

CONTINUUM LIMIT FOR THE ABLOWITZ–LADIK SYSTEM

ROWAN KILLIP, ZHIMENG OUYANG, MONICA VISAN, AND LEI WU

ABSTRACT. We show that solutions to the Ablowitz–Ladik system converge to solutions of the cubic nonlinear Schrödinger equation for merely L^2 initial data. Furthermore, we consider initial data for this lattice model that excites Fourier modes near both critical points of the discrete dispersion relation and demonstrate convergence to a *decoupled* system of nonlinear Schrödinger equations.

1. INTRODUCTION

The Ablowitz–Ladik system, introduced in [1], describes the evolution of a field $\alpha : \mathbb{Z} \rightarrow \mathbb{C}$. It comes in two variants, focusing and defocusing. Both may be written as

$$(AL) \quad i\partial_t \alpha_n = -(\alpha_{n-1} - 2\alpha_n + \alpha_{n+1}) + \alpha_n \beta_n (\alpha_{n-1} + \alpha_{n+1})$$

by adopting the expedient that $\beta_n := \bar{\alpha}_n$ in the defocusing case and $\beta_n := -\bar{\alpha}_n$ in the focusing case. (This convention will remain in force throughout the paper.)

Ablowitz and Ladik introduced this model as a discrete form of the one-dimensional cubic Schrödinger equation,

$$(NLS) \quad i\partial_t \psi = -\Delta \psi \pm 2|\psi|^2 \psi,$$

that preserves its complete integrability.

Given this connection, it is quite natural to imagine that solutions of (AL) provide an accurate means of simulating solutions of (NLS), at least for slowly-varying (high-regularity) initial data. Indeed, this intuition is backed up by a number of studies that we will discuss below. The question at the heart of this paper, however, is this: to what extent do solutions to (AL) and (NLS) parallel one another for low-regularity initial data?

We will affirm the intuition laid out above still more strongly than previous authors by showing convergence for merely L^2 initial data. This constitutes a significant expansion of the class of initial data relative to previous investigations such as [8, 14] which require the initial data to lie in H^1 . We achieve this through the introduction of a new method that synthesizes compactness and Strichartz-based techniques.

As a counter-point to our convergence result, we will also demonstrate a certain naiveté in the logic enunciated earlier by studying solutions to (AL) with initial data

$$(1.1) \quad \alpha_n(0) = h\psi_0(hn) + (-1)^n h\phi_0(hn),$$

which combines slowly varying and rapidly oscillating initial data. Here, h is the length scale associated to the continuum approximation; we are studying the $h \rightarrow 0$ limit. Note the prefactor h appearing in (1.1); this ensures the balance between

the dispersion and nonlinear effects. It is solely in this regime that one expects a nonlinear dispersive limit as $h \rightarrow 0$. In this regime, the natural time scale is also stretched — solutions of (NLS) over the time interval $[-T, T]$ correspond to those of (AL) over the longer period $[-h^{-2}T, h^{-2}T]$.

We will show that initial data of the form (1.1) leads to solutions to a system of nonlinear Schrödinger equations:

$$(1.2) \quad i\partial_t \psi = -\Delta \psi \pm 2|\psi|^2 \psi \quad \text{and} \quad -i\partial_t \phi = -\Delta \phi \pm 2|\phi|^2 \phi,$$

in which — and this was a surprise to us — the two frequency components evolve independently! As we will see, this absence of interactions is only observed asymptotically and will be traced to a certain nonresonance in spacetime. In our analysis of initial data of the form (1.1), we will allow data ϕ_0, ψ_0 that is merely L^2 , thereby unifying the two main themes of this paper.

An important phenomenological parallel to our discovery of (1.2) in the continuum limit of (AL) has appeared previously in the link between KdV and the Toda lattice. In [3, 4] it was demonstrated that the Toda lattice must be modeled by a *pair* of KdV equations. We also direct the reader to [18] which considers more general (non-integrable) lattice models.

Complete integrability of the Toda lattice is at the very heart of the methods of [3, 4], which focus on the description of action-angle variables. This is very different from what we shall be doing; indeed, much of our analysis is based directly on Strichartz estimates. The complete integrability of (AL) does play a small role in our arguments, namely, in demonstrating equicontinuity. Nevertheless, it is reasonable to imagine that our approach could be expanded to cover non-integrable lattice approximations to (NLS), obtaining equicontinuity via almost conservation laws and Strichartz estimates. We do not pursue this here because we are specifically interested in the Ablowitz–Ladik system due to its link to spin chain dynamics.

This link was first discovered in [11] and is a discrete analogue of the famous Hasimoto transform linking the continuum Heisenberg model to (NLS). Indeed, our desire to understand the continuum limit of low-regularity solutions to (AL) is fueled by the long-term goal, elaborated in [2], of constructing dynamics for the continuum Heisenberg model in its Gibbs state.

Via the discrete Hasimoto transform, the analysis of the spin model in the Gibbs state is converted to the study of (AL) with $\alpha_n(t)$ being certain specific (non-Gaussian) i.i.d. random variables. Such initial data excites Fourier modes throughout the circle to an equal degree and consequently both terms in (1.1) may be considered equally significant. In [2], the first and third author together with Angelopoulos constructed global dynamics to (AL) for this type of data, and proved invariance of this white noise measure under the (AL) flow. Moreover, the discrete spin chain model was shown to admit global solutions in the Gibbs state and these were shown to preserve the Gibbs measure.

In order to rigorously formulate our results, we need to explain how we pass between functions on the line \mathbb{R} and the lattice \mathbb{Z} . Clearly, the sampling formula (1.1) does not make sense in the L^2 setting we will be studying; indeed, point evaluation is not continuous on $H^s(\mathbb{R})$ unless $s > \frac{1}{2}$. The remedy is to perform a mild smoothing of the continuum initial data before sampling:

$$(1.3) \quad \alpha_n(0) = h[P_{\leq N} \psi_0](hn) + (-1)^n h[P_{\leq N} \phi_0](hn).$$

Here $P_{\leq N}$ denotes the sharp projection to frequencies $|\xi| \leq N$. For concreteness, we will relate h and N by a power law:

$$(1.4) \quad N = h^{-\gamma} \quad \text{with} \quad 0 < \gamma \leq \frac{13}{18}.$$

As $h \rightarrow 0$, so $N \rightarrow \infty$, thus revealing the full irregularity of the initial data; cf. (2.2).

The choice of sharp cutoffs is made for no other reason than expository simplicity. Such multipliers are much less local and are unbounded on ℓ^∞ . Our use of such cutoffs is indicative of the robustness of our arguments and also saves the reader from considering a multitude of bump functions concomitant with traditional Littlewood–Paley theory.

While a signal-processing philosophy would suggest choosing N as the Nyquist frequency (and no higher), dynamical considerations make $N \approx h^{-1}$ ill-suited to the problem at hand. The principal obstruction appears at the linear level: The dispersion relation for the discrete Schrödinger equation is $\omega = 2 - 2 \cos(\theta)$. This is only approximately quadratic near $\theta = 0$ and $\theta = \pi$ (modulo 2π), which are the only regimes where we may expect Schrödinger–like behavior. Near the inflection points $\theta = \pm\pi/2$, the discrete model has very weak dispersion. This in turn leads to a wholly different mKdV-like dynamics with a different characteristic time scale. Initial data with significant excitation near the inflection points will be treated in future work.

As the (AL) dynamics are nonlinear, the Fourier support of a solution is not preserved in time; one fully expects excitations to spread from the initial data to neighborhoods of the inflection points. Nevertheless, we are able to control the extent of such transfer and so demonstrate that the choice (1.4) suffices to suppress these unwanted mKdV dynamics; see Proposition 3.2.

Based on our analysis, which we believe gives an accurate estimate of the transfer of ℓ^2 norm to neighborhoods of the inflection points, we would argue that should one be employing (AL) as a numerical scheme for the simulation of NLS (an idea advocated for in [7], for example), then one should first band-limit the initial data in the manner indicated by (1.4) so as to avoid spurious results generated by the deficiencies of the discrete dispersion relation.

For readers with a particular interest in numerical schemes for NLS (rather than purely (AL)), we note that the inflection-point issue can also be avoided by combining Fourier truncation of the initial data in concert with a different discrete model. Two such approaches are discussed in [9, 10]: One may perform Fourier truncations on the nonlinearity (which preserves the Hamiltonian structure) or one may introduce a mild form of viscosity at the inflection points to suppress such unwanted excitations.

To compare the solution of (AL) with initial data (1.3) to that of (1.2), we must (at each moment of time) generate *two* functions, $\psi^h(t, x)$ and $\phi^h(t, x)$, on the real line from the single lattice function $\alpha_n(h^{-2}t)$. To do this, we simply split the Fourier transform $\widehat{\alpha}$ into two pieces using a sharp cutoff to either semicircle:

$$(1.5) \quad \begin{aligned} \widehat{\psi^h}(t, \xi) &= \widehat{\alpha}(h^{-2}t, h\xi) \chi_{(-\frac{\pi}{2}, \frac{\pi}{2})}(h\xi), \\ \widehat{\phi^h}(t, \xi) &= e^{4ih^{-2}t} \widehat{\alpha}(h^{-2}t, h\xi + \pi) \chi_{(-\frac{\pi}{2}, \frac{\pi}{2})}(h\xi). \end{aligned}$$

$$\begin{array}{ccc}
[\psi_0, \phi_0] & \xrightarrow{(1.3)} & \alpha_n(0) \\
\downarrow (1.2) & & \downarrow (\text{AL}) \\
[\psi, \phi](t) & \xleftarrow{(1.8)} & [\psi^h, \phi^h](t) \xleftarrow{(1.6)} \alpha_n(t)
\end{array}$$

FIGURE 1. A schematic representation of Theorem 1.1.

or equivalently,

$$\begin{aligned}
(1.6) \quad \psi^h(t, x) &= h^{-1} \int_{|\theta| < \frac{\pi}{2}} e^{i \frac{x}{h} \theta} \widehat{\alpha}(h^{-2}t, \theta) \frac{d\theta}{2\pi}, \\
\phi^h(t, x) &= h^{-1} e^{4ih^{-2}t} \int_{|\theta - \pi| < \frac{\pi}{2}} e^{i \frac{x}{h} (\theta - \pi)} \widehat{\alpha}(h^{-2}t, \theta) \frac{d\theta}{2\pi}.
\end{aligned}$$

One small quibble remains before we may present our main result: Do (AL) and (1.2) actually admit global solutions? In the case of (1.2), the answer is unequivocally yes: Tsutsumi [20] showed that these equations are globally well-posed on $L^2(\mathbb{R})$. Local well-posedness is shown by contraction mapping in Strichartz spaces; this is then rendered global using the conservation of the L^2 norm.

Regarding (AL), we see that local well-posedness in $\ell^2(\mathbb{Z})$ is trivial: RHS(AL) is locally Lipschitz! In Section 3 we will show how conservation laws guarantee that such solutions exist globally, at least for

$$(1.7) \quad h \leq h_0 := \min \left\{ 1, \frac{1}{100} [\|\psi_0\|_{L^2}^2 + \|\phi_0\|_{L^2}^2]^{-1} \right\}.$$

As we wish to send $h \rightarrow 0$ this hypothesis is of no real consequence; it exists solely to address a singularity in the natural conservation laws (and also the symplectic structure) in the defocusing case. Indeed, in the defocusing case, it is natural to regard the phase space as comprised only of maps $\alpha : \mathbb{Z} \rightarrow \mathbb{D}$.

The main result of this paper is the following, which we present schematically in Figure 1:

Theorem 1.1. *Fix $\psi_0, \phi_0 \in L^2(\mathbb{R})$ and let $\psi, \phi \in (C_t L_x^2 \cap L_{t, \text{loc}}^6 L_x^6)(\mathbb{R} \times \mathbb{R})$ denote the unique global solutions of (1.2) with this initial data.*

Given $h > 0$ satisfying (1.7), let $\alpha_n(t)$ be the global solution to (AL) with initial data specified by (1.3) and (1.4) and let $\psi^h, \phi^h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ be the corresponding spacetime representatives of this solution built via (1.6). Then as $h \rightarrow 0$,

$$(1.8) \quad \psi^h(t, x) \rightarrow \psi(t, x) \quad \text{and} \quad \phi^h(t, x) \rightarrow \phi(t, x)$$

in $C_t L_x^2([-T, T] \times \mathbb{R})$ for any $T > 0$.

1.1. Outline of the proof. Broadly speaking, our argument is one of compactness/uniqueness: we show that for every sequence of parameters $h \rightarrow 0$ there is a convergent subsequence of discrete solutions (transferred to the line via (1.6)). We then show that all such subsequential limits are identical because they obey the same integral equations and those integral equations have unique solutions.

The integral equations we refer to here are nothing more than the Duhamel formulas for solutions of (1.2); see (6.2) and (6.3). Naturally, these equations contain the cubic nonlinearity. This is a problem for initial data that is merely in L^2 — cubing an L^2 function need not yield a distribution. If we cannot even determine whether or not a $C_t L_x^2$ function is a solution of (NLS), uniqueness becomes hopeless!

