AN INVERSE PROBLEM FOR THE NON-LINEAR FRACTIONAL MAGNETIC SCHRÖDINGER EQUATION

RU-YU LAI AND TING ZHOU

Abstract. In this paper, we study the forward and the inverse problem for the fractional magnetic Schrödinger equation with a nonlinear electric potential. We first obtain the maximum principle for the linearized equation and apply it to show that the problem is well-posed under suitable assumptions for the exterior condition. Then we obtain the uniqueness in recovering both the magnetic and the nonlinear electric potentials, assumed to be analytic in terms of the solution, from the exterior data of the solution.

1. Introduction

We study the inverse problem for a nonlinear fractional magnetic Schrödinger equation (FMSE)

$$\begin{cases} (-\Delta)_A^s u + a(x,u) = 0 & \text{in } \Omega \\ u = g & \text{in } \Omega_e := \mathbb{R}^n \backslash \overline{\Omega}, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n, n \geq 2$ is a bounded domain with smooth boundary, $s \in (0,1)$, A represents the magnetic vector potential and a is the nonlinear electric potential. Here the operator $(-\Delta)_A^s$ is defined by $(\nabla^s + A)^2$ in Section 2.1 following the model introduced in [10].

The inverse problem considered in this paper is a natural generalization of the problems for *linear* fractional elliptic equations such as the fractional Schrödinger equation (FSE) $(-\Delta)^s u + qu = 0$ and the linear FMSE $(-\Delta)_A^s u + qu = 0$. The inverse problem for $(-\Delta)^s u + qu = 0$ was first considered in [18] and more results associated to the fractional Laplacian can be found in [3, 6, 16, 17, 21, 22, 37, 38, 51]. For the linear FMSE $(-\Delta)_A^s u + q(x)u = 0$, the inverse problems were studied in both [10] and [45], where the fractional magnetic laplacian $(-\Delta)_A^s$ were defined differently. We adopt the definition introduced in [10], given explicitly in (2.4) in Section 2 to form our problem. The alternative definition

$$(-\Delta)_A^s u(x) := C_{n,s} \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u(y)}{|x-y|^{n+2s}} \, dy, \tag{1.2}$$

was first introduced in [11] and then used in [45].

Below, we briefly discuss the difference between (2.4) and (1.2). Physically, the operator $(-\Delta)_A^s = (\nabla^s + A)^2$ is shown to arise as continuous limits of long jump random walks with weights (see [10]). For the definition (1.2), the motivation of its introduction is discussed in [11] and, in particular, relies on the Lévy-Khintchine formula for the generator associated to a general Lévy process. For the inverse problems, we point out that the FMSE associated to the definition (2.4) in [10] admits a gauge equivalence (see (1.6)) when considering the uniqueness of determining the potential pair (A, q) for the linear

equations, similar to the classical magnetic Schrödinger equations problem. During our tackling the problem for the nonlinear equations, we encountered the result in [46] based on the definition in (1.2). In comparison with [46], where only the nonlinear electric potential $a(x, u)$ is uniquely determined, we show that both the magnetic and nonlinear electric potentials can be recovered from the exterior measurements. Finally we mention that the inverse problems for the semilinear FSE, where the magnetic potential $A = 0$, were also studied in [35, 36].

For classical PDEs, the corresponding inverse problem for the linear magnetic Schrödinger equation (MSE) $(-i\nabla + A)^2 u + qu = 0$ has been investigated in [8, 13, 15, 19, 20, 24, 30, 34, 48, 49], where one aims to determine the vector magnetic potential $A(x)$ and the scalar electric potential $q(x)$ from the boundary Dirichlet-to-Neumann (DN) map. Specifically, due to a gauge invariance, one can only expect to recover uniquely the curl of magnetic field curl A and q. This forms a part of the inverse theory of Calderón problem for elliptic PDEs, originated from the 1980's. We omit the vast literature in this manuscript. (They can be found in the references of above mentioned results.) One regards the inverse problems described in the previous paragraph and those considered in this paper the fractional analogues of the Calderón problem. The nonlocal magnetic Schrödinger operator $(-\Delta)_A^s$ extends the classical diffusion process modeled by the Laplacian $-\Delta := \sum_{k=1}^n \partial_{x_k}^2$ and the nonlocal diffusion phenomena modeled by the fractional Laplacian $(-\Delta)^s$. We also mention that fractional Laplacian $(-\Delta)^s$ can be seen arising in stochastic theory as the operators associated with symmetric $2s$-stable Lévy processes, for example, in a pricing model in financial mathematics (see [9]). The equations with subcritical nonlinearities were studied in [4] from the variational point of view.

In this part, we briefly survey recent development of the (higher order) linearization approach for inverse problems of nonlinear PDEs. For example, in dealing with the Calderón problem for certain nonlinear equations, by taking the first order linearization of the DN-map, the result of the inverse problem for the linear PDEs can be applied directly to identify the full nonlinearity of the medium. See for instance [23, 25, 26, 27, 28, 53] for the demonstration of the method for certain semilinear, quasilinear elliptic and parabolic equations. Later by further expanding the terms in the data operators (such as the DN-map or the source-to-solution map), known as the *higher order linearization*, it is shown that several inverse problems for nonlinear hyperbolic equations on Lorentzian manifolds can be solved by combining the analysis of nonlinear interaction of waves [33]. In contrast the underlying problems for linear hyperbolic equations are still open in general, see [7, 44] and the references therein. In some other cases, higher order linearization reduces showing uniqueness for the inverse problem to proving certain density result of products of linear solutions. This strategy was successfully applied to elliptic equations with power-type nonlinearities in [14, 31, 32, 36, 41, 42, 47] (in particular, the second order linearization of the nonlinear boundary map was studied in [5, 29, 52, 53]); to Maxwell's equations with Kerr-type and the second harmonic generation nonlinearities in [1, 2]; to nonlinear kinetic equations in [39] and to semilinear wave equations in [43]. In [40], we solved an inverse problem for the magnetic Schrödinger equation with nonlinearity in both magnetic and electric potentials A and q.

In order to describe our main results, we first introduce some definitions, notations and assumptions on the potentials.

Definition 1. *Let $A \in C_c^s(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$. We define the symmetric A_s, antisymmetric A_a, parallel $A_{||}$ and perpendicular $A_\perp$ parts of A at point (x, y) as*

$$A_s(x,y) := \tfrac{A(x,y)+A(y,x)}{2}, \qquad A_a(x,y) := A(x,y) - A_s(x,y) = \tfrac{A(x,y)-A(y,x)}{2},$$
$$A_{||}(x,y) := \begin{cases} \frac{A(x,y)\cdot(x-y)}{|x-y|^2}(x-y) & \text{if } x \neq y \\ A(x,y) & \text{if } x = y \end{cases}, \qquad A_\perp(x,y) := A(x,y) - A_{||}(x,y).$$

For the inverse problem, we assume that $A(x,y) \in C_c^s(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ has compact support in $\Omega \times \Omega$ and satisfies

$$A_{s||} \in H^s(\mathbb{R}^{2n}), \qquad (\nabla\cdot)^s A_{s||} \in L^\infty(\mathbb{R}^n), \qquad A_{a||}(x,y)\cdot(y-x) \geq 0 \qquad \text{in } \mathbb{R}^n \times \mathbb{R}^n; \tag{1.3}$$

$a(x,z) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} a(x,0) = 0 \qquad \text{for all } x \in \overline{\Omega}, \\ \text{the map } z \mapsto a(\cdot, z) \text{ is analytic with values in } C^s(\overline{\Omega}). \end{cases} \tag{1.4}$$

Therefore, the potential a admits the following Taylor expansion

$$a(x,z) = \sum_{k=1}^{\infty} \partial_z^k a(x,0) \frac{z^k}{k!}, \qquad \partial_z^k a(x,0) \in C^s(\overline{\Omega}).$$

and the convergence of this series is in $C^s(\overline{\Omega})$ topology. Together they also satisfy

$$(\nabla\cdot)^s A_{s||} + \int_{\mathbb{R}^n} |A(x,y)|^2 \, dy + \partial_z a(x,z) \geq 0 \qquad \text{for } x \in \Omega, \ |z| < R_0 \tag{1.5}$$

for some constant $R_0 > 0$. Here we denote by $C^s(\overline{\Omega})$ the usual Hölder space.

Remark 1.1. *The above conditions posed on the potentials A and a are required for the definition of the FMSE to be well-defined and the properties such as the maximum principle to be satisfied in order to prove the well-posedness of both the linear and nonlinear FMSE, as well as the boundedness of the solutions. More discussions can be found in Remark 3.1 of* [10]*. Also, the alternative weak formulation of the operators in Remark* (2.1) *helps to understand these conditions better.*

In Theorem 2, it is shown that there exists a small constant $\varepsilon_0 > 0$ and a constant $C > 0$ such that when the exterior data $g \in \mathcal{S}_{\varepsilon_0}$, denoted by

$$\mathcal{S}_{\varepsilon_0} := \{g \in C_c^\infty(\Omega_e) : \ \|g\|_{C_c^\infty(\Omega_e)} \leq \varepsilon_0\},$$

the problem (1.1) has a small unique solution $u \in C^s(\mathbb{R}^n)$ satisfying $\|u\|_{C^s(\mathbb{R}^n)} \leq C\|g\|_{C_c^\infty(\Omega_e)}$. Therefore, we can define the Dirichlet-to-Neumann (DN) map $\Lambda_{A,a}^s$ via the bilinear form as in Lemma 4.

