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INVERSE PROBLEMS FOR THE FRACTIONAL LAPLACE EQUATION WITH LOWER ORDER NONLINEAR PERTURBATIONS

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Abstract. We study the inverse problem for the fractional Laplace equation with multiple nonlinear lower order terms. We show that the direct problem is well-posed and the inverse problem is uniquely solvable. More specifically, the unknown nonlinearities can be uniquely determined from exterior measurements under suitable settings.

1. Introduction.

We study the inverse problem for the fractional Laplace equation with lower order nonlinear perturbations. The problem setup is as follows. For $0<t<s<1$, let $\Omega\subset\mathbb{R}^n, n\geq 1$ be a bounded domain with smooth boundary $\partial\Omega$, and $\Omega_e:=\mathbb{R}^n\setminus\overline{\Omega}$ be the exterior domain of Ω. We consider the following fractional elliptic equation:

$$\begin{cases}(-\Delta)^s u+q(x,u,\nabla^t u)+a(x,u)=0 & \text{in } \Omega,\\ u=f & \text{in } \Omega_e,\end{cases}\tag{1.1}$$

where $a(x,u)$ is an unknown potential and the gradient term q takes the form

$$q(x,u,\nabla^t u):=b(x)\int_{\mathbb{R}^n}\nabla^t u(x,y)\cdot\nabla^t u(x,y)\,dy+u^m(x)\int_{\mathbb{R}^n}d(x,y)\cdot\nabla^t u(x,y)\,dy\tag{1.2}$$

for integer $m\geq 2$. Here the unknown scalar function $b(x)$ and vector-valued function $d(x,y)$, together with $a(x,u)$, are to be determined from the exterior measurement.

In (1.1), for $u\in H^s(\mathbb{R}^n)$, $0<s<1$, the fractional Laplacian is defined by

$$(-\Delta)^s u(x):=c_{n,s}\text{P.V.}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2s}}dy,\tag{1.3}$$

where the symbol P.V. denotes the principal value and

$$c_{n,s}=\frac{\Gamma(\frac{n}{2}+s)}{|\Gamma(-s)|}\frac{4^s}{\pi^{n/2}}$$

is a constant; see also [10]. The space $H^s(\mathbb{R}^n)$ is the standard fractional Sobolev space; see also Section 2. For $u\in H^s(\mathbb{R}^n)$, since $H^s(\mathbb{R}^n)\subset H^t(\mathbb{R}^n)$ for $0<t<$

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$s < 1$, u is also in $H^t(\mathbb{R}^n)$. Then the *fractional gradient* of u at points x and y is defined by

$$\nabla^t u(x,y) := \frac{c_{n,t}^{1/2}}{\sqrt{2}} \frac{y-x}{|x-y|^{n/2+t+1}}(u(x)-u(y)),$$

and the linear operator ∇^t maps $H^t(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$ [6]. Further discussion of notation will appear in Section 2.

For the coefficients $b(x)$ and $d(x,y)$, we assume that $b = b(x) : \Omega \to \mathbb{R}$ and $d = d(x,y) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy

$$b \in C(\Omega) \quad \text{and} \quad d \in C(\Omega \times \mathbb{R}^n) \text{ with compact support in } \Omega,\, \Omega \times \Omega, \tag{1.4}$$

respectively. Furthermore, we assume that the coefficient $a = a(x,z) : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

$$\begin{cases} \partial_z^k a(x,0) = 0 & \text{for all } x \in \overline{\Omega},\ 0 \le k \le m \\ \text{the map } z \mapsto a(\cdot, z) \text{ is holomorphic with values in } C^s(\overline{\Omega}), \end{cases} \tag{1.5}$$

where $C^s(\overline{\Omega})$ denotes the usual Hölder space; see also Section 2. Then the function a can be expanded into the following power series:

$$a(x,z) = \sum_{k=m+1}^{\infty} a_k(x) \frac{z^k}{k!}, \qquad a_k(x) := \partial_z^k a(x,0) \in C^s(\overline{\Omega}), \tag{1.6}$$

which converges in $C^s(\Omega \times \mathbb{R})$ space.

The exterior measurement is encoded in the *Dirichlet-to-Neumann* (DN) map:

$$\Lambda : H^s(\Omega_e) \to \left(H^s(\Omega_e)\right)^*,$$

where u is the solution to (1.1) with exterior data f and $\left(H^s(\Omega_e)\right)^*$ represents the dual space of $H^s(\Omega_e)$. For small data $f \in C_c^\infty(\Omega_e)$, we show in Section 2 that the problem (1.1) is well-posed and therefore the DN map is well-defined; indeed, it is defined through the integral (2.18) corresponding to the equation (1.1).

A fractional version of the well-known Calderón problem [2, 45] was first investigated in [14], in which the authors studied the inverse problem for the linear fractional Schrödinger equation (with $q = 0$ and $a(x,u) = a(x)u$ in (1.1)). Specifically, in [14] the potential $a(x)$ is uniquely determined from the associated DN map. The essential idea in obtaining this uniqueness result is to establish the strong uniqueness property of the fractional Laplacian $(-\Delta)^s$ (see Proposition 2.7) and the associated Runge approximation property. Since then, there have been many works concerning related inverse problems in various settings, including the problem with a single measurement [13, 39], unique determination for the (anisotropic) fractional Laplacian and conductivity equation [5, 7, 12], stability estimates [40], the inverse obstacle problem [3], monotonicity inversion [15, 16], nonlinear equations [25, 26, 33, 34], fractional parabolic equations [27], fractional magnetic equations [6, 31, 32], higher order operators [8, 9], as well as equations with lower order nonlocal perturbations [1].

1.1. **Main result.**

The main objective of this paper is to study the simultaneous reconstruction of three nonlinearities in a fractional equation. Due to the nonlocality, this is by nature a partial data inverse problem. The main result of the paper is stated below.

Theorem 1.1. *Let $0 < t < s < 1$ and let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded domain with smooth boundary. Let W_1, W_2 be two arbitrary open sets in Ω_e. Suppose that $b_j(x)$, $d_j(x,y)$, and $a_j(x,z)$ each satisfy the conditions (1.4) and (1.5) for $j = 1, 2$. Suppose furthermore that*

$$(d_1 - d_2)(x,y)|x-y|^{-n/2-t} \in L^2(\Omega) \qquad \text{for any fixed } x \in \Omega.$$

Let $\Lambda_j(f)$ be the DN map corresponding to (1.1) with a, b, d replaced by a_j, b_j, d_j, respectively, for $j = 1, 2$. Suppose that

$$\Lambda_1(f)|_{W_2} = \Lambda_2(f)|_{W_2} \qquad \text{for any } f \in C_c^\infty(W_1) \tag{1.7}$$

with $\|f\|_{C_c^\infty(W_1)} < \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Then

$$b_1(x) = b_2(x) \qquad \text{in } \Omega,$$

$$d_1(x,y) \cdot (x-y) = d_2(x,y) \cdot (x-y) \qquad \text{in } \Omega \times \Omega,$$

and

$$a_1(x,z) = a_2(x,z) \qquad \text{in } \Omega \times \mathbb{R}.$$

Remark 1.1. *We can fully recover the coefficient d only if d is of the form*

$$d(x,y) = d_0(x,y)(x-y)$$

for some scalar-valued function d_0 where $d_0(x,x)$ is known. This is due to the natural gauge enjoyed by equation (1.1); see [6]. In particular, if u satisfies (1.1) with $d = \underline{d}(x,y)$, then u also satisfies (1.1) for $d = \underline{d}(x,y) + d_\perp(x,y)$ for any $d_\perp$ satisfying $d_\perp \cdot (x-y) = 0$. See also [6].

The linearization scheme in [18] is a promising method for the study of inverse problems for local and nonlocal nonlinear elliptic equations. By performing a first order linearization of the DN map, one can reduce the inverse problem under study to the inverse problem for a linear equation. Then one can apply the available results for this linear case to recover the unknowns. The higher order linearization technique, in particular, uses nonlinearity as a tool in solving inverse problems for nonlinear equations. It involves introducing small parameters into the data, and then differentiating the nonlinear equation with respect to these parameters multiple times to obtain simpler linearized equations. Note that the application of this higher order linearization technique in treating local or nonlocal elliptic equations with power-type nonlinearities has been exploited in [11, 23, 24, 26, 28, 30, 29, 33, 34].

