2022 IEEE 61st Conference on Decision and Control (CDC)
December 6-9, 2022. Cancún, Mexico

# Approximate Submodularity of Maximizing Anticoordination in Network Games

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2022 IEEE 61st Conference on Decision and Control (CDC) | 978-1-6654-6761-2/22/$31.00 ©2022 IEEE | DOI: 10.1109/CDC51059.2022.9993180

***Abstract*— We consider decentralized learning dynamics for agents in an anti-coordination network game. In the anti-coordination network game, there is a preferred action in the absence of neighbors' actions, and the utility an agent receives from the preferred action decreases as more of its neighbors select the preferred action, potentially causing the agent to select a less desirable action. The decentralized dynamics that is based on the iterated elimination of dominated strategies converge for the considered game. Given a convergent action profile, we measure anti-coordination by the number of edges in the underlying graph that have at least one agent in either end of the edge not taking the preferred action. The maximum anti-coordination (MAC) problem seeks to find an optimal set of agents to control under a finite budget so that the overall network disconnect is maximized on game convergence as a result of the dynamics. In this paper we show that the MAC is approximately submodular in line networks for any realization of the utility constants in the population. Utilizing this result, we provide a performance guarantee for the greedy agent selection algorithm for MAC. Finally, we use a computational study to show the effectiveness of greedy node selection strategies to solve MAC on general bipartite networks.**

## I. Introduction

Anti-coordination games can be used to study competition among firms [1], [2], public goods scenarios [3], free-rider behavior during epidemics [4], [5], and network security [6]. In each of these scenarios, there is a desired action for each agent, e.g., not taking the costly preemptive measures during a disease outbreak, not investing in insurance/protection etc., in the absence of other agents. When other agents are around, they can affect the benefits of the desired action, providing incentives for agents to switch. Here we consider networked interactions, where the actions of an agent are only affected by its neighbors only (a subset of the population). Despite the peer effects, some of the agents may continue to take the individually desired action, endangering their peers and the rest of the population. That is, the rational behavior can lead to the failure of anti-coordination in the population, when anti-coordination is desirable for the well-being and safety of the system as a whole.

In such scenarios, we can envision the existence of a central coordinator with the goal to induce behavior that supports the well-being of the society. Here, we consider one such mechanism where the centralized coordinator intervenes by controlling a few agents in the network to incentivize anti-coordination among agents that repeatedly take actions to maximize individual payoffs. In particular, we consider decentralized learning dynamics inspired by the iterated elimination of dominated strategies [7]. A dominated strategy is an action that cannot be preferred under any circumstance. Thus a dominated strategy cannot be a rational action. In the learning dynamics considered here, agents eliminate certain actions by evaluating their individual utilities under the worst and best possible action profiles of their neighbors. The information about the elimination of an action by an agent can lead to a cascade of updates by other agents in the network. Indeed, we showed in [8] that such dynamics will converge in finite time and eliminate all dominated strategies of the anti-coordination network game considered here. Given such adaptive behavior of agents, the centralized player can steer the convergent action profile toward socially desirable outcomes by controlling the actions of a few players during the learning phase.

In our setting, we define the goal of the central coordinator as to maximize anti-coordinating connected pairs of agents upon convergence of the behavior. The MAC problem involves selecting a subset of the agents in the population, thus it is a combinatorial problem. We consider a greedy selection protocol for solving MAC, where at each selection epoch the agent that yields the highest number of anti-coordinating edges at the convergent action profile is added to the control set until the control budget is reached. We show that the MAC problem is approximately submodular and almost monotone in line networks implying that the worst case performance of the greedy selection protocol is bounded by a fraction of the optimal solution.

Eventually, for the general bipartite graph, we provide an inapproximability result, showing that the violation of submodularity on a specially designed bipartite graph instance of MAC can be of the order of the number of edges in the graph, thereby highlighting that our current performance guarantees will not be effective in the worst case. However, we back our results up with a simulation study that elucidates that the greedy selection protocol provides near-optimal results on average for general bipartite networks. Indeed, we show that MAC is submodular in expectation in dense bipartite networks in [9].

This work is most closely related to the following intervention mechanisms in games that aim to improve efficiency: nudging [10]–[12], influence maximization [13], and seeding in advertising [14]. All of these approaches aim to determine the emerging action profile resulting from an adaptive learning dynamics under repeated game play by either providing incentives or suggestions of "good" behavior to agents or by directly controlling a set of agents, as we do in this

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paper. Here, we aim to maximize anti-coordination instead of maximizing social welfare. Other forms of intervention mechanisms involve financial incentives in the form of taxations or rewards [15], and information design [16]. These mechanisms do not consider repeated game play, and instead focus on improving the efficiency of Nash equilibria. Here our control selection policy is dependent on the adaptive behavior of agents. Lastly, we considered a similar MAC problem for the same anti-coordination network game in [8]. This work provides a performance guarantee for the greedy selection algorithm when agents interact over a line network.

## II. Network games

We consider a graph $\mathcal{G}(V, E)$ where the set of vertices $V$ represent the agents, and the set of edges $E$ represent interactions between the agents. Each agent takes an action $a_i \in [0,1]$ to maximize its utility function

$$u_i(a_i, a_{-i}) = a_i\big(1 - c_i \sum_{j \in n(i)} a_j\big) \tag{1}$$

where $a_{-i} := \{a_j\}_{j \in V \setminus \{i\}}$ denotes the actions of all agents except agent $i$'s action, $n(i) := \{j : (i,j) \in E\}$ represents agent $i$'s neighbors, and $0 \le c_i < 1$ is a constant.

The utility function above captures scenarios where agent $i$ has a preferred action (action 1) but its incentive to choose this action decreases as more of its neighbors choose the preferred action. The decrease in incentive per neighbor is proportional to the constant $c_i$. The constant represents the sensitivity of agent $i$'s utility to its neighbor's actions.

