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Nuclear Physics B 984 (2022) 115947

NUCLEAR PHYSICS B

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Parton model description of quark and antiquark correlators and TMDs

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Received 1 July 2022; received in revised form 4 August 2022; accepted 17 August 2022
Available online 24 August 2022
Editor: Hong-Jian He

Abstract

Model studies play an important role for the understanding and elucidation of the nonperturbative properties of transverse momentum dependent parton distribution functions (TMDs). The parton model is often a helpful framework and starting point for first explorations of TMD properties and the description of deep-inelastic processes in which TMDs can be accessed. Based on a systematic exploration of the parton model concept, we reconcile the claims in literature that there are 2 independent structures in the quark correlator in the parton model vs the claim that there are 3, and explain the underlying assumptions leading to the different conclusions. We also systematically explore the antiquark correlator and, to the best of our knowledge, for the first time derive the model expressions for all T-even leading and subleading antiquark TMDs. We demonstrate the consistency of the framework which can be generalized in future studies for more sophisticated TMD modeling.

1. Introduction

Feynman's intuitive parton model concept [1,2] played an important role in establishing QCD as the theory of strong interactions. In many situations, the parton model can be considered a "zeroth order approximation" to QCD [3,4]. As such it constitutes a valuable starting point for explorations. This was also the case for TMDs. Based on a rigorous TMD factorization and

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https://doi.org/10.1016/j.nuclphysb.2022.115947

evolution framework [4–10], the recent past has witnessed impressive progress in modern phenomenology of deep-inelastic scattering (DIS) processes [11–19]. But the way to this progress was paved by, among others, important phenomenological work based on the "generalized parton model" in works like [20–24].

A systematic exploration of the parton model concept not for the purpose of describing DIS processes but the nonperturbative properties of TMDs *per se* was undertaken in Refs. [25–39], interestingly with conflicting results. Starting from the parton model concept in one approach, for which the name Covariant Parton Model (CMP) has been coined, the nucleon structure is described in terms of two independent covariant functions [25–38]. Starting from the very same parton model concept, the nucleon structure is described in terms of three such independent covariant functions in another parton model framework, namely that of Ref. [39].

While technical details and especially notations may easily differ in independent treatments in literature, one would expect agreement within a given approach on such an important point like the number of linearly independent structures in the quark correlator [38,39] in terms of which TMDs are defined. Surprisingly, there is a disagreement in literature on this fundamental question which remains not understood for more than a decade [33–39].

In this work we will show that both treatments are correct, but based on different assumptions about the state of polarization of a quark in the nucleon. For that we will systematically explore the consequences of the parton model concept for the description of the nucleon structure and show that there are not "different parton model approaches" but only one unifying parton model — in which, however, there is a choice on how to treat quark polarization effects. Our study will establish a bridge also to other works in literature, including applications of the parton model to studies of target mass corrections or weak structure functions [39–42], early studies of quark transverse motion [44,43] and extensions of the parton model concept to the free-quark ensemble model [45], gluon polarization effects [46], or the statistical parton model approach of Refs. [47–51].

The structure of this work is as follows. In Sec. 2 we will review the description of quark and antiquark correlators and TMDs in models with no explicit gauge degrees of freedom and discuss the simplifications compared to QCD. In general, even in simpler non-gauge field theories care might be needed due to complications from UV divergences, but in the parton model also this point is simplified and the correlators are UV-finite. In Sec. 3 we will introduce the parton model concept and explore the nontrivial formal consequences arising from the (free) equations of motion for the quark correlator in massive (Sec. 4) and massless (Sec. 5) case following [39] where, however, only the massless case was considered. In practice quark mass effects are negligible in DIS processes. But the distinction of massive vs massless partons is important when considering polarization effects which will turn out to be the key to understand and reconcile the conflicting results in literature. In Sec. 6, we will study quark TMDs in the parton model with due care to the treatment of polarization effects and reconcile the parton model approaches of [38,39]. In Sec. 7, we will repeat the above program for the antiquark correlator and antiquark TMDs. The quark and antiquark correlators are related to each other in a specific way in field theory. But we will treat the quark and antiquark cases independently, and use their field theoretic connection to demonstrate in Sec. 8 the consistency of the approach. In Sec. 9 we will solve the model and evaluate the parton model expressions reproducing prior results for quark TMDs in literature and presenting new results for antiquark TMDs. Finally, Sec. 10 contains the conclusions.

2. Quark and antiquark correlators and TMDs in quark models

In this section, we will review the properties of the correlators, and the definitions of TMDs. We will specifically indicate the simplifications arising in quark models by which we mean a model or effective theory with quark and antiquark degrees of freedom, but without explicit gauge field degrees of freedom.

2.1. Quark and antiquark correlators

In quark model approaches, the quark and antiquark correlation functions of the nucleon are defined as

$$\Phi^q_{ij}(k,P,S) = \int \frac{\mathrm{d}^4z}{(2\pi)^4}\, \mathrm{e}^{ik\cdot z}\, \langle N|\, \overline{\Psi}^q_j(0)\, \Psi^q_i(z)\, |N\rangle\,, \tag{1a}$$

$$\Phi^{\bar q}_{ij}(k,P,S) = \int \frac{d^4z}{(2\pi)^4}\, \mathrm{e}^{ik\cdot z}\, \langle N|\, \Psi^q_i(0)\, \overline{\Psi}^q_j(z)\, |N\rangle, \tag{1b}$$

where k^μ is the quark 4-momentum and $|N\rangle = |P,S\rangle$ denotes a covariantly normalized nucleon state with 4-momentum P^μ and polarization S^μ with $P^2 = M^2$, $S^2 = -1$, $P\cdot S = 0$. In QCD and in models, the correlators in (1) are connected to each other as $\Phi^{\bar q}_{ij}(k,P,S) = -\,\Phi^q_{ij}(-k,P,S)$. For our purposes it will, however, be convenient not to explore this connection and treat the quark and antiquark cases independently in the next sections, until in Sec. 8 we will come back to this connection and make use of it to demonstrate the consistency of the approach.

In QCD, the quark fields in the correlators (1) are connected by Wilson lines which can be chosen along process-dependent paths dictated by factorization theorems such that the correlators, upon integration over k^- and tracing with the relevant Dirac matrices Γ, yield TMDs describing semi-inclusive DIS, Drell-Yan or other processes [52–60]. In quark models, the Wilson lines are absent which brings simplifications to the structure of the correlators.

One of the simplifications is that the correlators $\Phi^a_{ij}(k,P,S)$ with $a = q,\, \bar q$ have expansions in terms of only 12 Lorentz-invariant amplitudes A^a_i as follows [54]

$$\begin{aligned}
\Phi^a(k,P,S) \,=\, & M A^a_1 + \not{P} A^a_2 + \not{k} A^a_3 + \frac{i}{2M}\,[\not{P},\not{k}]\, A^a_4 + i(k\cdot S)\gamma_5\, A^a_5 \\
& + M \not{S}\gamma_5\, A^a_6 + \frac{(k\cdot S)}{M}\not{P}\gamma_5\, A^a_7 + \frac{(k\cdot S)}{M}\not{k}\gamma_5\, A^a_8 \\
& + \frac{[\not{P},\not{S}]}{2}\gamma_5\, A^a_9 + \frac{[\not{k},\not{S}]}{2}\gamma_5\, A^a_{10} + \frac{(k\cdot S)}{2M^2}[\not{P},\not{k}]\gamma_5\, A^a_{11} \\
& + \frac{1}{M}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu P_\nu k_\rho S_\sigma\, A^a_{12}\,,
\end{aligned} \tag{2}$$

where $\varepsilon^{0123} = 1$. Notice that the correlators depend at most linearly on the nucleon polarization vector S^μ.

In QCD, besides k, P, S the correlators depend on an additional 4-vector n^μ characterizing the lightcone direction of the Wilson-lines (n^μ itself is slightly off-lightcone to regularize rapidity divergences) [4]. The presence of the vector n^μ in QCD makes more Lorentz structures possible in the decomposition (2) which are described in terms of 20 additional amplitudes, often called B^a_i amplitudes [60], which are absent in quark models.

The amplitudes A_i^a in (2) are real functions of the Lorentz scalars $P\cdot k$ and k^2 [53,54]. The amplitudes A_i^a for $i=2,\,3,\,6,\,7,\,8,\,12$ are chiral even, those for $i=1,\,4,\,5,\,9,\,10,\,11$ are chiral odd. Another simplification in quark models is that the T-odd amplitudes A_i^a for $i=4,\,5,\,12$ vanish [61]. We include them in (2) for completeness.

The correlators (1) can be used to define "fully unintegrated" parton distributions which have important applications [62–65]. In QCD the correlators contain divergences which simplifies in the parton model as will follow up in the next section where we will introduce TMDs.

2.2. Definition of TMDs

The TMDs of quarks and antiquarks are defined in terms of the correlators as follows. Introducing the lightcone coordinates, $k^\mu=(k^+,k^-,\vec{k}_T)$ with $k^\pm=\frac{1}{\sqrt{2}}(k^0\pm k^1)$ and analogously for other vectors, it is convenient to define

$$\Phi^{a[\Gamma]}\equiv\Phi^{a[\Gamma]}(x,\vec{k}_T,P,S)=\int\!\!\!\int dk^+dk^-\,\delta(k^+-xP^+)\,\frac{1}{2}\mathrm{tr}\Big[\Phi^a(k,P,S)\,\Gamma\Big]. \tag{3}$$

The leading twist TMDs are defined as

$$\Phi^{a[\gamma^+]}=\zeta_V\bigg[f_1^a-\frac{\varepsilon^{jk}k_T^jS_T^k}{M}\,f_{1T}^{\perp a}\bigg], \tag{4a}$$

$$\Phi^{a[\gamma^+\gamma_5]}=\zeta_A\bigg[S_L\,g_1^a+\frac{\vec{k}_T\cdot\vec{S}_T}{M}\,g_{1T}^{\perp a}\bigg], \tag{4b}$$

$$\Phi^{a[i\sigma^{j+}\gamma_5]}=\zeta_T\bigg[S_T^j\,h_1^a+S_L\,\frac{k_T^j}{M}\,h_{1L}^{\perp a}+\frac{\kappa^{jk}S_T^k}{M^2}\,h_{1T}^{\perp a}+\frac{\varepsilon^{jk}k_T^k}{M}\,h_1^{\perp a}\bigg], \tag{4c}$$

and the twist-3 quark TMDs are given by

$$\Phi^{a[\mathbb{1}]}=\zeta_S\,\frac{M}{P^+}\bigg[e^a-\frac{\varepsilon^{jk}k_T^jS_T^k}{M}\,e_T^{\perp a}\bigg], \tag{4d}$$

$$\Phi^{a[i\gamma^5]}=\zeta_P\,\frac{M}{P^+}\bigg[S_L\,e_L^a+\frac{\vec{k}_T\cdot\vec{S}_T}{M}\,e_T^a\bigg], \tag{4e}$$

$$\Phi^{a[\gamma^j]}=\zeta_V\,\frac{M}{P^+}\bigg[\frac{k_T^j}{M}\,f^{\perp a}+\varepsilon^{jk}S_T^k\,f_T^a+S_L\,\frac{\varepsilon^{jk}k_T^k}{M}\,f_L^{\perp a}-\frac{\kappa^{jk}\varepsilon^{kl}S_T^l}{M^2}\,f_T^{\perp a}\bigg], \tag{4f}$$

$$\Phi^{a[\gamma^j\gamma^5]}=\zeta_A\,\frac{M}{P^+}\bigg[S_T^j\,g_T^a+S_L\,\frac{k_T^j}{M}\,g_L^{\perp a}+\frac{\kappa^{jk}S_T^k}{M^2}\,g_T^{\perp a}+\frac{\varepsilon^{jk}k_T^k}{M}\,g^{\perp a}\bigg], \tag{4g}$$

$$\Phi^{a[i\sigma^{jk}\gamma^5]}=\zeta_T\,\frac{M}{P^+}\bigg[\frac{S_T^jk_T^k-S_T^kk_T^j}{M}\,h_T^{\perp a}-\varepsilon^{jk}\,h^a\bigg], \tag{4h}$$

$$\Phi^{a[i\sigma^{+-}\gamma^5]}=\zeta_T\,\frac{M}{P^+}\bigg[S_L\,h_L^a+\frac{\vec{k}_T\cdot\vec{S}_T}{M}\,h_T^a\bigg]. \tag{4i}$$

The spatial indices j, k are transverse with respect to the lightcone which is chosen along 0- and 1-directions, and we defined $\kappa^{jk}=(k_T^jk_T^k-\frac{1}{2}\delta^{jk}\vec{k}_T^2)$, and $\varepsilon^{23}=-\varepsilon^{32}=1$ and zero else.

The T-even TMDs are highlighted in blue color and can be computed in quark models. The T-odd TMDs highlighted in red color require explicit gauge field degrees of freedom, and are not studied in this work. The factors

$$\zeta_V = \zeta_T = \zeta_P = +1 \quad \text{for} \quad a = q, \bar{q},$$
$$\zeta_S = \zeta_A = \begin{cases} +1 & \text{for} \quad a = q, \\ -1 & \text{for} \quad a = \bar{q}, \end{cases} \tag{4j}$$

reflect the different C-parities of the quark bilinear operators $\bar{\Psi}\Gamma\Psi$ which are even in the vector (V), tensor (T) and pseudo-scalar (P) cases $\Gamma = \gamma^\mu,\ i\sigma^{\mu\nu}\gamma_5,\ i\gamma_5$, and odd in the scalar (S) and axial-vector (A) cases $\Gamma = \mathbb{1},\ \gamma^\mu\gamma_5$. The arguments of TMDs in (4) are omitted for brevity with the understanding that $f_1^a = f_1^a(x, k_T)$ where $k_T = |\vec{k}_T|$, etc.

We remark that in QCD factorization, TMDs depend on two scales commonly denoted as μ^2 (renormalization scale) and ζ (scale at which lightcone divergences are regulated). In general, also in models divergences may occur and require careful treatment [66]. But in the parton model, the correlators and TMDs are finite, and we will throughout refrain from indicating the dependence on the scales μ^2, ζ and comment on them when necessary. Because of these simplifications in contrast to QCD, in the parton model TMDs and colinear parton distribution functions (PDFs) are simply related as, e.g., $f_1^a(x) = \int d^2k_T\, f_1^a(x, k_T)$ where the integration over k_T is finite.

