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THE MODULI SPACE OF MARKED GENERALIZED CUSPS IN REAL PROJECTIVE MANIFOLDS

SAMUEL A. BALLAS, DARYL COOPER, AND ARIELLE LEITNER

Abstract. ln this paper, a generalized cusp is a properly convex manifold with strictly convex boundary that is diffeomorphic to $M \times [0, \infty)$ where M is a closed Euclidean manifold. These are classified by Ballas, Cooper, and Leitner [J. Topol. 13 (2020), pp. 1455-1496]. The marked moduli space is homeomorphic to a subspace of the space of conjugacy classes of representations of $\pi_1 M$. It has one description as a generalization of a trace-variety, and another description involving weight data that is similar to that used to describe semi-simple Lie groups. It is also a bundle over the space of Euclidean similarity (conformally flat) structures on M, and the fiber is a closed cone in the space of cubic differentials. For 3-dimensional orientable generalized cusps, the fiber is homeomorphic to a cone on a solid torus.

Introduction

A *generalized cusp* is a properly-convex real-projective manifold, C, such that C is diffeomorphic to $[0, 1) \times \partial C$, and $\pi_1 C$ is virtually-nilpotent, and ∂C contains no line segment.

From now on, in this paper, we use the term *generalized cusp* in the narrow sense that ∂C is also compact. It was shown in [2, (0.7)] this implies that $\pi_1 C$ is virtually abelian, and that C has a natural affine structure that is a stiffening of the projective structure.

Let $\mathbb{A}^n$ denote affine space, and $\mathrm{Aff}(n)$ the affine group. Then $C = \Omega/\rho(\pi_1 C)$ where $\Omega \subset \mathbb{A}^n$ is a non-compact, convex, closed set, bounded by a strictly-convex hypersurface that covers ∂C, and $\rho : \pi_1 C \to \mathrm{Aff}(n)$ is the holonomy.

The moduli space of marked generalized cusps turns out to be a beautiful object with interesting structure that admits several different descriptions. We concentrate on the case that $\partial C \cong \mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$, then the holonomy ρ extends over $V \cong \mathbb{R}^{n-1}$. In this case the moduli space $\mathcal{T}_n$ consists of all conjugacy classes of monomorphisms of $\mathbb{R}^{n-1}$ into $\mathrm{Aff}(n)$ such that the orbit of a generic point is a properly-embedded, strictly-convex hypersurface. It follows from (1.2) that

Theorem 0.1. *$\mathcal{T}_n \cong \mathcal{P} \times F$ where $\mathcal{P} \subset \mathrm{S}^2 V$ is the space of unimodular, positive-definite quadratic forms on V, and*

$$F = \{(v_1, \cdots, v_n) \in V^n : \ \exists \varpi \geq 0 \ \ \forall\, i \neq j \quad \langle v_i, v_j \rangle = -\varpi\}/\Sigma_n$$

where Σ_n permutes the summands of V^n.

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Thus one may view a generalized cusp as a Euclidean manifold with extra structure obtained by a deformation of a *standard* cusp i.e. equivalent to one in a hyperbolic manifold. The bundle structure on the moduli space admits several descriptions.

A generalized cusp is determined up to equivalence by the *complete invariant* $(\chi, [\beta])$ comprising the character $\chi : V \to \mathbb{R}$ of ρ, together with the projective class of a positive definite quadratic form β on V.

A generalized cusp is also determined by $[\beta]$ together with the Lie algebra weights $\xi_i : V \to \mathbb{R}$ of ρ that are arbitrary subject to a simple *geometric* constraint (1.1). The weights may be regarded as harmonic 1-forms representing elements of $H^1(\partial C)$. These 1-forms determine transversally measured foliations on ∂C which, together with the similarity structure, determine C. For non-diagonalizable holonomy, the cohomology classes are arbitrary subject to being pairwise orthogonal with respect to the dual of β.

The next description is differential-geometric: as the projective class of the sum of a quadratic and a cubic differential both defined on ∂C. This exhibits $\mathcal{T}_n$ as the product of the space of flat conformal structures on ∂C times a cone in the space of cubic polynomials on V. The second factor is a closed cone in $\mathrm{S}^3 V$ that is not a manifold. Points in the interior of this cone correspond to diagonalizable holonomy. The cone point corresponds to a *standard cusp*. The cubic is a weighted sum of the cubes of the weights, and it is *harmonic* if and only if $\partial\Omega$ is an affine sphere.

For three-manifolds this data is encoded by $(w, r, h) \in \mathbb{C}^3$ subject to $\operatorname{Im} w > 0$ and $|r| \leq 3|h|$. Here w determines the conformal structure on ∂C, and r, h are respectively the *radial* and *harmonic* components of the cubic polynomial. The generalized cusp is standard, with cusp shape w, if and only if $r = h = 0$.

1. Summary of results

Given $\psi \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ with $\psi(e_1) \geq \psi(e_2) \geq \cdots \geq \psi(e_n) \geq 0$ a *generalized cusp Lie group* $G(\psi) \subset \operatorname{Aff}(n)$ was defined in [2] and generalized cusps correspond to lattices in $G(\psi)$. Two generalized cusps are *equivalent* if they deformation-retract to affinely isomorphic cusps.

Every generalized cusp is equivalent to a *homogeneous* one for which $G(\psi)$ acts transitively on $\partial\Omega$. For these, there is a natural underlying Euclidean metric on ∂C. This metric is covered by one on $\partial\widetilde{C} = \partial\Omega \subset \mathbb{A}^n$ that is conformally equivalent to the second fundamental form, and is scaled so that $\operatorname{volume}(\partial C) = 1$. It follows from the Bieberbach theorems that C has a finite cover by a generalized cusp with boundary that is a torus $T^{n-1} = \mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$. These are called *torus cusps* and we concentrate on them. The general case reduces to this by (1.6).

Set $V = \mathbb{R}^{n-1}$. It is shown in [10] that $G(\psi)$ contains a unique closed subgroup $\operatorname{Tr}(\psi) \cong V$ called the *translation subgroup* that acts simply transitively on $\partial\Omega$. Moreover the image of the holonomy $\rho : \mathbb{Z}^{n-1} \to \operatorname{Tr}(\psi)$ is a lattice. Thus ρ extends to an isomorphism $\rho : V \to \operatorname{Tr}(\psi)$ called the *extended holonomy*.

The moduli space of equivalence classes of *marked* generalized cusps diffeomorphic to C is denoted $\mathcal{T}(C)$ and $\mathcal{T}_n := \mathcal{T}(T^{n-1} \times [0, \infty))$. It consists of equivalence classes of developing maps, see (4.4). The map that sends a point in $\mathcal{T}(C)$ to the conjugacy class of the extended holonomy identifies $\mathcal{T}(C)$ with the subspace $\operatorname{Rep}(C)$ of the quotient space $\operatorname{Hom}(V, \operatorname{Aff}(n))/\operatorname{Aff}(n)$ consisting of conjugacy classes of isomorphisms onto translation subgroups, see (4.5). Fenchel-Nielsen coordinates

provide a lift of Teichmuller space into the representation variety. However, we do not know if it is possible to lift $\mathcal{T}(C)$ into $\mathrm{Hom}(\pi_1 C, \mathrm{Aff}(n))$.

Let A_n be the closed Weyl chamber of $\mathrm{SL}(n+1,\mathbb{R})$. There is a family of representations parameterized by $A_n \times \mathrm{SL}\, V$. Theorem 4.6 says the holonomy map identifies $\mathcal{T}_n$ with the quotient of $A_n \times \mathrm{SL}\, V$ where (λ, A) is identified to (λ, A') whenever $A^{-1}A'$ lies in a certain orthogonal group that depends on λ.

The Euclidean structure on ∂C pulls back to give a unimodular positive definite quadratic form β_ρ on V. The *character* $\chi_\rho : V \to \mathbb{R}$ is given by $\chi_\rho(v) = \mathrm{trace}(\rho v)$. The *complete invariant* of ρ is $\eta(\rho) = (\chi_\rho, [\beta_\rho])$. It plays the role in our theory that the character plays in the theory of semi-simple representations, namely two representations have the same complete invariant if and only if they are conjugate. The *trace-variety* $\chi(V)$ is the set of all characters. Let X_n be the set of all $\eta(\rho)$ topologized as a subspace of $\chi(V) \times \mathbb{P}\,\mathrm{S}^2 V$.

Theorem 1.1. *The complete invariant $\eta : \mathcal{T}_n \to X_n$ is a homeomorphism.*

In [11] Dold studies the symmetric product $\mathrm{SP}^n X = \left(\prod_1^n X\right)/S_n$ of a topological space X, where the symmetric group S_n permutes factors. When $X = V$ and $n > 1$, this is distinct from the vector space, $\mathrm{S}^n V$, of symmetric tensors of degree n. The *linear part* of the holonomy ρ has n weights $\exp \xi_i$ (counted with multiplicity) where $\xi_i \in V^*$, and these give a point $\xi_\rho = [\xi_1, \cdots, \xi_n] \in \mathrm{SP}^n V^*$. The following description of the moduli space is reminiscent of the classification of semi-simple Lie groups via *roots*. Let $\mathcal{P} \subset \mathrm{S}^2 V$ be the space of unimodular positive definite quadratic forms on V. Define $\mathcal{R}_n$ to be the subspace of all $([\xi_1, \cdots, \xi_n], \beta)$ in $\mathrm{SP}^n(V^*) \times \mathcal{P}$ satisfying the *weights equation*

$$\exists\ \varpi \geq 0 \qquad \forall\ i \neq j \quad \langle \xi_i, \xi_j \rangle_{\beta^*} = -\varpi \tag{1}$$

where $\langle \cdot, \cdot \rangle_{\beta^*}$ is the inner product on V^* dual to β. A geometrical interpretation of this condition is given in (20).

Theorem 1.2. *The* weight data *$\nu : \mathcal{T}_n \longrightarrow \mathcal{R}_n$ is given by $\nu(\rho) = (\xi_\rho, \beta_\rho)$ and is a homeomorphism, and $\mathcal{R}_n$ is a semi-algebraic set. Moreover generalized cusps with non-diagonalizable holonomy form the subspace of $\mathcal{R}_n$ where $\varpi = 0$.*

Let $F_n = \{[v_1, \cdots, v_n] \in \mathrm{SP}^n V \ : \ \exists\ \varpi \geq 0 \ \ \forall\ i \neq j \ \ \langle v_i, v_i \rangle = -\varpi\}$ and $U_n \subset \mathrm{SL}\, V$ be the group of upper triangular unipotent matrices. There is a bundle isomorphism

$$\theta : U_n \times F_n \to \mathcal{R}_n \qquad \text{given by} \quad \theta(A, [v_1, \cdots, v_n]) = ([\xi_1, \cdots, \xi_n], A^t A)$$

where $\xi_i(v) = \langle v_i, Av \rangle$.