Our network game can be represented by the tuple $\Gamma = \{V, A, \{u_i\}_{i \in V}\}$ where $A = [0,1]^{|V|}$ is the set of actions available to all players.

Below we provide scenarios that can be captured by the network game $\Gamma$ with payoffs given by (1).

**Example 1 (Disease spread on networks [5])** *Consider a bipartite graph $\mathcal{G}_B$ where agents on one side are sick and the other side healthy [5]. The edges in $\mathcal{G}_B$ represent the network of interaction between them. The disease spreads when agents on either end of a interaction link do not follow healthcare protocols, such as wearing masks, vaccinating, social distancing etc. We model this using action level* $1$ *for the agent (the easier/preferred action). Action level* $0$ *represents following epidemic mitigation protocols (the costlier action), and all actions between* $0$ *and* $1$ *represents the relative importance given to disease prevention measures. When we have an agent playing* $0$ *on the end of an interaction link, we have deactivated a disease transmission pathway in society. The utilities of agents in the epidemic game are the anti-coordination type, i.e., its incentive to social distance increases with more of its neighbors flouting protocols and hence can be captured using the utility function in* (1). *The learning constants $c_i, i \in V$ represent the sensitivity of the agents to the neighbor influence.*

An example where polarization between opposing political parties is motivated using our anti-coordination framework is illustrated in [9].

## III. Local learning dynamics

We consider decentralized learning dynamics based on the notion of iterated elimination of dominated strategies [7]. At each stage $t = 1, 2, \ldots$, we assume agents observe the past actions of their neighbors $a^{t-1}_{n(i)}$, and determine its action $a^t_i$ according to the following rule

$$\begin{aligned} &a^t_i = 1, \textit{ if} \qquad 1 = BR_i(\lceil a^{t-1}_{n(i)} \rceil),\\ &a^t_i = 0, \textit{ if} \qquad 0 = BR_i(\lfloor a^{t-1}_{n(i)} \rfloor),\\ &a^t_i = \alpha, \text{ otherwise} \end{aligned} \tag{2}$$

where $BR_i(a_{n(i)}) := \operatorname{argmax}_{a_i \in [0,1]} u_i(a_i, a_{n(i)})$ is the best response action profile, and $\alpha \in (0,1)$ is an arbitrary action between 0 and 1. In (2), agent $i$ respectively evaluates the best response function given an overestimate (ceil) and an underestimate (floor) of the sum of its neighbors' actions. The best response action for the utility function in (1) is given by

$$BR_i(a_{n(i)}) = \mathbb{1}\big(1 > c_i \sum_{j \in n(i)} a_j\big) \tag{3}$$

where $\mathbb{1}(\cdot)$ is the indicator function. Accordingly, if the overestimate of neighbors' actions is less than 1 ($c_i \sum_{j \in n(i)} \lceil a_j \rceil < 1$), then agent $i$ would take the preferred action 1 regardless of its neighbors' future actions. Similarly, if the underestimate of neighbors' actions exceeds 1 ($c_i \sum_{j \in n(i)} \lfloor a_j \rfloor > 1$), then agent $i$ would take action 0 regardless of its neighbors future actions. In the former scenario, all actions are dominated by action 1, whereas in the latter scenario all actions are dominated by action 0. If neither of these conditions hold, i.e., when $c_i \sum_{j \in n(i)} \lfloor a_j \rfloor \le 1 \le c_i \sum_{j \in n(i)} \lceil a_j \rceil$, then agent $i$ cannot rule out any of the actions, thus it takes an arbitrary action $\alpha \in (0,1)$.

In [8], we show that the dynamics in (2) converges in at most $|V|$ iterations, eliminating all strictly dominated actions for the network game $\Gamma$ when the network $\mathcal{G}$ is bipartite and all agents play $\alpha$ (are undecided) initially, i.e. $a^0_i = \alpha$ for $i \in V$. These updates converge to a Nash equilibrium if the game is dominance solvable, i.e., if the game is such that a single action profile is left as a result of iterated elimination of dominated strategies. For instance, the anti-coordination game $\Gamma$ with utility function in (1) is dominance solvable given constants $c_i < \frac{1}{|n(i)|}$ for all $i \in V$. Indeed, all agents take action 1 after the first update in (2) because $c_i \sum_{j \in n(i)} \lceil a_j \rceil < 1$ for all $i \in V$. For general payoff constants, the game $\Gamma$ is not dominance solvable, i.e. some agents can continue to take action $\alpha$ at the end of $|V|$ iterations.

## IV. Maximum Anti-Coordination Problem

We define an edge between agents $i$ and $j$ ($(i,j) \in E$) to be inactive when at least one of the agents take action 0, i.e., when $a_i a_j = 0$. Our goal is to maximize the number of inactive edges by controlling a subset of the players (with set cardinality $r \in \mathbb{Z}^+$) to play action 0 during the learning dynamics in (2). We state this goal to maximize anti-coordination (MAC) as follows

$$\max_{X\subseteq V} f(X) := \sum_{(i,j)\in E} \mathbb{1}(a_i^\infty a_j^\infty = 0)$$
$$\text{subject to} \quad |X| = r, \tag{4}$$
$$a_j^0 = \alpha \quad \text{for all } j \in V,$$
$$(a^0, a^1, ...., a^\infty) = \Phi(a^0, X),$$

where $\Phi(a^0, X)$ represents the sequence of actions obtained when uncontrolled agents ($V \setminus X$) follow the learning dynamics in (2), and the actions of controlled agents are set to 0, i.e., $a_i^t = 0$ for all $t > 0$ and $i \in X$. By removing the agents that are controlled from the game, we can guarantee that the learning process converges in finite time as per the aforementioned convergence result in [8] for bipartite networks. The control budget for the planner is restricted, i.e., the planner can only control a given $r$ number of agents as indicated by the first constraint in (4).