In this work, we will focus on T-even TMDs which are expressed in terms of the A_i^q amplitudes as follows

$$f_1^q(x, k_T) = 2P^+ \int dk^- (A_2^q + x A_3^q), \tag{5a}$$
$$g_1^q(x, k_T) = 2P^+ \int dk^- \left(-A_6^q - \frac{P\cdot k - M^2 x}{M^2} (A_7^q + x A_8^q) \right), \tag{5b}$$
$$g_{1T}^{\perp q}(x, k_T) = 2P^+ \int dk^- (A_7^q + x A_8^q), \tag{5c}$$
$$h_1^q(x, k_T) = 2P^+ \int dk^- \left(-A_9^q - x A_{10}^q + \frac{\vec{k}_T^2}{2M^2} A_{11}^q \right), \tag{5d}$$
$$h_{1L}^{\perp q}(x, k_T) = 2P^+ \int dk^- \left(A_{10}^q - \frac{P\cdot k - M^2 x}{M^2} A_{11}^q \right), \tag{5e}$$
$$h_{1T}^{\perp q}(x, k_T) = 2P^+ \int dk^- A_{11}^q, \tag{5f}$$
$$e^q(x, k_T) = 2P^+ \int dk^- A_1^q, \tag{5g}$$
$$f^{\perp q}(x, k_T) = 2P^+ \int dk^- A_3^q, \tag{5h}$$
$$g_T^q(x, k_T) = 2P^+ \int dk^- \left(-A_6^q + \frac{\vec{k}_T^2}{2M^2} A_8^q \right), \tag{5i}$$
$$g_L^{\perp q}(x, k_T) = 2P^+ \int dk^- \left(-\frac{P\cdot k - M^2 x}{M^2} A_8^q \right), \tag{5j}$$

$$g_T^{\perp q}(x,k_T) = 2P^+\int dk^- A_8^q\,, \tag{5k}$$

$$h_L^q(x,k_T) = 2P^+\int dk^-\left(-A_9^q - \frac{P\cdot k}{M^2}\,A_{10}^q + \left(\frac{P\cdot k - M^2x}{M^2}\right)^2 A_{11}^q\right), \tag{5l}$$

$$h_T^q(x,k_T) = 2P^+\int dk^-\left(-\frac{P\cdot k - M^2x}{M^2}\,A_{11}^q\right), \tag{5m}$$

$$h_T^{\perp q}(x,k_T) = 2P^+\int dk^-(-A_{10}^q)\,, \tag{5n}$$

where it is understood that $k^+ = x\,P^+$ is fixed. We remind that these expressions are valid in quark models. In QCD also the B_i^q amplitudes contribute, see for instance Ref. [67]. The anti-quark TMDs are given by

$$f_1^{\bar q}(x,k_T) = 2P^+\int dk^-\left(A_2^{\bar q} + xA_3^{\bar q}\right), \tag{6a}$$

$$g_1^{\bar q}(x,k_T) = 2P^+\int dk^-\left(A_6^{\bar q} + \frac{P\cdot k - M^2x}{M^2}(A_7^{\bar q} + xA_8^{\bar q})\right), \tag{6b}$$

$$g_{1T}^{\perp\bar q}(x,k_T) = 2P^+\int dk^-\left(-A_7^{\bar q} - xA_8^{\bar q}\right), \tag{6c}$$

$$h_1^{\bar q}(x,k_T) = 2P^+\int dk^-\left(-A_9^{\bar q} - xA_{10}^{\bar q} + \frac{\vec k_T^{\,2}}{2M^2}\,A_{11}^{\bar q}\right), \tag{6d}$$

$$h_{1L}^{\perp\bar q}(x,k_T) = 2P^+\int dk^-\left(A_{10}^{\bar q} - \frac{P\cdot k - M^2x}{M^2}\,A_{11}^{\bar q}\right), \tag{6e}$$

$$h_{1T}^{\perp\bar q}(x,k_T) = 2P^+\int dk^- A_{11}^{\bar q}\,, \tag{6f}$$

$$e^{\bar q}(x,k_T) = 2P^+\int dk^-\left(-A_1^{\bar q}\right), \tag{6g}$$

$$f^{\perp\bar q}(x,k_T) = 2P^+\int dk^- A_3^{\bar q}\,, \tag{6h}$$

$$g_T^{\bar q}(x,k_T) = 2P^+\int dk^-\left(A_6^{\bar q} - \frac{\vec k_T^{\,2}}{2M^2}\,A_8^{\bar q}\right), \tag{6i}$$

$$g_L^{\perp\bar q}(x,k_T) = 2P^+\int dk^-\left(\frac{P\cdot k - M^2x}{M^2}\,A_8^{\bar q}\right), \tag{6j}$$

$$g_T^{\perp\bar q}(x,k_T) = 2P^+\int dk^-\left(-A_8^{\bar q}\right), \tag{6k}$$

$$h_L^{\bar q}(x,k_T) = 2P^+\int dk^-\left(-A_9^{\bar q} - \frac{P\cdot k}{M^2}\,A_{10}^{\bar q} + \left(\frac{P\cdot k - M^2x}{M^2}\right)^2 A_{11}^{\bar q}\right), \tag{6l}$$

$$h_T^{\perp\bar q}(x,k_T) = 2P^+\int dk^-\left(-A_{10}^{\bar q}\right), \tag{6m}$$

$$h_T^{\bar q}(x,k_T) = 2P^+\int dk^-\left(-\frac{P\cdot k - M^2x}{M^2}\,A_{11}^{\bar q}\right). \tag{6n}$$

3. Constraints on amplitudes from equations of motion in parton model

In this section we discuss how the equations of motion relate the amplitudes in the parton model. A similar analysis was presented in [39] for massless quarks. Here we keep track of mass terms and extend the analysis to antiquarks.

3.1. Quark case

In the parton model the quark fields satisfy the Dirac equation $(i\partial\!\!\!/ - m_q)\,\Psi^q(z) = 0$. Writing out all Dirac indices and performing an integration by parts, we obtain

$$\begin{aligned}
&\int \frac{\mathrm{d}^4 z}{(2\pi)^4}\, \mathrm{e}^{ikz}\, \langle N|\, \overline{\Psi}^q_j(0)\, \Gamma_{jl} \Big[(i\overrightarrow{\partial\!\!\!/} - m_q)_{li} \Psi^q_i(z) \Big] |N\rangle \\
&\quad = \Gamma_{jl} (k\!\!\!/ - m_q)_{li} \int \frac{\mathrm{d}^4 z}{(2\pi)^4}\, \mathrm{e}^{ikz}\, \langle N|\, \overline{\Psi}^q_j(0) \Psi^q_i(z)\, |N\rangle \\
&\quad = \Gamma_{jl} (k\!\!\!/ - m_q)_{li} \Phi^q_{ij}(k,P,S) = \mathrm{tr}\Big[\Gamma (k\!\!\!/ - m_q) \Phi^q(k,P,S) \Big] = 0\,,
\end{aligned} \tag{7}$$

where Γ can be any Dirac matrix. It is convenient to define identities for the quark correlator (2) as follows

$$\mathrm{Tr}^q[\Gamma] \equiv \frac{1}{4}\, \mathrm{tr}\Big[(k\!\!\!/ - m_q)\, \Phi^q(k,P,S) \Gamma \Big] = 0\,. \tag{8}$$

Exploring these identities for $\Gamma = \mathbb{1},\ \gamma_5,\ \gamma^\mu,\ \gamma^\mu\gamma_5,\ i\,\sigma^{\mu\nu}\gamma_5$ yields relations among the amplitudes in the parton model. It is instructive to show the derivation of these relations in detail. Let us consider first $\Gamma = \gamma^\mu$ which yields

$$\begin{aligned}
\mathrm{Tr}^q[\gamma^\mu] &= k^\mu \left\{ M A^q_1 - m_q A^q_3 + i\,\frac{(P\cdot k)}{M}\, A^q_4 \right\} - P^\mu m_q \left\{ A^q_2 + i\,\frac{m_q}{M}\, A^q_4 \right\} \\
&\quad + \varepsilon^{\mu\alpha\beta\gamma} k_\alpha P_\beta S_\gamma \left\{ i A^q_9 - \frac{m_q}{M}\, A^q_{12} \right\} = 0\,.
\end{aligned} \tag{9}$$

As the four-vectors k_μ, P_μ, S_μ and $\varepsilon_{\mu\rho\sigma\tau} k^\rho P^\sigma S^\tau$ are linearly independent, each of the expressions in the curly brackets must vanish separately. Since the A_i's are real, in each of the curly brackets the real and the imaginary parts must vanish separately. In this way, we find the relations

$$A^q_1 = \frac{m_q}{M}\, A^q_3, \quad A^q_2 = 0, \quad A^q_4 = 0, \quad A^q_9 = 0, \quad A^q_{12} = 0. \tag{10}$$

Exploring $\mathrm{Tr}^q[\mathbb{1}]$ yields no new information beyond what we found in (10). Considering $\Gamma = \gamma^\mu\gamma_5$ yields

$$\begin{aligned}
\mathrm{Tr}^q[\gamma^\mu\gamma_5] &= k^\mu (k\cdot S) \left\{ -i A^q_5 + \frac{m_q}{M}\, A^q_8 + A^q_{10} - \frac{(P\cdot k)}{M^2}\, A^q_{11} \right\} \\
&\quad + P^\mu (k\cdot S) \left\{ \frac{m_q}{M} A^q_7 + A^q_9 + \frac{k^2}{M^2}\, A^q_{11} \right\} \\
&\quad + S^\mu \left\{ m_q M A^q_6 - (P\cdot k) A^q_9 - k^2 A^q_{10} \right\} \\
&\quad = 0\,.
\end{aligned} \tag{11}$$

We again explore that the A_i^q's are real and k_μ, P_μ, S_μ, $\varepsilon_{\mu\rho\sigma\tau}k^\rho P^\sigma S^\tau$ are linearly independent such that the real and imaginary parts in each curly bracket in (11) vanish. Considering that $k^2 = m_q^2$ and using $A_9^q = 0$ from (10) we obtain

$$A_5^q = 0, \quad A_6^q = \frac{m_q}{M} A_{10}^q, \quad A_7^q = -\frac{m_q}{M} A_{11}^q, \quad A_{10}^q = \frac{(P\cdot k)}{M^2} A_{11}^q - \frac{m_q}{M} A_8^q . \tag{12}$$

Considering $\mathrm{Tr}^q[\gamma_5]$ or $\mathrm{Tr}^q[i\sigma^{\mu\nu}\gamma_5]$ gives no new information beyond (12).

Several comments are in order. The T-odd amplitudes A_4^q, A_5^q, A_{12}^q vanish in the parton model which is expected as we deal with a quark model without explicit gauge field degrees of freedom [61]. Interestingly, in the parton model also the T-even amplitudes A_2^q and A_9^q vanish. In QCD, these amplitudes receive contributions from "genuine twist-3" quark-gluon correlators and are non-zero. In the parton model, the amplitudes A_1^q, A_6^q, A_7^q are proportional to current quark masses. In QCD, these amplitudes contain, besides mass terms, also contributions from quark-gluon correlators.

For our purposes it is important to notice that, after exploring the equations of motion, all non-zero amplitudes are related in one or another way to A_3^q, A_8^q, A_{11}^q, i.e. to one unpolarized amplitude (A_3^q), one chiral even polarized amplitude (A_8^q), and one chiral odd polarized amplitude (A_{11}^q).

3.2. Antiquark case

In the antiquark case, we start from the free Dirac equation $\bar\Psi^q(z)(i\overleftarrow{\not\partial} + m_q) = 0$ where the arrow indicates which field is differentiated. Proceeding similarly to the quark case, we have

$$\begin{aligned}
&\int \frac{d^4z}{(2\pi)^4}\, \mathrm{e}^{ik\cdot z}\, \langle N|\, \Gamma_{li}\, \Psi_i^q(0) \left[\overline{\Psi}_j^q(z)(i\overleftarrow{\not\partial} + m_q)_{jl}\right]|N\rangle \\
&\quad = \Gamma_{li} \int \frac{d^4z}{(2\pi)^4}\, \mathrm{e}^{ik\cdot z}\, \langle N|\Psi_i^q(0)\, \overline{\Psi}_j^q(z)|N\rangle\, (\not k + m_q)_{jl} \\
&\quad = \Gamma_{li}\Phi_{ij}^{\bar q}(k,P,S)(\not k + m_q)_{jl} = \mathrm{tr}\Big[\Gamma\Phi^{\bar q}(k,P,S)(\not k + m_q)\Big] = 0 .
\end{aligned} \tag{13}$$

It is convenient to introduce the notation for the identities

$$\mathrm{Tr}^{\bar q}[\Gamma] \equiv \frac{1}{4}\,\mathrm{tr}\Big[\Gamma\,\Phi^{\bar q}(k,P,S)\,(\not k + m_q)\Big] = 0 , \tag{14}$$

where Γ can be again any Dirac matrix. Proceeding analog to the quark correlator case, we obtain from (14) the following relations among the antiquark amplitudes,

$$A_1^{\bar q} = -\frac{m_q}{M} A_3^{\bar q}, \quad A_2^{\bar q} = 0, \quad A_4^{\bar q} = 0, \quad A_5^{\bar q} = 0, \quad A_6^{\bar q} = -\frac{m_q}{M} A_{10}^{\bar q}, \tag{15}$$

$$A_7^{\bar q} = \frac{m_q}{M} A_{11}^{\bar q}, \quad A_9^{\bar q} = 0, \quad A_{10}^{\bar q} = \frac{(P\cdot k)}{M^2} A_{11}^{\bar q} + \frac{m_q}{M} A_8^{\bar q}, \quad A_{12}^{\bar q} = 0. \tag{16}$$

Analogously to the quark case, also the T-odd antiquark amplitudes $A_4^{\bar q}$, $A_5^{\bar q}$, $A_{12}^{\bar q}$ vanish and so do the T-even amplitudes $A_2^{\bar q}$ and $A_9^{\bar q}$. All non-zero amplitudes are related in one or another way to three amplitudes $A_3^{\bar q}$, $A_8^{\bar q}$, $A_{11}^{\bar q}$ which remain unconstrained by the equations of motion. The relations of the antiquark amplitudes in (15), (16) resemble those in the quark case in (10), (12) except that the current quark mass m_q enters with opposite sign.

4. Quark correlator in the massive case

Inserting the relations in (10), (12) in the quark correlator in (2) we obtain

$$\Phi^q(k,P,S) = \Phi^q(k,P,S)_{\rm unp} + \Phi^q(k,P,S)_{\rm pol} \tag{17}$$

where

$$\begin{aligned}
\Phi^q(k,P,S)_{\rm unp} &= (\not k + m_q)\, A^q_{\rm unp} \quad \text{with} \quad A^q_{\rm unp} = A^q_3\,,\\
\Phi^q(k,P,S)_{\rm pol} &= (\not k + m_q)\gamma_5 \bigg\{ \not P\, \frac{(k\cdot S)}{M^2} A^q_{11} - \not S\, \frac{(P\cdot k)\, A^q_{11} - m_q M\, A^q_8}{M^2} \\
&\qquad + \frac{(k\cdot S)}{M}\, A^q_8 \bigg\}\,.
\end{aligned} \tag{18}$$

In the parton model $\not k \not k = k^2 = m_q^2$ and the last term in the curly bracket of (18) can be written for $m_q \neq 0$ as

$$(\not k + m_q)\gamma_5\, \frac{(k\cdot S)}{M}\, A^q_8 = -(\not k + m_q)\gamma_5\, \frac{\not k}{m_q}\, \frac{(k\cdot S)}{M}\, A^q_8\,. \tag{19}$$

Inserting the relation (19) into (18) allows us to write the polarization-dependent part of the correlator as

$$\Phi^q(k,P,S)_{\rm pol} = (\not k + m_q)\gamma_5\, \not w\, A^q_{\rm pol} \tag{20}$$

where we introduce the polarized amplitude $A^q_{\rm pol}$ and the axial 4-vector w^μ_q defined as

$$A^q_{\rm pol} = -\,\frac{(P\cdot k)\, A^q_{11} - m_q M\, A^q_8}{M^2}\,, \tag{21a}$$

$$w^\mu_q = S^\mu - P^\mu \frac{(k\cdot S)\, A^q_{11}}{(P\cdot k)\, A^q_{11} - m_q M\, A^q_8} + k^\mu \frac{M}{m_q}\, \frac{(k\cdot S) A^q_8}{(P\cdot k)\, A^q_{11} - m_q M\, A^q_8}\,. \tag{21b}$$

We note that for w^μ_q to be well-defined in (21) it must be $m_q \neq 0$ (which is the case in this section) and the condition $(P\cdot k)\, A^q_{11} - m_q M\, A^q_8 \neq 0$ must hold. We will follow up on this shortly.

4.1. Quark polarization vector

The 4-vector w^μ_q has the following important properties. It has the transformation properties of a polarization vector, i.e. of an axial 4-vector, and satisfies the condition

$$w_q \cdot k = 0\,. \tag{22}$$

These 2 properties are necessary conditions for w^μ_q to be a candidate expression for a quark polarization vector. Remarkably, the condition $w_q \cdot k = 0$ holds for any A^q_8 and A^q_{11} without imposing a relation between these amplitudes.

In order to be a quark polarization vector, w^μ_q must in addition satisfy the requirements

$$\begin{aligned}
&\text{Condition (A)} \quad -1 < w_q^2 < 0 \qquad \text{mixed-spin state,}\\
&\text{Condition (B)} \quad\quad\; w_q^2 = -1 \qquad\; \text{pure-spin state.}
\end{aligned}$$

The square of the quark polarization vector is given by

$$w_q^2 = S^2 + \frac{(k\cdot S)^2 M^2}{[(P\cdot k)\, A_{11}^q - m_q M\, A_8^q]^2}\left((A_{11}^q)^2 - (A_8^q)^2\right). \tag{23}$$

In the following, we will explore both possibilities (A) to which we will refer as the mixed-spin version of the model, and (B) to which we will refer as the pure-spin version of the model.

