

Sufficient Conditions for Persistency of Excitation with Step and ReLU Activation Functions

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Abstract—This paper defines geometric criteria which are then used to establish sufficient conditions for persistency of excitation with vector functions constructed from single hidden-layer neural networks with step or ReLU activation functions. We show that these conditions hold when employing reference system tracking, as is commonly done in adaptive control. We demonstrate the results numerically on a system with linearly parameterized activations of this type and show that the parameter estimates converge to the true values with the sufficient conditions met.

I. INTRODUCTION

Persistency of excitation is a fundamental concept employed within contexts and applications related to parameter learning, such as system identification and adaptive control. It is often discussed, or at least mentioned, in adaptive control textbooks, such as in [1]–[3].

It was proven in [4], [5] that persistency of excitation is necessary and sufficient for the global uniform asymptotic stability of the linear time-varying (LTV) system

$$\dot{\tilde{\theta}}_t = -V_t V_t^\top \tilde{\theta}_t \quad (1)$$

where $\tilde{\theta}_t \in \mathbb{R}^n$ is the system state, and $V_t \in \mathbb{R}^{n \times d}$ is a vector ($d = 1$) or matrix ($d \geq 2$) function of time that is *regulated* (one-sided limits exist for all $t \in [0, \infty)$). Consider if $\tilde{\theta}_t = \hat{\theta}_t - \theta$ represents the error of a parameter estimate $\hat{\theta}_t$ from some fixed, unknown parameter values θ . Then if V_t is persistently exciting, the state of this system (ie, the error of the parameter estimates) converges globally uniformly asymptotically to zero.

In (non)linear systems with linear parameterizations, the parameter estimate error dynamics commonly have the form (1). For example, in [2] (sec 8.7), we see examples of systems that can be formed into a model $y_t = V_t \theta$, where the vector y_t and the matrix V_t are measurable, and then using a simple gradient-based update rule for the parameter estimate $\hat{\theta}_t$, within an estimator system $\hat{y}_t = V_t \hat{\theta}_t$, gives exactly these dynamics for the parameter estimate error $\hat{\theta}_t - \theta$. Thus, if V_t is persistently exciting, the parameter estimate error will converge to zero.

There have been many works since [4], [5] which utilize an assumption of persistency of excitation in order

to achieve results in parameter learning. In [6], a simple parameter learning scheme can be employed for a general class of nonlinear systems which have some kind of working (nonlinear) feedback controller. An integral condition similar to persistency of excitation is assumed to be satisfiable. This inspired the work in [7], which assumes a similar integral condition in order to identify, and then provide MRAC control for, an unknown MIMO LTI system. And in [3], [8] we see additional examples of MRAC control which assume persistency of excitation in order to achieve parameter convergence, while in [9], persistently exciting assumptions are made in reinforcement learning applications. In [10], a sufficient condition for windows of observed behavior of an LTI system (in discrete time) to span the space of possible windows, is for a component signal (like the input) to be persistently exciting. In [11], conditions for neural networks excitation are given to guarantee bounds on the function estimate error. Lastly, in [12] it is proven that for a general class of nonlinear systems which are feedback linearizable (see [13], [14]), global uniform asymptotic stability can be achieved for linearly parametrized vector functions meeting relaxed persistency of excitation conditions.

However, in these and other works which utilize persistency of excitation assumptions, there is often no explicit sufficient conditions provided for how to ensure that persistency of excitation is satisfied.

On the other hand, there have been some works which do provide these sufficient conditions. A classic result is that the state of an LTI system satisfies persistency of excitation if the (stationary) input to that system contains sufficient frequency content (“sufficient richness”, see [15]). In [16], [17] this is extended to certain linear time-varying systems, while in [18] frequency arguments for sufficient conditions for excitation are then extended to nonlinear systems in parametric-strict-feedback form and in [19] to the context of adaptive dynamic programming, in which optimal control value functions are approximated using polynomial basis functions. In [20], a rank condition is proven sufficient and necessary for the state of time-invariant systems to be persistently exciting, and in [21], strong Lyapunov functions are provided that are equivalent to the persistency of excitation condition. Finally, and closest in spirit to this paper, in [22] a sufficient condition, based on geometric criteria, is given for satisfying persistency of excitation with vector functions composed of radial basis functions.