We introduce the *gauge equivalence* $\sim$ defined in [10]. We say that two pairs of coefficients $(A_1(x,y), q_1(x))$ and $(A_2(x,y), q_2(x))$ satisfy $(A_1, q_1) \sim (A_2, q_2)$ if and only if

$$(-\Delta)_{A_1}^s u + q_1 u = (-\Delta)_{A_2}^s u + q_2 u \tag{1.6}$$

for all $u \in H^s(\mathbb{R}^n)$. [10, Lemma 3.8] further implies that $(A_1, q_1) \sim (A_2, q_2)$ holds if and only if

$$A_{1,a||}(x,y) = A_{2,a||}(x,y) \quad \text{in } \mathbb{R}^n \times \mathbb{R}^n,$$

and

$$\int_{\mathbb{R}^n} |A_1|^2 \, dy + (\nabla\cdot)^s A_{1,s||} + q_1 = \int_{\mathbb{R}^n} |A_2|^2 \, dy + (\nabla\cdot)^s A_{2,s||} + q_2 \quad \text{in } \Omega.$$

Our main result is stated here.

Theorem 1. *Let $0 < s < 1$ and let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with smooth boundary. Let W_1 and W_2 be two arbitrary nonempty open subsets in Ω_e. Suppose that (A_j, a_j) satisfy (1.3), (1.4) and (1.5) for $j = 1, 2$. Then if*

$$\Lambda^s_{A_1,a_1}[g]\big|_{W_2} = \Lambda^s_{A_2,a_2}[g]\big|_{W_2} \qquad \text{for any } g \in \mathcal{S}_{\varepsilon_0} \cap C_c^\infty(W_1), \tag{1.7}$$

where $\varepsilon_0 > 0$ is sufficiently small (Here $[g]$ is the equivalence class of g in $H^s(\mathbb{R}^n)\backslash\widetilde{H}^s(\Omega)$), we have

$$(A_1, \partial_z a_1(x,0)) \sim (A_2, \partial_z a_2(x,0)) \tag{1.8}$$

and

$$a_1(x,z) - \partial_z a_1(x,0) z = a_2(x,z) - \partial_z a_2(x,0) z \quad \text{in } \Omega \times \mathbb{R}. \tag{1.9}$$

The proof of Theorem 1 is built upon several preliminary results: the well-posedness for the problem, the Runge approximation property and maximum principle. We start by investigating the forward problem since the study of the inverse problem stands on it. To this end, we formulate the maximum principle and a barrier function, which can be applied to prove the boundedness of solution to the FMSE. Together with the benefit introduced by the nonlinearity of the equation, they guarantee the effectiveness of the fixed point theorem. This then leads to the well-posedness result for the nonlinear equation under study. Moreover, to reconstruct unknown potentials, we apply the higher order linearization scheme. By taking derivatives of the integral identity for the DN map multiple times, the Runge approximation property, which states that the set of solutions is dense in $L^2(\Omega)$, then plays in to extract the information of unknown potentials out of the integral.

This theorem guarantees uniqueness of higher order term of $a(x,z)$. However, similar to the inverse problem for the linear FMSE, it is expected that one can only determine the magnetic and linear electric potentials up to a gauge as shown in (1.8). In fact, in Section 3, we found that only coefficients A and $\partial_z a(x,0)$ appear in the linearized equation and then they are recovered up to the gauge by applying the available inverse problem result for the linear equation. While in the higher order linearization steps, the higher order coefficients $\partial_z^k a(x,0), k \geq 2$ can be viewed as a part of the source term in the k-th linearized equation. Therefore, this explains the unique reconstruction of these terms without any gauge in (1.9).

The structure of the paper is as follows. In Section 2, we establish the well-posedness of the problem (1.1) for a small enough exterior data g by deriving the maximum principle

and the barrier function in order to obtain the C^s regularity of the solution for $(-\Delta)^s_A + q$. In Section 3, we determine the potentials A and $a(x,z)$ using the linearization steps.

2. The forward problem

2.1. **Notations.** We introduce the notations and properties below. We define $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ to be the standard L^2-based Sobolev space with norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \|\mathcal{F}^{-1}((1+|\xi|^2)^{s/2}\mathcal{F}u)\|_{L^2(\mathbb{R}^n)},$$

where $\mathcal{F}$ is the Fourier transform defined as

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx.$$

Let U be an open set in $\mathbb{R}^n$. For scalar $\beta \in \mathbb{R}$, we define the following spaces:

$$\begin{aligned} H^\beta(U) &:= \left\{ u|_U : u \in H^\beta(\mathbb{R}^n) \right\}, \\ \widetilde{H}^\beta(U) &:= \text{closure of } C_c^\infty(U) \text{ in } H^\beta(\mathbb{R}^n), \\ H_0^\beta(U) &:= \text{closure of } C_c^\infty(U) \text{ in } H^\beta(U) \end{aligned}$$

where $\|u\|_{H^\beta(U)} := \inf\{\|w\|_{H^\beta(\mathbb{R}^n)} : w \in H^\beta(\mathbb{R}^n), w|_U = u\}$.

Following the notations in [10], the magnetic fractional Laplacian $(-\Delta)^s_A$ is an operator mapping from $H^s(\mathbb{R}^n)$ to $H^{-s}(\mathbb{R}^n)$, such that for all $u, v \in H^s(\mathbb{R}^n)$,

$$\langle (-\Delta)^s_A u, v\rangle = \langle \nabla^s_A u, \nabla^s_A v\rangle.$$

The magnetic fractional gradient operator ∇^s_A is defined by $\nabla^s + A(x,y)$ with the fractional gradient $\nabla^s : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$ extends the definition

$$\nabla^s u(x,y) = \sqrt{\frac{C_{n,s}}{2}} \frac{u(x)-u(y)}{|y-x|^{n/2+s+1}}(y-x).$$

Then the fractional divergence $(\nabla\cdot)^s : L^2(\mathbb{R}^{2n}) \to H^{-s}(\mathbb{R}^n)$ is defined by

$$\langle (\nabla\cdot)^s u, v\rangle = \langle u, \nabla^s v\rangle \qquad \text{for } v \in H^s(\mathbb{R}^n).$$

Recall that the fractional Laplacian $(-\Delta)^s : H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$ is defined by

$$(-\Delta)^s u(x) := C_{n,s}\ \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy,$$

where the constant $C_{n,s}$ depends only on n, s, see for instance [12], and p.v. stands for the principal value. Then we have $(-\Delta)^s = (\nabla\cdot)^s\nabla^s$ in weak sense, that is, $\langle (-\Delta)^s u, v\rangle = \langle \nabla^s u, \nabla^s v\rangle$ for $u, v \in H^s(\mathbb{R}^n)$.

2.2. **Preliminary results.** The proof of the following result can be found in [10, Lemma 3.15], where the regularity of A can be relaxed if certain integrability conditions are imposed.

Proposition 1 (The Runge approximation)**.** *Suppose that $q \in L^p(\Omega)$, $p := \max\{2, n/(2s)\}$ and $A \in C^s_c(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{C}^n)$ has compact support in $\Omega \times \Omega$ and satisfies (2.1). Let W be an open set in Ω_e and u_g be the solution to $(-\Delta)^s_A u_g + q u_g = 0$ in Ω with $u_g = g$ in Ω_e. Then the set $\mathcal{R} = \{u_g|_\Omega : g \in C_c^\infty(W)\}$ is dense in $L^2(\Omega)$.*

The Runge approximation will be applied to recover the unknowns A and $a(x,z)$ in Section 3.

2.3. **Boundedness of solutions.** We will follow the steps in [35] to prove an L^∞-bound for the weak solution of (2.5) in Proposition 3. To this end, we prove a maximum principle in Proposition 2 and construct a barrier function in Lemma 1.

Proposition 2 (Maximum principle for the weak solution of FMSE)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and $A(x,y) \in C_c^s(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ satisfy*

$$A_{s||} \in H^s(\mathbb{R}^{2n}), \qquad A_{a||}(x,y)\cdot(y-x) \geq 0 \qquad \text{in } \mathbb{R}^n \times \mathbb{R}^n; \tag{2.1}$$

and

$$(\nabla\cdot)^s A_{s||} + \int_{\mathbb{R}^n} |A(x,y)|^2 \, dy \geq 0 \qquad \text{for } x \in \Omega. \tag{2.2}$$

Suppose $u \in H^s(\Omega)$ is a weak solution of

$$\begin{cases} (-\Delta)_A^s u = F & \text{in } \Omega, \\ u = g & \text{in } \Omega_e. \end{cases}$$

Then if $0 \leq F \in L^\infty(\Omega)$ and $0 \leq g \in L^\infty(\Omega_e)$, we have $u \geq 0$ in Ω, hence in $\mathbb{R}^n$.

Proof. Using the weak formulation, we obtain for $\phi \in H_0^s(\Omega)$ satisfying $\phi \geq 0$,

$$\langle \nabla^s u(x) + A(x,y)u(x), \nabla^s \phi(x) + A(x,y)\phi(x)\rangle = \int_\Omega F\phi \, dx \geq 0, \tag{2.3}$$

where we used $F \geq 0$ in Ω. Let $u^- := \max\{-u, 0\}$. Then by $u \in H^s(\Omega)$, we can take $\phi = u^- \in H_0^s(\Omega)$ as a test function. We want to show that if $\phi \not\equiv 0$, the left hand side of (2.3) turns out to be negative, in order to draw a contradiction. We rewrite the left hand side of (2.3) as

$$\begin{aligned} \langle \nabla^s u + Au, \nabla^s \phi + A\phi\rangle &= \langle \nabla^s u, \nabla^s \phi\rangle + [\langle \nabla^s u, A(x,y)\phi(x)\rangle - \langle \nabla^s u, A(y,x)\phi(x)\rangle] \\ &\quad + [\langle \nabla^s u, A(y,x)\phi(x)\rangle + \langle \nabla^s \phi, A(x,y)u(x)\rangle + \langle Au, A\phi\rangle] \\ &=: \langle \nabla^s u, \nabla^s \phi\rangle + I + II. \end{aligned}$$

It was shown in [35] (see the proof of Proposition 3.1 in [35]) that the fraction Laplacian term $\langle \nabla^s u, \nabla^s \phi\rangle < 0$, where $g \geq 0$ in Ω_e is used. We will discuss below that $I + II$ is indeed nonpositive, that is $I + II \leq 0$, which then leads to a contradiction to (2.3).

