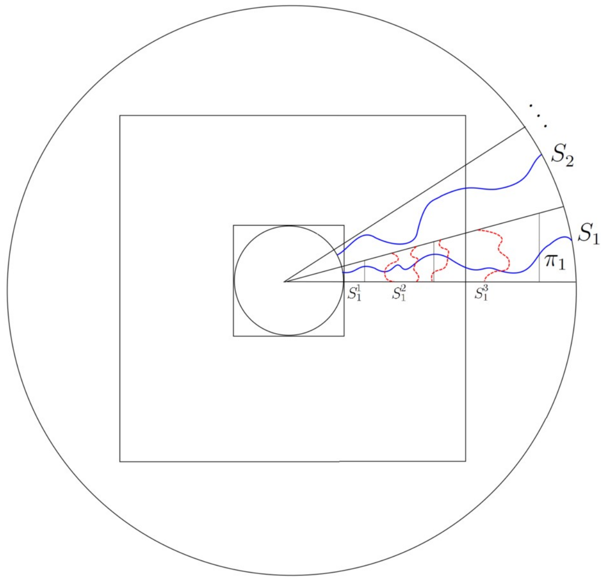

FIG. 2. *The blue solid lines are the π_i's, paths connecting the inner boundary of S_i to the outer boundary with the minimal number N_i of nonzero-weight vertices. On S_1 (say), for each nonzero-weight vertex that π_1 passes through, by minimality and planar duality, there must be a closed path containing that vertex and connecting the bottom of S_1 to the top of S_1. These closed paths are the red dotted curves in the figure.*

Now to bound $\mathbf{P}(N_i \geq jK)$, we note that if $N_i \geq jK$, then by planar duality, there must be at least jK many disjoint paths connecting the side boundaries of S_i to each other, and consisting only of nonzero-weight vertices. Splitting S_i into at most $\lceil \log_2(n/m) \rceil$ many sets of the form $S_i \cap [B(2^k) \setminus B(2^{k-1})]$ and writing these sets as $\{S_i^r\}_r$, we see that at least $\lfloor jK/\lceil \log_2(n/m) \rceil \rfloor$ many of these paths must intersect some S_i^r. By the RSW theorem, the probability that there exists at least one such path is bounded above by $1 - c_{14}$ for some c_{14} positive, uniformly in r, m, n. By the BK inequality, we therefore obtain

$$\begin{aligned}
\mathbf{P}(N_i \geq jK) &\leq \sum_r \mathbf{P}\big(\text{at least } \lfloor jK/\lceil \log_2(n/m) \rceil \rfloor \text{ such paths intersect } S_i^r\big) \\
&\leq \lceil \log_2(n/m) \rceil (1 - c_{14})^{\lfloor jK/\lceil \log_2(n/m) \rceil \rfloor} \\
&\leq C_{13} e^{-c_{15} j}.
\end{aligned}$$

We plug this estimate back into (5.6) to obtain

$$\mathbf{E}T(\pi_i) \leq C_{14} \log^3(n/m),$$

and then back in (5.5) to obtain

$$\mathbf{E}T\big(B(m), \partial B(n)\big)^p \leq C_{15} \log^{3R}(n/m).$$

This shows (5.4).

The next step is to extend (5.4) in the case when $p = 2$ to all m, n with $n \geq m$. We claim that there are numbers C_{16}, C_{17}, C_{18} such that

$$\mathbf{E}T\big(B(m), \partial B(n)\big)^2 \leq C_{16} + C_{17}\left(\log \frac{n}{m}\right)^{C_{18}} \quad \text{if } n \geq m \geq 0. \tag{5.7}$$

To do this, we note that if $n \geq m \geq m_0$ (where m_0 is from (5.4)), then the inequality follows from (5.4). Otherwise $m < m_0$ and either $n \leq m_0 + 100$, say, or $n > m_0 + 100$. In the first case, we upper bound

$$\mathbf{E}T\big(B(m), \partial B(n)\big)^2 \leq \mathbf{E}T\big(0, \partial B(m_0 + 100)\big)^2. \tag{5.8}$$

In the second case, we let π be a geodesic between $B(m_0)$ and $\partial B(n)$ (chosen in some deterministic way) and π' be the portion of π from its initial point to its first intersection with $\partial B(m_0+100)$. Then

$$(5.9)\quad \begin{aligned} \mathbf{E}T\big(B(m),\partial B(n)\big)^2 &\le 2\mathbf{E}T\big(0,\pi'\big)^2+2\mathbf{E}T\big(B(m_0),\partial B(n)\big)^2 \\ &\le 2\sum_P \mathbf{E}T\big(0,P\big)^2+2\mathbf{E}T\big(B(m_0),\partial B(n)\big)^2, \end{aligned}$$

where the sum is over all paths P that start at $\partial B(m_0)$ and end at $\partial B(m_0+100)$.

We bound the terms in (5.8) and (5.9) similarly. For example, for (5.9), a moment's reflection shows that one can construct six deterministic paths $\rho_1,\ldots,\rho_6$ that start at 0 and end at P, and are vertex disjoint except for their initial points. The upper bound we obtain is then (using (5.4))

$$\mathbf{E}T\big(B(m),\partial B(n)\big)^2 \le 2\sum_P \mathbf{E}\min\{T(\rho_1),\ldots,T(\rho_6)\}^2 + C_{17}\Big(\log\frac{n}{m}\Big)^{C_{18}}.$$

The argument of [1], Lemma 3.1, shows that since we have assumed $\mathbf{E}\min\{t_1,\ldots,t_6\}^2<\infty$, then the first term on the right is bounded by a constant. The bound on (5.8) using this method is just a constant, so this implies (5.7).

We now move to showing that for some k_0,

$$(5.10)\qquad \mathbf{E}\tilde{Y}_k^4 \le C_9 \quad \text{for all } k\ge k_0.$$

Given (5.4), the claimed inequality will follow forthwith. Indeed, since $T^{\mathrm{B}}\le T$, we write the left side as

$$(5.11)\quad \begin{aligned} &\mathbf{E}\Big[T\big(\mathcal{O}_k(\omega),\mathcal{O}_{(k,\omega,\omega)}(\omega)\big)(\omega)-T^{\mathrm{B}}\big(\mathcal{O}_k(\omega),\mathcal{O}_{(k,\omega,\omega)}(\omega)\big)(\omega)\Big]^4 \\ &\quad\le \mathbf{E}\Big[T\big(\mathcal{O}_k(\omega),\mathcal{O}_{(k,\omega,\omega)}(\omega)\big)(\omega)\Big]^4 \\ &\quad= \sum_{r=k}^{\infty}\sum_{s=r+1}^{\infty}\mathbf{E}\Big[T\big(\mathcal{O}_k(\omega),\mathcal{O}_{(k,\omega,\omega)}(\omega)\big)(\omega)\Big]^4\mathbf{1}_{\{m(k)=r,(k,\omega,\omega)=s\}} \\ &\quad\le \sum_{r=k}^{\infty}\sum_{s=r+1}^{\infty}\mathbf{E}T\big(B\big(2^r\big),\partial B\big(2^{s+1}\big)\big)^4\mathbf{1}_{\{m(k)=r,(k,\omega,\omega)=s\}} \end{aligned}$$