The remedy is provided by Strichartz estimates. As shown in [20], solutions with $L^2(\mathbb{R})$ initial data can be constructed via contraction mapping in $L_{t,x}^6([-T, T] \times \mathbb{R})$; these solutions are unique in this class and the cubic nonlinearity makes sense as a spacetime distribution. To exploit this form of uniqueness, we must prove $L_{t,x}^6([-T, T] \times \mathbb{R})$ bounds for the functions ψ^h, ϕ^h .

As mentioned earlier and thoroughly discussed in [8], the presence of inflection points in the discrete dispersion relation presents a major obstacle to controlling the discrete solutions in Strichartz spaces. Our new remedy to this problem is to begin our analysis by controlling the dynamical redistribution of ℓ^2 norm among frequencies; see Proposition 3.2. In Proposition 4.4, we show that the frequency control we achieve is strong enough to recover from the inevitable losses in the discrete Strichartz estimates originating from the inflection points.

Let us turn now to the question of compactness, which is addressed in Section 5; see Theorem 5.1. As we review at the beginning of that section, precompactness rests on three pillars: boundedness, equicontinuity, and tightness. Boundedness is deduced easily from the conservation laws of (AL), specifically, from the analogue of L^2 conservation.

There are actually two equicontinuity requirements because we must allow translations both in space and in time. Equicontinuity in space can also be formulated as tightness of the Fourier transform; see (5.4). In this guise, it is easier to see that it is amenable to attack via conservation laws and indeed, this is how we prove it.

Both equicontinuity in time and tightness rely on the high-frequency control provided by equicontinuity in space. Even under the linear flow, high-frequency wave packets travel very fast, which is antithetical to both tightness and equicontinuity in time. Of the two, tightness is the more delicate to prove because the natural microscopic conservation law does not interact well with frequency cutoffs. This is a peculiarity of (AL) not present in (NLS) or other discrete analogues.

Section 6 is primarily devoted to showing that any subsequential limits of discrete solutions actually solve the Duhamel integral equation. This relies on all of the preceding. Strichartz control is essential to overcome the cubic power in the nonlinearity, for example, while equicontinuity in space is needed to replace the nonlinearity in (AL), which involves values at adjacent lattice points, with a purely on-site nonlinearity. Perhaps the most surprising element of this analysis is the fact that the two frequency components ψ^h and ϕ^h of the solution do not interact. The explanation for this has both algebraic and analytic components. The fact that the (AL) nonlinearity involves adjacent sites produces a key sign change in (6.11) which cancels the naive interactions $|\psi|^2 \phi$ and $|\phi|^2 \psi$. However, two prima facie significant interactions remain. We prove that these drop out of the $h \rightarrow 0$ limit due to a nonresonance phenomenon — they oscillate in time at a frequency far removed from that of the linear solutions; see Lemma 6.4.

At the end of Section 6, we close the paper with a quick review of how these results complete the proof of Theorem 1.1.

Acknowledgements. Rowan Killip is supported by NSF grants DMS-1856755 and DMS-2154022. Monica Visan is supported by NSF grants DMS-1763074 and DMS-2054194. Lei Wu is supported by NSF grant DMS-2104775.

2. PRELIMINARIES

Throughout this paper, C will denote a constant that does not depend on the initial data or on h , and which may vary from one line to another. We write $A \lesssim B$ or $B \gtrsim A$ whenever $A \leq CB$ for some constant $C > 0$. We write $A \simeq B$ whenever $A \lesssim B$ and $B \lesssim A$.

We use $L_t^q L_x^r$ to denote the spacetime norm

$$\|F\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R})} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(t, x)|^r dx \right)^{q/r} dt \right)^{1/q},$$

with the usual modifications when q or r is infinity, or when the domain $\mathbb{R} \times \mathbb{R}$ is replaced by some smaller spacetime region. When $q = r$ we abbreviate $L_t^q L_x^r$ by $L_{t,x}^q$. The discrete version of the spacetime norm $L_t^q L_x^r(\mathbb{R} \times \mathbb{R})$ is $L_t^q \ell_n^r(\mathbb{R} \times \mathbb{Z})$.

The Hilbert–Schmidt norm of a bounded linear operator $\Omega : \ell_n^2(\mathbb{Z}) \rightarrow \ell_n^2(\mathbb{Z})$ is the ℓ^2 norm of its matrix entries:

$$\|\Omega\|_{\mathcal{J}_2}^2 := \sum_{n,m \in \mathbb{Z}} |\Omega_{nm}|^2.$$

By the Hölder inequality, if $\Omega_1, \Omega_2, \dots, \Omega_m$ with $m \geq 2$ are Hilbert–Schmidt operators, then their product is trace class and

$$|\text{tr}\{\Omega_1 \Omega_2 \cdots \Omega_m\}| \leq \prod_{i=1}^m \|\Omega_i\|_{\mathcal{J}_2}.$$

Our convention for the Fourier transform on the line will be

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx \quad \text{so that} \quad f(x) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{ix\xi} \frac{d\xi}{2\pi},$$

while in the discrete case we will use

$$\widehat{f}(\theta) = \sum_{n \in \mathbb{Z}} f(n) e^{-in\theta} \quad \text{so that} \quad f(n) = \int_{-\pi}^{\pi} \widehat{f}(\theta) e^{in\theta} \frac{d\theta}{2\pi}.$$

These definitions of the Fourier transform yield the Plancherel identities

$$\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \frac{d\xi}{2\pi} \quad \text{and} \quad \sum_{n \in \mathbb{Z}} |f(n)|^2 = \int_{-\pi}^{\pi} |\widehat{f}(\theta)|^2 \frac{d\theta}{2\pi}.$$

With these conventions, the Poisson summation formula takes the form

$$\sum_{n \in \mathbb{Z}} h f(nh) e^{-in\theta} = \sum_{m \in \mathbb{Z}} \widehat{f}\left(\frac{\theta+2\pi m}{h}\right) \quad \text{for any} \quad f \in \mathcal{S}(\mathbb{R}).$$

Therefore, if $\text{supp}(\widehat{f}) \subset [-\frac{\pi}{h}, \frac{\pi}{h}]$ and $|\theta| \leq \pi$, then

$$\sum_{n \in \mathbb{Z}} h f(nh) e^{-in\theta} = \widehat{f}\left(\frac{\theta}{h}\right).$$

Consequently, for the initial data (1.3), we have

$$(2.1) \quad \widehat{\alpha}(0, \theta) = \sum_{n \in \mathbb{Z}} \alpha(0, n) e^{-in\theta} = \widehat{P}_{\leq N} \widehat{\psi}_0\left(\frac{\theta}{h}\right) + \widehat{P}_{\leq N} \widehat{\phi}_0\left(\frac{\theta-\pi}{h}\right)$$

for $\theta \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ and $2Nh < \pi$. In particular, recalling that $N = h^{-\gamma}$, we have

$$(2.2) \quad \lim_{h \rightarrow 0} h^{-1} \|\alpha(0)\|_{\ell^2}^2 = \lim_{h \rightarrow 0} \|P_{\leq N} \psi_0\|_{L^2}^2 + \|P_{\leq N} \phi_0\|_{L^2}^2 = \|\psi_0\|_{L^2}^2 + \|\phi_0\|_{L^2}^2.$$

Lemma 2.1. *If $f, g \in L^2(\mathbb{R})$ satisfy $\text{supp}(\widehat{f}) \subseteq [-\frac{\pi}{h}, \frac{\pi}{h}]$ and $\text{supp}(\widehat{g}) \subseteq [-\frac{\pi}{h}, \frac{\pi}{h}]$, then*

$$\int f(x) \overline{g(x)} dx = h \sum_n f(nh) \overline{g(nh)}.$$

Proof. Expanding \widehat{f} and \widehat{g} with respect to the orthonormal basis $\{h^{\frac{1}{2}} e^{-inh\xi} : n \in \mathbb{Z}\}$ of $L^2([-\frac{\pi}{h}, \frac{\pi}{h}]; \frac{d\xi}{2\pi})$, we obtain

$$\begin{aligned} \int f(x) \overline{g(x)} dx &= \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} \frac{d\xi}{2\pi} = \sum_n \int h^{\frac{1}{2}} e^{inh\xi} \widehat{f}(\xi) \frac{d\xi}{2\pi} \overline{\int h^{\frac{1}{2}} e^{inh\eta} \widehat{g}(\eta) \frac{d\eta}{2\pi}} \\ &= h \sum_n f(nh) \overline{g(nh)}. \end{aligned} \quad \square$$

It is not difficult to upgrade the previous argument to show that $f \mapsto \sqrt{h}f(nh)$ is actually a unitary map — a result known as Shannon's sampling theorem. For what follows, we will also need to understand the $L_x^p \rightarrow \ell_n^p$ properties of this mapping (on band limited functions). This was addressed already by Plancherel and Polya in [15]:

Lemma 2.2. *Fix $1 < p < \infty$ and let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a Schwartz function with $\text{supp}(\widehat{f}) \subseteq [-\frac{\pi}{h}, \frac{\pi}{h}]$. Then*

$$(2.3) \quad h \sum_{n \in \mathbb{Z}} |f(nh)|^p \simeq \int_{\mathbb{R}} |f(x)|^p dx.$$

It is well-known that operators defined by sharp (as opposed to smooth) Fourier cutoffs are L^p -bounded for $1 < p < \infty$; indeed, this follows immediately from the well-known L^p -boundedness of the Hilbert transform,

$$\widehat{Hf}(\xi) = -i \text{sign}(\xi) \widehat{f}(\xi) \quad \text{or equivalently, } [Hf](x) = \text{P.V.} \int \frac{f(y)}{x-y} \frac{dy}{\pi},$$

which was proved by M. Riesz [16]. In Section 23 of this very same paper, Riesz considered the discrete Hilbert transform and showed that it is ℓ^p -bounded for all $1 < p < \infty$. It follows that for any arc $[a, b]$ on the circle, the mapping $\widehat{\alpha}(\theta) \mapsto \chi_{[a,b]}(\theta) \widehat{\alpha}(\theta)$ is bounded on $\ell^p(\mathbb{Z})$.

Our next two results concern the linear operator arising in (1.6), which transfers solutions of (AL) to functions on \mathbb{R} . Specifically, we mean the mapping

$$(2.4) \quad [\mathcal{R}c](x) := h^{-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\frac{x}{h}\theta} \widehat{c}(\theta) \frac{d\theta}{2\pi} = \text{P.V.} \sum_n \frac{\sin(\frac{\pi}{2h}[x-nh])}{\pi(x-nh)} c_n$$

which we will show maps ℓ_n^2 to L_x^2 . Note that (1.6) can be rewritten as

$$\psi^h(t) = \mathcal{R}[\alpha_n(h^{-2}t)] \quad \text{and} \quad \phi^h(t) = e^{4ih^{-2}t} \mathcal{R}[(-1)^n \alpha_n(h^{-2}t)].$$

Lemma 2.3. *The operator \mathcal{R} is a bounded operator from $\ell_n^2(\mathbb{Z})$ to $L_x^2(\mathbb{R})$ with norm*

$$\|\mathcal{R}\|_{\ell_n^2(\mathbb{Z}) \rightarrow L_x^2(\mathbb{R})} \lesssim h^{-\frac{1}{2}}.$$

Proof. The result is an immediate consequence of Lemma 2.1. Indeed,

$$\|\mathcal{R}c\|_{L_x^2} = h^{\frac{1}{2}} \|[\mathcal{R}c](nh)\|_{\ell_n^2} = h^{-\frac{1}{2}} \|\widehat{c}\|_{L^2([- \frac{\pi}{2}, \frac{\pi}{2}])} \leq h^{-\frac{1}{2}} \|c\|_{\ell_n^2}. \quad \square$$

Lemma 2.4. *Let $c \in \ell_n^2$ be a sequence supported in the interval $[-h^{-1}L, h^{-1}L]$. Then for any $L' > 0$ we have*

$$(2.5) \quad \int_{|x| > L+L'} |\mathcal{R}c(x)|^2 dx \lesssim \frac{L}{hL'} \|c_n\|_{\ell_n^2}^2.$$

Proof. The result follows immediately from two simple observations:

$$\int_{|x-nh| > L'} \left| \frac{2 \sin(\frac{\pi}{2h}[x-nh])}{x-nh} \right|^2 dx \lesssim \frac{1}{L'} \quad \text{and} \quad \|c\|_{\ell_n^1} \lesssim h^{-\frac{1}{2}} L^{\frac{1}{2}} \|c\|_{\ell_n^2}. \quad \square$$

3. CONSERVATION LAWS

As a completely integrable system, (AL) enjoys infinitely many conservation laws. Of these, we mention just three:

$$\begin{aligned} M(\alpha) &= - \sum_{n \in \mathbb{Z}} \ln(1 - \alpha_n \beta_n), \\ H(\alpha) &= - \sum_{n \in \mathbb{Z}} \alpha_n \beta_{n+1} + \alpha_{n+1} \beta_n + 2 \ln(1 - \alpha_n \beta_n), \\ H_2(\alpha) &= - \sum_n 2 \ln(1 - \alpha_n \beta_n) + 2 \operatorname{Re} \left[\alpha_{n+2} \beta_n - \frac{1}{2} \alpha_n^2 \beta_{n-1}^2 - \alpha_{n+1} \alpha_n \beta_n \beta_{n-1} \right]. \end{aligned}$$

Note that H also serves as the Hamiltonian for (AL) with respect to the Poisson structure

$$\{F, G\} := \frac{1}{i} \sum_{n \in \mathbb{Z}} (1 - \alpha_n \beta_n) \left(\frac{\partial F}{\partial \alpha_n} \frac{\partial G}{\partial \beta_n} - \frac{\partial F}{\partial \beta_n} \frac{\partial G}{\partial \alpha_n} \right).$$

Proposition 3.1. *Under the assumption (1.7), the evolution (AL) with initial data (1.3) has a global solution in $C_t \ell_n^2$. Moreover,*

$$(3.1) \quad |M(\alpha(t))| \simeq \|\alpha(t)\|_{\ell_n^2}^2 \lesssim h [\|\psi_0\|_{L^2}^2 + \|\phi_0\|_{L^2}^2] \quad \text{uniformly for } t \in \mathbb{R}.$$

Proof. Local well-posedness in ℓ^2 is guaranteed by Picard's theorem. Under the assumption (1.7), the quantity $M(\alpha(t))$ is initially well-defined; moreover, it is conserved by these local solutions. We will show that such solutions can be extended globally in time by demonstrating the equivalence stated in (3.1). The inequality in (3.1) then follows from Lemma 2.2.