The inverse boundary value problem (IBVP) for nonlocal elliptic equations with nonlinearities was investigated in [25, 26, 34] for $(-\Delta)^s u + a(x,u) = 0$. In particular, when $b = 0$, $d = 0$ in (1.1), $a(x,u)$ is uniquely determined from an exterior measurement in [25] based on first order linearization. The necessary condition $W_1 = W_2$ in [25] was removed later in [26], which also showed the well-posedness of the equation using higher order linearization. Moreover, in [33], the problem for the nonlinear fractional magnetic equation was studied by applying first order linearization.

We shall next discuss the IBVP for local nonlinear elliptic equations. This problem has been extensively studied in the literature. For instance, $-\Delta u + a(x,u) = 0$ was studied in [20, 21, 43] for the full data problem and [24, 29] for the partial data setting when $n \geq 2$. The quasilinear equation $-\Delta u + a(u, \nabla u) = 0$ was studied in [19] when $n = 3$ and $-\Delta u + a(x, \nabla u) = 0$ was investigated in [42] when $n = 2$. It was however noted in [42] that the uniqueness of recovery of more general nonlinearity $a(x,u,\nabla u)$ in $-\Delta u + a(x,u,\nabla u) = 0$ in general fails. We refer the interested reader to [4, 17, 18, 22, 41, 44] for related results.

In this paper, we apply the higher order linearization technique to prove the well-posedness of (1.1) and reconstruct the unknown coefficients when the data is sufficiently small ($\|f\|_{C_c^\infty(W_1)} < \varepsilon$ for some $\varepsilon > 0$). We consider $m \geq 2$ so that the nonlinear terms in (1.1) have different degree of nonlinearity, which helps in separating the unknown terms when performing the linearization scheme, see Section 3 for details. More specifically, in our setting, differentiating (1.1) w.r.t. to the small parameter ε yields the equation $(-\Delta)^s u^{(1)} = 0$, whose solution is independent of unknown coefficients. Differentiating (1.1) twice leads to $(-\Delta)^s u^{(2)} + b(x)h(x;u^{(1)}) = 0$, which specifically contains only the unknown b with $h(x;u^{(1)})$ acting as a source term. We can then determine b uniquely from the exterior data; see Section 3 for notation and details. Finally, let us remark that the nonlinearities here indeed help by reducing the nonlinear equation to $(-\Delta)^s u^{(1)} = 0$ after the first linearization. This then enables the use of both strong uniqueness property (Proposition 2.7) and the Runge approximation property for $(-\Delta)^s$.

As mentioned above, when $s = 1$, $a(x,u,\nabla u)$ in $-\Delta u + a(x,u,\nabla u) = 0$ cannot be fully determined in general, which inspires us to consider the nonlocal setting as in (1.1). We may think of the three nonlinear terms in (1.1) as an example of the general nonlinear term $a(x,u,\nabla^t u)$. We show that they can be recovered simultaneously in Theorem 1.1.

Finally, for the local equations, when $s = 1$, the determination of multiple nonlinear terms was investigated in [23] for $-\Delta u + q(x)\nabla u \cdot \nabla u + a(x,u) = 0$ and in [28] for the magnetic Schrödinger equation with nonlinear terms like $a_1(x,u) + a_2(u,\nabla u)$. Both [23] and [28] applied the higher order linearization and the density result for harmonic functions to solve the inverse problem. Here we apply an analogous density result, the Runge approximation, characterizing the density of the collection of solutions to the fractional Laplace equation in L^2 space. This density result is crucial to recovering the coefficient d; see Section 3 for details.

The paper is organized as follows. Section 2 introduces notation and several previous results, including the unique continuation property and the maximum principle. The well-posedness result for (1.1) is also stated and proven in Section 2. Finally in Section 3 we use the results of Section 2 to show Theorem 1.1.

2. **Preliminaries.**

In this section, we introduce notation and the well-posedness result for the problem (1.1).

2.1. **Function spaces.**

We start by defining the Hölder spaces. Let $U \subset \mathbb{R}^n$ be an open set and k a nonnegative integer. For a given $0 < \alpha < 1$, the Hölder space $C^{k,\alpha}(U)$ is defined by

$$C^{k,\alpha}(U) := \left\{ f : U \to \mathbb{R} : \ \|f\|_{C^{k,\alpha}(U)} < \infty \right\},$$

where

$$\|f\|_{C^{k,\alpha}(U)} := \sum_{|\beta|\leq k} \|\partial^\beta f\|_{L^\infty(U)} + \sup_{x\neq y,\ x,y\in U} \sum_{|\beta|=k} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x-y|^\alpha}.$$

Here $\beta = (\beta_1, \ldots, \beta_n)$ is a multi-index with $\beta_i \in \mathbb{N}^+ \cup \{0\}$ and $|\beta| = \beta_1 + \ldots + \beta_n$. When $k = 0$, we simply set $C^\alpha(U) \equiv C^{0,\alpha}(U)$. We use $C_c^k(U)$ to denote the space of functions on $C^k(U)$ with compact support in U. Note that the above notation applies similarly for the closed set $\overline{U}$.

Next, following the notation in [14], for $0<s<1$, we use $H^s(\mathbb{R}^n):=W^{s,2}(\mathbb{R}^n)$ to denote the L^2-based Sobolev space with the following norm:

$$\|u\|^2_{H^s(\mathbb{R}^n)}=\|u\|^2_{L^2(\mathbb{R}^n)}+\|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^n)}.$$

Here, by the Parseval identity, the semi-norm $\|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^n)}$ can be expressed as

$$\|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^n)}=((-\Delta)^s u,u)_{\mathbb{R}^n}\,,$$

where the operator $(-\Delta)^s$ is as defined in (1.3) and we write $(u,v)_U:=\int_U uv dx$ if U is an open set in $\mathbb{R}^n$ and $u,v\in L^2(\mathbb{R}^n)$.

For scalar $\beta\in\mathbb{R}$, we define the following Sobolev spaces and follow the notation of [35]:

$$\begin{aligned}
H^\beta(U)&:=\left\{u|_U:\,u\in H^\beta(\mathbb{R}^n)\right\},\\
\widetilde{H}^\beta(U)&:=\text{closure of } C_c^\infty(U) \text{ in } H^\beta(\mathbb{R}^n),\\
H_0^\beta(U)&:=\text{closure of } C_c^\infty(U) \text{ in } H^\beta(U),
\end{aligned}$$

and

$$H^\beta_{\overline{U}}(\mathbb{R}^n):=\left\{u\in H^\beta(\mathbb{R}^n):\,\mathrm{supp}(u)\subset\overline{U}\right\}.$$

The Sobolev space $H^\beta(U)$ is complete under the graph norm

$$\|u\|_{H^\beta(U)}:=\inf\left\{\|v\|_{H^\beta(\mathbb{R}^n)}:\,v\in H^\beta(\mathbb{R}^n)\text{ and } v|_U=u\right\}.$$

It is known that $\widetilde{H}^\beta(U)\subsetneq H_0^\beta(U)$, and $H^\beta_{\overline{U}}(\mathbb{R}^n)$ is a closed subspace of $H^\beta(\mathbb{R}^n)$. Moreover,

$$(H^\beta(U))^*=\widetilde{H}^{-\beta}(U),\qquad (\widetilde{H}^\beta(U))^*=H^{-\beta}(U),\quad \beta\in\mathbb{R}.$$

If U is also a bounded Lipschitz domain, the spaces and dual spaces can be expressed as

$$H^\beta_{\overline{U}}(\mathbb{R}^n)=\widetilde{H}^\beta(U),\quad\text{and}\quad (H^\beta_{\overline{U}}(\mathbb{R}^n))^*=H^{-\beta}(U),\quad\text{and}\quad (H^\beta(U))^*=H^{-\beta}_{\overline{U}}(\mathbb{R}^n).$$

For more details on fractional Sobolev spaces, we refer to [10, 14, 35].

