Law of large numbers and central limit theorem for ergodic quantum processes

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Abstract

A discrete quantum process is represented by a sequence of quantum operations, which are completely positive maps that are not necessarily trace preserving. We consider quantum processes that are obtained by repeated iterations of a quantum operation with noise. Such ergodic quantum processes generalize independent quantum processes. An ergodic theorem describing convergence to equilibrium for a general class of such processes was recently obtained by Movassagh and Schenker. Under irreducibility and mixing conditions we obtain a central limit type theorem describing fluctuations around the ergodic limit.

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1 Introduction and main results

A *quantum channel* (QC) is a linear, completely positive, and trace preserving map on the trace class operators, where the state of the system is represented by a non-negative operator of trace one — a *density matrix*. Such maps can describe the evolution of an open quantum system over a discrete unit of time, including averaged effects of measurements and environmental noise. More generally, one introduces *quantum operations*(QOs) — completely positive and trace non-increasing maps — to describe processes with loss or which happen only with a certain probability. A *quantum process* is a sequence of QOs describing the evolution of the system over a consecutive sequence of time intervals. Quantum processes represent the most general description of the average evolution of an open quantum system neglecting memory effects in the environment.

In a pair of recent papers [22, 21], Movassagh and the second author formulated the notion of an *ergodic quantum process* in which the individual QOs are obtained by sampling a QO valued function along a trajectory of an ergodic dynamical system. For processes on a finite dimensional Hilbert space and satisfying a physically natural decoherence condition, they proved convergence of the density matrix to a stationary, ergodic sequence of density matrices as time goes to infinity. This theorem of [21] generalizes a result of Hennion [14] on products of non-negative random matrices and is closely related Oseledec's multiplicative ergodic theorem [23].

The results of [21] require essentially only decoherence and ergodicity. In the present paper, we examine processes that satisfy stronger integrability and mixing conditions. We prove a law of large numbers and a central limit theorem for the expectation values of observables in states evolving under such a processes. Although our main interest is in the application of these results to quantum processes, the results themselves do not require the maps to be trace non-increasing and require only *positivity* (not complete positivity).

This paper is organized as follows:

1. In §2 we state our main results after formulating certain background notions.
2. In §3, review some definitions and arguments from [22] that are fundamental to the proofs of our main results.
3. In §4, we prove Theorem 1 - Law of Large Numbers.
4. In §5, we prove Theorem 2 - Central Limit Theorem.
5. In §6, we prove Theorem 3, which gives sufficient conditions for the main hypothesis of Theorem 2 - Central Limit Theorem to hold.

2 Formal statement of the main results

2.1 Positive Linear Maps

Let $\mathbb{M}_D = \mathbb{C}^{D\times D}$ denote the space of $D \times D$ matrices. We consider the space $\mathbb{M}_D$ with its standard topology as a finite-dimensional vector space. For definiteness, we take this to be the norm topology generated by the *trace norm*, $||A|| := \operatorname{Tr}\sqrt{(A^*A)}$ for any $A \in \mathbb{M}_D$, but of course the topology is independent of the norm (since $\mathbb{M}_D$ is finite dimensional). For any matrix $A \in \mathbb{M}_D$ we denote by A^* the adjoint matrix (conjugate transpose).

The space of linear operators on $\mathbb{M}^D$ will be denoted by $\mathcal{L}(\mathbb{M}_D)$. We equip the space $\mathcal{L}(\mathbb{M}_D)$ with the operator norm induced by the trace norm on $\mathbb{M}_D$. That is, for $\phi \in \mathcal{L}(\mathbb{M}_D)$:

$$||\phi|| = \sup\{||\phi(A)|| : A \in \mathbb{M}_D, ||A|| = 1\} . \tag{2.1}$$

For any $\phi \in \mathcal{L}(\mathbb{M}_D)$ the adjoint of ϕ is the unique map $\phi^* \in \mathcal{L}(\mathbb{M}_D)$ determined by the identity:

$$\langle A, \phi(B)\rangle = \langle \phi^*(A), B\rangle \text{ for all } A, B \in \mathbb{M}_D , \tag{2.2}$$

where $\langle A, B\rangle$ denotes the Hilbert-Schmidt inner product,

$$\langle A, B\rangle = \mathrm{tr} A^* B \ . \tag{2.3}$$

We recall that a map $\phi \in \mathcal{L}(\mathbb{M}_D)$ is *positive*, if it maps the set of positive semi-definite matrices to itself. It is convenient to introduce notation for certain subsets of positive semi-definite matrices as follows:

1. POS_D is the set of all positive semi-definite $D \times D$ matrices,
2. POS_D^0 is the set of all positive definite $D \times D$ matrices,
3. $\mathbb{S}_D$ is the set of positive semi-definite $D \times D$ matrices with trace one, and
4. $\mathbb{S}_D^\circ$ is the set of positive definite $D \times D$ matrices with trace one.

The subset $\mathbb{S}_D$, being bounded and closed, is compact by the Heine-Borel theorem. Note that ϕ is positive if and only if $\phi(\mathbb{S}_D) \subset \mathrm{POS}_D$. We call ϕ *strictly positive* if $\phi(\mathbb{S}_D) \subset \mathrm{POS}_D^0$.

Positive maps satisfy a generalization of the Perron-Frobenius Theorem (see [18, 10]): every such map ϕ has an eigenmatrix $R \in \mathbb{S}_D$ with eigenvalue equal to the spectral-radius $r(\phi)$. The map ϕ is called *irreducible* if $(\mathbf{1} + \phi)^n$ is strictly positive for some n.[1] By [10, Theorems 2.3 & 2.4] we have the following

Proposition 2.1. *If ϕ is an irreducible positive map, then there is a unique $R \in \mathbb{S}_D$ such that $\phi(R) = \Lambda R$ for some $\Lambda \in \mathbb{C}$. Furthermore, the eigen-matrix R is non-singular ($R \in \mathbb{S}_D^\circ$) and the eigenvalue $\Lambda = r(\phi) > 0$ is the spectral radius of ϕ.*

We call the unique eigenmatrix $R \in \mathbb{S}_D$ of an irreducible map ϕ the *right Perron-Frobenius eigenmatrix of ϕ*. The map ϕ also has a *left Perron-Frobenius eigenmatrix*, which is the Perron-Frobenius eigenmatrix of ϕ^*. (Note that ϕ is irreducible if and only if ϕ^* is.)

The Perron-Frobenius eigenmatrix R of an irreducible map ϕ may be interpreted as a fixed point of the *projective action of ϕ* :

$$\phi \cdot X = \frac{\phi(X)}{\mathrm{tr}\phi(X)} \ . \tag{2.4}$$

For a general map, the projective action is defined for $X \in \mathbb{S}_D \setminus \ker \phi$. However, if $\ker \phi \cap \mathbb{S}_D = \emptyset$ then the projective action is defined on all of $\mathbb{S}_D$. As this condition will play a key role in our analysis, we make the following

Definition 1. A positive linear map $\phi \in \mathcal{L}(\mathbb{M}_D)$ is *non-destructive* if $\ker \phi \cap \mathbb{S}_D = \emptyset$. If ϕ^* is non-destructive, we say that ϕ is *non-transient*.

The terminology *non-transient* stems from the fact that if $\rho \in \ker \phi^* \cap \mathbb{S}_D$ and P is the projection onto $\mathrm{ran}\, \rho$, then $\phi^*(P) = 0$ and $\mathrm{ran}\, \phi$ is contained in the hereditary sub-algebra $P^\perp \mathbb{M}_D P^\perp$ where $P^\perp = I - P$. Thus the subspace corresponding to $\mathrm{ran}\, P$ is a "transient subspace" for ϕ.

A sufficient condition for ϕ to be non-destructive *and* non-transient is that ϕ^n be strictly positive for some $n > 0$. This condition is, in turn, equivalent to ϕ being *irreducible and aperiodic*, i.e., irreducible and having no eigenvalues on the circle $\{|z| = r(\phi)\}$ except for the Perron-Frobenius eigenvalue.

2.2 Limiting results for eigenmatrices of ergodic quantum processes

As in [22], we are interested in sequences $\Phi^{(n)}$ such that

$$\Phi^{(n)} = \phi_n \circ \dots \phi_1 \quad \text{with} \quad \phi_n = \phi_{0;\theta^n\omega} \ , \tag{2.5}$$

where $\omega \mapsto \phi_{0;\omega}$ is a positive map valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\theta : \Omega \to \Omega$ is an ergodic map. We recall that a measurable map $\theta : \Omega \to \Omega$ is

[1] Equivalently, no *hereditary sub-algebra*, $P\mathbb{M}_D P$ with P an orthogonal projection, is invariant under ϕ. See [10].

1. *measure preserving* if $\mathbb{P}(\theta^{-1}(A)) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$, and

2. *ergodic* if it is measure preserving and $\mathbb{P}(A) = 0$ or 1 whenever $\theta^{-1}(A) = A$.

We further recall that either of the following two conditions is sufficient for a measure preserving map θ to be ergodic:

1. *essentially θ-invariant sets have measure* 0 *or* 1, i.e., $\mathbb{P}(A) = 0$ or 1 whenever $A \in \mathcal{F}$ with $\mathbb{P}(A \Delta \theta^{-1}(A)) = 0$.

2. *essentially θ-invariant functions are almost surely constant*, i.e., if $f \circ \theta = f$ almost surely, then there is $c \in \mathbb{R}$ such that $f = c$ almost surely.

See [24] for proofs of these facts and further discussion of ergodic maps.

Now fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an ergodic map $\theta : \Omega \to \Omega$. For a random variable $X : \Omega \to \mathcal{S}$, with $\mathcal{S}$ some measurable space, we denote the value of X at $\omega \in \Omega$ by X_ω, and will often omit ω from the notation for simplicity. This subscript notation is convenient as we consider map valued random variables which take a matrix as an argument. Let $\varphi_0 : \Omega \to \mathcal{L}(\mathbb{M}_D)$ be a positive map valued random variable, where we take the Borel σ-algebra on $\mathcal{L}(\mathbb{M}_D)$. For each $n \in \mathbb{N}$, define $\varphi_{n;\omega} = \varphi_{0;\theta^n(\omega)}$. Let

$$\Phi_\omega^{(n)} = \varphi_{n;\omega} \circ \varphi_{n-1;\omega} \circ \cdots \varphi_{1;\omega}. \tag{2.6}$$

For $k \geq 0$, we have

$$\Phi_{\theta^k(\omega)}^{(n)} = \varphi_{n;\theta^k(\omega)} \circ \ldots \varphi_{1,\theta^k(\omega)} = \varphi_{n+k;\omega} \circ \cdots \varphi_{1+k;\omega} \ ; \tag{2.7}$$

as above we may omit ω from the notation and simply write this as $\Phi_{\theta^k}^{(n)} = \varphi_{n+k} \circ \cdots \varphi_{1+k}$.