From the power series expansion of \ln , we have

$$(3.2) \quad \left| |M(\alpha(t))| - \|\alpha(t)\|_{\ell_n^2}^2 \right| \leq \sum_{\ell=2}^{\infty} \frac{1}{\ell} \|\alpha(t)\|_{\ell_n^2}^{2\ell}.$$

On any time interval where

$$(3.3) \quad \|\alpha(t)\|_{\ell_n^2}^2 \leq \frac{1}{20},$$

the series on the right-hand side of (3.2) converges and yields

$$\left| |M(\alpha(t))| - \|\alpha(t)\|_{\ell_n^2}^2 \right| \leq \frac{1}{20} \|\alpha(t)\|_{\ell_n^2}^2$$

and consequently,

$$(3.4) \quad \frac{1}{2} |M(\alpha(t))| \leq \|\alpha(t)\|_{\ell_n^2}^2 \leq 2 |M(\alpha(t))|.$$

By Lemma 2.2, (1.7) guarantees

$$\|\alpha_0\|_{\ell_n^2}^2 \leq \frac{1}{100}.$$

As trajectories are ℓ_n^2 -continuous in time, (3.4) and the conservation of mass show that (3.3), and so also (3.4), hold for all time. \square

It is elementary to verify that the solutions constructed in Proposition 3.1 also conserve H and H_2 , which are readily seen to be continuous functionals on ℓ_n^2 . Our next result demonstrates suppression of the influence of the inflection points $\pm\frac{\pi}{2}$ on the solution to (AL) for all times.

Proposition 3.2. *Let α be the solution to (AL) with initial data (1.3) and h satisfying (1.7). For $0 < \delta < 1$, let P_δ be the sharp Fourier cutoff defined via*

$$(3.5) \quad \widehat{P_\delta f}(\theta) = \chi_{\mathcal{G}_\delta}(\theta) \widehat{f}(\theta) \quad \text{with} \quad \mathcal{G}_\delta = \{\theta \in \mathbb{R}/2\pi\mathbb{Z} : \sin^2(\theta) < \delta^2\}.$$

Then

$$(3.6) \quad \sup_t \| [1 - P_\delta] \alpha(t) \|_{\ell_n^2} \lesssim \delta^{-1} (hN + h^{\frac{1}{2}}) \|\alpha(0)\|_{\ell_n^2}.$$

Proof. A straightforward computation reveals that the quadratic part of the conserved quantity $H_2(\alpha)$ is given by

$$(3.7) \quad H_2^{[2]}(\alpha(t)) = \pm \int_{-\pi}^{\pi} 4 \sin^2(\theta) |\widehat{\alpha}(t, \theta)|^2 \frac{d\theta}{2\pi},$$

while the higher order terms can be estimated using Proposition 3.1 by

$$\left| H_2(\alpha(t)) - H_2^{[2]}(\alpha(t)) \right| \lesssim \|\alpha(t)\|_{\ell_n^2}^4 + \sum_{\ell \geq 2} \|\alpha(t)\|_{\ell_n^2}^{2\ell} \lesssim \|\alpha(t)\|_{\ell_n^2}^4 \lesssim h \|\alpha(0)\|_{\ell_n^2}^2.$$

Using the conservation of H_2 and recalling (2.1), we therefore deduce that

$$\begin{aligned} \int_{-\pi}^{\pi} 4 \sin^2(\theta) |\widehat{\alpha}(t, \theta)|^2 \frac{d\theta}{2\pi} &\lesssim \int_{-\pi}^{\pi} 4 \sin^2(\theta) |\widehat{\alpha}(0, \theta)|^2 \frac{d\theta}{2\pi} + h \|\alpha(0)\|_{\ell_n^2}^2 \\ &\lesssim [(hN)^2 + h] \|\alpha(0)\|_{\ell_n^2}^2. \end{aligned}$$

This completes the proof of the proposition. \square

A very similar argument using only the Hamiltonian H (rather than H_2) may be employed in the case that $\phi_0 \equiv 0$. In contrast to the symbol of the quadratic part of H_2 given in (3.7), the symbol of $H^{[2]}$ is $4 \sin^2(\theta/2)$ and so vanishes only at $\theta = 0 \pmod{2\pi}$.

A convenient way of understanding the whole family of conservation laws is through their generating function, which we will discuss next, following the paradigm set forth in [6].

For $z \in \mathbb{C}$ with $|z| > 1$, we define

$$(3.8) \quad \mathbf{A}(z; \alpha) := \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} \text{tr} \{ (\Lambda \Gamma)^\ell \}$$

where

$$\Lambda(z; \alpha) := \alpha(S - z^{-1})^{-1}, \quad \Gamma(z; \alpha) := \beta(z - S)^{-1},$$

and S denotes the shift operator on $\ell^2(\mathbb{Z})$ given by $(S[f])_n = f_{n+1}$.

In view of these definitions of Λ and Γ , we see that (3.8) has the form of a power series in α and $\bar{\alpha}$. The convergence of this series will be addressed in Proposition 3.3. In understanding what the conservation of \mathbf{A} expresses about solutions to (AL), it is useful to compute the quadratic term $\mathbf{A}^{[2]}$ exactly:

$$\mathbf{A}^{[2]}(z; \alpha) = \text{tr}\{\Lambda\Gamma\} = \pm \int_{-\pi}^{\pi} \frac{|\widehat{\alpha}(\theta)|^2}{z^2 e^{-i\theta} - 1} \frac{d\theta}{2\pi}.$$

Although this quantity is not coercive, one may remedy this by taking linear combinations with M . The particular linear combination that will be useful to us is

$$(3.9) \quad \mathbf{G}(\kappa h; \alpha) = \pm \frac{2}{e^{4\kappa h} + 1} M(\alpha) \mp \tanh(2\kappa h) \text{Re} \left[\mathbf{A}(e^{\kappa h}; \alpha) + \mathbf{A}(ie^{\kappa h}; \alpha) \right],$$

which has quadratic part

$$(3.10) \quad \mathbf{G}^{[2]}(\kappa h; \alpha) = \int_{-\pi}^{\pi} \frac{\sin^2(\theta) |\widehat{\alpha}(\theta)|^2}{\sinh^2(2\kappa h) + \sin^2(\theta)} \frac{d\theta}{2\pi}.$$

Proposition 3.3. *Let α be the solution to (AL) with initial data (1.3) and h satisfying (1.7). Then there exists $\kappa_0 > 0$ depending only on $\|\psi_0\|_{L_x^2}$ and $\|\phi_0\|_{L_x^2}$ so that for all $t \in \mathbb{R}$ and all $\kappa \geq \kappa_0$, the series defining $\mathbf{G}(\kappa h; \alpha(t))$ converges and is independent of t . Moreover,*

$$(3.11) \quad \left| \mathbf{G}(\kappa h; \alpha(t)) - \mathbf{G}^{[2]}(\kappa h; \alpha(t)) \right| \lesssim \sinh^{-1}(\kappa h) \|\alpha(0)\|_{\ell_n^2}^4,$$

with an implicit constant independent of h and t .

Proof. It was observed in [6, Lemma 5.2] that Λ and Γ are Hilbert–Schmidt operators with

$$(3.12) \quad \|\Lambda(z; \alpha)\|_{\mathfrak{I}_2}^2 = \frac{|z|^2}{|z|^2 - 1} \|\alpha\|_{\ell_n^2}^2 \quad \text{and} \quad \|\Gamma(z; \alpha)\|_{\mathfrak{I}_2}^2 = \frac{1}{|z|^2 - 1} \|\alpha\|_{\ell_n^2}^2.$$

Moreover, by [6, Theorem 5.1], $\mathbf{A}(z; \alpha)$ converges and is conserved under the (AL) flow provided that

$$\frac{|z|}{|z|^2 - 1} \|\alpha\|_{\ell_n^2}^2 < 1.$$

Combining this (3.2), (3.8), and Proposition 3.1, we may estimate

$$\begin{aligned} \text{LHS}(3.11) &\leq \tanh(2\kappa h) \sum_{\ell \geq 2} \frac{1}{\ell} \|\Lambda(e^{\kappa h}; \alpha(t))\|_{\mathfrak{I}_2}^\ell \|\Gamma(e^{\kappa h}; \alpha(t))\|_{\mathfrak{I}_2}^\ell \\ &\quad + \tanh(2\kappa h) \sum_{\ell \geq 2} \frac{1}{\ell} \|\Lambda(ie^{\kappa h}; \alpha(t))\|_{\mathfrak{I}_2}^\ell \|\Gamma(ie^{\kappa h}; \alpha(t))\|_{\mathfrak{I}_2}^\ell \\ &\quad + \frac{2}{e^{4\kappa h} + 1} \sum_{\ell \geq 2} \frac{1}{\ell} \|\alpha(t)\|_{\ell_n^2}^{2\ell} \\ &\leq \tanh(2\kappa h) \sum_{\ell \geq 2} \left(\frac{e^{\kappa h}}{e^{2\kappa h} - 1} \|\alpha(t)\|_{\ell_n^2}^2 \right)^\ell + \frac{1}{\cosh(2\kappa h)} \|\alpha(t)\|_{\ell_n^2}^4 \lesssim \text{RHS}(3.11), \end{aligned}$$

provided $\kappa \geq \kappa_0$ for some κ_0 depending only on the L_x^2 norms of ϕ_0, ψ_0 . \square

4. STRICHARTZ ESTIMATES

Our main goal in this section is to prove certain Strichartz estimates for α , ψ^h , and ϕ^h that are uniform in the parameter h . We start by recording the Strichartz estimates for the continuum and discrete Schrödinger propagators.

Proposition 4.1 (Strichartz estimates for $e^{it\Delta}$; [5]). *Let (q, r) and (\tilde{q}, \tilde{r}) be two pairs satisfying*

$$q, r, \tilde{q}, \tilde{r} \geq 2 \quad \text{and} \quad \frac{2}{q} + \frac{1}{r} = \frac{1}{2} = \frac{2}{\tilde{q}} + \frac{1}{\tilde{r}}.$$

If ψ is a solution to the Schrödinger equation

$$i\partial_t \psi = -\Delta \psi + F$$

with initial data $\psi_0 \in L_x^2(\mathbb{R})$, then

$$\|\psi\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R})} \lesssim \|\psi_0\|_{L_x^2(\mathbb{R})} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R} \times \mathbb{R})}.$$

As shown in [19, Theorem 3], Strichartz estimates for the discrete Schrödinger propagator $e^{it\Delta_d}$, where Δ_d denotes the discrete Laplacian

$$[\Delta_d \alpha]_n = \alpha_{n-1} - 2\alpha_n + \alpha_{n+1},$$

can be derived via the same techniques used in the continuum case, [12]. As the discrete dispersion relation has inflection points (unlike in the continuum case), the estimates are more closely related to those familiar from the Airy propagator:

Proposition 4.2 (Strichartz estimates for $e^{it\Delta_d}$; [19]). *Let (q, r) and (\tilde{q}, \tilde{r}) be two pairs satisfying*

$$q, r, \tilde{q}, \tilde{r} \geq 2, \quad \frac{1}{q} + \frac{1}{3r} \leq \frac{1}{6}, \quad \text{and} \quad \frac{1}{\tilde{q}} + \frac{1}{3\tilde{r}} \leq \frac{1}{6}.$$

If α is a solution to the discrete Schrödinger equation

$$i\partial_t \alpha_n = -(\Delta_d \alpha)_n + F_n$$

with initial data $\alpha_0 \in \ell_n^2(\mathbb{Z})$, then

$$\|\alpha\|_{L_t^q \ell_n^r(\mathbb{R} \times \mathbb{Z})} \lesssim \|\alpha_0\|_{\ell_n^2(\mathbb{Z})} + \|F\|_{L_t^{\tilde{q}'} \ell_n^{\tilde{r}'}(\mathbb{R} \times \mathbb{Z})}.$$

However, if we project to frequencies away from the inflection points, we recover the same dispersive decay we find in the continuum. Consequently, we have

Proposition 4.3 (Frequency-localized Strichartz estimates for $e^{it\Delta_d}$). *Let (q, r) and (\tilde{q}, \tilde{r}) be two pairs satisfying*

$$q, r, \tilde{q}, \tilde{r} \geq 2 \quad \text{and} \quad \frac{2}{q} + \frac{1}{r} = \frac{1}{2} = \frac{2}{\tilde{q}} + \frac{1}{\tilde{r}}.$$

The solution to the discrete Schrödinger equation

$$i\partial_t \alpha_n = -(\Delta_d \alpha)_n + F_n$$

with initial data $\alpha_0 \in \ell_n^2(\mathbb{Z})$ satisfies

$$\|P\alpha\|_{L_t^q \ell_n^r(\mathbb{R} \times \mathbb{Z})} \lesssim \|P\alpha_0\|_{\ell_n^2(\mathbb{Z})} + \|PF\|_{L_t^{\tilde{q}'} \ell_n^{\tilde{r}'}(\mathbb{R} \times \mathbb{Z})},$$

where $P = P_{\frac{3}{4}}$ is the projection operator defined in Proposition 3.2.