2.2. **Well-posedness.** Let $0<t<s<1$ and let $\Omega\subset\mathbb{R}^n$, $n\geq 1$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following Dirichlet problem with exterior data:

$$\begin{cases}(-\Delta)^s u+q(x,u,\nabla^t u)+a(x,u)=0 & \text{in } \Omega,\\ u=f & \text{in } \Omega_e,\end{cases}\tag{2.1}$$

where $f\in C_c^\infty(\Omega_e)$, and q and a are as in (1.2) and (1.6).

For notational brevity, we define the function h as

$$h(x;u,v):=\int_{\mathbb{R}^n}\nabla^t u(x,y)\cdot\nabla^t v(x,y)\,dy,$$

and, in particular, when $u=v$, we denote

$$h(x;u):=\int_{\mathbb{R}^n}\nabla^t u(x,y)\cdot\nabla^t u(x,y)\,dy.\tag{2.2}$$

We also define

$$\psi(x;d,u) := u^m(x)\int_{\mathbb{R}^n} d(x,y)\cdot\nabla^t u(x,y)\,dy. \tag{2.3}$$

Then q can be expressed as $q(x,u,\nabla^t u) = b(x)h(x;u) + \psi(x;d,u)$.

In the following lemma, we analyze the boundness of h and ψ, which will be a crucial ingredient in proving the well-posedness result.

Lemma 2.1. *Let $0 < t < s < 1$ and $u, v \in C^s(\mathbb{R}^n)$. For a fixed constant $R > 0$, we have*

$$\begin{aligned}&\int_{\mathbb{R}^n}\frac{|(u(x)-u(y))(v(x)-v(y))|}{|x-y|^{n+2t}}\,dy\\ &\le C_n\|u\|_{C^s(\mathbb{R}^n)}\|v\|_{C^s(\mathbb{R}^n)}\left(\frac{1}{2s-2t}R^{2s-2t}+\frac{2}{t}R^{-2t}\right)\end{aligned} \tag{2.4}$$

for all $x \in \overline{\Omega}$. In particular, when $u = v$, we have

$$\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2t}}\,dy \le C_n\|u\|^2_{C^s(\mathbb{R}^n)}\left(\frac{1}{2s-2t}R^{2s-2t}+\frac{2}{t}R^{-2t}\right) \tag{2.5}$$

for all $x \in \overline{\Omega}$. Here the constant C_n only depends on n.

Proof. We first denote $M := \|u\|_{C^s(\mathbb{R}^n)}$ and $\widetilde{M} := \|v\|_{C^s(\mathbb{R}^n)}$ and note that $u, v \in C^s(\mathbb{R}^n)$ yields

$$|u(x)-u(y)| \le M|x-y|^s, \qquad |v(x)-v(y)| \le \widetilde{M}|x-y|^s \tag{2.6}$$

for all x, $y \in \mathbb{R}^n$.

To show (2.4), we note that for any fixed $x \in \overline{\Omega}$ we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\frac{|u(x)-u(y)||v(x)-v(y)|}{|x-y|^{n+2t}}\,dy\\ &=\int_{|x-y|\le R}\frac{|u(x)-u(y)||v(x)-v(y)|}{|x-y|^{n+2t}}\,dy+\int_{|x-y|>R}\frac{|u(x)-u(y)||v(x)-v(y)|}{|x-y|^{n+2t}}\,dy\\ &\le M\widetilde{M}\int_{|x-y|\le R}|x-y|^{-n-2t+2s}\,dy+(2M)(2\widetilde{M})\int_{|x-y|>R}|x-y|^{-n-2t}\,dy.\end{aligned}$$

Here we used (2.6) to derive the first term in the inequality. Applying a change of variables to spherical coordinates and recalling that $t < s$, we then obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}\frac{|u(x)-u(y)||v(x)-v(y)|}{|x-y|^{n+2t}}\,dy\\ &\le C_nM\widetilde{M}\int_0^R\rho^{2s-2t-1}\,d\rho+C_n4M\widetilde{M}\int_R^\infty\rho^{-2t-1}\,dy\\ &=C_n\|u\|_{C^s(\mathbb{R}^n)}\|v\|_{C^s(\mathbb{R}^n)}\left(\frac{1}{2s-2t}R^{2s-2t}+\frac{2}{t}R^{-2t}\right),\end{aligned}$$

which completes the proof of (2.4). Finally, the estimate (2.4) implies (2.5) when $u = v$. $\square$

We note that Lemma 2.1 implies that

$$
\begin{aligned}
\|h(x;u,v)\|_{L^\infty(\Omega)} &= \|\int_{\mathbb{R}^n} \nabla^t u(x,y)\cdot\nabla^t v(x,y)\,dy\|_{L^\infty(\Omega)} \\
&\le \|\frac{c_{n,t}}{2}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)||v(x)-v(y)|}{|x-y|^{n+2t}}\,dy\|_{L^\infty(\Omega)} \\
&\le C\|u\|_{C^s(\mathbb{R}^n)}\|v\|_{C^s(\mathbb{R}^n)}\left(\frac{1}{2s-2t}R^{2s-2t}+\frac{2}{t}R^{-2t}\right), \qquad (2.7)
\end{aligned}
$$

where the constant C depends on n and t, and thus $h(x;u,v)$ is in $L^\infty(\Omega)$.

Similarly, Lemma 2.1 also implies that

$$
\begin{aligned}
\|\psi(x;d,u)\|_{L^\infty(\Omega)} &\le \|u\|^m_{L^\infty(\Omega)}\|\int_{\mathbb{R}^n}|d(\cdot,y)|^2\,dy\|^{1/2}_{L^\infty(\Omega)}\|\int_{\mathbb{R}^n}|\nabla^t u(x,y)|^2\,dy\|^{1/2}_{L^\infty(\Omega)} \\
&\le C\|u\|^m_{L^\infty(\Omega)}\|\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2t}}\,dy\|^{1/2}_{L^\infty(\Omega)} \\
&\le C\|u\|^{1+m}_{C^s(\mathbb{R}^n)}\left(\frac{1}{2s-2t}R^{2s-2t}+\frac{2}{t}R^{-2t}\right)^{1/2}. \qquad (2.8)
\end{aligned}
$$

Here C depends on Ω, n, t, and the coefficient d.

Remark 2.2. *Lemma 2.1 suggests that in order to have pointwise control on the terms $h(x;u)$ and $\psi(x;d,u)$, we must consider t satisfying $0<t<s<1$, as the above arguments fail when $t=s$.*

The following lemma will also be useful for showing a contraction property in the proof of well posedness (Theorem 2.1).

Lemma 2.3. *Let $0<t<s<1$ and $u_1,u_2\in C^s(\mathbb{R}^n)$. We have the following two estimates:*

$$
\|h(x;u_1)-h(x;u_2)\|_{L^\infty(\Omega)} \le C\|u_1-u_2\|_{C^s(\mathbb{R}^n)}\|u_1+u_2\|_{C^s(\mathbb{R}^n)}
$$

and

$$
\begin{aligned}
&\|\psi(x;d,u_1)-\psi(x;d,u_2)\|_{L^\infty(\Omega)} \\
&\le C\|u_1-u_2\|_{C^s(\mathbb{R}^n)}\left(\|u_1\|_{C^s(\mathbb{R}^n)}\sum_{k=1}^m\|u_1\|^{m-k}_{C^s(\mathbb{R}^n)}\|u_2\|^{k-1}_{C^s(\mathbb{R}^n)}+\|u_2\|^m_{C^s(\mathbb{R}^n)}\right).
\end{aligned}
$$

Here the constant C depends only on n, t, s, d, and Ω.

Proof. First, from the definition of h and (2.7) with $R=1$, we derive

$$
\begin{aligned}
&\|h(x;u_1)-h(x;u_2)\|_{L^\infty(\Omega)} \\
&=\|\int_{\mathbb{R}^n}\nabla^t u_1(x,y)\cdot\nabla^t u_1(x,y)\,dy-\int_{\mathbb{R}^n}\nabla^t u_2(x,y)\cdot\nabla^t u_2(x,y)\,dy\|_{L^\infty(\Omega)} \\
&=\|\int_{\mathbb{R}^n}(\nabla^t u_1-\nabla^t u_2)\cdot(\nabla^t u_1+\nabla^t u_2)(x,y)\,dy\|_{L^\infty(\Omega)} \\
&=\|h(x;u_1-u_2,u_1+u_2)\|_{L^\infty(\Omega)} \\
&\le C\|u_1-u_2\|_{C^s(\mathbb{R}^n)}\|u_1+u_2\|_{C^s(\mathbb{R}^n)},
\end{aligned}
$$

where C is a constant depending on s, t and n.