The *type* of ρ is the number of non-trivial distinct *weights* of ρ, and can be any integer $0 \leq \mathbf{t} \leq n$. It equals the number of non-zero coordinates of ψ and also of ξ_ρ. There is an affine projection $\pi : \Omega \to (0,\infty)^{\mathbf{t}}$. Each fiber has the geometry of horoball in $\mathbb{H}^{n-\mathbf{t}}$. The geometry transverse to the fibers is *Hex geometry*: the projective geometry of an open simplex, see [2, Section 1.5].

The similarity structure is part of a certain kind of geometric structure on ∂C, called a *cusp geometry*, that uniquely determines the cusp up to equivalence. The extra structure consists of $\mathbf{t}$ transversally measured codimension-1 foliations with flat leaves. The foliations are the preimages of foliations of $(0,\infty)^{\mathbf{t}}$ by coordinate hyperplanes. When $\mathbf{t} < n$ then these foliations are arbitrary, subject to being

pairwise orthogonal. The transverse measures are harmonic 1-forms representing the cohomology classes ξ_i given by the weights.

The cusp geometry is also encoded by a polynomial, J, called the *shape invariant*, defined up to scaling, that is the sum of the quadratic, β_ρ, and a cubic. This gives an embedding of the marked moduli space into the vector space of such polynomials. Projection onto the quadratic term exhibits the moduli space as a bundle over $\mathcal{P}$. The fiber is a cone in the space of cubic differentials. The cubic is a linear combination of the cubes of the weights (39).

This is reminiscent of the result of Hitchin [15], Labourie [17], and Loftin [19], that the moduli space of properly convex structures on a closed surface is a vector bundle over the space of conformal structures, with fiber the space of *holomorphic* cubic differentials. However, in general the cubic differentials for generalized cusps are not holomorphic.

The polynomial J is defined as follows. Choose a basepoint $b \in \partial\Omega \subset \mathbb{R}^n$ and an affine map $\tau : \mathbb{R}^n \to \mathbb{R}$ so that $\tau(b) = 0$ and $\tau(\operatorname{int}\Omega) > 0$. The hyperplane $H = \tau^{-1}(0)$ is then tangent to Ω at b. The hypersurface $\partial\Omega$ is parameterized by the function $\mu : V \to \partial\Omega$ given by the orbit, $\mu(v) = \rho(v)(b)$ of b. The function $h = \tau \circ \mu$ can be thought of as the height of points in $\partial\Omega$ above H. *However $\partial\Omega$ is* **not** *the graph of h, see* (6.16). Then $J : V \to \mathbb{R}$ is the 3-Jet of h, normalized so the quadratic term is unimodular. The cubic is zero if and only if C is equivalent to a cusp in a hyperbolic manifold. This is similar to [20, Thm 4.5], that an affine hypersurface is quadratic if and only if a certain cubic differential form vanishes identically. There is a subspace $\mathcal{J}_n \subset \mathbb{P}(\mathrm{S}^2 V \oplus \mathrm{S}^3 V)$ defined in (6.3) and

Theorem 1.3. *If $n \geq 3$ then the shape invariant $J : \mathcal{T}_n \to \mathcal{J}_n$ is a homeomorphism. Moreover, the projection $\pi : \mathcal{J}_n \to \mathcal{P}$ is a trivial bundle with fiber homeomorphic to a closed cone in $\mathrm{S}^3 V$.*

The cubic is harmonic if and only if $\partial\Omega$ is an affine sphere. The moduli space $\mathcal{T}_n$ is stratified by type. The stratum for each type is a manifold whose dimension increases with type, see Proposition 4.8. The frontier of the stratum of type $\mathbf{t}$ consists of the union of strata of smaller type. The largest type corresponds to diagonalizable holonomy. In particular:

Corollary 1.4. *Every generalized cusp is a geometric limit of diagonalizable cusps.*

It seems hard to show this directly. Another consequence is:

Theorem 1.5. *$\mathcal{T}_n$ is contractible, of dimension $k = n^2 - n$, and is manifold if and only if $n = 2$.*

Suppose $M = \mathbb{E}^n/G$ is a closed Euclidean manifold with holonomy $\rho : \pi_1 M \to \mathrm{Isom}(\mathbb{E}^n)$. Using the decomposition $\mathrm{Isom}(\mathbb{E}^n) = O(n) \ltimes \mathbb{R}^n$ gives a surjection $R : \mathrm{Isom}(\mathbb{E}^n) \to O(n)$ called the *rotational part*. By the Bieberbach theorems [3], [6] M has a finite cover by a torus $T^n = \mathbb{E}^n/H$ where H is a lattice in $\mathbb{R}^n$. Thus $R \circ \rho(\pi_1 M)$ is a finite subgroup $F \subset O(n)$ and we may choose $H = \ker R \circ \rho$. Applying this to the generalized cusp $C \cong M \times [0, \infty)$ shows that there is a finite cover $p : \widetilde{C} \to C$ corresponding to H, and $\widetilde{C} \cong T^n \times [0, \infty)$.

This cover induces a map $p^* : \mathcal{T}(C) \to \mathcal{T}(\widetilde{C})$ that sends an affine structure on C to the structure on $\widetilde{C}$ that covers it. This structure on $\widetilde{C}$ is preserved by the action of F by covering transformations. Using the identification of a structure with its holonomy gives an algebraic formulation. Since H is an abelian normal

subgroup of G, the action of G on H by conjugation determines a homomorphism $\theta : F \to \mathrm{Aut}(H)$. Define

$$\mathrm{Rep}(\widetilde{C};\theta) = \{[\rho] \in \mathrm{Rep}(\widetilde{C}) : \forall\ f \in F \quad \rho \sim \rho \circ (\theta f)\}$$

where $\sim$ denotes conjugate representations.

Theorem 1.6. *The map* $\mathrm{hol} \circ p^* : \mathcal{T}(C) \to \mathrm{Rep}(\widetilde{C};\theta)$ *is a homeomorphism.*

A generalized cusp C in a 3-manifold is determined by three complex numbers (w, h, r) subject to $\mathrm{Im}\, w > 0$ and $|c| \leq 3|h|$. The conformal structure on ∂C is $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}w)$. The parameter w was used by Thurston to describe cusps in hyperbolic 3-manifolds. There is a unique upper-triangular matrix $A = A_w \in \mathrm{SL}(2,\mathbb{R})$ with positive eigenvalues such that the Mobius transformation α corresponding to A satisfies $\alpha(w) = i$. Then the quadratic term in J is $q_w = A^t A \in \mathrm{S}^2\,\mathbb{R}^2$.

After identifying $\mathbb{R}^2 \equiv \mathbb{C}$ a cubic $p \in \mathrm{S}^3\,\mathbb{R}^2$ is uniquely expressible as $p = \mathrm{Re}(hz^3) + \mathrm{Re}(rz|z|^2)$ for some $h, r \in \mathbb{C}$. The first term is harmonic and the second is called radial.

Theorem 1.7. *There is a homeomorphism*

$$\Theta : \mathcal{T}_3 \longrightarrow \{(w, h, r) \in \mathbb{C}^3 : \mathrm{Im}(w) > 0, \quad |r| \leq 3|h|\}$$

If $\Theta(x) = (w, h, r)$ *then* $J(x) = [q_w + c]$ *with* q_w, A_w *as above, and* $c = \mathrm{Re}(hz^3 + rz|z|^2) \circ A_w$.

This result determines exactly which cubic differentials appear. One may regard the generalized cusp for (w, h, r) as a deformation of the hyperbolic cusp corresponding to $(w, 0, 0)$. The generalized cusps with a fixed conformal structure, w, on the boundary are parameterized by a point in $\{(h, r) \in \mathbb{C}^2 : \ |r| \leq 3|h|\}$. This is a cone on a solid torus. The cubic is harmonic if and only if $r = 0$, in which case either the cusp holonomy is conjugate in $\mathrm{GL}(4,\mathbb{R})$ into a unipotent subgroup of $O(3,1)$ or into the diagonal subgroup of $\mathrm{Aff}(\mathbb{R}^3)$ where the determinant is one.

We assume the reader is familiar with the main results and definitions up to the end of Section 1 from [2]. Each facet of the closed Weyl chamber $A_n \subset \mathbb{R}^n$ parameterizes those translation groups $\mathrm{Tr}(\psi)$ of a fixed type. The main new ingredient, (3.4), is a connected set $\widetilde{A}_n$ of representations that give conjugates of generalized cusps of all types.

The set $\widetilde{A}_n$ is obtained by a kind of iterated *blowup* of A_n in the sense of algebraic geometry, and each fiber of each blowup consists of pairwise conjugate representations. There seems to be no obvious way to replace $\widetilde{A}_n$ by a continuous family containing only one representative of each conjugacy class. The subspace of $\widetilde{A}_n$ consisting of cusps of type $\mathbf{t}$ is the interior of a compact manifold, M, with boundary. The *direction* that a sequence $\rho_n \in \mathrm{int}(M) \subset \widetilde{A}_n$ converges to a point $p \in \partial M$ determines a point in $\widetilde{A}_n$ that is some conjugate of some representation corresponding to p.

The paper is organized as follows. In Section 2 we review the translation groups $\mathrm{Tr}(\psi)$ and show that a marked translation group is uniquely determined by the complete invariant. In Section 3 we introduce a connected space $\widetilde{A}_n$ that continuously parameterizes translation groups of all types. In Section 4 we prove the complete invariant provides an embedding of the marked moduli space $\mathcal{T}_n$. In Section 5 we obtain the characterization (1) of the weights of marked translation groups. In

Section 6 we show that a marked translation group is determined by the sum of a quadratic and a cubic differential. In Section 7 we compute $\mathcal{T}_3$, the marked moduli space for 3-manifolds. Various routine computational proofs were moved into Section 8 to avoid disrupting the flow of ideas. Section 9 is a summary of notation.

The proof that the shape invariant determines a marked translation group that is unique up to conjugacy is a rather long and technical computation in Lemma 6.9 that is an ad hoc algebraic argument. Perhaps there is a better way to establish this with some differential geometry. The various descriptions of the moduli space only gradually emerged as we stumbled upon various clues. In particular, the new parameters in Section 2 were discovered by a very circuitous route.

2. The complete invariant

Throughout $V \equiv \mathbb{R}^{n-1}$ denotes the *extended domain* of the holonomy of a marked generalized cusp, and $\{e_1, \cdots, e_k\}$ is the standard basis of $\mathbb{R}^k$, and $\{e_1^*, \cdots, e_k^*\}$ is the dual basis of the dual vector space. If $X \subset \mathbb{R}^n$ then $\mathrm{GL}(X) \subset \mathrm{GL}(n, \mathbb{R})$ is the subgroup that preserves X. Affine space is denoted $\mathbb{A}^n := \mathbb{R}^n \times 1 \subset \mathbb{R}^{n+1}$ and the affine group is $\mathrm{Aff}(n) := \mathrm{GL}(\mathbb{A}^n) \subset \mathrm{GL}(n+1, \mathbb{R})$. If $X \subset \mathbb{A}^n$ then $\mathrm{Aff}(X) \subset \mathrm{Aff}(n)$ is the subgroup that preserves X. What follows, up to Theorem 2.5, is from [2].