4.2. Mixed-spin model

Let us first discuss the mixed-spin version of the model which requires $-1 < \omega_q^2 < 0$. The condition $\omega_q^2 > -1$ is equivalent to $|A_{11}^q| > |A_8^q|$. The condition $\omega_q^2 < 0$ is satisfied as long as $(P\cdot k\, A_{11}^q - m_q M A_8^q)^2 > 0$ which is always the case because we had to assume in (21) that $(P\cdot k)A_{11}^q - m_q M A_8^q \neq 0$. Thus, in the mixed-spin case, besides the inequality $|A_{11}^q| > |A_8^q|$, the amplitudes A_8^q and A_{11}^q remain unrelated.

In this approach, the quark correlator is described in terms of 3 independent amplitudes, namely A_3^q, A_8^q, A_{11}^q, i.e. in terms of one unpolarized amplitude and two polarized amplitudes: one chiral even and one chiral odd.

4.3. Pure-spin state model

If we demand the quarks to be in a pure-spin state, w_q^μ must be normalized as $w_q^2 = -1$ in (23). This means that the second term on the right-hand-side of (23) must vanish which implies the following condition

$$w_q^2 = -1 \quad \Leftrightarrow \quad A_{11}^q = \pm A_8^q\,. \tag{24a}$$

The 2 solutions in (24a) lead to 2 different solutions for $A_{\rm pol\pm}^q$ and $w_{q\pm}^\mu$. One way of writing the pertinent solutions consists in eliminating the chiral odd amplitude A_{11}^q which yields

$$A_{\rm pol\pm}^q = -\frac{(\pm P\cdot k) - m_q M}{M^2}\, A_8^q\,, \tag{24b}$$

$$w_{q\pm}^\mu = S^\mu - \frac{(\pm k\cdot S)}{(\pm P\cdot k) - m_q M}\, P^\mu + \frac{M}{m_q}\,\frac{(\pm k\cdot S)}{(\pm P\cdot k) - m_q M}\,(\pm k^\mu)\,. \tag{24c}$$

This is not the most economic notation, but we have chosen it to avoid the usage of $\mp$ and in this way the $\pm$ always appear together with k. We will show below in Sec. 6 that one of the 2 solutions is physical, and the other one is unphysical. Before that, however, we will discuss the massless case.

5. Quark correlator in the massless case

When $m_q = 0$, the analysis of the previous section cannot be carried out because for massless quarks a polarization vector w_q^μ cannot be defined, see e.g. [68,4]. This can be seen directly in (24b) where for $m_q \to 0$ one would encounter a $1/m_q$-singularity. In the case $m_q = 0$, one has to proceed in a different way as shown below.

The relations derived from the equations of motion in (10), (12) are of course valid also for $m_q = 0$. Inserting these relations for $m_q = 0$ in the quark correlator in (2) yields an unpolarized and a polarized contribution to the quark correlator in (17) which are given by

$$\Phi^q(k,P,S)_{\rm unp} = \not{k}\, A^q_{\rm unp}\,,$$
$$\Phi^q(k,P,S)_{\rm pol} = \not{k}\,\gamma_5 \Big\{ -\lambda\, + \not{b}_T \Big\}\, A^q_{\rm pol} \tag{25}$$

where, assuming $A^q_8 \neq 0$, we defined

$$A^q_{\rm unp} = A^q_3,$$
$$A^q_{\rm pol} = -\,\frac{(P\cdot k)}{M^2}\, A^q_8\,, \quad \lambda = \frac{M(k\cdot S)}{(P\cdot k)}\,, \quad b^\mu_T = \left(S^\mu - P^\mu\, \frac{(k\cdot S)}{(P\cdot k)} \right) \frac{A^q_{11}}{A^q_8}\,. \tag{26}$$

The quantities λ and b^μ_T are defined in (26) following the common conventions in literature, see for instance (2.3.8) in Ref. [68] or Appendix A of Ref. [4]. Their properties are discussed in the next section.

5.1. Spin density matrix of massless quarks

The quantities λ and b^μ_T in (26) have the following properties.

(i) In the high-energy limit when the nucleon moves very fast, i.e. $P^\mu \to \infty$, and is polarized along its direction of motion, the nucleon polarization vector is given by $S^\mu = \lambda_N P^\mu/M + \ldots$ where $\lambda_N = \pm 1$ is (twice) the helicity of the nucleon [68] (see Sec. 9.5 for a reminder) and the dots indicate terms suppressed in the high-energy limit. In such an infinite momentum frame, the quantity λ defined in (26) is given by $\lambda = \lambda_N$, i.e. depending on the sign of $A^q_{\rm pol}$, the massless quark has the same or opposite helicity as the nucleon. Thus in the infinite momentum frame, λ has the expected intuitive interpretation as (twice) the quark helicity.

(ii) In the general case, a quark may of course have transverse polarization. In the massless case, this is described by the vector b^μ_T which is transverse with respect to the quark momentum k^μ. It is important to stress that b^μ_T introduced in (26) has the property $k\cdot b_T = 0$ for any A^q_8 and A^q_{11}.

(iii) Also in the massless case, we can distinguish pure-spin and mixed-spin states. They are defined as follows:

$$\begin{aligned} &\text{Condition (A)} \quad -1 < b_T^2 - \lambda^2 < 0 \qquad \text{mixed-spin state,} \\ &\text{Condition (B)} \qquad\quad b_T^2 - \lambda^2 = -1 \qquad \text{pure-spin state.} \end{aligned} \tag{27}$$

5.2. Massless mixed-spin model

If one chooses to work with massless quarks in a mixed-spin state, then $-1 < b_T^2 - \lambda^2 < 0$ must hold. The condition $b_T^2 - \lambda^2 > -1$ will always be satisfied as long as $|A^q_{11}| > |A^q_8|$. The condition $b_T^2 - \lambda^2 < 0$ is trivial and always satisfied without imposing any new requirements. Recalling that we had to exclude the case $A^q_8 = 0$ from the very beginning, we conclude that the mixed-spin parton model is consistently defined provided $0 < |A^q_8| < |A^q_{11}|$. Similarly to the massive case, in this approach the quark correlator is described in terms of 3 independent amplitudes, namely the unpolarized A^q_3, chiral even polarized A^q_8, and chiral odd polarized A^q_{11} amplitude. In practice, these 3 independent amplitudes can be determined from, for instance $f^q_1(x)$, $g^q_1(x)$, $h^q_1(x)$ at some scale which is part of the model. This corresponds to the parton model version discussed in Ref. [39].

5.3. Massless pure-spin model

If we choose to work with a parton model of massless quarks in a pure-spin state, then the condition (B) in (27) implies

$$b_T^2 - \lambda^2 = -\frac{M^2(k\cdot S)^2}{(P\cdot k)^2}\left(1 - \frac{(A_{11}^q)^2}{(A_8^q)^2}\right) + S^2 = -1 \quad \Leftrightarrow \quad A_8^q = \pm A_{11}^q\,. \tag{28}$$

Also in the massless case, we have two choices for a pure-spin state parton model, one of which will turn out to be physical and the other unphysical. We will discuss this in the next section.

6. Quark TMDs

In this section we will discuss the results for TMDs following from the quark correlator in the parton model derived in the previous sections. We will keep $m_q \neq 0$ and comment on the massless case where necessary.

6.1. Unpolarized TMDs

The results for the T-even unpolarized TMDs are of course independent of the (mixed-spin or pure-spin state) polarization of partons, and can be uniquely expressed in terms of the amplitude A_3^q in the following way

$$f_1^q(x,k_T) = 2P^+\int dk^-\Big[x A_3^q(P\cdot k)\Big]_{k^+=xP^+}, \tag{29a}$$

$$f^{\perp q}(x,k_T) = 2P^+\int dk^-\Big[A_3^q(P\cdot k)\Big]_{k^+=xP^+}, \tag{29b}$$

$$e^q(x,k_T) = 2P^+\int dk^-\Big[\frac{m_q}{M}\,A_3^q(P\cdot k)\Big]_{k^+=xP^+}. \tag{29c}$$

Here and in the following we shall abbreviate the notation for the amplitudes as $A_3^q(P\cdot k,\, k^2) = A_3^q(P\cdot k)$ because in the parton model $k^2 = m_q^2$ is fixed (but we will reinstate the notation $A_3^q(P\cdot k,\, k^2)$ when it will become important to stress the specific k^2 dependence in the parton model).

We see that $e^q(x,k_T)$ is proportional to the current quark mass and becomes zero if one considers the parton model with massless quarks. This is the only TMD in the parton model with this property.

6.2. Polarized TMDs for partons in mixed-spin state

The T-even chiral even polarized TMDs in the mixed-spin state parton model with massive quarks are expressed in terms of the chiral even amplitude A_8^q and the chiral odd amplitude A_{11}^q entering as current quark mass effect. The model expressions are given by

$$g_1^q(x,k_T) = 2P^+\int dk^-\left[\frac{x^2M^2 - x\,P\cdot k + m_q^2}{M^2}\,A_8^q(P\cdot k) - \frac{m_q}{M}\,x\,A_{11}^q(P\cdot k)\right]_{k^+=xP^+}, \tag{30a}$$

$$g_{1T}^{\perp q}(x,k_T) = 2P^+ \int dk^- \left[x A_8^q(P\cdot k) - \frac{m_q}{M} A_{11}^q(P\cdot k) \right]_{k^+=xP^+}, \tag{30b}$$

$$g_T^q(x,k_T) = 2P^+ \int dk^- \left[\frac{\vec{k}_T^2 + 2m_q^2}{2M^2}\, A_8^q(P\cdot k) - \frac{m_q}{M}\,\frac{P\cdot k}{M^2}\, A_{11}^q(P\cdot k) \right]_{k^+=xP^+}, \tag{30c}$$

$$g_L^{\perp q}(x,k_T) = 2P^+ \int dk^- \left[\frac{x\,M^2 - P\cdot k}{M^2}\, A_8^q(P\cdot k) \right]_{k^+=xP^+}, \tag{30d}$$

$$g_T^{\perp q}(x,k_T) = 2P^+ \int dk^- \left[A_8^q(P\cdot k) \right]_{k^+=xP^+}. \tag{30e}$$

For the T-even chiral odd polarized TMDs the situation is opposite: they are given in terms of the chiral odd amplitude A_{11}^q while the chiral even amplitude A_8^q enters as current quark mass effect. The model expressions read

$$h_1^q(x,k_T) = 2P^+ \int dk^- \left[\frac{\vec{k}_T^2 - 2x\,P\cdot k}{2M^2}\, A_{11}^q(P\cdot k) + x\,\frac{m_q}{M}\, A_8^q(P\cdot k) \right]_{k^+=xP^+}, \tag{31a}$$

$$h_{1L}^{\perp q}(x,k_T) = 2P^+ \int dk^- \left[x A_{11}^q(P\cdot k) - \frac{m_q}{M} A_8^q(P\cdot k) \right]_{k^+=xP^+}, \tag{31b}$$

$$h_{1T}^{\perp q}(x,k_T) = 2P^+ \int dk^- \left[A_{11}^q(P\cdot k) \right]_{k^+=xP^+}, \tag{31c}$$

$$h_L^q(x,k_T) = 2P^+ \int dk^- \left[\frac{x^2M^2 - 2x\,P\cdot k}{M^2}\, A_{11}^q(P\cdot k) + \frac{m_q}{M}\,\frac{P\cdot k}{M^2}\, A_8^q(P\cdot k) \right]_{k^+=xP^+}, \tag{31d}$$

$$h_T^q(x,k_T) = 2P^+ \int dk^- \left[\frac{x\,M^2 - P\cdot k}{M^2}\, A_{11}^q(P\cdot k) \right]_{k^+=xP^+}, \tag{31e}$$

$$h_T^{\perp q}(x,k_T) = 2P^+ \int dk^- \left[-\frac{P\cdot k}{M^2} A_{11}^q(P\cdot k) + \frac{m_q}{M} A_8^q(P\cdot k) \right]_{k^+=xP^+}. \tag{31f}$$

For massless partons the situation simplifies. Then all chiral even polarized TMDs are expressed in terms of the amplitude A_8^q, while all chiral odd polarized TMDs are expressed in terms of the amplitude A_{11}^q.

6.3. Polarized TMDs for partons in pure-spin state

In the pure-spin state parton model all polarized (chiral even *and* chiral odd) TMDs, can be expressed in terms of one single amplitude. We choose A_8^q for that, and replace A_{11}^q by $A_{11}^q = \pm A_8^q$. In the following the upper (lower) sign in $(\pm)$ is associated with the upper (lower) sign in the two solutions $w_\pm^q$ for the quark polarization vector and polarized amplitude $A_{\text{pol}\pm}^q$ which correspond to the choices $A_{11}^q = \pm A_8^q$. For massive partons in a pure-spin state, the model expressions for the chiral even polarized TMDs are given by

$$g_1^q(x,k_T) = 2P^+ \int dk^- \left[\frac{x^2M^2 - x\,P\cdot k + m_q^2}{M^2}\, A_8^q(P\cdot k) - \frac{m_q}{M}\, x \left[\pm A_8^q(P\cdot k) \right] \right]_{k^+ = x P^+}, \tag{32a}$$

$$g_{1T}^{\perp q}(x,k_T) = 2P^+ \int dk^- \left[x\, A_8^q(P\cdot k) - \frac{m_q}{M} \left[\pm A_8^q(P\cdot k) \right] \right]_{k^+ = x P^+}, \tag{32b}$$

$$g_T^q(x,k_T) = 2P^+ \int dk^- \left[\frac{\vec{k}_T^{\,2} + 2m_q^2}{2M^2}\, A_8^q(P\cdot k) - \frac{m_q}{M} \left[\pm \frac{P\cdot k}{M^2}\, A_8^q(P\cdot k) \right] \right]_{k^+ = x P^+}, \tag{32c}$$

where we do not quote the expressions for $g_L^{\perp q}(x,k_T)$ and $g_T^{\perp q}(x,k_T)$ since they do not depend on A_{11}^q and are the same as in (30d), (30e). Since $m_q \ll M$, the difference between the two solutions for $w_\pm^q$ and $A_{\mathrm{pol}\pm}^q$ is in practice numerically negligibly small, and disappears in the massless case. In fact, in the massless case the mixed-spin and pure-spin versions of the parton model give exactly the same results for the chiral even polarized TMDs.

The model expressions for the polarized chiral odd TMDs are given by

$$h_1^q(x,k_T) = 2P^+ \int dk^- \left[\pm \frac{\vec{k}_T^{\,2} - 2\,x\,P\cdot k}{2M^2}\, A_8^q(P\cdot k) + \frac{m_q}{M}\, x\, A_8^q(P\cdot k) \right]_{k^+ = x P^+}, \tag{33a}$$

$$h_{1L}^{\perp q}(x,k_T) = 2P^+ \int dk^- \left[\pm x\, A_8^q(P\cdot k) - \frac{m_q}{M}\, A_8^q(P\cdot k) \right]_{k^+ = x P^+}, \tag{33b}$$

$$h_{1T}^{\perp q}(x,k_T) = 2P^+ \int dk^- \left[\pm A_8^q(P\cdot k) \right]_{k^+ = x P^+}, \tag{33c}$$

$$h_L^q(x,k_T) = 2P^+ \int dk^- \left[\pm \frac{x^2M^2 - 2\,x\,P\cdot k}{M^2}\, A_8^q(P\cdot k) + \frac{m_q}{M}\, \frac{P\cdot k}{M^2}\, A_8^q(P\cdot k) \right]_{k^+ = x P^+}, \tag{33d}$$

$$h_T^q(x,k_T) = 2P^+ \int dk^- \left[\pm \frac{x\,M^2 - P\cdot k}{M^2}\, A_8^q(P\cdot k) \right]_{k^+ = x P^+}, \tag{33e}$$

$$h_T^{\perp q}(x,k_T) = 2P^+ \int dk^- \left[\pm \left(-\frac{P\cdot k}{M^2}\, A_8^q(P\cdot k) \right) + \frac{m_q}{M}\, A_8^q(P\cdot k) \right]_{k^+ = x P^+}. \tag{33f}$$

If we consider that quark mass effects are negligible in practical applications, we notice that the signs $\pm$ associated with the solutions for $w_\pm^q$ and $A_{\mathrm{pol}\pm}^q$ yield different predictions for the overall signs of the chiral odd polarized TMDs. For massless quarks, the two choices give exactly opposite signs for chiral odd polarized TMDs.