To this end, the term I is actually $2\langle \nabla^s u, A_a(x,y)\phi(x)\rangle$ and

$$I = 2\sqrt{\frac{C_{n,s}}{2}} \int_{\mathbb{R}^{2n}} \frac{A_a(x,y)\cdot(y-x)}{|x-y|^{n/2+s+1}} \left(u(x) - u(y)\right)\phi(x) \, dxdy.$$

Note that the integrand function vanishes on $S_0 := \{(x,y) \mid u(x) \geq 0\}$ since $\phi = 0$ on S_0. On the set $S_1 := \{(x,y) \mid u(x) < 0 \text{ and } u(y) \geq 0\}$, we have

$$(u(x) - u(y))\phi(x) < 0 \qquad \text{in } S_1,$$

by (2.1) (which is equivalent to $A_a(x,y)\cdot(y-x) \geq 0$), and thus the integrand function is ≤ 0. On the remaining set $S_2 := \{(x,y) \mid u(x) < 0 \text{ and } u(y) < 0\}$, we have

$$\phi(x) = u^- = -u(x) \qquad \text{in } S_2.$$

We denote $\Omega^- := \{x \mid u(x) < 0\}$. Then it is sufficient to consider the integral over $\Omega^- \times \Omega^-$:

$$\begin{aligned}
&-2\sqrt{\frac{C_{n,s}}{2}} \int_{\Omega^-\times\Omega^-} \frac{A_a(x,y)\cdot(y-x)}{|x-y|^{n/2+s+1}} \left(u(x)-u(y)\right) u(x)\ dxdy \\
=&-\sqrt{\frac{C_{n,s}}{2}} \int_{\Omega^-\times\Omega^-} \frac{A_a(x,y)\cdot(y-x)}{|x-y|^{n/2+s+1}} \left(u(x)-u(y)\right) u(x) \\
&\qquad + \frac{A_a(y,x)\cdot(x-y)}{|x-y|^{n/2+s+1}} \left(u(y)-u(x)\right) u(y)\ dxdy \\
=&-\sqrt{\frac{C_{n,s}}{2}} \int_{\Omega^-\times\Omega^-} \frac{A_a(x,y)\cdot(y-x)}{|x-y|^{n/2+s+1}} \left(u(x)-u(y)\right)^2\ dxdy \le 0,
\end{aligned}$$

since $A_a(y,x) = -A_a(x,y)$ and also by (2.1). Combining these estimates together, we have $I \le 0$.

For the term II, one first check that, using the definition of $\nabla^s u$, we have

$$\langle \nabla^s u, A(y,x)\phi(x) - A(x,y)\phi(y)\rangle = 0.$$

Moreover,

$$\begin{aligned}
&\langle \nabla^s u, A(x,y)\phi(y)\rangle + \langle \nabla^s \phi, A(x,y)u(x)\rangle \\
=&\int_{\mathbb{R}^{2n}} A(x,y)\cdot\left(\phi(y)\nabla^s u(x,y) + u(x)\nabla^s\phi(x,y)\right) dx\,dy \\
=&\int_{\mathbb{R}^{2n}} \sqrt{\frac{C_{n,s}}{2}} A(x,y)\cdot\frac{(y-x)}{|y-x|^{\frac{n}{2}+s+1}}\big(\phi(y)(u(x)-u(y)) + u(x)(\phi(x)-\phi(y))\big) dx\,dy \\
=&\int_{\mathbb{R}^{2n}} \sqrt{\frac{C_{n,s}}{2}} A_{s||}\cdot\frac{(y-x)}{|y-x|^{\frac{n}{2}+s+1}}\big(u(x)\phi(x) - u(y)\phi(y)\big) dx\,dy \\
=&\langle A_{s||}, \nabla^s(u\phi)\rangle.
\end{aligned}$$

Combining these together, we obtain

$$II = \langle A_{s||}, \nabla^s(u\phi)\rangle + \langle Au, A\phi\rangle = \int_\Omega \left((\nabla\cdot)^s A_{s||}(x) + \int_{\mathbb{R}^n} |A(x,y)|^2\ dy\right)\phi(x)u(x)\ dx \le 0$$

by the assumption (2.2) and the fact $u(x)\phi(x) \le 0$ in $\mathbb{R}^n$. Note that the fourth line is due to that the integrand is the product of a symmetric, parallel vector and A.

This completes the proof of $\langle \nabla^s u + Au, \nabla^s\phi + A\phi\rangle < 0$, which in turn concludes the proof of the proposition by contradiction. □

Remark 2.1. *In the proof above, it actually verifies a more accessible format of* $(-\Delta)^s_A$ *in weak sense (was also presented in Lemma 3.3 of* [10]*) given by*

$$(-\Delta)^s_A u = (-\Delta)^s u + 2\int_{\mathbb{R}^n} \left(A_{a||}\cdot\nabla^s u\right)\ dy + \left((\nabla\cdot)^s A_{s||} + \int_{\mathbb{R}^n} |A|^2\ dy\right) u \tag{2.4}$$

for $u \in H^s(\mathbb{R}^n)$.

We now build a barrier function.

Lemma 1 (Barrier)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Assume that $A \in C_c^s(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$ satisfies* (2.1) *and* (2.2)*. Then there exists $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfying*

$$\begin{cases} (-\Delta)_A^s \varphi \geq 1 & \text{in } \Omega, \\ \varphi \geq 0 & \text{in } \mathbb{R}^n, \\ \varphi \leq C & \text{in } \Omega, \end{cases}$$

for some constant C depending on n, s and Ω.

Proof. We will show the barrier function φ in Lemma 3.4 in [35] qualifies. More specifically, let B_R be a large ball such that $\Omega \Subset B_R$ and $\eta \in C_c^\infty(B_R)$ be a smooth cutoff function satisfying

$$0 \leq \eta \leq 1 \quad \text{in } \mathbb{R}^n, \qquad \eta \equiv 1 \quad \text{in } \Omega.$$

We directly use the format (2.4). By the definition of $(-\Delta)^s$ and the fact that η has maximum value 1 in Ω, one has when $x \in \Omega$

$$\begin{aligned} (-\Delta)^s \eta(x) &= C_{n,s} \int_{\mathbb{R}^n} \frac{\eta(x) - \eta(z)}{|x-z|^{n+2s}}\, dz \\ &\geq C_{n,s} \int_{\mathbb{R}^n \setminus B_R} \frac{1}{|x-z|^{n+2s}}\, dz \\ &\geq \frac{C_{n,s}}{2} \int_{\mathbb{R}^n \setminus B_R} \frac{1}{(R+|z|)^{n+2s}}\, dz := \lambda > 0, \end{aligned}$$

where λ depends only on n, s and Ω. By the condition (2.2) and A is compactly supported in $\Omega \times \Omega$, we only need to show

$$2 \int_{\mathbb{R}^n} \left(A_{a||} \cdot \nabla^s \eta\right)\, dy = 2 \int_{\Omega} \left(A_{a||} \cdot \nabla^s \eta\right)\, dy \geq 0$$

when $x \in \Omega$. This is verified because the integrand

$$\left(A_{a||} \cdot \nabla^s \eta\right)(x,y) = \sqrt{\frac{C_{n,s}}{2}} \frac{A_a(x,y) \cdot (y-x)}{|x-y|^{n/2+s+1}} (\eta(x) - \eta(y)) \geq 0 \qquad \text{when } x \in \Omega.$$

Here we applied (2.1) to obtain $A_a(x,y) \cdot (y-x) = A_{a||}(x,y) \cdot (y-x) \geq 0$ and also observed that $\eta(x) - \eta(y) = 1 - \eta(y) \geq 0$ for $x \in \Omega$ and $y \in \mathbb{R}^n$. Thus we have shown that $(-\Delta)_A^s \eta \geq \lambda$. Finally, set $\varphi(x) = \frac{\eta(x)}{\lambda}$. The upper bound C is $1/\lambda$, hence depends on n, s and Ω. □

With the maximum principle and the barrier function, we can show an L^∞-estimate for the weak solution.

Proposition 3 (L^∞-bound for the weak solution of FMSE)**.** *Let Ω be a bounded smooth domain and $A \in C_c^s(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ satisfy* (2.1) *and* (2.2)*. For $F \in L^\infty(\Omega)$ and $g \in L^\infty(\Omega_e)$, assume that $u \in H^s(\Omega)$ is a weak solution of*

$$\begin{cases} (-\Delta)_A^s u = F & \text{in } \Omega, \\ u = g & \text{in } \Omega_e. \end{cases} \tag{2.5}$$

Then

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq \|g\|_{L^\infty(\Omega_e)} + C\|F\|_{L^\infty(\Omega)}$$

for some constant $C > 0$ depending on n, s and Ω. (It can be the same constant as in Lemma 1.)