Definition 2.1. Suppose $\Omega \subset \mathbb{A}^n$ is a closed, convex, subset bounded by a non-compact, properly embedded, strictly convex hypersurface $\partial\Omega$. Also suppose $\mathrm{Aff}(\Omega)$ contains a subgroup $T = T(\Omega) \cong (V, +)$ that acts simply-transitively on $\partial\Omega$. Then T is called a *translation group* and the group $G(\Omega) \subset \mathrm{Aff}(\Omega)$ that preserves each T-orbit is called a *cusp Lie group*.

The subgroup T is unique. The T-orbit of a point in Ω is called a *horosphere*. Horospheres are smooth, strictly-convex hypersurfaces that foliate Ω. In particular $\partial\Omega$ is a horosphere. Moreover $G(\Omega) = \mathrm{Aff}(\Omega)$ unless $\Omega \cong \mathbb{H}^n$, in which case $G(\Omega)$ is conjugate into a subgroup $\mathrm{PO}(n, 1)$. A *generalized cusp* is an affine manifold Ω/Γ where $\Gamma \subset G(\Omega)$ is a torsion-free lattice. Choose a *basepoint* $b \in \partial\Omega$. The subgroup $\mathrm{O}(\Omega, b) \subset G(\Omega)$ that fixes b is called a *cusp orthogonal group*, and is compact, and $G(\Omega) = \mathrm{O}(\Omega, b) \ltimes T$. Different notation was used for this in [2, Definition 1.45]. We focus on torus cusps. Then the holonomy is an isomorphism $\theta' : \mathbb{Z}^{n-1} \to \Gamma \subset T$. The *extended holonomy* is the extension of this homomorphism to an isomorphism $\theta : V \to T$.

Definition 2.2. A *marked translation group* is an isomorphism $\theta : V \to T$ where $T \subset \mathrm{Aff}(n)$ is a translation group.

Given a marked translation group θ, there is a direct sum decomposition

$$V = D \oplus U \tag{2}$$

where $\theta(U)$ is the subgroup of unipotent elements, and $\theta(D)$ is the subgroup of elements for which the largest Jordan block has size 2. Thus $\theta(D)$ contains the diagonalizable subgroup. In the notation of [2, (1.41)] $U = P(\psi)$ and $D = T_2$.

Definition 2.3. The *type* $\mathbf{t} : \mathbb{R}^n \to \mathbb{Z}$, the *unipotent rank* $\mathbf{u} : \mathbb{R}^n \to \mathbb{Z}$ and the *rank* $\mathbf{r} : \mathbb{R}^n \to \mathbb{Z}$ are defined for $x = (x_1, \cdots, x_n)$ by

$$\mathbf{t}(x) = |\{i : x_i \neq 0\}| \qquad \mathbf{r}(x) = \min(\mathbf{t}(x), n-1) \qquad \mathbf{u}(x) + \mathbf{r}(x) = n-1$$

These functions are used in the context of two families of marked translation groups that involve a parameter $x \in \mathbb{R}^n$ and for these, $\mathbf{r}(x) = \dim D$, and $\mathbf{u}(x) = \dim U$, and $\mathbf{t}(x)$ is the number of non-constant weights of θ. If $\psi : \mathbb{R}^n \to \mathbb{R}$ is a homomorphism we will often identify ψ with $(\psi_1, \cdots, \psi_n) \in \mathbb{R}^n$ where $\psi_i = \psi(e_i)$.

Definition 2.4. The group $\mathrm{Tr}(\psi) = \zeta_\psi(V)$ is defined as follows.

$A_n(\Psi) := \{(\psi_1, \cdots, \psi_n) : \psi_1 \geq \psi_2 \geq \cdots \geq \psi_n \geq 0\}$

$A_n^u(\Psi) := \{(\psi_1, \cdots, \psi_n) : \ \forall\ i\ \psi_i \geq 0\ \&\ \exists\ t\ (\psi_i > 0 \Leftrightarrow i \leq t)\ \}\}$ (u=unordered)

If $\psi \in A_n^u(\Psi) \cup A_n(\Psi)$ set $\mathbf{t} = \mathbf{t}(\psi)$ and $\mathbf{u} = \mathbf{u}(\psi)$ and $\mathbf{r} = \mathbf{r}(\psi)$. If $\mathbf{t} = 0$ set $E = \emptyset$ and $\psi^- = 0$, otherwise define $\psi^- \in V^*$ and E by

$$\psi^-(v_1, \cdots, v_{n-1}) = -\psi(v_1, \cdots, v_{n-1}, 0), \qquad E = \psi_{\mathbf{t}} \cdot \mathrm{Diag}(v_1, \cdots, v_{\mathbf{r}})$$

Define $\zeta_\psi : V \to \mathrm{Aff}(n)$ by $\zeta_\psi(v) = \exp f_\psi(v)$ where $f_\psi(v) =$

$\mathbf{t} < n-1$ $\qquad$ $\mathbf{t} = n-1$ $\qquad$ $\mathbf{t} = n$

$$\begin{pmatrix} E & & & 0 & \\ & \begin{pmatrix} 0 & v_{\mathbf{r}+1} & \cdots & v_{\mathbf{r}+\mathbf{u}} & \psi^-(v) \\ 0 & \cdots & & 0 & v_{\mathbf{r}+1} \\ \vdots & & & & \vdots \\ 0 & \cdots & & 0 & v_{\mathbf{r}+\mathbf{u}} \\ 0 & \cdots & & 0 & 0 \end{pmatrix} \\ 0 & \end{pmatrix} \qquad \begin{pmatrix} E & 0 & 0 \\ 0 & 0 & \psi^-(v) \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} E & 0 & 0 \\ 0 & \psi^-(v) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Observe that $\mathbf{r} + \mathbf{u} = n - 1$.

Since all the eigenvalues are positive, $\zeta_\psi : V \to \mathrm{Tr}(\psi)$ is an isomorphism, so $\mathrm{Tr}(\psi) \cong \mathbb{R}^{n-1}$. It follows from [2, Theorem 0.2], and we show below that ζ_ψ is conjugate to $\zeta_{\psi'}$ if and only if $\psi = \psi'$. However $\mathrm{Tr}(\psi)$ and $\mathrm{Tr}(\psi')$ are conjugate subgroups if and only if $\psi = s\psi'$ for some $s > 0$.

Theorem 2.5.

(a) $\mathrm{Tr}(\psi)$ *is a translation group.*
(b) *If* $s > 0$ *then* $\zeta_{s\psi} = \zeta_\psi \circ ((s\,\mathrm{I}_{\mathbf{r}}) \oplus \mathrm{I}_{\mathbf{u}})$ *where* I_k *is the identity map on* $\mathbb{R}^k$.
Suppose $\theta : V \to \mathrm{Aff}(n)$ *is a marked translation group; then*
(c) $\exists!\ \psi \in A_n(\Psi)$ *and* $\exists\ B \in \mathrm{SL}^{\pm}(V)$ *such that* θ *is conjugate to* $\zeta_\psi \circ B$.
(d) $\exists\ \psi' \in A_n^u(\Psi)$ *and* $\exists\ B' \in \mathrm{SL}(V)$ *such that* θ *is conjugate to* $\zeta_{\psi'} \circ B'$.

Proof. (b) The definition shows $f_{s\psi}(v_1, \cdots, v_{n-1}) = f(sv_1, \cdots, sv_{\mathbf{r}}, v_{\mathbf{r}+1}, \cdots, v_{\mathbf{r}+\mathbf{u}})$.

(a) Given a marked translation group $\rho : V \to \mathrm{Aff}(n)$ then, by [2, Theorem 0.1], there is $\psi \in A_n(\Psi)$ such that $\rho(V)$ is conjugate into the group $T(\psi)$ defined in [2, Definition 1.32]. Moreover if $\psi \neq 0$ we may choose $\psi_{\mathbf{t}} = 1$ by [2, Theorem 0.2(iii)] and then $T(\psi) = \mathrm{Tr}(\psi)$ as defined in (2.4). This proves (a).

It follows that $\rho = \zeta_\psi \circ A$ for some $A \in \mathrm{GL}(V)$. If $\mathbf{r} > 0$ then there is $s > 0$ so that $A = ((s\,\mathrm{I}_{\mathbf{r}}) \oplus \mathrm{I}_{\mathbf{u}})B$ with $B \in \mathrm{SL}^{\pm}(V)$. Then $\rho = \zeta_{s\psi} \circ B$ by (b). If $\mathbf{r} = 0$ then $\psi = 0$ and $\zeta_0 \circ (s\,\mathrm{I})$ is conjugate to ζ_0. Thus in this case we may also choose $B \in \mathrm{SL}^{\pm}(V)$.

To show ψ is unique, by [2, Theorem 0.2] ψ is unique up to multiplication by some $s > 0$. Suppose $\zeta_\psi \circ B$ is conjugate to $\zeta_{s\psi} \circ B'$. Then ζ_ψ is conjugate to $\zeta_{s\psi} \circ (B'B^{-1})$, and thus to $\zeta_\psi \circ ((s\,\mathrm{I}_{\mathbf{r}}) \oplus \mathrm{I}_{\mathbf{u}})B'B^{-1})$. By [2, Theorem 0.2(v)]

$((s\,\mathrm{I}_{\mathbf{r}})\oplus \mathrm{I}_{\mathbf{u}})B'B^{-1} \in \mathrm{O}(\mathbb{E}^{n-1},\psi)$. By [2, (1.44)] this is a subgroup of the orthogonal group, thus $s=1$. This proves (c).