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Our primary contributions are sufficient conditions, based on geometric criteria, for satisfying persistency of excitation with vector functions $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^N$ which are composed of ReLU or step activation functions, together with affine transformations of the state space. We then demonstrate this using a simulated MRAC control application with the parameter estimates converging to the true values.

The organization of the remaining parts of the paper are as follows. Section II provides preliminary notation. Section III sets up problem definitions and geometry. Section IV presents the main theoretical results, while Section V presents numerical results, and closing remarks are given in Section VI.

II. NOTATION

We interpret $w, x \in \mathbb{R}^n$ as column vectors and denote their inner product as $w^\top x$. Similarly, we denote the product of matrix $W \in \mathbb{R}^{n \times N}$ with vector x as $W^\top x$, which is a length N vector where the i th (row) element is the inner product $W_i^\top x$. Index subscripts on vectors and matrices denote the row index, for example W_i^\top is the i th row of W^\top . The standard Euclidean norm and Frobenius matrix norm are respectively denoted $\|w\|_2$ and $\|W\|_F$. The integer set $\{1, \dots, k\}$ is denoted by $[k]$. We use t subscripts on time-dependent variables to reduce parentheses, for example $\phi(x(t))$ is instead denoted $\phi(x_t)$. The i th row of a time-varying vector or matrix is thus denoted with t, i subscript. For square $n \times n$ matrix A , we use $aI_n \preceq A \preceq bI_n$ to denote that $\text{eig}(A) \in [a, b]$, where I_n denotes the $n \times n$ identity matrix. We denote the n -length zeros vector as 0_n .

III. SETUP

This section formally defines the nonlinear vector function ϕ and what persistency of excitation means with regards to this definition, then describes the geometry induced on the state space \mathbb{R}^n by this construction.

A. Nonlinear, Positive Semidefinite Activation Functions

Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^N$ be a vector function defined as

$$\phi(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_N(x) \end{bmatrix} = \begin{bmatrix} \sigma(w_1^\top x + b_1) \\ \vdots \\ \sigma(w_N^\top x + b_N) \end{bmatrix}, \quad (2)$$

where $\phi_1, \dots, \phi_N : \mathbb{R}^n \rightarrow \mathbb{R}$ are composed of *nonlinear*, piecewise continuous functions $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ together with affine transformations $w_1^\top x + b_1, \dots, w_N^\top x + b_N : \mathbb{R}^n \rightarrow \mathbb{R}$. We allow $w_i \in \mathbb{R}^n \setminus \{0_n\}$ and $b_i \in \mathbb{R}$ to be arbitrary for all $i \in [N]$, except we assume each $w_i^\top x + b_i = 0$ hyperplane in \mathbb{R}^n is unique with dimension $n - 1$. Let $W = [w_1 \ \dots \ w_N]$ and $b = [b_1 \ \dots \ b_N]^\top$. For any $\mathcal{S} \subset [N]$, let $W_{\mathcal{S}}^\top$ and be the submatrix of W^\top with rows given by w_i^\top for $i \in \mathcal{S}$. Define $b_{\mathcal{S}}$ similarly for b .

Note then that (2) is equivalent to the output of a single hidden neural network layer with N neurons fully connected to the input $x \in \mathbb{R}^n$, having nonlinear (eg, ReLU)

activations, and being initialized with weights and biases defining unique hyperplanes. Hence why we refer to σ as an *activation* function or simply an activation. This paper will focus on the following activations:

$$\sigma_{cs}(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ c & \text{if } y > 0 \end{cases} \quad (\text{scaled step}) \quad (3)$$

$$\sigma_r(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ y & \text{if } y > 0 \end{cases} \quad (\text{ReLU}), \quad (4)$$

where $c > 0$ is an arbitrary positive scalar.

B. Persistency of Excitation

Let $x : [0, \infty) \rightarrow \mathbb{R}^n$ be some continuous trajectory in the state space. Then, $\phi(x_t)$ is a piecewise continuous (and regulated) vector function of time, mapping $[0, \infty) \rightarrow \mathbb{R}^N$. For any time window $t \in [\tau, \tau + T]$ with $\tau \geq 0$ and $T > 0$, the integral

$$\int_{\tau}^{\tau+T} \phi(x_t) \phi(x_t)^\top dt \quad (5)$$

defines a $N \times N$ *Gramian matrix* since the corresponding i, j -th entry $\int_{\tau}^{\tau+T} \phi_i(x_t) \phi_j(x_t) dt$ is an inner product of the composition functions $\phi_1(x_t), \dots, \phi_N(x_t) : [0, \infty) \rightarrow \mathbb{R}$. Gramian matrices are always positive semidefinite, which can be shown using the bi-linearity of inner products.