Proof. It is a standard proof such as in [35]. For completeness we prove it here for $(-\Delta)^s_A$. We set

$$v(x) = \|g\|_{L^\infty(\Omega_e)} + \|F\|_{L^\infty(\Omega)}\varphi(x) \qquad \text{in } \mathbb{R}^n,$$

where φ is the barrier function in Lemma 1. Since $(-\Delta)^s_A\varphi \geq 1$ in Ω,

$$(-\Delta)^s_A(\|g\|_{L^\infty(\Omega_e)}) = \left((\nabla\cdot)^s A_{s||} + \int_{\mathbb{R}^n} |A|^2 \, dy\right)\|g\|_{L^\infty(\Omega_e)}$$

and (2.2), we have

$$(-\Delta)^s_A v \geq \|F\|_{L^\infty(\Omega)} \geq F = (-\Delta)^s_A u,$$

in Ω. Moreover, we get $v \geq \|g\|_{L^\infty(\Omega_e)} \geq u$ in Ω_e due to $\varphi \geq 0$ in Ω_e. Applying Proposition 2 we obtain $v - u \geq 0$ in $\mathbb{R}^n$. This proves

$$u(x) \leq \|g\|_{L^\infty(\Omega_e)} + C\|F\|_{L^\infty(\Omega)} \qquad \text{in } \Omega.$$

Similarly, the same would hold for $-u$. This completes the proof. □

Remark 2.2. *Here we can certainly add a scalar potential $q \in L^\infty(\Omega, \mathbb{R})$ to obtain the above results (Proposition 2, Lemma 1, Proposition 3) for equation $(-\Delta)^s_A u + q(x)u = F$ with the condition* (2.2) *replaced by*

$$(\nabla\cdot)^s A_{s||} + \int_{\mathbb{R}^n} |A(x,y)|^2 \, dy + q(x) \geq 0 \qquad \text{for } x \in \Omega. \tag{2.6}$$

We also need the following strong maximum principle in the proof of Theorem 1.

Proposition 4 (Strong maximum principle)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $A(x,y) \in C^s_c(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and $q \in L^\infty(\Omega, \mathbb{R})$ satisfy conditions* (2.1) *and* (2.6)*. Suppose $u \in H^s(\Omega)$ is a weak solution of*

$$\begin{cases} (-\Delta)^s_A u + qu = F & \text{in } \Omega, \\ u = g & \text{in } \Omega_e. \end{cases}$$

Then if $0 \leq F \in L^\infty(\Omega)$ and $0 \leq g \in L^\infty(\Omega_e)$ with $g \not\equiv 0$, we have $u > 0$ in Ω.

Proof. Proposition 2 yields that $u \geq 0$ in $\mathbb{R}^n$. Suppose that u is not strictly positive in Ω. Then there must exists a nonempty subset $B \subset \Omega$ with positive measure so that $u = 0$ in B. We take a smooth function $0 \neq \varphi \in C^\infty_c(B)$ satisfying $\varphi \geq 0$ in B. Since u is the weak solution, by (2.4), we have

$$\begin{aligned} 0 &\leq \int_B F(x)\varphi(x)\,dx = \int_B ((-\Delta)^s_A u(x) + q(x)u(x))\varphi(x)\,dx \\ &= \int_B [-C_{n,s}\int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n+2s}}\,dy - \sqrt{2C_{n,s}}\int_{\mathbb{R}^n} \frac{A_a(x,y)\cdot(y-x)u(y)}{|x-y|^{n/2+s+1}}\,dy]\varphi(x)\,dx \\ &\leq 0, \end{aligned} \tag{2.7}$$

where the last inequality follows by the fact that $u, \varphi \geq 0$, the constant $C_{n,s} > 0$ and also $A_a(x,y)\cdot(y-x) \geq 0$ based on the assumption. This further yields that the first term in (2.7) satisfies

$$\int_B -C_{n,s}\int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n+2s}}\,dy\varphi(x)dx = 0.$$

Now due to $C_{n,s} > 0$, $u, \varphi \geq 0$, from the above identity, we can then derive $\int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n+2s}}\,dy = 0$, which leads to $u \equiv 0$ in $\mathbb{R}^n$ and contradicts the assumption that $g \not\equiv 0$ in Ω_e. □

2.4. **Well-posedness of a nonlinear fractional MSE.** With the L^∞-bound, we will show that the solution of the fractional magnetic equation (2.5) indeed has C^s regularity. This regularity is essential to prove the well-posedness result later.

Lemma 2 (C^s-estimate)**.** *Let Ω be a bounded smooth domain in $\mathbb{R}^n$ ($n \geq 2$). Suppose $A \in C_c^s(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and $q \in L^\infty(\Omega, \mathbb{R})$ satisfy conditions* (2.1) *and* (2.6)*. Suppose also that $(\nabla\cdot)^s A_{s||} \in L^\infty(\mathbb{R}^n)$. Given $F \in L^\infty(\Omega)$ and $g \in C_c^\infty(\Omega_e)$, if $u \in H^s(\mathbb{R}^n)$ is a weak solution of*

$$\begin{cases} (-\Delta)_A^s u + qu = F & \text{in } \Omega, \\ u = g & \text{in } \Omega_e, \end{cases} \tag{2.8}$$

then we have $u \in C^s(\mathbb{R}^n)$ and u satisfies

$$\|u\|_{C^s(\mathbb{R}^n)} \leq C\left(\|F\|_{L^\infty(\Omega)} + \|g\|_{C^s(\Omega_e)}\right), \tag{2.9}$$

where the constant C depends on $A, q, n, s,$ and Ω.

Proof. We first extend $g \in C_c^\infty(\Omega_e)$ to the whole $\mathbb{R}^n$ by zero and denote this extension by g as well. Then $\widetilde{u} := u - g$ satisfies

$$\begin{cases} (-\Delta)_A^s \widetilde{u} + q\widetilde{u} = F - [(-\Delta)_A^s + q]g =: \widetilde{F} \in L^\infty(\Omega) & \text{in } \Omega, \\ \widetilde{u} = 0 & \text{in } \Omega_e. \end{cases} \tag{2.10}$$

Note that since $n \geq 2$, the change of variables yields that $(-\Delta)_A^s g \in L^\infty(\Omega)$. By Proposition 3 and Remark 2.2, one has

$$\|\widetilde{u}\|_{L^\infty(\mathbb{R}^n)} \leq C_{n,s,\Omega}\|\widetilde{F}\|_{L^\infty(\Omega)},$$

where the constant $C_{n,s,\Omega}$ depends on n, s, Ω. Also, from (2.4), we have

$$(-\Delta)^s\widetilde{u} = \widetilde{F} - H_{\widetilde{u}}, \qquad \widetilde{u}|_{\Omega_e} = 0, \tag{2.11}$$

where

$$H_{\widetilde{u}} := 2\int_{\mathbb{R}^n} \left(A_{a||}\cdot\nabla^s\widetilde{u}\right)\,dy + \left((\nabla\cdot)^s A_{s||} + \int_{\mathbb{R}^n}|A|^2\,dy + q\right)\widetilde{u}.$$

Note that since A is C^s and compactly supported in $\Omega\times\Omega$ and $n/2\geq 1>s>0$, we can derive that $\frac{A_a(x,y)\cdot(y-x)}{|x-y|^{n/2+s+1}}$ is integrable in $\mathbb{R}^n$ and thus

$$\begin{aligned}\left|2\int_{\mathbb{R}^n}\left(A_{a||}\cdot\nabla^s\widetilde{u}\right)\,dy\right| &= \left|\sqrt{2C_{n,s}}\int_{\mathbb{R}^n}\left(A_a(x,y)\cdot(y-x)\frac{\widetilde{u}(x)-\widetilde{u}(y)}{|x-y|^{n/2+s+1}}\right)\,dy\right|\\ &\leq 2\sqrt{2C_{n,s}}\|\widetilde{u}\|_{L^\infty(\Omega)}\int_{\mathbb{R}^n}\frac{|A_a\cdot(y-x)|}{|x-y|^{n/2+s+1}}\,dy\\ &\leq C_{A,n,s,\Omega}\|\widetilde{F}\|_{L^\infty(\Omega)};\end{aligned}$$

and $\left((\nabla\cdot)^sA_{s||}+\int_{\mathbb{R}^n}|A|^2\,dy+q\right)$ is bounded in Ω. This yields that

$$\left|\left((\nabla\cdot)^sA_{s||}+\int_{\mathbb{R}^n}|A|^2\,dy+q\right)\widetilde{u}\right|\leq C_{A,q,n,s,\Omega}\|\widetilde{F}\|_{L^\infty(\Omega)}\qquad\text{in }\Omega.$$

Therefore,

$$\|H_{\widetilde{u}}\|_{L^\infty(\Omega)}\leq C_{A,q,n,s,\Omega}\|\widetilde{F}\|_{L^\infty(\Omega)}.$$

We apply to (2.11) the classical C^s elliptic regularity (see [50, Proposition 1.1]) for the fractional laplacian, that is, the solution to $(-\Delta)^su=g$ in Ω with $u=0$ in $\mathbb{R}^n\backslash\Omega$ belongs to $C^s(\mathbb{R}^n)$ and satisfies $\|u\|_{C^s(\mathbb{R}^n)}\leq C\|g\|_{L^\infty(\Omega)}$, given $g\in L^\infty(\Omega)$. Then we have $\widetilde{u}\in C^s(\mathbb{R}^n)$ and

$$\|\widetilde{u}\|_{C^s(\mathbb{R}^n)}\leq C_{\Omega,s}\|\widetilde{F}-H_{\widetilde{u}}\|_{L^\infty(\Omega)}\leq C_{A,q,n,s,\Omega}\|\widetilde{F}\|_{L^\infty(\Omega)},$$

which proves (2.9). □

We are ready to prove unique existence of solutions to the nonlinear FMSE.

Theorem 2 (Well-posedness for the nonlinear equation)**.** *Let Ω be a bounded smooth domain and $A\in C^s_c(\mathbb{R}^n\times\mathbb{R}^n,\mathbb{C}^n)$ and $a(x,z)$ satisfy (1.3), (1.4) and (1.5). Moreover, assume that 0 is not an eigenvalue for the linear operator $(-\Delta)^s_A+\partial_z a(x,0)$. Then there exists $\varepsilon>0$ small enough such that when $g\in C^\infty_c(\Omega_e)$ with $\|g\|_{C^\infty_c(\Omega_e)}<\varepsilon$, the boundary value problem*

$$\begin{cases}(-\Delta)^s_Au+a(x,u)=0 & \text{in }\Omega,\\ u=g & \text{in }\Omega_e,\end{cases}\tag{2.12}$$

admits a unique solution $u\in C^s(\mathbb{R}^n)\cap H^s(\mathbb{R}^n)$ satisfying

$$\|u\|_{C^s(\mathbb{R}^n)}\leq C\|g\|_{C^\infty_c(\Omega_e)},\tag{2.13}$$

where C is a constant depending on $A,\partial_z a(x,0),n,s,$ and Ω.