In the context of disease spread in a population (Example 1), maximizing anti-coordination in the underlying relationship network by inactivating disease transmission links is highly desirable as an effective means of curbing spread of the disease between members of society. The decentralized learning dynamics do not inactivate all edges on convergence and thereby a central planner would need to control/enforce certain agents to coordinate with policy guidelines (playing action level 0) so that the maximum number of transmission pathways shall be dismissed.

## V. Approximate Submodularity of MAC

In a greedy approach, we obtain a solution to a cardinality constrained maximization problem $\max_{X\subseteq V, |X|\le r} f(X)$ by selecting one element at a time, i.e.,

$$u = \operatorname*{argmax}_{w\in N} f(G_{j-1} \cup \{w\}) \tag{5}$$
$$G_j = G_{j-1} \cup \{u\}, \text{ for } 1 \le j \le r$$

where $G_0 = \emptyset$. The greedy approach is a computationally tractable way to build a set of maximum cardinality $r$ for solving MAC in (4) when, in addition, we consider the learning dynamics. If the MAC is approximately submodular, then the greedy approach can obtain a solution comparable to the optimal set $X^*$. In the following, we provide preliminary definitions of approximate submodularity and monotonicity, and then characterize the optimality loss when we implement a greedy selection (5).

**Definition 1** *A set function $f(X) : 2^{|V|} \to \mathbb{R}$ is $\epsilon$-submodular if the following condition holds for any $X \subseteq Y \subseteq V$ and $u \in V \setminus Y$*

$$\Delta_u f(X) + \epsilon \ge \Delta_u f(Y) \tag{6}$$

*where $\epsilon \ge 0$ and $\Delta_u f(X) = f(X \cup \{u\}) - f(X)$ is the discrete derivative of $f$ at $X$ with respect to $u$.*

When $\epsilon = 0$, the function $f$ is submodular. Submodularity here refers to a diminishing returns property, i.e., in the context of MAC (4), the amount we gain in the objective by additionally controlling an element $u$ on top of $X$ is non-increasing as we control elements in set $V \setminus X$.

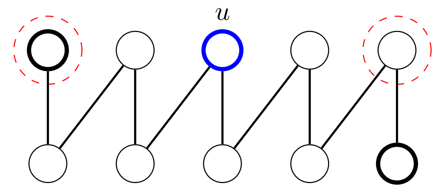


Fig. 1. A $u$-snippet with cardinality $C(4, 5)$. The bold end-agents are insensitive. Dashed lines encircle agents in set $Y \setminus X$. Here, when agents in $Y$ is controlled, we have $\Delta_u f(Y) = 3 + 3 = 6$ which is the maximum marginal gain we can achieve on the line topology by additionally controlling $u$ given the updates in (2). $\Delta_u f(X) = 2$, leading to $\epsilon$-gap= 4.

**Definition 2** *A set function is almost monotone increasing if*

$$f(X) \le f(Y) + k_{X,Y} \tag{7}$$

*for all $X \subseteq Y$, and $k_{X,Y} : (2^{|V|} \times 2^{|V|}) \to \mathbb{R}_+$.*

When $k_{X,Y} = 0$, the function is monotone. Note that in (4) if $f(X)$ were monotone, then all discrete derivatives $\Delta_u f(X)$, $u \in V \setminus X$ are required to be non-negative. That property does not hold for MAC (see Table I), and hence we relax monotonicity to accept violations in this paper.

A celebrated result by Nemhauser et al. [17] proves that monotone submodular set functions lend themselves to efficient greedy maximization with worst case performance lower-bounded by $1 - e^{-1}$ of the optimal objective. Here we show that even though the properties of monotonicity and submodularity hold only approximately for MAC instances defined on a line graph, the violations are limited and the performance guarantees are preserved with slight modifications.

### A. Approximate Submodularity of MAC in line networks

We evaluate the violation of submodularity and monotonicity in MAC for line networks and any utility constant values $\{c_i\}_{i\in V}$, $c_i \in [0, 1)$ for all $i \in V$. In a line graph $\mathcal{G}(V, E)$ with $k = |V|$ agents, an agent $i \in V \setminus \{1, k\}$ has its neighbors defined as $n(i) = \{i - 1, i + 1\}$, and the endpoint agents $\{1, k\}$ have a single neighbor $n(1) = \{2\}$ and $n(k) = \{n - 1\}$–see Fig. 1 for an illustration. The particular structure of the line-graph yields that agents with $c_i < \frac{1}{2}$ and the endpoint agents $\{1, k\}$ are *insensitive* to their neighbors' actions as per the update in (2), i.e., $1 > c_i \sum_{j\in n(i)} \lceil a_j \rceil$. If $c_i \ge \frac{1}{2}$, then agent $i$ is *sensitive* to the actions of their neighbors (unless it is an endpoint agent), and remain undecided in the initial step of the learning dynamics.

Our analysis relies on identifying the worst-case scenario of control sets $X \subseteq Y$ such that the violation of the submodularity property ($\epsilon$) is maximized for a given graph topology and given values of the learning constants $c_i \in [0, 1)$ for $i \in V$. In identifying the worst-case scenario, we consider subgraphs of the line graph that have insensitive nodes on both ends. We formally define such subgraph next.