By the Cauchy–Schwarz inequality, we obtain the upper bound

$$\sum_{r=k}^{\infty}\sum_{s=r+1}^{\infty}\sqrt{\mathbf{E}T\big(B\big(2^r\big),\partial B\big(2^{s+1}\big)\big)^8\mathbf{P}\big(m(k)=r,(k,\omega,\omega)=s\big)}.$$

Assuming that $k\ge k_0$, where k_0 is large enough so that $B(2^{k_0})\supseteq B(m_0)$ (here m_0 is from (5.4) with $p=8$), we can use (5.4) to further bound this by

$$(5.12)\quad \begin{aligned} &C_{18}\sum_{r=k}^{\infty}\sum_{s=r+1}^{\infty}(s+1-r)^{C_{19}}\sqrt{\mathbf{P}\big(m(k)=r,(k,\omega,\omega)=s\big)} \\ &\quad\le C_{19}\sum_{r=k}^{\infty}e^{-c_{16}(r-k)}\sum_{s=r+1}^{\infty}(s+1-r)^{C_{20}}e^{-c_{17}(s-r)} \\ &\quad\le C_{21}\sum_{r=k}^{\infty}e^{-c_{18}(r-k)} \\ &\quad\le C_{22}. \end{aligned}$$

In the second line, we have used the RSW theorem. This proves (5.10).

Last, to show $\mathbf{E}\tilde{Y}^2 < \infty$ for all k, due to (5.10), we need only consider the case when $k < k_0$. Then we move to (5.11) with an exponent 2 instead of 4:

$$\mathbf{E}\tilde{Y}^2 \le \sum_{r=k}^{\infty}\sum_{s=r+1}^{\infty} \mathbf{E}T\big(B(2^r), \partial B(2^{s+1})\big)^2 \mathbf{1}_{\{m(k)=r, (k,\omega,\omega)=s\}}$$

The proof from here follows similar lines to that of the above, so we only briefly indicate the idea. For values of r, s such that $r \le k_0$ and $s \le k_0 + 100$, we upper bound by removing the indicator and summing over these (finitely many) values with the bound (5.7) to obtain a finite number. For values of r, s which are $> k_0$, we apply the Cauchy–Schwarz inequality to obtain a sum over such r, s of

$$\sqrt{\mathbf{E}T\big(B(2^r), \partial B(2^{s+1})\big)^4 \mathbf{P}\big(m(k) = r, (k, \omega, \omega) = s\big)}$$

and sum this as in the case of bounding $\mathbf{E}\tilde{Y}^4$. Last, if $r \le k_0$ but $s > k_0 + 100$, we bound $T(B(2^r), \partial B(2^{s+1}))$ above by the sum of $T(0, \tilde{\pi})$ and $T(B(2^{k_0}), \partial B(2^{s+1}))$ (where $\tilde{\pi}$ is a geodesic connecting $B(2^{k_0})$ to $\partial B(2^{k_0+100})$ chosen analogously to that above (5.9)), and note that the first term has finite second moment. The second term is bounded as in the case where $r, s > k_0$. Combining the cases will produce the final inequality, $\mathbf{E}\tilde{Y}^2 < \infty$.

To bound $\mathbf{E}\tilde{Y}^2$ from (5.3) more tightly, we introduce two events. Choose c_1, c_3 and c_1', c_3' with corresponding $c_2 = c_2'$ as dictated by Lemma 2.6. Also fix $c \in (c_1, c_1' c_3')$, and $\hat{c} \in (0, c_1 c_3)$. Write $\tilde{\mathcal{C}}_k$ for the first circuit in the sequence in Lemma 2.3 with $\mathcal{O}_k$ in the interior of $\tilde{\mathcal{C}}_k$, and let

$$F_k = \big\{\exists v \in \mathcal{O}_k \text{ such that } d(v, \tilde{\mathcal{C}}_k) < 2\, \mathrm{diam}(\tilde{\mathcal{C}}_k)^{c_1}\big\}.$$

Also, for $C_6, C_7, c_7 > 0$, let G_k be the event that at least one of the following fails:

- each circuit from the sequence in Lemma 2.3 with diameter at least 2^k is (c_1, c_2, c_3)-good and (c_1', c_2', c_3')-good,
- $N(j, c) \le j^2$, $N(j, \hat{c}) \le j^2$, and $N^{(3)}(j, c_3') = 0$ for all $j \ge k$,
- $\mathrm{diam}(\mathcal{C}_{i+2}) \le (i+1)^{C_6}\, \mathrm{diam}(\mathcal{C}_{i+1})$ and $c_7 i \le \log(\mathrm{diam}(\mathcal{C}_{i+2})) \le C_7 i$ for all i such that $\mathcal{C}_i$ has diameter at least 2^k.

Then,

$$\mathbf{E}\mathbf{E}_{\mathcal{Y}}\, Y^2 = \mathbf{E}\tilde{Y}^2 = \mathbf{E}\tilde{Y}^2\mathbf{1}_{F_k^c \cap G_k^c} + \mathbf{E}\tilde{Y}^2\mathbf{1}_{F_k \cup G_k}. \tag{5.13}$$

LEMMA 5.3. *For C_6 and C_7 chosen to be large enough and c_7 chosen to be small enough, there exist $C_{23}, C_{24}, c_{19} > 0$ such that*

$$\mathbf{P}(G_k) \le C_{23}/k$$

and

$$\mathbf{P}(F_k) \le C_{24}e^{-c_{19}k}. \tag{5.14}$$

PROOF. The bound on $\mathbf{P}(G_k)$ follows from Lemma 2.6, (2.2), (2.4), and the fact that for all $k \ge 1$,

$$\mathbf{P}\big(\log \mathrm{diam}(\mathcal{C}_{j+2}) < c_7 j \text{ or } \ge C_7 j \text{ for some } j \ge k\big) \le \frac{C_{25}}{k}, \tag{5.15}$$

and

$$\mathbf{P}\big(\mathrm{diam}(\mathcal{C}_{j+2}) > (j+1)^{C_6}\, \mathrm{diam}(\mathcal{C}_{j+1}) \text{ for some } j \ge k\big) \le \frac{C_{26}}{k}. \tag{5.16}$$

Here C_6, C_7 are chosen large enough and c_7 is chosen small enough. One can show both (5.15) and (5.16) hold for all k by following the proof of [7], Lemma 4. We omit the details here.

We now move to the proof of the second statement, the bound on $\mathbf{P}(F_k)$. In $A(r)$ $(r \geq 0)$, let $\hat{\mathcal{O}}_r$ be the innermost open circuit (if there exists one) and let $\hat{\mathcal{C}}_r$ be the first circuit in the sequence in Lemma 2.3 with $\hat{\mathcal{O}}_r$ in the interior of $\hat{\mathcal{C}}_r$. Because $\mathcal{O}_k \subseteq A(m(k))$ the probability $\mathbf{P}(F_k)$ equals

$$\sum_{=k}^{\infty} \mathbf{P}\ \exists v \in \mathcal{O}_k \text{ with } d(v, \tilde{\mathcal{C}}_k) < 2 \text{ diam}(\tilde{\mathcal{C}}_k)^{c_1} \mid m(k) = \ \mathbf{P}\ m(k) =$$

$$(5.17) \qquad \leq \sum_{=k}^{\infty} \mathbf{P}\ \exists v \in \hat{\mathcal{O}} \text{ with } d(v, \hat{\mathcal{C}}) < 2 \text{ diam}(\hat{\mathcal{C}})^{c_1} \mid$$

$$\exists \text{ open circuit around 0 in } A() \quad e^{-c_{20}(-k)}.$$

To change the conditioning above, we used independence of the site variables in disjoint annuli, and to bound $\mathbf{P}(m(k) =)$, we used the RSW theorem (see [9], equation (2.28)).