Proof. The well-posedness of the direct problem for FMSE (see Lemma 2.6 from [51]) and the regularity estimate in Lemma 2 yield that there exists a unique solution $u_0\in C^s(\mathbb{R}^n)\cap H^s(\mathbb{R}^n)$ to the linear equation

$$\begin{cases}(-\Delta)^s_Au_0+\partial_z a(x,0)u_0=0 & \text{in }\Omega,\\ u_0=g & \text{in }\Omega_e,\end{cases}\tag{2.14}$$

such that

$$\|u_0\|_{C^s(\mathbb{R}^n)} \le C\|g\|_{C_c^\infty(\Omega_e)}. \tag{2.15}$$

Then looking for a solution of (2.12) is equivalent to solving for $v := u - u_0$ in

$$\begin{cases} (-\Delta)_A^s v + \partial_z a(x,0)v = -\left(a(x,u_0+v) - \partial_z a(x,0)(u_0+v)\right) & \text{in } \Omega, \\ v = 0 & \text{in } \Omega_e. \end{cases} \tag{2.16}$$

where we use the assumption $a(x,0) = 0$. To achieve this, let us define the set

$$X_\delta = \left\{\phi \in C^s(\mathbb{R}^n) :\ \phi|_{\Omega_e} = 0,\ \|\phi\|_{C^s(\mathbb{R}^n)} \le \delta\right\},$$

where $0 < \delta < 1$ will be determined later. It is easy to see that X_δ is a Banach space. We also define the map $\mathcal{F} : X_\delta \to L^\infty(\Omega)$, by

$$\mathcal{F}(v) := -\left(a(x,u_0+v) - \partial_z a(x,0)(u_0+v)\right),$$

and note that the operator

$$\mathcal{L}_s^{-1} : F \in L^\infty(\Omega) \ \mapsto\ u_{F,0} \in C^s(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$$

is bounded, where $u_{F,0}$ denotes the solution of $\left[(-\Delta)_A^s + \partial_z a(x,0)\right] u_{F,0} = F$ in Ω and $u_{F,0} = 0$ in Ω_e. It suffices to show that $\mathcal{L}_s^{-1} \circ \mathcal{F}$ is a contraction map on X_δ.

To show $\mathcal{L}_s^{-1} \circ \mathcal{F}$ is contractive, we first apply Taylor's theorem and the assumption $a(x,0) = 0$ to obtain

$$a(x,z) = \partial_z a(x,0)z + a_r(x,z)z^2, \qquad (x,z) \in \Omega \times \mathbb{R}$$

with

$$a_r(x,z) := \int_0^1 \partial_z^2 a(x,tz)(1-t)\, dt.$$

Therefore,

$$\mathcal{F}(v) = -a_r(x,u_0+v)(u_0+v)^2.$$

In Ω, it is not hard to see that $a_r(x,u_0+v)$ is bounded by a constant. Then

$$\|\mathcal{F}(v)\|_{L^\infty(\Omega)} \le C\left(\|u_0\|_{C^s(\mathbb{R}^n)} + \|v\|_{C^s(\mathbb{R}^n)}\right)^2 \le C(\varepsilon+\delta)^2 \tag{2.17}$$

for $v \in X_\delta$. This implies

$$\|\mathcal{L}_s^{-1} \circ \mathcal{F}(v)\|_{C^s(\mathbb{R}^n)} \le C(\varepsilon+\delta)^2 \le \delta$$

when $\varepsilon < \mathcal{C}\delta$ for some $\mathcal{C} > 0$ (this is to say that δ cannot be arbitrarily small, but depending on ε) and δ is small enough, that is, $\mathcal{L}_s^{-1} \circ \mathcal{F}$ maps X_δ to itself.

To show it is a contraction, we derive for $v_1, v_2 \in X_\delta$

$$\begin{aligned} \|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_{L^\infty(\Omega)} &\le \|a_r(x,u_0+v_1) - a_r(x,u_0+v_2)\|_{L^\infty(\Omega)} \left(\|u_0\|_{C^s(\mathbb{R}^n)} + \|v_1\|_{C^s(\mathbb{R}^n)}\right)^2 \\ &\quad + \|a_r(x,u_0+v_2)\|_{L^\infty(\Omega)} \|2u_0 + v_1 + v_2\|_{C^s(\mathbb{R}^n)} \|v_1 - v_2\|_{C^s(\mathbb{R}^n)} \\ &\le C[(\varepsilon+\delta)^2 + 2\varepsilon + 2\delta]\|v_1 - v_2\|_{C^s(\mathbb{R}^n)}, \end{aligned}$$

where we used the fact that $a_r(x,z)$ is Lipchitz in z. This shows

$$\|\mathcal{L}_s^{-1}(\mathcal{F}(v_1) - \mathcal{F}(v_2))\|_{C^s(\mathbb{R}^n)} \le C[(\varepsilon+\delta)^2 + 2\varepsilon + 2\delta]\|v_1 - v_2\|_{C^s(\mathbb{R}^n)},$$

hence $\mathcal{L}_s^{-1} \circ \mathcal{F}$ is a contraction when ε and δ are small enough.

Therefore, the contraction mapping principle yields that there exists a fixed point $v \in X_\delta$ so that $v = \mathcal{L}_s^{-1} \circ \mathcal{F}(v)$ satisfies (2.16). Lastly, we have by (2.15) and (2.17)

$$\|v\|_{C^s(\mathbb{R}^n)} = \|\mathcal{L}_s^{-1} \circ \mathcal{F}(v)\|_{C^s(\mathbb{R}^n)} \leq C\|\mathcal{F}(v)\|_{L^\infty(\Omega)} \leq C(\varepsilon+\delta)(\|g\|_{C_c^\infty(\Omega_e)} + \|v\|_{C^s(\mathbb{R}^n)}),$$

which gives

$$\|v\|_{C^s(\mathbb{R}^n)} \leq C\|g\|_{C_c^\infty(\Omega_e)}$$

if ε, δ are sufficiently small. Combining the above estimate for v with (2.15), this gives (2.13). □

Corollary 1. *Assume that Ω, A and a are as in Theorem 2. Let u be the unique solution to (2.12) for $g \in C_c^\infty(\Omega_e)$ with $\|g\|_{C_c^\infty(\Omega_e)} < \varepsilon$. Then for $\varepsilon > 0$ small enough, we have*

$$\|u\|_{H^s(\mathbb{R}^n)} \leq C\|g\|_{H^s(\mathbb{R}^n)}.$$

Proof. In the above proof of Theorem 2, indeed we also have

$$\|u_0\|_{H^s(\mathbb{R}^n)} \leq C\|g\|_{H^s(\mathbb{R}^n)}$$

from (2.14) and the well-posedness result of [10] for the linear FMSE. Then using the C^s bounds of u_0 and v from the theorem, we obtain

$$\|\mathcal{F}(v)\|_{H^{-s}(\Omega)} \leq \|\mathcal{F}(v)\|_{H^s(\Omega)} \leq C(\varepsilon+\delta)(\|g\|_{H^s(\mathbb{R}^n)} + \|v\|_{H^s(\mathbb{R}^n)})$$

for δ, ε as in the proof of the theorem. Then the well-posedness result of [10] also implies the solution of (2.16) has H^s bound, which satisfies

$$\|v\|_{H^s(\mathbb{R}^n)} \leq C\|\mathcal{F}(v)\|_{H^{-s}(\Omega)} \leq C(\varepsilon+\delta)(\|g\|_{H^s(\mathbb{R}^n)} + \|v\|_{H^s(\mathbb{R}^n)}).$$

Lastly, by choosing the δ, ε pair in the proof of Theorem 2 to be further small to obtain

$$\|v\|_{H^s(\mathbb{R}^n)} \leq C\|g\|_{H^s(\mathbb{R}^n)}.$$

From $u = u_0 + v$ and the estimates for u_0 and v above, the proof is complete. □

3. The inverse problem

In this section, we will reconstruct the magnetic potential and nonlinear electric potential. We have showed in Theorem 2 that for any

$$g \in \mathcal{S}_{\varepsilon_0} := \{g \in C_c^\infty(\Omega_e) : \|g\|_{C_c^\infty(\Omega_e)} \leq \varepsilon_0\},$$

with $\varepsilon_0 > 0$ small enough, there is a unique small solution $u_g \in C^s(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ to the problem

$$\begin{cases} (-\Delta)_A^s u + a(x,u) = 0 & \text{in } \Omega, \\ u = g & \text{in } \Omega_e. \end{cases} \tag{3.1}$$

To show that the map $g \mapsto u_g$ is differentiable in $\mathcal{S}_{\varepsilon_0}$, we consider for sufficiently small $\varepsilon > 0$ and $f \in C_c^\infty(\Omega_e)$, let $u_{\varepsilon f} = u_{\varepsilon f}(x;\varepsilon)$ be the unique small solution to the problem

$$\begin{cases} (-\Delta)_A^s u + a(x,u) = 0 & \text{in } \Omega, \\ u = \varepsilon f & \text{in } \Omega_e. \end{cases} \tag{3.2}$$

3.1. **Linearization.** We define the k-th derivative of the solution $u_{\varepsilon f}$ with respect to ε by

$$u_\varepsilon^{(k)}(x) := \frac{d^k}{d\varepsilon^k} u_{\varepsilon f}(x;\varepsilon)$$

for any positive integer k. We show that $u_\varepsilon^{(k)}$ satisfies various linear equations.