In the present work, we study sequences $\Phi^{(n)}$ with the property that $\Phi^{(n)}$ is eventually strictly positive. We denote by τ_ω the time at which $\Phi_\omega^{(n)}$ becomes strictly positive and stays strictly positive thereafter:

$$\tau_\omega = \inf\{n \geq 1 : \Phi_\omega^{(n+k)} \text{ is strictly positive } \forall k \geq 0\} \ . \tag{2.8}$$

Our first assumption is that $\tau < \infty$ almost surely:

Assumption 1. *We have $\mathbb{P}\{\tau < \infty\} = 1$, i.e., the sequence $\Phi^{(n)}$ is* almost surely eventually strictly positive.

Assumption 1 was also the main assumption of [22], where it was shown to be equivalent to the following two conditions provided that θ is invertible (see [22, Lemma 2.1]):

1. there exists $N_0 \in \mathbb{N}$ such that $\mathbb{P}(\Phi^{(N_0)}$ is strictly positive$) > 0$, and

2. $\mathbb{P}\{\varphi_0$ is non-destructive and non-transient $\} = 1$.

One consequence of this equivalence is that, if θ is invertible and Assumption 1 holds, then τ can be expressed as

$$\tau = \inf\{n \geq 1 : \Phi^{(n)} \text{ is strictly positive }\} \ . \tag{2.9}$$

In particular, τ is then a *stopping time* with respect to the filtration $(\mathcal{F}_n)_{n=0}^\infty$ where $\mathcal{F}_n$ denotes the σ-algebra generated by $\phi_0, \ldots, \phi_n$.

Since any strictly positive map is irreducible, Assumption 1 guarantees that the left and right Perron-Frobenius eigenmatrices, R_n and L_n, exist for for sufficiently large n:

$$\Phi^{(n)}(R_n) = \Lambda_n R_n \quad \text{ and } \quad \Phi^{(n)*}(L_n) = \Lambda_n L_n \ . \tag{2.10}$$

Here $\Lambda_n = \Lambda_{n;\omega}$ denotes the spectral radius of $\Phi^{(n)}$ and L_n, R_n are $\mathbb{S}_D^\circ$ valued random variables, i.e., they are $D \times D$ positive definite matrix valued random variables with $\text{tr}R_n = \text{tr}L_n = 1$. We have the following

Lemma 2.2 ([22, Theorem 1])**.** *Let $(\varphi_n)_{n\geq 1}$ and $\Phi^{(n)}$ be as in eq. (2.6) and let L_n be as in eq. (2.10). If Assumption 1 holds, then there is an $\mathbb{S}_D^\circ$ valued random variable Z_1' such that*

$$Z_1 \overset{a.s.}{=} \lim_{n\to\infty} L_n \tag{2.11}$$

and, with $Z_k := Z_1 \circ \theta^{k-1}$, we have for every $k \in \mathbb{N}$, $\varphi_k^* \cdot Z_{k+1} = Z_k$ a.s..

Remark 2.3. This is half of [22, Theorem 1]. The other half involves the convergence of the right eigenvectors and requires invertibility of the ergodic map θ. A close reading of the proof (see [22, Lemma 3.12]) shows that invertibility of θ is not necessary for the portion stated here.

2.3 Law of Large Numbers

Our first main result is concerned with expectations of the form $\langle Y, \Phi^{(n)}(X)\rangle$ with $X, Y \in \mathbb{S}_D$. The main idea here is that for large n, the Perron-Frobenius eigenvalue Λ_n of $\Phi^{(n)}$ typically exhibits exponential growth or decay and dominates the expression, so that we expect

$$\langle Y, \Phi_n(X)\rangle \;\approx\; \Lambda_n \frac{\langle Y, R_n\rangle\langle L_n, X\rangle}{\langle L_n, R_n\rangle} \;+\; \text{lower order terms}, \tag{2.12}$$

where L_n and R_n are the left and right Perron-Frobenius eigenmatrices, respectively, normalized so that $\mathrm{tr}L_n = \mathrm{tr}R_n = 1$. Under Assumption 1, L_n and R_n are positive definite, so $\langle Y, R_n\rangle\langle L_n, X\rangle \neq 0$ and eq. (2.12) suggests that

$$\ln\langle Y, \Phi^{(n)}(X)\rangle \;\approx\; \ln \Lambda_n \;+\; O(1).$$

Thus we expect a Law of Large Numbers, $\frac{1}{n}\ln\langle Y, \Phi^{(n)}(X)\rangle \to l$, where $l = \lim_n \frac{1}{n}\ln \Lambda_n$.

To obtain this Law of Large Numbers, we require an integrability assumption for $\ln\|\varphi_0^*\|$ and for $\ln v(\varphi_0^*)$, where for $\phi \in \mathcal{L}(\mathbb{M}_D)$ we define

$$v(\phi) \;:=\; \inf\{\|\phi(X)\| : X \in \mathbb{S}_D\}\ . \tag{2.13}$$

Assumption 2. *We have $\mathbb{E}[|\ln\|\varphi_0^*\||] < \infty$ and $\mathbb{E}[|\ln v(\varphi_0^*)|] < \infty$.*

Remark 2.4. We note that any non-destructive map ϕ (in particular, any strictly positive map) must have $v(\phi) > 0$ because $\mathbb{S}_D$ is a compact set and the map $A \mapsto ||\phi(A)||$ is continuous.

With Assumptions 1 and 2 we have the following

Theorem 1 - Law of Large Numbers. *Let $\Phi^{(n)}$ be a random sequence of positive maps as in eq. (2.6). If Assumptions 1 and 2 hold then*

$$\lim_{n\to\infty} \sup_{X,Y\in\mathbb{S}_D} \left|\frac{1}{n}\ln\langle Y, \Phi^{(n)}(X)\rangle - l\right| \;=\; 0 \quad a.s., \tag{2.14}$$

where $l = \mathbb{E}[\ln\|\varphi_0^*(Z_1)\|]$ with $Z_1 = \lim_n L_n$. Furthermore

$$\lim_{n\to\infty} \frac{1}{n}\ln||\Phi^{(n)}|| \;=\; \lim_{n\to\infty}\frac{1}{n}\ln\Lambda_n = l \quad a.s.\ , \tag{2.15}$$

with Λ_n the Perron-Frobenius eigenvalue of $\Phi^{(n)}$.

Remark 2.5. We take $\ln\langle Y, \Phi^{(n)}(X)\rangle = -\infty$ if $\langle Y, \Phi^{(n)}(X)\rangle = 0$; by Assumption 1 this happens for at most finitely many n. By Assumption 2, $l = \mathbb{E}[\ln\|\varphi_0^*(Z_1)\|]$ is finite.

Theorem 1 - Law of Large Numbers is closely related in spirit to the Furstenberg-Kesten theorem [11] and Oseledet's Theorem [23] (see also [12]). By the Furstenberg-Kesten Theorem, the following limit exsists

$$\lim_{n\to\infty}\frac{1}{n}\ln||\Phi^{(n)}|| \quad \text{a.s.} \;=\; \lambda \quad a.s.,$$

where λ is a deterministic quantity called the *top Lyapunov exponent* of the cocycle $(X, n) \mapsto \Phi^{(n)}(X)$. By Oseledet's Theorem, there is a (random) proper subspace $L \subset \mathbb{M}_D$ such that for $X \in \mathbb{M}_D \setminus L_{j+1}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \ln ||\Phi^{(n)}(X)|| \ = \ \lambda \ .$$

The identity eq. (2.14) is the key result in Theorem 1 - Law of Large Numbers. Indeed, since $\Lambda_n = \langle L_n, \Phi^{(n)}(\mathbb{I})\rangle$ it follows directly from eq. (2.14) that $l = \lim_n \frac{1}{n} \ln \Lambda_n$ almost surely. Furthermore, as the proof of eq. (2.14) will make clear (see Lemma 4.1), we also have $\lim_n \frac{1}{n} \ln ||\Phi^{(n)*}(Y)|| \ = \ l$ a.s. for any $Y \in \mathbb{S}_D$. Since $\operatorname{span} \mathbb{S}_D = \mathbb{M}_D$, it follows from Oseledet's Theorem that $l = \lambda$, the top Lyapunov exponent, and thus that $l = \lim_n \frac{1}{n} \ln ||\Phi^{(n)}||$. Therefore eq. (2.15) is a consequence of eq. (2.14). Thus to prove Theorem 1 - Law of Large Numbers it suffices to prove eq. (2.14). This is accomplished in §4 below.

2.4 Central Limit Theorem

Our second main result is a central limit theorem for the fluctuations of $\ln\langle Y, \Phi^{(n)}(X)\rangle$ around its asymptotic value nl. For this result we require additional integrability for $\ln \|\varphi^*_{0;\omega}\|$ and $\ln v(\phi^*_{0;\omega})$:

Assumption 2$_p$. *For $p > 1$, the random variables $\ln \|\varphi^*_{0;\omega}\|$ and $\ln v(\varphi^*_{0;\omega})$ are in L^p .*

To obtain a central limit theorem, we require the ergodic map θ to be invertible, and extend the definition of φ_k to $k < 0$ by $\varphi_{k;\omega} = \varphi_{0;\theta^k\omega}$, just as for $k \geq 0$. Similarly we define $Z_{k;\omega} = Z_{1;\theta^{k-1}\omega}$ for $k \leq 0$. The key quantities that describe the fluctuations are the deviations of $\ln ||\varphi^*_k(Z_{k+1})||$ from its mean:

$$\xi_k \ := \ \ln ||\varphi^*_k(Z_{k+1})|| - l \quad , \tag{2.16}$$

where l is as in Theorem 1 - Law of Large Numbers. We also introduce the following reverse filtration $(\mathcal{F}^n)_{n\in\mathbb{Z}}$ on the probability space:

$$\mathcal{F}^n \ := \ \text{ sigma algebra generated by } (\varphi_k)_{k\geq n} \ . \tag{2.17}$$

With these preliminaries, we have the following

Theorem 2 - Central Limit Theorem. *Let $\Phi^{(n)}$ be a random sequence of positive maps as in eq. (2.6). Suppose that the ergodic map θ is invertible, that Assumption 1 holds, and that Assumption 2_p holds for some $p \geq 2$. If*

$$\sum_{n=1}^{\infty} ||\mathbb{E}[\xi_0|\mathcal{F}^n]||_q < \infty \tag{2.18}$$

with $1/p + 1/q = 1$, then for any sequences $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ in $\mathbb{S}_n$, the random sequence

$$\left(\frac{1}{\sqrt{n}} \big(\ln\langle Y_n, \Phi^{(n)}(X_n)\rangle - nl \big) \right)_{n\geq 1} \tag{2.19}$$

converges in distribution to a centered normal random variable with variance

$$\sigma^2 \ := \ \mathbb{E}\left[\left(\sum_{k\geq 0} \big(\mathbb{E}[\xi_{-k}|\mathcal{F}^0] - \mathbb{E}[\xi_{-k}|\mathcal{F}^1] \big) \right)^2 \right] \ \geq \ 0 \ . \tag{2.20}$$

Remark 2.6. The proof will show that $\sigma < \infty$, but we have allowed the possibility that $\sigma = 0$. If $\sigma = 0$, the sequence in 2.19 converges to 0 in distribution (and hence in probability). Else, the sequence in 2.19 converges to a centered normal law with variance $\sigma^2 > 0$.