**Definition 3** *An $u$-snippet is a segment of the line graph that contains a candidate control agent $u \in V \setminus Y$ and has insensitive agents exclusively on either end. We refer to the cardinality of the $u$-snippet as $C(x, y)$ indicating that there are $x$ and $y$ agents respectively on each side of $u$ in the line topology, where $x, y \in \mathbb{Z}^+ \cup \{0\}, x + y > 0$. When $u$ is additionally insensitive, i.e., $c_u < \frac{1}{2}$, we say the $u$-snippet is constrained.*

Fig. 1 shows an u-snippet with degree $C(4, 5)$. The influence of controlling $u$ in an $u$-snippet cannot go beyond the insensitive end-nodes of it, except for the case when $u$ is an end-node. Thus, our analysis focuses on finding the worst-case scenario $u$-snippet where $\Delta_u f(Y)$ is the largest and $\Delta_u f(X)$ is the smallest possible. Indeed, the u-snippet shown in fig. 1 corresponds to a worst case scenario $\epsilon$-gap $= 4$, where no agent from $X$ is in the u-snippet, and the agents in $Y$ belonging to the u-snippet are encircled by dashed lines. The following result states the approximate-submodularity of the MAC problem on a line graph.

**Proposition 1** *The MAC problem in a line graph for any set of payoff constants $\{c_i\}_{i \in V}, c_i \in [0, 1)$ for all $i \in V$ is $\epsilon = 2d_{max}$-submodular in the worst case where $d_{max}(=2)$ is the maximum degree.*

The proof (given in the appendix) first shows that the approximate submodularity is $\epsilon = 4$ via providing the example in Fig. 1. Note that this example is not unique and can happen given a range of payoff constant values. Then we show that no other $u$-snippet can attain a worse $\epsilon$ value by eliminating other scenarios of $u$-snippets.

Let us define now the graph combination operation which will be used in the remainder of the text.

**Definition 4** *The graph $\bar{\mathcal{G}} = (V, E)$ is a combination of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ at nodes $u \in V(\mathcal{G}_1)$ and $u' \in V(\mathcal{G}_2)$ (denoted as $\bar{\mathcal{G}} = \mathcal{G}_1 + \mathcal{G}_2|_{(u,u')}$) where $V = V(\mathcal{G}_1) \cup V(\mathcal{G}_2) \setminus u'$ and $E = E(\mathcal{G}_1) \cup E(\mathcal{G}_2) \cup \{(v, u) : v \in V(\mathcal{G}_2), (v, u') \in E(\mathcal{G}_2)\}$.*

In this operation we merge the two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ at nodes $u$ and $u'$, with the *two* nodes becoming *one* in the resultant graph. Observe here that the graph combination operation is symmetric, i.e $\mathcal{G}_1 + \mathcal{G}_2|_{(u,u')} = \mathcal{G}_2 + \mathcal{G}_1|_{(u',u)}$.

**Proposition 2** *The MAC problem is almost monotone increasing for the line-network. Given selections of control sets $X \subseteq Y \subseteq V$ and $f(\cdot)$ defined as in* (4)

$$f(X) \leq f(Y) + k_{X,Y} \tag{8}$$

*where the violation of monotonicity $k_{X,Y} = 2|Y \setminus X|$.*

*Proof:* Consider an extension of Def. 3 where we do not have a candidate control agent $u$. A *segment* is a subgraph of the line network that has *insensitive* agents exclusively on either end. The *segment* is *constrained* when some node other than the end-nodes is *insensitive*. Given a selection of set $X$, we can bifurcate the line network into multiple mutually independent *segments* that do not contain agents in set $X$. Recall that an agent $j$ is *insensitive* if $c_j < 1/2$, if agent $j$ has a neighbor playing 0, or if it is an end-node.

Let the notation $C_n$ refer to a *segment* containing $n$ nodes. Take the family of *segments* $\mathcal{F} = \{C_n | n \in \mathbb{Z}^+, n \neq 3\}$. For $C_n \in \mathcal{F}$, for control set $X$, we achieve a convergent action profile $1 - \alpha - ... - \alpha - 1$, following (2). That is, on convergence, all edges are active.

Consider $y \in Y \setminus X$ that is located inside the *segment* $C_n \in \mathcal{F}$. Controlling $y$ can inactivate certain edges, and therefore the objective function evaluated on this segment $f(\cdot)|_{C_n}$ yields that $f(X)|_{C_n} \leq f(X \cup \{y\})|_{C_n}$, and the contribution of this arrangement towards violation of overall monotonicity is negative. Instead, consider $C_3$. On convergence of (2), we get $1 - 0 - 1$, all edges in $C_3$ already inactive. Take $y \in Y \setminus X$ to be one of the end-nodes. Then for control set $X \cup \{y\}$, we achieve $1 - 1 - 0$ on convergence of (2). Thus, we have done worse by controlling a larger set, i.e., $f(X)|_{C_3} \leq f(X \cup \{y\})|_{C_3} + 1$. Thus for all *segments*, $-1$ is the absolute worst we can do.

See that the set of all *constrained segments* can be obtained using graph combination on $C_x, C_y \in \mathcal{F} \cup \{C_3\}$ at the end-nodes. Now, use graph combination (Definition 4) on two $C_3$ *segments* at one of the end-nodes to construct a $C_5$ *constrained* at the central node. For control set $X$, the convergent action profile is $1 - 0 - 1 - 0 - 1$. When we additionally control $y \in Y \setminus X$, the central node, then we get $1 - 1 - 0 - 1 - 1$, thereby violating monotonicity by $-2$ by controlling a single element. Doing worse, for any other *constrained segment* is impossible, as that would imply a violation of at least $-2$ for some *snippet* which is one of the components of the *constrained* segment which we showed is not possible.