To bound (5.17), we first show that for any choice of $\nu \in (c_1, 1)$, one has $2(\text{diam}(\hat{\mathcal{C}}))^{c_1} < 2^{\nu}$ with high probability. To see this, fix $\alpha > 1$ and consider $\mathbf{P}(\text{diam}(\hat{\mathcal{C}}) \geq 2^{\alpha})$. For $r \geq 0$, define E_r to be the event that:

- there exists a closed circuit surrounding 0 in $A(r)$, and
- there exists an open circuit surrounding the above closed circuit in $A(r)$.

By the RSW theorem, there exists $\kappa > 0$ such that $\mathbf{P}(E_r) > \kappa$ for all $r \geq 0$. Because there is no closed circuit surrounding 0 strictly between $\hat{\mathcal{O}}$ and $\hat{\mathcal{C}}$, the event $\{\text{diam}(\hat{\mathcal{C}}) \leq 2^{d}\}$ contains E_d. Therefore, by independence,

$$(5.18) \qquad \mathbf{P}\ \text{diam}(\hat{\mathcal{C}}) \geq 2^{\alpha} \mid \exists \text{open circuit around 0 in } A() \quad \leq \mathbf{P} \bigcap_{k=1}^{p()} E^{c}_{+k} \leq e^{-\beta p()}$$

for some $\beta > 0$ depending only on α, and $p()$ is the integer such that

$$\text{diam } B\ 2^{+p()} \leq 2^{\alpha} < \text{diam } B\ 2^{+p()+1}.$$

Solving for $p()$, we see that $p() > (\alpha - 1) - 5/2$. Thus

$$\mathbf{P}\ \text{diam}(\hat{\mathcal{C}}) \geq 2^{\alpha} \leq e^{-\beta}$$

for some $\beta > 0$ depending only on α.

In particular, for any $\nu \in (c_1, 1)$, we can apply (5.18) with $\alpha = \frac{1+\nu/c_1}{2}$ to obtain

$$\mathbf{P}\ \exists v \in \hat{\mathcal{O}} \text{ such that } d(v, \hat{\mathcal{C}}) < 2 \text{ diam}(\hat{\mathcal{C}})^{c_1} \mid \exists \text{open circuit around 0 in } A()$$

$$(5.19) \qquad \leq \mathbf{P}\ \exists v \in \hat{\mathcal{O}} \text{ such that } d(v, \hat{\mathcal{C}}) < 2^{\nu} \mid \exists \text{open circuit around 0 in } A() \quad + e^{-\beta}$$

$$\leq C_{27} \mathbf{P}\ \exists v \in \hat{\mathcal{O}} \text{ such that } d(v, \hat{\mathcal{C}}) < 2^{\nu} + e^{-\beta}$$

for some $\beta > 0$ depending on ν and c_1. Here, we have used the RSW theorem to remove the conditioning.

We now cover $A()$ by the squares $\tau(r, s)$ from Lemma 2.4. These were defined as

$$\tau(r, s) = \tau(r, s; \nu,) = r 2^{\nu(+1)}, (r+3)2^{\nu(+1)} \times s 2^{\nu(+1)}, (s+3)2^{\nu(+1)},$$

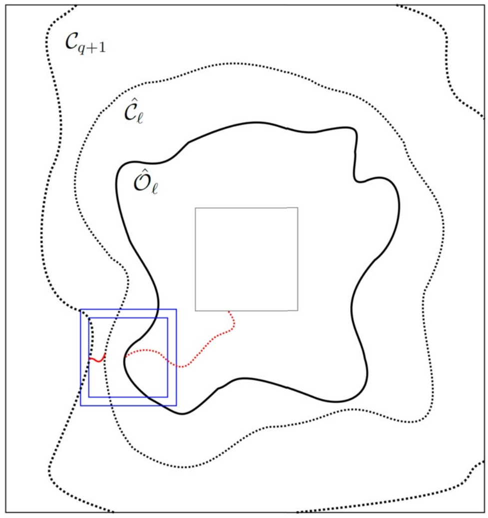

FIG. 3. *If $\hat{O}$ is too close to $\hat{C} = C_q$, then locally there is a 6-arm event (see the blue annulus). Four of the six arms are furnished by the circuits $\hat{O}$ and $\hat{C}$ themselves, and the other two, drawn in red, exist because $\hat{O}$ is innermost and $\hat{C}$ is outermost in the interior of C_{q+1}. If C_{q+1} is also close to $\hat{C}$, there is even a 7-arm event.*

where ν (as above) is any number in $(c_1, 1)$. These squares are defined so that any $v \in A(\ell)$ is in the central square $\tau(r, s)$ of sidelength $2^{\nu(\ell+1)}$ of some $\tau(r, s)$.

To bound the probability in (5.19), we need the notion of arm events. For any $j \geq 1$, any sequence $\sigma = (\sigma_1, \ldots, \sigma_j)$, where $\sigma_i \in \{\text{open}, \text{closed}\}$, and any positive integers n, N with $n \leq N$, we define the j-arm event, $A_{j,\sigma}(n, N)$ to be the event that there exist j disjoint paths from $B(n)$ to $\partial B(N)$, and the ith path has occupation status σ_i (either open or closed), taken in counterclockwise order. Similarly, we define $A^{1/2}_{j,\sigma}(n, N)$ and $A^{3/4}_{j,\sigma}(n, N)$ in the same way, but further restrict the paths to lie entirely in the upper half plane and in the union of the first three quadrants respectively. It is known (see, for instance, [10] or [11]) that if the σ_i's are not all open or all closed (this is irrelevant except in the full plane),

$$
\begin{aligned}
\mathbf{P}\big(A_{j,\sigma}(n, N)\big) &= \left(\frac{N}{n}\right)^{-(j^2-1)/12+o(1)}, \\
\mathbf{P}\big(A^{1/2}_{j,\sigma}(n, N)\big) &= \left(\frac{N}{n}\right)^{-j(j+1)/6+o(1)} \quad \text{as } N/n \to \infty.
\end{aligned}
\tag{5.20}
$$

Further, by conformal invariance of limiting crossing probabilities, one can also deduce that

$$
\mathbf{P}\big(A^{3/4}_{j,\sigma}(n, N)\big) = \left(\frac{N}{n}\right)^{-j(j+1)/9+o(1)} \quad \text{as } N/n \to \infty.
\tag{5.21}
$$

For our chosen $\nu \in (c_1, 1)$, now pick $\nu' \in (\nu, 1)$, and suppose that the event

$$\exists v \in \hat{\mathcal{O}} \text{ such that } d(v, \hat{\mathcal{C}}) < 2^{\nu'\ell}$$

occurs and select such a v. We will sketch the idea of how arm events help us to bound the probability of this event. Because it is a standard "arms reckoning" argument, we omit details and refer the reader to similar arguments in [7], Lemma 6. Choose a central square $\tau(r, s)$ that contains v. If the distance between $\tau(r, s)$ and the box $B(2^\ell)$ is at least $2^{\nu'\ell}$, then there is a 6-arm event in $B(v, 2^{\nu'\ell}) \setminus \tau(r, s)$ (here $B(v, q) = v + B(q)$). Otherwise, it will yield a 5-arm event on a half plane or on a $3/4$-plane, depending on the position of $\tau(r, s)$.