We prove Theorem 2 - Central Limit Theorem in §5 below.

The hypothesis eq. (2.18) of Theorem 2 - Central Limit Theorem may not be easy to verify directly. We close this section by introducing *mixing conditions* that are sufficient for eq. (2.18) to hold. Let

$$\mathcal{F}_n \;:=\; \text{sigma algebra generated by } (\varphi_k)_{k\le n} \;. \tag{2.21}$$

Note that $(\mathcal{F}_n)_{n\in\mathbb{Z}}$ is a filtration, i.e., $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, while $(\mathcal{F}^n)_{n\in\mathbb{Z}}$ (defined above in eq. (2.17)) is a reverse filtration, i.e., $\mathcal{F}^n \supset \mathcal{F}^{n+1}$. We introduce the following *mixing coefficients*:

$$\alpha_n \;:=\; \sup_{k\ge 0}\; \sup\left\{ \left|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)\right| \;:\; A\in\mathcal{F}_k,\; B\in\mathcal{F}^{n+k}\right\} \tag{2.22}$$

$$\rho_n \;:=\; \sup_{k\ge 0}\; \sup\left\{ \left|\frac{\mathbb{E}[(Y-\mathbb{E}[Y])(X-\mathbb{E}[X])]}{\sigma(Y)\sigma(X)}\right| \;:\; Y\in L^2(\mathcal{F}_k),\; X\in L^2(\mathcal{F}^{n+k}),\; X,Y\neq 0\right\} \tag{2.23}$$

We have the following:

Theorem 3. *If Assumption 2_p holds with $p>2$ and $\sum_{n\ge1}\alpha_n^{(p-2)/p}<\infty$, then*

$$\sum_{n=1}^{\infty} ||\mathbb{E}[\xi_0|\mathcal{F}^n]||_q < \infty \;,$$

with q the conjugate exponent to p. If Assumption 2_p holds with $p=2$ and $\sum_{n\ge1}\rho_n<\infty$, then

$$\sum_{n=1}^{\infty} ||\mathbb{E}[\xi_0|\mathcal{F}^n]||_2 < \infty$$

Theorem 3 is proved in §6 below.

3 Background results: geometry of $\mathbb{S}_D$, contraction for positive maps, and ergodic arguments

In this section we review some definitions and arguments from [22] that are fundamental to the proofs below.

3.1 A metric on $\mathbb{S}_D$

Following [22], we define the following metric on $\mathbb{S}_D$:

$$\mathrm{d}(A,B) \;:=\; \frac{1-m(A,B)m(B,A)}{1+m(A,B)m(B,A)} \;, \tag{3.1}$$

where

$$m(A,B) \;=\; \sup\{\lambda : \lambda B \le A\}] \tag{3.2}$$

for $A,B\in\mathbb{S}_D$. The following lemma lists key properties of this metric (see [22, Lemma 3.3, 3.8, 3.9] for further details and proofs):

Lemma 3.1. *The function* d *defined in eq. (3.1) is a metric on $\mathbb{S}_D$ satisfying:*

1. $\frac{1}{2}||A-B|| \le \mathrm{d}(A,B) \le 1$ *for* $A,B\in\mathbb{S}_D$.
2. $\mathrm{d}(A,B) < 1$ *for* $A,B\in\mathbb{S}_D^{\circ}$.
3. *If* $A\in\mathbb{S}_d^{o}$, *then* $\mathrm{d}(A,B)=1$ *if and only if* $B\in\mathbb{S}_D\setminus\mathbb{S}_D^{\circ}$.

4. *The set* $\mathbb{S}_D^\circ$ *is open in the metric topology generated by* d *and* $(\mathbb{S}_D^\circ, d)$ *is homeomorphic to* $\mathbb{S}_D^\circ$ *in the standard topology (generated by* $\mathrm{d}_1(A,B) = ||A-B||$*).*

In the proofs below, the following simple consequence of the lower bound $\frac{1}{2}||A-B|| \leq \mathrm{d}(A,B)$ will be useful.

Lemma 3.2. *Let* $\phi \in \mathcal{L}(\mathbb{M}_D)$ *be a positive map with the property that* $\ker\phi \cap \mathbb{S}_D = \emptyset$*. Then for all* $X, Y \in \mathbb{S}_D$*;*

$$\big|\ln||\phi(X)|| - \ln||\phi(Y)||\big| \leq 2\frac{||\phi||}{v(\phi)}\mathrm{d}(X,Y) \ , \tag{3.3}$$

with $v(\phi)$ *as in eq.* (2.13).

Remark 3.3. For $\phi = \varphi_n^*$, we have $\ker\phi \cap \mathbb{S}_D = \emptyset$ with probability one under the Assumption 1, see [22, Lemma 2.1]. Under Assumption 2, $v(\phi)$ is non-zero with probability 1 and the right-hand-side of eq. (3.3) is finite almost surely.

Proof. Let $g : (\mathbb{S}_D, ||\cdot||) \to \mathbb{R}$ be defined as $g(X) = ||\phi(X)||$. Since ϕ is positive with no matrix in $\mathbb{S}_D$ in its kernel we must have that $g(X) > 0$ for all $X \in \mathbb{S}_D$. Since $\mathbb{S}_D$ is compact in the standard topology, we have that

$$v(\phi) = \min\{||\phi(Z)|| : Z \in \mathbb{S}_D\} > 0 \ . \tag{3.4}$$

It follows from the mean value inequality, applied to ln, that

$$\big|\ln||\phi(X)|| - \ln||\phi(Y)||\big| \leq \frac{\big|||\phi(X)|| - ||\phi(Y)||\big|}{v(\phi)} \leq \frac{||\phi||||X-Y||}{v(\phi)} \tag{3.5}$$

The results follows from lemma 3.1 as $||X-Y|| \leq 2\mathrm{d}(X,Y)$. ■

3.2 Contraction Coefficient for ϕ

For any non-destructive positive map $\phi \in \mathcal{L}(\mathbb{M}_D)$ we define the *contraction coefficient* of ϕ, denoted $c(\phi)$, as follows:

$$\mathrm{c}(\phi) = \sup\{\mathrm{d}(\phi\cdot A, \phi\cdot B) : A, B \in \mathbb{S}_D\} \ . \tag{3.6}$$

We have the following properties of the contraction coefficient:

Lemma 3.4 ([22, Lemma 3.14])**.** *If* $\phi \in \mathcal{L}(\mathbb{M}_D)$ *be a non-destructive positive map, then*

1. $\mathrm{d}(\phi\cdot X, \phi\cdot Y) \leq c(\phi)\mathrm{d}(X,Y)$ *for all* $X, Y \in \mathbb{S}_D$ *.*
2. $c(\phi) \leq 1$ *and if* ϕ *is strictly positive then* $c(\phi) < 1$*.*
3. *If there exist* X, Y *such that* $\phi\cdot X \in \mathbb{S}_D^\circ$ *and* $\phi\cdot Y \in \mathbb{S}_D \setminus \mathbb{S}_D^\circ$*, then* $c(\phi) = 1$*.*
4. *For any non-destructive positive map* ψ*, we have* $c(\phi\circ\psi) \leq c(\phi)c(\psi)$*.*
5. *If* ϕ *is also non-transient, then* $c(\phi) = c(\phi^*)$*.*

Remark 3.5. We note that the lemma above is stated slightly differently than [22, Lemma 3.14]. However a close reading of the proof in [22] shows that the above version holds.

Under Assumption 1, the maps Φ^n defined as in eq. (2.6) become strictly positive in finite time. As a consequence the following result was proved in [22] using Kingman's sub additive ergodic theorem [16, 17, 19]:

Lemma 3.6 ([22, Lemma 3.11])**.** *Let* $(\varphi_n)_{n\geq 1}$ *and* Φ^n *be as in eq.* (2.6)*. If Assumption 1 holds, then there exists a deterministic constant* $\kappa \in [0,1)$ *such that almost surely*

$$\ln \kappa = \lim_{n\to\infty} \frac{1}{n} \ln c(\Phi^{(n)})$$

and

$$\ln \kappa = \lim_{n\to\infty} \frac{1}{n} \mathbb{E} \ln c(\Phi^{(n)}) = \inf_{n\in\mathbb{N}} \frac{1}{n} \mathbb{E} \ln c(\Phi^{(n)}) \ .$$

Remark 3.7. In [22] the ergodic map θ is assumed to be invertible. However, a close reading of the proof of [22, Lemma 3.11] shows that invertibility of θ is not required.

Lemma 3.6 directly yields the following corollary:

Corollary 3.8. $\lim_{n\to\infty} c(\Phi^{(n)}) = 0$ *almost surely.*

The contraction provided by Lemma 3.6 is the driving force behind the convergence $L_n \to Z_1$ state in Lemma 2.2. In fact this convergence can be made more quantitative:

Lemma 3.9 ([22, Lemma 3.12])**.** *Let* $(\varphi_n)_{n\geq 1}$ *and* $\Phi^{(n)}$ *be as in eq.* (2.6) *and suppose that Assumption 1 holds. Let* L_n *be as in eq.* (2.10) *and let* $Z_1 = \lim_n L_n$ *and* $Z_k = Z_1 \circ \theta^{k-1}$ *be as in Lemma 2.2. Then, for each* $Y \in \mathbb{S}_D$ *and* $k \in \mathbb{N}$,

$$\mathrm{d}((\varphi_k^* \circ \ldots \circ \varphi_n^*) \cdot Y, Z_k) \ \leq \ c(\varphi_k^* \circ \ldots \circ \varphi_n^*)$$

for all sufficiently large n*. In particular, we have* $\lim_n (\varphi_k^* \circ \ldots \circ \varphi_n^*) \cdot Y = Z_k$ *with probability one.*

Below it will be useful to consider the contraction obtained from only a fraction of the process. This is described in the following

Lemma 3.10. *Let* $(\varphi_n)_{n\geq 1}$ *and* Φ^n *be as in eq.* (2.6)*. Let* $\alpha \in (0,1)$ *and let* $n_\alpha = \lfloor (1-\alpha)n \rfloor$*, the integer part of* $(1-\alpha)n$*. If Assumption 1 holds, then*

$$\lim_{n\to\infty} \frac{1}{n} \ln c(\varphi_n \circ \cdots \circ \varphi_{n_\alpha+1}) \ = \ \alpha \ln \kappa \quad \textit{almost surely,} \tag{3.7}$$

where κ *is the deterministic constant in Lemma 3.6.*

Proof. First note that, by Part 4 of Lemma 3.4, we have

$$\ln \mathrm{c}(\varphi_n \circ \ldots \circ \varphi_{n_\alpha+1}) \ \geq \ \ln \mathrm{c}(\varphi_n \circ \ldots \circ \varphi_1) - \ln \mathrm{c}(\varphi_{n_\alpha} \circ \ldots \circ \varphi_1) \ . \tag{3.8}$$

Thus, by Lemma 3.6,

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mathrm{c}(\varphi_n \circ \ldots \circ \varphi_{n_\alpha+1}) \ \geq \ \alpha \ln \kappa \quad \text{almost surely.} \tag{3.9}$$

To prove the complementary upper bound, i.e., that

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mathrm{c}(\varphi_n \circ \ldots \circ \varphi_{n_\alpha+1}) \ \leq \ \alpha \ln \kappa \ , \tag{3.10}$$

we will show that for each $m \in \mathbb{N}$

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mathrm{c}(\varphi_n \circ \ldots \circ \varphi_{n_\alpha+1}) \ \leq \ \alpha \frac{1}{m} \mathbb{E}[\ln \mathrm{c}(\Phi^{(m)})] \quad \text{almost surely.} \tag{3.11}$$

Eq. (3.10) will then follow by Lemma 3.4.