Our main result in this section is the following:

Proposition 4.4. *Let α be the solution to (AL) with initial data (1.3) and h satisfying (1.7). Then for any $T > 0$ we have*

$$(4.1) \quad \|\alpha\|_{L_t^6 \ell_n^6([-h^{-2}T, h^{-2}T] \times \mathbb{Z})} + \|\alpha\|_{L_t^4 \ell_n^\infty([-h^{-2}T, h^{-2}T] \times \mathbb{Z})} \lesssim_T \|\alpha_0\|_{\ell_n^2},$$

where the implicit constant is independent of h . Consequently,

$$\|\psi^h\|_{L_{t,x}^6([-T, T] \times \mathbb{R})} + \|\phi^h\|_{L_{t,x}^6([-T, T] \times \mathbb{R})} \lesssim_T 1 \quad \text{uniformly in } h,$$

where ψ^h and ϕ^h are as defined in (1.5).

Proof. First note that by Hölder's inequality and Proposition 3.1,

$$\begin{aligned} \|\alpha\|_{L_t^6 \ell_n^6([-h^{-2}T, h^{-2}T] \times \mathbb{Z})} + \|\alpha\|_{L_t^4 \ell_n^\infty([-h^{-2}T, h^{-2}T] \times \mathbb{Z})} \\ \lesssim (h^{-2}T)^{\frac{1}{6}} \|\alpha\|_{L_t^\infty \ell_n^2(\mathbb{R} \times \mathbb{Z})} + (h^{-2}T)^{\frac{1}{4}} \|\alpha\|_{L_t^\infty \ell_n^2(\mathbb{R} \times \mathbb{Z})} \\ (4.2) \quad \lesssim [(h^{-2}T)^{\frac{1}{6}} + (h^{-2}T)^{\frac{1}{4}}] \|\alpha(0)\|_{\ell_n^2}. \end{aligned}$$

In order to eliminate the dependence on h in the inequality above, we will run a bootstrap argument combined with Strichartz estimates.

Let $P = P_{\frac{1}{100}}$ and $\tilde{P} = P_{\frac{3}{4}}$ denote the sharp Fourier cutoffs introduced in (3.5). Writing $F(\alpha)$ for the nonlinearity in (AL) and exploiting that $\tilde{P}[F(P\alpha)] = F(P\alpha)$, we obtain the following Duhamel representation of the solution:

$$\begin{aligned} \alpha(t) &= e^{it\Delta_d} \alpha(0) - i \int_0^t e^{i(t-\tau)\Delta_d} F(\alpha(\tau)) d\tau \\ &= e^{it\Delta_d} \tilde{P}\alpha(0) + e^{it\Delta_d} (1 - \tilde{P})\alpha(0) - i \int_0^t \tilde{P} e^{i(t-\tau)\Delta_d} F(P\alpha(\tau)) d\tau \\ &\quad - i \int_0^t e^{i(t-\tau)\Delta_d} [F(\alpha(\tau)) - F(P\alpha(\tau))] d\tau. \end{aligned}$$

With a view to closing a bootstrap argument, we will estimate these terms on the time interval $[-h^{-2}T_0, h^{-2}T_0]$ with $T_0 \leq T$, which may later be chosen small. (Recall that T was arbitrarily large.) For two of the terms, the presence of \tilde{P} allows us to employ Proposition 4.3 and so treat both spacetime norms of interest simultaneously. First,

$$(4.3) \quad \|e^{it\Delta_d} \tilde{P}\alpha(0)\|_{L_t^6 \ell_n^6 \cap L_t^4 \ell_n^\infty} \lesssim \|\tilde{P}\alpha(0)\|_{\ell_n^2} \lesssim \|\alpha(0)\|_{\ell_n^2}.$$

Secondly, using Proposition 3.1, we have

$$\begin{aligned} \left\| \int_0^t \tilde{P} e^{i(t-\tau)\Delta_d} F(P\alpha(\tau)) d\tau \right\|_{L_t^6 \ell_n^6 \cap L_t^4 \ell_n^\infty} &\lesssim \|F(P\alpha)\|_{L_t^{\frac{6}{5}} \ell_n^{\frac{6}{5}}} \\ &\lesssim (h^{-2}T_0)^{\frac{1}{2}} \|\alpha\|_{L_t^\infty \ell_n^2} \|\alpha\|_{L_t^6 \ell_n^6}^2 \\ (4.4) \quad &\lesssim (h^{-2}T_0)^{\frac{1}{2}} \|\alpha(0)\|_{\ell_n^2} \|\alpha\|_{L_t^6 \ell_n^6}^2. \end{aligned}$$

Turning to the remaining terms in the Duhamel expansion, we first narrow our focus to just the $L_t^6 \ell_n^6$ norm. By Propositions 4.2 and 3.2,

$$\begin{aligned} \|e^{it\Delta_d} (1 - \tilde{P})\alpha(0)\|_{L_t^6 \ell_n^6} &\lesssim (h^{-2}T_0)^{\frac{1}{18}} \|e^{it\Delta_d} (1 - \tilde{P})\alpha(0)\|_{L_t^9 \ell_n^6} \\ &\lesssim (h^{-2}T_0)^{\frac{1}{18}} \|(1 - \tilde{P})\alpha(0)\|_{\ell_n^2} \\ (4.5) \quad &\lesssim (h^{-2}T_0)^{\frac{1}{18}} (Nh + h^{\frac{1}{2}}) \|\alpha(0)\|_{\ell_n^2}. \end{aligned}$$

Proceeding in a parallel fashion, we find

$$\begin{aligned}
& \left\| \int_0^t e^{i(t-\tau)\Delta_d} [F(\alpha(\tau)) - F(P\alpha(\tau))] d\tau \right\|_{L_t^6 \ell_n^6} \\
& \lesssim (h^{-2} T_0)^{\frac{1}{18}} \left\| \int_0^t e^{i(t-\tau)\Delta_d} [F(\alpha(\tau)) - F(P\alpha(\tau))] d\tau \right\|_{L_t^9 \ell_n^6} \\
& \lesssim (h^{-2} T_0)^{\frac{1}{18}} \|F(\alpha) - F(P\alpha)\|_{L_t^{\frac{9}{8}} \ell_n^{\frac{6}{5}}} \\
& \lesssim (h^{-2} T_0)^{\frac{11}{18}} \|(I - P)\alpha\|_{L_t^\infty \ell_n^2} \|\alpha\|_{L_t^6 \ell_n^6}^2 \\
& \lesssim (h^{-2} T_0)^{\frac{11}{18}} (Nh + h^{\frac{1}{2}}) \|\alpha(0)\|_{\ell_n^2} \|\alpha\|_{L_t^6 \ell_n^6}^2 \\
(4.6) \quad & \lesssim (h^{-2} T_0)^{\frac{11}{18}} (Nh + h^{\frac{1}{2}}) h^{\frac{1}{2}} [\|\psi_0\|_{L^2} + \|\phi_0\|_{L^2}] \|\alpha\|_{L_t^6 \ell_n^6}^2.
\end{aligned}$$

The last step here was an application of (3.1).

Combining (4.3) through (4.6), recalling that $N \leq h^{-\frac{7}{9}}$, and using Proposition 3.1, we deduce that

$$\begin{aligned}
& \|\alpha\|_{L_t^6 \ell_n^6([-h^{-2} T_0, h^{-2} T_0] \times \mathbb{Z})} \\
& \lesssim (1 + T_0^{\frac{1}{18}}) \|\alpha(0)\|_{\ell_n^2} + h^{-\frac{1}{2}} (T_0^{\frac{1}{2}} + T_0^{\frac{11}{18}}) \|\alpha\|_{L_t^6 \ell_n^6([-h^{-2} T_0, h^{-2} T_0] \times \mathbb{Z})}^2,
\end{aligned}$$

where the implicit constant depends only on ψ_0 and ϕ_0 . Taking T_0 sufficiently small, a bootstrap argument yields

$$\|\alpha\|_{L_t^6 \ell_n^6([-h^{-2} T_0, h^{-2} T_0] \times \mathbb{Z})} \lesssim \|\alpha(0)\|_{\ell_n^2}.$$

Note that (4.2) guarantees that the quantity being bootstrapped is initially finite.

Iterating this argument $(1 + \frac{T}{T_0})$ many times, we conclude that

$$(4.7) \quad \|\alpha\|_{L_t^6 \ell_n^6([-h^{-2} T, h^{-2} T] \times \mathbb{Z})} \lesssim_T \|\alpha(0)\|_{\ell_n^2}.$$

In particular, in view of Lemma 2.2 and Proposition 3.1, this yields

$$\|\psi^h\|_{L_{t,x}^6([-T, T] \times \mathbb{R})} + \|\phi^h\|_{L_{t,x}^6([-T, T] \times \mathbb{R})} \lesssim h^{-\frac{1}{2}} \|\alpha\|_{L_t^6 \ell_n^6([-h^{-2} T, h^{-2} T] \times \mathbb{Z})} \lesssim_T 1.$$

We turn now to the $L_t^4 L_x^\infty$ norm, which we will treat using (4.7) and so not need to argue via bootstrap. Henceforth, all norms will be taken over the full time interval $[-h^{-2} T, h^{-2} T]$.

Mimicking (4.5), we find

$$\begin{aligned}
& \left\| e^{it\Delta_d} (1 - \tilde{P}) \alpha(0) \right\|_{L_t^4 \ell_n^\infty} \lesssim (h^{-2} T)^{\frac{1}{12}} \left\| e^{it\Delta_d} (1 - \tilde{P}) \alpha(0) \right\|_{L_t^6 \ell_n^\infty} \\
& \lesssim (h^{-2} T)^{\frac{1}{12}} \|(1 - \tilde{P}) \alpha(0)\|_{\ell_n^2} \\
(4.8) \quad & \lesssim (h^{-2} T)^{\frac{1}{12}} (Nh + h^{\frac{1}{2}}) \|\alpha(0)\|_{\ell_n^2}.
\end{aligned}$$

Likewise, paralleling (4.6), we find

$$\begin{aligned}
& \left\| \int_0^t e^{i(t-\tau)\Delta_d} [F(\alpha(\tau)) - F(P\alpha(\tau))] d\tau \right\|_{L_t^4 \ell_n^\infty} \\
& \lesssim (h^{-2} T_0)^{\frac{1}{12}} \left\| \int_0^t e^{i(t-\tau)\Delta_d} [F(\alpha(\tau)) - F(P\alpha(\tau))] d\tau \right\|_{L_t^6 \ell_n^\infty} \\
& \lesssim (h^{-2} T_0)^{\frac{1}{12}} \|F(\alpha) - F(P\alpha)\|_{L_t^{\frac{9}{8}} \ell_n^{\frac{6}{5}}}
\end{aligned}$$

$$(4.9) \quad \lesssim (h^{-2}T_0)^{\frac{23}{36}} (Nh + h^{\frac{1}{2}}) h^{\frac{1}{2}} [\|\psi_0\|_{L^2} + \|\phi_0\|_{L^2}] \|\alpha\|_{L_t^6 \ell_n^6}^2.$$

Combining (4.3), (4.4), (4.8), (4.9), (4.7), and the fact that $N \lesssim h^{-\frac{13}{18}}$ yields

$$\|\alpha\|_{L_t^4 \ell_n^\infty([-h^{-2}T, h^{-2}T] \times \mathbb{Z})} \lesssim_T \|\alpha(0)\|_{\ell_n^2}$$

and so completes the proof of the proposition. \square

5. PRECOMPACTNESS IN $C_t L_x^2$

This section is dedicated to the proof of the following precompactness result:

Theorem 5.1. *Let α be the solution to (AL) with initial data (1.3) and h satisfying (1.7) and let ψ^h and ϕ^h be as in (1.5). For $T > 0$ fixed, the two families of functions $\Psi = \{\psi^h(t, x) : 0 < h \leq h_0\}$ and $\Phi = \{\phi^h(t, x) : 0 < h \leq h_0\}$ are precompact in $C([-T, T]; L_x^2(\mathbb{R}))$.*

By a generalization of the Arzelà–Ascoli theorem due to M. Riesz [17], precompactness of the two families will follow once we establish uniform boundedness, equicontinuity, and tightness properties. Specifically, we will demonstrate:

Uniform boundedness: there exists $C > 0$ such that

$$(5.1) \quad \sup_{0 < h \leq h_0} \left[\|\psi^h(t)\|_{L_t^\infty L_x^2([-T, T] \times \mathbb{R})} + \|\phi^h(t)\|_{L_t^\infty L_x^2([-T, T] \times \mathbb{R})} \right] \leq C.$$

Equicontinuity: for any $\varepsilon > 0$, there exists $\delta > 0$ so that whenever $|s| + |y| < \delta$,

$$(5.2) \quad \|\psi^h(t+s, x+y) - \psi^h(t, x)\|_{L_x^2} + \|\phi^h(t+s, x+y) - \phi^h(t, x)\|_{L_x^2} < \varepsilon$$

uniformly for $t \in [-T, T]$ (with $t+s \in [-T, T]$) and for $0 < h \leq h_0$.