Persistency of excitation is the requirement that the Gramian matrix (5) must be strictly positive definite, with eigenvalues in some bounded interval $[\alpha_1, \alpha_2]$, over all shifts $\tau \geq 0$ of the sliding time window $[\tau, \tau + T]$ for some window length $T > 0$. Formally, persistency of excitation requires existence of constants $\alpha_1, \alpha_2, T > 0$ such that

$$\alpha_1 I_N \preceq \int_{\tau}^{\tau+T} \phi(x_t) \phi(x_t)^\top dt \preceq \alpha_2 I_N \quad (6)$$

holds for all $\tau \geq 0$. An equivalent scalar requirement is that

$$\alpha_1 \|v\|_2^2 \leq \int_{\tau}^{\tau+T} (v^\top \phi(x_t))^2 dt \leq \alpha_2 \|v\|_2^2 \quad (7)$$

must hold for all $v \in \mathbb{R}^N$ and $\tau \geq 0$. This follows since $v^\top \phi(x_t) = \phi(x_t)^\top v$.

Note that if these hold for some $T > 0$, then they also hold for all $\tilde{T} > T$. This follows because the integrals can be broken into a sum of two integrals over $t \in [\tau, \tau + T]$ and $t \in [\tau + T, \tau + \tilde{T}]$, with the former being strictly positive (definite) and the latter positive semidefinite (nonnegative).

C. Activation Geometry

We define the following subsets for all activations $i \in [N]$:

$$X_i^+ = \{x \in \mathbb{R}^n \mid w_i^\top x + b_i > 0\} \quad (\text{active})$$

$$X_i^0 = \{x \in \mathbb{R}^n \mid w_i^\top x + b_i \leq 0\} \quad (\text{zero}) .$$

Each is a half-space of \mathbb{R}^n formed by the hyperplane $w_i^\top x + b_i = 0$. These are then used to define the *activation regions* \mathcal{A}_j , with indices corresponding to a binary string indicating

which active (binary 1) and zero (binary 0) half-spaces are in the intersection:

$$\begin{aligned}\mathcal{A}_0 &= X_N^0 \cap X_{N-1}^0 \cap \dots \cap X_2^0 \cap X_1^0 \\ \mathcal{A}_1 &= X_N^0 \cap X_{N-1}^0 \cap \dots \cap X_2^0 \cap X_1^+ \\ &\vdots \\ \mathcal{A}_{2N-1} &= X_N^+ \cap X_{N-1}^+ \cap \dots \cap X_2^+ \cap X_1^+.\end{aligned}$$

These partition \mathbb{R}^n into, at most, 2^N convex polytopes. It is likely that some of the regions will be infeasible ($\mathcal{A}_j = \emptyset$). For example, in the $n = 1$ case, there are always only $N+1$ feasible activation regions, since there will be N unique (by assumption) points partitioning the \mathbb{R} line. In higher dimensions, more feasible regions are possible.

For all $j \in \{0, \dots, 2N-1\}$, we define the *active set* \mathcal{S}_j as

$$\mathcal{S}_j = \{i \in [N] \mid w_i^\top x + b_i > 0 \quad \forall x \in \mathcal{A}_j\}.$$

This captures which of the N activations are active in a particular activation region. For any two activation regions $j, k \in \{0, \dots, 2N-1\}$ with a nonempty intersection $\mathcal{A}_j \cap \mathcal{A}_k \neq \emptyset$, their intersection defines a *border* between the two regions. We define a *nondegenerate border* to mean $\dim(\mathcal{A}_j \cap \mathcal{A}_k) = n-1$. In this case, only one activation is different (active to zero or vice-versa) between those regions. This is because the borders between activation regions must be (subsets of) the hyperplanes that define the half-spaces X_i , and the intersection of more than one unique hyperplane with dimension $n-1$ must have a dimension *less than* $n-1$. Thus, a nondegenerate border must be a subset of (or equal to) a single hyperplane, meaning only one X_i half-space can flip from active to zero or vice-versa. We then define a *degenerate border* to mean $\dim(\mathcal{A}_j \cap \mathcal{A}_k) < n-1$. In this case, the border is a (subset of) the lower dimensional intersection of two or more unique hyperplanes. Thus, multiple activations are different between the regions.