Let $m \in \mathbb{N}$ be fixed and consider $n \in \mathbb{N}$ large enough that $n - n_\alpha > 2m$. Let $p(n) = \lfloor \frac{n_\alpha + m}{m} \rfloor$ and let $q = q(n) \in \mathbb{N}$ and $r = r(n) \in \{0, 1, \ldots, m-1\}$ be defined by $n = qm + r$. Then,

$$n_\alpha + 1 \;\le\; p(n)m + 1 \;\le\; n_\alpha + m \;<\; n - m + 1 \le\; q(n)m \;. \tag{3.12}$$

Since $\ln \mathrm{c}(\varphi) \le 0$ for any $\varphi \in \mathcal{L}(\mathbb{M}_D)$, we have, using lemma 3.4, that

$$\ln \mathrm{c}(\varphi_n \circ \ldots \circ \varphi_{n_\alpha+1}) \;\le\; \ln \mathrm{c}(\varphi_{q(n)m+j} \circ \ldots \circ \varphi_{p(n)m+j+1}) \tag{3.13}$$

for any $0 \le j \le m-1$, where eq. (3.12) guarantees that $p(n)m + j + 1 \ge 1$ and the composition on the right hand side has non-zero number of factors. Using, 3.4 again we find that

$$\ln \mathrm{c}(\varphi_n \circ \ldots \circ \varphi_{n_\alpha+1}) \;\le\; \sum_{k=p(n)}^{q(n)-1} \ln \mathrm{c}(\varphi_{km+j+m} \circ \ldots \circ \varphi_{km+j+1}) \;=\; \sum_{k=p(n)}^{q(n)-1} \ln \mathrm{c}(\varphi_m \circ \ldots \circ \varphi_1) \circ \theta^{km+j} \;.$$

Since this holds for any $j \in \{0, 1, \ldots, m-1\}$, we have

$$\begin{aligned} \ln \mathrm{c}(\varphi_n \circ \ldots \circ \varphi_{n_\alpha+1}) \;&\le\; \frac{1}{m} \sum_{j=0}^{m-1} \sum_{k=p(n)}^{q(n)-1} \ln \mathrm{c}(\varphi_m \circ \ldots \circ \varphi_1) \circ \theta^{km+j} \\ &=\; \frac{1}{m} \sum_{i=p(n)m}^{q(n)m-1} \ln \mathrm{c}(\varphi_m \circ \ldots \circ \varphi_1) \circ \theta^i \\ &=\; \sum_{i=0}^{q(n)m-1} \frac{1}{m} \ln \mathrm{c}(\Phi^{(m)}) \circ \theta^i \;-\; \sum_{i=0}^{p(n)m-1} \frac{1}{m} \ln \mathrm{c}(\Phi^{(m)}) \circ \theta^i \;. \end{aligned}$$

Since $\left(\frac{1}{m} \ln \mathrm{c}(\varphi_m \circ \ldots \circ \varphi_1)\right)^+ \in L^1(\Omega)$ (where $(\cdot)^+$ denotes the positive part), eq. (3.10) follows from the Birchoff ergodic theorem. ■

3.3 Invertible ergodic dynamics

In this section, we assume that θ is an invertible ergodic map. It is often possible to replace the original dynamical system by a natural extension on which θ is invertible; for instance this is possible if θ is *essentially surjective*, i.e. if $\Omega \setminus \theta(\Omega)$ is a sub-null set —see [7]. We will denote this extension also by $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ and note that the previously stated results still hold.

Since θ is invertible and measure preserving, the inverse map θ^{-1} is also a measure preserving ergodic transformation. We extend the definition of $(\Phi^{(n)})$ to include negative indices as follows

$$\Phi^{(n)}(\omega) = \begin{cases} \varphi_n(\omega) \circ \ldots \circ \varphi_1(\omega) & \text{for } n \ge 1 \;, \\ \varphi_0 & \text{for } n = 0 \;, \\ \varphi_{-1}(\omega) \circ \ldots \circ \varphi_n(\omega) & \text{for } n \le -1 \;, \end{cases} \tag{3.14}$$

where $\varphi_n := \varphi_{\theta^n}$ for all n. When θ is invertible, Assumption 1 guarantees that with probability one $(\Phi^{(-n)})_{n\ge1}$ is almost surely eventually strictly positive — see [22, Lemma 3.13].

With this extended dynamical system, we introduce some new notation. Let $n \in \mathbb{N}$ and define

$$\psi_n = \varphi^*_{-n} \quad \text{and} \quad \Psi^{(n)} = \psi_n \circ \ldots \circ \psi_1 \;. \tag{3.15}$$

Note that $\Psi^{(n)*} = \Phi^{(-n)}$. We see that $(\Psi^{(n)})_{n\in\mathbb{N}}$ is almost surely eventually strictly positive. This allows us to define a new stopping time τ' as:

$$\tau' \;=\; \inf\{n \ge 1 : \Phi^{(n+k)} \text{ and } \Psi^{(n+k)} \text{ are strictly positive } \forall k \ge 0\} \;, \tag{3.16}$$

satisfying $\mathbb{P}[\tau' < \infty] = 1$ if θ is invertible and Assumption 1 holds.

We have the following result analogous to Lemma 3.6 for the sequence $(\Psi^{(n)})_{n\ge1}$:

Lemma 3.11. *If θ is invertible and $(\Phi_{(n)})_{n\geq 1}$ satisfies Assumption 1, then*

$$\ln \kappa \overset{a.s}{=} \lim_{n\to\infty} \frac{1}{n} \ln c(\Psi^{(n)}), \tag{3.17}$$

where $\Psi^{(n)}$ is as in eq. (3.15) and κ is the deterministic constant appearing in lemma 3.6. In particular, $\lim_n c(\Psi^{(n)}) = 0$ *almost surely.*

Remark 3.12. The existence of the deterministic limit on the right hand side of eq. (3.17) follows directly from Lemma 3.6 applied with the sequence $\Psi^{(n)}$ in place of $\Phi^{(n)}$. That the limit equals κ follows from the identity

$$\mathbb{E}\ln \mathrm{c}(\Psi^{(n)}) = \mathbb{E}\ln \mathrm{c}(\Phi^{(n)*}_{\theta^{-n-1}}) = \mathbb{E}\ln \mathrm{c}(\Phi^{(n)}) \ ,$$

where we have used the facts that θ is measure preserving and that $\mathrm{c}(\phi^*) = \mathrm{c}(\phi)$ for any ϕ.

If θ is invertible and Assumption 1 holds, then the left and right Perron-FRobenius eigenmatrices R_n and L_n for $\Phi^{(n)}$ exist also for large negative n. As a result we have the following lemma for the convergence of the right eigenvectors:

Lemma 3.13 ([22, Lemma 3.14])**.** *Let $(\varphi_n)_{n\geq 1}$ and Φ^n be as in eq. (2.6) and let R_n be the right Perron-Frobenius eigenmatrix for $\Phi^{(n)}$, see eq. (2.10). If θ is invertible an Assumption 1 holds, then there is an $\mathbb{S}_D^\circ$ valued random variable Z_1' such that*

$$\lim_{n\to -\infty} R_n \overset{a.s}{=} Z_1' \tag{3.18}$$

and, with $Z_k' := Z_1' \circ \theta^{-k+1}$, we have:

1. *for every $k \in \mathbb{N}$, $\psi_k^* \cdot Z_{k+1}' = Z_k'$ a.s., and*
2. *for each $Y \in \mathbb{S}_D$ and $k \in \mathbb{N}$,*

$$\mathrm{d}((\psi_k^* \circ \ldots \circ \psi_n^*) \cdot Y, Z_k') \ \leq \ c(\psi_k^* \circ \ldots \circ \psi_n^*)$$

   for all sufficiently large n. In particular, we have $\lim_n (\psi_k^* \circ \ldots \circ \psi_n^*) \cdot Y = Z_k'$ a.s..

If instead we take $n \to \infty$, we do not have almost sure convergence of R_n. However, we do have convergence in distribution:

Corollary 3.14. *We have that*

$$R_n \xrightarrow[n\to\infty]{d} Z_1' \quad \textit{and} \quad L_n \xrightarrow[n\to -\infty]{d} Z_1 \ , \tag{3.19}$$

where $\xrightarrow{d}$ denotes convergence in distribution.

Proof. Note that $R_n = R_{-n;\theta^{n+1}}$, so that $R_n \overset{d}{=} R_{-n}$. Since $\lim_{n\to\infty} R_{-n} = Z_1'$ a.s., the first limit holds. The proof for the second limit is similar. ∎

4 Proof of the Law of Large Numbers

We now describe the proof of Theorem 1 - Law of Large Numbers. Recall from the discussion following the statement of the theorem above, that it suffices to prove eq. (2.14), which states that

$$\lim_{n\to\infty} \sup_{X,Y\in\mathbb{S}_D} \left| \frac{1}{n} \ln\langle Y, \Phi^{(n)}(X)\rangle - l \right| \ = \ 0 \quad \text{a.s..}$$

To this end, note that by Assumption 2 we have $\mathbb{E}[\ln ||\varphi_k^*(Z_{k+1})||] < \infty$ for each $k \in \mathbb{N}$. Thus by Birkhoff's ergodic theorem we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln ||\varphi_k^*(Z_{k+1})|| \overset{a.s}{=} \mathbb{E} \ln ||\varphi_0^*(Z_1)|| := l.$$

Thus eq. (2.14), and therefore Theorem 1 - Law of Large Numbers, follows from the following

Lemma 4.1. *Suppose that Assumption 1 holds and let*

$$D_n = \sup_{X,Y\in\mathbb{S}_D} \left\{ \left| \ln\langle Y, \Phi^{(n)}(X)\rangle - \ln ||\Phi^{(n)*}(Y)|| \right| \right\} , \tag{4.1}$$

and

$$E_n := \sup_{Y\in\mathbb{S}_D} \left\{ \left| \ln ||\Phi^{(n)*}(Y)|| - \sum_{k=1}^{n} \ln ||\varphi_{k,\omega}^*(Z_{k+1})|| \right| \right\} . \tag{4.2}$$

for $n \geq 1$. Then, with probability one,

1. *D_n is eventually bounded, i.e., $\limsup_{n\to\infty} D_n < \infty$, and*
2. *$\lim_{n\to\infty} \frac{1}{n} E_n = 0$.*

Remark 4.2. Note that from $\lim_n \frac{1}{n} E_n$ we conclude directly that $\lim_n \frac{1}{n} \ln ||\Phi^{(n)*}(Y)|| = l$ for every $Y \in \mathbb{S}_D$. In particular $l = \lambda$, the top Lyapunov exponent of $\Phi^{(n)}$, as claimed in the discussion following Theorem 1 - Law of Large Numbers above.