For (d), when $n=2$ the result is easy, so assume $n\geq 3$ and $\det B=-1$. There are two coordinates ψ_i,ψ_{i+1} of ψ that are either both zero or both non-zero. Swapping columns i and $i+1$ of B gives $B'\in \mathrm{SL}(V)$ and swapping ψ_i and ψ_{i+1} gives $\psi'\in A^u_n(\Psi)$. Then $\zeta_\psi\circ B$ is conjugate to $\zeta_{\psi'}\circ B'$ by swapping the i and $i+1$ coordinates in $\mathbb{R}^{n+1}$. □

We regard the second symmetric power, $\mathrm{S}^2 V$, as the vector space of homogeneous polynomials $\beta : V\to\mathbb{R}$ of degree two. The subspace $\widetilde{\mathcal{P}}(V)\subset \mathrm{S}^2 V$ consists of positive definite forms and $\mathcal{P}(V)\subset\widetilde{\mathcal{P}}(V)$ is the subspace of unimodular forms. Let $\pi_{\mathcal{P}}:\widetilde{\mathcal{P}}(V)\to\mathcal{P}(V)$ be the projection

$$\pi_{\mathcal{P}}(\beta)=(\det\beta)^{-1/(n-1)}\beta$$

The notation $\beta\sim\beta'$ means there is $\lambda>0$ with $\beta'=\lambda\beta$. Given a marked translation group $\theta: V\to\mathrm{Aff}(n)$ the *orbit map* $\mu_{\theta,b}:V\to\partial\Omega$ is the homeomorphism defined by

$$\mu_{\theta,b}(v)=(\theta v)\,b \tag{3}$$

where $b\in\partial\Omega$ is some choice of basepoint. Since $\partial\Omega$ is smooth and strictly convex, there is a unique affine hyperplane $H_b\subset\mathbb{A}^n$ with $H_b\cap\Omega=b$. There is an affine map $\tau:\mathbb{A}^n\to\mathbb{R}$ with $\tau(H_b)=0$ and $\tau(\operatorname{int}\Omega)>0$. The *height function*

$$h_\theta=\tau\circ\mu_{\theta,b}:V\to\mathbb{R} \tag{4}$$

is only defined up to multiplication by a positive real. Observe that $\partial\Omega$ is **not** the graph of h_θ, see (6.16). Note that if b' is a different choice of basepoint, then there is unique element $A\in T(\Omega)$ such that $Ab=b'$. In this case $\tau'=\tau\circ A^{-1}$ is an affine map such that $\tau'(H_{b'})=0$ and $\tau'(\operatorname{int}\Omega)>0$. Furthermore, $\mu_{\theta,b'}=A\circ\mu_{\theta,b}$, and so $\tau'\circ\mu_{\theta,b'}=\tau\circ\mu_{\theta,b}$. It follows that the height function is independent of the choice of basepoint.

Since $\partial\Omega$ is strictly convex one obtains positive definite quadratic forms

$$\widetilde{\beta}(\theta)=\mathrm{D}^2 h_\theta,\qquad \beta(\theta)=\pi_{\mathcal{P}}(\widetilde{\beta}(\theta))\in\mathcal{P}(V) \tag{5}$$

After rescaling, the orbit map is an isometry from (V,β) to $\partial\Omega$ with the horosphere metric [2, (2.14)]. The form $\widetilde{\beta}$ is only defined up to scaling. To emphasize this we usually work with $[\beta]\in\mathbb{P}\mathcal{P}$. However it is sometimes convenient to use the natural identification $\mathcal{P}\equiv\mathbb{P}\mathcal{P}$. Then one must remember that preserving β only means β is preserved up to rescaling.

Writing $v=\sum_{i=1}^{n-1}v_ie_i$ and $u_i=(\partial\mu_{\theta,b}/\partial v_i)_{v=0}\in\mathbb{R}^n$ then $(u_1,\cdots,u_{n-1})$ is a basis of the tangent space $\mathrm{T}_b\,\partial\Omega\cong H_b$. We may use $\tau(x)=\pm\det(u_1,\cdots,u_{n-1},x)$ and a height function is then given by

$$h_\theta(v)=\pm\det(u_1,\cdots,u_{n-1},\mu_{\theta,b}(v)-b) \tag{6}$$

where the sign is chosen so that $\tau(\Omega)\geq 0$.

The space $\mathrm{Hom}(V,\mathrm{Aff}(n))$ is given the weak topology. This coincides with the Euclidean topology when it is realized as an algebraic subset of Euclidean space. The space $\mathrm{Hom}(V,\mathrm{Aff}(n))/\mathrm{Aff}(n)$ is the quotient space under the action of conjugacy. It is not Hausdorff.

Definition 2.6. $\widetilde{\mathrm{Rep}}(V) \subset \mathrm{Hom}(V, \mathrm{Aff}(n))$ is the subspace of marked translation groups, and $\mathrm{Rep}(V) = \widetilde{\mathrm{Rep}}(V)/\mathrm{Aff}(n)$ is the space of conjugacy classes with the quotient topology.

In (4.4) we define $\mathcal{T}(V)$ as equivalence classes of developing maps and show it is homeomorphic to $\mathrm{Rep}(V)$. Various functions defined on $\mathrm{Rep}(V)$ in this section can then be re-interpreted as functions on $\mathcal{T}(V)$.

Lemma 2.7. *$\widetilde{\beta} : \widetilde{\mathrm{Rep}}(V) \to \mathcal{P}(V)$ is smooth and covers a continuous map $\beta : \mathrm{Rep}(V) \to \mathcal{P}(V)$.*

Proof. By the discussion above $\widetilde{\beta}$ does not depend on the choice of basepoint b or height function used above. Given a marked translation group θ every choice of basepoint b has orbit a convex hypersurface unless b lies in a projective subspace preserved by θ. Thus in a neighborhood of θ in $\widetilde{\mathrm{Rep}}(V)$ a fixed choice of basepoint b can be used for the orbit map. Then the function $\mu : \widetilde{\mathrm{Rep}}(V) \times V \to \mathbb{R}$ given by $\mu(\theta, v) = \mu_{\theta,b}(v)$ is smooth near (θ, v). Equations (3) and (6) then imply h_θ is smooth near θ, so $\widetilde{\beta}$ is smooth. It is clear that $\widetilde{\beta}(\theta)$ is invariant under conjugation of θ. Therefore $\widetilde{\beta}$ covers a map $\beta : \mathrm{Rep}(V) \to \mathcal{P}(V)$ which is continuous by properties of the quotient topology. □

The *character* of a homomorphism $\rho : V \to \mathrm{GL}(n+1, \mathbb{R})$ is $\chi(\rho) : V \to \mathbb{R}$ given by $\chi(\rho) = \mathrm{trace} \circ \rho$. The *trace-variety*, $\chi(V)$, is the set of characters of all such homomorphisms. $\mathrm{Hom}(V, \mathrm{Aff}_n)$ is a real algebraic variety, and $\chi(V)$ is its image under a polynomial map. Thus $\chi(V)$ is a semi-algebraic set, and in particular is homeomorphic to a subspace of Euclidean space.

By (2.5) a marked translation group is conjugate to an upper triangular group. The character is not changed by conjugation. The character of an upper-triangular representation is a function on V that is the sum of $(n+1)$ functions, each of which is the exponential of an element of V^*. Thus the subspace of $\chi(V)$ consisting of characters of marked translation groups is homeomorphic to a subspace of $\mathrm{SP}^{n+1} V^*$.

Definition 2.8. Given a marked translation group $\theta : V \to \mathrm{Aff}(n)$ then

- The *horosphere metric* is the unimodular quadratic form $\beta(\theta) \in \mathcal{P}(V)$
- The *complete invariant* is $\eta(\theta) = (\chi(\theta), [\beta(\theta)])$.

Also $\mathrm{O}(\eta(\theta)) \subset \mathrm{GL}(V)$ is the subgroup that preserves both $\chi(\theta)$ and $[\beta(\theta)]$.

Lemma 2.14 implies $\mathrm{O}(\eta(\theta))$ is a subgroup of the orthogonal group of β unless $\mathbf{t}(\theta) = 0$, in which case it is the group of Euclidean similarities fixing 0.

Proposition 2.9. *The complete invariant $\eta : \mathrm{Rep}(V) \to \chi(V) \times \mathbb{P}(\mathrm{S}^2 V)$ is continuous.*

Proof. It is well known that χ is continuous, and β is continuous by (2.7). □

Definition 2.10. The *complete invariant variety* is $X_n = \mathrm{Im}(\eta)$. This is homeomorphic a subspace of Euclidean space.

A dual vector $\xi \in V^*$ is a *Lie-algebra weight* of $\theta : V \to \mathrm{GL}(n+1, \mathbb{R})$, and $\exp \circ \xi$ is a *weight* of θ, if the *weight space*

$$V(\theta, \xi) := \bigcap_{v \in V} \ker(\theta(v) - \exp \circ \xi(v)) \neq 0$$

Let $\langle\cdot,\cdot\rangle_\beta$ be the inner product on V given by β. Let $\beta^* \in \mathrm{S}^2 V^*$ denote the dual quadratic form defined by $\beta^*(\phi) = \beta(v)$ if $\phi(x) = \langle v, x\rangle$. Let $\langle\cdot,\cdot\rangle_{\beta^*}$ be inner product on V^* given by β^*. Next we compute the complete invariant, the details are in Section 8.

Proposition 2.11. *Given $\psi \in A_n^u(\Psi)$ the decomposition $V = D \oplus U$ for ζ_ψ is orthogonal with respect to $\beta(\zeta_\psi)$. Set $\mathbf{u} = \mathbf{u}(\psi)$ and $\mathbf{t} = \mathbf{t}(\psi)$, then $\beta(\zeta_\psi) \sim \beta'$ where for $v \in V$ and $\mathbf{t} < n$*

$$\beta'(v) = \sum_{i=1}^{\mathbf{t}} \psi_i v_i^2 + \psi_{\mathbf{t}}^{-1} \sum_{i=\mathbf{t}+1}^{n-1} v_i^2 \quad \chi(\zeta_\psi)(v) = 2 + \mathbf{u} + \sum_{i=1}^{\mathbf{t}} \exp(\psi_{\mathbf{t}} v_i)$$

and for $\mathbf{t} = n$

$$\beta'(v) = \sum_{i=1}^{n-1} \psi_i v_i^2 + \psi_n^{-1} \left(\sum_{i=1}^{n-1} \psi_i v_i \right)^2 \quad \chi(\zeta_\psi)(v) = 1 + \sum_{i=1}^{n-1} \exp(\psi_n v_i) + \exp\left(-\sum_{i=1}^{n-1} \psi_i v_i \right)$$

Moreover, when $\mathbf{t} < n$ then $\det \beta' = \psi_1 \cdots \psi_{\mathbf{t}-1} \psi_{\mathbf{t}}^{\mathbf{t}+2-n}$ and the non-zero Lie algebra weights of ζ_ψ are $\{\xi_i = \psi_{\mathbf{t}} e_i^* : 1 \le i \le \mathbf{t}\}$, and their duals are an orthogonal basis of D, and $\beta^*(\xi_i) = \psi_{\mathbf{t}}^2 (\det \beta')^{-1/(n-1)} \psi_i^{-1}$. Also when $\mathbf{t} = n$ then $\det \beta' = \psi_1 \cdots \psi_{n-1} \psi_n^{-1} \sum_{i=1}^n \psi_i$.