The negative-sign solution for $w_\pm^q$ and $A_{\mathrm{pol}\pm}^q$ in (24) coincides with the conventions in [26–38] and predicts transversity and helicity quark TMDs to have equal signs in agreement with

other models and lattice QCD [69–94,96,95,98,97].[1] We therefore conclude that the negative-sign solution in (24) is physical. From the positive-sign solution w^q_+ and $A^q_{\rm pol+}$ in (24), one would obtain transversity and helicity quark TMDs with opposite signs. This has not been observed in any model or lattice study we are aware of. We therefore discard the positive-sign solution in (24) as unphysical.

It is natural to encounter two solutions because the pure-spin state condition $w_q^2 = -1$ is a quadratic equation. Interestingly, the pure-spin state parton model by itself cannot distinguish which solution is physical and which one is unphysical, and thus does not predict the sign of transversity and other chiral odd TMDs. We have to determine the physical solution using results from other models or lattice QCD as a guideline.[2]

Thus, the physical (negative-sign) solution corresponds to the covariant parton model in [26–38] and is given by

$$A^q_8 = -A^q_{11}\,, \quad A^q_{\rm pol-} = \frac{(P\cdot k)+m_q M}{M^2}\,A^q_8\,,$$
$$w^\mu_{q-} = S^\mu - \frac{(k\cdot S)}{(P\cdot k)+m_q M}\,P^\mu - \frac{M}{m_q}\,\frac{(k\cdot S)}{(P\cdot k)+m_q M}\,k^\mu\,. \tag{34}$$

The final model expressions for the polarized TMDs in the pure-spin state version of the model are given by

$$g^q_1(x,k_T) = 2P^+\int dk^-\left[\frac{x^2M^2 - x\,P\cdot k + x\,m_q M + m_q^2}{M^2}\,A^q_8(P\cdot k)\right]_{k^+=xP^+}, \tag{35a}$$

$$g^{\perp q}_{1T}(x,k_T) = 2P^+\int dk^-\left[\frac{x\,M+m_q}{M}\,A^q_8(P\cdot k)\right]_{k^+=xP^+}, \tag{35b}$$

$$g^q_T(x,k_T) = 2P^+\int dk^-\left[\frac{M\,\vec{k}_T^{\,2} + 2m_q\,P\cdot k + 2m_q^2 M}{2M^3}\,A^q_8(P\cdot k)\right]_{k^+=xP^+}, \tag{35c}$$

$$g^{\perp q}_L(x,k_T) = 2P^+\int dk^-\left[\frac{x\,M^2 - P\cdot k}{M^2}\,A^q_8(P\cdot k)\right]_{k^+=xP^+}, \tag{35d}$$

$$g^{\perp q}_T(x,k_T) = 2P^+\int dk^-\left[A^q_8(P\cdot k)\right]_{k^+=xP^+}. \tag{35e}$$

$$h^q_1(x,k_T) = 2P^+\int dk^-\left[\frac{2x\,P\cdot k - \vec{k}_T^{\,2} + 2xm_q M}{2M^2}\,A^q_8(P\cdot k)\right]_{k^+=xP^+}, \tag{35f}$$

$$h^{\perp q}_{1L}(x,k_T) = 2P^+\int dk^-\left[-\frac{x\,M+m_q}{M}\,A^q_8(P\cdot k)\right]_{k^+=xP^+}, \tag{35g}$$

[1] We remark that one can determine the physical sign solution for $h^q_1(x)$ also from model and lattice QCD studies of the nucleon tensor charge $\int_0^1 dx(h^q_1 - h^{\bar q}_1)(x)$ [99–108], under the safe assumption that the sign of the tensor charge is determined by $h^q_1(x)$.

[2] Let us recall that the sign of $h^q_1(x)$ cannot be inferred from experiment because chiral odd (TMDs, fragmentation, etc) functions enter observables paired with other chiral odd functions. The positive sign of $h^u_1(x)$ in parametrizations (see, e.g., [15] for a recent extraction) is a convention based models and lattice QCD predictions [69–94,96,95,98,97,99–108].

$$h_{1T}^{\perp q}(x,k_T) = 2P^+ \int dk^- \Big[-A_8^q(P\cdot k) \Big]_{k^+=xP^+}, \tag{35h}$$

$$h_L^q(x,k_T) = 2P^+ \int dk^- \left[\frac{2\,x\,M\,P\cdot k - x^2M^3 + m_q\,P\cdot k}{M^3}\, A_8^q(P\cdot k) \right]_{k^+=xP^+}, \tag{35i}$$

$$h_T^q(x,k_T) = 2P^+ \int dk^- \left[\frac{P\cdot k - x\,M^2}{M^2}\, A_8^q(P\cdot k) \right]_{k^+=xP^+}, \tag{35j}$$

$$h_T^{\perp q}(x,k_T) = 2P^+ \int dk^- \left[\frac{P\cdot k + m_q M}{M^2}\, A_8^q(P\cdot k) \right]_{k^+=xP^+}. \tag{35k}$$

Above, we explored theoretical studies of h_1^q to determine the physical sign solution, because transversity is arguably the most widely studied chiral odd nucleon distribution function. For completeness, let us remark that in principle we could have used any other chiral odd quark distribution function, for instance $h_L^q(x)$ for which some model and lattice studies are available in literature [69,72,73,89,109–111]. One obtains the same result from $h_L^q(x)$. It is important to remark that there is more than one way to determine the physical sign — including further possibilities to use chiral odd TMDs like pretzelosity $h_{1T}^{\perp q}$, the gear-worm function $h_{1L}^{\perp q}$, or other polarized chiral odd TMDs. The fact that one concludes unanimously the same relative (negative) sign between the amplitudes A_8^q and A_{11}^q using the different methods supports the consistency of the approach.

7. Antiquarks in the parton model

The discussion of the antiquark correlator parallels that of the quark correlator such that we can abbreviate many details and focus on showing the main results.

7.1. Correlator for massive antiquarks

Inserting the relations in (15), (16) in the antiquark correlator in (2) and performing exactly the same steps as in the polarized correlator as in (17)-(21) we obtain

$$\Phi^{\bar q}(k,P,S) = \Phi^{\bar q}(k,P,S)_{\rm unp} + \Phi^{\bar q}(k,P,S)_{\rm pol} \tag{36a}$$

with

$$\Phi^{\bar q}(k,P,S)_{\rm unp} = (\not k - m_q)\, A_{\rm unp}^{\bar q}\,, \tag{36b}$$

$$\Phi^{\bar q}(k,P,S)_{\rm pol} = (\not k - m_q)\gamma_5\, \not w_{\bar q}\, A_{\rm pol}^{\bar q} \tag{36c}$$

where we introduced $A_{\rm unp}^{\bar q}$, $A_{\rm pol}^{\bar q}$, $w_{\bar q}^\mu$ defined as[3]

$$A_{\rm unp}^{\bar q} = A_3^{\bar q}\,,$$

[3] We recall that by convention antifermions are polarized opposite to the polarization vector, cf. Sec. 9.5. We stress that this has no practical relevance for our results and the TMDs of both quarks and antiquarks have their usual partonic interpretations.

$$A^{\bar q}_{\rm pol} = -\frac{(P\cdot k)\,A^{\bar q}_{11} + m_q M\,A^{\bar q}_8}{M^2}\,, \tag{36d}$$

$$w^\mu_{\bar q} = S^\mu - P^\mu \frac{(k\cdot S)\,A^{\bar q}_{11}}{(P\cdot k)\,A^{\bar q}_{11} + m_q M\,A^{\bar q}_8} - k^\mu \frac{M}{m_q}\,\frac{(k\cdot S)A^{\bar q}_8}{(P\cdot k)\,A^{\bar q}_{11} + m_q M\,A^{\bar q}_8}\,. \tag{36e}$$

In (36e) we assumed that $m_q \neq 0$ and $(P\cdot k)\,A^{\bar q}_{11} + m_q M\,A^{\bar q}_8 \neq 0$. The axial 4-vector $w_{\bar q}$ plays the role of the antiquark polarization vector. It satisfies $k\cdot w_{\bar q} = 0$ for any $A^{\bar q}_8$ and $A^{\bar q}_{11}$.

In the mixed-spin version of the parton model, we have

$$-1 < w^2_{\bar q} < 0 \tag{37}$$

where the upper bound is always satisfied. The lower bound is satisfied provided $|A^{\bar q}_{11}| < |A^{\bar q}_8|$. In this version of the model, the antiquark correlator is described in terms of three independent amplitudes $A^{\bar q}_3$, $A^{\bar q}_8$, $A^{\bar q}_{11}$.

In the pure-spin state version of the model, the polarized amplitudes $A^{\bar q}_8$ and $A^{\bar q}_{11}$ are related as

$$w^2_{\bar q} = -1 \quad \Leftrightarrow \quad A^{\bar q}_{11} = \pm A^{\bar q}_8\,. \tag{38a}$$

The two solutions $A^{\bar q}_8 = \pm A^{\bar q}_{11}$ imply

$$A^{\bar q}_{{\rm pol}\pm} = -\frac{(\pm P\cdot k) + m_q M}{M^2}\,A^{\bar q}_8, \tag{38b}$$

$$w^\mu_{\bar q\pm} = S^\mu - P^\mu \frac{(\pm k\cdot S)}{(\pm P\cdot k) + m_q M} - \frac{M}{m_q}\,\frac{(\pm k\cdot S)}{(\pm P\cdot k) + m_q M}\,(\pm k^\mu)\,. \tag{38c}$$

7.2. Correlator for massless antiquarks

In the massless case, the unpolarized and polarized parts of the antiquark correlator in (36b) are described as[4]

$$\Phi^{\bar q}(k,P,S)_{\rm unp} = \not k\, A^{\bar q}_{\rm unp}\,, \quad \Phi^{\bar q}(k,P,S)_{\rm pol} = \not k\,\gamma_5 \Big\{ -\lambda\, + \not b_T \Big\}\, A^{\bar q}_{\rm pol} \tag{39a}$$

where, assuming $A^{\bar q}_8 \neq 0$, we introduced

$$A^{\bar q}_{\rm unp} = A^{\bar q}_3,$$
$$A^{\bar q}_{\rm pol} = -\frac{(P\cdot k)}{M^2}\,A^{\bar q}_8\,, \quad \lambda = \frac{M(k\cdot S)}{(P\cdot k)}\,, \quad b^\mu_T = \left(S^\mu - P^\mu\,\frac{(k\cdot S)}{(P\cdot k)}\right)\frac{A^{\bar q}_{11}}{A^{\bar q}_8}\,. \tag{39b}$$

The condition $-1 < b^2_T - \lambda^2 < 0$ defining the massless spin state model holds provided $|A^{\bar q}_{11}| < |A^{\bar q}_8|$. In this case, the model is characterized in terms of three independent amplitudes $A^{\bar q}_3$, $A^{\bar q}_8$, $A^{\bar q}_{11}$.

[4] In our notation, the helicity of the antiquark is given by $(-\lambda)$, cf. footnote 3, the reminder in Sec. 9.5, and the "mental health warning" in Sec. 6.4 of Ref. [4].

In the pure-spin case for antiquarks, it is required that $b_T^2 - \lambda^2 = -1$ which implies $A_{11}^{\bar q} = \pm A_8^{\bar q}$. The description of the antiquark correlator proceeds then in terms of two independent amplitudes which can be chosen to be $A_3^{\bar q}$, $A_8^{\bar q}$.

7.3. Unpolarized antiquark TMDs

The T-even unpolarized antiquark TMDs are uniquely expressed in terms of the amplitude $A_3^{\bar q}$ as follows, cf. (6),

$$f_1^{\bar q}(x,k_T) = 2P^+ \int dk^- \Big[x A_3^{\bar q}(P\cdot k)\Big]_{k^+=xP^+}, \tag{40a}$$

$$f^{\perp \bar q}(x,k_T) = 2P^+ \int dk^- \Big[A_3^{\bar q}(P\cdot k)\Big]_{k^+=xP^+}, \tag{40b}$$

$$e^{\bar q}(x,k_T) = 2P^+ \int dk^- \Big[\frac{m_q}{M} A_3^{\bar q}(P\cdot k)\Big]_{k^+=xP^+}. \tag{40c}$$

7.4. Polarized chiral even antiquark TMDs in mixed-spin case

The polarized chiral even TMDs in the mixed-spin model are given by

$$g_1^{\bar q}(x,k_T) = 2P^+ \int dk^- \Big[\frac{x\,P\cdot k - x^2M^2 - m_q^2}{M^2} A_8^{\bar q} - \frac{m_q}{M} x\, A_{11}^{\bar q}\Big]_{k^+=xP^+}, \tag{41a}$$

$$g_{1T}^{\perp \bar q}(x,k_T) = 2P^+ \int dk^- \Big[-x\, A_8^{\bar q} - \frac{m_q}{M} A_{11}^{\bar q}\Big]_{k^+=xP^+}, \tag{41b}$$

$$g_T^{\bar q}(x,k_T) = 2P^+ \int dk^- \Big[-\frac{\vec k_T^{\,2} + 2m_q^2}{2M^2} A_8^{\bar q} - \frac{m_q}{M}\frac{P\cdot k}{M^2} A_{11}^{\bar q}\Big]_{k^+=xP^+}, \tag{41c}$$

$$g_L^{\perp \bar q}(x,k_T) = 2P^+ \int dk^- \Big[-\frac{xM^2 - P\cdot k}{M^2} A_8^{\bar q}\Big]_{k^+=xP^+}, \tag{41d}$$

$$g_T^{\perp \bar q}(x,k_T) = 2P^+ \int dk^- \Big[- A_8^{\bar q}\Big]_{k^+=xP^+}. \tag{41e}$$

7.5. Polarized chiral odd antiquark TMDs in mixed-spin state case

The polarized chiral odd TMDs in the mixed-spin model are given by

$$h_1^{\bar q}(x,k_T) = 2P^+ \int dk^- \Big[\frac{\vec k_T^{\,2} - 2x\,P\cdot k}{2M^2} A_{11}^{\bar q} - \frac{m_q}{M} x\, A_8^{\bar q}\Big]_{k^+=xP^+}, \tag{42a}$$

$$h_{1L}^{\perp \bar q}(x,k_T) = 2P^+ \int dk^- \Big[x\, A_{11}^{\bar q} + \frac{m_q}{M} A_8^{\bar q}\Big]_{k^+=xP^+}, \tag{42b}$$

$$h_{1T}^{\perp \bar q}(x,k_T) = 2P^+ \int dk^- \Big[A_{11}^{\bar q}\Big]_{k^+=xP^+}, \tag{42c}$$

$$h_L^{\bar q}(x,k_T) \;=\; 2P^+\int dk^-\left[\frac{x^2M^2-2\,x\,P\cdot k}{M^2}A_{11}^{\bar q}-\frac{m_q}{M}\,\frac{P\cdot k}{M^2}\,A_8^{\bar q}\right]_{k^+=xP^+}, \tag{42d}$$

$$h_T^{\perp\bar q}(x,k_T) \;=\; 2P^+\int dk^-\left[-\frac{P\cdot k}{M^2}\,A_{11}^{\bar q}-\frac{m_q}{M}\,A_8^{\bar q}\right]_{k^+=xP^+}, \tag{42e}$$

$$h_T^{\bar q}(x,k_T) \;=\; 2P^+\int dk^-\left[\frac{x\,M^2-P\cdot k}{M^2}\,A_{11}^{\bar q}\right]_{k^+=xP^+}. \tag{42f}$$

7.6. Polarized chiral even antiquark TMDs in pure-spin state case

The polarized chiral even TMDs in the mixed-spin model are given by

$$g_1^{\bar q}(x,k_T) = 2P^+\int dk^-\left[\frac{x\,P\cdot k-x^2M^2-m_q^2}{M^2}\,A_8^{\bar q}-\frac{m_q}{M}\,x\left(\pm A_8^{\bar q}\right)\right]_{k^+=xP^+}, \tag{43a}$$

$$g_{1T}^{\perp\bar q}(x,k_T) \;=\; 2P^+\int dk^-\left[-x\,A_8^{\bar q}-\frac{m_q}{M}\left(\pm A_8^{\bar q}\right)\right]_{k^+=xP^+}, \tag{43b}$$

$$g_T^{\bar q}(x,k_T) \;=\; 2P^+\int dk^-\left[-\frac{\vec k_T^{\,2}+2m_q^2}{2M^2}\,A_8^{\bar q}-\frac{m_q}{M}\,\frac{P\cdot k}{M^2}\left(\pm A_8^{\bar q}\right)\right]_{k^+=xP^+}, \tag{43c}$$

while $g_L^{\perp\bar q}(x,k_T)$ and $g_T^{\perp\bar q}(x,k_T)$ which do not depend on the amplitude $A_{11}^{\bar q}$ are still given by the expressions (41d), (41e).