Now consider a continuous state trajectory $x_t \in \mathbb{R}^n$ visiting a sequence of activation regions over the time window $[\tau, \tau+T]$, for some $\tau \geq 0$ and $T > 0$. Assume x_t only crosses nondegenerate borders and that the number of regions visited is $\mathcal{L} \geq 2$. Let us denote $\mathcal{A}_1, \dots, \mathcal{A}_{\mathcal{L}}$ as the activation region indices of the visited sequence and $\mathcal{I}_1, \dots, \mathcal{I}_{\mathcal{L}-1}$ as the sequence of activation indices corresponding to the hyperplanes crossed in order to visit that sequence of activation regions. For all $s \in [\mathcal{L}]$, we define time window subsets $\mathcal{T}_s \subset [\tau, \tau+T]$ as

$$\mathcal{T}_s = \{t \in [\tau, \tau+T] \mid x_t \in \mathcal{A}_{\mathcal{A}_s}\}. \quad (8)$$

Since we assume x_t crosses only nondegenerate borders when visiting the activation regions during the time window, we have by definition over all $s \in [\mathcal{L}-1]$ that

$$\mathcal{S}_{\mathcal{A}_{s+1}} = \begin{cases} \mathcal{S}_{\mathcal{A}_s} \cup \{\mathcal{I}_s\} & \text{if } \mathcal{A}_{\mathcal{A}_{s+1}} \subseteq X_{\mathcal{I}_s}^+ \\ \mathcal{S}_{\mathcal{A}_s} \setminus \{\mathcal{I}_s\} & \text{if } \mathcal{A}_{\mathcal{A}_{s+1}} \subseteq X_{\mathcal{I}_s}^0 \end{cases}. \quad (9)$$

IV. THEORETICAL RESULTS

This section presents our main theoretical results, which provide sufficient conditions for satisfying persistency of excitation with (scaled) step or ReLU activations.

A. Main Results

Theorem 1: *Let state trajectory x_t be continuous and stay within some compact set $\mathcal{B} \subset \mathbb{R}^n$ for all $t \geq 0$, and let $\phi(x_t) = [\sigma_{cs}(w_1^\top x_t + b_1), \dots, \sigma_{cs}(w_N^\top x_t + b_N)]$ be composed of (scaled) step functions (3) with positive scalars $c = [c_1, \dots, c_N]^\top$ together with N unique affine transformations of \mathbb{R}^n according to $w_1, \dots, w_N \in \mathbb{R}^n \setminus \{0_n\}$ and $b_1, \dots, b_N \in \mathbb{R}$. If x_t over $t \geq 0$ is such that there exists a window length $T^* > 0$ whereby the sequence $s \in [\mathcal{L}]$ of activation regions visited during any shift $\tau \geq 0$ of the time window $[\tau, \tau+T^*]$ always satisfies the following two conditions:*

1. *all $i \in [N]$ hyperplanes $w_i^\top x_t + b_i = 0$ are crossed*
 2. *only nondegenerate borders are crossed,*
- then $\phi(x_t)$ satisfies the persistency of excitation conditions (6) and (7).*

Proof: Given in Appendix I of [23]. \blacksquare

Theorem 2: *Let state trajectory x_t be continuous and stay within some compact set $\mathcal{B} \subset \mathbb{R}^n$ for all $t \geq 0$, and let $\phi(x_t) = [\sigma_r(w_1^\top x_t + b_1), \dots, \sigma_r(w_N^\top x_t + b_N)]$ be comprised of ReLU functions (4) together with N unique affine transformations of \mathbb{R}^n according to $w_1, \dots, w_N \in \mathbb{R}^n \setminus \{0_n\}$ and $b_1, \dots, b_N \in \mathbb{R}$. If x_t over $t \geq 0$ is such that there exists a window length $T^* > 0$ whereby the sequence $s \in [\mathcal{L}]$ of activation regions visited during any shift $\tau \geq 0$ of the time window $[\tau, \tau+T^*]$ always satisfies the following three conditions:*