Next, for any $x \in \Omega$, we consider

$$
\begin{aligned}
&\|\psi(x;d,u_1) - \psi(x;d,u_2)\|_{L^\infty(\Omega)} \\
=\ & \|u_1^m(x)\int_{\mathbb{R}^n} d(x,y)\cdot\nabla^t u_1(x,y)\,dy - u_2^m(x)\int_{\mathbb{R}^n} d(x,y)\cdot\nabla^t u_2(x,y)\,dy\|_{L^\infty(\Omega)} \\
\leq\ & \|(u_1^m(x) - u_2^m(x))\int_{\mathbb{R}^n} d(x,y)\cdot\nabla^t u_1(x,y)\,dy\|_{L^\infty(\Omega)} \\
&+ \|u_2^m\|_{L^\infty(\Omega)}\|\int_{\mathbb{R}^n} d(x,y)\cdot\nabla^t u_1(x,y)\,dy - \int_{\mathbb{R}^n} d(x,y)\cdot\nabla^t u_2(x,y)\,dy\|_{L^\infty(\Omega)} \\
\leq\ & \|u_1 - u_2\|_{L^\infty(\Omega)}\|\sum_{k=1}^m u_1^{m-k}u_2^{k-1}\|_{L^\infty(\Omega)}\|\int_{\mathbb{R}^n} d(x,y)\cdot\nabla^t u_1(x,y)\,dy\|_{L^\infty(\Omega)} \\
&+ \|u_2^m\|_{L^\infty(\Omega)}\|\int_{\mathbb{R}^n} d(x,y)\cdot\nabla^t (u_1-u_2)(x,y)\,dy\|_{L^\infty(\Omega)}.
\end{aligned}
$$

Application of a similar argument as in (2.8) gives the following upper bounds:

$$
\|\int_{\mathbb{R}^n} d(x,y)\cdot\nabla^t u_1(x,y)\,dy\|_{L^\infty(\Omega)} \leq C\|u_1\|_{C^s(\mathbb{R}^n)}
$$

and

$$
\|u_2^m\|_{L^\infty(\Omega)}\|\int_{\mathbb{R}^n} d(x,y)\cdot\nabla^t(u_1-u_2)(x,y)\,dy\|_{L^\infty(\Omega)} \leq C\|u_2\|^m_{C^s(\mathbb{R}^n)}\|u_1-u_2\|_{C^s(\mathbb{R}^n)}.
$$

Combining these estimates, we obtain the desired estimate for ψ. ◻

We are now ready to show the well-posedness result.

Theorem 2.1 (Well-posedness). *Let $0 < t < s < 1$ and let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded domain with smooth boundary $\partial\Omega$. Suppose that $b(x)$, $d(x,y)$, and $a(x,z)$ satisfy the conditions (1.4) - (1.6). Then there exists a small parameter $0 < \varepsilon < 1$ such that when*

$$
f \in \mathcal{X} := \left\{ f \in C_c^\infty(\Omega_e) : \ \|f\|_{C_c^\infty(\Omega_e)} \leq \varepsilon \right\}, \tag{2.9}
$$

the boundary value problem (2.1) has a unique small solution $u \in C^s(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$. Moreover, the solution u satisfies the estimate

$$
\|u\|_{C^s(\mathbb{R}^n)} \leq C\|f\|_{C_c^\infty(\Omega_e)},
$$

where the constant $C > 0$ is independent of u and f.

Proof. Suppose that $\|f\|_{C_c^\infty(\Omega_e)} \leq \varepsilon$ for some sufficiently small $\varepsilon > 0$. We may extend f to the whole space $\mathbb{R}^n$ by zero so that $\|f\|_{C_c^\infty(\mathbb{R}^n)} \leq \varepsilon$.

Before getting into the proof, we recall the following result of [14]. For $g \in L^\infty(\Omega)$, there exists a unique solution $\tilde{v} \in H^s(\mathbb{R}^n)$ to the problem

$$
\begin{cases}
(-\Delta)^s \tilde{v} = g & \text{in } \Omega, \\
\tilde{v} = 0 & \text{in } \Omega_e.
\end{cases} \tag{2.10}
$$

Moreover, by [38, Proposition 1.1], we have

$$
\|\tilde{v}\|_{C^s(\mathbb{R}^n)} \leq C\|g\|_{L^\infty(\Omega)}
$$

for some constant $C > 0$ depending on s and Ω. This enables us to define the solution operator

$$
\mathcal{L}_s^{-1} : g \in L^\infty(\Omega) \to \tilde{v} \in C^s(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)
$$

to (2.10). The solution $\mathcal{L}_s^{-1}(g)$ to (2.10) then satisfies

$$\|\mathcal{L}_s^{-1}(g)\|_{C^s(\mathbb{R}^n)} \le C\|g\|_{L^\infty(\Omega)}. \tag{2.11}$$

We may now proceed to the linearization procedure.

Step 1: The linearized problem. We first consider the linear part of (2.1), given by

$$\begin{cases} (-\Delta)^s u_0 = 0 & \text{in } \Omega, \\ u_0 = f & \text{in } \Omega_e. \end{cases} \tag{2.12}$$

Due to [14], there exists a unique solution $u_0 \in H^s(\mathbb{R}^n)$ to (2.12) satisfying

$$\|u_0\|_{H^s(\mathbb{R}^n)} \le C\|f\|_{H^s(\mathbb{R}^n)}.$$

By considering $(-\Delta)^s(u_0 - f) = -(-\Delta)^s f$ with $(u_0 - f)|_{\Omega_e} = 0$, we may then apply (2.11) to obtain

$$\|u_0 - f\|_{C^s(\mathbb{R}^n)} \le C\|(-\Delta)^s f\|_{L^\infty(\Omega)},$$

which implies that

$$\|u_0\|_{C^s(\mathbb{R}^n)} \le C\|f\|_{C_c^\infty(\Omega_e)}, \tag{2.13}$$

where the constant $C > 0$ depends only on s and Ω.

We next consider $v := u - u_0$, where u_0 satisfies (2.12) and u satisfies the original nonlinear equation (2.1). If such a function v exists, then v satisfies the following problem:

$$\begin{cases} (-\Delta)^s v = G(v) & \text{in } \Omega, \\ v = 0 & \text{in } \Omega_e, \end{cases} \tag{2.14}$$

where $G(\phi)$ is defined by

$$G(\phi) := -b(x)h(x; u_0 + \phi) - \psi(x; d, u_0 + \phi) - a(x, u_0 + \phi).$$

We now construct a contraction map and establish the unique existence of a solution v to (2.14) by the contraction mapping principle.

Step 2: Construct a contraction map. Let us define the set

$$\mathcal{M} = \left\{\phi \in C^s(\mathbb{R}^n) : \ \phi|_{\Omega_e} = 0, \ \|\phi\|_{C^s(\mathbb{R}^n)} \le \delta\right\},$$

where $0 < \delta < 1$ will be determined later (by choosing sufficiently small δ to satisfy the specific inequalities below).

We define the map $\mathcal{F}$ on $\mathcal{M}$ by

$$\mathcal{F} := \mathcal{L}_s^{-1} \circ G.$$

We will show below that $\mathcal{F}$ is indeed a contraction map on $\mathcal{M}$.