7.7. Polarized chiral odd antiquark TMDs in pure-spin state case

The polarized chiral odd TMDs in the pure-spin state model are given by

$$h_1^{\bar q}(x,k_T) \;=\; 2P^+\int dk^-\left[\frac{\vec k_T^{\,2}-2\,x\,P\cdot k}{2M^2}\left(\pm A_8^{\bar q}\right)-\frac{m_q}{M}\,x\,A_8^{\bar q}\right]_{k^+=xP^+}, \tag{44a}$$

$$h_{1L}^{\perp\bar q}(x,k_T) \;=\; 2P^+\int dk^-\left[x\left(\pm A_8^{\bar q}\right)+\frac{m_q}{M}\,A_8^{\bar q}\right]_{k^+=xP^+}, \tag{44b}$$

$$h_{1T}^{\perp\bar q}(x,k_T) \;=\; 2P^+\int dk^-\left[\left(\pm A_8^{\bar q}\right)\right]_{k^+=xP^+}, \tag{44c}$$

$$h_L^{\bar q}(x,k_T) \;=\; 2P^+\int dk^-\left[\frac{x^2M^2-2\,x\,P\cdot k}{M^2}\left(\pm A_8^{\bar q}\right)-\frac{m_q}{M}\,\frac{P\cdot k}{M^2}\,A_8^{\bar q}\right]_{k^+=xP^+}, \tag{44d}$$

$$h_T^{\perp\bar q}(x,k_T) \;=\; 2P^+\int dk^-\left[-\frac{P\cdot k}{M^2}\left(\pm A_8^{\bar q}\right)-\frac{m_q}{M}\,A_8^{\bar q}\right]_{k^+=xP^+}, \tag{44e}$$

$$h_T^{\bar q}(x,k_T) \;=\; 2P^+\int dk^-\left[\frac{x\,M^2-P\cdot k}{M^2}\left(\pm A_8^{\bar q}\right)\right]_{k^+=xP^+}. \tag{44f}$$

Keeping in mind that quark mass effects are practically negligible for the light u- and d-flavors, we see that the two $\pm$ solutions for $w_\pm^{\bar q}$ and $A_{\rm pol\pm}^{\bar q}$ in (38a)–(38c), predict opposite signs of the

chiral odd polarized antiquark TMDs. In order to determine the physical solution, we will again rely on results from model and lattice calculations.

In order to determine the physical solution for $w^{\bar q}_{\pm}$ and $A^{\bar q}_{\rm pol\pm}$ in (38a)–(38c) we proceed analog to the quark case and consult theory results from literature. Far fewer studies are available for antiquark distributions as compared to quarks, but they unanimously yield $h_1^{\bar q}$ and $g_1^{\bar q}$ of opposite signs for a given flavor $\bar q$ [71,78,80,90,92,96,95,98,97]. Now, the positive-sign solution $w^{\bar q}_{+}$ and $A^{\bar q}_{+}$ yields transversity and helicity antiquark TMDs of equal signs, i.e. this is the unphysical solution in antiquark case. The negative-sign solution $w^{\bar q}_{-}$ and $A^{\bar q}_{-}$ in (38a)–(38c) gives transversity and helicity antiquark TMDs of opposite sign which is in agreement with results from literature. We therefore choose the negative-sign solution in (38a)–(38c) as the physical solution which is given by

$$\begin{aligned}
A^{\bar q}_{11} &= -\,A^{\bar q}_{8}\,,\\
A^{\bar q}_{\rm pol-} &= \frac{P\cdot k - m_q M}{M^2}\,A^{\bar q}_{8},\\
w^{\mu}_{\bar q -} &= S^\mu - \frac{k\cdot S}{P\cdot k - m_q M}\,P^\mu + \frac{M}{m_q}\,\frac{k\cdot S}{P\cdot k - m_q M}\,k^\mu\,.
\end{aligned} \tag{45}$$

The final model expressions in the pure-spin state model are given by

$$g_1^{\bar q}(x,k_T) = 2P^+\int dk^-\left[\frac{x\,P\cdot k - x^2M^2 + x\,m_q M - m_q^2}{M^2}\,A_8^{\bar q}\right]_{k^+=xP^+}, \tag{46a}$$

$$g_{1T}^{\perp\bar q}(x,k_T) = 2P^+\int dk^-\left[-\frac{x\,M - m_q}{M}\,A_8^{\bar q}\right]_{k^+=xP^+}, \tag{46b}$$

$$g_T^{\bar q}(x,k_T) = 2P^+\int dk^-\left[-\frac{M\,\vec k_T^{\,2} - 2m_q\,P\cdot k + 2m_q^2 M}{2M^3}\,A_8^{\bar q}\right]_{k^+=xP^+}, \tag{46c}$$

$$g_L^{\perp\bar q}(x,k_T) = 2P^+\int dk^-\left[\frac{P\cdot k - x\,M^2}{M^2}\,A_8^{\bar q}\right]_{k^+=xP^+}, \tag{46d}$$

$$g_T^{\perp\bar q}(x,k_T) = 2P^+\int dk^-\left[-A_8^{\bar q}\right]_{k^+=xP^+}, \tag{46e}$$

$$h_1^{\bar q}(x,k_T) = 2P^+\int dk^-\left[\frac{2x\,P\cdot k - \vec k_T^{\,2} - 2x\,m_q M}{2M^2}\,A_8^{\bar q}\right]_{k^+=xP^+}, \tag{46f}$$

$$h_{1L}^{\perp\bar q}(x,k_T) = 2P^+\int dk^-\left[-\frac{x\,M - m_q}{M}\,A_8^{\bar q}\right]_{k^+=xP^+}, \tag{46g}$$

$$h_{1T}^{\perp\bar q}(x,k_T) = 2P^+\int dk^-\left[-A_8^{\bar q}\right]_{k^+=xP^+}, \tag{46h}$$

$$h_L^{\bar q}(x,k_T) = 2P^+\int dk^-\left[\frac{2x\,M\,P\cdot k - x^2M^3 - m_q\,P\cdot k}{M^3}\,A_8^{\bar q}\right]_{k^+=xP^+}, \tag{46i}$$

$$h_T^{\perp\bar{q}}(x,k_T) \;=\; 2P^+\int dk^-\left[\frac{P\cdot k - m_q M}{M^2}\,A_8^{\bar{q}}\right]_{k^+=xP^+}, \tag{46j}$$

$$h_T^{\bar{q}}(x,k_T) \;=\; 2P^+\int dk^-\left[\frac{P\cdot k - M^2 x}{M^2}\,A_8^{\bar{q}}\right]_{k^+=xP^+}. \tag{46k}$$

8. Consistency of the approach

In this section we demonstrate the internal consistency of the approach.

8.1. Sum rule of $e^a(x)$

As a first consistency test we consider the sum rule for the twist-3 function $e^q(x)$ [69,113]. Introducing the integrated functions $e^a(x) = \int d^2k_T\, e^a(x,k_T)$ and $f_1^a(x) = \int d^2k_T\, f_1^a(x,k_T)$ (we recall that in the parton model TMDs and PDFs are simply related, see Sec. 2.2), we obtain from (29, 40) the result

$$\int_0^1 dx\Big(x\,e^q(x) + x\,e^{\bar{q}}(x)\Big) = \frac{m_q}{M}\int_0^1 dx\Big(f_1^q(x) - f_1^{\bar{q}}(x)\Big). \tag{47}$$

Using the customary "continuation" to negative x according to $e^{\bar{q}}(x) = e^q(-x)$ and $f_1^{\bar{q}}(x) = -f_1^q(-x)$, (47) can be expressed as

$$\int_{-1}^1 dx\, x\, e^q(x) = \frac{m_q}{M}\,N_q\,, \tag{48}$$

where $N_q = \int_{-1}^1 dx f_1^q(x)$ is the number of valence quarks of flavor q in the nucleon. This is a consistency test for the amplitude relations derived from the equations of motion for respectively Ψ_q and $\bar{\Psi}_q$ in Secs. 3.1 and 3.2. The sum rule in (48) is correctly satisfied due to the opposite signs in the relations $A_1^q = \frac{m_q}{M}A_3^q$ and $A_1^{\bar{q}} = -\frac{m_q}{M}A_3^{\bar{q}}$. We will follow up below on the negative-x continuation also for the other TMDs.

8.2. Equation of motion (EOM) relations

From the QCD equations of motion, one obtains the so-called EOM relations. Below we quote only the EOM relation for T-even functions relevant in this work. The EOM relations among the unpolarized TMDs are given by

$$x\,f^{\perp a}(x,k_T) = x\,\tilde{f}^{\perp a}(x,k_T) + f_1^q(x,k_T), \tag{49a}$$

$$x\,e^a(x,k_T) = x\,\tilde{e}^a(x,k_T) + \frac{m_q}{M}\,f_1^a(x,k_T), \tag{49b}$$

where $a = q,\ \bar{q}$. The EOM relations among the polarized TMDs read

$$x g_L^{\perp a}(x,k_T) = x\tilde{g}_L^{\perp a}(x,k_T) + g_1^a(x,k_T) + \frac{m_q}{M}\,h_{1L}^{\perp a}(x,k_T), \tag{50a}$$

$$x g_T^a(x,k_T) = x\tilde{g}_T^a(x,k_T) + g_{1T}^{\perp(1)a}(x,k_T) + \frac{m_q}{M}\,h_1^a(x,k_T), \tag{50b}$$

$$x g_T^{\perp a}(x,k_T) = x\tilde{g}_T^{\perp a}(x,k_T) + g_{1T}^{\perp a}(x,k_T) + \frac{m_q}{M}\, h_{1T}^{\perp a}(x,k_T), \tag{50c}$$

$$x h_L^a(x,k_T) = x\tilde{h}_L^a(x,k_T) - 2\, h_{1L}^{\perp(1)a}(x,k_T) + \frac{m_q}{M}\, g_1^a(x,k_T), \tag{50d}$$

$$x h_T^a(x,k_T) = x\tilde{h}_T^a(x,k_T) - h_1^a(x,k_T) - h_{1T}^{\perp(1)a}(x,k_T) + \frac{m_q}{M}\, g_{1T}^{\perp a}(x,k_T), \tag{50e}$$

$$x h_T^{\perp a}(x,k_T) = x\tilde{h}_T^{\perp a}(x,k_T) + h_1^a(x,k_T) - h_{1T}^{\perp(1)a}(x,k_T), \tag{50f}$$

where the (1)-moments of TMDs are defined, for instance, as

$$g_{1T}^{\perp(1)q}(x,k_T) = \frac{\vec{k}_T^2}{2M^2}\, g_{1T}^{\perp q}(x,k_T) \tag{51}$$

and analogously for other TMDs. In QCD, the tilde terms in (49, 50) are related to quark-gluon correlators [59]. In quark models, despite the absence of gluonic degrees of freedom, the tilde terms are in general nonzero as they can arise from the respective model interactions.

In the parton model, the tilde terms are expected to be zero. One readily verifies that the model expressions for quark and antiquark TMDs in (29, 30, 35, 40, 41, 42, 43, 44) in mixed-spin and pure-spin version of the model satisfy the EOM relations with the tilde terms set to zero. This is another important consistency test of the approach.

8.3. Relation between quark and antiquark correlators and its consequences

In order to carry out further tests of the consistency of the approach, we explore the field-theoretical relation between the quark and antiquark correlators in (1) given by [45,53]

$$\Phi_{ij}^{\bar{q}}(k,P,S) = -\,\Phi_{ij}^{q}(-k,P,S) \tag{52}$$

The relation in (52) shows that the amplitudes A_i^q and $A_i^{\bar{q}}$ are related to each other by a "continuation" of $P\cdot k$ to $(-P\cdot k)$. From the expansions of the correlators $\Phi^a(k,P,S)$ in terms of the amplitudes in (2) and the relation in (52) we read off

$$A_i^{\bar{q}}(P\cdot k) = \begin{cases} +A_i^q(-P\cdot k) & \text{for} \quad i = 3,\,4,\,5,\,7,\,10,\,12, \\ -A_i^q(-P\cdot k) & \text{for} \quad i = 1,\,2,\,6,\,8,\,9,\,11. \end{cases} \tag{53}$$

As a first consistency test, let us rederive the relations among the antiquark amplitudes in (15), (16) from the relations among the quark amplitudes $A^q(P\cdot k)$ in (10), (12). This is accomplished by continuing $k \to (-k)$ in (10), (12). Using (53) to express the $A_i^q(-P\cdot k)$ in terms of the $A_i^{\bar{q}}(P\cdot k)$ yields the antiquark amplitude relations in (15), (16). This test demonstrates the consistency of the derivations of the relations (10), (12) among the quark amplitudes and the derivations of the relations (15), (16) among antiquark amplitudes.

As a second test related to the amplitudes, let us consider the final results for the correlators in the pure-spin state model with massive partons which are given by

$$\Phi^q(k,P,S) = (\not{k} + m_q)\left[A_{\rm unp}^q(P\cdot k) + \gamma_5\, \not{\psi}_q(k)\, A_{\rm pol}^q(P\cdot k)\right] \tag{54a}$$

$$\Phi^{\bar{q}}(k,P,S) = (\not{k} - m_q)\left[A_{\rm unp}^{\bar{q}}(P\cdot k) + \gamma_5\, \not{\psi}_{\bar{q}}(k)\, A_{\rm pol}^{\bar{q}}(P\cdot k)\right] \tag{54b}$$

$$A_{\rm unp}^a(P\cdot k) = A_3^a(P\cdot k)\,, \tag{54c}$$

$$A^q_{\rm pol}(P\cdot k) = \frac{P\cdot k + m_q M}{M^2}\, A^q_8(P\cdot k)\,, \tag{54d}$$

$$A^{\bar q}_{\rm pol}(P\cdot k) = \frac{P\cdot k - m_q M}{M^2}\, A^{\bar q}_8(P\cdot k)\,, \tag{54e}$$

$$w^\mu_q(k) = S^\mu - \frac{k\cdot S}{P\cdot k + m_q M}\, P^\mu - \frac{M}{m_q}\,\frac{k\cdot S}{P\cdot k + m_q M}\, k^\mu\,, \tag{54f}$$

$$w^\mu_{\bar q}(k) = S^\mu - \frac{k\cdot S}{P\cdot k - m_q M}\, P^\mu + \frac{M}{m_q}\,\frac{k\cdot S}{P\cdot k - m_q M}\, k^\mu = w^\mu_q(-k)\,. \tag{54g}$$

From (53), (54) we see that $A^{\bar q}_{\rm unp}(P\cdot k) = A^q_{\rm unp}(-P\cdot k)$ and $A^{\bar q}_{\rm pol}(P\cdot k) = A^q_{\rm pol}(-P\cdot k)$ while $w^\mu_{\bar q}(k) = w^\mu_q(-k)$. With these preparations, we see that the quark and antiquark correlators are related to each other by the field-theoretical relation in (52). This test is non-trivial, because we have independently chosen the signs of the solutions in the quark and antiquark cases in (24, 38). In the same way, one can show that the quark and antiquark correlators are consistently described also in the mixed-spin state and massless versions of the model.