Proof. First note that for any $X, Y \in \mathbb{S}_D$,

$$\langle Y, \Phi^{(n)}(X)\rangle = \langle \Phi^{(n)*}(Y), X\rangle \leq \langle \Phi^{(n)*}(Y), \mathbb{I}\rangle = ||\Phi^{(n)*}(Y)|| , \tag{4.3}$$

Here we have used that $X \leq \mathbb{I}$ and $\mathrm{tr}M = ||M||$ for any positive semi-definite matrix.

For the rest of the proof, we restrict to a configuration ω such that $\tau = \tau_\omega < \infty$. Such configurations form a full measure set by Assumption 1.

Because $\Phi^{(\tau)}$ is strictly positive, we have $\min \sigma(\Phi^{(\tau)}(P)) > 0$ for any projection P, where $\sigma(\Phi^{(\tau)}(P))$ denotes the spectrum of $\Phi^{(\tau)}(P)$. Thus the map $P \mapsto \min \sigma(\Phi^{(\tau)}(P))$ is a continuous function from the set of rank-1 projections into $(0, \infty)$. Since the set of rank-1 projections is compact, we have

$$a := \min \left\{ \min(\sigma(\Phi^{(\tau)}(P))) \ : \ P \text{ is a rank-1 projection} \right\} > 0.$$

Given $X, Y \in \mathbb{S}_D$ and $n > \tau$, let $W = \varphi_{\tau+1}^* \circ \ldots \circ \varphi_n^*(Y)$. Because X has at least one eigenvalue greater than or equal to $\frac{1}{D}$, we have $X \geq \frac{1}{D} P$ for some rank-1 projection P, and thus

$$\langle Y, \Phi^{(n)}(X)\rangle = \langle W, \Phi^{(\tau)}(X)\rangle \geq \frac{1}{D} \langle W, \Phi^{(\tau)}(P)\rangle \geq \frac{a}{D} \langle W, \mathbb{I}\rangle = \frac{a}{D} ||W|| .$$

Since $||\Phi^{(n)*}(Y)|| = ||\Phi^{(\tau)*}(W)|| \leq ||\Phi^{(\tau)*}|| \, ||W||$, we have

$$\langle Y, \Phi^{(n)}(X)\rangle \geq \frac{a}{D||\Phi^{(\tau)*}||} ||\Phi^{(\tau)*}(Y)|| . \tag{4.4}$$

Putting eqs. (4.3) and (4.4) together, we see that

$$\ln a - \ln D - \ln ||\Phi^{(\tau)*}|| \leq \ln\langle Y, \Phi^{(n)}(X)\rangle - \ln ||\Phi^{(\tau)*}(Y)|| \leq 0$$

for $X, Y \in \mathbb{S}_D$ and $n > \tau$. It follows that $\limsup_n D_n \leq \ln D + \ln ||\Phi^{(\tau)*}|| - \ln a < \infty$ whenever $\tau < \infty$.

Turning now to the proof that $\lim_n \frac{1}{n} E_n = 0$, consider $n > \tau$. Note that

$$||\Phi^{(n)*}(Y)|| \;=\; ||\phi_1^*(\phi_2^* \circ \cdots \circ \phi_n^*(Y))|| \;=\; ||\phi_1^*\left((\phi_2^* \circ \cdots \circ \phi_n^*) \cdot Y\right)||\, ||\phi_2^* \circ \cdots \circ \phi_n^*(Y)||\ ,$$

where in the final expression we have introduced the projective action by multiplying and dividing by $||\phi_2^* \circ \cdots \circ \phi_n^*(Y)|| = \mathrm{tr}\phi_2^* \circ \cdots \circ \phi_n^*(Y)$. Taking logarithms and iterating, we find that

$$\ln ||\Phi^{(n)*}(Y)|| \;=\; \sum_{k=1}^{n} \ln ||\varphi_k^*((\varphi_{k+1}^* \circ \ldots \circ \varphi_n^*) \cdot Y)||\ ,$$

where the empty composition $\varphi_{n+1}^* \circ \ldots \circ \varphi_n^*$ is understood as the identity map. Thus

$$E_n(Y) \;:=\; \left|\ln ||\Phi^{(n)*}(Y)|| - \sum_{k=1}^{n} \ln ||\varphi_k^*(Z_{k+1})||\right| \;\le\; \sum_{k=1}^{n} E_n^k(Y)\ ,$$
$$\text{with}\quad E_n^k(Y) \;:=\; \left|\ln ||\varphi_k^*((\varphi_{k+1}^* \circ \ldots \circ \varphi_n^*) \cdot Y)|| - \ln ||\varphi_k^*(Z_{k+1})||\right|\ . \tag{4.5}$$

Using Lemma 3.2, Remark 3.3, and Lemma 3.9 we may bound $E_n^k(Y)$ as follows

$$E_n^k(Y) \;\le\; 2\frac{||\varphi_k^*||}{v(\varphi_k^*)} \mathrm{c}(\varphi_{k+1}^* \circ \ldots \circ \varphi_n^*)\ . \tag{4.6}$$

Now let $\alpha \in (0,1)$ and let n_α be the integer part of $(1-\alpha)n$. We will bound the terms on the right hand side of (4.5) differently according to if $k < n_\alpha$ or $k \ge n_\alpha$. For $k < n_\alpha$, we have

$$E_n^k(Y) \;\le\; 2\frac{||\varphi_k^*||}{v(\varphi_k^*)} \mathrm{c}(\varphi_{n_\alpha}^* \circ \ldots \circ \varphi_n^*)\ ,$$

where we have used eq. (4.6) and applied Lemma 3.4 to bound $\mathrm{c}(\varphi_{k+1}^* \circ \ldots \circ \varphi_n^*) \le \mathrm{c}(\varphi_{n_\alpha}^* \circ \ldots \circ \varphi_n^*)$. For $k \ge n_\alpha$, on the other hand, we have

$$E_n^k(Y) \;\le\; \left|\ln ||\varphi_k^*((\varphi_{k+1}^* \circ \ldots \circ \varphi_n^*) \cdot Y)||\right| + \left|\ln ||\varphi_k^*(Z_{k+1})||\right| \;\le\; 2\Big(\left|\ln v(\varphi_k^*)\right| + \left|\ln ||\varphi_k^*||\right|\Big)\ .$$

Thus

$$E_n \;=\; \sup_{Y \in \mathbb{S}_D} E_n(Y) \;\le\; S_n^< + S_n^> \tag{4.7}$$

with

$$S_n^< \;= 2\sum_{k=1}^{n_\alpha - 1} \frac{||\varphi_k^*||}{v(\varphi_k^*)} \mathrm{c}(\varphi_{n_\alpha}^* \circ \ldots \circ \varphi_n^*)\ ,$$

and

$$S_n^> \;= 2\sum_{k=n_\alpha}^{n} \Big(\left|\ln v(\varphi_k^*)\right| + \left|\ln ||\varphi_k^*||\right|\Big)\ .$$

We will prove that $\lim_n S_n^< = 0$ and $\lim_n \frac{1}{n} S_n^> = O(\alpha)$.

Note that by Assumption 2 we have $\mathbb{E}[|\ln(\frac{||\varphi_0^*||}{v(\varphi_0^*)})|] < \infty$. Thus, for any $\epsilon > 0$,

$$\infty \;>\; \frac{1}{\epsilon}\mathbb{E}[|\ln(\tfrac{||\varphi_0^*||}{v(\varphi_0^*)})|] \;\ge\; \sum_{k=1}^{\infty} \mathbb{P}\big(\ln(\tfrac{||\varphi_0^*||}{v(\varphi_0^*)}) > k\epsilon\big)$$
$$=\; \sum_{k=1}^{\infty} \mathbb{P}\big(\ln(\tfrac{||\varphi_0^*||}{v(\varphi_0^*)}) > k\epsilon\big) \;=\; \sum_{k=1}^{\infty} \mathbb{P}\big(\tfrac{||\varphi_k^*||}{v(\varphi_k^*)} > e^{k\epsilon}\big)\ .$$

Hence, by the Borel-Cantelli Lemma, we find that $\limsup_k e^{k\epsilon} \frac{||\varphi_k^*||}{v(\varphi_k^*)} \leq 1$ with probability one. Taking $\epsilon < \alpha|\ln \kappa|$, we conclude from Lemma 3.10 that

$$\limsup_{n\to\infty} S_n^< \;\leq\; \limsup_{n\to\infty} n_\alpha e^{\epsilon n_\alpha} \mathrm{c}((\varphi_{n_\alpha}^* \circ \ldots \circ \varphi_n^*)) \;=\; 0\;.$$

In particular, we also have $\lim_n \frac{1}{n} S_n^< = 0$.

Now consider $S_n^>$. Since $\ln v(\phi_0^*)$ and $\ln ||\phi_0^*||$ are L^1 random variables by Assumption 1, we conclude from the Birkhoff ergodic theorem [3] that

$$\lim_n \frac{1}{n} S_n^> \;=\; 2\alpha\left[\mathbb{E}|\ln ||\varphi_0^*||| + \mathbb{E}|\ln v(\varphi_0^*)|\right]\;.$$

We conclude that $\limsup_n \frac{1}{n} E_n \;=\; O(\alpha)$. Since $\alpha \in (0,1)$ was arbitrary, we have $\lim_n \frac{1}{n} E_n = 0$. ■

5 Proof the Central Limit Theorem

In this section we prove Theorem 2 - Central Limit Theorem. Let $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ be sequences in $\mathbb{S}_n$. Then

$$\frac{1}{\sqrt{n}}\big(\ln\langle Y_n, \Phi^{(n)}(X_n)\rangle - nl\big) \;=\; \frac{1}{\sqrt{n}}\big(\ln\langle Y_n, \Phi^{(n)}(X_n)\rangle - \ln ||\Phi^{(n)*}(Y_n)||\big) + \frac{1}{\sqrt{n}}\Big(\ln ||\Phi^{(n)*}(Y_n)|| - \sum_{k=1}^n \ln ||\varphi_k^*(\omega)(Z_{k+1}(\omega))||\Big) + \frac{1}{\sqrt{n}}\sum_{k=1}^n \xi_k\;,$$

where $\xi_k = \ln ||\varphi_k^*(Z_{k+1})|| - l$. Thus

$$\left|\frac{1}{\sqrt{n}}\big(\ln\langle Y_n, \Phi^{(n)}(X_n)\rangle - nl\big) - \frac{1}{\sqrt{n}}\sum_{k=1}^n \xi_k\right| \;\leq\; \frac{1}{\sqrt{n}}(D_n + E_n)$$

with D_n and E_n as in eqs. (4.1) and (4.2), respectively. By Lemma 4.1, D_n is almost surely eventually bounded. Thus to prove that $\big(\frac{1}{\sqrt{n}}\ln\langle Y_n, \Phi^{(n)}(X_n)\rangle\big)_{n\geq 1}$ converges in distribution to a centered normal variable, it suffices to prove the following two results:

1. $\frac{1}{\sqrt{n}}E_n$ converges to 0 in probability, and
2. $Q_n := \dfrac{1}{\sqrt{n}}\displaystyle\sum_{k=1}^n \xi_k$ converges in distribution to a centered normal variable with variance given by eq. (2.20) above.