1. *all $i \in [N]$ hyperplanes $w_i^\top x_t + b_i = 0$ are crossed*
2. *only nondegenerate borders are crossed*
3. *for each $s \in [\mathcal{L}]$, there are times $t_1, \hat{t}_1, \dots, t_M, \hat{t}_M \in \mathcal{T}_s$ such that*

$$\text{rank}(W_{\mathcal{A}_{\mathcal{A}_s}}^\top) = \text{rank} \left(W_{\mathcal{A}_{\mathcal{A}_s}}^\top [x_{t_1} - x_{\hat{t}_1} \dots x_{t_M} - x_{\hat{t}_M}] \right),$$

then $\phi(x_t)$ satisfies the persistency of excitation conditions (6) and (7).

Proof: Given in Appendix I of [23]. \blacksquare

Note that a sufficient condition for property (iii) is that $[x_{t_1} - x_{\hat{t}_1} \dots x_{t_M} - x_{\hat{t}_M}]$ has rank n . This can be achieved, for example, if x_t is the state trajectory of a system that satisfies a suitable (local) controllability property.

B. Proof Sketch

Both proofs rely on the same overall contradiction method. That is, we assume there exists a nonzero vector $v \in \mathbb{R}^N \setminus \{0_N\}$ such that

$$v^\top \left(\int_{\tau}^{\tau+T} \phi(x_t) \phi(x_t)^\top dt \right) v = 0. \quad (10)$$

We show that if the state trajectory x_t meets certain requirements over any shift $\tau \geq 0$ of the window $[\tau, \tau+T^*]$, for

some window length $T^* > 0$, then in fact (10) can only hold if $v = 0_N$. This is a contradiction and thus proves that the LHS integral must be strictly positive definite, uniformly over all windows $[\tau, \tau + T^*]$ for all $\tau \geq 0$.

V. NUMERICAL RESULTS

In this section we provide a numerical simulation¹ of the theoretical results, using a MRAC application which is a variation on the setup from Chapter 9 of [3]. The plant and reference systems have $n = 2$ states, allowing convenient visualization of the hyperplanes and state space.

A. Setup

The plant is given by

$$\dot{x}_t = A x_t + B(u_t + \Theta^\top \phi(x_t)) , \quad (11)$$

where A is a known $n \times n$ state matrix for the plant state $x_t \in \mathbb{R}^n$, B is a known $n \times \ell$ input matrix for the input $u_t \in \mathbb{R}^\ell$, and Θ is an unknown $N \times \ell$ matrix which linearly parameterizes the known vector function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^N$ defined by (2). The setup in [3] also includes an unknown diagonal scaling matrix Λ , such that the overall input matrix is $B\Lambda$. We have omitted this for simplicity.

The control input u_t will be designed in order to force the plant states x_t to track the states of a reference system x_t^r that is driven by a bounded reference input r_t . The reference system is given by

$$\dot{x}_t^r = A_r x_t^r + B_r r_t , \quad (12)$$

where A_r and B_r are known reference matrices, with A_r Hurwitz, and $r_t \in \mathbb{R}^\ell$ is a bounded reference input.

We assume there exists an $n \times \ell$ matrix of feedback gains K_x and an $\ell \times \ell$ matrix of feedforward gains K_r satisfying the *matching conditions*

$$A + BK_x^\top = A_r \quad \text{and} \quad BK_r^\top = B_r . \quad (13)$$

The setup in [3] has A and Λ as unknown, and thus K_x and K_r need to be estimated. For this simulation, we will assume that K_x and K_r can be directly calculated from known A and B , and used within the control law.