Tightness: for any $\varepsilon > 0$, there exists $R > 0$ such that

$$(5.3) \quad \sup_{0 < h \leq h_0} \sup_{|t| \leq T} \int_{|x| \geq R} |\psi^h(t, x)|^2 + |\phi^h(t, x)|^2 dx < \varepsilon.$$

As $h \mapsto (\psi^h, \phi^h)$ defines a continuous mapping from $(0, h_0]$ to $C([-T, T]; L_x^2(\mathbb{R}))$, these three conditions automatically hold on any interval of the form $[h_1, h_0]$. Correspondingly, it suffices to prove (5.2) and (5.3) only for $0 < h \leq h_1$ where h_1 may depend on ε .

Combining Lemma 2.2 and Proposition 3.1, we find

$$\begin{aligned} & \|\psi^h(t)\|_{L_t^\infty L_x^2([-T, T] \times \mathbb{R})} + \|\phi^h(t)\|_{L_t^\infty L_x^2([-T, T] \times \mathbb{R})} \\ & \lesssim h^{-\frac{1}{2}} \|\alpha\|_{L_t^\infty \ell_n^2([-h^{-2}T, h^{-2}T] \times \mathbb{Z})} \lesssim 1, \end{aligned}$$

uniformly for $0 < h \leq h_0$, which settles (5.1).

We turn now to the equicontinuity property, starting with equicontinuity in the spatial variable. By the uniform boundedness property (5.1) and Plancherel, this is equivalent to tightness on the Fourier side, that is, for any $\varepsilon > 0$ there exists $\kappa > 0$ such that

$$(5.4) \quad \sup_{\substack{|\xi| \geq \kappa \\ |t| \leq T}} \|P_{|\xi| \geq \kappa} \psi^h(t)\|_{L_x^2} + \|P_{|\xi| \geq \kappa} \phi^h(t)\|_{L_x^2} < \varepsilon$$

uniformly for $0 < h \leq h_0$.

A straightforward computation using (1.5) shows that for $\kappa h < \frac{\pi}{2}$ we have

$$\|P_{|\xi| \geq \kappa} \psi^h(t)\|_{L_x^2}^2 + \|P_{|\xi| \geq \kappa} \phi^h(t)\|_{L_x^2}^2 = \int_{\kappa h \leq |h\xi| \leq \pi - \kappa h} |\widehat{\alpha}(h^{-2}t, h\xi)|^2 \frac{d\xi}{2\pi}$$

$$\begin{aligned} &\leq \frac{C}{h} \int_{\kappa h \leq |\theta| \leq \pi - \kappa h} \frac{\sin^2(\theta) |\widehat{\alpha}(h^{-2}t, \theta)|^2}{\sinh^2(2\kappa h) + \sin^2(\theta)} \frac{d\theta}{2\pi} \\ &\leq \frac{C}{h} \mathbf{G}^{[2]}(\kappa h; \alpha(h^{-2}t)) \end{aligned}$$

for a universal constant $C > 0$ and $\mathbf{G}^{[2]}$ as defined in (3.10). Using Proposition 3.3 followed by (2.1) and Proposition 3.1, we may thus estimate

$$\begin{aligned} &\sup_{|t| \leq T} \left[\|P_{|\xi| \geq \kappa} \psi^h(t)\|_{L_x^2}^2 + \|P_{|\xi| \geq \kappa} \phi^h(t)\|_{L_x^2}^2 \right] \\ &\lesssim h^{-1} [\mathbf{G}^{[2]}(\kappa h, \alpha(0)) + \sinh^{-1}(\kappa h) \|\alpha(0)\|_{\ell_n^2}^4] \\ &\lesssim \int_{-N}^N \frac{\sin^2(h\xi)}{\sinh^2(2\kappa h) + \sin^2(h\xi)} [|\widehat{\psi}_0(\xi)|^2 + |\widehat{\phi}_0(\xi)|^2] d\xi + \kappa^{-1} \\ &\lesssim \|P_{|\xi| \geq \sqrt{\kappa}} \psi_0\|_{L_x^2}^2 + \|P_{|\xi| \geq \sqrt{\kappa}} \phi_0\|_{L_x^2}^2 + \kappa^{-1} [\|\psi_0\|_{L_x^2}^2 + \|\phi_0\|_{L_x^2}^2] + \kappa^{-1}. \end{aligned}$$

Choosing $\kappa = \kappa(\varepsilon)$ sufficiently large we can guarantee that (5.4) holds whenever $0 < h < h_1$ for $h_1 = h_1(\varepsilon)$. Note that the restriction $0 < h < h_1$ ensures that $\kappa h < \frac{\pi}{2}$, which allowed for the computations above. Recall that spatial equicontinuity in the regime $h \in [h_1, h_0]$ is a consequence of the compactness of the interval $[h_1, h_0]$ and the continuity of the mapping $h \mapsto (\psi^h, \phi^h)$.

We now turn to the second half of (5.2), namely, equicontinuity in the time variable. By (1.5) and Plancherel,

$$\begin{aligned} \|\psi^h(t+s) - \psi^h(t)\|_{L_x^2}^2 &= \frac{1}{h} \|P_{|\theta| < \frac{\pi}{2}} [\alpha(h^{-2}(t+s)) - \alpha(h^{-2}t)]\|_{\ell_n^2}^2, \\ \|\phi^h(t+s) - \phi^h(t)\|_{L_x^2}^2 &= \frac{1}{h} \|P_{|\theta - \pi| < \frac{\pi}{2}} [e^{4ih^{-2}s} \alpha(h^{-2}(t+s)) - \alpha(h^{-2}t)]\|_{\ell_n^2}^2. \end{aligned}$$

To estimate the right-hand sides above, we will rely on Duhamel's formula

$$\alpha(h^{-2}(t+s)) = e^{ih^{-2}s\Delta_d} \alpha(h^{-2}t) - i \int_{h^{-2}t}^{h^{-2}(t+s)} e^{i[h^{-2}(t+s) - \tau]\Delta_d} F(\alpha(\tau)) d\tau.$$

Using Plancherel and evaluating the contributions of the regions $|\theta| < \kappa h$ and $\kappa h < |\theta| < \frac{\pi}{2}$ separately, we find

$$\begin{aligned} &\|P_{|\theta| < \frac{\pi}{2}} [e^{ih^{-2}s\Delta_d} \alpha(h^{-2}t) - \alpha(h^{-2}t)]\|_{\ell_n^2}^2 \\ &= \int_{|\theta| < \frac{\pi}{2}} |e^{-4ih^{-2}s \sin^2(\frac{\theta}{2})} - 1|^2 |\widehat{\alpha}(h^{-2}t, \theta)|^2 \frac{d\theta}{2\pi} \\ &\lesssim \kappa^4 |s|^2 \|\alpha(h^{-2}t)\|_{\ell_n^2}^2 + h \|P_{|\xi| \geq \kappa} \psi^h(t)\|_{L_x^2}^2. \end{aligned}$$

In view of Proposition 3.1 and (5.4), we may choose $\kappa = \kappa(\varepsilon)$ sufficiently large, followed by $\delta = \delta(\varepsilon)$ sufficiently small to guarantee that

$$(5.5) \quad \|P_{|\theta| < \frac{\pi}{2}} [e^{ih^{-2}s\Delta_d} \alpha(h^{-2}t) - \alpha(h^{-2}t)]\|_{\ell_n^2}^2 \leq \frac{\varepsilon^2 h}{10} \quad \text{for all } |s| \leq \delta.$$

Arguing similarly, we find

$$\begin{aligned} &\|P_{|\theta - \pi| < \frac{\pi}{2}} [e^{4ih^{-2}s + ih^{-2}s\Delta_d} \alpha(h^{-2}t) - \alpha(h^{-2}t)]\|_{\ell_n^2}^2 \\ &= \int_{|\theta| < \frac{\pi}{2}} |e^{4ih^{-2}s \sin^2(\frac{\theta+\pi}{2})} - 1|^2 |\widehat{\alpha}(h^{-2}t, \theta + \pi)|^2 \frac{d\theta}{2\pi} \\ &\lesssim \kappa^4 |s|^2 \|\alpha(h^{-2}t)\|_{\ell_n^2}^2 + h \|P_{|\xi| \geq \kappa} \phi^h(t)\|_{L_x^2}^2 \leq \frac{\varepsilon^2 h}{10} \end{aligned} \tag{5.6}$$

for $\kappa = \kappa(\varepsilon)$ sufficiently large and $|s| \leq \delta = \delta(\varepsilon)$.

It remains to estimate the contribution of the nonlinearity. To this end, let $P = P_{\frac{1}{100}}$ and $\tilde{P} = P_{\frac{3}{4}}$ denote the sharp Fourier cutoffs introduced in (3.5). We will again use that $\tilde{P}[F(P\alpha)] = F(P\alpha)$ to estimate the contribution of $F(P\alpha)$ using the frequency-localized Strichartz estimates of Proposition 4.3, followed by Propositions 3.1 and 4.4:

$$\begin{aligned} & \left\| \int_{h^{-2}t}^{h^{-2}(t+s)} e^{i[h^{-2}(t+s)-\tau]\Delta_d} F(P\alpha(\tau)) d\tau \right\|_{\ell_n^2}^2 \\ & \lesssim \|F(P\alpha)\|_{L_t^{\frac{6}{5}} \ell_n^{\frac{6}{5}}([-h^{-2}t, h^{-2}(t+s)] \times \mathbb{Z})}^2 \\ (5.7) \quad & \lesssim h^{-2}|s| \|\alpha\|_{L_t^6 \ell_n^6([-h^{-2}T, h^{-2}T] \times \mathbb{Z})}^4 \|\alpha\|_{L_t^\infty \ell_n^2([-h^{-2}T, h^{-2}T] \times \mathbb{Z})}^2 \lesssim_T h|s| \leq \frac{\varepsilon^2 h}{10}, \end{aligned}$$

provided $\delta = \delta(\varepsilon, T)$ is chosen sufficiently small.

Finally we estimate the contribution of $F(\alpha) - F(P\alpha)$ using the discrete Strichartz estimates from Proposition 4.2, followed by Propositions 4.4 and 3.2:

$$\begin{aligned} & \left\| \int_{h^{-2}t}^{h^{-2}(t+s)} e^{i[h^{-2}(t+s)-\tau]\Delta_d} (F(\alpha) - F(P\alpha)) d\tau \right\|_{\ell_n^2}^2 \\ & \lesssim \|F(\alpha) - F(P\alpha)\|_{L_t^{\frac{9}{8}} \ell_n^{\frac{6}{5}}([-h^{-2}t, h^{-2}(t+s)] \times \mathbb{Z})}^2 \\ & \lesssim (h^{-2}|s|)^{\frac{10}{9}} \|\alpha\|_{L_t^6 \ell_n^6([-h^{-2}T, h^{-2}T] \times \mathbb{Z})}^4 \|(I - P)\alpha\|_{L_t^\infty \ell_n^2([-h^{-2}T, h^{-2}T] \times \mathbb{Z})}^2 \\ (5.8) \quad & \lesssim_T (h^{-2}|s|)^{\frac{10}{9}} (Nh + h^{\frac{1}{2}})^2 \|\alpha_0\|_{\ell_n^2}^6 \leq \frac{\varepsilon^2 h}{10}, \end{aligned}$$

since $N \leq h^{-\frac{8}{9}}$ and $\delta = \delta(\varepsilon, T)$ is chosen sufficiently small.

Collecting (5.5) through (5.8), we conclude that for all $|s| \leq \delta = \delta(\varepsilon, T)$,

$$(5.9) \quad \|\psi^h(t+s) - \psi^h(t)\|_{L_x^2} + \|\phi^h(t+s) - \phi^h(t)\|_{L_x^2} < \varepsilon,$$

uniformly for $t \in [-T, T]$ (with $t+s \in [-T, T]$) and for $0 < h \leq h_0$. This expresses equicontinuity in the time variable and combined with (5.4) settles (5.2).

Lastly, we will demonstrate the tightness property (5.3). To this end, let $\chi(x)$ be a smooth cutoff function satisfying

$$\chi(x) = \begin{cases} 1 & : |x| \leq 1 \\ 0 & : |x| \geq 2 \end{cases}$$

from which we build a cutoff function to large n on the lattice via $\varphi_R(n) := 1 - \chi(nh/R)$ for some $R \geq 1$ to be chosen later.