These results are proved in Lemma 5.1 and Lemma 5.2 below, respectively.

Lemma 5.1. *Suppose that θ is invertible and that Assumption 1 holds. Let $(E_n)_{n=1}^\infty$ be the variables defined in eq. (4.2). Then $(E_n)_{n=1}^\infty$ is tight. In particular, $(\frac{1}{\sqrt{n}}E_n)_{n=1}^\infty$ converges to 0 in probability.*

Proof. Following the proof of eq. (4.7) above, but applying in the proof of Lemma 4.1, we have

$$E_n \;\leq\; S_n \;:=\; 2\sum_{k=1}^n \frac{||\varphi_k^*||}{v(\varphi_k^*)}\mathrm{c}(\varphi_{k+1}^* \circ \ldots \circ \varphi_n^*)\;.$$

We prove that E_n are tight by showing that $S_n \overset{d}{=} S_n'$ where the random variables S_n' satisfy $\sup_n S_n' < \infty$ almost surely.

Consider the variables $S'_n = S_{n;\theta^{-n}}$. Since $\mathrm{c}(\phi^*) = \mathrm{c}(\phi)$, we have

$$S'_n \;=\; 2\sum_{k=0}^{n-1} \frac{||\varphi^*_{-k}||}{v(\varphi^*_{-k})}\mathrm{c}(\varphi_0 \circ \ldots \circ \varphi_{1-k})\;.$$

As above the empty composition appearing at $k = 0$ is understood as the identity map. By the Borel-Cantelli similar to that used to bound $S_n^<$ in the proof of Lemma 4.1 we see that

$$\limsup_{k\to\infty} \frac{1}{k}\ln\left(\frac{||\varphi^*_{-k}||}{v(\varphi^*_{-k})}\right) \;=\; 0 \quad \text{a.s..}$$

On the other hand by Lemma 3.11 we have

$$\limsup_{k\to\infty} \frac{1}{k}\ln \mathrm{c}(\varphi_0 \circ \ldots \circ \varphi_{1-k}) \;=\; \ln\kappa \;<\; 0 \quad \text{a.s..}$$

It follows that

$$\lim_{n\to\infty} S'_n \;=\; 2\sum_{k=0}^{n} \frac{||\varphi^*_{-k}||}{v(\varphi^*_{-k})}\mathrm{c}(\varphi_0 \circ \ldots \circ \varphi_{1-k}) \;=:\; S'_\infty$$

is finite almost surely. Clearly $S'_\infty = \sup_n S'_n$.

Since $S_n \overset{d}{=} S'_n$ we have

$$\mathbb{P}\left[E_n > \epsilon\right] \;\le\; \mathbb{P}\left[S_n > \epsilon\right] \;=\; \mathbb{P}\left[S'_n > \epsilon\right] \;\le\; \mathbb{P}\left[S'_\infty > \epsilon\right]\;,$$

so $(E_n)_{n=1}^\infty$ is tight as claimed. It follows that

$$\mathbb{P}\left[\tfrac{1}{\sqrt{n}}E_n > \epsilon\right] \;\le\; \mathbb{P}\left[S'_\infty > \sqrt{n}\epsilon\right] \;\to\; 0\;,$$

so $\frac{1}{\sqrt{n}}E_n$ converges to zero in probability. ∎

To prove the convergence of $Q_n = \frac{1}{\sqrt{n}}\sum_{k=1}^n \xi_k$ to a centered normal law in distribution, we use the martingale approximation method of Gordin [13]. The following proof is adapted from the proof of [14, Lemma 9.2] and is similar to the proof of [20, Theorem 1.1]. The key idea is to find a *reverse martingale difference* with respect to the filtration $(\mathcal{F}^n)_{n\ge1}$ and use the Central Limit Theorem for (reverse) martingale differences [2, 4, 6] which was proved independently by Billingsly [1] and [15] for the ergodic case:

Martingale Difference Central Limit Theorem. *Let $(X_n)_{n\ge1}$ be a stationary ergodic direct or reversed martingale difference with respect to a filtration $\{\mathcal{A}_n\}_{n\ge1}$. If $X_1 \in L^2$, then $\left(\frac{1}{\sqrt{n}}\sum_{k=1}^n X_k\right)_{n\ge1}$ converges in distribution to a centered normal random variable with variance $\sigma^2 = \mathbb{E}(X_1^2)$.*

Lemma 5.2. *Suppose that θ is invertible, that Assumption 1 holds, and Assumption 2_p holds for some $p \ge 2$. Let $\xi_k = \ln||\varphi^*_k(Z_{k+1})|| - l$ for $k \in \mathbb{Z}$. If*

$$\sum_{n=1}^{\infty} ||\mathbb{E}[\xi_0|\mathcal{F}^n]||_q < \infty\;, \tag{5.1}$$

with $\frac{1}{p}+\frac{1}{q} = 1$, then the sequence $(Q_n)_{n=1}^\infty$ given by

$$Q_n \;=\; \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\xi_k \tag{5.2}$$

converges in distribution to a centered normal law with variance $\sigma^2 < \infty$. Furthermore $\sigma = 0$ if and only if there exists stationary sequence $(g_n)_{n\ge1}$ such that

$$g_n \in L^q(\mathcal{F}^n) \quad \text{and} \quad \xi_n = g_{n+1} - g_n \tag{5.3}$$

Proof. Let $M := \sum_{k=1}^{\infty} ||\mathbb{E}[\xi_0|\mathcal{F}^k]||_q < \infty$ by eq. (5.1). We define

$$g_0 \;:=\; \sum_{k=1}^{\infty} \mathbb{E}[\xi_{-k}|\mathcal{F}^0] \;, \tag{5.4}$$

and note that

$$||g_0||_q \;\le\; \sum_{k=1}^{\infty} ||\mathbb{E}[\xi_{-k}|\mathcal{F}^0]|| \;=\; M \;,$$

since θ is measure preserving. Since $||\cdot||_1 \le ||\cdot||_q$, the series defining g_0 converges in L^1 and hence absolutely, almost everywhere.

We define

$$\zeta_0 \;=\; \sum_{k=0}^{\infty} \left(\mathbb{E}[\xi_{-k}|\mathcal{F}^0] - \mathbb{E}[\xi_{-k}|\mathcal{F}^1]\right) \;, \tag{5.5}$$

and note that $\zeta_0 \;=\; \xi_0 + g_0 - g_0 \circ \theta$. For $n \in \mathbb{Z}$, we now define $\zeta_n = \zeta_0 \circ \theta^n$ and $g_n = g_0 \circ \theta^n$, so that

$$\xi_n \;=\; \zeta_n + g_{n+1} - g_n \;. \tag{5.6}$$

Since

$$|\xi_n| \;\le\; \ln ||\phi_n^*|| + \ln v(\phi_n^*) + |l| \;, \tag{5.7}$$

we have $\xi_n \in L_p \subset L_q$ by Assumption 2_p, so $\zeta_n = \xi_n - g_{n+1} + g_n \in L_q \subset L^1$. Taking conditional expectation with respect to $\mathcal{F}^{n+1}$ in eq. (5.5), we see that

$$\mathbb{E}[\zeta_n|\mathcal{F}^{n+1}] = 0 \;, \tag{5.8}$$

i.e., $(\zeta_n)_{n\ge1}$ is a reverse martingale difference (*reverse* because $(\mathcal{F}^n)_{n\ge1}$ is a reverse filtration). Now eq. (5.6) shows that

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n} \xi_k \;=\; \frac{1}{\sqrt{n}}\sum_{k=1}^{n} \zeta_n + \frac{1}{\sqrt{n}}(g_{n+1} - g_1) \;. \tag{5.9}$$

Since $g_{n+1} = g_1 \circ \theta^n$, we see that $g_{n+1} - g_1$ is tight and thus $\frac{1}{\sqrt{n}}(g_{n+1} - g_1)$ converges to 0 in probability. Therefore, by the Martingale Difference Central Limit Theorem, we will have the required convergence in distribution if we establish that $\zeta_0 \in L^2$.

Since $\zeta_0 = \xi_0 - (g_1 - g_0)$ and $\xi_0 \in L_p \subset L_2$ by eq. (5.7), it suffices to show that $g_1 - g_0 \in L^2$. We have $g_n \in L^q(\mathcal{F}^n)$, but this does not suffice as $q < 2$. To show that $g_1 - g_0 \in L^2$ we need to exploit cancellation between the two terms. To this end, let $\lambda \in (0,1)$ and define

$$g_0^\lambda \;=\; \sum_{k=1}^{\infty} \lambda^{k-1}\mathbb{E}[\xi_{-k}|\mathcal{F}^0] \;, \tag{5.10}$$

and define $g_n^\lambda = g_0^\lambda \circ \theta^n$ for $n \in \mathbb{Z}$. Since $||\mathbb{E}[\xi_{-k}|\mathcal{F}^0]||_p \le ||\xi_{-k}||_p = ||\xi_0||_p$, the convergence factor λ^{k-1} in eq. (5.10) guarantees that $g_0^\lambda \in L^p \subset L^2$. Furthermore, we have

$$||g_0^\lambda||_q \;\le\; \sum_{k=1}^{\infty} \lambda^{k-1}||\mathbb{E}[\xi_{-k}|\mathcal{F}^0]||_q \;\le\; M \;, \tag{5.11}$$

since $\lambda \le 1$.