Next, we introduce parameter estimates $\hat{\Theta}_t$, which will be dynamically updated to estimate true parameter values Θ . Thus, by applying to the plant (18) the feedback control law $u_t = K_x^\top x_t + K_r^\top r_t - \hat{\Theta}_t^\top \phi(x_t)$, the plant dynamics become

$$\dot{x}_t = A x_t + B_r r_t - B(\hat{\Theta}_t - \Theta)^\top \phi(x_t) . \quad (14)$$

This in turn gives the dynamics of the state tracking error $e_t = x_t - x_t^r$ as

$$\dot{e}_t = \dot{x}_t - \dot{x}_t^r = A_r e_t - B(\hat{\Theta}_t - \Theta)^\top \phi(x_t) . \quad (15)$$

In [3], it is then shown that these state tracking error dynamics \dot{e}_t are globally uniformly asymptotically stable,

such that $\lim_{t \rightarrow \infty} \|e_t\|_2 = 0$, if the parameter estimates are dynamically updated as

$$\dot{\hat{\Theta}}_t = \Gamma \phi(x_t) e_t^\top P_x B . \quad (16)$$

This is shown by analyzing the Lyapunov function $\mathcal{V} = e_t^\top P_x e_t + \text{tr}((\hat{\Theta}_t - \Theta)^\top \Gamma^{-1}(\hat{\Theta}_t - \Theta))$, along with using Barbalat's lemma, such that the update rule (16) results in $\dot{\mathcal{V}} \leq -e_t^\top Q_x e_t$ for all values of e_t and $\hat{\Theta}_t - \Theta$. Here, P_x is the unique symmetric, positive-definite $n \times n$ matrix that solves the algebraic Lyapunov equation $P_x A_r + A_r^\top P_x = -Q_x$ for some symmetric, positive-definite $n \times n$ matrix Q_x , and *adaptation rates* Γ is some symmetric, positive-definite $N \times N$ matrix, where we denote $\|\Gamma\|_F := \mathcal{G}$.

Remark 1: In the case where K_x and K_r are also being estimated by $\hat{K}_{x,t}$ and $\hat{K}_{r,t}$ respectively, the true values can be appended to Θ , the estimates can be appended to $\hat{\Theta}_t$, and an overall vector function $\Phi(x_t, r_t)$ which combines x_t , r_t , and $\phi(x_t)$, can be formed. However, such Φ then do not strictly meet the definition of (2), and are thus beyond the scope of this paper.

B. Persistency of Excitation

The dynamic update rule (16) only guarantees asymptotic convergence of the state tracking error e_t to zero. We now show that the parameter estimation error, which we will denote compactly as $\tilde{\Theta}_t = \hat{\Theta}_t - \Theta$, also goes to zero if $\phi(x_t)$ is persistently exciting.

Since A_r is Hurwitz, it is guaranteed invertible. And so, from (15) we have

$$e_t^\top P_x B = \dot{e}_t^\top A_r^{-1\top} P_x B + \phi(x_t)^\top \tilde{\Theta}_t B^\top A_r^{-1\top} P_x B$$

and then combined with (16) we get

$$\begin{aligned} \dot{\tilde{\Theta}}_t &= \dot{\hat{\Theta}}_t = \Gamma \phi(x_t) e_t^\top P_x B = \\ &= \Gamma \phi(x_t) \phi(x_t)^\top \tilde{\Theta}_t B^\top A_r^{-1\top} P_x B + \Gamma \phi(x_t) \dot{e}_t^\top A_r^{-1\top} P_x B . \end{aligned}$$

Note that the second term asymptotically goes to zero with \dot{e}_t and we have in the first term $B^\top A_r^{-1\top} P_x B = -\frac{1}{2} B^\top A_r^{-1\top} Q_x A_r^{-1} B$ by the Lyapunov equation definition for P_x .

Let us now restrict to the case of $\ell = 1$. Since Q_x is positive-definite we have the positive scalar $c = \frac{1}{2} B^\top A_r^{-1\top} Q_x A_r^{-1} B$, and obtain that the dynamics of the parameter estimation error vector $\tilde{\Theta}_t \in \mathbb{R}^N$ asymptotically tend to the linear time-varying system

$$\tilde{\Theta}_t = -\Gamma_c \phi(x_t) \phi(x_t)^\top \tilde{\Theta}_t . \quad (17)$$

$\Gamma_c = c\Gamma$ is symmetric, positive-definite with $\|\Gamma_c\|_F = c\mathcal{G}$.

Therefore, if $\phi(x_t)$ satisfies the persistency of excitation condition (6), then the parameter estimation error dynamics (17) are globally uniformly asymptotically stable such that we have $\lim_{t \rightarrow \infty} \|\tilde{\Theta}_t\|_2 = 0$. We show explicitly in Appendix II of [23] how this follows from Lemmas 1 and 2 and Theorem 1 of [5].