We first claim that $\mathcal{F} : \mathcal{M} \to \mathcal{M}$. By (2.7), (2.8), (2.11), and the Taylor expansion of a (1.6), for any $\phi \in \mathcal{M}$, we obtain $\mathcal{F}(\phi) \in C^s(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$, and

$$\begin{aligned} \|\mathcal{F}(\phi)\|_{C^s(\mathbb{R}^n)} &\le C\|G(\phi)\|_{L^\infty(\Omega)} \\ &= C\|b(x)h(x; u_0 + \phi) + \psi(x; d, u_0 + \phi) + a(x, u_0 + \phi)\|_{L^\infty(\Omega)} \\ &\le C\|b\|_{C(\Omega)}\|u_0 + \phi\|^2_{C^s(\mathbb{R}^n)} + C\|u_0 + \phi\|^{m+1}_{C^s(\mathbb{R}^n)} + C\|u_0 + \phi\|^{m+1}_{C^s(\overline{\Omega})} \\ &\le C\|b\|_{C(\Omega)}(\delta + \varepsilon)^2 + C(\delta + \varepsilon)^{m+1} + C(\delta + \varepsilon)^{m+1}, \end{aligned} \tag{2.15}$$

where the constant C depends on s, t, n and Ω. This indicates that the function $G(\phi) \in L^\infty(\Omega)$. When $\varepsilon < C\delta$ for some $C > 0$ and δ is small enough, we then have

$$\|\mathcal{F}(\phi)\|_{C^s(\mathbb{R}^n)} \leq C(\varepsilon+\delta)^2 + C(\varepsilon+\delta)^{1+m} + C(\delta+\varepsilon)^{m+1} < \delta.$$

This yields that $\mathcal{F}$ maps $\mathcal{M}$ into itself.

We also need to show that $\mathcal{F}$ is contractive. For any $\phi_1, \phi_2 \in \mathcal{M}$, we apply Lemma 2.3, (1.6), and (2.11) to obtain

$$\begin{aligned}
\|\mathcal{F}(\phi_1) - \mathcal{F}(\phi_2)\|_{C^s(\mathbb{R}^n)} &= \|(\mathcal{L}_s^{-1} \circ G)(\phi_1) - (\mathcal{L}_s^{-1} \circ G)(\phi_2)\|_{C^s(\mathbb{R}^n)} \\
&\leq C\|G(\phi_1) - G(\phi_2)\|_{L^\infty(\Omega)} \\
&\leq C\|b(x)(h(x; u_0+\phi_1) - h(x; u_0+\phi_2))\|_{L^\infty(\Omega)} \\
&\quad + C\|\psi(x; d, u_0+\phi_1) - \psi(x; d, u_0+\phi_2)\|_{L^\infty(\Omega)} \\
&\quad + C\|a(x, u_0+\phi_1) - a(x, u_0+\phi_2)\|_{L^\infty(\Omega)} \\
&\leq C(\varepsilon+\delta)\|\phi_1-\phi_2\|_{C^s(\mathbb{R}^n)} + C(\varepsilon+\delta)^m\|\phi_1-\phi_2\|_{C^s(\mathbb{R}^n)} \\
&\quad + C(\varepsilon+\delta)^m\|\phi_1-\phi_2\|_{C^s(\mathbb{R}^n)},
\end{aligned} \tag{2.16}$$

where C is independent of ε, δ.

By further taking ε, δ sufficiently small so that $C(\varepsilon+\delta)+C(\varepsilon+\delta)^m+C(\varepsilon+\delta)^m < 1$, the following estimate also holds:

$$\|\mathcal{F}(\phi_1) - \mathcal{F}(\phi_2)\|_{C^s(\mathbb{R}^n)} < \|\phi_1-\phi_2\|_{C^s(\mathbb{R}^n)}.$$

Combining these results, we have shown that $\mathcal{F}$ is a contraction mapping on $\mathcal{M}$.

Finally, the contraction mapping principle gives that there is a fixed point $v \in \mathcal{M}$ such that $\mathcal{F}(v) = v$ and thus $v \in H^s(\mathbb{R}^n)$ as well. This v is the solution to the equation (2.14) and also satisfies

$$\|v\|_{C^s(\mathbb{R}^n)} \leq C(\|u_0\|^2_{C^s(\overline{\Omega})} + \|v\|^2_{C^s(\overline{\Omega})}) \leq C\left(\varepsilon\|f\|_{C_c^\infty(\Omega_e)} + \delta\|v\|_{C^s(\overline{\Omega})}\right) \tag{2.17}$$

due to (2.15). For δ small enough, by absorbing $C\delta\|v\|_{C^s(\overline{\Omega})}$ into the left-hand side of (2.17), we then have

$$\|v\|_{C^s(\mathbb{R}^n)} \leq C\varepsilon\|f\|_{C_c^\infty(\Omega_e)}.$$

As a result, we obtain the solution $u = u_0 + v \in C^s(\mathbb{R}^n)$ to (2.1) and it satisfies

$$\|u\|_{C^s(\mathbb{R}^n)} \leq C\|f\|_{C_c^\infty(\Omega_e)}$$

for some constant $C > 0$ independent of u and f. This completes the proof of well-posedness for the boundary value problem (2.1). ☐

Lemma 2.4. *Under the same assumption as in Theorem 2.1, we have*

$$\|u\|_{H^s(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}.$$

Proof. In the proof of Theorem 2.1, we already have that $\|u_0\|_{H^s(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}$. We only need to estimate $\|v\|_{H^s(\mathbb{R}^n)}$. Since v is the solution to (2.14), it satisfies

$$\|v\|_{H^s(\mathbb{R}^n)} \leq C\|G(v)\|_{H^{-s}(\Omega)} \leq C(\varepsilon+\delta)(\|u_0\|_{H^s(\mathbb{R}^n)} + \|v\|_{H^s(\mathbb{R}^n)}),$$

where in the second inequality we applied small C^s bound for v and u_0 due to (2.13) and $v \in \mathcal{M}$. Combining with the H^s estimate for u_0, we then obtain

$$\|v\|_{H^s(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}$$

when ε, δ are chosen small enough. ☐

2.3. **The DN map.** In this subsection, we will define the corresponding DN map for the equation (2.1).

We define the operator $B : H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{R}$ by

$$B[u, v] = \int_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} v \, dx + \int_\Omega (q(x, u, \nabla^t u)v + a(x, u)v) \, dx.$$

By Theorem 2.1, for $f \in \mathcal{X}$, there exists a unique (small) solution $u_f \in C^s(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ to (2.1) with $u_f - f \in \widetilde{H}^s(\Omega)$. We define the DN map $\Lambda : \mathcal{X} \cap X \to X^*$ as follows:

$$\langle \Lambda[f], [v] \rangle := B[u_f, v] \tag{2.18}$$

for $v \in H^s(\mathbb{R}^n)$, where q and a are as defined in (1.2) and (1.6). Here X is the quotient space $H^s(\mathbb{R}^n) \setminus \widetilde{H}^s(\Omega)$. Note that (2.18) is not a bilinear form as in [14] due to the nonlinear terms q and a.

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ for $n \geq 1$, $0 < t < s < 1$. Suppose that b, d, and $a = a(x, z)$ satisfy the conditions* (1.4) *-* (1.6)*. Then the DN map defined in* (2.18) *is bounded.*

Proof. The definition of the DN map depends only on the equivalence classes. To see this, we take any $\phi, \psi \in \widetilde{H}^s(\Omega)$, the well-posedness result implies that $u_{f+\phi} = u_f$ in $\mathbb{R}^n$ for $f + \phi, f \in \mathcal{X}$. Then

$$B[u_{f+\phi}, v + \psi] = B[u_f, v + \psi] = B[u_f, v] + B[u_f, \psi] = B[u_f, v],$$

where in the last identity we used the fact that u_f is the solution to $(-\Delta)^s u_f + q(x, u, \nabla^t u_f) + a(x, u_f) = 0$ in Ω and also $\psi \in \widetilde{H}^s(\Omega)$.

We now show that Λ is bounded. By using Lemma 2.4, we have

$$\begin{aligned}
|B[u_f, v]| &\leq \|(-\Delta)^{s/2} u_f\|_{L^2(\mathbb{R}^n)} \|(-\Delta)^{s/2} v\|_{L^2(\mathbb{R}^n)} \\
&\quad + \|q(x, u, \nabla^t u_f) + a(x, u_f)\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)} \\
&\leq \|u_f\|_{H^s(\mathbb{R}^n)} \|v\|_{H^s(\mathbb{R}^n)} + C\|u_f\|_{H^s(\mathbb{R}^n)} \|v\|_{H^s(\mathbb{R}^n)} \\
&\leq C\|f\|_{H^s(\mathbb{R}^n)} \|v\|_{H^s(\mathbb{R}^n)},
\end{aligned}$$

where C depends on b, d, a, ε. This completes the proof.