The reason that a 6-arm event will occur is as follows. First, since $\hat{\mathcal{O}}$ is the innermost open circuit in $A()$, for any vertex in $\hat{\mathcal{O}}$, in particular v, there exists a closed path from that vertex to the inner boundary of $A()$. Since $B(v, 2^{\nu})\subseteq A()$, there is a closed path from $\tau(r, s)$ to $\partial B(v, 2^{\nu})$. The circuit $\hat{\mathcal{O}}$ furnishes two more disjoint open paths. Similarly, because $\hat{\mathcal{C}} = \mathcal{C}_q$ is a circuit in the sequence of Lemma 2.3, for any vertex on $\hat{\mathcal{C}}$, there exists an open path from that vertex to the next circuit in the sequence, $\mathcal{C}_{q+1}$. If $\mathcal{C}_{q+1}$ does not intersect $B(v, 2^{\nu}) \setminus \tau(r, s)$, then we choose for our other three paths the open path mentioned in the previous sentence, and two disjoint closed paths furnished by the circuit $\hat{\mathcal{C}}$. These would give the final three of the required six arms.

If, on the other hand, $\mathcal{C}_{q+1}$ does intersect $B(v, 2^{\nu}) \setminus \tau(r, s)$, then we have six arms from $\tau(r, s)$ to the boundary of some intermediate annulus (between $\tau(r, s)$ and $B(v, 2^{\nu})$), and then seven arms from the boundary of this annulus to $\partial B(v, 2^{\nu})$ (see Figure 3). (The circuits $\hat{\mathcal{C}}$ and $\mathcal{C}_{q+1}$ furnish four more arms.) Writing $\pi_j(n_1, n_2)$ for the probability of the j-arm event in $B(2^{\nu+(\nu-\nu)n_2}) \setminus B(2^{\nu+(\nu-\nu)n_1})$, where $n_2 > n_1$, and summing over possible positions of the intermediate annulus, we have the following probability bound corresponding to cases in which $\tau(r, s)$ has distance at least 2^{ν} from $B(2)$:

$$C_{28}\#\ \tau(r, s)\text{ intersecting } B\ 2^{+1} \sum_{r=1}^{-1} \pi_6(0, r)\pi_7(r+1,).$$

The sum is further bounded above by

$$\begin{aligned} &C_{29}2^{(2-2\nu)} \sum_{r=1}^{-1} 2^{-(35/12+o(1))(\nu-\nu)r}2^{-(4+o(1))(\nu-\nu)(-r)} \\ &\leq C_{30}2^{(-(35/12+o(1))(\nu-\nu)+(2-2\nu))}, \end{aligned}$$

where the exponents $-35/12$ and -4 come from (5.20) (putting $j = 6$ and 7 respectively). The right side goes to 0 if $\nu \in (\frac{24}{35} + \frac{11}{35}\nu, 1)$.

To deal with the cases where $\tau(r, s)$ is close to either a corner or a side of $B(2)$, we use a similar argument, but further decomposing the arm events according to their distance to $B(2)$. Because the exponents for the 5-arm event in the half plane and in the $3/4$-plane are 5 and $10/3$ respectively (putting $j = 5$ in (5.20) and (5.21)), and both are greater than 2, we are able to choose ν close enough to 1 so that the probability corresponding to such $\tau(r, s)$ is also small. Putting together all the cases, we have

$$\mathbf{P}\ \exists v \in \hat{\mathcal{O}} \text{ such that } d(v, \hat{\mathcal{C}}) < 2^{\nu} \leq C_{31}e^{-C_{21}}.$$

Plugging this back into (5.19), and then back into (5.17) completes the proof.

We return to (5.13). First, the inequality

$$\mathbf{E}\tilde{Y}^2\mathbf{1}_{F_k\cup G_k} \leq C_{32}/\sqrt{k} \tag{5.22}$$

follows directly from Lemmas 5.2 and 5.3. (For small k, we remove the indicator and use Lemma 5.2 only, and for larger k, we bound by $\sqrt{\mathbf{E}\tilde{Y}^4\mathbf{P}(F_k \cup G_k)}$ and apply both lemmas.)

Next, we bound the second term of (5.13) by showing that for large k,

$$\mathbf{E}\tilde{Y}^2\mathbf{1}_{F_k^c\cap G_k^c} \leq C_{33}a_k^2 + C_{34}e^{-C_8k/2}. \tag{5.23}$$

To do this, we apply Remark 2. Since $\tilde{\mathcal{C}}_k$ is the first circuit in the sequence from Lemma 2.3 outside of $\mathcal{O}_k$, there are no closed circuits in the region strictly between them. By planar duality, there must be an open path $\mathcal{D}$ connecting a neighbor of $\mathcal{O}_k$ to a neighbor of $\tilde{\mathcal{C}}_k$. Choose $v^{(k)}$ to be any vertex of $\mathcal{O}_k$ adjacent to this open path. Because the event F_k^c occurs, $v^{(k)}$ must have distance at least $2(\mathrm{diam}(\tilde{\mathcal{C}}_k))^{c_1}$ from $\tilde{\mathcal{C}}_k$, and so the diameter of the open path $\mathcal{D}$ is at least $(\mathrm{diam}(\tilde{\mathcal{C}}_k))^{c_1}$. Together with the conditions comprising the event G_k^c, this is sufficient to invoke the remark, and to deduce that, with conditional probability $\geq 1 - e^{-C_8 k}$ (conditioned on η), there are sequences $(v^{(i)})_{i \geq k+1}$ and $(\mathcal{D}^{(i)})_{i\geq k}$ as in the conclusion of Lemma 3.2. In particular, we may find an infinite path γ starting from a neighbor of $v^{(k)}$ which consists of only zero-weight vertices or low-weight vertices which are on the circuits $\mathcal{C}_j$ (at most one from each $\mathcal{C}_j$) from Lemma 2.3. Letting Υ be the event that this γ exists, we obtain by the Cauchy–Schwarz inequality that for large k,

$$\begin{aligned}
\mathbf{E}\tilde{Y}^2 \mathbf{1}_{F_k^c\cap G_k^c} &\leq \mathbf{E}\tilde{Y}^2\mathbf{1}_\Upsilon + \mathbf{E}\big[\mathbf{E}\big[\tilde{Y}^2\mathbf{1}_{\Upsilon^c} \mid \eta\big]\mathbf{1}_{F_k^c\cap G_k^c}\big] \\
&\leq \mathbf{E}\tilde{Y}^2\mathbf{1}_\Upsilon + e^{-C_8 k/2}\mathbf{E}\Big[\sqrt{\mathbf{E}\big[\tilde{Y}^4 \mid \eta\big]}\mathbf{1}_{F_k^c\cap G_k^c}\Big] \qquad (5.24)\\
&\leq \mathbf{E}\tilde{Y}^2\mathbf{1}_\Upsilon + C_{34}e^{-C_8 k/2},
\end{aligned}$$

where we used Lemma 5.2 in the last line.