Given sets $X$ and $Y$, in the worst case, for every agent $y \in Y \setminus X$, we can find a $C_5$ *constrained segment* where the central node is $y$, thereby generating $f(Y) = f(X) - 2|Y \setminus X|$. Thus, the result follows. ■

**Theorem 1** *Let $f(\cdot)$ be defined as in* (4). *Let $G_r$ be the control set obtained after $r$ steps of the greedy algorithm in* (5). *Then, for a line-network,*

$$f(G_r) \geq (1 - e^{-1}) f(X^*) + e^{-1}(1 - \frac{1}{r})(f(\emptyset) + 2r) - 4r \tag{9}$$

The proof follows almost identically the performance guarantee proof in our work on MAC in dense bipartite networks (Theorem 4, [9]). The fact that monotonicity and submodularity hold only approximately with bounded violations in line networks leads to additional terms which when summed up produce $e^{-1}(1 - \frac{1}{r})2r - 4r$ in (9). If the function $f(\cdot)$ is monotonic increasing and purely submodular ($\epsilon = 0$), then we recover the standard suboptimality of the greedy approach where $f(G_r) \geq (1 - e^{-1}) f(X^*)$ [17]. In fact we can do better than the rate $(1 - e^{-1})$ because of the term $f(\emptyset)$ is non-zero. The MAC objective value $f(\emptyset)$ is non-zero even

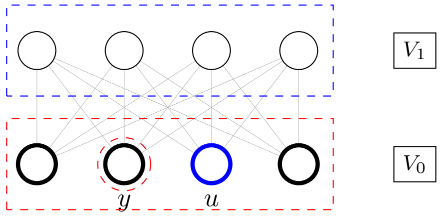


Fig. 2. We have a $4 \times 4$ complete bipartite network, with selection for agent $u$ shown in blue. Sets $V_0$ and $V_1$ are marked in red and blue boxes

before applying any control, because the greedy approach benefits from the deactivation of edges due to the preferences of agents to anti-coordinate in (1).

### B. Approximate Submodularity of MAC in bipartite networks

**Theorem 2** *MAC problem in a general bipartite network can be $\epsilon$-submodular where $\epsilon \sim O(N^2)$ is proportional to the number of edges for a certain range of utility constants.*

*Proof:* Consider a complete bipartite graph $K_{N,N} = \mathcal{G}(V, E)$ such that $V = V_1 \cup V_0, V_1 \cap V_0 = \emptyset$ and $E = V_1 \times V_0$. $|V_0| = |V_1| = N$. Let the learning constants $c_i < \frac{1}{N}$ for all $i \in V_0$ and $c_i \in (\frac{1}{N}, \frac{1}{N-1})$ for $i \in V_1$. Since $1 > Nc_i$ for $i \in V_0$, we have $a_i = 1$ for $i \in V_0$ by (2).

Consider the empty control set $X = \emptyset$. All agents in set $V_1$ play 0 since $1 < Nc_i$ for $i \in V_1$. The number of inactive edges $f(X) = N^2$. When we control agent $u \in V_0$ to play 0, agents in set $V_1$ switch to playing 1 ($a_i = 1$ for $i \in V_1$ since $1 > (N-1)c_i$. Agents in $V_0$ continue playing 1, barring agent $u$ that is controlled. The number of inactive edges is equal to the number of edges connected to agent $u$, i.e., $f(u) = N$. Thus, $\Delta_u f(X) = N - N^2$.

Consider $Y = \{y\}$ where $y \in V_0$. Agents in $V_0$ play 1 except agent $y$, and agents in set $V_1$ play 1 since $1 > (N-1)c_i$ for $i \in V_1$. Thus $f(Y) = N$. If $Y \cup \{u\}$ is controlled where $u \neq y$, then $f(Y \cup \{u\}) = 2N$ by the same previous reasoning. Thus, we have $\Delta_u f(Y) = N$. Then the submodularity gap according to this scenario is given by $\epsilon = N - (N - N^2) = N^2$ which is equal to the number of edges. ■

We see that in the worst case, for some specific instances, the maximum violation of the submodularity property for *MAC* can equal to the number of edges in the network. However, the performance we obtain from the greedy selection approach often performs comparable to the optimal solution as we show computationally in the next section.

## VI. Simulation

MAC is concerned with the deactivation of as many edges as possible on convergence of learning dynamics, perturbed by controlling a select few agents. We define the *Inactivation Ratio* as the ratio of the number of edges inactivated on convergence to the number of active edges in the network before the dynamics progress. *Inactivation Ratio*, therefore, is a measure of how successful MAC is on the particular graph instance, given the control.

For our simulations we consider random bipartite graph instances with edge formation probability equal to 0.3 and 0.8. Every realization of a network for given network sizes ($\{4, 8, 12, .., 40\}$) has a random topology with random learning constants for the agents. We sample the learning constants $c$ for every agent from a uniform distribution between the limits $[0, 1)$. The control budget is fixed at $\lceil \frac{N}{10} \rceil$, where $N = |V|$ is the number of nodes in the graph. Given the budget, we select the control profile using a greedy cascade based algorithm (5). We compare its anti-coordination performance with a control set generated using brute force search. In the brute force approach, we go over all the possible control sets for the budget specified and find the one that maximizes the number of edges deactivated. For every network instance, we calculate the *Inactivation Ratio* for both the control sets found using the greedy algorithm (5) and brute force search. For a given network size, we sample 40 instances of random bipartite graphs and evaluate the performance of the greedy algorithm.

We plot the average *Inactivation Ratio* against the size of network in Figure 3. The *Inactivation Ratio*, on average, for the greedy algorithm is close to the optimal inactivation at the current control budget for every network size. The maximum inactivation ratio gap for our simulations stands at 0.106 for the sparse networks and 0.095 for the dense ones, further highlighting the good performance of (5) in selecting control agents to induce anti-coordination. All our simulations have been performed on Apple(R) M1 CPU (Arm(R) based, 8-core) with 16GB of RAM.