¹All code is available at: <https://github.com/tylerlekang/CDC2022>

C. Simulation

We perform the numerical simulation in the $x_t \in \mathbb{R}^2$ state space. Euler integration with a timestep of $\delta t = 0.001$ sec. is used to obtain all solutions to differential equations. We use a basic controllable canonical form for the plant:

$$\dot{x}_t = \begin{bmatrix} 0 & 1 \\ -a_1 & 2a_2 \end{bmatrix} \begin{bmatrix} x_{t,1} \\ x_{t,2} \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} (u_t + \Theta^\top \phi(x_t)) , \quad (18)$$

where $a_1, a_2, \beta > 0$ are positive scalars, $u_t \in \mathbb{R}$ is the $\ell = 1$ dimensional input, and Θ is a fixed vector of nonzero scalars which linearly parameterizes the known nonlinear vector function $\phi(x_t) = [\sigma_R(w_1^\top x_t + b_1) \ \dots \ \sigma_R(w_4^\top x_t + b_4)]^\top$ with ReLU activation functions, meeting the definition (2). This plant model is an $n = 2$ example of a general class of single-input systems with dynamics characterized by an n th order nonlinear ordinary differential equation. See section 9.5 in [3] and section 4.1 in [14] for examples of physical systems in this form using various nonlinear functions.

The reference system is a unity-gain damped harmonic oscillator with a bounded reference input $r_t \in \mathbb{R}$:

$$\dot{x}_t^r = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\xi\omega_0 \end{bmatrix} \begin{bmatrix} x_{t,1}^r \\ x_{t,2}^r \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_0^2 \end{bmatrix} r_t , \quad (19)$$

where $\omega_0, \xi > 0$ are the natural frequency and damping ratio. This gives the plant and reference eigenvalues as $\lambda = a_2 \pm \sqrt{a_2^2 - a_1}$ and $\lambda^r = (-\xi \pm \sqrt{\xi^2 - 1})\omega_0$, thus A is unstable with oscillations if $a_1 > a_2^2$ and A_r is always Hurwitz, and without oscillations if $\xi \geq 1$. Thus, we can always directly calculate the required feedback and feedforward gains that satisfy the matching conditions (13) as

$$K_x^\top = \begin{bmatrix} -\omega_0^2 - a_1 & -2\xi\omega_0 + 2a_2 \end{bmatrix} \quad \text{and} \quad K_r^\top = \begin{bmatrix} \omega_0^2 \\ \beta \end{bmatrix} .$$

For the plant system, we use $a_1 = 2$, $a_2 = 0.5$, $\beta = 0.75$. This results in an unstable, oscillatory A matrix with $\text{eig}(A) = 0.5 \pm j\sqrt{1.75}$. The reference system uses $\omega_0 = 2$ and $\xi = 1$, ensuring that A_r is Hurwitz and nonoscillatory with $\text{eig}(A_r) = -2$. The matching conditions are thus satisfied with $K_x^\top = [-2.667 \ -6.667]$ and $K_r^\top = [5.333]$. For the parameter estimate dynamic updates we use:

$$\Gamma = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad Q_x = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \quad P_x = \begin{bmatrix} 5.625 & 0.125 \\ 0.125 & 1.281 \end{bmatrix} ,$$

and for the linearly parameterized nonlinear vector function $\Theta^\top \phi(x_t)$ we use:

$$\Theta = \begin{bmatrix} -1.2 \\ 2.7 \\ 0.8 \\ -3.2 \end{bmatrix} \quad W^\top = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 1.5 & -0.5 \\ 0.5 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 2 \\ 2.5 \\ 3 \end{bmatrix} .$$

Defining parameter estimates $\hat{\Theta}_t = [\hat{\theta}_{t,1} \ \dots \ \hat{\theta}_{t,4}]^\top$ and using the known K_x and K_r , we apply the feedback control law $u_t = K_x x_t + K_r r_t - \hat{\Theta}_t^\top \phi(x_t)$ to the plant and update the parameter estimates according to the update law (16).