Lemma 3. *Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary. Assume $A \in C_c^s(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$ and $a = a(x,z) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfy (1.3), (1.4) and (1.5). Let $f \in C_c^\infty(\Omega_e)$, $f \neq 0$ and $\varepsilon \in \mathbb{R}$ satisfy $|\varepsilon| < \frac{\varepsilon_0}{\|f\|_{C_c^\infty(\Omega_e)}}$ for above $\varepsilon_0 > 0$. Then we have $u_{\varepsilon f}$ is infinitely many times differentiable in ε in $\left(-\frac{\varepsilon_0}{\|f\|_{C_c^\infty(\Omega_e)}}, \frac{\varepsilon_0}{\|f\|_{C_c^\infty(\Omega_e)}}\right)$. Moreover, we have for $k = 1$,*

$$(-\Delta)_A^s u_\varepsilon^{(1)} + \partial_z a(x, u_{\varepsilon f}) u_\varepsilon^{(1)} = 0 \quad \text{in } \Omega, \qquad u_\varepsilon^{(1)} = f \quad \text{in } \Omega_e,$$

and for $k = 2, 3, \ldots$,

$$\begin{cases} (-\Delta)_A^s u_\varepsilon^{(k)} + \partial_z a(x, u_{\varepsilon f}) u_\varepsilon^{(k)} + \partial_z^k a(x, u_{\varepsilon f})(u_\varepsilon^{(1)})^k + R_{k-1}(a, u_{\varepsilon f}) = 0 & \text{in } \Omega, \\ u_\varepsilon^{(k)} = 0 & \text{in } \Omega_e, \end{cases} \tag{3.3}$$

where the term $R_{k-1}(a, u_{\varepsilon f})$ only involves $\partial_z^2 a(x, u_{\varepsilon f}), \ldots, \partial_z^{k-1} a(x, u_{\varepsilon f})$ and $u_\varepsilon^{(1)}, \ldots, u_\varepsilon^{(k-1)}$.

Proof. For $\varepsilon \in \left(-\frac{\varepsilon_0}{\|f\|_{C_c^\infty(\Omega_e)}}, \frac{\varepsilon_0}{\|f\|_{C_c^\infty(\Omega_e)}}\right)$ and $\Delta\varepsilon \neq 0$, set

$$\widetilde{u} = \frac{u_{(\varepsilon+\Delta\varepsilon)f} - u_{\varepsilon f}}{\Delta\varepsilon}.$$

By Taylor's formula, $\widetilde{u}$ is the solution to

$$(-\Delta)_A^s \widetilde{u} + a^*(x)\widetilde{u} = 0 \qquad \text{in } \Omega$$

with $\widetilde{u} = f$ in Ω_e, where

$$a^*(x) := \int_0^1 \partial_z a(x, s u_{(\varepsilon+\Delta\varepsilon)f} + (1-s) u_{\varepsilon f})\, ds$$

belongs to $L^\infty(\Omega)$ satisfies $\|a^*\|_{L^\infty(\Omega)}$ is bounded by a constant independent of the solution $u_{\varepsilon f}$ and

$$(\nabla\cdot)^s A_{s||} + \int_{\mathbb{R}^n} |A(x,y)|^2 \, dy + a^*(x) \geq 0$$

for $x \in \Omega$ due to (1.5). By Lemma 2,

$$\|\widetilde{u}\|_{C^s(\mathbb{R}^n)} \leq C \|f\|_{C_c^\infty(\Omega_e)},$$

which implies that

$$\|u_{(\varepsilon+\Delta\varepsilon)f} - u_{\varepsilon f}\|_{C^s(\mathbb{R}^n)} \leq C|\Delta\varepsilon| \|f\|_{C_c^\infty(\Omega_e)}. \tag{3.4}$$

By (3.4), we have

$$\begin{aligned}\|a^*(x)-\partial_z a(x,u_{\varepsilon f})\|_{L^\infty(\Omega)} &= \|\int_0^1 (\partial_z a(x, su_{(\varepsilon+\Delta\varepsilon)f}+(1-s)u_{\varepsilon f})-\partial_z a(x,u_{\varepsilon f}))\,ds\|_{L^\infty(\Omega)}\\ &\le \sup_{0<s<1}\|\partial_z a(x, su_{(\varepsilon+\Delta\varepsilon)f}+(1-s)u_{\varepsilon f})-\partial_z a(x,u_{\varepsilon f})\|_{L^\infty(\Omega)}\\ &\le \|\partial_z^2 a(x,z)\|_{L^\infty(\Omega\times(-c\varepsilon_0,c\varepsilon_0))}\|u_{(\varepsilon+\Delta\varepsilon)f}-u_{\varepsilon f}\|_{C^s(\mathbb{R}^n)}\\ &\le \|\partial_z^2 a(x,z)\|_{L^\infty(\Omega\times(-c\varepsilon_0,c\varepsilon_0))}|\Delta\varepsilon|\|f\|_{C_c^\infty(\Omega_e)}\end{aligned}$$

where $c>0$ is a uniform constant. This shows that as $\Delta\varepsilon\to 0$, we have $\widetilde{u}\to u_\varepsilon^{(1)}$ in $C^s(\mathbb{R}^n)$, where $u_\varepsilon^{(1)}$ is the solution to

$$(-\Delta)_A^s u_\varepsilon^{(1)}+\partial_z a(x,u_{\varepsilon f})u_\varepsilon^{(1)}=0 \quad \text{in } \Omega, \qquad u^{(1)}=f \quad \text{in } \Omega_e.$$

In fact, if we denote $w:=\widetilde{u}-u_\varepsilon^{(1)}$, it is then a solution to

$$(-\Delta)_A^s w+a^*(x)w=(\partial_z a(x,u_{\varepsilon f})-a^*(x))u_\varepsilon^{(1)} \quad \text{in } \Omega, \qquad w=0 \quad \text{in } \Omega_e.$$

By Lemma 2 we have the estimate

$$\begin{aligned}\|w\|_{C^s(\mathbb{R}^n)} &\le C\|(\partial_z a(x,u_{\varepsilon f})-a^*(x))u_\varepsilon^{(1)}\|_{L^\infty(\Omega)}\\ &\le C\|\partial_z a(x,u_{\varepsilon f})-a^*(x)\|_{L^\infty(\Omega)}\|u_\varepsilon^{(1)}\|_{C^s(\mathbb{R}^n)}\\ &\le \|\partial_z^2 a(x,z)\|_{L^\infty(\Omega\times(-c\varepsilon_0,c\varepsilon_0)}|\Delta\varepsilon|\|f\|^2_{C_c^\infty(\Omega_e)}\to 0, \qquad \text{as } \Delta\varepsilon\to 0.\end{aligned}$$

Similarly, set $\widetilde{u}^{(1)}:=\frac{u_{\varepsilon+\Delta\varepsilon}^{(1)}-u_\varepsilon^{(1)}}{\Delta\varepsilon}$. Then

$$(-\Delta)_A^s\widetilde{u}^{(1)}+\frac{1}{\Delta\varepsilon}\left[\partial_z a\left(x,u_{(\varepsilon+\Delta\varepsilon)f}\right)u_{\varepsilon+\Delta\varepsilon}^{(1)}-\partial_z a\left(x,u_{\varepsilon f}\right)u_\varepsilon^{(1)}\right]=0,$$

giving

$$(-\Delta)_A^s\widetilde{u}^{(1)}+a_1^*(x)\widetilde{u}u_{\varepsilon+\Delta\varepsilon}^{(1)}+\partial_z a(x,u_{\varepsilon f})\widetilde{u}^{(1)}=0,$$

where

$$a_1^*(x):=\int_0^1 \partial_z^2 a(x, su_{(\varepsilon+\Delta\varepsilon)f}+(1-s)u_{\varepsilon f})\,ds\to\partial_z^2 a(x,u_{\varepsilon f})$$

in L^∞ as $\Delta\varepsilon\to 0$. Similar to above, as $\Delta\varepsilon\to 0$, we have $\widetilde{u}^{(1)}\to u_\varepsilon^{(2)}$ where

$$(-\Delta)_A^s u_\varepsilon^{(2)}+\partial_z a(x,u_{\varepsilon f})u_\varepsilon^{(2)}+\partial_z^2 a(x,u_{\varepsilon f})(u_\varepsilon^{(1)})^2=0.$$

Here we used the continuity of $u_\varepsilon^{(1)}$ in ε which can be derived by following a similar argument in the derivation of (3.4) above. We apply the induction argument. Suppose (3.3) is true for index k and thus we have

$$\widetilde{u}^{(\ell)}:=\frac{u_{\varepsilon+\Delta\varepsilon}^{(\ell)}-u_\varepsilon^{(\ell)}}{\Delta\varepsilon}\to u_\varepsilon^{(\ell+1)} \tag{3.5}$$

for $\ell = 0, \dots, k-1$. Here we denote $\widetilde{u}^{(0)} := \widetilde{u}$. Set $\widetilde{u}^{(k)} := \frac{u^{(k)}_{\varepsilon+\Delta\varepsilon} - u^{(k)}_{\varepsilon}}{\Delta\varepsilon}$. Then one can check for $k \geq 2$,

$$\begin{aligned}(-\Delta)^s_A \widetilde{u}^{(k)} &+ \frac{1}{\Delta\varepsilon}\left[\partial_z a(x, u_{(\varepsilon+\Delta\varepsilon)f}) u^{(k)}_{\varepsilon+\Delta\varepsilon} - \partial_z a(x, u_{\varepsilon f}) u^{(k)}_{\varepsilon}\right] \\ &+ \frac{1}{\Delta\varepsilon}\left[\partial^k_z a(x, u_{(\varepsilon+\Delta\varepsilon)f})(u^{(1)}_{\varepsilon+\Delta\varepsilon})^k - \partial^k_z a(x, u_{\varepsilon f})(u^{(1)}_{\varepsilon})^k\right] \\ &+ \widetilde{R}_{k-1} = 0,\end{aligned} \tag{3.6}$$

where the term $\widetilde{R}_1 = 0$ and $\widetilde{R}_{k-1} := R_{k-1}(a, u_{(\varepsilon+\Delta\varepsilon)f}) - R_{k-1}(a, u_{\varepsilon f})$ involves $a^*_2(x), \dots, a^*_{k-1}(x)$, $\widetilde{u}, \widetilde{u}^{(1)}, \dots, \widetilde{u}^{(k-1)}$, $\partial^2_z a(x, u_{\varepsilon f}), \dots, \partial^{k-1}_z a(x, u_{\varepsilon f})$, $u^{(1)}_{\varepsilon}, \dots, u^{(k-1)}_{\varepsilon}$ and $u^{(1)}_{\varepsilon+\Delta\varepsilon}, \dots, u^{(k-1)}_{\varepsilon+\Delta\varepsilon}$ where

$$a^*_l(x) := \int_0^1 \partial^{l+1}_z a(x, s u_{(\varepsilon+\Delta\varepsilon)f} + (1-s) u_{\varepsilon f})\, ds.$$