On the event Υ, write γ_k for the segment of γ beginning at $v^{(k)}$ and ending at the first intersection of γ with $\mathcal{O}_{(k,\omega,\omega)}$. Then

$$T^{\mathrm{B}}(\mathcal{O}_k, \mathcal{O}_{(k,\omega,\omega)}) \leq T(\mathcal{O}_k, \mathcal{O}_{(k,\omega,\omega)}) \leq T(\gamma_k).$$

All vertices w on γ_k which have nonzero weight are of low weight, and so such w satisfy $t_w \leq I + a_k$. (Here we use that the sequence (a_j) is nonincreasing.) Distinct w's on γ_k correspond to distinct circuits $\mathcal{C}_j$. Therefore if we define

$$N_L = \text{maximal number of disjoint closed circuits around 0 intersecting } B(2^L)\setminus B(2^k),$$

then we have

$$\mathbf{E}\tilde{Y}^2\mathbf{1}_\Upsilon \leq a_k^2\mathbf{E}N^2_{(k,\omega,\omega)}. \qquad (5.25)$$

Next, we use [15], Lemma 2, which, in our context, states that $\mathbf{E}N_L^4 \leq C_{35}\log^4(2^L/2^k)$, and this is bounded by $C_{36}(L-k)^4$. Therefore, the expectation in (5.25) equals

$$\begin{aligned}
\mathbf{E}N^2_{(k,\omega,\omega)} &= \sum_{m=k}^{\infty}\sum_{t=0}^{\infty}\mathbf{E}N^2_{(k,\omega,\omega)}\mathbf{1}_{\{m(k,\omega)=m\}}\mathbf{1}_{\{(k,\omega,\omega)=m+t+1\}} \\
&\leq \sum_{m=k}^{\infty}\sum_{t=0}^{\infty}\big(\mathbf{E}N^4_{m+t+1}\big)^{1/2}\mathbf{P}\big(m(k,\omega)=m, (k,\omega,\omega)=m+t+1\big)^{1/2} \\
&\leq \sum_{m=k}^{\infty}\sum_{t=0}^{\infty}C_{36}^{1/2}(1+t+m-k)^2\mathbf{P}\big(m(k,\omega)=m, (k,\omega,\omega)=m+t+1\big)^{1/2}.
\end{aligned}$$

Since $\mathbf{P}(m(k,\omega)=m, (k,\omega,\omega)=m+t+1) \leq C_{37}e^{-C_{22}(t+m-k)}$ by the RSW theorem, the above expression is summable and independent of k. Returning to (5.25), and placing this in (5.24), we obtain $\mathbf{E}\tilde{Y}^2\mathbf{1}_{F_k^c\cap G_k^c} \leq C_{33}a_k^2 + C_{34}e^{-C_8 k/2}$. This shows (5.23).

Together, (5.22) and (5.23) show that $\mathbf{E}\tilde{Y}^2 \leq C_{38}\max\{a_k^2, k^{-1/2}\}$. X and Z from (5.2) can be bounded similarly, so returning to (5.1) yields

$$\big|\mathrm{Var}\, T(0,\mathcal{O}_n) - \mathrm{Var}\, T^{\mathrm{B}}(0,\mathcal{O}_n)\big| \leq C_{39}\sum_{k=1}^{n}\max\big\{a_k^2 + a_k, k^{-1/2} + k^{-1/4}\big\} = o(n). \qquad (5.26)$$

Finally, we argue that the above inequality implies

$$\operatorname{Var} T\big(0, \partial B(n)\big) - \operatorname{Var} T^{\mathrm{B}}\big(0, \partial B(n)\big) = o(\log n), \tag{5.27}$$

which, along with Yao's results quoted above Theorem 1.1, gives (1.4). The main ingredient in the proof of (5.27) is the following moment bound. (Our argument is modified from [2], Lemma 5.7). There exists C_{38} such that for all sufficiently large n, $q \geq 1$ such that $2^{q-1} \leq n < 2^q$,

$$\mathbf{E}\big[T\big(0, \partial B(n)\big) - T(0, \mathcal{O}_q)\big]^2 \leq C_{40}. \tag{5.28}$$

The same method can be used (or [2], Lemma 5.7, can be used directly) to show the corresponding statement for T^{B} in place of T. Observe that for $1 \leq \ell \leq q$, on the event $\{m(q-\ell) \geq q+\ell > m(q-\ell-1)\} \cap \{m(q) = q+t\}$, we have

$$T\big(0, \partial B(n)\big) - T(0, \mathcal{O}_q) \leq T\big(\partial B\big(2^{q-\ell-1}\big), \partial B\big(2^{q+t+1}\big)\big)$$

and on the event $\{m(q-\ell) \geq q+\ell > m(q-\ell-1)\}$, we have

$$T\big(0, \partial B(n)\big) - T(0, \mathcal{O}_q) \leq T\big(\partial B\big(2^{q-\ell-1}\big), \mathcal{O}_q\big).$$

Then define the events $A_\ell := \{m(q-\ell) \geq q+\ell > m(q-\ell-1)\}$, for $1 \leq \ell \leq q$, and $B_t := \{m(q) = q+t\}$ for $t \geq 0$. Using the above inequalities and the fact that the events $A_\ell \cap B_t$ (over all t, ℓ) cover the whole probability space, we have

$$\begin{aligned}\mathbf{E}\big[T\big(0, \partial B(n)\big) - T(0, \mathcal{O}_q)\big]^2 \leq{}& \sum_{\ell=1}^{q-q_0}\sum_{t=0}^{\infty} \mathbf{E}\big[T^2\big(\partial B\big(2^{q-\ell-1}\big), \partial B\big(2^{q+t+1}\big)\big)\mathbf{1}_{A_\ell \cap B_t}\big] \\ &+ \sum_{\ell=q-q_0+1}^{q} \mathbf{E}\big[T^2\big(\partial B\big(2^{q-\ell-1}\big), \mathcal{O}_q\big)\mathbf{1}_{A_\ell}\big].\end{aligned} \tag{5.29}$$

Here, q_0 is equal to $\lceil \log_2 m_0 \rceil + 1$, where m_0 is from (5.4). For summands in the first line, we use the Cauchy–Schwarz inequality to bound them by

$$\sqrt{\mathbf{E}T^4\big(\partial B\big(2^{q-\ell-1}\big), \partial B\big(2^{q+t+1}\big)\big)}\big(\mathbf{P}(A_\ell)\mathbf{P}(B_t)\big)^{1/4} \leq C_{41}(t+\ell+2)^{C_{42}}\big(\mathbf{P}(A_\ell)\mathbf{P}(B_t)\big)^{1/4}.$$

For summands in the second line, assuming that $2^q \geq m_0 + 100$, we replicate the argument leading to (5.9). Specifically, letting π be a geodesic between $B(m_0)$ and $\mathcal{O}_q$, and $\tilde{\pi}$ be the portion of π from its initial point to its first intersection with $\partial B(m_0 + 100)$, then the summand of (5.29) is at most

$$2\mathbf{E}T^2(0, \tilde{\pi}) + 2\sum_{t=0}^{\infty} \mathbf{E}\big[T^2\big(B(m_0), \partial B\big(2^{q+t+1}\big)\big)\mathbf{1}_{A_\ell \cap B_t}\big]. \tag{5.30}$$