□

Remark 2.6. *In this remark, we will discuss the differentiability of the solution of* (2.1) *with respect to the given exterior data. To this end, we first define spaces*

$$V_1 := \left\{ u \in H^s(\mathbb{R}^n) : (-\Delta)^s u \in L^\infty(\Omega), u|_\Omega \in C^s(\Omega) \text{ and } u|_{\Omega_e} \in C^{1,s}(\Omega_e) \right\}$$

equipped with the norm $\|u\|_{V_1} := \|u\|_{H^s(\mathbb{R}^n)} + \|(-\Delta)^s u\|_{L^\infty(\Omega)} + \|u\|_{C^s(\Omega)} + \|u\|_{C^{1,s}(\Omega_e)}$ *and*

$$V_2 := C^{1,s}(\Omega_e) \cap H^s(\Omega_e)$$

equipped with the norm $\|u\|_{V_2} := \|u\|_{C^{1,s}(\Omega_e)} + \|u\|_{H^s(\Omega_e)}$*. Then V_1 and V_2 are Banach spaces. We now consider the map $F : V_2 \times V_1 \to L^\infty(\Omega) \times V_2$ defined by*

$$F : (f, u) \mapsto ((-\Delta)^s u + q(x, u, \nabla^t u) + a(x, u), \, u|_{\Omega_e} - f).$$

Note that $F(0, 0) = 0$. A similar discussion as in the proof of Theorem 2.1 and (1.4) *yield that $\partial_u F(0, 0) : V_1 \to L^\infty(\Omega) \times V_2$ is linear isomorphism, and in particular,* *for any* $(w,g)\in L^\infty(\Omega)\times V_2$*, one can find a unique solution* $v\in V_1$ *satisfying the linearized equation*

$$\begin{cases} (-\Delta)^s v = w & \text{in } \Omega, \\ v = g & \text{in } \Omega_e. \end{cases} \tag{2.19}$$

By the implicit function theorem for Banach spaces (see for instance, [36, Chapter 10]*), we have that there exists an open neighborhood* $\mathcal{O}$ *of* 0 *in* V_2 *and a unique analytic function* $h:\mathcal{O}\to V_1$ *such that* $h(0)=0$ *and also* $F(f,h(f))=(0,0)$ *for all* $f\in\mathcal{O}$*. Therefore, we have the solution* $u=h(f)$ *to the problem* (2.1) *and, moreover, it is infinitely differentiable with respect to the data* f *in* $\mathcal{O}$*.*

Based on the above discussion, one can take a smaller set contained in both $\mathcal{X}$ *and* $\mathcal{O}$ *so that the differentiability of the solution, Theorem 2.1 and Proposition 2.5 hold.*

2.4. **Known results.** We next state two known results which are crucial in the proof of Theorem 1.1.

The first is the unique continuation property (UCP) for the fractional Laplacian [14, Theorem 1.2].

Proposition 2.7 (UCP)**.** *Suppose that* U *is a nonempty open subset of* $\mathbb{R}^n$, $n\geq 1$*. Let* $0<s<1$ *and* $v\in H^r(\mathbb{R}^n)$ *for* $r\in\mathbb{R}$*. If* $v=(-\Delta)^s v=0$ *in* U*, then* $v\equiv 0$ *in* $\mathbb{R}^n$*.*

The second result is the maximum principle for the fractional Laplacian. The proof of the following proposition can be found in [26] and [25], which extends the result in [37] to include a non-negative potential term.

Proposition 2.8 (Maximum principle)**.** *Let* $\Omega\subset\mathbb{R}^n$, $n\geq 1$ *be a bounded domain with* C^1 *boundary* $\partial\Omega$*, and* $0<s<1$*. Suppose that* $w(x)\in L^\infty(\Omega)$ *is a nonnegative potential. Let* $u\in H^s(\mathbb{R}^n)$ *be the unique solution of*

$$\begin{cases} (-\Delta)^s u + w(x)u = F & \text{in } \Omega, \\ u = f & \text{in } \Omega_e. \end{cases}$$

Suppose that $0\leq F\in L^\infty(\Omega)$ *in* Ω *and* $0\leq f\in L^\infty(\Omega_e)$ *with* $f\not\equiv 0$ *in* Ω_e*. Then* $u>0$ *in* Ω*.*

3. **Proof of Theorem 1.1.**

Using the results of Section 2, we proceed to show the main theorem. Let $u=u(x;\varepsilon)$ be the solution to the exterior boundary value problem

$$\begin{cases} (-\Delta)^s u + q(x,u,\nabla^t u) + a(x,u) = 0 & \text{in } \Omega, \\ u = \epsilon f & \text{in } \Omega_e. \end{cases} \tag{3.1}$$

Recall that

$$q(x,u,\nabla^t u) = b(x)h(x;u) + \psi(x;d,u),$$

where h and ψ are defined in (2.2) and (2.3), respectively.

For notational simplicity, we denote the k^{th} derivative of u with respect to ε by

$$u_\epsilon^{(k)}(x;\epsilon) := \frac{\partial^k u}{\partial\epsilon^k}(x;\epsilon),$$

and at $\epsilon=0$ we simply denote

$$u^{(k)}(x) := \partial_\epsilon^k\big|_{\varepsilon=0} u(x;\varepsilon).$$

Moreover, the differentiability of u with respect to ε allows us to take the k-th derivative $\Lambda^{(k)}$ of the map Λ with respect to ε at $\varepsilon = 0$, and therefore we have $\Lambda_1^{(k)}(f)|_{W_2} = \Lambda_2^{(k)}(f)|_{W_2}$ provided that $\Lambda_1(f)|_{W_2} = \Lambda_2(f)|_{W_2}$. We then have the following result.

Proposition 3.1. *Let $0 < t < s < 1$ and let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain with smooth boundary. Let ϵ be a small parameter and let $f \in C_c^\infty(W_1)$. For $j = 1, 2$, consider b_j, d_j, and a_j satisfying (1.4) - (1.6), and let u_j denote the solution to (3.1) with b, d, and a replaced by b_j, d_j, and a_j, respectively.*

Suppose that

$$\Lambda_1(f)|_{W_2} = \Lambda_2(f)|_{W_2} \qquad \text{for any } f \in C_c^\infty(W_1). \tag{3.2}$$

Then if $b_1 = b_2$ in Ω, we have $u_1^{(1)} = u_2^{(1)}$ and $u_1^{(2)} = u_2^{(2)}$ in $\mathbb{R}^n$.

Moreover, given $N \geq 3$, if

$$b_1 = b_2, \quad d_1 \cdot (x-y) = d_2 \cdot (x-y), \quad \partial_z^\ell a_1(x,0) = \partial_z^\ell a_2(x,0) \text{ for any } 3 \leq \ell \leq N, \tag{3.3}$$

then

$$u_1^{(k)} = u_2^{(k)} \qquad \text{in } \mathbb{R}^n \qquad \text{for any } 3 \leq k \leq N. \tag{3.4}$$

Proof. For clarity, we present the proof in the case $m = 2$ in the nonlinear terms ψ and a. The proof for more general $m > 2$ follows a similar outline.
Fixing an arbitrary positive integer N, it is sufficient to show that $u_1^{(k)} = u_2^{(k)}$ in $\mathbb{R}^n$ for all $1 \leq k \leq N$.