8.4. Continuation of quark TMDs to negative x

The model expression for $f^q_1(x,k_T)$ in (29a) can be expressed as

$$f^q_1(x,k_T) = 2P^+ \int dk^- \int dk^+\, \delta(k^+ - xP^+)\Big[x A^q_3(P\cdot k)\Big]. \tag{55a}$$

In this expression, we replace x by $(-x)$, and perform the substitutions[5] $k^\pm \to -k^\pm$. This yields

$$\begin{aligned} f^q_1(-x,k_T) &= -2P^+ \int dk^- \int dk^+\, \delta(k^+ - xP^+)\Big[x A^q_3(-P\cdot k)\Big] \\ &= -2P^+ \int dk^- \int dk^+\, \delta(k^+ - xP^+)\Big[x A^{\bar q}_3(P\cdot k)\Big] \end{aligned} \tag{55b}$$

where in the last step we made use of (53). Comparing to (40a) we recognize the expression for $(-1)\, f^{\bar q}_1(x,k_T)$. We can proceed analogously with the other TMDs. Summarizing the results, we find the familiar relations

$$f^{\bar q}_1(x,k_T) = -\ f^q_1(-x,k_T)\,, \tag{56a}$$

$$g^{\bar q}_1(x,k_T) = \ \ g^q_1(-x,k_T)\,, \tag{56b}$$

$$h^{\bar q}_1(x,k_T) = -\ h^q_1(-x,k_T)\,, \tag{56c}$$

$$g^{\perp \bar q}_{1T}(x,k_T) = -\, g^{\perp q}_{1T}(-x,k_T)\,, \tag{56d}$$

$$h^{\perp \bar q}_{1L}(x,k_T) = \ \ h^{\perp q}_{1L}(-x,k_T)\,, \tag{56e}$$

$$h^{\perp \bar q}_{1T}(x,k_T) = -\, h^{\perp q}_{1T}(-x,k_T)\,, \tag{56f}$$

$$e^{\bar q}(x,k_T) = \ \ e^q(-x,k_T)\,, \tag{56g}$$

$$f^{\perp \bar q}(x,k_T) = \ \ f^{\perp q}(-x,k_T)\,, \tag{56h}$$

[5] Notice that under the substitutions $k^\pm \to -k^\pm$ in the integrals in (55a), the product $P\cdot k$ changes sign $P\cdot k \to -P\cdot k$. This is so because $P\cdot k = P^+k^- + P^-k^+$ is independent of k_T as by definition the nucleon momentum has no transverse component.

$$g_T^{\bar q}(x,k_T) = \quad g_T^q(-x,k_T)\,, \tag{56i}$$
$$g_T^{\perp\bar q}(x,k_T) = \quad g_T^{\perp q}(-x,k_T)\,, \tag{56j}$$
$$g_L^{\perp\bar q}(x,k_T) = -\, g_L^{\perp q}(-x,k_T)\,, \tag{56k}$$
$$h_L^{\bar q}(x,k_T) = -\, h_L^q(-x,k_T)\,, \tag{56l}$$
$$h_T^{\perp\bar q}(x,k_T) = \quad h_T^{\perp q}(-x,k_T)\,, \tag{56m}$$
$$h_T^{\bar q}(x,k_T) = \quad h_T^q(-x,k_T)\,. \tag{56n}$$

In these relations x is always in the range $0<x<1$. We remark that based on these relations it is customary to introduce TMDs defined on the domain $-1<x<1$ with the understanding that quark TMD functions of $(-x)$ mean (±1) of the respective antiquark TMD functions of x with the signs as specified in (56), e.g., $h_1^q(x,k_T)$ for $0<(-x)<1$ means $-h_1^{\bar q}(x,k_T)$ for $0<x<1$, etc, cf. also Sec. 8.1.

9. Evaluation of TMDs in covariant parton model

In this section we will show that the pure-spin state parton model derived in this work corresponds to the CPM of Refs. [25–38]. For that we will introduce the notation of [25–27], explore consequences of the equation of motion [39], consider kinematic DIS constraints, and rederive the model expressions for quark TMDs from prior studies [30,33,38] and present new results for antiquark TMDs.

9.1. Model expressions

Starting from the equation of motion $(i\partial\!\!\!/ - m_q)\,\Psi^q(z)=0$, one obtains for the quark correlator the relation [39],

$$\begin{aligned}
0 &= \int \frac{\mathrm{d}^4z}{(2\pi)^4}\, \mathrm{e}^{ikz}\, \langle N|\,\overline{\Psi}_j^q(0)\left[(i\overrightarrow{\partial\!\!\!/} + m_q)_{ik}(i\overrightarrow{\partial\!\!\!/} - m_q)_{kl}\Psi_l^q(z)\right]|N\rangle \\
&= \int \frac{\mathrm{d}^4z}{(2\pi)^4}\, \mathrm{e}^{ikz}\, \langle N|\,\overline{\Psi}_j^q(0)\left[(-\Box - m_q^2)\,\mathbb{1}_{il}\Psi_l^q(z)\right]|N\rangle\,, \\
&= (k^2-m^2)\,\Phi_{ij}^q(k,P,S)\,,
\end{aligned} \tag{57}$$

where in the intermediate step we have performed twice integration by parts. The antiquark correlator satisfies an analogous relation. Since the Lorentz structures in the decompositions of the correlators are linearly independent, the result (57) and the analogous result for the antiquark correlator imply that the amplitudes satisfy

$$(k^2-m^2)\,A_i^a(P\cdot k,\,k^2)=0, \quad a=q,\,\bar q\,. \tag{58}$$

Keeping in mind that in the pure-spin-state version of the model we only have 2 independent amplitudes which can be expressed as $A^a_{\rm unp}(P\cdot k,k^2)$ and $A^a_{\rm pol}(P\cdot k,k^2)$, cf. (54), the solutions to (58) can be stated as

$$\begin{aligned}
A^a_{\rm unp}(P\cdot k,k^2) &= M\,\delta(k^2-m_q^2)\,\Theta_{\rm kin}(P\cdot k)\,\mathcal{G}^a(P\cdot k)\,, \\
A^a_{\rm pol}(P\cdot k,k^2) &= M\,\delta(k^2-m_q^2)\,\Theta_{\rm kin}(P\cdot k)\,\mathcal{H}^a(P\cdot k)\,,
\end{aligned} \tag{59}$$

where $a = q,\ \bar{q}$. Several comments are in order. The delta-function $\delta(k^2 - m_q^2)$ ensures that (58) is satisfied, and puts the partons on shell. The functions $\mathcal{G}^q(P\cdot k)$ and $\mathcal{H}^q(P\cdot k)$ are Lorentz-invariant functions of the variable $P\cdot k$ and are defined following the notation of [25–27]. In (59), the factor M was introduced for convenience such that $\mathcal{G}^q(P\cdot k)$ and $\mathcal{H}^q(P\cdot k)$ have the dimension (mass)$^{-3}$ and can be interpreted as 3D momentum densities [25–27]. Finally, the function $\Theta_{\rm kin}(P\cdot k)$ incorporates the kinematic constraints on the partons and is defined as

$$\Theta_{\rm kin}(P\cdot k) = \Theta^+_{\rm kin}(P\cdot k) + \Theta^-_{\rm kin}(P\cdot k), \quad \Theta^\pm_{\rm kin}(P\cdot k) = \Theta(\pm P\cdot k)\,\Theta\Big((P\mp k)^2\Big)\,. \tag{60}$$

Below we shall see that only $\Theta^+_{\rm kin}(P\cdot k)$ in (60) contributes to quark or antiquark TMDs. However, although it drops out from TMDs, the presence of $\Theta^-_{\rm kin}(P\cdot k)$ in (60) is nevertheless of importance for the analytic structure of the amplitudes and completeness of the model. We will follow up on this shortly. Let us mention here merely that due to $\Theta(P\cdot k) = \Theta(k^0)$, the role of $\Theta(P\cdot k)$ in $\Theta^+_{\rm kin}(P\cdot k)$ is to project out positive energy solutions [38], while $\Theta((P-k)^2)$ ensures that the nucleon remnant has positive energy and constitutes a physical state [39,112].

The scalar product $P\cdot k$ is positive (since it can be evaluated in any frame including nucleon rest frame where it is Mk^0; and for a real, on-shell parton the energy k^0 is of course positive). But it is convenient to define the "analytical continuation" of the covariant functions at negative values of $P\cdot k$ as follows

$$\mathcal{G}^q(-P\cdot k) = \mathcal{G}^{\bar q}(P\cdot k), \quad \mathcal{H}^q(-P\cdot k) = \mathcal{H}^{\bar q}(P\cdot k). \tag{61a}$$

With this definition, we see that the amplitudes defined in (59) satisfy

$$A^q_{\rm unp}(-P\cdot k, k^2) = A^{\bar q}_{\rm unp}(P\cdot k, k^2)\,, \quad A^q_{\rm pol}(-P\cdot k, k^2) = A^{\bar q}_{\rm pol}(P\cdot k, k^2)\,. \tag{61b}$$

We also see that with these definitions the model expressions for the quark and anti-quark correlators are given by

$$\Phi^q(k,P,S) = (\not{k} + m_q)\Big(\mathcal{G}^q(P\cdot k) + \mathcal{H}^q(P\cdot k)\,\gamma_5\,\not{w}_q(k)\Big) \times M\,\delta(k^2 - m_q^2)\,\Theta_{\rm kin}(P\cdot k) \tag{61c}$$

$$\Phi^{\bar q}(k,P,S) = (\not{k} - m_q)\Big(\mathcal{G}^{\bar q}(P\cdot k) + \mathcal{H}^{\bar q}(P\cdot k)\,\gamma_5\,\not{w}_{\bar q}(k)\Big) \times M\,\delta(k^2 - m_q^2)\,\Theta_{\rm kin}(P\cdot k) \tag{61d}$$

with $w^\mu_a(k)$ defined in (54), and satisfy (52). This is important for the internal consistency of the model. Notice that a consistent description of the quark and antiquark correlators is guaranteed by the specific structure of $\Theta_{\rm kin}(P\cdot k)$ as defined in (60).

In the quark case, the result in (61c) coincides with the expression for the quark correlator in Ref. [38].[6] Notice that the step function $\Theta((P-k)^2)$ was implicitly understood in Refs. [25–38] (and explicitly formulated in [39]). With the above remarks in mind, the result in (61c) practically reproduces the results for quark TMDs from [38]. In the antiquark case, the result in (61c) is new and was not given in prior studies.

[6] At this occasion, let us remark that in Ref. [38] instead of nucleon mass M the nucleon energy P^0 was used to give $\mathcal{G}^q(P\cdot k)$ and $\mathcal{H}^q(P\cdot k)$ the desired dimension. This choice is incorrect as it would imply incorrect Lorentz-transformation properties for the amplitudes. But in [38] the TMD expressions were evaluated in the nucleon rest frame where $P^0 = M$, so the practical results from [38] are correct.

Let us explicitly carry out the calculation of the unpolarized leading-twist quark TMD. Starting from (29a) and inserting for $A_3^q(P\cdot k, k^2) = A_{\rm unp}^q(P\cdot k, k^2)$ the result in (59) we obtain

$$f_1^q(x,k_T) = 2P^+ \int dk^- \int dk^+ \,\delta(k^+ - xP^+)\, xM\,\delta(k^2 - m_q^2)\,\Theta_{\rm kin}(P\cdot k)\,\mathcal{G}^q(P\cdot k)$$
$$= (\text{part 1}) + (\text{part 2}) \tag{62a}$$

with the two parts arising from respectively the two contributions $\Theta_{\rm kin}^{\pm}(P\cdot k)$ in (60). The first part in (62a) is evaluated by noticing that $\Theta(P\cdot k) = \Theta(k^0)$ and using $\delta(k^2 - m_q^2)\,\Theta(k^0) = \delta(k^0 - E_q)\,/\,(2E_q)$ with

$$E_q = \sqrt{\vec{k}^2 + m_q^2} \tag{62b}$$

denoting the parton energy. Recalling that $k^{\pm} = (k^0 \pm k^1)/\sqrt{2}$, changing the integration variables $dk^+dk^- \to dk^0 dk^1$ and choosing for convenience to work in the nucleon rest frame, we obtain

$$(\text{part 1}) = xM \int dk^0 \int dk^1\, \delta\Big(x - \frac{k^+}{P^+}\Big)\, \frac{\delta(k^0 - E_q)}{E_q}\, \Theta_{\rm kin}^{+}(P\cdot k)\, \mathcal{G}^q(P\cdot k) \tag{62c}$$
$$= x\,M \int \frac{dk^1}{E_q}\, \delta\Big(x - \frac{E_q + k^1}{M}\Big)\, \Theta\left(M^2 + m_q^2 - 2ME_q\right)\, \mathcal{G}^q(ME_q)\,. \tag{62d}$$

The "part 2" is evaluated similarly except that now we pick $\Theta(-P\cdot k) = \Theta(-k^0)$ from $\Theta_{\rm kin}(P\cdot k)$ in (60). We then get $\delta(k^2 - m_q^2)\,\Theta(-k^0) = \delta(k^0 + E_q)\,/\,(2E_q)$ which yields, using the nucleon rest frame for convenience, the result

$$(\text{part 2}) = xM \int \frac{dk^1}{E_q}\, \delta\Big(x + \frac{E_q - k^1}{M}\Big) \Theta(M^2 + m_q^2 + 2ME_q)\, \mathcal{G}^q(-ME_q) = 0\,. \tag{62e}$$

The "part 2" vanishes because x is positive and $(E_q - k^1)/M$ is also always positive for an on-shell particle such that the delta-function under the k^1-integral in (62e) is always zero.

The result for $f_1^q(x,k_T)$ due to "part 1" in (62a) coincides exactly with the expression from prior studies [25–38] (notice that in prior studies $\mathcal{G}^q(P\cdot k)$ was often denoted in the nucleon rest frame as $\mathcal{G}^q(E_q)$ or $\mathcal{G}^q(k^0)$ for simplicity). The calculation of $f_1^{\bar q}(x,k_T)$ is analog to the above calculation of the quark TMD with the labels $q \leftrightarrow \bar q$ interchanged, and other quark and antiquark TMDs are evaluated analogously. In order to list the final results and abbreviate the model expressions, it is convenient to introduce a compact notation for the flavor-dependent integration measures [30]

$$\{dk^1\}_{\rm unp}^a = \frac{dk^1}{E_q}\, \frac{\mathcal{G}^a(ME_q)}{E_q + m_q}\, \delta\Big(x - \frac{E_q + k^1}{M}\Big)\, \Theta(M^2 + m_q^2 - 2ME_q)\,, \tag{63a}$$

$$\{dk^1\}_{\rm pol}^a = \frac{dk^1}{E_q}\, \frac{\mathcal{H}^a(ME_q)}{E_q + m_q}\, \delta\Big(x - \frac{E_q + k^1}{M}\Big)\, \Theta(M^2 + m_q^2 - 2ME_q)\,, \tag{63b}$$

Summarizing all results in the compact notation of (63), we obtain

$$f_1^a(x,k_T) = \int \{dk^1\}_{\rm unp}^a \Big[\, xM(E_q + m_q)\,\Big], \tag{64a}$$

$$g_1^a(x,k_T) = \int \{dk^1\}_{\rm pol}^a \Big[\, x^2M^2 - xE_qM + xm_qM + m_q^2\,\Big], \tag{64b}$$

$$g_{1T}^{\perp a}(x,k_T) = \int \{dk^1\}^a_{\rm pol} \Big[xM^2 + m_q M \Big], \tag{64c}$$

$$h_1^a(x,k_T) = \int \{dk^1\}^a_{\rm pol} \Big[2xE_q M - \vec{k}_T^2 + 2xm_q M \Big], \tag{64d}$$

$$h_{1L}^{\perp a}(x,k_T) = \int \{dk^1\}^a_{\rm pol} \Big[-xM^2 - m_q M \Big], \tag{64e}$$

$$h_{1T}^{\perp a}(x,k_T) = \int \{dk^1\}^a_{\rm pol} \Big[-M^2 \Big], \tag{64f}$$

$$e^a(x,k_T) = \int \{dk^1\}^a_{\rm unp} \Big[m_q(E_q + m_q) \Big], \tag{64g}$$

$$f^{\perp a}(x,k_T) = \int \{dk^1\}^a_{\rm unp} \Big[M(E_q + m_q) \Big], \tag{64h}$$

$$g_T^a(x,k_T) = \int \{dk^1\}^a_{\rm pol} \Big[\vec{k}_T^2 + 2m_q E_q + 2m_q^2 \Big], \tag{64i}$$

$$g_L^{\perp a}(x,k_T) = \int \{dk^1\}^a_{\rm pol} \Big[xM^2 - E_q M \Big], \tag{64j}$$

$$g_T^{\perp a}(x,k_T) = \int \{dk^1\}^a_{\rm pol} \Big[M^2 \Big], \tag{64k}$$

$$h_L^a(x,k_T) = \int \{dk^1\}^a_{\rm pol} \Big[2xME_q - x^2M^2 + m_q E_q \Big], \tag{64l}$$

$$h_T^a(x,k_T) = \int \{dk^1\}^a_{\rm pol} \Big[E_q M - xM^2 \Big], \tag{64m}$$

$$h_T^{\perp a}(x,k_T) = \int \{dk^1\}^a_{\rm pol} \Big[M(E_q + m_q) \Big]. \tag{64n}$$

These results are valid for $a = q, \bar{q}$. The results for quark TMDs were obtained before in [30] in twist-2 and in [38] in twist-3 case. The results for antiquark TMDs are presented for the first time in this work. It should be noted that equivalent expressions can be obtained for the TMDs [30] by exploring $E_q^2 = k_1^2 + \vec{k}_T^2 + m_q^2$ and $E_q + k^1 = xM$ due to the on-shell condition and the delta-function present in the compact definition of the integration measure (63).