We can then, as before, sum over all possible values of $\tilde{\pi} = P$, and bound the passage time from 0 to P using six disjoint (except for their initial points) deterministic paths. This leads to the bound $\mathbf{E}T^2(0, \tilde{\pi}) \leq C_{41}$. Applying the Cauchy–Schwarz inequality to the other term gives the following bound for (5.30):

$$C_{43} + C_{41}\sum_{t=0}^{\infty}(q+t+1-\log_2 m_0)^{C_{42}}\big(\mathbf{P}(A_\ell)\mathbf{P}(B_t)\big)^{1/4}.$$

Using these inequalities in (5.29) gives

$$
\begin{aligned}
&\mathbf{E}\left[\left(T\left(0, \partial B(n)\right) - T(0, \mathcal{O}_q)\right)^2\right] \\
&\quad \leq \left(\sum_{\ell=1}^{q-q_0} \sum_{t=0}^{\infty} C_{39}(t+\ell+2)^{C_{42}} \left(\mathbf{P}(A_\ell)\mathbf{P}(B_t)\right)^{1/4} + q_0 C_{43}\right) \\
&\qquad + C_{41} \sum_{\ell=q-q_0+1}^{q} \sum_{t=0}^{\infty} (q+t+1-\log m_0)^{C_{42}} \left(\mathbf{P}(A_\ell)\mathbf{P}(B_t)\right)^{1/4}.
\end{aligned}
$$

By the RSW theorem, there is c_{23} such that $\mathbf{P}(A_\ell) \leq e^{-c_{23}\ell}$ and $\mathbf{P}(B_t) \leq e^{-c_{23}t}$, so this leads to (5.28).

Now that we have proved (5.28), we can quickly derive (5.27). Indeed, we estimate as follows, with $2^{q-1} \leq n < 2^q$:

$$
\begin{aligned}
&\left|\operatorname{Var} T\left(0, \partial B(n)\right) - \operatorname{Var} T^{\mathrm{B}}\left(0, \partial B(n)\right)\right| \\
&\quad \leq \left|\operatorname{Var} T\left(0, \partial B(n)\right) - \operatorname{Var} T(0, \mathcal{O}_q)\right| && (5.31)\\
&\qquad + \left|\operatorname{Var} T^{\mathrm{B}}\left(0, \partial B(n)\right) - \operatorname{Var} T^{\mathrm{B}}(0, \mathcal{O}_q)\right| && (5.32)\\
&\qquad + \left|\operatorname{Var} T(0, \mathcal{O}_q) - \operatorname{Var} T^{\mathrm{B}}(0, \mathcal{O}_q)\right|. && (5.33)
\end{aligned}
$$

By (5.26), (5.33) is $o(q) = o(\log n)$. The term (5.32) is a special case of (5.31) (with Bernoulli weights). To bound (5.31), we use the inequality

$$
\left|\operatorname{Var}(X_1) - \operatorname{Var}(X_2)\right| \leq \operatorname{Var}(X_1 - X_2) + 2\sqrt{\operatorname{Var}(X_1 - X_2)\operatorname{Var}(X_2)}.
$$

To derive this, one sets $\hat{X} = X - \mathbf{E}X$ for a random variable X and writes the left side as

$$
\left|\|\hat{X}_1\|_2^2 - \|\hat{X}_2\|_2^2\right| \leq \|\hat{X}_1 - \hat{X}_2\|_2 \left(\|\hat{X}_1 - \hat{X}_2\|_2 + 2\|\hat{X}_2\|_2\right).
$$

We put $X_1 = T(0, \partial B(n))$ and $X_2 = T(0, \mathcal{O}_q)$ to bound (5.31), noting that (5.28) implies that $\operatorname{Var}(X_1 - X_2) \leq C_{40}$. We obtain

$$
C_{40} + 2\sqrt{C_{40} \operatorname{Var} T(0, \mathcal{O}_q)}.
$$

Finally, $\operatorname{Var} T(0, \mathcal{O}_q) = \sum_{k=0}^{q} \mathbf{E}\Delta_k^2$, which, by Lemma 5.2, is bounded by $C_{44}q$. Therefore the sum of (5.31), (5.32), and (5.33) is bounded above by

$$
2C_{40} + 4\sqrt{C_{40}C_{44}q} + o(\log n) = o(\log n).
$$

This completes the proof of (1.4).

APPENDIX: SKETCH OF [7], PROPOSITION 1

In this appendix, we sketch the idea of [7], Proposition 1, which we used in Lemma 2.6 to construct "good" circuits. We provide this for the reader's convenience, as a guide to reading [7], so we use their notation, although it may conflict with some of ours from previous sections. In particular, we will use the terms "occupied" and "vacant" for "open" and "closed." Furthermore, we will reverse the roles of open and closed (this is fine by symmetry) and assume that the sequence $(\mathcal{C}_k)$ of circuits consists of both occupied and vacant circuits. Last, we use $S(n)$ in place of $B(n)$.

Let us first recall the statement. Let C be a circuit surrounding 0. For certain constants $0 < c_{30}, c_{31}, c_{33} < 1$, $c_{32} > 0$, still to be determined, we consider vacant connected sets $\mathcal{D}$

with some or all of the following properties:

(A.1) $\mathcal{D} \subset \mathring{C}$ and $\mathcal{D}$ contains exactly one vertex adjacent to C;

(A.2) $$\operatorname{diam}(\mathcal{D}) \geq \operatorname{diam}(C)^{c_{30}};$$

(A.3) the vacant cluster of $\mathcal{D}$ in $\mathring{C}$ contains at least $[\operatorname{diam}(C)]^{c_{31}}$ selfavoiding paths θ_m which are adjacent to C, have length$(\theta_m) \geq c_{32} \log\log(\operatorname{diam}(C))$ and satisfy $d(\theta_p, \theta_q) \geq [\operatorname{diam}(C)]^{c_{33}c_{30}}$ for $p \neq q$; moreover, there exists a vertex $z \in \mathcal{D}$ and for each of the θ_m a vacant path from z to θ_m and such that only its endpoint on θ_m is adjacent to C.

We call an occupied circuit C good (or more explicitly (c_{30}-c_{33}-good) if it has the following property:

(A.4) Any vacant connected set $\mathcal{D}$ with the properties (A.1) and (A.2) also satisfies (A.3).

Then [7], Proposition 1, says:

PROPOSITION A.1. *For any* $0 < c_{33} < 1$, *and* $0 < c_{30} < 1$, *but* c_{30} *sufficiently close to* 1, *we have*

$$\text{(A.5)}\qquad \mathbf{P}\left\{\begin{array}{c}\textit{there exists an occupied circuit } \mathcal{C}_k \textit{ with } 2^j \leq \operatorname{diam}(\mathcal{C}_k) < 2^{j+1}\\ \textit{and which is not good}\end{array}\right\} \leq c_{34}\exp(-c_{35} j).$$

PROOF. Most of the proof serves to reduce the number of possible variables involved: the different circuits $\mathcal{C}_k$, and the different possible sets $\mathcal{D}$. We will also need to use the "outermost" property of the $\mathcal{C}_k$'s in a crucial way, to independently make constructions in their interiors.