We first apply the operator $\partial_\varepsilon|_{\varepsilon=0}$ to (3.1). Using that $u(x;0) = 0$ by the well-posedness of (3.1), we obtain

$$\begin{cases} (-\Delta)^s u_j^{(1)} = 0 & \text{in } \Omega, \\ u_j^{(1)} = f & \text{in } \Omega_e. \end{cases} \tag{3.5}$$

Since $u_1^{(1)} = u_2^{(1)} = f$ in Ω_e, the well-posedness of the problem (Theorem 2.1) implies that

$$u_1^{(1)} = u_2^{(1)} =: u^{(1)} \qquad \text{in } \mathbb{R}^n. \tag{3.6}$$

Next we apply $\partial_\varepsilon^2|_{\varepsilon=0}$ to (3.1). Then $u_j^{(2)}$ satisfies the following problem:

$$\begin{cases} (-\Delta)^s u_j^{(2)} + b_j(x) h(x; u^{(1)}) = 0 & \text{in } \Omega, \\ u_j^{(2)} = 0 & \text{in } \Omega_e. \end{cases} \tag{3.7}$$

Since $b_1 = b_2$ and $u_1^{(1)} = u_2^{(1)}$, both $u_j^{(2)}$ for $j = 1, 2$ satisfy the same equation (3.7) with trivial exterior data. Thus we have $u_1^{(2)} = u_2^{(2)}$ in $\mathbb{R}^n$.

Recalling that we have set $m = 2$ in ψ, we next apply $\partial_\varepsilon^3|_{\varepsilon=0}$ to (3.1) to get that $u_j^{(3)}$ satisfies

$$\begin{cases} (-\Delta)^s u_j^{(3)} + 2b_j h(x; u^{(1)}, u^{(2)}) + 2\psi(x; d_j, u^{(1)}) + \partial_z^3 a_j(x,0) \left(u^{(1)}\right)^3 = 0 & \text{in } \Omega, \\ u_j^{(3)} = 0 & \text{in } \Omega_e. \end{cases} \tag{3.8}$$

By (3.3), we also have that both $u_j^{(3)}$ for $j = 1, 2$ satisfy the same problem (3.8) and thus $u_1^{(3)} = u_2^{(3)}$ in $\mathbb{R}^n$.

Next, by an induction argument, we suppose that when $N \geq 3$, (3.3) holds for $3 \leq \ell \leq N+1$ and $u_1^{(k)} = u_2^{(k)}$ for $1 \leq k \leq N$. Now we perform $\partial_\varepsilon^{N+1}|_{\varepsilon=0}$ on (3.1), which gives

$$(-\Delta)^s u_j^{(N+1)} + R_N(u_j, a_j, b_j, d_j) + \partial_z^{N+1} a_j(x,0) \left(u_j^{(1)}\right)^{N+1} = 0 \qquad \text{in } \Omega, \quad (3.9)$$

with boundary data $u_1^{(N+1)} = u_2^{(N+1)} = 0$ in Ω_e. Here $R_N(u_j, a_j, b_j, d_j)$ involves only the functions $b_j(x)$, $d_j(x,y)$, and $\partial_z^\beta a_j(x,0)$ for $3 \leq \beta \leq N$ and $u_j^{(k)}(x)$ for $1 \leq k \leq N$. Thus we have $R_N(u_1, a_1, b_1, d_1) = R_N(u_2, a_2, b_2, d_2)$. As a result, both $u_j^{(N+1)}$ for $j = 1, 2$ satisfy the same equation with trivial data and thus $u_1^{(N+1)} = u_2^{(N+1)}$ in $\mathbb{R}^n$. This completes the induction proof. □

With Proposition 3.1, we are now ready to show the main result. The outline of the proof of Theorem 1.1 is as follows. We will first show that $b_1 = b_2$ and then $\partial_z^3 a_1(x,0) = \partial_z^3 a_2(x,0)$. Using these equalities, we can show $d_1 \cdot (x-y) = d_2 \cdot (x-y)$. Finally, to fully recover a, we rely on an induction argument.

Proof of Theorem 1.1. We again present the proof for the case $m = 2$ in the nonlinear terms ψ and a. For more general $m > 2$, the proof can be shown in a similar manner.

The proof is completed in 3 steps.

Step 1. Recover b**.** Let ϵ be sufficiently small and let $f \in C_c^\infty(W_1)$ be a nonconstant function. For $j = 1, 2$, let u_j be the solution to the following exterior boundary value problem:

$$\begin{cases} (-\Delta)^s u_j + b_j(x) h(x; u_j) + \psi(x; d_j, u_j) + a_j(x, u_j) = 0 & \text{in } \Omega, \\ u_j = \epsilon f & \text{in } \Omega_e. \end{cases} \quad (3.10)$$

Using (2.18) and $u_j^{(N)} = 0$ in Ω_e for $N \geq 2$, one can derive that

$$\begin{aligned} \left\langle \Lambda_j^{(N)}(f), f_2 \right\rangle &= \int_{\mathbb{R}^n} (-\Delta)^{s/2} u_j^{(N)} (-\Delta)^{s/2} v \, dx + \int_\Omega [\partial_z^N a_j(x,0) (u_j^{(1)})^N \\ &\quad + R_{N-1}(u_j, a_j, b_j, d_j)] v \, dx \\ &= \int_\Omega [\partial_z^N a_j(x,0) (u_j^{(1)})^N + R_{N-1}(u_j, a_j, b_j, d_j)] v \, dx, \end{aligned}$$

where $v \in H^s(\mathbb{R}^n)$ is the solution to $(-\Delta)^s v = 0$ in Ω with $v = f_2$ in W_1. In particular, since $\partial_z^2 a(x,0) = 0$, when $N = 2$, we simply have

$$\left\langle \Lambda_j^{(2)}(f), f_2 \right\rangle = \int_\Omega b_j(x) h(x; u_j^{(1)}) v \, dx.$$

Thus since $\Lambda_1^{(2)}(f) = \Lambda_2^{(2)}(f)$ for any small $f \in C_c^\infty(W_1)$ and since $u^{(1)} := u_1^{(1)} = u_2^{(1)}$ by Proposition 3.1, we then have

$$0 = \int_\Omega (b_1 - b_2)(x) h(x; u^{(1)}) v \, dx. \quad (3.11)$$

By the Runge approximation property (see [14, Lemma 4.1] with $q = 0$), for any $g \in L^2(\Omega)$, there exists a sequence of solutions $v_j \in H^s(\mathbb{R}^n)$ to $(-\Delta)^s v_j = 0$ in Ω

with exterior data in $C_c^\infty(W_1)$ such that $v_j|_\Omega \to g$ in $L^2(\Omega)$. Replacing v in (3.11) by v_j and letting $j \to \infty$, we have

$$0 = \int_\Omega (b_1 - b_2)(x) h(x; u^{(1)}) g \, dx,$$

which further leads to

$$(b_1 - b_2)(x) h(x; u^{(1)}) = 0 \qquad \text{in } \Omega, \tag{3.12}$$

since g is arbitrary. Note that by the definition of h,

$$h(x; u^{(1)}) = \frac{c_{n,t}}{2} \int_{\mathbb{R}^n} \frac{|u^{(1)}(x) - u^{(1)}(y)|^2}{|x-y|^{n+2t}} \, dy \geq 0 \qquad \text{for all } x \in \Omega.$$

We will show that in fact $h > 0$ in Ω. By contradiction, suppose that $h(x_0; u^{(1)}) = 0$ for some point $x_0 \in \Omega$. This implies that $u^{(1)} \equiv u^{(1)}(x_0)$ in $\mathbb{R}^n$, which contradicts that the chosen exterior data f is not a constant function. Therefore $h(x; u^{(1)}) \neq 0$ for any point x in Ω. Thus (3.12) implies that

$$b_1 = b_2 \qquad \text{in } \Omega.$$

Moreover, Proposition 3.1 yields that $u^{(2)} := u_1^{(2)} = u_2^{(2)}$.

Step 2. Recover d and $\partial_z^3 a(x, 0)$. We will use that $b := b_1 = b_2$.