The model expression for quark and antiquark TMDs satisfy (56). This follows immediately from the properties of the amplitudes in (61b). It can also be verified explicitly by working with the model expressions at the level of (62c) before the k^0-integration is carried out and the contribution of "part 2" explicitly drops out. In fact, when we continue the expression for the quark TMD to negative x values, then the roles of $\Theta^+_{\rm kin}$ and $\Theta^-_{\rm kin}$ in (60) interchange: when evaluating $f_1^q(-x,k_T)$ the contribution from $\Theta^+_{\rm kin}$ drops out and that of $\Theta^-_{\rm kin}$ yields a non-zero result. In order to show that $f_1^q(-x,k_T) = -f_1^{\bar{q}}(x,k_T)$ one needs to perform substitutions $k^0 \to -k^0$ and $k^1 \to -k^1$ (or equivalently $k^\pm \to -k^\pm$ in lightcone coordinates). Under these substitutions $P\cdot k \to -P\cdot k$, cf. footnote 5, such that $\Theta^+_{\rm kin}(P\cdot k)$ is transformed into $\Theta^-_{\rm kin}(P\cdot k)$ and vice versa. Thus, we see that the full analytical structure of $\Theta_{\rm kin}(P\cdot k)$ is crucial for the proof of the relations in (56). Once the integration over k^0 is carried out and the "part 2" contribution drops out in (62e), the relations (56) are implicit and cannot be easily verified.

9.2. Relations among antiquark TMDs in the CPM

In QCD all TMDs are independent functions. But simpler model dynamics or additional model symmetries can generate relations between TMDs in models. In the quark case, relations among TMDs were found in several models. Based on the results of the previous section we present relations among antiquark TMDs which to the best of our knowledge have not been derived before. In fact, as shown in the previous section, the antiquark TMDs are formally given by same expressions as the quark TMDs with $\mathcal{G}^q(P\cdot k)$ replaced by $\mathcal{G}^{\bar q}(P\cdot k)$ and analog for $\mathcal{H}^{\bar q}(P\cdot k)$. Therefore, the antiquark TMDs and quark TMDs satisfy in the CPM the same model relations. Considering how little is known from models about nonperturbative properties of antiquark TMDs, the new relations are of interest.

In Sec. 8.2 we already discussed the EOM relations (49, 50), which are satisfied in the model with the tilde terms absent also in the antiquark case. Next, we consider the so-called quark model Lorentz invariance relations (qLIRs) which arise in effective theories without gluonic degrees of freedom where T-odd A_i^q amplitudes are absent and the 14 T-even TMDs are described in terms of 9 T-even amplitudes implying 5 relations given for antiquarks by [53]

$$g_T^{\bar q}(x) \overset{\text{qLIR}}{=} g_1^{\bar q}(x) + \frac{\mathrm{d}}{\mathrm{d}x} g_{1T}^{\perp(1)\bar q}(x)\,, \tag{65a}$$

$$h_L^{\bar q}(x) \overset{\text{qLIR}}{=} h_1^{\bar q}(x) - \frac{\mathrm{d}}{\mathrm{d}x} h_{1L}^{\perp(1)\bar q}(x)\,, \tag{65b}$$

$$h_T^{\bar q}(x) \overset{\text{qLIR}}{=} -\frac{\mathrm{d}}{\mathrm{d}x} h_{1T}^{\perp(1)\bar q}(x)\,, \tag{65c}$$

$$g_L^{\perp\bar q}(x) + \frac{\mathrm{d}}{\mathrm{d}x} g_T^{\perp(1)\bar q}(x) \overset{\text{qLIR}}{=} 0\,, \tag{65d}$$

$$h_T^{\bar q}(x,p_T) - h_T^{\perp\bar q}(x,p_T) \overset{\text{qLIR}}{=} h_{1L}^{\perp\bar q}(x,p_T)\,. \tag{65e}$$

The qLIRs are written such that twist-3 TMDs appear on the left-hand sides and twist-2 TMDs (if any) on the right-hand-sides. The CPM model expressions for antiquark TMDs (64) satisfy the qLIRs (65). The proofs are identical to the proofs of the corresponding qLIRs for quark TMDs and can be found in Ref. [30].

While the qLIRs (65) must be valid in all models with no explicit gluon degrees of freedom, in specific models further model relations may hold. In the CPM, the antiquark TMDs obey the following model-specific relations

$$g_{1T}^{\perp\bar q}(x,p_T) = -h_{1L}^{\perp\bar q}(x,p_T), \tag{66a}$$

$$g_T^{\perp\bar q}(x,p_T) = -h_{1T}^{\perp\bar q}(x,p_T), \tag{66b}$$

$$g_L^{\perp\bar q}(x,p_T) = -h_T^{\bar q}(x,p_T), \tag{66c}$$

$$g_1^{\bar q}(x,p_T) - h_1^{\bar q}(x,p_T) = h_{1T}^{\perp(1)\bar q}(x,p_T), \tag{66d}$$

$$g_T^{\bar q}(x,p_T) - h_L^{\bar q}(x,p_T) = h_{1T}^{\perp(1)\bar q}(x,p_T), \tag{66e}$$

and one more relation which coincides with the qLIR (65e). The quark-model relations are verified by directly inserting the model expressions (64). The analogous relations among quark TMDs were derived in CPM in [89] and are valid also in spectator, bag, and light-front constituent quark model [73,88,89,85,114]. The CPM also supports the following non-linear relations among antiquark TMDs

$$\frac{1}{2}\left[h_{1L}^{\perp\bar{q}}(x,p_T)\right]^2 = -\,h_1^{\bar{q}}(x,p_T)\,h_{1T}^{\perp\bar{q}}(x,p_T)\,, \tag{66f}$$

$$\frac{1}{2}\left[g_{1T}^{\perp\bar{q}}(x,p_T)\right]^2 = g_{1T}^{\perp\bar{q}}(x,p_T)\,g_L^{\perp\bar{q}}(x,p_T) + g_T^{\bar{q}}(x,p_T)\,g_T^{\perp\bar{q}}(x,p_T)\,. \tag{66g}$$

The linear and nonlinear relations were derived for quark TMDs in [30] in twist-2 and in [38] in twist-3 case.

For twist-2 quark TMDs, the deeper reason underlying the model relations (66a), (66d), (66g), can be traced back to a rotational symmetry of the lightcone wave functions in independent-particle models with quarks bound by mean fields [114]. It will be interesting to see whether the same arguments can be generalized to antiquark TMDs. Not all models support such relations with quark-target models [82] being one counter-example. However, we see that in the CPM the quark model relations are satisfied not only by quark TMDs but also by antiquark TMDs.

Let us remark that if one would impose in addition the SU(4) spin-flavor symmetry of the nucleon wave function, then additional relations would hold in the CPM for antiquark TMDs the same way they hold for quark TMDs [30,38]. In the quark case, the SU(4) spin-flavor symmetry is a useful approximate concept, but it is unclear whether it is a useful concept for antiquark TMDs. We therefore refrain from showing the results here, but it will be interesting to test the SU(4) symmetry in antiquark TMDs in future studies.

Last not least, let us remark that the Wandzura-Wilczek (WW) approximations for the PDFs $g_T^{\bar{q}}(x)$ and $h_L^{\bar{q}}(x)$ [115,69] are exact in the CPM in the antiquark case analogously to the quark case [26,38]. Similarly, the similar approximate "WW-type" relations for the transverse moments of the TMDs $g_{1T}^{\perp\bar{q}}$ and $h_{1L}^{\perp\bar{q}}$ [116] hold exactly in the CPM. Again, the proof is analogous to the quark case [30].

9.3. *Kinematic constraints, and limitations of the approach*

The partonic interpretation of TMDs in QCD is done in infinite momentum frame where $f_1^a(x,k_T)dx$ is interpreted as the probability to find a parton of flavor a carrying a fraction of the longitudinal nucleon momentum in the interval $[x,\, x+dx]$ and a transverse momentum $k_T = |\vec{k}_T|$. The other twist-2 TMDs have analogous interpretations, albeit involving polarization of the parton and/or the nucleon [54]. Bjorken-x is a Lorentz scalar, and transverse momenta are not affected by longitudinal boosts. Thus, we of course have the same x and k_T also in the nucleon rest frame (where, however, we no longer a partonic interpretation is applicable in QCD). If x and k_T are specified for an on-shell parton, then the parton 4-momentum $k^\mu = (E_q,\, k^1,\, \vec{k}_T)$ with $\vec{k}_T = (k^2,\, k^3)$ is completely fixed, and we have [34]

$$E_q = \frac{xM}{2} + \frac{\vec{k}_T^{\,2} + m_q^2}{2xM}\,, \tag{67a}$$

$$k^1 = \frac{xM}{2} - \frac{\vec{k}_T^{\,2} + m_q^2}{2xM}\,. \tag{67b}$$

In the nucleon rest frame, the step function $\Theta_+(P\cdot k)$ introduced in (60) can be expressed and rewritten thank to the on-shell condition as follows

$$\Theta_+(P\cdot k) = \Theta(M^2 + m_q^2 - 2M E_q) = \Theta(M^2 - m_q^2 - 2M|\vec{k}|)\,. \tag{68}$$

Hence we see that $E_q < (M^2+m_q^2)/(2M)$ and $|\vec{k}| \le (M^2-m_q^2)/(2M)$, i.e. the parton energy and parton 3-momentum are constrained in the nucleon rest frame [34]. From these bounds and

the relations (67) we see that x and k_T constrain each other. For instance, for a given k_T the Bjorken variable is in the range

$$\frac{1+x_{\min}}{2} - \sqrt{\Big(\frac{1-x_{\min}}{2}\Big)^2 + \frac{k_T^2}{M^2}} \;<\; x \;<\; \frac{1+x_{\min}}{2} + \sqrt{\Big(\frac{1-x_{\min}}{2}\Big)^2 + \frac{k_T^2}{M^2}} \tag{69a}$$

where $x_{\min}$ is the smallest kinematically possible x-value

$$x_{\min} = \frac{m_q^2}{M^2}\,. \tag{69b}$$

The extreme x-values in (69a) are assumed only when $k_T = 0$. The allowed x-range is then

$$x_{\min} \;<\; x \;<\; 1\,. \tag{69c}$$

This is also the x-range in which colinear PDFs have a finite support. In QCD, the range is $0 < x < 1$. Neglecting quark masses (as it is done in all practical applications in QCD and in the CMP), we see that the model respects this constraint. When m_q is finite or k_T is non-zero, then the x-range is more restricted according to (69a). Considering a fixed value of x, we also obtain a bound on the allowed transverse parton momenta, namely [34]

$$k_T^2 \;<\; (1-x)\,(x-x_{\min})\,M^2\,. \tag{69d}$$

This result shows a limitation of the model. In QCD, a necessary condition for the applicability of TMD factorization is that $k_T \ll Q$. The hard scale Q can in practice be large enough, such that transverse momenta $k_T \sim M$ and larger may be relevant for the phenomenological description of a DIS reaction [117]. However, from (69d) we see that in the CMP the transverse parton momenta cannot exceed the bound $k_T < \frac{1}{2}M$. In fact, the CMP yields for the mean transverse momenta $\langle k_T \rangle \sim 0.1\,\text{GeV}$ [31]. The consideration of offshellness effects is of importance for a more realistic description of nonperturbative TMD properties.

9.4. Determination of covariant functions, and the scale of the model

For completeness, let us briefly review the determination of the covariant functions $\mathcal{G}^a(P\cdot k)$ and $\mathcal{H}^a(P\cdot k)$ needed to obtain model predictions. The covariant functions are Lorentz-scalars and can be evaluated in any frame. It is convenient to work in the nucleon rest frame where $P\cdot k = M E_q$. The parton energy defined in (62b) depends only on the modulus of the parton 3-momentum. Thus, $\mathcal{G}^a(P\cdot k)$ and $\mathcal{H}^a(P\cdot k)$ are effectively functions of $|\vec{k}|$, and the CMP exhibits a 3D symmetry in momentum space which tightly connects longitudinal and transverse parton momenta[7] and gives predictive power to the approach allowing one, e.g., to make predictions for the x- and k_T-dependence of TMDs based on the knowledge of the x-dependence of the corresponding parton distribution functions.

More precisely, the knowledge of the x-dependence of two PDFs (for each of the flavors $a = u,\, d,\, \bar{u},\, \bar{d},\, \ldots$) is required to determine the covariant functions in the CMP and hence to predict all TMDs. The obvious choice are $f_1^a(x)$ and $g_1^a(x)$. From $f_1^a(x)$ one can uniquely determine $\mathcal{G}^a(P\cdot k)$ and from $g_1^a(x)$ one can uniquely determine $\mathcal{H}^a(P\cdot k)$. The corresponding inversion

[7] The underlying symmetry is a 3D symmetry in the nucleon rest frame. In any other frame, longitudinal and transverse parton momenta are still tightly connected, but we would not call it a 3D symmetry.

formulas have been derived in 3 independent ways, in Refs. [31,32] and [33] as well as in [39]. For the light flavors $a = u,\ d,\ \bar{u},\ \bar{d}$ the parton masses can be neglected, and one obtains

$$\mathcal{G}^q(P\cdot k) = -\frac{1}{\pi M^3}\left[\frac{d}{dx}\,\frac{f_1^a(x)}{x}\right] \tag{70a}$$

$$\mathcal{H}^q(P\cdot k) = \frac{1}{\pi x^2 M^3}\left[3g_1^a(x) + 2\int\limits_x^1 \frac{dy}{y}\,g_1^a(y) - x\,\frac{dg_1^a(x)}{dx}\right] \tag{70b}$$

where it is understood that $P\cdot k = \frac{1}{2}xM^2$. Due to the kinematics constraints (67), the variable $P\cdot k$ is for massless parton $0 < P\cdot k < \frac{1}{2}M^2$ when m_q is neglected (or $m_q M < P\cdot k < \frac{1}{2}(M^2 - m_q^2)$ if we keep track of parton masses).

Numerical predictions for quark TMDs in the pure-spin version of the model were presented in [33,38]. In the mixed-spin version of the parton model, one needs to introduce one additional covariant function which describes the chiral odd polarized TMDs and which can be determined from, e.g., transversity [39] similarly to (70).

In order to determine the covariant functions in (70) it is necessary to use $f_1^a(x)$ and $g_1^a(x)$ from a phenomenological parametrization at some chosen scale μ^2 which must be high enough for the parton model concept to be valid. But its exact value of this scale is unknown. In [33] the scale was chosen to be $\mu^2 = 4\,\mathrm{GeV}^2$ and in [38] it was chosen to be $\mu^2 = 2.5\,\mathrm{GeV}^2$. As different TMDs obey different evolution equations, the model and the relations among TMDs are valid only at this scale. TMDs strictly speaking depend on two scales, cf. Sec. 2.2, but one choice is $\zeta = \mu^2$.