Note that if any $\mathcal{C}_k$ surrounds 0 and has $2^j \leq \operatorname{diam}(\mathcal{C}_k) < 2^{j+1}$ then it must be contained in $S(2^{j+1})$ but must also contain points outside $S(2^{j-2})$. To fix the offending $\mathcal{C}_k$, we let $\mathcal{C}_\tau$ be the last circuit in our sequence $(\mathcal{C}_k)$ which is contained in $S(2^{j+1})$ and write

$$\text{(A.6)}\qquad \text{(A.5)} \leq \sum_{0 \leq p < j^2} \mathbf{P}\left\{\begin{array}{c} p < \tau, \mathcal{C}_{\tau-p} \text{ is occupied, has}\\ \text{diameter } \geq 2^j, \text{ but is not good}\end{array}\right\}$$

$$\text{(A.7)}\qquad + \mathbf{P}\left\{\tau > j^2, \mathcal{C}_{\tau-j^2} \text{ is not contained in } S\left(2^{j-2}\right)\right\}.$$

Using the RSW theorem, one can show (see [7], equation (4.8))

$$\text{(A.8)}\qquad \text{(A.7)} \leq (2c_3 + 1)e^{-c_{36} j}.$$

For (A.6), we sum over all possible values of $\mathcal{C}_{\tau-p}$ with the notation

$$E(p, C) = \{\tau > p, \mathcal{C}_{\tau-p} \text{ is occupied and equals } C\}.$$

We obtain

$$\begin{aligned}\text{(A.6)} &= \sum_{0 \leq p < j^2} \sum_{C} \mathbf{P}\{C \text{ is not good} \mid E(p, C)\}\mathbf{P}\{E(p, C)\}\\ &\leq j^2 \sup_{p,C} \mathbf{P}\{C \text{ is not good} \mid E(p, C)\},\end{aligned}$$

where the supremum is over all circuits C surrounding 0 in $S(2^{j+1})$ but with diameter $\geq 2^j$, and over all $p \in [0, j^2)$. Putting this back in (A.6), we find that our desired probability is

$$\text{(A.5)} \leq j^2 \sup_{p,C} \mathbf{P}\big(C \text{ is not good} \mid E(p,C)\big) + (2c_3 + 1)e^{-c_{36}j}. \tag{A.9}$$

To analyze the supremum in (A.9), we use the outermost property of $\mathcal{C}_{\tau-p}$. Specifically, its definition entails that the event $E(p, C)$ depends only on the state of vertices in $C^{\text{ext}} \cup C$ (the exterior of C union with C). From the paragraph above [7], equation (4.37), we quote "(Note that the condition $\{p < \tau, \mathcal{C}_{\tau-p} = C\}$ merely says that C is one of our $\mathcal{C}_k$ and there are exactly p of the circuits $\mathcal{C}_k$ outside C but inside $S(2^{j+1})$; this only involves vertices on or outside C.) Given $E(p, C)$, the further conditions for $\mathcal{C}_{\tau-p}$ to be good, depend only on the vertices in $\mathring{C}$." This means that to calculate $\mathbf{P}(C$ is not good $\mid E(p, C))$, "even with the conditioning on $E(p, C)$, we may assume that all vertices in the interior of C are still independently occupied or vacant with probability $1/2$."

In addition to the independence mentioned in the last paragraph, we note that the vacant set $\mathcal{D}$ in the definition of good circuit can be replaced by a path. That is (see [7], p. 33), if $\mathcal{C}_{\tau-p} = C$ and $\mathcal{C}_{\tau-p}$ is occupied but not good, with associated vacant set $\mathcal{D}$, then there is a vertex selfavoiding vacant path $\xi = (v_1, \ldots, v_q) \subset \mathcal{D}$ such that

$$v_1 \text{ is the unique vertex of } \xi \text{ adjacent to } C \text{ and } \xi \subset \mathring{C}; \tag{A.10}$$

$$\operatorname{diam}(\xi) \geq \frac{1}{2} \operatorname{diam}(C)^{c_{30}} \tag{A.11}$$

and such that (A.3) with $\mathcal{D}$ replaced by ξ fails.

Using the results of the last two paragraphs in (A.6), we obtain

$$\begin{aligned} \text{(A.5)} \leq{}& (2c_3 + 1)e^{-c_{36}j} \\ &+ j^2 \sup_C \mathbf{P}\left(\begin{array}{c} \exists \text{ a vacant path } \xi = (v_1, \ldots, v_q) \\ \text{which satisfies (A.10) and (A.11)} \\ \text{but for which (A.3) fails} \end{array}\right), \end{aligned} \tag{A.12}$$

where the supremum is over all circuits C surrounding 0 in $S(2^{j+1})$ but with diameter $\geq 2^j$. To bound the supremum in (A.12), we apply the method of [6], Lem. 8.2. (It is done in steps 4 and 5 of [7], in [7], p. 37-52.) This is an inductive argument and in our context constructs for a given "extremal" candidate vacant path ξ, order 2^{cj} many vacant paths connecting ξ to the circuit C, along with the required vacant paths θ_m that run along the inner boundary of C. The induction is carried out by comparison to a branching process, and roughly speaking, goes as follows. In the first step, one attempts to construct a vacant arc ξ connecting ξ to C in an annulus $S(2^{\ell}) \setminus S(2^{\ell-1})$ with ℓ of order j centered at v_1 whose endpoint w_1 adjacent to C is the beginning of a path θ_m as in the definition of good circuit. If this is possible, we assign two children to the root node of our branching process, and otherwise, we assign only one child. At the next step, we attempt to construct similar vacant arcs corresponding now to grandchildren of the root node, but this time in smaller annuli: one centered at v_1 with ℓ replaced by ℓ minus a constant (connecting ξ to C), and another of the same size, but centered at w_1 (and connecting ξ to C). In this step, we find either 0, 1, or 2 many vacant paths; we assign 2 children to either node whose corresponding annulus contains such a vacant path, and 1 child otherwise. This procedure continues, with each node in our branching process having 1 or 2 children, and each node at a given level in the process corresponds to a path θ_m as in the definition of good circuit. One can show that the probability that a node has 2 children is uniformly positive, over all possible histories of the construction (see [7], equations (4.62), (4.63)), and since there are order j many stages in the construction, with exponentially high probability, the process produces order $\exp(cj)$ many paths. This implies in the end that

$$(A.12) \le Ce^{-cj} + (2c_3 + 1)e^{-c_{36}j}.$$

(An inequality of this type eventually arises from substituting the bound [7], equation (4.91), on the moment generating type function (α, h, D) into [7], equation (4.79).)

There are a number of difficulties in implementing this program and they consume most of the 27-page proof. The major one is how to enumerate candidate vacant paths ξ in a sensible way. This is done by finding a suitable smaller-scale annulus in which to exhibit ξ as an occupied crossing from the inner boundary of C to a boundary point of the annulus. To do this, the box $S(2^{j+1})$ is tiled by squares $S(r, s)$ of the form

$$S(r, s) = r2, (r+1)2 \times s2, (s+1)2, \quad -2^{j+1-} \le r, s \le 2^{j+1-}$$

and surrounding squares

$$S(r, s) = (r-1)2, (r+2)2 \times (s-1)2, (s+2)2,$$

where $= c_{30}j - 2\log j/\log 2 - 4$, so that the total number of squares is at most $c_{37}j^4 2^{2(1-c_{30})j}$. Any path ξ as in (A.12) originates at a vertex v_1 which is in some $S(r, s)$ adjacent to C, and so is on a certain arc F of a component J of the inner boundary of C which connects two vertices outside of but adjacent to $S(r, s)$. One defines a certain class $\mathfrak{C}$ of circuits C which have too many crossings of any one of the $S(r, s)$ (more than $c_{38}j^4 2^{2(1-c_{30})j}$) and shows that (see [7], equation (4.13))

$$\mathbf{P}(\exists \text{ circuit in } \mathfrak{C}) \le c_{39} \exp -c_{40}j^4 2^{2(1-c_{30})j}.$$

This means that in the initial decomposition ((A.6) and (A.7)), we may discard the small probability that our C_k is in $\mathfrak{C}$, allowing us to restrict our supremum in (A.12) to circuits C that are not in the class $\mathfrak{C}$. (This class is further enlarged to forbid too many crossings of more, similarly defined annuli below [7], equation (4.36).)