We will now show that $||g_1^\lambda - \lambda g_0^\lambda||_2^2$ is bounded uniformly in λ. We start with the estimate

$$\begin{aligned} ||g_1^\lambda - \lambda g_0^\lambda||_2^2 \;&=\; (1+\lambda^2)||g_1^\lambda||_2^2 - 2\lambda\mathbb{E}[g_0^\lambda g_1^\lambda] \\ &\le\; 2\left[||g_1^\lambda||_2^2 - \lambda\mathbb{E}[g_0^\lambda g_1^\lambda]\right] \;=\; 2\mathbb{E}\left[g_1^\lambda\left(g_1^\lambda - \lambda\mathbb{E}[g_0^\lambda|\mathcal{F}^1]\right)\right] \;, \end{aligned}$$

where we have noted that $||g_1^\lambda||_2 = ||g_0^\lambda||_2$ (since $(g_n^\lambda)_{n=1}^\infty$ is stationary) and that g_1^λ is $\mathcal{F}^1$ measurable. Note that

$$g_1^\lambda - \lambda\mathbb{E}[g_0^\lambda|\mathcal{F}^1] \;=\; \sum_{k=1}^\infty \lambda^{k-1}\mathbb{E}[\xi_{-k+1}|\mathcal{F}^1] - \lambda\sum_{k=1}^\infty \lambda^{k-1}\mathbb{E}[\xi_{-k}|\mathcal{F}^1] \;=\; \mathbb{E}[\xi_0|\mathcal{F}^1]\ .$$

Thus

$$||g_1^\lambda - \lambda g_0^\lambda||_2^2 \;\leq\; 2\int_\Omega \mathbb{E}[\xi_0|\mathcal{F}^1]g_1^\lambda d\mathbb{P} \;\leq\; 2||\mathbb{E}[\xi_0|\mathcal{F}^1]||_p||g_1^\lambda||_q \;\leq\; 2||\xi_0||_p M\ ,$$

where we have used Hölder's inequality and eq. (5.11).

Since $g_1 - g_0 = \lim_{\lambda\uparrow 1} g_1^\lambda - \lambda g_0^\lambda$, we have

$$\mathbb{E}[(g_1-g_0)^2] \;=\; \mathbb{E}\big[\lim_{\lambda\uparrow 1}(g_1^\lambda - \lambda g_0^\lambda)^2\big] \;\leq\; \liminf_{\lambda\uparrow 1}\mathbb{E}[(g_1^\lambda - \lambda g_0^\lambda)^2] \;\leq\; 2||\xi_0||_p M\ ,$$

by Fatou's Lemma. Therefore $g_1 - g_0 \in L^2$. Thus $\zeta_n \in L^2$ for each n and the martingale difference central limit theorem implies that $(\frac{1}{\sqrt{n}}\sum_{k=1}^n \zeta_k)_{n\geq 1}$ (and thus $(\frac{1}{\sqrt{n}}\sum_{k=1}^n \xi_k)_{n\geq 1}$) converges in distribution to a centered normal random variable with variance $\sigma^2 = \mathbb{E}[\zeta_0^2]$.

If $\sigma = 0$ then we have that $\zeta_n = 0$ a.s. for each $n \in \mathbb{Z}$. In this case, we have $\xi_n = g_{n+1} - g_n$ for the stationary processes $(g_n)_{n\in\mathbb{Z}}$ defined above. This concludes the proof of lemma 5.2 ■

This completes the proof of Theorem 2 - Central Limit Theorem. In the next section we discuss the mixing conditions sufficient to prove the hypothesis eq. (2.18).

6 Mixing Conditions

In this section we prove Theorem 3, which provides sufficient conditions for the main hypothesis eq. (2.18) of Theorem 2 - Central Limit Theorem. The arguments in this section are based on similar results in [9] and [14]. We rely on the following estimate on averages of sub-multiplicative random variables that combines [14, Lemma 6.2 & Lemma 6.3] — see also [5, Lemma 3 & Lemma 4].

Lemma 6.1 ([14]). *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with an ergodic measure preserving map $\theta : \Omega \to \Omega$, a filtration $(\mathcal{F}_n)_{n\geq 0}$, and a reverse filtration $(\mathcal{F}^n)_{n\geq 0}$, such that $\theta^{-1}(\mathcal{F}_{n+1}) = \mathcal{F}_n$ and $\theta^{-1}(\mathcal{F}^{n+1}) = \mathcal{F}^n$ for each $n \geq 0$. Let α_n and ρ_n be mixing coefficients defined as defined in eqs. (2.22) and (2.23), respectively. Let $(M_n)_{n\geq 1}$ be a sequence of $[0,1]$-valued random variables with the following* sub-multiplicative property

$$M_{m+n} \;\leq\; M_m\ M_n \circ \theta^n. \tag{6.1}$$

If for each $0 \leq m < n$ it holds that $M_{n-m}\circ\theta^m$ is both $\mathcal{F}_n$ and $\mathcal{F}^m$ measurable, then we have:

1. *If $\alpha_n \leq cn^{-\lambda}$ with $c, \lambda > 0$, then $\mathbb{E}[M_n]$ almost vanishes to order $n^{-\lambda}$,*

$$\mathbb{E}[M_n] \;=\; O\left(\frac{a_n}{n}\right)^\lambda \tag{6.2}$$

   for any sequence $(a_n)_{n\geq 1}$ of real numbers such that

$$\lim_{n\to\infty}\frac{\ln n}{a_n} \;=\; \lim_{n\to\infty}\frac{a_n}{n} \;=\; 0\ . \tag{6.3}$$

2. *If $\lim_{n\to\infty}\rho_n = 0$, then $\mathbb{E}[M_n]$ vanishes faster than any polynomial, i.e.,*

$$\mathbb{E}[M_n] \;=\; O\left(\frac{1}{n^k}\right)\ . \tag{6.4}$$

   for each $k \in \mathbb{N}$,

Lemma 6.1 directly implies bounds on $\mathbb{E}[c(\Phi^{(n)})]$, stated in the following

Lemma 6.2. *Suppose that Assumptions 1 and 2 hold, and let α_n and ρ_n be mixing coefficients defined as defined in eqs. (2.22) and (2.23), respectively. For $r \in (0,1)$ define*

$$\tau_r \;=\; \inf\{n \geq 1 : c(\Psi^{(n)}) \leq r \;\&\; c(\Phi^{(n)}) \leq r\} \;.$$

Then we have that $\tau_r < \infty$ almost surely. Moreover

1. *If $\sum_{k=1}^{\infty} \alpha_k^{1/\lambda} < \infty$, for some $\lambda > 0$, then*

$$\max\Big\{\mathbb{P}[\tau_r > n]\;,\;\mathbb{E}[c(\Phi^{(n)})]\Big\} \;=\; O\left(\frac{a_n}{n}\right)^{\lambda} \tag{6.5}$$

for any sequence $(a_n)_{n\geq 1}$ satisfying eq. (6.3).

2. *If $\lim_{n\to\infty} \rho_n = 0$, then*

$$\max\Big\{\mathbb{P}[\tau_r > n]\;,\;\mathbb{E}[c(\Phi^{(n)})]\Big\} \;=\; O\left(\frac{1}{n^k}\right) \tag{6.6}$$

for any $k \geq 1$.

Proof. From Corollary 3.8 and Lemma 3.11 we see that $\mathbb{P}[\tau_r < \infty] = 1$. We also have, by Assumption 2, that ϕ_n is non-destructive and non-transient for all $n \geq 1$, with probability one. Therefore, we have that almost surely for all $n \in \mathbb{Z}$, $\mathrm{c}(\Psi^{\tau_r+n}), \mathrm{c}(\Phi^{\tau_r+n}) < r$.

Suppose that $\sum_{k=1}^{\infty} \alpha_k^{1/\lambda} < \infty$. We start with the observation that α_n is non-increasing in n; this can be seen directly from the definition (2.22) of α_n using the fact that $(\mathcal{F}^n)_{n\geq 1}$ is decreasing in n. Since $\alpha_n^{1/\lambda}$ is a non-increasing sequence of positive numbers with $\sum_n \alpha_n^{1/\lambda} < \infty$, we have $\lim_{n\to\infty} n\alpha_n^{\lambda} = 0$. Therefore we have $\alpha_n \leq cn^{-\lambda}$. Now notice that the sequence $M_n = c(\Phi^{(n)})$, for $n \geq 1$, satisfies the sub-multiplicative condition in Lemma 6.1. Therefore we obtain

$$\mathbb{E}[\mathrm{c}(\Phi^{(n)})] \;\leq\; c_1\left(\frac{a_n}{n}\right)^{\lambda} \tag{6.7}$$

for any sequence $(a_n)_{n\geq 1}$ satisfying eq. (6.3). A similar analysis can be applied to $(c(\Psi^n)_{n\in\mathbb{N}}$, resulting in

$$\mathbb{E}[\mathrm{c}(\Phi^{(n)})] \;\leq\; c_2\left(\frac{a_n}{n}\right)^{\lambda}\;. \tag{6.8}$$

Since

$$\mathbb{P}[\tau_r > n] \;\leq\; \mathbb{P}[c(\Psi^{(n)}) > r] + \mathbb{P}[c(\Phi^{(n)}) > r] \;\leq\; \frac{1}{r}\mathbb{E}[c(\Phi^{(n)})] + \frac{1}{r}\mathbb{E}[c(\Psi^{(n)})]\;, \tag{6.9}$$

we see that eq. (6.5) holds.

If $\lim_{n\to\infty} \rho_n = 0$, then the second part of Lemma 6.1 applies and eq. (6.9) still holds. These two combined give us eq. (6.6). ■

We are now ready to state the main technical estimate of this section:

Lemma 6.3. *Suppose that Assumptions 1 and 2 hold, and let α_n and ρ_n be mixing coefficients defined in eqs. (2.22) and (2.23), respectively. Let $r \in (0,1)$ and let τ_r be as defined in Lemma 6.2. Let n_α denote the integer part of $(1-\alpha)n$, for $\alpha \in (0,1)$.*

1. *If Assumption 2_p holds with $p > 2$ then there is $K < \infty$ such that*

$$||\mathbb{E}[\xi_0|\mathcal{F}^n]||_q \;\leq\; K\left[\alpha_{n-n_\alpha}^{(p-2)/p} + \mathbb{E}[c(\Phi^{(n_\alpha)*})] + (\mathbb{P}[\tau_r > n_\alpha])^{1/q}\right]\;, \tag{6.10}$$

with q the conjugate exponent to p.