We will also be localizing in frequency: We define $P : \ell_n^2 \rightarrow \ell_n^2$ via

$$(5.10) \quad \widehat{P\alpha}(\theta) = [\chi\left(\frac{\theta}{\kappa h}\right) + \chi\left(\frac{\theta-\pi}{\kappa h}\right)]\widehat{\alpha}(\theta) \quad \text{where } -\frac{\pi}{2} \leq \theta < \frac{3\pi}{2} \text{ and } \kappa h < \frac{\pi}{4}.$$

By Schur's test, we obtain the commutator bound

$$(5.11) \quad \|[P, \varphi_R]\|_{\ell_n^2 \rightarrow \ell_n^2} \lesssim \frac{1}{\kappa R}.$$

We will prove tightness of the orbit of the solution to (AL). Specifically, we will show that for any $\varepsilon > 0$ there exists $R \geq 1$ such that

$$(5.12) \quad \sup_{|t| \leq h^{-2}T} \|\varphi_R \alpha(t)\|_{\ell_n^2}^2 < \varepsilon h$$

uniformly for $0 < h \leq h_0$. The tightness of $\alpha_n(t)$ transfers to ψ^h and ϕ^h as a consequence of Lemmas 2.3 and 2.4.

By the equicontinuity in the space variable (5.4), we have

$$(5.13) \quad \|(1-P)\alpha(t)\|_{\ell_n^2}^2 < \varepsilon^4 h \quad \text{uniformly for } |t| \leq h^{-2}T \text{ and } 0 < h \leq h_0,$$

provided $\kappa \geq 1$ is chosen sufficiently large depending only on ε . To maintain the condition $\kappa h < \frac{\pi}{4}$ from (5.10), we will restrict attention to $0 < h < h_1 = h_1(\varepsilon)$ and prove (5.12) in this regime. Recall that tightness in the regime $h \in [h_1, h_0]$ is an immediate consequence of the compactness of the interval $[h_1, h_0]$ and the continuity of the mapping $h \mapsto \alpha$ with α being the solution to (AL) with initial data as in (1.3).

In view of (5.13) and the preceding discussion, (5.12) will thus follow from the statement that for any $\varepsilon > 0$ there exists $R \geq 1$ such that

$$(5.14) \quad \sup_{|t| \leq h^{-2}T} \|\varphi_R P\alpha(t)\|_{\ell_n^2}^2 < \varepsilon h \quad \text{uniformly for } 0 < h < h_1.$$

From (AL) we have

$$(5.15) \quad \partial_t \|\varphi_R P\alpha(t)\|_{\ell_n^2}^2 = -2 \operatorname{Im} \sum_n \varphi_R^2(n) \overline{P\alpha_n(t)} \cdot P\{(\Delta_d \alpha)_n(t) - F_n[\alpha(t)]\}.$$

To control the contribution of the quadratic term above we rewrite

$$\begin{aligned} & 2 \operatorname{Im} \sum_n \varphi_R^2(n) P\beta_n(t) \cdot (\Delta_d P\alpha)_n(t) \\ &= 2 \operatorname{Im} \sum_n \varphi_R^2(n) P\beta_n(t) \cdot P\alpha_{n+1}(t) - 2 \operatorname{Im} \sum_n \varphi_R^2(n) P\alpha_n(t) \cdot P\beta_{n-1}(t) \\ &= 2 \operatorname{Im} \sum_n [\varphi_R^2(n) - \varphi_R^2(n+1)] P\beta_n(t) \cdot P\alpha_{n+1}(t) \\ &= \operatorname{Im} \sum_n [\varphi_R^2(n) - \varphi_R^2(n+1)] P\beta_n(t) \cdot P[\alpha_{n+1}(t) - \alpha_{n-1}(t)] \\ & \quad + \operatorname{Im} \sum_n [2\varphi_R^2(n) - \varphi_R^2(n+1) - \varphi_R^2(n-1)] P\beta_n(t) \cdot P\alpha_{n-1}(t). \end{aligned}$$

By Plancherel, (5.10), and Proposition 3.1,

$$\begin{aligned} \|P[\alpha_{n+1}(t) - \alpha_{n-1}(t)]\|_{\ell_n^2} &\lesssim \|[\chi(\frac{\theta}{\kappa h}) + \chi(\frac{\theta-\pi}{\kappa h})](e^{i\theta} - e^{-i\theta})\widehat{\alpha}(t, \theta)\|_{L_\theta^2} \\ &\lesssim \kappa h \|\alpha(t)\|_{\ell_n^2} \lesssim \kappa h \|\alpha(0)\|_{\ell_n^2}. \end{aligned}$$

Thus, using Proposition 3.1 we may estimate

$$\begin{aligned} & \left| 2 \operatorname{Im} \sum_n \varphi_R^2(n) P\beta_n(t) \cdot (\Delta_d P\alpha)_n(t) \right| \\ & \lesssim \kappa h \|\varphi_R^2(n) - \varphi_R^2(n+1)\|_{\ell_n^\infty} \|\alpha(0)\|_{\ell_n^2}^2 \\ & \quad + \|2\varphi_R^2(n) - \varphi_R^2(n+1) - \varphi_R^2(n-1)\|_{\ell_n^\infty} \|\alpha(0)\|_{\ell_n^2}^2 \\ (5.16) \quad & \lesssim \left(\frac{\kappa h^2}{R} + \frac{h^2}{R^2} \right) \|\alpha(0)\|_{\ell_n^2}^2. \end{aligned}$$

We turn now to the contribution of the nonlinearity and decompose $\alpha = P\alpha + (1-P)\alpha$. The contribution of the nonlinearity containing $(1-P)\alpha$ can be estimated using (5.13), as follows:

$$\left| \operatorname{Im} \sum_n \varphi_R^2(n) P\beta_n(t) \cdot P\{(1-P)\alpha_n(t) \cdot \beta_n(t) [\alpha_{n+1}(t) + \alpha_{n-1}(t)]\} \right|$$

$$(5.17) \quad \lesssim \|(1 - P)\alpha(t)\|_{\ell_n^2} \|\alpha(t)\|_{\ell_n^6}^3 \lesssim \varepsilon^2 h^{\frac{1}{2}} \|\alpha(t)\|_{\ell_n^6}^3.$$

Using (5.11) and Proposition 3.1, we bound the contribution of the remaining term via

$$\begin{aligned} & \left| \operatorname{Im} \sum_n \varphi_R^2(n) P \beta_n(t) \cdot P \left\{ P \alpha_n(t) \cdot \beta_n(t) [\alpha_{n+1}(t) + \alpha_{n-1}(t)] \right\} \right| \\ & \lesssim \|\varphi_R P \alpha(t)\|_{\ell_n^2}^2 \|\alpha(t)\|_{\ell_n^\infty}^2 + \|\varphi_R P \alpha(t)\|_{\ell_n^2} \|P, \varphi_R\|_{\ell_n^2 \rightarrow \ell_n^2} \|\alpha(t)\|_{\ell_n^6}^3 \\ (5.18) \quad & \lesssim \|\varphi_R P \alpha(t)\|_{\ell_n^2}^2 \|\alpha(t)\|_{\ell_n^\infty}^2 + \frac{h^{\frac{1}{2}}}{\kappa R} \|\alpha(t)\|_{\ell_n^6}^3. \end{aligned}$$

Combining (5.15) through (5.18), we find that for $\kappa h < \frac{\pi}{4}$ with $\kappa \geq 1$ large depending only on ε ,

$$\partial_t \|\varphi_R P \alpha(t)\|_{\ell_n^2}^2 \lesssim \frac{\kappa h^2}{R} \|\alpha(0)\|_{\ell_n^2}^2 + (\varepsilon^2 + \frac{1}{\kappa R}) h^{\frac{1}{2}} \|\alpha(t)\|_{\ell_n^6}^3 + \|\varphi_R P \alpha(t)\|_{\ell_n^2}^2 \|\alpha(t)\|_{\ell_n^\infty}^2.$$

By Gronwall and Propositions 4.4 and 3.1, this yields

$$\begin{aligned} & \sup_{|t| \leq h^{-2}T} \|\varphi_R P \alpha(t)\|_{\ell_n^2}^2 \\ & \lesssim \left[\|\varphi_R P \alpha(0)\|_{\ell_n^2}^2 + \frac{\kappa h^2}{R} \|\alpha(0)\|_{\ell_n^2}^2 h^{-2}T + (\varepsilon^2 + \frac{1}{\kappa R}) h^{\frac{1}{2}} \|\alpha\|_{L_t^6 \ell_n^6}^3 (h^{-2}T)^{\frac{1}{2}} \right] \\ & \quad \times \exp \left\{ C \|\alpha\|_{L_t^4 \ell_n^\infty}^2 (h^{-2}T)^{\frac{1}{2}} \right\} \\ & \lesssim_T \|\varphi_R P \alpha(0)\|_{\ell_n^2}^2 + \frac{\kappa T}{R} h + (\varepsilon^2 + \frac{1}{\kappa R}) T^{\frac{1}{2}} h, \end{aligned}$$

where all spacetime norms are over $[-h^{-2}T, h^{-2}T] \times \mathbb{Z}$. The right-hand side here can be made smaller than εh by first choosing κ large (to ensure (5.13)) and then choosing R sufficiently large. Note that by Lemma 2.2, monotone convergence, and the L^2 -boundedness of the Hardy-Littlewood maximal operator \mathcal{M} , we have

$$\limsup_{R \rightarrow \infty} h^{-1} \|\varphi_R P \alpha(0)\|_{\ell_n^2}^2 \lesssim \limsup_{R \rightarrow \infty} \|[1 - \chi(\frac{x}{R})][\mathcal{M}\psi_0 + \mathcal{M}\phi_0]\|_{L^2} = 0.$$

This completes the proof of (5.14) and so that of (5.12).

6. CONVERGENCE OF THE FLOWS

As a consequence of Theorem 5.1, every sequence $h_n \rightarrow 0$ admits a subsequence $h_{n_j} \rightarrow 0$ such that $(\psi^{h_{n_j}}, \phi^{h_{n_j}})$ converges in $C([-T, T]; L_x^2(\mathbb{R}))$ to some (ψ, ϕ) . As a first step toward proving Theorem 1.1, we will show that all such subsequential limits are solutions to (1.2) with initial data (ψ_0, ϕ_0) and satisfy certain spacetime bounds. For notational simplicity, we will omit the subscripts on h in what follows.

Proposition 6.1. *Let α be the solution to (AL) with initial data (1.3) and h satisfying (1.7) and let ψ^h and ϕ^h be as in (1.5). Let $\psi, \phi \in C([-T, T]; L_x^2(\mathbb{R}))$ so that*

$$(6.1) \quad \psi^h \rightarrow \psi \quad \text{and} \quad \phi^h \rightarrow \phi \quad \text{in} \quad C([-T, T]; L_x^2(\mathbb{R}))$$

along some sequence of $h \rightarrow 0$. Then $\psi, \phi \in L_{t,x}^6([-T, T] \times \mathbb{R})$ and for any $|t| \leq T$ we have

$$(6.2) \quad \psi(t) = e^{it\Delta} \psi_0 \mp 2i \int_0^t e^{i(t-s)\Delta} |\psi(s)|^2 \psi(s) ds,$$

$$(6.3) \quad \phi(t) = e^{-it\Delta} \phi_0 \pm 2i \int_0^t e^{-i(t-s)\Delta} |\phi(s)|^2 \phi(s) ds.$$

Proof. As ψ^h and ϕ^h converge in $C([-T, T]; L_x^2(\mathbb{R}))$, they converge distributionally on $[-T, T] \times \mathbb{R}$. Consequently, by Proposition 4.4, ψ and ϕ satisfy

$$(6.4) \quad \|\psi\|_{L_{t,x}^6([-T,T]\times\mathbb{R})} + \|\phi\|_{L_{t,x}^6([-T,T]\times\mathbb{R})} \lesssim_T 1.$$

To prove (6.2) and (6.3), our starting point is the Duhamel formula satisfied by the solution α of (AL): for any $|t| \leq T$,

$$(6.5) \quad \alpha_n(h^{-2}t) = e^{ih^{-2}t\Delta_d} \alpha_n(0) - \frac{i}{h^2} \int_0^t e^{ih^{-2}(t-s)\Delta_d} F_n(\alpha(h^{-2}s)) ds.$$

We reconstitute $\psi^h(t, x)$ and $\phi^h(t, x)$ from the left-hand side above using (1.6). Employing the notation introduced in (2.4) we may write

$$\psi^h(t) = \mathcal{R}[\alpha_n(h^{-2}t)] \quad \text{and} \quad \phi^h(t) = e^{4ih^{-2}t} \mathcal{R}[(-1)^n \alpha_n(h^{-2}t)].$$

Given the hypothesis (6.1), it thus suffices to prove that

$$(6.6) \quad \lim_{h \rightarrow 0} \mathcal{R}[\text{RHS}(6.5)] = \text{RHS}(6.2) \quad \text{in } C([-T, T]; L_x^2(\mathbb{R}))$$

and

$$(6.7) \quad \lim_{h \rightarrow 0} e^{4ih^{-2}t} \mathcal{R}[(-1)^n \text{RHS}(6.5)] = \text{RHS}(6.3) \quad \text{in } C([-T, T]; L_x^2(\mathbb{R})).$$

We first address the convergence of the linear terms on the left-hand sides of (6.6) and (6.7).