In Steps 2 and 3 of [7], it is argued by topological considerations that the total number of choices (given that C is not in the class $\mathfrak{C}$) of boundary components J, arcs F, and squares $S(r, s)$ is at most $c_{42}j^4 2^{2(1-c_{30})j}$. This leads to introducing the conditions

$$v_1 \in S(r, s), v_q \notin S(r, s), \text{ but } v_i \in S(r, s) \text{ for } 1 \le i \le q-1 \tag{A.13}$$

and

$$v_q \in \hat{S}, \tag{A.14}$$

where $\hat{S}$ is the topological boundary of

$$\hat{S} = \hat{S}(r, s) = (r-1)2 - 1, (r+2)2 + 1 \times (s-1)2 - 1, (s+2)2 + 1$$

(which just surrounds $\hat{S}$), and replacing the first summand on the right of (A.12) by

$$\begin{aligned} &c_{42}j^4 2^{2(1-c_{30})j} j^2 \\ &\times \sup \mathbf{P} \begin{array}{c} \exists \text{ a vacant path } \xi = (v_1, \ldots, v_q) \text{ which satisfies (A.10)}, \\ \text{(A.13), (A.14) and } v_q \in \mathring{J}, v_1 \in F, \text{ but for which (A.3) fails} \end{array}. \end{aligned} \tag{A.15}$$

Here, the supremum is over all choices of C, all (r, s), and choices of J and F.

It is in the setting of the new supremum (A.15) that one can define suitable ξ, called the "permissible" paths (see [7], equation (4.40)). These paths can be ordered as $\xi^{(1)}, \xi^{(2)}, \ldots$ according to a partial ordering so that they have an extremal property: for a given γ, the event that $\xi^{(i)} = \gamma$ depends only on vertices on γ and on one side (properly defined). This extremal property is again important so we can make constructions on one side (associated to the previously-described branching process). In addition to this, if a ξ exists as in (A.15), then for some choice of C, (r,s), J, and F, there is a path $\xi^{(i)}$ in the ordering that does not satisfy (A.3). Therefore we bound (A.15) by

$$c_{42}j^4 2^{2(1-c_{30})j} j^2 \sup_{i\geq 1} \mathbf{P}\big(\xi^{(i)} \text{ exists, but (A.3) fails for } \xi^{(i)}\big)$$

(see [7], equation (4.78)). By now running the branching process argument with $\xi^{(i)}$ in place of ξ and using the independence due to the extremal property described, one obtains the bound

$$c_{42}j^4 2^{2(1-c_{30})j} j^2 \sup_{i\geq 1} \big[\mathbf{P}\big(\xi^{(i)} \text{ exists}\big) \times Ce^{-cj}\big].$$

However (see [7], equation (4.14)), $\mathbf{P}(\xi^{(i)} \text{ exists}) \leq (1-c_{41})^i$, and so this sum gives a bound of

$$c_{42}j^4 2^{2(1-c_{30})j} j^2 Ce^{-cj} \sum_{i\geq 1} (1-c_{41})^i.$$

Placing this back in (A.15) finally gives the stated inequality in (A.5), so long as c_{30} is sufficiently close to 1.

Acknowledgments. The authors wish to thank an anonymous referee for many helpful suggestions that led to a more readable exposition.

Funding. The first author was supported by an NSF CAREER grant.
The second author funded in part by NSF Grant DMS-1612921.

REFERENCES

[1] COX, J. T. and DURRETT, R. (1981). Some limit theorems for percolation processes with necessary and sufficient conditions. *Ann. Probab.* **9** 583–603. MR0624685

[2] DAMRON, M., LAM, W.-K. and WANG, X. (2017). Asymptotics for 2D critical first passage percolation. *Ann. Probab.* **45** 2941–2970. MR3706736 https://doi.org/10.1214/16-AOP1129

[3] GRIMMETT, G. (1999). *Percolation*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **321**. Springer, Berlin. MR1707339 https://doi.org/10.1007/978-3-662-03981-6

[4] JIANG, J. and YAO, C.-L. (2019). Critical first-passage percolation starting on the boundary. *Stochastic Process. Appl.* **129** 2049–2065. MR3958423 https://doi.org/10.1016/j.spa.2018.06.008

[5] KESTEN, H. (1980). The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. *Comm. Math. Phys.* **74** 41–59. MR0575895

[6] KESTEN, H. (1982). *Percolation Theory for Mathematicians. Progress in Probability and Statistics* **2**. Birkhäuser, Boston, MA. MR0692943

[7] KESTEN, H., SIDORAVICIUS, V. and ZHANG, Y. (1998). Almost all words are seen in critical site percolation on the triangular lattice. *Electron. J. Probab.* **3** no. 10, 75. MR1637089 https://doi.org/10.1214/EJP.v3-32

[8] KESTEN, H. and ZHANG, Y. (1987). Strict inequalities for some critical exponents in two-dimensional percolation. *J. Stat. Phys.* **46** 1031–1055. MR0893131 https://doi.org/10.1007/BF01011155

[9] KESTEN, H. and ZHANG, Y. (1997). A central limit theorem for "critical" first-passage percolation in two dimensions. *Probab. Theory Related Fields* **107** 137–160. MR1431216 https://doi.org/10.1007/s004400050080

[10] NOLIN, P. (2008). Near-critical percolation in two dimensions. *Electron. J. Probab.* **13** 1562–1623. MR2438816 https://doi.org/10.1214/EJP.v13-565

[11] SMIRNOV, S. and WERNER, W. (2001). Critical exponents for two-dimensional percolation. *Math. Res. Lett.* **8** 729–744. MR1879816 https://doi.org/10.4310/MRL.2001.v8.n6.a4

[12] VAN DEN BERG, J. and KESTEN, H. (1993). Inequalities for the time constant in first-passage percolation. *Ann. Appl. Probab.* **3** 56–80. MR1202515

[13] YAO, C.-L. (2014). Law of large numbers for critical first-passage percolation on the triangular lattice. *Electron. Commun. Probab.* **19** no. 18, 14. MR3183571 https://doi.org/10.1214/ECP.v19-3268

[14] YAO, C.-L. (2018). Limit theorems for critical first-passage percolation on the triangular lattice. *Stochastic Process. Appl.* **128** 445–460. MR3739504 https://doi.org/10.1016/j.spa.2017.05.002

[15] YAO, C.-L. (2019). Asymptotics for 2D critical and near-critical first-passage percolation. *Probab. Theory Related Fields* **175** 975–1019. MR4026611 https://doi.org/10.1007/s00440-019-00908-2