Theorem 2.16 shows that the complete invariant determines a marked translation group up to conjugacy. Theorem 6.14 shows the same for the shape invariant. The strategy is the same in both cases. One argument shows the invariant determines the translation group up to conjugacy. A second discussion shows that the invariant determines a basis of a lattice in this translation group by determining a coset of $\mathrm{O}(\partial\Omega, b)$ in $G(\Omega)$.

Corollary 2.12. *Suppose $n \ge 3$ and $\theta : V \to \mathrm{Aff}(n)$ is a marked translation group. Then θ is conjugate to $\zeta_\psi \circ B$ for some $\psi = (\psi_1, \cdots, \psi_n) \in A_n$ and $B \in \mathrm{SL}^\pm V$, and the complete invariant $\eta(\theta)$ uniquely determines ψ.*

Proof. By (2.5)(b) θ is conjugate to some $\zeta_\psi \circ B$ with $B \in \mathrm{SL}^\pm V$, and ψ is uniquely determined by the conjugacy class of θ. If $\mathbf{t} = n$ then ζ_ψ is diagonal, so $\chi(\theta)$ determines θ up to conjugacy, and hence determines ψ by (2.5)(b). So suppose $\mathbf{t} < n$. It follows immediately from the definitions that $\beta(\zeta_\psi \circ B) = \beta(\zeta_\psi) \circ B$, and $\xi_i(\zeta_\psi \circ B) = \xi_i(\zeta_\psi) \circ B$. Hence $\beta^*(\xi_i \circ B) = \beta^*(\xi_i)$. By (2.11) it follows that $\eta(\theta)$ determines

$$(\beta^*\xi_1, \cdots, \beta^*\xi_{\mathbf{t}}) = \psi_{\mathbf{t}}^2 \left(\psi_1 \cdots \psi_{\mathbf{t}-1} \psi_{\mathbf{t}}^{\mathbf{t}+2-n}\right)^{-1/(n-1)} (\psi_1^{-1}, \cdots, \psi_{\mathbf{t}}^{-1})$$

up to permutations. Let $x_i = \log \psi_i$ and $y_i = \log \beta^* \xi_i$ and $x = (x_1, \cdots, x_{\mathbf{t}})$ and $y = (y_1, \cdots, y_{\mathbf{t}})$. Define $v : \mathbb{R}^{\mathbf{t}} \to \mathbb{R}$ by

$$\begin{aligned} v(x) &= \log \left[\psi_{\mathbf{t}}^2 \left(\psi_1 \cdots \psi_{\mathbf{t}-1} \psi_{\mathbf{t}}^{\mathbf{t}+2-n}\right)^{-1/(n-1)} \right] \\ &= -(n-1)^{-1}(x_1 + \cdots + x_{\mathbf{t}-1} + (\mathbf{t} + 4 - 3n) x_{\mathbf{t}}) \end{aligned}$$

Let $e = (1, \cdots, 1)$; then $y = x + (v(x))e = (I + G)x$ where $G = e \otimes v$. Then $\eta(\theta)$ determines y, and recovering the ψ_i amounts to finding x that solves the linear equation $y = (I + G)x$.

We claim that $I + G$ is invertible. For the sake of contradiction assume that $0 \ne w \in \ker(I + G)$, then $w + v(w)e = 0$. This implies that $w = \alpha e$ for some $\alpha \ne 0$. Since all non-zero multiples of w are also in the kernel there is no loss of generality

in assuming that $\alpha = 1$. This implies that $e + v(e)e = 0$ and so $v(e) = -1$. From the definition of v the equation $v(e) = -1$ becomes $-(n-1)^{-1}(2\mathbf{t} + 3 - 3n) = -1$, or equivalently that $\mathbf{t} = 2n - 2$. However, since $n \geq 3$ this implies that $\mathbf{t} > n$, which is a contradiction. It follows that the x_i can be recovered from the y_i, and by exponentiating we recover the ψ_i. □

The characteristic polynomial of a square matrix A is $c(A) = \det(x\,\mathrm{I} - A)$. An affine automorphism of $\mathbb{R}^n$ is given by $f(x) = Ax + b$ with *linear part* $A \in \mathrm{GL}(n, \mathbb{R})$ and also given by $B \in \mathrm{Aff}(n) \subset \mathrm{GL}(n+1, \mathbb{R})$. Then $c(B) = (x-1)c(A)$. This means that a translation group has one more zero Lie-algebra weight than the linear part. The character of a marked translation group determines the weights, thus the complete invariant determines the weight data.

Lemma 2.13. *Suppose $\theta : V \to \mathrm{Aff}(n)$ is a marked translation group. Let $\xi_\theta = [\xi_1, \cdots, \xi_n] \in \mathrm{SP}^n V^*$ be the Lie-algebra weights of the linear part of θ. Then the characteristic polynomial $c_\theta : V \to \mathbb{R}[x]$ given by*

$$c_\theta = \det(xI - \theta) = (x-1)\prod_{i=1}^{n}(x - \exp \circ \xi_i)$$

is uniquely determined by $\chi(\theta)$. Moreover there is $f : X_n \to \mathcal{R}_n$ with $\nu = f \circ \eta$, where $\nu : \mathcal{T}_n \longrightarrow \mathcal{R}_n$ is the weight data $\nu(\rho) = (\xi_\rho, [\beta_\rho])$.

Proof. Suppose $A = \theta(v)$. Then $\theta(kv) = A^k$ so $\chi(\theta)(kv) = \mathrm{trace}\, A^k$. If A has eigenvalues $\mu_0, \cdots, \mu_n$ counted with multiplicity then $p_k := \mathrm{trace}(A^k) = \sum \mu_i^k$ is a symmetric polynomial function of the eigenvalues. Every symmetric polynomial is a polynomial in the p_k, and in particular the coefficients of $c(A)$ have this property. Hence $\chi(\theta)$ determines the characteristic polynomial of $\theta(v)$ for every $v \in V$. Thus $\chi(\theta)$ determines the function $c_\theta = c \circ \theta : V \to \mathbb{R}[x]$ which sends $v \in V$ to the characteristic polynomial of $\theta(v)$. Since all the eigenvalues of $\theta(v)$ are positive, there are $\xi_i \in V^*$ with $c_\theta = \prod_{i=0}^{n}(x - \exp \circ \xi_i)$. Hence $\chi(\theta)$ determines the Lie algebra weights ξ_i. The factorization of a polynomial into linear factors is unique up to order and scaling. It follows that ξ_θ is also uniquely determined, and thus f exists. □

Recall that $\mathrm{O}(\Omega, b) \subset G(\Omega)$ is the group of affine maps that fixes the basepoint b, and acts on $\mathbb{R}^n$, preserving $\partial\Omega$. The orbit map $\mu_{\theta,b}$ identities $V \cong \mathbb{R}^{n-1}$ with $\partial\Omega$, therefore $O(\Omega, b)$ also acts on V. Under this identification $\mathrm{O}(\Omega, b) \subset \mathrm{Aff}(\mathbb{R}^n)$ is conjugate to $\mathrm{O}(\eta(\theta)) \subset \mathrm{GL}(V)$ when $\mathbf{t} > 0$. The group $\mathrm{Sim}(\beta) \subset \mathrm{GL}\, V$ is the group of similarities that preserve $[\beta]$.

Lemma 2.14. *Suppose θ is a marked translation group. If $\mathbf{t}(\theta) > 0$ then there is an isomorphism $f : \mathrm{O}(\Omega, b) \to \mathrm{O}(\eta(\theta))$ given by $f(A) = \mu^{-1} A \mu$ where $\mu = \mu_{\theta,b} : V \to \partial\Omega$ is the orbit map. If $\mathbf{t}(\theta) = 0$ then $\mathrm{O}(\eta) = \mathrm{Sim}(\beta)$.*

Proof. Let $\eta = \eta(\theta) = (\chi, [\beta])$ and $\mathbf{t} = \mathbf{t}(\theta)$. By definition $\mathrm{O}(\eta)$ is the subgroup of $\mathrm{Sim}(\beta)$ that preserves $\chi(\theta)$. If $\mathbf{t} = 0$ then θ is unipotent so χ is constant and the result follows. Now assume $\mathbf{t} > 0$, thus χ is not constant.

We claim that $\mathrm{O}(\eta)$ is a subgroup of $\mathrm{O}(\beta)$. The character $\chi : V \to \mathbb{R}$ is preserved by the action of $O(\eta)$. Now $\mathrm{O}(\eta) \subset \mathrm{Sim}(\beta)$, so if the claim is false there is $A \in \mathrm{O}(\eta)$ that moves all points in V closer to 0. It follows that $\chi(v) = \lim_{n\to\infty} \chi(A^n v) = \chi(0)$ so χ is constant, which contradicts $\mathbf{t} > 0$.

We claim that f maps into $\mathrm{O}(\eta)$. The orbit map $\mu = \mu_{\theta,b}$ defined in (3) is given by $\mu(v) = \theta(v)b$. Recall that $\partial\Omega$ is the orbit of b under $\operatorname{Im}\theta$. Given $A \in \mathrm{O}(\Omega, b)$ and $v \in V$ then $A(\mu(v)) = \mu(u)$ for some $u \in V$, and $(fA)(v) = u$. Since A fixes b it follows that

$$(\theta u)b = \mu(u) = A(\mu v) = (A\theta v)b = (A(\theta v)A^{-1})Ab = (A(\theta v)A^{-1})b$$

Now $A(\theta v)A^{-1} \in T(\Omega)$, and the action of V on $\partial\Omega$ is free, thus $\theta u = A(\theta v)A^{-1}$, so

$$(fA)(v) = u = \theta^{-1}\left(A(\theta v)A^{-1}\right)$$

Now θ and conjugation by A are both group isomorphisms, thus fA is a group automorphism of $(V, +)$, and it is continuous; thus $fA \in \mathrm{GL}\, V$. Now

$$\operatorname{trace}\theta((fA)(v)) = \operatorname{trace}\theta u = \operatorname{trace} A(\theta v)A^{-1} = \operatorname{trace}\theta v$$

Thus $\chi \circ (fA) = \chi$. It is clear that fA preserves β, hence $fA \in \mathrm{O}(\eta)$, which proves the claim.

The lemma is true for θ if and only if it is true for a conjugate of $\theta \circ B$ for some $B \in \mathrm{GL}(V)$. By Theorem 2.5 it suffices to prove the result when $\theta = \zeta_\psi$. Set $\mathbf{t} = \mathbf{t}(\psi)$. First consider the case $0 < \mathbf{t} < n$ and define

$$B = \operatorname{Diag}(\psi_1^{-1/2}, \cdots, \psi_{\mathbf{t}}^{-1/2}, \psi_{\mathbf{t}}^{1/2}, \cdots, \psi_{\mathbf{t}}^{1/2}) \in \mathrm{GL}(V)$$

It suffices to assume $\theta = \zeta_\psi^\perp := \zeta_\psi \circ B$. By (2.11)

$$\beta(\zeta_\psi^\perp)(v) = \langle v, v\rangle \qquad \chi(\zeta_\psi^\perp)(v) = 2 + \mathbf{u} + \sum_{i=1}^{\mathbf{t}} \exp(\psi_i^{-1/2}\psi_{\mathbf{t}} v_i)$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^{n-1}$.