## VII. Conclusion

We defined the combinatorial problem of selecting agents to control to maximize anti-coordination among rational agents in a network game. Anti-coordination is measured as the number of edges deactivated from the network on convergence of decentralized learning dynamics. Firstly, we showed that MAC is approximately submodular when the underlying interaction network is a line graph. Our proof technique utilized fragmenting the network into subgraphs or snippets which are self-contained units, and further reasoning to find the maximum violation of submodularity in these snippets. Moreover, we also show that MAC behaves almost like a monotone increasing function in the set of control agents. Using these results in conjunction, we derived the approximation guarantee for greedy node selection for MAC. Our computational results indicate that greedy selection strategies may be effective in producing near-optimal control sets for MAC on general bipartite network scenarios.

## VIII. Appendix

### A. Proof of Proposition 1

From Lemmas 2 and 3, we have that the maximum value of $\epsilon$-gap equals 4 for all $u$-snippets. Given any selection of $X$, $Y$, and $u$, we can always construct an $u$-snippet (or a constrained $u$-snippet), and the action switches triggered by controlling $u$ stay within the snippet and have no effects outside of the snippet. Hence the maximum violation of the

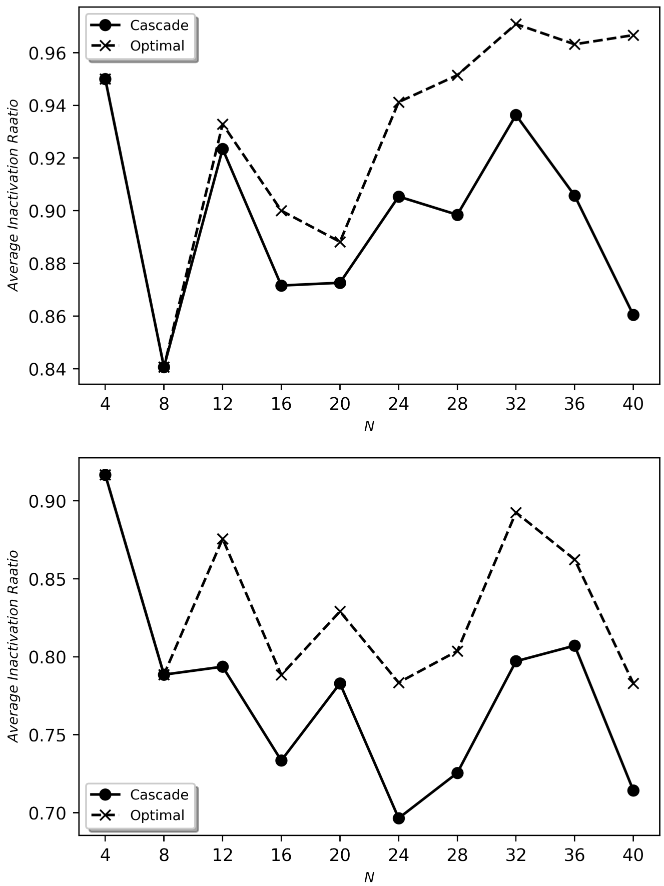


Fig. 3. Average Inactivation Ratio vs Cardinality ($N$) of the Bipartite Network for connection probability equal to 0.3 and 0.8 (sparse and dense networks) respectively. Control budget set at $\lceil \frac{N}{10} \rceil$ for every network realization.

submodularity property in case of line graphs is 4. The result follows by the fact that maximum node-degree is $d_{max} = 2$. ■

Next we provide a few definitions and results that will be useful in proving the two main lemmas used in the above proof.

**Definition 5** *A degree $(x, y)$ $u$-snippet, referred to as $D(x, y)$ has control agents in set $Y \setminus X$ at a minimum distance of $x$ and $y$ edges respectively on either side of $u$ in the snippet, where the indexes $x, y \in \mathbb{Z}^+ \cup \{0\}, x + y > 0$.*

To accommodate for cases where we do not have any agent in set $Y/X$ in one-*half* of the $u$-snippet, we set the corresponding index ($x$ or $y$) equal to $1 + \delta$, where $\delta$ is the diameter of the induced subgraph which corresponds to that half of the snippet.

If a $D(m, n)$ degree configuration is realized on some $u$-snippet due to selections of agents in set $Y/X$, we say that the corresponding snippet *invoked* $D(m, n)$. For a given $C(x, y)$, the set of *degree* configurations, represented by $D_{C(x,y)}$, is given by

$$D_{C(x,y)} := \{D(k,l) | k, l \in \mathbb{Z}^+, k \leq x+1, l \leq y+1, \\ (k,l) \neq (x+1, y+1)\} \quad (10)$$

We do not allow both $k$ and $l$ in the previous expression to be their limiting values at the same time, as $D(x+1, y+1)$ is a meaningless degree configuration. We have control agents in both *halves* of the snippet or at least in one *half*.

See that no matter the cardinality of the underlying $u$-snippet that *invokes* $D(m, n)$, the value of $\Delta_u f(Y)|_{D(m,n)}$ stays the same. Moreover, we are not concerned with *degree* configurations where there are multiple controlled agents in either *half* of the snippet. This is because the value of $\Delta_u f(Y)$ depends only on the nearest controlled agent $y \in Y \setminus X$ to $u$ in either *half*. Since agents in $Y \setminus X$ are playing 0 for both control sets $Y$ and $Y \cup \{u\}$, the network $y - \ldots - u$ behaves as a self contained unit.