Lemma 6.2. *Under the hypotheses of Proposition 6.1, we have*

$$\begin{aligned} \lim_{h \rightarrow 0} \left\| \mathcal{R}[e^{ih^{-2}t\Delta_d} \alpha_n(0)] - e^{it\Delta} \psi_0 \right\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R})} &= 0, \\ \lim_{h \rightarrow 0} \left\| e^{4ih^{-2}t} \mathcal{R}[(-1)^n e^{ih^{-2}t\Delta_d} \alpha_n(0)] - e^{-it\Delta} \phi_0 \right\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R})} &= 0. \end{aligned}$$

Proof. Performing the change of variables $\theta = h\xi$ and using Plancherel and (2.1), we estimate

$$\begin{aligned} &\left\| \mathcal{R}[e^{ih^{-2}t\Delta_d} \alpha_n(0)] - e^{it\Delta} \psi_0 \right\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R})} \\ &\leq \left\| \int \left[\exp\left\{ ix\xi - i \frac{4 \sin^2(\frac{h\xi}{2})}{h^2} t \right\} - \exp\{ix\xi - it|\xi|^2\} \right] \widehat{P_{\leq N} \psi_0}(\xi) \frac{d\xi}{2\pi} \right\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R})} \\ &\quad + \left\| e^{it\Delta} \psi_0 - e^{it\Delta} P_{\leq N} \psi_0 \right\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R})} \\ &\lesssim \left\| \left[\exp\left\{ it|\xi|^2 - i \frac{4 \sin^2(\frac{h\xi}{2})}{h^2} t \right\} - 1 \right] \widehat{P_{\leq N} \psi_0}(\xi) \right\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R})} + \left\| \widehat{P_{\geq N} \psi_0} \right\|_{L_x^2}. \end{aligned}$$

The first claim now follows from the dominated convergence theorem, recalling that $N = h^{-\gamma}$. The second claim is proved analogously. \square

We now turn to the convergence of the nonlinear terms on the left-hand sides of (6.6) and (6.7). We will only present the details for (6.6); the treatment of (6.7) is analogous. To complete the proof of (6.2), we must show that

$$(6.8) \quad \lim_{h \rightarrow 0} \mathcal{R} \left[\frac{i}{h^2} \int_0^t e^{ih^{-2}(t-s)\Delta_d} F_n(\alpha(h^{-2}s)) ds \right] = \pm 2i \int_0^t e^{i(t-s)\Delta} |\psi(s)|^2 \psi(s) ds$$

in $C([-T, T]; L_x^2(\mathbb{R}))$. Recall that by Lemma 2.3, we have $\|\mathcal{R}\|_{\ell_n^2(\mathbb{Z}) \rightarrow L_x^2(\mathbb{R})} \lesssim h^{-\frac{1}{2}}$.

In view of (1.5),

$$(6.9) \quad \alpha_n(h^{-2}t) = h[\psi^h(t, nh) + (-1)^n e^{-4ih^{-2}t} \phi^h(t, nh)].$$

Employing the sharp Fourier cutoff P_h to $|\xi| < h^{-\frac{1}{2}}$, we define

$$\tilde{\psi}^h = P_h \psi^h \quad \text{and} \quad \tilde{\phi}^h = P_h \phi^h$$

and then

$$(6.10) \quad \tilde{\alpha}_n(h^{-2}t) = h[\tilde{\psi}^h(t, nh) + (-1)^n e^{-4ih^{-2}t} \tilde{\phi}^h(t, nh)].$$

We will show that as $h \rightarrow 0$, the nonlinearity in (AL) can be replaced by an on-site nonlinearity based solely on this more narrowly Fourier localized sequence. One reason for introducing this additional localization is to ensure that Lemma 2.2 may be applied to such nonlinear functions.

Lemma 6.3. *Adopting the notation*

$$(6.11) \quad \tilde{F}_n(t) = \pm 2h |\tilde{\alpha}_n(h^{-2}t)|^2 [\tilde{\psi}^h(t, nh) - (-1)^n e^{-4ih^{-2}t} \tilde{\phi}^h(t, nh)]$$

(notice the sign flip relative to (6.10)), we have

$$\left\| F(\alpha(h^{-2}t)) - \tilde{F}(t) \right\|_{L_t^1 \ell_n^2([-T, T] \times \mathbb{Z})} = o(h^{\frac{5}{2}}) \quad \text{as } h \rightarrow 0.$$

In particular,

$$\lim_{h \rightarrow 0} \left\| \mathcal{R} \left[\frac{i}{h^2} \int_0^t e^{ih^{-2}(t-s)\Delta_d} [F(\alpha(h^{-2}s)) - \tilde{F}(s)] ds \right] \right\|_{L_t^\infty L_x^2([-T, T] \times \mathbb{R})} = 0.$$

Proof. In view of (6.9), Lemma 2.2, and Theorem 5.1 (specifically (5.4)), we may estimate

$$\begin{aligned} & \left\| \alpha_n(h^{-2}t) - \tilde{\alpha}_n(h^{-2}t) \right\|_{L_t^\infty \ell_n^2([-T, T] \times \mathbb{Z})} \\ & \leq h \left\| (1 - P_h) \psi^h(t, nh) \right\|_{L_t^\infty \ell_n^2([-T, T] \times \mathbb{Z})} + h \left\| (1 - P_h) \phi^h(t, nh) \right\|_{L_t^\infty \ell_n^2([-T, T] \times \mathbb{Z})} \\ & \lesssim h^{\frac{1}{2}} \left\| (1 - P_h) \psi^h(t) \right\|_{L_t^\infty L_x^2([-T, T] \times \mathbb{R})} + h^{\frac{1}{2}} \left\| (1 - P_h) \phi^h(t) \right\|_{L_t^\infty L_x^2([-T, T] \times \mathbb{R})} \\ & = o(h^{\frac{1}{2}}) \quad \text{as } h \rightarrow 0. \end{aligned}$$

Consequently, by Proposition 4.4, we get

$$\begin{aligned} & \left\| F(\alpha(h^{-2}t)) - F(\tilde{\alpha}(h^{-2}t)) \right\|_{L_t^1 \ell_n^2([-T, T] \times \mathbb{Z})} \\ & \leq \left\| \alpha(h^{-2}t) \right\|_{L_t^4 \ell_n^\infty([-T, T] \times \mathbb{Z})}^2 \left\| \alpha_n(h^{-2}t) - \tilde{\alpha}_n(h^{-2}t) \right\|_{L_t^\infty \ell_n^2([-T, T] \times \mathbb{Z})} \\ (6.12) \quad & = o(h^{\frac{5}{2}}) \quad \text{as } h \rightarrow 0. \end{aligned}$$

To continue, we use Lemma 2.2 and Theorem 5.1 to estimate

$$\begin{aligned} & \left\| [\tilde{\alpha}_{n-1}(h^{-2}t) + \tilde{\alpha}_{n+1}(h^{-2}t)] - 2h [\tilde{\psi}^h(t, nh) - (-1)^n e^{-4ih^{-2}t} \tilde{\phi}^h(t, nh)] \right\|_{L_t^\infty \ell_n^2} \\ & \leq 2h \left\| \tilde{\psi}^h(t, (n+1)h) - \tilde{\psi}^h(t, nh) \right\|_{L_t^\infty \ell_n^2} + 2h \left\| \tilde{\phi}^h(t, (n+1)h) - \tilde{\phi}^h(t, nh) \right\|_{L_t^\infty \ell_n^2} \\ & \leq 2h \left\| \psi^h(t, (n+1)h) - \psi^h(t, nh) \right\|_{L_t^\infty \ell_n^2} + 2h \left\| \phi^h(t, (n+1)h) - \phi^h(t, nh) \right\|_{L_t^\infty \ell_n^2} \\ & \lesssim h^{\frac{1}{2}} \left\| \psi^h(t, \cdot + h) - \psi^h(t) \right\|_{L_t^\infty L_x^2} + h^{\frac{1}{2}} \left\| \phi^h(t, \cdot + h) - \phi^h(t) \right\|_{L_t^\infty L_x^2} = o(h^{\frac{1}{2}}) \end{aligned}$$

as $h \rightarrow 0$. Consequently, using Proposition 4.4 as in (6.12), we obtain

$$(6.13) \quad \left\| F(\tilde{\alpha}(h^{-2}t)) - \tilde{F}(t) \right\|_{L_t^1 \ell_n^2([-T, T] \times \mathbb{Z})} = o(h^{\frac{5}{2}}) \quad \text{as } h \rightarrow 0.$$

Combining (6.12) and (6.13) settles the lemma. \square

Expanding out the definition, we have

$$\begin{aligned} \tilde{F}_n(t) &= 2h^3 |\tilde{\psi}^h(t, nh)|^2 \tilde{\psi}^h(t, nh) - 2h^3 e^{-8ih^{-2}t} (\tilde{\phi}^h(t, nh))^2 \overline{\tilde{\psi}^h(t, nh)} \\ &\quad - 2(-1)^n h^3 e^{-4ih^{-2}t} \left[|\tilde{\phi}^h(t, nh)|^2 \tilde{\phi}^h(t, nh) - e^{8ih^{-2}t} (\tilde{\psi}^h(t, nh))^2 \overline{\tilde{\phi}^h(t, nh)} \right]. \end{aligned}$$

By looking at the Fourier supports, we see that only the top row of terms contributes to the left-hand side of (6.6) and the second row contributes only to that of (6.7).

Our next result shows how the temporal non-resonance of the unexpected terms in the expansion for \tilde{F}_n (namely those involving both $\tilde{\psi}^h$ and $\tilde{\phi}^h$) cause them to drop out in the limit $h \rightarrow 0$.

Lemma 6.4. *Let \mathcal{E}_n denote a cubic polynomial in $\tilde{\psi}^h(nh)$, $\tilde{\phi}^h(nh)$, and their complex conjugates and let m be a non-zero integer. Then*

$$\left\| \int_0^t e^{ih^{-2}(t-s)\Delta_d} e^{imh^{-2}s} \mathcal{E}_n(s) ds \right\|_{L_t^\infty \ell_n^2([-T, T] \times \mathbb{Z})} = o(h^{-\frac{1}{2}}) \quad \text{as } h \rightarrow 0.$$

In particular,

$$\lim_{h \rightarrow 0} \left\| \mathcal{R} \left[\frac{i}{h^2} \int_0^t e^{ih^{-2}(t-s)\Delta_d} e^{imh^{-2}s} h^3 \mathcal{E}_n(s) ds \right] \right\|_{L_t^\infty L_x^2([-T, T] \times \mathbb{R})} = 0.$$

Proof. We decompose

$$\begin{aligned} &\int_0^t e^{ih^{-2}(t-s)\Delta_d} e^{imh^{-2}s} \mathcal{E}_n(s) ds \\ &= \frac{1}{2} \int_0^t e^{ih^{-2}(t-s)\Delta_d} e^{imh^{-2}s} \mathcal{E}_n(s) ds - \frac{1}{2} \int_0^t e^{ih^{-2}(t-s)\Delta_d} e^{imh^{-2}(s+\frac{\pi h^2}{m})} \mathcal{E}_n(s) ds \\ &= \frac{1}{2} \int_0^t e^{imh^{-2}s+ih^{-2}(t-s)\Delta_d} \left[1 - e^{i\frac{\pi}{m}\Delta_d} \right] \mathcal{E}_n(s) ds \\ &\quad + \frac{1}{2} \int_0^t e^{ih^{-2}(t+\frac{\pi h^2}{m}-s)\Delta_d} e^{imh^{-2}s} \left[\mathcal{E}_n(s) - \mathcal{E}_n(s - \frac{\pi h^2}{m}) \right] ds \\ &\quad + \frac{1}{2} \int_0^{\frac{\pi h^2}{m}} e^{ih^{-2}(t+\frac{\pi h^2}{m}-s)\Delta_d} e^{imh^{-2}s} \mathcal{E}_n(s - \frac{\pi h^2}{m}) ds \\ &\quad - \frac{1}{2} \int_t^{t+\frac{\pi h^2}{m}} e^{ih^{-2}(t+\frac{\pi h^2}{m}-s)\Delta_d} e^{imh^{-2}s} \mathcal{E}_n(s - \frac{\pi h^2}{m}) ds. \end{aligned}$$

Using the discrete Strichartz inequality Proposition 4.2 followed by Lemma 2.2 and Proposition 4.4, we estimate the contribution of the last two terms in our decomposition by

$$\begin{aligned} \left(\frac{\pi h^2}{m} \right)^{\frac{1}{2}} \|\mathcal{E}_n\|_{L_t^2 \ell_n^2([-T, T] \times \mathbb{Z})} &\lesssim h \left[\|\tilde{\psi}^h(nh)\|_{L_t^6 \ell_n^6([-T, T] \times \mathbb{Z})}^3 + \|\tilde{\phi}^h(nh)\|_{L_t^6 \ell_n^6([-T, T] \times \mathbb{Z})}^3 \right] \\ &\lesssim h^{\frac{1}{2}} \left[\|\tilde{\psi}^h\|_{L_{t,x}^6([-T, T] \times \mathbb{R})}^3 + \|\tilde{\phi}^h\|_{L_{t,x}^6([-T, T] \times \mathbb{R})}^3 \right] \lesssim h^{\frac{1}{2}}, \end{aligned}$$

which is acceptable.