By (2.13) $\chi(\theta)$ determines, and is uniquely determined by, the Lie algebra weights of θ. Thus $O(\eta)$ is the subgroup of $\mathrm{O}(\beta)$ that preserves the Lie-algebra weights. Hence it is the subgroup that preserves the subset of V consisting of the vectors that are dual with respect to β of these weights in V^*. By (24) the non-zero duals are $\{(\gamma\psi_i)^{-1}\psi_{\mathbf{t}} e_i : 1 \le i \le \mathbf{t}\} \subset V$. The action of $\mathrm{O}(\eta)$ permutes this set, but preserves the lengths of vectors. Thus $\mathrm{O}(\eta)$ is the subgroup of $\mathrm{O}(\beta)$ that permutes $\{e_i : 1 \le i \le \mathbf{t}\}$ and preserves the vector

$$\gamma^{-1}\psi_{\mathbf{t}}(\psi_1^{-1}, \cdots, \psi_{\mathbf{r}}^{-1}, 0, \cdots, 0) \in V$$

where the last $\mathbf{u}$ coordinates are 0. Clearly this is the same as preserving

$$(\psi_1, \cdots, \psi_{\mathbf{r}}, 0, \cdots, 0) \in V$$

Let $S(\psi)$ be the group of coordinate permutations of $\mathbb{R}^{\mathbf{r}}$ that preserve $(\psi_1, \cdots, \psi_{\mathbf{r}})$, then $\mathrm{O}(\eta(\zeta_\psi)) = S(\psi) \oplus O(\mathbf{u})$. When $\mathbf{t} < n$ it follows from [2, Proposition 1.44] that $f(\mathrm{O}(\Omega, b)) = S(\psi) \oplus O(\mathbf{u})$ which gives the result.

The remaining case is that $\mathbf{t} = n$, and then ζ_ψ has n non-zero Lie-algebra weights $\xi_i \in V^*$ and $\sum \psi_i \xi_i = 0$. Observe that ψ is determined up to scaling by this equation. If $B \in \mathrm{O}(\eta(\zeta_\psi))$ then it preserves $\chi(\zeta_\psi)$, and therefore, by (2.13), permutes these weights, so that $\xi_i \circ B = \xi_{\sigma i}$ for some permutation σ of $\{1, \cdots, n\}$. However $\sum \psi_i \xi_{\sigma i} = 0$ so $\psi_i = \psi_{\sigma i}$. Thus $\mu \cdot B \cdot \mu^{-1} = A \in \mathrm{Aff}(n)$ permutes the coordinate axes of $\mathbb{R}^n$ and preserves ψ. Again by [2, Proposition 1.44] $A \in \mathrm{O}(\Omega, b)$. It follows that $\mathrm{O}(\eta(\zeta_\psi)) \subset \mu^{-1} \cdot \mathrm{O}(\Omega, b) \cdot \mu$. It is clear that $\mathrm{O}(\eta(\zeta_\psi)) \supset \mu^{-1} \cdot \mathrm{O}(\Omega, b) \cdot \mu$. □

Suppose $\theta : V \to \mathrm{Aff}(n)$ is a marked translation group. If we consider a generalized cusp as a projective manifold, instead of as an affine one, then the holonomy might be given as $\theta_* : V \to \mathrm{SL}(n+1, \mathbb{R})$ where

$$\theta_*(v) = \alpha(v) \cdot \theta(v) \qquad \text{and} \quad \alpha(v) = (\det \theta(v))^{-1/n+1} \tag{7}$$

It follows from [2, Prop. 1.29(c)] that if two marked translation groups are conjugate in $\mathrm{GL}(n+1, \mathbb{R})$ then they are conjugate in $\mathrm{Aff}(n)$, and therefore have the same complete invariant. In (2.15) we show if $\theta_* : V \to \mathrm{SL}(n+1, \mathbb{R})$ is the corresponding projective translation group then $\chi(\theta_*)$ determines $\chi(\theta)$. However the computations are simpler using $\chi(\theta)$.

We now explain how to recover θ from θ_*. The idea is that to recover the affine action amounts to determining the weight of θ_* that corresponds to the hyperplane at infinity for affine space. Suppose $\theta : V \to \mathrm{GL}(n+1, \mathbb{R})$ and every weight is real and positive. Let $\mathcal{W}(\theta) = (\xi_0, \xi_1, \cdots, \xi_n)$ be the Lie algebra weights of θ counted with multiplicity. The Lie algebra weight ξ_i is called *a middle weight* if

$$\forall\ v \in V \quad \xi_i(v) \leq \max\{\xi_j(v) \ : \ j \neq i\ \}$$

Applied to diagonalizable representations, this is the *middle eigenvalue condition* of Choi [8]. It follows that a Lie algebra weight with multiplicity larger than 1 is a middle weight.

If $\theta : V \to \mathrm{Aff}(n)$ then ξ_i is a middle weight of θ if and only if $\xi_i = 0$. From (7) it follows that if $\mathcal{W}(\theta) = (\xi_0, \xi_1, \cdots, \xi_n)$ then $\mathcal{W}(\theta_*) = (\xi_0 - \mu, \cdots, \xi_n - \mu)$ where $\mu = (n+1)^{-1} \sum \xi_i$. The characterization above implies that ξ is a middle weight for θ if and only if $\xi - \mu$ is a middle weight for θ_*. Since the middle weight of θ_* only depends on θ_*, this shows θ_* determines θ.

Proposition 2.15. *Let $\theta : V \to \mathrm{Aff}(n)$ be a marked translation group and $\theta_* : V \to \mathrm{SL}(n+1, \mathbb{R})$ as above. Then $\chi(\theta_*)$ determines $\chi(\theta)$ and vice versa.*

Proof. The characteristic polynomial c_θ is determined by $\chi(\theta)$ using (2.13). The constant term of c_θ determines $\det \theta : V \to \mathbb{R}$, and therefore $\chi(\theta_*) = \chi(\theta)\,(\det)^{-1/n+1}$ is determined. Conversely, given $\chi(\theta_*)$ the characteristic polynomial c_{θ_*} is determined by (2.13), and so the Lie-algebra weights $\{\xi_i : 1 \leq i \leq n\}$ of θ_* are determined. Thus the middle weight ξ of θ_* is determined by $\chi(\theta_*)$, and $\theta = \exp(-\xi)\theta_*$ has middle weight 0. □

Theorem 2.16. *If $\theta, \theta' : V \to \mathrm{Aff}(n)$ are marked translation groups, then $\eta(\theta) = \eta(\theta')$ if and only if θ and θ' are conjugate in $\mathrm{Aff}(n)$.*

Proof. It is clear that the complete invariant is a conjugacy invariant. We show that if $\eta(\theta) = \eta(\theta')$ then θ and θ' are conjugate. By (2.13) $\chi(\theta)$ determines the characteristic polynomial and weights of θ, counted with multiplicity. The type of θ is the maximum over $v \in V$ of the number of eigenvalues of $\theta(v)$ that are not equal to 1. This is determined by $\chi(\theta)v$, so $\chi(\theta)$ determines $\mathbf{t}(\theta)$. In particular $\mathbf{t}(\theta) = \mathbf{t}(\theta')$.

The first case is that $\mathbf{t}(\theta) = n$ so θ is diagonalizable. Since $\mathbf{t}(\theta') = n$ then θ' is also diagonalizable, and therefore semisimple. The character of a semisimple representation determines the representation up to conjugacy, see for example [18, pp. 650]. Hence θ and θ' are conjugate in $\mathrm{GL}(n+1, \mathbb{R})$. This implies they are conjugate in $\mathrm{Aff}(n)$. If $\mathbf{t} = 0$ then the generalized cusps are equivalent to cusps in

hyperbolic manifolds. It is well known that these are determined by the Euclidean similarity structure on the boundary, and hence by $[\beta]$.

Now assume $0 < \mathbf{t}(\theta) < n$. By (2.5)(c) every marked translation group is conjugate in $\mathrm{Aff}(n)$ to some $\zeta_\psi \circ B$ where $B \in \mathrm{SL}^{\pm} V$ and $\psi \in A_n(\Psi)$. After conjugacies in $\mathrm{Aff}(n)$ we may assume $\theta = \zeta_\psi \circ B$ and $\theta' = \zeta_{\psi'} \circ B'$ are both of this form. Observe that θ and θ' are conjugate if and only if $\theta \circ (B^{-1})$ and $\theta' \circ (B)^{-1}$ are conjugate. Thus it suffices to assume that $\theta = \zeta_\psi$ and $\theta' = \zeta_{\psi'} \circ B' \circ B^{-1}$. We now change notation by replacing $B' \circ B^{-1}$ in this formula with B, so we work with $\theta = \zeta_\psi$ and $\theta' = \zeta_{\psi'} \circ B$.

By (2.12) ψ is determined by the complete invariant, hence $\psi = \psi'$, so $\theta' = \theta \circ B$. Thus $\eta(\theta') = \eta(\theta) \circ B$. We are given that $\eta(\theta) = \eta(\theta')$, so it follows that $B \in \mathrm{O}(\eta(\theta))$. Then by Lemma 2.14 $B = \mu^{-1} P \mu$ for some $P \in \mathrm{O}(\Omega, b)$.

Claim. $\theta' = P\theta P^{-1}$.

Since $\theta' = \theta \circ (\mu^{-1} P \mu)$, given $v \in V$, and recalling $b \in \partial\Omega$ is the basepoint, and using $\mu(v) = (\theta v)(b)$ gives

$$\theta'(v) = \theta \circ (\mu^{-1} P \mu)(v) = \theta(\mu^{-1} P \theta(v) b) = \theta(u), \quad \text{where } u = \mu^{-1}\left(P((\theta v)b)\right) \in V$$

Now $P \in \mathrm{O}(\Omega, b)$ fixes the basepoint b so

$$P((\theta v)b) = P((\theta v)P^{-1}b) = (P(\theta v)P^{-1})(b)$$

Thus

$$(\theta u)b = \mu(u) = P(\theta(v)b) = (P(\theta v)P^{-1})(b)$$

Now $\theta(u)$ and $P\theta(v)P^{-1}$ are both in $T(\Omega)$ which acts freely on $\partial\Omega$. Thus $\theta(u) = P\theta(v)P^{-1}$. Now $\theta'(v) = \theta(u)$ so $\theta' = P\theta P^{-1}$ as claimed. □

There is an interpretation of the complete invariant as a geometric structure on the boundary of a generalized cusp.