2. *If Assumption 2_p holds with $p = 2$ then there is $K < \infty$ such that*

$$||\mathbb{E}[\xi_0|\mathcal{F}^n]||_2 \ \le \ K \left[\rho_{n-n_\alpha} + \mathbb{E}[c(\Phi^{(n_\alpha)*})] + (\mathbb{P}(\tau_r > n_\alpha))^{1/2}\right] \ . \tag{6.11}$$

Before proving Lemma 6.3, let us show how it implies Theorem 3. First note that if $(b_n)_{n\ge 1}$ is a sequence of non-negative numbers then

$$\sum_{n=1}^{\infty} b_{n_\alpha} \ \le \ \frac{1}{1-\alpha}\sum_{n=1}^{\infty} b_n \ , \tag{6.12}$$

$$\sum_{n=1}^{\infty} b_{n-n_\alpha} \ \le \ \frac{1}{\alpha}\sum_{n=1}^{\infty} b_n \ . \tag{6.13}$$

To see that eq. (6.12) holds, note that given $m \in \mathbb{N}$, the number of integers n such that $n_\alpha = m$ is bounded by $\frac{1}{1-\alpha}$. The proof of eq. (6.13) is similar. Now suppose that Assumption 2_p holds with $p > 2$ and $\sum_n \alpha_n^{(p-2)/p} < \infty$. Then by Lemma 6.2, Lemma 6.3, and eqs. (6.12, 6.13), we have

$$\sum_{n=1}^{\infty} ||\mathbb{E}[\xi_0|\mathcal{F}^n]||_q \ \le \ K' \sum_{n\ge 1}\left[\alpha_n^{\frac{p-2}{p}} + \left(\frac{a_n}{n}\right)^{\frac{p}{p-2}} + \left(\frac{a_n}{n}\right)^{\frac{p}{p-2}\frac{1}{q}}\right]$$

for a suitable $K' < \infty$ and a slowly increasing sequence $(a_n)_{n\ge 1}$ satisfying eq. (6.3). Since $\frac{p}{p-2} > 1$ and $\frac{p}{p-2}\frac{1}{q} = \frac{p-1}{p-2} > 1$ the right hand side is finite. Similarly, if $\sum_n \rho_n < \infty$, then we have

$$\sum_{n=1}^{\infty} ||\mathbb{E}[\xi_0|\mathcal{F}^n]||_q \ \le \ K' \sum_{n\ge 1}\left[\rho_n + \frac{1}{n^k}\right]$$

for any k, which is clearly finite. This completes the proof of Theorem 3.

We now turn to the proof of Lemma 6.3:

Proof of Lemma 6.3. By Lemma 2.2, we have $\Phi^{(n_\alpha)} \cdot Z_{n_\alpha+1} = Z_1$. Therefore

$$\xi_0 \ = \ A_n + B_n - \mathbb{E}[A_n] \ ,$$

where

$$A_n = \ln ||\varphi_0^*(\Phi^{(n_\alpha)} \cdot Z_{n_\alpha+1})|| - \ln ||\varphi_0^*(\Phi^{(n_\alpha)} \cdot \tfrac{1}{D}\mathbb{I}||$$

and

$$B_n = \ln ||\varphi_0^*(\Phi^{(n_\alpha)} \cdot \tfrac{1}{D}\mathbb{I})|| - \mathbb{E}\left[\ln ||\varphi_0^*(\Phi^{(n_\alpha)} \cdot \tfrac{1}{D}\mathbb{I})||\right] \ .$$

Now consider the event $\{\tau_r \le n_\alpha\}$. On this event, $\mathrm{c}(\Phi^{(n_\alpha)*}) \le r$. To bound A_n on this event we will use the following proposition which we prove below after completing the present proof.

Proposition 6.4. *Let $\psi, \phi \in \mathcal{L}(\mathbb{M}_D)$. Suppose that ψ is a positive map and ϕ is a strictly positive map with $c(\phi) \le r < 1$. If ψ is non-transient, then for any $A, B \in \mathbb{S}_D$ we have*

$$\big|\ln ||\psi(\phi \cdot A)|| - \ln ||\psi(\phi \cdot B)||\big| \ \le \ c(\phi)\frac{2}{r}\ln\frac{1}{1-r}$$

Using Proposition 6.4, we see that

$$|A_n| \ \le \ \frac{2}{r}\ln\frac{1}{1-r}\mathrm{c}(\Phi^{(n_\alpha)*})1_{\tau_r \le n_\alpha} + 2\left(|\ln ||\varphi_0^*||| + |\ln v(\varphi_0^*)|\right)1_{\tau_r > n_\alpha} \ =: \ A_n'.$$

Therefore, using Hölder's inequality and Assumption 2_p, we have

$$\mathbb{E}|A_n| \;\le\; \mathbb{E}A'_n \;\le\; C\left(\mathbb{E}[\mathrm{c}(\Phi^{(n_\alpha)*})] + (\mathbb{P}(\tau_r > n_\alpha))^{(p-1)/p}\right) \tag{6.14}$$

with $C < \infty$. Furthermore we also have that

$$\sup_n ||A_n||_p \;\le\; \sup_n ||A'_n||_p \;<\; \infty \text{ and } \sup_n ||B_n||_p \;<\; \infty\,. \tag{6.15}$$

Now, for $\frac{1}{p} + \frac{1}{q} = 1$ we have that

$$||\mathbb{E}[\xi_0|\mathcal{F}^n]||_q = \sup_{\{f\in L^p(\mathcal{F}^n):||f||_p\le 1\}} \left|\mathbb{E}[f\xi_0]\right|\,. \tag{6.16}$$

Hence to bound $||\mathbb{E}[\xi_0|\mathcal{F}^n]||_q < \infty$ it suffices to find a uniform upper bound for $\mathbb{E}[\xi f]$ as f ranges over the unit ball in $L^p(\mathcal{F}^n)$. Since $\xi_0 = A_n + B_n - \mathbb{E}[A_n]$ and $\mathbb{E}[B_n] = 0$, we have

$$\begin{aligned} |\mathbb{E}[\xi_0 f]| \;&\le\; \left|\mathbb{E}[A_n f]\right| + \left|\mathbb{E}[B_n f]\right| + \left|\mathbb{E}[A_n]\mathbb{E}[f]\right| \\ &\le\; \mathbb{E}[A'_n\,|f|] + \left|\mathbb{E}[B_n f] - \mathbb{E}[B_n]\mathbb{E}[f]\right| + \mathbb{E}[A'_n]\mathbb{E}[|f|] \\ &\le\; \left|\mathbb{E}[A'_n\,|f|] - \mathbb{E}[A'_n]\mathbb{E}[|f|]\right| + \left|\mathbb{E}[B_n f] - \mathbb{E}[B_n]\mathbb{E}[f]|\right| + 2\mathbb{E}[A'_n]\mathbb{E}[|f|] \end{aligned} \tag{6.17}$$

To estimating the right hand side we use the following covariance inequalities involving the mixing coefficients α_n and ρ_n.

Lemma 6.5 ([8, §1.2 Theorem 3] —see also [14, §6.2]). *For each $n \in \mathbb{N}$, Let α_n and ρ_n be as defined in eqs. (2.22) and (2.23), respectively. For each $n, k \in \mathbb{N}$, we have*

$$\left|\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\right| \;\le\; \rho_n ||X||_2 ||Y||_2$$

whenever $X \in L^2(\mathcal{F}_k)$ and $Y \in L^2(\mathcal{F}^{n+k})$, and

$$\left|\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\right| \;\le\; 8\alpha_n^{1/r} ||X||_p ||Y||_q$$

whenever $X \in L^p(\mathcal{F}_k)$ and $Y \in L^q(\mathcal{F}^{k+n})$ with $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

We note that $A_n, B_n \in L^p(\mathcal{F}_{n_\alpha})$. If $p > 2$, then eq. (6.17) and Lemma 6.5 (with $q = p$ and $r = \frac{p}{p-2}$) together imply that

$$|\mathbb{E}[\xi_0 f]| \;\le\; 8\alpha_{n-n_\alpha}^{\frac{p}{p-2}} \left(||A'_n||_p + ||B_n||_p\right) ||f||_p + 2\mathbb{E}[A'_n]||f||_p\,,$$

where we have used the estimate $\mathbb{E}[|f|] \le ||f||_p$ in the last term. Eq. (6.10) follows this inequality together with eqs. (6.14, 6.15, 6.16). If $p = 2$, then eq. (6.17) and Lemma 6.5 together imply that

$$|\mathbb{E}[\xi_0 f]| \;\le\; \rho_{n-n_\alpha} \left(||A'_n||_2 + ||B_n||_2\right) ||f||_2 + 2\mathbb{E}[A'_n]||f||_2\,.$$

Eq. (6.11) follows from this inequality, again using eqs. (6.14, 6.15, 6.16). This completes the proof of Lemma 6.3. ■

It remains to prove Proposition 6.4:

Proof of Proposition 6.4. From [22, Lemma 3.3], the quantity $m(A, B)$ appearing in the definition (3.1) of the metric $\mathrm{d}(A, B)$ can be expressed as

$$m(A, B) \;=\; \min\left\{\frac{\mathrm{tr}[XA]}{\mathrm{tr}[XB]} \;:\; X \in \mathbb{S}_D \text{ and } \mathrm{tr}[XA] \ne 0\right\}\,.$$

Since

$$\frac{||\psi(\phi\cdot A)||}{||\psi(\phi\cdot B)||} = \frac{\mathrm{tr}\psi^*(\mathbb{I})\phi\cdot A}{\mathrm{tr}\psi^*(\mathbb{I})\phi\cdot B} = \frac{\mathrm{tr}\psi^*(\frac{1}{D}\mathbb{I})\phi\cdot A}{\mathrm{tr}\psi^*(\frac{1}{D}\mathbb{I})\phi\cdot B},$$

we see that

$$m(\phi\cdot A,\phi\cdot B) \le \frac{||\psi(\phi\cdot A)||}{||\psi(\phi\cdot B)||} \le \frac{1}{m(\phi\cdot B,\phi\cdot A)}.$$

Since $\phi\cdot A$, $\phi\cdot B$ are positive definite (because ϕ is strictly positive), the various terms appearing in this inequality are all finite and non-zero. Taking logarithms yields

$$\begin{aligned}\big|\ln||\psi(\phi\cdot A)|| - \ln||\psi(\phi\cdot B)||\big| &\le -\ln m(\phi\cdot A,\phi\cdot B) - \ln m(\phi\cdot B,\phi\cdot A)\\ &\le \ln\frac{1+\mathrm{d}(\phi\cdot A,\phi\cdot B)}{1-\mathrm{d}(\phi\cdot A,\phi\cdot B)} \le \ln\frac{1+\mathrm{c}(\phi)}{1-\mathrm{c}(\phi)},\end{aligned}$$

where we have used the definition eq. (3.1) of $\mathrm{d}(\cdot,\cdot)$ and Lemma 3.1 to obtain $\mathrm{d}(\phi\cdot A,\phi\cdot B)\le c(\phi)$. Now for $x\in[0,1)$ we have

$$\frac{1+x}{1-x} \le \frac{1}{(1-x)^2}$$

As $x=\mathrm{c}(\phi)\in(0,1)$ (since ϕ is strictly positive) we have that

$$\big|\ln||\psi(\phi\cdot A)|| - \ln||\psi(\phi\cdot B)||\big| \le 2\ln\frac{1}{1-\mathrm{c}(\phi)}.$$

Now consider the convex function $f(x)=\ln 1/(1-x)$ for $x\in[0,1)$. Since f is convex and $f(0)=0$, we have $f(tr)\le tf(r)$ for any $t,r\in[0,1)$. Hence, $f(\lambda)\le f(r)\lambda/r$ for any $\lambda\in[0,r]$. Thus

$$\big|\ln||\psi(\phi\cdot A)|| - \ln||\psi(\phi\cdot B)||\big| \le c(\phi)\frac{2}{r}\ln\frac{1}{1-r}$$

if $c(\phi)\le r$. ■

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