**Lemma 1** *$\Delta_u f(Y)$ for a $D(x, y)$ $u$-snippet is equal to $\Delta_u f(X)$ for a $C(x-1, y-1)$ $u$-snippet, where $x, y \in \mathbb{Z}^+$.*

*Proof:* Let agents on $D(x, y)$ be $i_x - \ldots - i_1 - u - j_1 - \ldots - j_y$, where $i_x \in Y \setminus X$, and $j_y \in Y \setminus X$. According to the updates (2), neighbors of $i_x$ and $j_y$, which are $i_{x-1}$ and $j_{y-1}$ respectively, will play 1, i.e., they will be insensitive. Thus, the effects of controlling agent $u$ cannot affect $i_{x-1}$ and $j_{y-1}$. Then, the segment $i_{x-1} - \cdots - i_1 - u - j_1 - \cdots - j_{y-1}$ is an $u$-snippet with cardinality $C(x-1, y-1)$ (Definition 3). Since there are no agents in $Y \setminus X$ in this $u$-snippet, the result follows. ■

**Lemma 2** *The maximum value of $\epsilon$ as defined in* (6) *for any $u$-snippet, no matter the selection of control sets $X \subseteq Y \subseteq \mathcal{V}$ and $u \in \mathcal{V} \setminus Y$ cannot exceed 4.*

*Proof:* In finding the maximum $\epsilon$ value, we will focus on $u$-snippets that do not contain an element of $X$. This is because we can always compute $\Delta_u f(X)$ by considering an $u$-snippet that does not contain an element of $X$. To see this, consider an agent $x \in X$ that lies within an $u$-snippet. By definition, agent $x$ plays 0 resulting its neighbors $n(x)$ to play 1. Thus, the $u$-snippet $x - x^{'} - .. - u$ behaves identical as the $u$-snippet $x' - .. - u$ in terms of computing $\Delta_u f(X)$.

Consider the following set of $u$-snippets with at least 4 agents in one half and zero in the other half, $\mathcal{K}_1 = \{C(x, 0), x \geq 4, x \in \mathbb{Z}^+\}$. For all $Z \in \mathcal{K}_1$, the action profiles for control sets $X$ and $X \cup \{u\}$ will be $1 - \alpha - \ldots - \alpha - \alpha - 1$ or $1 - \alpha - \ldots - \alpha - 1 - 0$ following the updates in (2). Thus, $\Delta_u f(X)|_Z = 1$ for all $Z \in \mathcal{K}_1$. Next, define $\mathcal{K}_4 = \{C(x, 3), x \geq 4, x \in \mathbb{Z}^+\}$. We have $\Delta_u f(X)|_Z = 4$ for all $Z \in \mathcal{K}_4$, as the $u$-triad on one *half* of the snippet contributes $+3$ in the marginal gain calculation, and we have $+1$ from the other half of length 4 or more. In an $u$-triad the initial action profile is $1 - \alpha - \alpha - \alpha$ where one end is insensitive, i.e., plays 1 and the other end is the sensitive agent $u$ (See Fig. 4). In this network configuration, the action profile converges to $1 - 0 - 1 - 0$

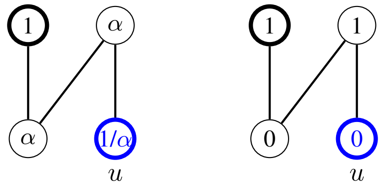


Fig. 4. An $u$-triad, with convergent action profiles for control sets $X$ and $X \cup \{u\}$ respectively. We have $u$ play 1 or $\alpha$ depending on whether $u$ is insensitive or not in the $u$-snippet. Regardless, $\Delta_u f(X) = f(X \cup u) - f(X) = 3 - 0 = 3$

creating three anti-coordinating links when the controlled agent $u$ plays 0. Next, define $\mathcal{K}_2 = \mathcal{N} \setminus \{\mathcal{K}_1 \cup \mathcal{K}_4\}$. Equivalently $\mathcal{K}_2 = \{C(x,y) | x \geq 4, y \in \mathbb{Z}^+ \setminus \{3\}, y \leq x\}$. We have $\Delta_u f(X)|_Z = 2$ for all $Z \in \mathcal{K}_2$, i.e. controlling $u$ only causes its neighbors to switch to playing 1.

**Claim 1** *For all $u$-snippets, we have*

$$\max_{D \in D_Z, Z \in \mathcal{N}_\infty} \Delta_u f(Y)|_D = \Delta_u f(Y)|_{D(4,4)} = 6 \qquad (11)$$

*where $\mathcal{N}_\infty := \{\bigcup_{i>=1} \{C(i,j) | j \leq i, i, j \in \mathbb{Z}^+ \cup \{0\}\}\}$.*

*(Proof of Claim 1) Observe that the degree configuration $\Delta_u f(Y)|_{D(4,4)} = \Delta_u f(X)|_{C(3,3)} = 6$ via Lemma 1 and the fact that controlling agent $u$ leads to three anti-coordinating links on either end of the $u$-snippet with cardinality $C(3,3)$, amounting to a total of six edges inactivated. Assume now there exists $D \in D_Z$ for some $Z \in \mathcal{N}_\infty$ such that $\Delta_u f(Y)|_D > 6$. Using Lemma 1, we have $\Delta_u f(X)|_{C(x-1,y-1)} > 6$ for some $x, y \in \mathbb{Z}$. Prior to the claim, we established that $\Delta_u f(X)|_Z \leq 4$ for $Z \in \mathcal{N}_{\geq 4} := \{C(x,y) | x \geq 4, y \leq x, x, y \in \mathbb{Z}^+ \cup \{0\}\}$. And for $u$-snippets with halves of length three or less, ($Z \in \mathcal{N}_{\leq 3} := \{C(x,y) | x \leq 3, y \leq x, x, y \in \mathbb{Z}^+ \cup \{0\}\}$), we have that the number edges is at most six. Thus, we arrive at a contradiction proving the claim in* (11).