Definition 2.17. A *cusp geometry* on a torus $T \cong \mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$ is $(\beta, \mathcal{C})$ where β is a Euclidean metric on T with volume 1, and $\mathcal{C} \subset H^1(T; \mathbb{R}) \setminus 0$. The *type* of the geometry is $\mathbf{t} = |\mathcal{C}|$.

If $\theta : V \to \mathrm{Aff}(n)$ is a marked translation group then there is a properly convex set $\Omega \subset \mathbb{R}^n$ that is preserved by θV and $C = \Omega/\theta(\mathbb{Z}^{n-1})$ is a generalized cusp. Given $b \in \partial\Omega$ the orbit map $\mu_{\theta,b} : V \to \partial\Omega$ is a homeomorphism. Let $\pi : \Omega \to C$ be projection. Then $\pi_C := \pi \circ \mu_{\theta,b} : V \to \partial C$ can be regarded as the universal cover of ∂C. A cusp geometry $(\beta, \{\alpha_1, \cdots, \alpha_{\mathbf{t}}\})$ of type $\mathbf{t} = \mathbf{t}(\theta)$ on ∂C is defined as follows.

The metric β on ∂C is as defined above. The character $\chi(\theta)$ determines Lie-algebra weights of the representation $\xi_i : V \to \mathbb{R}$ for $1 \le i \le \mathbf{t}(\psi)$, and $\alpha_i = [\omega_i] \in H^1(\partial C; \mathbb{R})$ is determined by $\pi^* \omega_i = \xi_i$.

Thus ω_i is the harmonic representative of the de-Rham class α_i. Generalized cusps with type $\mathbf{t} < n$ correspond to choices of non-zero cohomology classes that are orthogonal with respect to the dual of β, and all such cusp geometries are realized by generalized cusps. Those of type $\mathbf{t} = n$ are determined by (1). Observe that one can recover the complete invariant from the cusp geometry.

Proposition 2.18. *Suppose $\theta_1, \theta_2 : V \to \mathrm{Aff}(n)$ are marked translation groups and $C_i = \Omega_i/\theta_i(V)$ are corresponding generalized cusps. Then θ_1 and θ_2 are conjugate if and only if there is a map $f : \partial C_1 \to \partial C_2$ that preserves the cusp geometries defined above, and f is in the correct homotopy class, as determined by the marking.*

Proof. The existence of f implies the two generalized cusps have the same complete invariant. Then θ_1 and θ_2 are conjugate by (2.16). Conversely, if θ_1 and θ_2 are conjugate, then C_1 and C_2 are equivalent cusps and so have the same cusp geometry. □

3. New parameters

In this section we define another family of translation groups in (3.4). First we motivate the definition in dimension $n = 4$. The reader may choose to replace 4 by n in what follows, and introduce $\cdots$ in the formulae.

The goal is to construct a connected algebraic family of Lie groups that gives conjugates of all the translation groups $\mathrm{Tr}(\psi)$, and such that the diagonalizable ones are dense. Recall that $\mathbf{t} = n$ is diagonalizable, and $\mathbf{t} < n$ is non-diagonalizable.

Refer to (2.11) for the following discussion. If we reparameterize ζ_ψ in the diagonal case using $t_i = \sqrt{\psi_i} v_i$ then $\beta(\zeta_\psi)(t) = \|t\|^2 + \delta^2$ where $\delta = \psi_n^{-1/2} \sum_{i=1}^{n-1} \sqrt{\psi_i} t_i$. When $\psi_n = \max_i \psi_i$ then $|\delta| \leq n\|t\|$, so β varies in a compact subset of $\mathrm{S}^2 V$. Hence, if the character remains bounded along a sequence in this subspace, there is a subsequence for which the complete invariants converge. Then, after a suitable conjugacy, the limit should be a marked generalized cusp of smaller type. To obtain an algebraic family set $\psi_i = 1/\lambda_i^2$, then $v_i = \lambda_i t_i$. The diagonal group $\mathrm{Tr}(\psi)$ consists of the matrices $\exp(M)$, for those M shown below, satisfying (9).

(8)
$$M = \begin{pmatrix} \lambda_1 t_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 t_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 t_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 t_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & -\lambda_2^{-1} & -\lambda_3^{-1} & -\lambda_4^{-1} & \lambda_1^{-2} \\ 0 & 1 & 0 & 0 & \lambda_2^{-1} \\ 0 & 0 & 1 & 0 & \lambda_3^{-1} \\ 0 & 0 & 0 & 1 & \lambda_4^{-1} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$0 = \sum \psi_i v_i = \sum (1/\lambda_i^2)(\lambda_i t_i) = \sum \lambda_i^{-1} t_i \tag{9}$$

The orbits flatten in the directions for which $\lambda_i \to 0$. To prevent this, conjugate M by the matrix P in (8) to get:

(10)
$$R := P^{-1} M P = \begin{pmatrix} 0 & t_2 & t_3 & t_4 & 0 \\ 0 & \lambda_2 t_2 & 0 & 0 & t_2 \\ 0 & 0 & \lambda_3 t_3 & 0 & t_3 \\ 0 & 0 & 0 & \lambda_4 t_4 & t_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \lambda_1 t_1 \begin{pmatrix} 1 & -\lambda_2^{-1} & -\lambda_3^{-1} & -\lambda_4^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We may assume that ψ_i decreases with i, and it follows that λ_i increases with i. We want this new family to contain only polynomials (rather than rational functions) in the parameters, so that they are defined whenever

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \tag{11}$$

To do this we introduce extra parameters κ_i for $2 \leq i \leq 4$, and require

$$\lambda_i \kappa_i = \lambda_1 \tag{12}$$

then

$$R = \begin{pmatrix} 0 & t_2 & t_3 & t_4 & 0 \\ 0 & \lambda_2 t_2 & 0 & 0 & t_2 \\ 0 & 0 & \lambda_3 t_3 & 0 & t_3 \\ 0 & 0 & 0 & \lambda_4 t_4 & t_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + t_1 \begin{pmatrix} \lambda_1 & -\kappa_2 & -\kappa_3 & -\kappa_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Using (9) we replace t_1 by

$$t_1 = -\lambda_1 \left(\lambda_2^{-1} t_2 + \lambda_3^{-1} t_3 + \lambda_4^{-1} t_4\right) = -\left(\kappa_2 t_2 + \kappa_3 t_3 + \kappa_4 t_4\right)$$

and this gives a family of representations

$$\Phi_{\lambda,\kappa} : \mathbb{R}^3 \to \mathrm{Aff}(4), \qquad \Phi_{\lambda,\kappa}(t_2, t_3, t_4) = \exp R$$

parameterized by those (λ, κ) satisfying (11) and (12). When $\lambda_1 > 0$ then $\kappa_i = \lambda_1/\lambda_i \in [0,1]$ so λ determines $\kappa \in [0,1]^3$. We will see that the conjugacy class of the image group only depends on λ. Thus the same collection of conjugacy classes of groups is obtained by restricting to $\kappa_i \in [0,1]$. Restricting κ to a compact set helps later with the point-set topology, when we quotient out by this compact set. Finally, since t_1 is expressed in terms of the other t_i, the terms for $i = 1$ are different to the other terms. Thus we replace the index set $1 \le i \le 4$ by $0 \le i \le 3$, to emphasize the special role of λ_0. This leads to the following definitions.

Given $\lambda \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ define $\lambda_{i-1} = \lambda(e_i)$. The subspace

$$A_n = \{(\lambda_0, \ldots, \lambda_{n-1}) \mid 0 \le \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n-1}\} \subset \mathbb{R}^n \tag{13}$$

is called the *(closed) Weyl chamber*. It is a fundamental domain for the action by signed coordinate permutations on $\mathbb{R}^n$. Observe that $\lambda_i = 0$ if and only if $\mathbf{t} < n$ and $i \le \mathbf{u}(\lambda)$.

The *blown up Weyl chamber* is

$$\widetilde{A}_n = \{(\lambda, \kappa) \in A_n \times [0,1]^{n-1} \ : \ \lambda_0 = \lambda_i \kappa_i\} \tag{14}$$

The projections $p_1 : \widetilde{A}_n \to A_n$ and $p_2 : \widetilde{A}_n \to [0,1]^{n-1}$ are defined by $p_1(\lambda, \kappa) = \lambda$ and $p_2(\lambda, \kappa) = \kappa$. Since $\lambda_i \ge \lambda_0$ it follows that p_1 is surjective. When $\lambda_i \ne 0$ then $\kappa_i = \lambda_0/\lambda_i$ is determined by λ_i. However when $\lambda_i = 0$ then $\lambda_0 = 0$ also, thus $\kappa_i \in [0,1]$ is arbitrary. One may regard $\widetilde{A}_n$ as obtained from A_n by a kind of *blowup* of the subset of A_n where $\lambda_0 = 0$, and the κ coordinates record certain tangent directions when some of the coordinates of λ are zero.

We make frequent use of the following *inverse function theorem*.

Lemma 3.1 ([13, Corollary 10.1.6])**.** *Let $f : X \to Y$ be a continuous bijection between locally compact spaces. If Y is Hausdorff and f is a proper map, then f is a homeomorphism.*

Let $D_n = \{(\lambda, \kappa) \in (0,\infty)^n \times [0,1]^{n-1} \ : \ \lambda_0 = \lambda_i \kappa_i\}$. A point in D_n determines a *diagonalizable* marked translation group via (3.4), however the coordinates of λ are in arbitrary order subject only to $\lambda_0 = \min \lambda_i$, rather than non-increasing.

Lemma 3.2. *Given $(\lambda, \kappa) \in \widetilde{A}_n$ set $\mathbf{t} = \mathbf{t}(\lambda)$ and $\mathbf{u} = \mathbf{u}(\lambda)$. If $\mathbf{t} = n$ then $p_2(p_1^{-1}\lambda) = \kappa$. If $\mathbf{t}(\lambda) < n$ then $p_2(p_1^{-1}\lambda) = [0,1]^{\mathbf{u}} \times \mathbf{0}$ where $\mathbf{0} = (0, \cdots, 0) \in [0,1]^{n-1-\mathbf{u}}$. Moreover*

(a) *$\widetilde{A}_n \subset \mathrm{cl}\, D_n$*
(b) *p_1 has compact fibers*

(c) $p_1 : \widetilde{A}_n \to A_n$ *is a quotient map.*

Proof. If $\mathbf{t} = n$ then all $\lambda_i > 0$ and λ determines κ. Otherwise $\mathbf{t} < n$ and $\lambda_i = 0$ if and only if $i \le \mathbf{u}$. For $i \ge 1$ then κ_i is the set of solutions in $[0,1]$ of $0 = \lambda_0 = \kappa_i \lambda_i$. For $1 \le i \le \mathbf{u}$ then $\lambda_i = 0$ and $\kappa_i \in [0,1]$ is arbitrary. For $\mathbf{u} < i \le n-1$ then $\lambda_i > 0$, so $\kappa_i = 0$. This gives the formula for $p_2(p_1^{-1}\lambda)$, and (b) is an immediate consequence.