We use the following bounded reference inputs to drive the reference system as two different scenarios:

$$r_t^{(1)} = 10 \sin(0.5t) \quad r_t^{(2)} = 40 + \sum_{k=1}^2 10 \sin(0.25k t) .$$

The resulting plant and reference state trajectories x_t and x_t^r are plotted in state space along with the hyperplanes $W^\top x + b = 0_4$, in Figure 1 and Figure 2 respectively for the two scenarios. We see clearly that the limit cycles in both cases stay within a compact set $\mathcal{B} \in \mathbb{R}^2$, and so the ReLU activations within ϕ are bounded on this \mathcal{B} , and the trajectories never maintain a linear path. The parameter estimation error $\|\hat{\Theta}_t - \Theta\|_2$ for both cases is plotted over simulation time in Fig. 3. For the case with $r_t^{(1)}$, we see that the error converges to zero as the parameter estimators converge to the true values, while for the case with $r_t^{(2)}$ the error does not converge and is well above zero.

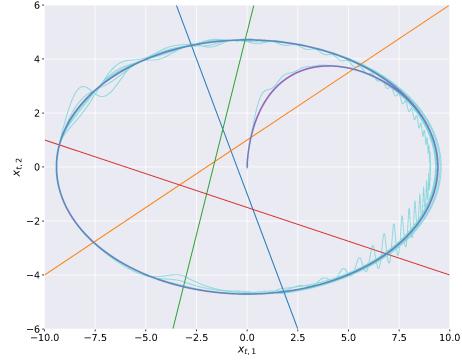


Fig. 1. Phase plot of x_t (cyan), tracking into x_t^r (magenta) driven by $r_t^{(1)}$. Crosses all N hyperplanes at nondegenerate borders along the limit cycle.

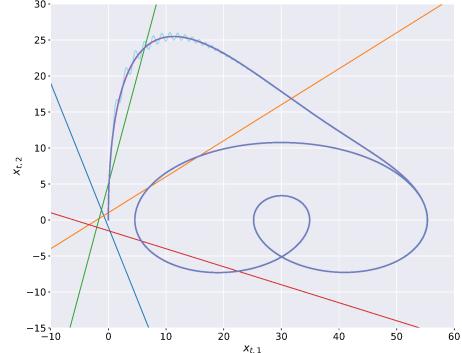


Fig. 2. Phase plot of x_t (cyan), tracking into x_t^r (magenta) driven by $r_t^{(2)}$. Does not cross all N hyperplanes along the limit cycle.

VI. CONCLUSION

In this paper, we defined a geometric criteria that leads to sufficient conditions for persistency of excitation with single hidden-layer neural networks of step or ReLU activation functions. Future work will focus on using the function approximation properties of ReLU activations to obtain similar results when the plant nonlinearity is *not known* but can be *approximated* by $\Theta^\top \phi(x_t)$ with ReLU's.

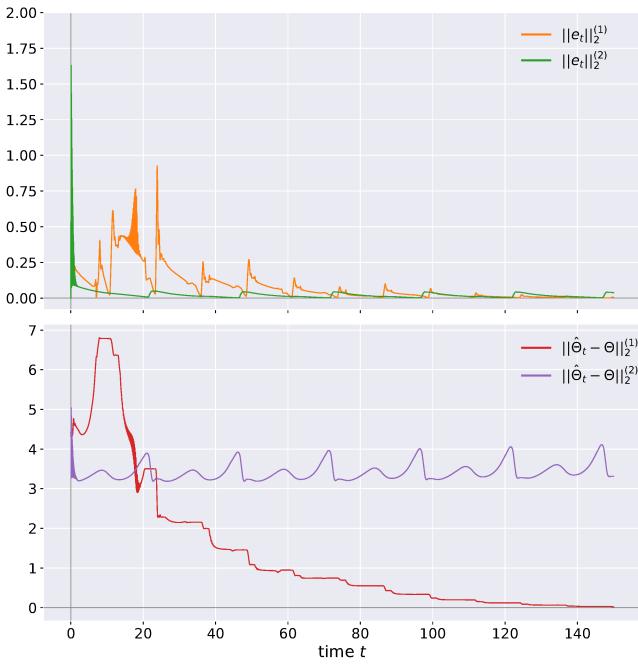


Fig. 3. Norm of state tracking error $\|e_t\|_2$ converges to zero in both cases, while the parameter estimation error $\|\hat{\Theta}_t - \Theta\|_2$ converges to zero for the case with $r_t^{(1)}$ and but not for the case with $r_t^{(2)}$.

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