As $\Delta\varepsilon \to 0$, we can derive $\tilde{u}^{(k)} \to u^{(k+1)}_{\varepsilon}$ due to (1.4), (3.5) and a similar argument above. Here $u^{(k+1)}_{\varepsilon}$ satisfies

$$(-\Delta)^s_A u^{(k+1)}_{\varepsilon} + \partial_z a(x, u_{\varepsilon f}) u^{(k+1)}_{\varepsilon} + \partial^{k+1}_z a(x, u_{\varepsilon f})(u^{(1)}_{\varepsilon})^{k+1} + R_k(a, u_{\varepsilon f}) = 0, \tag{3.7}$$

where

$$R_k(a, u_{\varepsilon f}) = \partial^2_z a(x, u_{\varepsilon f}) u^{(1)}_{\varepsilon} u^{(k)}_{\varepsilon} + k \partial^k_z a(x, u_{\varepsilon f})(u^{(1)}_{\varepsilon})^{k-1} u^{(2)}_{\varepsilon} + \lim_{\Delta\varepsilon \to 0} \widetilde{R}_{k-1}$$

only involves $\partial^2_z a(x, u_{\varepsilon f}), \dots, \partial^k_z a(x, u_{\varepsilon f})$ and $u^{(1)}_{\varepsilon}, \dots, u^{(k)}_{\varepsilon}$. This completes the proof. □

3.2. **The DN map.** We now define the operator $B^s_{A,a} : H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{R}$ by

$$B^s_{A,a}[u, v] := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \nabla^s_A u \cdot \nabla^s_A v \, dydx + \int_\Omega a(x, u) v dx.$$

Now we give a definition of the DN map $\Lambda^s_{A,a}$.

Lemma 4. *Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary. Assume $A \in C^s_c(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$ and $a = a(x, z) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfy (1.3), (1.4) and (1.5). There exists a bounded map $\Lambda^s_{A,a} : \{[g] \in X : g \in \mathcal{S}_{\varepsilon_0}\} \to X^*$ defined by*

$$\langle \Lambda^s_{A,a}[g], [v] \rangle := B^s_{A,a}[u_g, v] \qquad \forall\ v \in H^s(\mathbb{R}^n),\ g \in \mathcal{S}_{\varepsilon_0},$$

where X is the quotient space $H^s(\mathbb{R}^n)/\widetilde{H}^s(\Omega)$ and $u_g \in H^s(\mathbb{R}^n)$ solves $(-\Delta)^s_A u_g + a(x, u_g) = 0$ in Ω with $u_g - g \in \widetilde{H}^s(\Omega)$.

Proof. We first show that the definition of the DN map only depend on the equivalence classes. To this end, for any ϕ, ψ in $\widetilde{H}^s(\Omega)$, we have $u_{g+\phi} = u_g$ on $\mathbb{R}^n$ by uniqueness of the solution. Also, since $B^s_{A,s}$ is linear in the second component,

$$B^s_{A,a}[u_{g+\phi}, v + \psi] = B^s_{A,a}[u_g, v + \psi] = B^s_{A,a}[u_g, v] + B^s_{A,a}[u_g, \psi],$$

where

$$B^s_{A,a}[u_g, \psi] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \nabla^s_A u_g \cdot \nabla^s_A \psi \, dydx + \int_\Omega a(x, u_g)\psi dx = 0$$

by the weak formulation of the equation $(-\Delta)^s_A u_g + a(x, u_g) = 0$ in Ω and $\psi \in \widetilde{H}^s(\Omega)$.

Next the map is bounded because

$$\begin{aligned}|B^s_{A,a}[u_g,v]| &\le C\|\nabla^s_A u_g\|_{L^2(\mathbb{R}^{2n})}\|\nabla^s_A v\|_{L^2(\mathbb{R}^{2n})} + \|a(x,u_g)\|_{L^2(\Omega)}\|v\|_{L^2(\mathbb{R}^n)}\\ &\le C\|u_g\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}\\ &\le C\|g\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)},\end{aligned}$$

by Corollary 1. Here we used the fact that $a(x,0)=0$ and Taylor's Theorem yield

$$\begin{aligned}\|a(x,u_g)\|_{L^2(\Omega)} &= \left\| \left(\int_0^1 \partial_z a(x, su_g(x))\, ds \right)\, u_g(x) \right\|_{L^2(\Omega)}\\ &\le \|\partial_z a(x,z)\|_{L^\infty(\Omega\times B_{C\varepsilon_0})}\|u_g\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

where $B_{C\varepsilon_0}$ stands for a ball with center at the origin and radius $C\varepsilon_0>0$. This completes the proof. □

Lemma 3 implies that the solution $u_{\varepsilon f}=u_{\varepsilon f}(x;\varepsilon)$ to (3.2) is differentiable with respect to ε in the space $C^s(\mathbb{R}^n)$. To simplify the notation, we now denote the k-th derivative of the solution u with respect to ε at $\varepsilon=0$ by

$$u^{(k)}(x) = \frac{d^k}{d\varepsilon^k}\Big|_{\varepsilon=0} u_{\varepsilon f}(x;\varepsilon). \tag{3.8}$$

Moreover, this allows us to take the k-th derivative $\Lambda^{(k),s}_{A,a}$ of the map $\Lambda^s_{A,a}$ with respect to ε at $\varepsilon=0$:

$$\langle \Lambda^{(k),s}_{A,a}[f],[v]\rangle := \lim_{\varepsilon\to 0}\frac{d^k}{d\varepsilon^k} B^s_{A,a}[u_{\varepsilon f}(x;\varepsilon),v], \qquad f\in C^\infty_c(\Omega_e),\ v\in H^s(\mathbb{R}^n).$$

Lemma 5. *Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary. Assume $A\in C^s_c(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$ and $a=a(x,z):\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ satisfy* (1.3), (1.4) *and* (1.5). *Then we have*

$$\langle \Lambda^{(1),s}_{A,a}[f],[v]\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \nabla^s_A u^{(1)}\cdot\nabla^s_A v\, dydx + \int_\Omega \partial_z a(x,0)u^{(1)}v\, dx,$$

and for $k\ge 2$,

$$\begin{aligned}\langle \Lambda^{(k),s}_{A,a}[f],[v]\rangle = &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \nabla^s_A u^{(k)}\cdot\nabla^s_A v\, dydx\\ &+\int_\Omega \left(\partial_z a(x,0)u^{(k)} + \partial^k_z a(x,0)(u^{(1)})^k + R_{k-1}(a,u)\right) v\, dx,\end{aligned}$$

for $v\in H^s(\mathbb{R}^n)$, where the term $R_1=0$ and $R_{k-1}(a,u)$ only contains $\partial^2_z a(x,0),\dots,\partial^{k-1}_z a(x,0)$ and $u^{(1)},\dots,u^{(k-1)}$.

Proof. For any $v\in H^s(\mathbb{R}^n)$, by the C^s regularity of A and a, we can justify passing the limits into the integral to obtain

$$\begin{aligned}&\frac{d^k}{d\varepsilon^k}\Big|_{\varepsilon=0} B^s_{A,a}[u_{\varepsilon f}(x,\varepsilon),v]\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\nabla^s_A u^{(k)}(x)\cdot\nabla^s_A v\, dydx + \int_\Omega \left(\partial_z a(x,0)u^{(k)} + \ldots + \partial^k_z a(x,0)(u^{(1)})^k\right) v\, dx.\end{aligned}$$

□

3.3. **Proof of Theorem 1.** Below are the steps to prove the Theorem 1. Assume

$$\Lambda^s_{A_1,a_1}[g]\big|_{W_2} = \Lambda^s_{A_2,a_2}[g]\big|_{W_2} \qquad \text{for any } g \in \mathcal{S}_{\varepsilon_0} \cap C_c^\infty(W_1). \tag{3.9}$$

where W_1, W_2 are two arbitrary open subsets in Ω_e. For $f \in C_c^\infty(W_1)$ and $\varepsilon > 0$ small enough, let u_j be the solution to (1.1) with the exterior data εf and A, a replaced by A_j, a_j for $j = 1, 2$. The hypothesis (3.9) yields

$$0 = \langle (\Lambda^s_{A_1,a_1} - \Lambda^s_{A_2,a_2})[\varepsilon f], [v]\rangle = B^s_{A_1,a_1}[u_1(x,\varepsilon), v] - B^s_{A_2,a_2}[u_2(x,\varepsilon), v]$$

for any $v \in H^s(\mathbb{R}^n)$ satisfying $\mathrm{supp}(v\chi_{\Omega_e}) \subset W_2$ where χ_{Ω_e} denotes the characteristic function of Ω_e. Immediately, we have the k-derivatives of $\Lambda^s_{A_j,a_j}$ satisfying

$$\Lambda^{(k),s}_{A_1,a_1}[f]\Big|_{W_2} = \Lambda^{(k),s}_{A_2,a_2}[f]\Big|_{W_2} \qquad \text{for } \ f \in C_c^\infty(W_1).$$

3.3.1. *The determination of the first order term.* We can recover A and $\partial_z a$ up to a gauge.