In this step, we also let ϵ be sufficiently small and f be any function in $C_c^\infty(W_1)$. For $j = 1, 2$, we also let u_j be the solution to the following exterior boundary value problem:

$$\begin{cases} (-\Delta)^s u_j + b(x) h(x; u_j) + \psi(x; d_j, u_j) + a_j(x, u_j) = 0 & \text{in } \Omega, \\ u_j = \epsilon f & \text{in } \Omega_e. \end{cases} \tag{3.13}$$

Similarly to above, since $\Lambda_1^{(3)}(f) = \Lambda_2^{(3)}(f)$ and $u^{(k)} := u_1^{(k)} = u_2^{(k)}$ for $k = 1, 2$, we have

$$\int_\Omega \Big(2 \int_{\mathbb{R}^n} (d_1 - d_2) \cdot \nabla^t u^{(1)}(x, y) \, dy + (\partial_z^3 a_1(x, 0) - \partial_z^3 a_2(x, 0)) u^{(1)} \Big) (u^{(1)})^2 v \, dx = 0. \tag{3.14}$$

By applying the Runge approximation property as above, we obtain

$$\Big(2 \int_{\mathbb{R}^n} (d_1 - d_2) \cdot \nabla^t u^{(1)}(x, y) \, dy + (\partial_z^3 a_1(x, 0) - \partial_z^3 a_2(x, 0)) u^{(1)} \Big) (u^{(1)})^2 = 0. \tag{3.15}$$

Here $u^{(1)}$ is the solution to (3.5) with $u^{(1)}|_{\Omega_e} = f$ for any $f \in C_c^\infty(W_1)$. We can also apply the Runge approximation property to find a sequence of solutions w_k to (3.5) such that $w_k \to 1$ in $L^2(\Omega)$ as $k \to \infty$. Then there is a subsequence w_{k_j} which converges pointwise almost everywhere (a.e.) to 1 as $j \to \infty$. Note, then, that since we assume $(d_1 - d_2)(x, y)|x - y|^{-n/2-t} \in L^2(\Omega)$ for any fixed $x \in \Omega$, we have that $\int_{\mathbb{R}^n} (d_1 - d_2)(x, y) \cdot \nabla^t w_{k_j}(x, y) \, dy \to 0$ as $j \to \infty$. Replacing $u^{(1)}$ by w_{k_j} in (3.15) and taking $j \to \infty$, the first term thus vanishes, yielding

$$\partial_z^3 a_1(x, 0) = \partial_z^3 a_2(x, 0).$$

With this, we now turn back to (3.15) and get that

$$(u^{(1)})^2(x) \int_{\mathbb{R}^n} (d_1 - d_2)(x, y) \cdot (y - x) \frac{u^{(1)}(x) - u^{(1)}(y)}{|x - y|^{n/2+t+1}} \, dy = 0. \tag{3.16}$$

For any fixed $x_0 \in \Omega$, since $(d_1 - d_2)(x_0, y) \cdot (y - x_0)$ is continuous in Ω, we may define the following two open subsets of Ω:

$$A_+ := \{y \in \Omega \setminus \{x_0\} : (d_1 - d_2)(x_0, y) \cdot (y - x_0) > 0\}$$

and

$$A_- := \{y \in \Omega \setminus \{x_0\} : (d_1 - d_2)(x_0, y) \cdot (y - x_0) < 0\}.$$

We will show by contradiction that $(d_1 - d_2)(x, y) \cdot (y - x) = 0$. Suppose that at least one of $A_\pm$ is not empty.

We define the function φ_{x_0} by

$$\varphi_{x_0}(y) = \begin{cases} \frac{1}{1+|x_0-y|^2} & \text{if } y \in A_+, \\ \frac{1+2|x_0-y|^2}{1+|x_0-y|^2} & \text{if } y \in A_-, \\ 1 & \text{if } y \in \Omega \setminus (A_+ \cup A_-). \end{cases}$$

Since Ω is bounded, φ_{x_0} is in $L^2(\Omega)$. It is clear that $\varphi_{x_0}(x_0) = 1$ since $x_0 \notin A_\pm$. Then we have

$$\begin{cases} \varphi_{x_0}(x_0) = 1 > \varphi_{x_0}(y) & \text{for all } y \in A_+, \\ \varphi_{x_0}(x_0) = 1 < \varphi_{x_0}(y) & \text{for all } y \in A_-, \end{cases}$$

and thus

$$(d_1 - d_2)(x_0, y) \cdot (y - x_0) \frac{\varphi_{x_0}(x_0) - \varphi_{x_0}(y)}{|x_0 - y|^{n/2+t+1}} > 0 \qquad \text{for all } y \in A_\pm. \tag{3.17}$$

Again by the Runge approximation property, there exists a sequence of solutions $\tilde{w}_k$ to (3.5) such that $\tilde{w}_k \to \varphi_{x_0}$ in $L^2(\Omega)$ as $k \to \infty$, which implies that there exists a subsequence $\tilde{w}_{k_j} \to \varphi_{x_0}$ a.e. as $j \to \infty$. Since $(d_1 - d_2)(x, y)|x - y|^{-n/2-t} \in L^2(\Omega)$ for any fixed $x \in \Omega$, we may replace $u^{(1)}$ by $\tilde{w}_{k_j}$ in (3.16) and take $j \to \infty$ to obtain

$$\varphi_{x_0}^2(x_0) \int_{\mathbb{R}^n} (d_1 - d_2)(x_0, y) \cdot (y - x_0) \frac{\varphi_{x_0}(x_0) - \varphi_{x_0}(y)}{|x_0 - y|^{n/2+t+1}} \, dy = 0. \tag{3.18}$$

However, since $0 \neq \varphi_{x_0}(x_0)$, by (1.4) and (3.17), the integral in (3.18) must be strictly positive for any nonempty $A_\pm$, which is a contradiction. Therefore, both $A_\pm$ must be empty sets, which implies that

$$d_1(x_0, y) \cdot (x_0 - y) = d_2(x_0, y) \cdot (x_0 - y) \qquad \text{for all } y \in \Omega.$$

Since $x_0 \in \Omega$ is arbitrary, we then have

$$d_1(x, y) \cdot (x - y) = d_2(x, y) \cdot (x - y) \qquad \text{for each } (x, y) \in \Omega \times \Omega.$$

Thus we uniquely determine the $(x - y)$-direction component of $d(x, y)$.

Now the problem boils down to showing the uniqueness of the potential a. It is then sufficient to show that $\partial_z^k a_1(x, 0) = \partial_z^k a_2(x, 0)$ for $k > 3$.

Step 3. Recover higher order terms $\partial_z^k a(x, 0)$, $k > 3$. Step 1 and Step 2 have shown that

$$b_1 = b_2, \;\; \psi(x; d_1, u^{(1)}) = \psi(x; d_2, u^{(1)}), \;\; \partial_z^3 a_1(x, 0) = \partial_z^3 a_2(x, 0). \tag{3.19}$$

By induction, for any fixed $N \in \mathbb{N}$, suppose that

$$\partial_z^j a_1(x, 0) = \partial_z^j a_2(x, 0) \qquad \text{for } 3 \le j \le N - 1, \tag{3.20}$$

and thus $u^{(k)} := u_1^{(k)} = u_2^{(k)}$ for $1 \le k \le N - 1$ by Proposition 3.1. It is sufficient to show that $\partial_z^N a_1(x, 0) = \partial_z^N a_2(x, 0)$ holds as well. From now on, we will use j subscripts on a_j only since the coefficients b, d have been recovered.

From $\Lambda_1^{(N)}(f) = \Lambda_2^{(N)}(f)$, we have

$$0 = \int_\Omega [\partial_z^N (a_1(x,0) - a_2(x,0))(u^{(1)})^N + R_{N-1}(u, a_j, b, d)] v \, dx. \tag{3.21}$$

Recall that $R_{N-1}(u, a_j, b, d)$ only consists of the functions $b(x)$, $d(x,y)$, and $\partial_z^\beta a_j(x,0)$ for $3 \leq \beta \leq N-1$ and $u^{(k)}(x)$ for all $1 \leq k \leq N-1$. Then (3.20) implies that

$$R_{N-1}(u, a_1, b, d) = R_{N-1}(u, a_2, b, d),$$

and therefore, by applying the Runge approximation property as in Step 1 and 2 to remove the integrand in (3.21), we have

$$\partial_z^N a_1(x,0)(u^{(1)})^N = \partial_z^N a_2(x,0)(u^{(1)})^N.$$

Choosing exterior data $f > 0$ in (3.5) and using the maximum principle (Proposition 2.8), we have $u^{(1)} \neq 0$. This gives $\partial_z^N a_1(x,0) = \partial_z^N a_2(x,0)$. Finally, by the uniqueness of the expansion (1.6), we obtain $a_1(x,z) = a_2(x,z)$. The proof is complete. □

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