In view of the Fourier localization imposed on the functions $\tilde{\psi}^h$ and $\tilde{\phi}^h$, we have $\text{supp}(\widehat{\mathcal{E}_n}) \subset [-3h^{\frac{1}{2}}, 3h^{\frac{1}{2}}]$. Thus, estimating as above, we may bound the contribution of the first term in our decomposition by

$$\begin{aligned} & \left\| \int_0^t e^{ih^{-2}s+ih^{-2}(t-s)\Delta_d} [1 - e^{i\frac{\pi}{m}\Delta_d}] \mathcal{E}_n(s) ds \right\|_{L_t^\infty \ell_n^2([-T, T] \times \mathbb{Z})} \\ & \lesssim \| [1 - e^{i\frac{\pi}{m}\Delta_d}] \mathcal{E}_n \|_{L_t^1 \ell_n^2([-T, T] \times \mathbb{Z})} \lesssim h \| \mathcal{E}_n \|_{L_t^1 \ell_n^2([-T, T] \times \mathbb{Z})} \lesssim h^{\frac{1}{2}}, \end{aligned}$$

which is acceptable.

This leaves us to estimate the second term in our decomposition, which represents the central question at hand. The key idea here is to exploit the equicontinuity in time of the functions $\tilde{\psi}^h$ and $\tilde{\phi}^h$ proved in Theorem 5.1. As $\text{supp}(\widehat{\mathcal{E}_n}) \subset [-3h^{\frac{1}{2}}, 3h^{\frac{1}{2}}]$, we may employ the frequency-localized Strichartz estimates from Proposition 4.3 and scaling to estimate

$$\begin{aligned} & \left\| \int_0^t e^{ih^{-2}(t+\frac{\pi h^2}{m}-s)\Delta_d} e^{ih^{-2}s} [\mathcal{E}_n(s) - \mathcal{E}_n(s - \frac{\pi h^2}{m})] ds \right\|_{L_t^\infty \ell_n^2} \\ & \lesssim h^{\frac{1}{3}} \| \mathcal{E}_n(s) - \mathcal{E}_n(s - \frac{\pi h^2}{m}) \|_{L_t^{\frac{6}{5}} \ell_n^{\frac{6}{5}}([-T, T] \times \mathbb{Z})} \\ & \lesssim T^{\frac{1}{2}} h^{-\frac{1}{2}} \left[\| \psi^h \|_{L_{t,x}^6([-T, T] \times \mathbb{R})}^2 + \| \phi^h \|_{L_{t,x}^6([-T, T] \times \mathbb{R})}^2 \right] \\ & \quad \times \left[\| \psi^h - \psi^h(\cdot - \frac{\pi h^2}{m}) \|_{L_t^\infty L_x^2([-T, T] \times \mathbb{R})} + \| \phi^h - \phi^h(\cdot - \frac{\pi h^2}{m}) \|_{L_t^\infty L_x^2([-T, T] \times \mathbb{R})} \right] \\ & = o(h^{-\frac{1}{2}}) \quad \text{as } h \rightarrow 0 \end{aligned}$$

where we used Lemma 2.2, Proposition 4.4, and Theorem 5.1 in the last two lines.

This completes the proof of the lemma. \square

We finally consider the main contribution to the left-hand side of (6.8).

Lemma 6.5. *We have*

$$\lim_{h \rightarrow 0} \mathcal{R} \left[\frac{1}{h^2} \int_0^t e^{ih^{-2}(t-s)\Delta_d} 2h^3 (|\tilde{\psi}^h|^2 \tilde{\psi}^h)(s, nh) ds \right] = 2 \int_0^t e^{i(t-s)\Delta} |\psi(s)|^2 \psi(s) ds$$

in $C([-T, T]; L_x^2(\mathbb{R}))$.

Proof. Combining hypothesis (6.1) and Lemmas 6.2 through 6.4, we know that the term on the left-hand side above converges in $C([-T, T]; L_x^2(\mathbb{R}))$. Thus, it suffices to identify its limit via duality. Let $f \in \mathcal{S}(\mathbb{R})$ be L^2 -normalized and satisfy $\widehat{f} \in C_c^\infty(\mathbb{R})$. Note that for h sufficiently small we have $\text{supp} \widehat{f} \subseteq \{\xi : |h\xi| < \frac{\pi}{2}\}$. Thus, we may use Plancherel to compute

$$\begin{aligned} & \left\langle \mathcal{R} \left[\frac{1}{h^2} \int_0^t e^{ih^{-2}(t-s)\Delta_d} 2h^3 (|\tilde{\psi}^h|^2 \tilde{\psi}^h)(s, nh) ds \right], f \right\rangle \\ & = 2h \int_0^t \left\langle \widehat{|\tilde{\psi}^h|^2 \tilde{\psi}^h}(s, h\xi), \exp \left\{ i \frac{4 \sin^2(\frac{h\xi}{2})}{h^2} (t-s) \right\} \widehat{f}(\xi) \right\rangle ds \\ & = 2h \int_0^t \sum_n (|\tilde{\psi}^h|^2 \tilde{\psi}^h)(s, nh) \overline{[e^{-i(t-s)\Delta} f]}(nh) ds \\ & \quad + 2h \int_0^t \sum_n (|\tilde{\psi}^h|^2 \tilde{\psi}^h)(s, nh) \int e^{inh\xi} \left[e^{i \frac{4 \sin^2(\frac{h\xi}{2})}{h^2} (t-s)} - e^{i|\xi|^2(t-s)} \right] \widehat{f}(\xi) d\xi ds. \end{aligned}$$

Using Plancherel, Lemma 2.2, and Proposition 4.4 together with the dominated convergence theorem, we may bound the second summand above by

$$\begin{aligned} h \int_0^t \| (|\tilde{\psi}^h|^2 \tilde{\psi}^h)(s, nh) \|_{\ell_n^2} h^{-\frac{1}{2}} \left\| \left[e^{i[\xi^2 - \frac{4 \sin^2(\frac{h\xi}{2})}{h^2}](t-s)} - 1 \right] \tilde{f} \right\|_{L_\xi^2} ds \\ \lesssim T^{\frac{1}{2}} \|\tilde{\psi}^h\|_{L_{t,x}^6([-T,T] \times \mathbb{R})}^3 o(1) = o(1) \quad \text{as } h \rightarrow 0. \end{aligned}$$

Noting that the Fourier support of $|\tilde{\psi}^h|^2 \tilde{\psi}^h$ is contained in $[-3h^{-\frac{1}{2}}, 3h^{-\frac{1}{2}}]$ and invoking Lemma 2.1, we see that the first summand equals

$$2 \int_0^t \int (|\tilde{\psi}^h|^2 \tilde{\psi}^h)(s, x) \overline{[e^{-i(t-s)\Delta} f](x)} dx ds = 2 \int_0^t \langle e^{i(t-s)\Delta} (|\tilde{\psi}^h|^2 \tilde{\psi}^h)(s), f \rangle ds.$$

That this converges to the desired limit as $h \rightarrow 0$ follows readily from the Strichartz inequality Proposition 4.1, hypothesis (6.1), Theorem 5.1, Proposition 4.4 and (6.4):

$$\begin{aligned} & \left\| \int_0^t \langle e^{i(t-s)\Delta} [|\tilde{\psi}^h|^2 \tilde{\psi}^h - |\psi|^2 \psi](s), f \rangle ds \right\|_{L_t^\infty([-T,T])} \\ & \lesssim T^{\frac{1}{2}} \|\tilde{\psi}^h - \psi\|_{L_t^\infty L_x^2} [\|\tilde{\psi}^h\|_{L_{t,x}^6}^2 + \|\psi\|_{L_{t,x}^6}^2] \|f\|_{L_x^2} \\ & \lesssim_T \|\psi^h - \psi\|_{L_t^\infty L_x^2} + \|P_{|\xi| \geq h^{-\frac{1}{2}}} \psi^h\|_{L_t^\infty L_x^2} = o(1) \quad \text{as } h \rightarrow 0, \end{aligned}$$

where all spacetime norms are taken over $[-T, T] \times \mathbb{R}$. \square

The proof of Proposition 6.1 is now complete. Indeed, Lemmas 6.2 through 6.5 show that ψ satisfies (6.2). The proof that ϕ satisfies (6.3) follows from parallel arguments. \square

We are finally ready to prove our main result:

Proof of Theorem 1.1. As noted at the beginning of this section, Theorem 5.1 guarantees that every sequence $h \rightarrow 0$ admits a subsequence so that both ψ^h and ϕ^h converge in $C([-T, T]; L_x^2(\mathbb{R}))$. By Proposition 6.1, the limiting functions lie in $L_{t,x}^6$ and solve the integral equations (6.2) and (6.3).

These integral equations admit only one solution in $C_t L_x^2 \cap L_{t,x}^6$ as is easily shown by contraction mapping using the estimates recalled in Proposition 4.1. Originating in [20], this is now the textbook approach to the construction of solutions to (1.2).

As all subsequential limits agree, it follows that the original sequences ψ^h and ϕ^h converge as $h \rightarrow 0$ without passing to subsequences at all. Moreover, as noted above, the resulting limits are the unique solutions to the evolutions (1.2) with initial data ψ_0, ϕ_0 . \square

REFERENCES

- [1] M. J. Ablowitz and J. F. Ladik, *Nonlinear differential-difference equations*. J. Mathematical Phys. **16** (1975), 598–603.
- [2] Y. Angelopoulos, R. Killip, and M. Visan, *Invariant measures for integrable spin chains and an integrable discrete nonlinear Schrödinger equation*. SIAM J. Math. Anal. **52** (2020), no. 1, 135–163.
- [3] D. Bambusi, T. Kappeler, and T. Paul, *From Toda to KdV*. Nonlinearity **28** (2015), no. 7, 2461–2496.
- [4] D. Bambusi, T. Kappeler, T. Paul, *Dynamics of periodic Toda chains with a large number of particles*. J. Differential Equations **258** (2015), no. 12, 4209–4274.
- [5] J. Ginibre and G. Velo, *Smoothing properties and retarded estimates for some dispersive evolution equations*. Comm. Math. Phys. **144** (1992), no. 1, 163–188.

- [6] B. Harrop-Griffiths, R. Killip, and M. Visan, *Microscopic conservation laws for integrable lattice models*. Monatsh. Math. **196** (2021), no. 3m 477–504.
- [7] B. M. Herbst, F. Váradi, and M. J. Ablowitz, *Symplectic methods for the nonlinear Schrödinger equation*. In “Solitons, nonlinear wave equations and computation (New Brunswick, NJ, 1992)”. Math. Comput. Simulation **37** (1994), no. 4–5, 353–369.
- [8] Y. Hong and C. Yang, *Strong convergence for discrete nonlinear Schrödinger equations in the continuum limit*. SIAM J. Math. Anal. **51** (2019), no. 2, 1297–1320.
- [9] L. I. Ignat and E. Zuazua, *Numerical dispersive schemes for the nonlinear Schrödinger equation*. SIAM Journal of Numerical Analysis **47** (2009), no. 2, 1366–1390.
- [10] L. I. Ignat and E. Zuazua, *Convergence rates for dispersive approximation schemes to nonlinear Schrödinger equations*. J. Math. Pures Appl. (9) **98** (2012), no. 5, 479–517.
- [11] Y. Ishimori, *An integrable classical spin chain*. J. Phys. Soc. Jpn. **51** (1982), no. 11, 3417–3418.
- [12] M. Keel and T. Tao, *Endpoint Strichartz estimates*. Amer. J. Math. **120** (1998), no. 5, 955–980.
- [13] R. Killip, M. Visan, and X. Zhang, *Low regularity conservation laws for integrable PDE*. Geom. Funct. Anal. **28** (2018), 1062–1090.
- [14] K. Kirkpatrick, E. Lenzmann, and G. Staffilani, *On the continuum limit for discrete NLS with long-range lattice interactions*. Comm. Math. Phys. **317** (2013), no. 3, 563–591.
- [15] M. Plancherel and G. Pólya, *Fonctions entières et intégrales de Fourier multiples*. Comment. Math. Helv. **10** (1937), no. 1, 110–163.
- [16] M. Riesz, *Sur les fonctions conjuguées*. Math. Z. **27** (1928), no. 1, 218–244.
- [17] M. Riesz, *Sur les ensembles compacts de fonctions sommable*. Acta Sci.Math. (Szeged) **6** (1933), 136–142.
- [18] G. Schneider and C. Wayne, *Counter-propagating waves on fluid surfaces and the continuum limit of the Fermi-Pasta-Ulam model*. International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999), 390–404, World Sci. Publ., River Edge, NJ, 2000.
- [19] A. Stefanov and P. G. Kevrekidis, *Asymptotic behaviour of small solutions for the discrete nonlinear Schrödinger and Klein-Gordon equations*. Nonlinearity **18** (2005), 1841–1857.
- [20] Y. Tsutsumi, *L^2 -solutions for nonlinear Schrödinger equations and nonlinear groups*. Funkcial. Ekvac. **30** (1987), no. 1, 115–125.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CA 90095, USA
Email address: killip@math.ucla.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, IL 60637, USA
Email address: ouyangzm9386@uchicago.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CA 90095, USA
Email address: visan@math.ucla.edu

DEPARTMENT OF MATHEMATICS, LEHIGH UNIVERSITY
Email address: lew218@lehigh.edu