Proposition 5. *Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary. Assume $A_j \in C_c^s(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$ with support in $\Omega \times \Omega$ and $a_j = a_j(x,z) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfy (1.3), (1.4) and (1.5) for $j = 1, 2$. Suppose $\Lambda^s_{A_1,a_1}[g]|_{W_2} = \Lambda^s_{A_2,a_2}[g]|_{W_2}$ for $g \in \mathcal{S}_{\varepsilon_0} \cap C_c^\infty(W_1)$. Then we have*

$$(A_1, \partial_z a_1(\cdot, 0)) \sim (A_2, \partial_z a_2(\cdot, 0)).$$

Proof. It is shown in [10] that $\Lambda^{(1),s}_{A,a}$ is linear, symmetric and bounded on X, and $\Lambda^{(1),s}_{A,a}[f]|_{W_2}$ for all $f \in C_c^\infty(W_1)$ determines $(A(x,y), q(x))$ up to the gauge $\sim$, where $q(x) = \partial_z a(x,0)$ here. □

We need the following lemma for determining the full nonlinear potential $a(x,u)$.

Lemma 6. *Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary. Assume $A \in C_c^s(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$ and $a = a(x,z) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfy (1.3), (1.4), (1.5) and*

$$(A_1, \partial_z a_1(\cdot, 0)) \sim (A_2, \partial_z a_2(\cdot, 0)).$$

Given $f \in C_c^\infty(\Omega_e)$ and $\varepsilon > 0$ small, let $u_j = u_j(x;\varepsilon)$ be the solution to

$$\begin{cases} (-\Delta)^s_{A_j} u + a_j(x,u) = 0 & \text{in } \Omega, \\ u = \varepsilon f & \text{in } \Omega_e, \end{cases} \tag{3.10}$$

for $j = 1, 2$. Then we have $u_1^{(1)} = u_2^{(1)}$. Moreover, for $k \geq 2$, if

$$\partial_z^\ell a_1(x,0) = \partial_z^\ell a_2(x,0) \ \text{ in } \Omega \qquad \text{for any } 2 \leq \ell \leq k,$$

then

$$u_1^{(\ell)} = u_2^{(\ell)} \ \text{ in } \mathbb{R}^n \qquad \text{for any } 2 \leq \ell \leq k, \tag{3.11}$$

where $u_j^{(\ell)}$ is defined as in (3.8).

Proof. First, by Lemma 3, we have

$$(-\Delta)^s_{A_j} u_j^{(1)} + \partial_z a_j(x,0) u_j^{(1)} = 0 \quad \text{in } \Omega, \qquad u_j^{(1)} = f \quad \text{in } \Omega_e.$$

Since $(A_1, \partial_z a_1(x,0)) \sim (A_2, \partial_z a_2(x,0))$, the definition of the equivalence relation, (2.4) and the uniqueness of the solution to the linear FMSE yield that

$$u_1^{(1)} = u_2^{(1)} \qquad \text{in } \mathbb{R}^n.$$

For $k = 2$, by Lemma 3 again, $u_j^{(2)}$ are solutions to

$$(-\Delta)^s_{A_j} u_j^{(2)} + \partial_z a_j(x,0) u_j^{(2)} = -\partial_z^2 a_j(x,0)(u_j^{(1)})^2$$

When $\partial_z^2 a_1(x,0) = \partial_z^2 a_2(x,0)$ in Ω, the equations for $u_j^{(2)}$, $j = 1,2$ are identical and both $u_1^{(2)}$ and $u_2^{(2)}$ are zero in Ω_e. Then we obtain $u_1^{(2)} = u_2^{(2)}$ by the well-posedness.

Now suppose for $k \geq 2$, $\partial_z^\ell a_1(x,0) = \partial_z^\ell a_2(x,0)$ in Ω for $2 \leq \ell \leq k+1$ and $u_1^{(\ell)} = u_2^{(\ell)}$ for $2 \leq \ell \leq k$. The equations for $u_j^{(k+1)}$ is given by

$$(-\Delta)^s_{A_j} u_j^{(k+1)} + \partial_z a_j(x,0) u_j^{(k+1)} = -\partial_z^{k+1} a_j(x,0)(u_j^{(1)})^{k+1} + R_k(a_j, u_j).$$

Here $R_k(a_j, u_j)$ only involves $\partial_z^2 a_j(x,0), \dots, \partial_z^k a_j(x,0)$ and $u_j^{(1)}, \dots, u_j^{(k)}$ and thus we have

$$R_k(a_1, u_1) = R_k(a_2, u_2).$$

By assumption, the equations for $u_1^{(k+1)}$ and $u_2^{(k+1)}$ are identical in Ω and both $u_1^{(k+1)}$ and $u_2^{(k+1)}$ are zero in Ω_e, hence $u_1^{(k+1)} = u_2^{(k+1)}$ in $\mathbb{R}^n$. This completes the induction proof. □

3.3.2. *The proof of main result.* Now we are ready to complete the proof of the main theorem by reconstructing the higher order term of Taylor expansion of a.

Proof of Theorem 1. From Proposition 5, it remains to show $\partial_z^k a_1(x,0) = \partial_z^k a_2(x,0)$ for $x \in \Omega$ and $k \geq 2$ due to (1.4).

First, let $v \in H^s(\mathbb{R}^n)$ be a function satisfying

$$(-\Delta)^s_{A_1} v + \partial_z a_1(x,0) v = 0 \qquad \text{in } \Omega \tag{3.12}$$

and $\operatorname{supp}(v\chi_{\Omega_e}) \subset W_2$. Note that here $(A_1, \partial_z a_1(\cdot,0)) \sim (A_2, \partial_z a_2(\cdot,0))$ implies $(-\Delta)^s_{A_2} v + \partial_z a_2(x,0) v = 0$ in Ω. Then by Lemma 5 and that $u_j^{(2)} = u_j^{(3)} = \cdots = 0$ in Ω_e, we have

$$\begin{aligned}
\langle \Lambda^{(k+1),s}_{A_j,a_j}[f], [v] \rangle &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \nabla^s_{A_j} u_j^{(k+1)} \cdot \nabla^s_{A_j} v \, dy \, dx + \int_\Omega \partial_z a_j(x,0) u_j^{(k+1)} v \, dx \\
&\quad + \int_\Omega \left[\partial_z^{k+1} a_j(x,0)(u_j^{(1)})^{k+1} + R_k(a_j, u_j) \right] v \, dx \\
&= \int_\Omega \left[\partial_z^{k+1} a_j(x,0)(u_j^{(1)})^{k+1} + R_k(a_j, u_j) \right] v \, dx, \qquad \text{for } k \geq 1.
\end{aligned}$$

By Lemma 6, we first have $u_1^{(1)} = u_2^{(1)}$. From $\Lambda^{(2),s}_{A_1,a_1}[f]|_{W_2} = \Lambda^{(2),s}_{A_2,a_2}[f]|_{W_2}$ for any $f \in C_c^\infty(W_1)$, we have

$$0 = \langle (\Lambda^{(2),s}_{A_1,a_1} - \Lambda^{(2),s}_{A_2,a_2})[f], [v] \rangle = \int_\Omega (\partial_z^2 a_1(x,0) - \partial_z^2 a_2(x,0))(u_1^{(1)})^2 v \, dx,$$

since $R_1(a_j, u_j) = 0$.

By the Runge approximation property in Proposition 1, for any $g \in L^2(\Omega)$, we can find a sequence of linear solutions $v_i \in H^s(\mathbb{R}^n)$, whose restriction to Ω_e is compactly supported in W_2, to (3.12) such that $v_i \to g$ in $L^2(\Omega)$. Plugging v_i and let $i \to \infty$, we obtain

$$\int_\Omega (\partial_z^2 a_1(x,0) - \partial_z^2 a_2(x,0))(u_1^{(1)})^2 g \, dx = 0$$

for any $g \in L^2(\Omega)$, which yields

$$(\partial_z^2 a_1(x,0) - \partial_z^2 a_2(x,0))(u_1^{(1)})^2 = 0, \qquad x \in \Omega.$$

Now based on the strong maximum principle proved in Proposition 4, we can choose suitable exterior data $u_1^{(1)} = f \geq 0$ and $f \not\equiv 0$ in W_1 so that $u_1^{(1)} > 0$ in Ω. Then $(\partial_z^2 a_1(x,0) - \partial_z^2 a_2(x,0))$ must be zero in Ω.

Now for any fixed positive integer $k > 1$, we suppose that

$$\partial_z^\ell a_1(x,0) = \partial_z^\ell a_2(x,0) \quad \text{in } \Omega, \qquad \text{for } \ 2 \leq \ell \leq k. \tag{3.13}$$

Then Lemma 6 yields that

$$u_1^{(\ell)} = u_2^{(\ell)} \quad \text{in } \mathbb{R}^n, \qquad \text{for } \ 1 \leq \ell \leq k. \tag{3.14}$$

Combining $\Lambda_{A_1,a_1}^{(k+1),s}[f]|_{W_2} = \Lambda_{A_2,a_2}^{(k+1),s}[f]|_{W_2}$ for $f \in C_c^\infty(W_1)$ with (3.13) and (3.14), one has

$$0 = \int_{\mathbb{R}^n} (\partial_z^{k+1} a_1(x,0) - \partial_z^{k+1} a_2(x,0))(u_1^{(1)})^{k+1} v \, dx$$

for $u_1^{(1)}$ and v as above, which similarly proves $\partial_z^{k+1} a_1(x,0) = \partial_z^{k+1} a_2(x,0)$ in Ω. By induction, this completes the proof. □

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School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
Email address: rylai@umn.edu

School of Mathematical Science, Zhejiang University, Hangzhou, China
Email address: ting_zhou@zju.edu.cn