For maximizing $\epsilon$ for all $Z \in \mathcal{N}_{\geq 4}$, we select $Z$ such that we can *invoke* $D(4,4)$ on it while minimizing the value of $\Delta_u f(X)|_Z$. We can only invoke $D(4,4)$ for $Z \in \mathcal{K}_2' = \{C(x,y) | x \geq 4, y \geq 3, y \leq x, x, y \in \mathbb{Z}\}$. To maximize $\epsilon$-gap, we therefore select $Z$ from the set $\mathcal{K}_2 \cap \mathcal{K}_2'$. Since for all $Z \in \mathcal{K}_2$, $\Delta_u f(X)|_Z = 2$, we get an $\epsilon$-gap$= 4$. See Figure 1 for an example. Now, for all $Z \in \mathcal{N}_{\leq 3}$, $m|_Z - \Delta_u f(X)|_Z < 4$ holds, where $m|_Z$ is the number of edges in $Z$ (See table I and II, $m|_{C(x,y)} = x + y$). Since $m|_Z$ is an obvious upper-bound for $\Delta_u f(Y)|_D$ for all $D \in D_Z$ for any $Z$, we cannot exceed $\epsilon$-gap$= 4$ for $Z \in \mathcal{N}_{\leq 3}$.

Thus, we see for cardinality configurations $\mathcal{N}_\infty$, $\epsilon$ in (6) can at most be 4. Note that the values of $\Delta_u f(X)$ for $C(x,y)$ and $C(y,x)$ are equal for any $x, y \in \mathbb{Z}$. Thus the bound for $\epsilon$ holds for all $u$-snippets. ∎

**Lemma 3** *The maximum value of $\epsilon$ as defined in* (6) *for any constrained $u$-snippet, no matter the selection of control sets $X \subseteq Y \subseteq \mathcal{V}$ and $u \in \mathcal{V} \setminus Y$ cannot exceed 4.*

| $Z$ | $C(1,0)$ | $C(1,1)$ | $C(2,0)$ | $C(2,1)$ | $C(2,2)$ |
|---|---|---|---|---|---|
| $\Delta_u f(X)\|_Z$ | 1 | 0 | -1 | 2 | 2 |

TABLE I
$\Delta_u f(X)|_Z$ FOR $Z \in \mathcal{N}_{\leq 2}$.

| $Z$ | $C(3,0)$ | $C(3,1)$ | $C(3,2)$ | $C(3,3)$ |
|---|---|---|---|---|
| $\Delta_u f(X)\|_Z$ | 3 | 4 | 4 | 6 |

TABLE II
$\Delta_u f(X)|_Z$ FOR $Z \in \mathcal{N}_3$.

*Proof:* Every constrained snippet has insensitive $u$, which implies all such snippets can be represented as a graph combination of two $u$-snippets of the form $C(x,0), x > 0$. The rest follows similarly from the proof of Lemma 2. ∎

## References

[1] Y. Bramoullé and R. Kranton, "Public goods in networks," *Journal of Economic theory*, vol. 135, no. 1, pp. 478–494, 2007.
[2] Y. Bramoullé, "Anti-coordination and social interactions," *Games and Economic Behavior*, vol. 58, no. 1, pp. 30–49, 2007.
[3] J. Hirshleifer, "From weakest-link to best-shot: The voluntary provision of public goods," *Public choice*, vol. 41, no. 3, pp. 371–386, 1983.
[4] C. T. Bauch and A. P. Galvani, "Social factors in epidemiology," *Science*, vol. 342, no. 6154, pp. 47–49, 2013.
[5] C. Eksin, J. S. Shamma, and J. S. Weitz, "Disease dynamics in a stochastic network game: a little empathy goes a long way in averting outbreaks," *Scientific reports*, vol. 7, p. 44122, 2017.
[6] P. Naghizadeh and M. Liu, "Exit equilibrium: Towards understanding voluntary participation in security games," in *IEEE INFOCOM 2016-The 35th Annual IEEE International Conference on Computer Communications*. IEEE, 2016, pp. 1–9.
[7] I. Menache and A. Ozdaglar, "Network games: Theory, models, and dynamics," *Synthesis Lectures on Communication Networks*, vol. 4, no. 1, pp. 1–159, 2011.
[8] C. Eksin and K. Paarporn, "Control of learning in anticoordination network games," *IEEE Transactions on Control of Network Systems*, vol. 7, no. 4, pp. 1823–1835, 2020.
[9] S. Das and C. Eksin, "Average submodularity of maximizing anticoordination in network games," *arXiv preprint arXiv:2207.00379*, 2022.
[10] Y. Xiao, J. Park, and M. Van Der Schaar, "Intervention in power control games with selfish users," *IEEE Journal of Selected Topics in Signal Processing*, vol. 6, no. 2, pp. 165–179, 2011.
[11] R. Guers, C. Langbort, and D. Work, "On informational nudging and control of payoff-based learning," *IFAC Proceedings Volumes*, vol. 46, no. 27, pp. 69–74, 2013.
[12] J. Riehl, P. Ramazi, and M. Cao, "Incentive-based control of asynchronous best-response dynamics on binary decision networks," *IEEE Transactions on Control of Network Systems*, vol. 6, no. 2, pp. 727–736, 2018.
[13] D. Kempe, J. Kleinberg, and É. Tardos, "Maximizing the spread of influence through a social network," in *Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining*, 2003, pp. 137–146.
[14] M.-F. Balcan, S. Krehbiel, G. Piliouras, and J. Shin, "Minimally invasive mechanism design: Distributed covering with carefully chosen advice," in *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*. IEEE, 2012, pp. 2690–2695.
[15] P. N. Brown and J. R. Marden, "Studies on robust social influence mechanisms: Incentives for efficient network routing in uncertain settings," *IEEE Control Systems Magazine*, vol. 37, no. 1, pp. 98–115, 2017.
[16] F. Sezer, H. Khazaei, and C. Eksin, "Social welfare maximization and conformism via information design in linear-quadratic-gaussian games," *arXiv preprint arXiv:2102.13047*, 2021.
[17] G. L. Nemhauser, L. A. Wolsey, and M. L. Fisher, "An analysis of approximations for maximizing submodular set functions—i," *Mathematical programming*, vol. 14, no. 1, pp. 265–294, 1978.