9.5. Comparison to other approaches in literature

It is instructive to review first the spinor description in free theory. The spinors of a spin-$\frac{1}{2}$ fermion or antifermion of mass m_q and momentum k^μ polarized along the spacelike vector n^μ with $k\cdot n = 0$ are customarily denoted by $u(k,n)$ and $v(k,n)$ where we suppress an additional index which can assume 2 values and indicates a spin-up or spin-down state. If $n^2 = -1$ it is a pure-spin state, and if $-1 < n^2 < 0$ it is a mixed-spin state. The spinors satisfy

$$(\not{k} - m_q)\,u(k,n) = 0\,, \quad (\not{k} + m_q)\,v(k,n) = 0\,. \tag{71}$$

We choose the normalization $\bar{u}(k,n)u(k,n) = -\bar{v}(k,n)v(k,n) = 2m_q$. The polarization is revealed by acting on the spinors with the Pauli-Lubanski vector $W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\hat{J}^{\nu\rho}\hat{P}^\sigma$, where $\hat{J}^{\mu\nu}$ and $\hat{P}^\nu$ are respectively the generators of rotations and translations of the Poincaré group, projected on n^μ as follows

$$\frac{(-W\cdot n)}{m_q}\,u(k,n) = \pm\frac{1}{2}\,u(k,n)\,, \qquad \frac{(-W\cdot n)}{m_q}\,v(k,n) = \pm\frac{1}{2}\,v(k,n)\,, \tag{72}$$

with the two signs depending on whether the particles are spin-up or spin-down. If in the particle rest frame one chooses $n^\mu = (0,0,0,1)$, then the operator $(-W\cdot n)/m_q$ coincides with the z-component of the familiar intrinsic spin operator. The spinors satisfy the completeness relations

$$u(k,n)\otimes\bar{u}(k,n) = (\not{k} + m_q)P(n)\ , \tag{73a}$$

$$v(k,n)\otimes\bar{v}(k,n) = (\not{k} - m_q)P(n)\ . \tag{73b}$$

In (73a), (73b) a summation over spin-up and spin-down states is understood (we recall that we suppress the corresponding index, see above), and we introduced the matrix

$$P(n) = \frac{1}{2}(1 + \gamma_5 \not{n}) \tag{73c}$$

For a pure-spin state, $P(n)$ is a projector, i.e. $P(n)^2 = P(n)$. Notice that $P(n)$ projects out a fermion in a spin-up state with respect to n^μ *but* an antifermion in a spin-down state, cf. the footnotes 3 and 4. For a mixed-spin state, $P(n)$ is a spin density matrix with the properties $\mathrm{tr}\, P(n) = 2$ and $\mathrm{tr}\, P(n)^2 < 4$.

At this occasion it is useful to review also the concept of helicity defined as expectation value of the Pauli-Lubanski spin operator projected on $\vec{k}/|\vec{k}|$ for a particle with non-zero 3-momentum [4]. A spin-$\frac{1}{2}$ particle polarized along (opposite to) its direction of motion has the helicity $+\frac{1}{2}$ ($-\frac{1}{2}$). Thus, λ in (26) denotes twice the helicity of a quark, while in (39a) it denotes (-1)×twice the helicity, see footnote 4. Notice that longitudinal boosts do not affect helicity with the obvious exception, for massive particles, of boosts into frames where the particles move backwards (in which case the helicity flips sign) and boosts into particle rest frames (in which case helicity is undefined).

We shall not review the massless case here in detail, but only remark that λ is a Lorentz-scalar for massless particles, while the transverse spin is not affected by longitudinal boost. This leads to the description of massless quarks and antiquarks introduced (26) and (39a) where λ is twice the helicity of a quark and b_T^μ the transverse polarization vector for a quark (with opposite signs for an antiquark).

This mini-review paves the way to the description of the quark correlator $\Phi^q(k, P, S)$ in the free-quark-target model [45] where P^μ, S^μ denote momentum and spin vector of the quark target while k^μ and n^μ are the momentum and spin vector of the parton inside the quark target. Based on (73a), the quark correlator is given by [45]

$$\Phi^q(k, P, S) = u(k, n) \otimes \bar{u}(k, n)\, \delta^{(4)}(P - k) = (\not{k} + m_q) \frac{1 + \gamma_5 \not{n}}{2}\, \delta^{(4)}(P - k) \tag{74}$$

The antiquark correlator $\Phi^{\bar{q}}(k, P, S)$ is zero, because the probability to find an antiquark in a quark target is of course zero in a free theory. The quark target model becomes non-trivial and then much more interesting when gluon interactions are included and treated with perturbative QCD methods [82].

From (74) one can obtain a formulation of Feynman's parton model [1,2] by replacing the quark target with a nucleon target. This is customarily formulated in terms of lightcone coordinates and sometimes referred to as the "free quark ensemble model" [45]. Instead of the $\delta^4(P - k)$ in (74) one has some probabilities $\mathcal{P}_q(k)$ to find quarks with momentum k^μ and polarization $s_q^\mu(k)$ inside the nucleon target with momentum P^μ and polarization S^μ. Now there is also a non-zero chance to find antiquarks with the corresponding antiquark probabilities described by $\mathcal{P}_{\bar{q}}(k)$ and a polarization vector $s_{\bar{q}}^\mu(k)$. For quarks, the polarization 4-vector is given by

$$s_q^\mu(k) = \lambda_q\, n_q^\mu + s_{qT}^\mu \tag{75}$$

where λ_q is the lightcone helicity, and the helicity vector n_q^μ satisfies $n_q^2 = -1$ and $n_q \cdot k = 0$, and it is also $k \cdot s_{qT} = 0$. For antiquarks, one has analogous definitions. λ_q and s_{qT}^μ may depend on parton momenta. The correlators of quarks and antiquarks can be compactly expressed as [45]

$$\Phi^q(k,P,S) = \delta(k^2 - m_q^2)\Big[\Theta(k^+)\,\mathcal{P}_q(k)\Big(\not{k} - m_q\Big)\Big(1 + \gamma_5 \not{S}_q(k)\Big) + \Theta(-k^+)\,\mathcal{P}_{\bar{q}}(k)\Big(\not{k} + m_q\Big)\Big(1 + \gamma_5 \not{S}_{\bar{q}}(k)\Big)\Big]. \tag{76}$$

Except for the lightcone formulation and different notation, this is equivalent to the description of quark and antiquark correlators in this work: $\mathcal{P}_q(k)$ is basically equivalent to A_3^q in our work, $\lambda_q \mathcal{P}_q$ to A_8^q and the momentum dependence of $s_{qT}^\mu \mathcal{P}_q(k)$ is contained in A_{11}^q, and similarly for antiquarks. However, to the best of our knowledge no attempt was made in the free-quark ensemble model to consequently explore the free equation of motion and practically determine the covariant functions.

The program of practically exploring the parton model to describe the nucleon structure was pioneered in [25–27]. In these works, the starting point was a covariant description of the hadronic tensor and unpolarized and polarized DIS structure functions considering exact kinematics of free, relativistic, on-shell partons whose momentum distributions are governed by covariant functions $\mathcal{G}^a(P\cdot k)$ (for unpolarized partons) and $\mathcal{H}^a(P\cdot k)$ (for polarized partons). The polarization vector w_q^μ was constructed starting from $w_q^\mu = A\,P^\mu + B\,C^\mu + C\,k^\mu$ and the coefficients A, B, C (in general functions of $k\cdot S$ and/or $P\cdot k$) were determined by imposing the constraints $k\cdot w_q = 0$ and $w_q^2 = -1$ (which corresponds to a pure-spin state) [26], see also [35]. The Callan-Gross relation between unpolarized structure functions and the WW relation between polarized structure functions, which are both approximations in QCD, become exact in the approach [25–29] which also generates the Cahn effect [31] and makes specific predictions about quark orbital angular momentum solely from the relativistic motion of quarks [35].

In Ref. [28], by exploring an auxiliary polarized process due to the interference of vector and scalar currents, the approach was used to compute a hypothetical chiral odd structure function and the transversity PDF $h_1^q(x)$. In [30] the model was extended to TMDs, by introducing the concept of "unintegrated structure functions" which lead to the description of twist-2 T-even TMDs $f_1^q(x,k_T)$, $g_1^q(x,k_T)$, $h_1^q(x,k_T)$, $g_{1T}^{\perp q}(x,k_T)$, $h_{1L}^{\perp q}(x,k_T)$, $h_{1T}^{\perp q}(x,k_T)$, and the twist-3 TMD $g_T^q(x,k_T)$ which has a colinear counterpart but it was unclear how to describe other twist-3 TMDs. This was accomplished and a systematic description of all twist-2 and twist-3 TMDs in [38], where the starting point was the quark spinor description (73a) in combination with the covariant functions $\mathcal{G}^a(P\cdot k)$ and $\mathcal{H}^a(P\cdot k)$.

Prior to the latter work, in Ref. [39] a parton model framework was developed based on the same concepts as in Ref. [25–38], but centered around a consequent exploration of the equation of motion in the quark correlator language with the puzzling result that in "one parton model" the nucleon structure is described in terms of 2 independent covariant functions [25–38] in "another parton model" 3 independent covariant functions are necessary for that [39]. (As explained in Sec. 1, the motivation of our study was to resolve the puzzle and we extended [39] by systematically including quark mass effects, the antiquark correlator, and pure-spin vs mixed-spin parton polarization states.)

The exploration of covariant parton models dates back earlier works [40–42,44,43] where structure functions in electron-nucleon DIS or electro-weak reactions as well as target mass corrections were studied. An interesting extension of the parton model was carried out in the quantum statistical approach of Refs. [47–51] where the nucleon was treated as a gas of massless partons (quarks, antiquarks, gluons) in a finite size volume in thermal equilibrium at a common temperature. The model exhibits in the nucleon case a total of 8 parameters which were fixed

through fits to DIS data. This number of free parameters should be compared with the typically $\mathcal{O}$(20-25) free parameters in global fits of PDF parametrizations, and it was argued that the statistical model provides a physically motivated and streamlined Ansatz for global fits [47–51]. The possibility to associate partonic motion and mean transverse momenta $\langle k_T \rangle$ with temperature was explored in [118].

10. Conclusions

We have studied the description of TMDs in the parton model. We have explored the equations of motion to show that the quark correlator can be expressed in terms of a spin density matrix which must be treated differently in the cases of massive and massless quarks. In either case, one has a choice how to describe the quark polarization state.

If one chooses to work with quarks in a pure-spin state, the nonperturbative information content of the quark correlator is described in terms of two independent amplitudes, which can be chosen to be A_3^q describing unpolarized TMDs and A_8^q describing polarized TMDs. If one chooses to work with quarks in a mixed-spin state, the nonperturbative information is described in terms of three independent amplitudes, which can be chosen to be A_3^q describing unpolarized TMDs and A_8^q and A_{11}^q describing, respectively, polarized chiral even and chiral odd TMDs.

One central result of our work is that we have reconciled conflicting results in literature regarding how many independent covariant functions are needed to describe the nonperturbative information contained in the quark correlator in the parton model, namely 2 vs 3 in Refs. [38] vs [39]. In fact, there really is only one unifying parton model framework to which we refer as Covariant Parton Model (CPM) where, however, one can choose to work with quarks in pure- or mixed-spin states. The pure-spin state version of the CPM was explored in Refs. [25–38] and describes the nucleon structure in terms of two independent amplitudes. The mixed-spin version of the CPM was studied in Ref. [39] for massless partons and describes quark TMDs in terms of 3 independent amplitudes.

The assumption of massless quarks is natural since in DIS processes current quark mass effects are suppressed by powers of $m_q/Q \ll 1$ where Q is the hard scale of the process. In this work, we have considered $m_q \neq 0$ and $m_q = 0$. The results for TMDs are the same whether one keeps $m_q \neq 0$ and neglects current quark mass effects at the end [25–38] or works with massless quarks from the very beginning [39]. But the description of the quark spin-density matrix differs in the two cases. Keeping track of current quark mass effects has the advantage that one can use the QCD equations-of-motion (EOM) relations to check consistency. In the CPM, the quarks are non-interacting and TMDs must satisfy the EOM relations with (pure twist-3) tilde terms neglected which we have shown to be the case. In the quark case, we have rederived previous model results [25–39].

Interestingly, the CPM cannot predict the sign of polarized chiral odd TMDs. Information from other nonperturbative methods (models, lattice) is necessary to inform the CPM and choose the physical sign for chiral odd polarized TMDs. Once the physical solution is chosen using one function as input, the CPM predicts unambiguously the signs of all other polarized chiral odd TMDs in agreement with other models. The results from many quark models refer to a low initial scale. The precise scale at which, e.g., PDFs should be evaluated in the CPM is not known. But the parton model concept is valid at high energies, and the initial scale of the CPM was, e.g., chosen to be $\mu_0^2 = 4\,\text{GeV}^2$ [25–38]. The choice of the scale is part of the model.

Another important result is that we have extended the treatment to include antiquarks. The quark and antiquark correlators are connected to each other by a field theoretical relation which,

however, we have not imposed. Rather, we have studied the TMDs of quarks and antiquarks independently and used the field theoretical connection in the end of day to verify the theoretical consistency. To the best of our knowledge, we have presented for the first time a complete discussion of all T-even leading and subleading antiquark TMDs in a consistent framework. We have shown that in the CPM the antiquark TMDs satisfy the same linear and non-linear relations as quark TMDs. This result might be of interest for modeling of antiquark effects in phenomenology. It will be interesting to use the model for numerical predictions of antiquark TMDs and study their impact in phenomenology which we leave to future studies.

The simple covariant parton model for quark and antiquark correlators obtained in this work may provide the basis for the modeling of unintegrated parton densities and may help to consistently implement TMD effects in Monte Carlo event generators [63–65]. The approach receives general support from the fact that the WW approximation for $g_T^a(x)$ is supported by data with an accuracy of 40 % or better [119], and according to first phenomenological studies the WW-type approximation for $g_{1T}^{\perp a}$ seems to work similarly well [120]. Further phenomenological applications of the Wandzura-Wilczek-type approximation were practically implemented in [121]. It will be interesting to see whether the TMDs in nature can be better approximated in terms of a pure-spin or mixed-spin state model.

The model results may also be of interest to study TMDs at small transverse momenta $k_T \ll M$ where M denotes the nucleon mass. In the Collins-Soper-Sterman equations governing the evolution of TMDs, the region of small k_T is correlated with large impact parameters b_T. Not much is known about TMDs in b_T-space in the region of large $b_T \gtrsim 1$ fm. Here the model results could provide useful insights which will be explored elsewhere.

When k_T is not small, the model becomes less realistic. In fact, due to the absence of interactions, the partons are on mass-shell and as a consequence of that their transverse momenta are bound from above by $k_T < \frac{1}{2} M$. It would be interesting to explore the possibility of introducing in the CPM a way to include offshellness effects and investigate whether this can lead to a more realistic modeling of k_T-dependencies of TMDs.

The model also does not exhibit the initial- or final-state interactions as encoded in Wilson lines which can generate phases and give rise to T-odd TMDs like Sivers function [52] or Boer-Mulders function [54]. The modeling of T-odd TMDs is therefore beyond the scope of the CPM. It would be very interesting to explore the possibility of introducing the necessary phases and extend the approach to the modeling of T-odd TMDs. These topics will be left to future investigations.

CRediT authorship contribution statement

F. Aslan: Conceptualization, Methodology, Writing – review & editing. **S. Bastami:** Conceptualization, Methodology, Writing – review & editing. **P. Schweitzer:** Conceptualization, Funding acquisition, Methodology, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

This work is dedicated to the memory of Anatoli Vasilievich Efremov, an outstanding physicist, dear colleague and wise mentor who made numerous important contributions to hadronic and spin physics in his fruitful career over more than six decades.

This work was supported by the National Science Foundation under the Award No. 1812423 and Award No. 2111490, by the U.S. Department of Energy, under the contract no. DE-AC05-06OR23177, under which Jefferson Science Associates, LLC operates Jefferson Lab, and within the framework of the TMD Collaboration.

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