For (a), we prove there is a sequence $(\lambda(m), \kappa(m)) \in D_n$ that converges to $(\lambda, \kappa) \in \widetilde{A}_n$. If $\mathbf{t} = n$ then $(\lambda, \kappa) \in D_n$ so a constant sequence suffices. Otherwise $\lambda_0 = 0$. Since $\kappa \in [0,1]^{n-1}$ there is a sequence $\kappa(m) \in (0,1]^{n-1}$ that converges to κ. Now define $\lambda_0(m) = m^{-1}$ and $\lambda_j(m)$ by $\lambda_0(m) = \lambda_j(m)\kappa_j(m)$ for $j > 0$. Then $(\lambda(m), \kappa(m)) \in D_n$, and converges to (λ, κ). When $\lambda_i = 0$ for all $i \le \mathbf{u}$ then the coordinates of κ need not be monotonic. This is where we exploit that there is no ordering requirement for the λ coordinates in D_n.

For (c), let $B = \widetilde{A}_n / \sim$ be the space of fibers of p_1 equipped with the quotient topology. The map $f : B \to A_n$ induced by p_1 is a proper continuous bijection. Moreover A_n is compact and Hausdorff. Also B is locally compact because $p_1^{-1}(K)$ is compact whenever K is compact. Hence f is a homeomorphism by Lemma 3.1. □

Remark 3.3. (c) is where $[0,1]^{n-1}$ is compact is needed. The reader might like to consider what B becomes if $[0,1]^n$ is replaced by $[0,\infty)^n$ in the definition of $\widetilde{A}_n$.

We now define another family of Lie groups $T(\lambda, \kappa)$ that varies continuously with $(\lambda, \kappa) \in \widetilde{A}_n$. Theorem 3.10 shows that the families of Lie groups $T(\lambda, \kappa)$ and $\mathrm{Tr}(\psi)$ are *conjugate*.

Definition 3.4. For each $(\lambda, \kappa) \in \widetilde{A}_n \cup D_n$ define $\Phi_{\lambda,\kappa} := \exp \circ \phi_{\lambda,\kappa} : V \to \mathrm{Aff}(n)$ where $\phi_{\lambda,\kappa} : V \to \mathfrak{aff}(n)$ is given by

$$\phi_{\lambda,\kappa}(v) = \begin{pmatrix} 0 & v_1 & v_2 & \cdots & v_{n-1} & 0 \\ 0 & \lambda_1 v_1 & 0 & \cdots & 0 & v_1 \\ \vdots & & \ddots & & & \vdots \\ & & & & \lambda_{n-1} v_{n-1} & v_{n-1} \\ 0 & & & & \cdots & 0 \end{pmatrix} + \langle v, \kappa \rangle \begin{pmatrix} -\lambda_0 & \kappa_1 & \cdots & \kappa_{n-1} & 0 \\ 0 & \cdots & & & 0 \\ \vdots & & & & \vdots \\ & & & & \\ 0 & \cdots & & & 0 \end{pmatrix}$$

and $v = (v_1, \cdots, v_{n-1}) \in V$, and $\lambda = (\lambda_0, \cdots, \lambda_{n-1})$, and $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$. Also $\mathfrak{t}(\lambda, \kappa) := \mathrm{Im}(\phi_{\lambda,\kappa})$ and $T(\lambda, \kappa) := \mathrm{Im}(\Phi_{\lambda,\kappa})$.

If $(\lambda, \kappa) \in D_n$ then $\Phi_{\lambda,\kappa}$ is diagonalizable. It follows if $(\lambda, \kappa) \in \widetilde{A}_n$ then $\Phi_{\lambda,\kappa}$ is the limit of these diagonalizable representations by (3.2)(a). This fact is exploited to prove that $T(\lambda, \kappa)$ is a translation group. The proof of the following is routine and in Section 8.

Proposition 3.5.

(a) *Given* $(\lambda, \kappa) \in D_n$ *let* $\psi_i = \lambda_i^{-2}$ *for* $1 \le i \le n-1$ *and* $\psi_n = \lambda_0^{-2}$. *Then there is* $Q \in \mathrm{SL}(n+1, \mathbb{R})$ *and* $\mathfrak{f} \in \mathrm{GL}(V)$ *given by* $\mathfrak{f}(v_1, \cdots, v_{n-1}) = \lambda_0^2(\lambda_1 v_1, \cdots, \lambda_{n-1} v_{n-1})$ *such that* $Q \Phi_{\lambda,\kappa} Q^{-1} = \zeta_\psi \circ \mathfrak{f}$, *and* $Q T(\lambda, \kappa) Q^{-1} = \mathrm{Tr}(\psi)$.

(b) $T(\lambda,\kappa)$ *is a translation group that preserves a convex set* $\Omega(\lambda,\kappa)\subset\mathbb{R}^n$, *and* $\partial\Omega(\lambda,\kappa)=T(\lambda,\kappa)\cdot 0$. *Also* $\eta(\Phi_{\lambda,\kappa})=(\chi_{\lambda,\kappa},[\beta'_\kappa])$ *where*

$$\beta'_\kappa=\mathrm{I}+\kappa\otimes\kappa=\begin{bmatrix}1+\kappa_1^2 & \kappa_1\kappa_2 & \cdots & \kappa_1\kappa_{\mathbf{u}} & \\ \kappa_2\kappa_1 & 1+\kappa_2^2 & \cdots & \kappa_2\kappa_{\mathbf{u}} & \\ \vdots & & & \vdots & 0\\ \kappa_{\mathbf{u}}\kappa_1 & \kappa_{\mathbf{u}}\kappa_2 & \dots & 1+\kappa_{\mathbf{u}}^2 & \\ & 0 & & & \mathrm{I}_{\mathbf{r}}\end{bmatrix}$$

$$\chi_{\lambda,\kappa}(v_1,\cdots,v_{n-1})=1+\exp\left(-\lambda_0\langle\kappa,v\rangle\right)+\sum_{i=1}^{n-1}\exp(\lambda_i v_i)$$

Define $\varkappa=(1+\|\kappa\|^2)^{1/(n-1)}$; *then* $\det\beta'_\kappa=\varkappa^{n-1}$ *and* $\beta_\kappa=\varkappa^{-1}\beta'_\kappa$ *is unimodular.*

Definition 3.6. If $Q=\mathrm{I}+M\in\mathrm{GL}(k,\mathbb{R})$ and $M^2=\alpha M$ then the *preferred square root of* Q is

$$\widetilde{S}(Q)=\mathrm{I}+\alpha^{-1}(\sqrt{1+\alpha}-1)M$$

This *is* a square root since $(\mathrm{I}+xM)^2=\mathrm{I}+(2x+\alpha x^2)M=Q$ when $2x+\alpha x^2=1$. If $v\in\mathbb{R}^k$ then $M=v\otimes v$ has rank 1 and the condition holds with $\alpha=\|v\|^2$. Moreover, if Q is symmetric and positive definite, then so is $\widetilde{S}$. Observe that $\beta'_0=\beta_0=\|v\|^2$ is given by the identity matrix in the standard basis.

Lemma 3.7. *If* $\widetilde{S}=\widetilde{S}(\mathrm{I}+\kappa\otimes\kappa)$ *then* $\widetilde{S}^{-1}:(V,\beta_0)\to(V,\beta'_\kappa)$ *is an isometry, where* β'_κ *is defined in* (3.5). *Moreover* $\widetilde{S}^{-1}$ *varies continuously with* κ.

This gives a re-parameterization of $\Phi_{\lambda,\kappa}$ that makes the horosphere metric standard.

Definition 3.8. $\Phi^\perp_{\lambda,\kappa}:V\to\mathrm{Aff}(n)$ is given by $\Phi^\perp_{\lambda,\kappa}=\Phi_{\lambda,\kappa}\circ\widetilde{S}^{-1}$ where $\widetilde{S}=\widetilde{S}(I+\kappa\otimes\kappa)\in\mathrm{GL}(V)$.

If $\mathbf{t}<n$ then $\kappa_i=0$ whenever $\lambda_i\neq 0$. By (3.5) this re-parameterization does not change the character. However if $\mathbf{t}=n$ the character of $\Phi^\perp_{\lambda,\kappa}$ is more complicated. Fortunately we will not need an explicit formula for it in this case. It follows from (3.7) that

Corollary 3.9. *The map* $\widetilde{A}_n\to\mathrm{Hom}(V,\mathrm{Aff}_n)$ *given by* $(\lambda,\kappa)\mapsto\Phi^\perp_{\lambda,\kappa}$ *is continuous. The complete invariant of* $\Phi^\perp_{\lambda,\kappa}$ *is given by* $\beta(\Phi^\perp_{\lambda,\kappa})(v)=\langle v,v\rangle$, *and if* $\mathbf{t}(\lambda)<n$ *then*

$$\chi(\Phi^\perp_{\lambda,\kappa})(v_1,\cdots,v_{n-1})=2+\mathbf{u}+\sum_{i=\mathbf{u}+1}^{n-1}\exp(\lambda_i v_i)$$

The next result shows that the conjugacy classes of the family of groups $\mathrm{Tr}(\psi)$ coincide with the conjugacy classes of the groups $T(\lambda,\kappa)$, and that the conjugacy class of $T(\lambda,\kappa)$ only depends on λ. Changing κ but keeping λ fixed changes the conjugacy class of $\Phi_{\lambda,\kappa}$ (as detected by the horosphere metric) without changing the conjugacy class of $T(\lambda,\kappa)$.

Theorem 3.10. *Given* $(\lambda,\kappa)\in\widetilde{A}_n$ *then* $T(\lambda,\kappa)$ *is conjugate to* $\mathrm{Tr}(\psi)$ *in* $\mathrm{Aff}(n)$ *where* ψ *is defined in terms of* λ *as follows.*

Set $\mathbf{u} = \mathbf{u}(\lambda)$ *and* $\mathbf{t} = \mathbf{t}(\lambda)$. *When* $\mathbf{t} = 0$ *then* $\lambda = 0$ *and define* $\psi = 0$. *When* $\mathbf{t} = n$ *define* ψ *as in* (3.5)*(a). When* $0 < \mathbf{t} < n$ *then* $\mathbf{t} + \mathbf{u} = n - 1$ *and*

$$\begin{aligned} &\textit{given} && \lambda = (\lambda_0, \cdots, \lambda_{n-1}) = (0, \cdots, 0, \lambda_{\mathbf{u}+1}, \cdots \lambda_{\mathbf{u}+\mathbf{t}}) \in \mathbb{R}^n \\ &\textit{define} && \psi = (\psi_1, \cdots, \psi_n) = (\lambda_{\mathbf{u}+1}^{-2}, \cdots, \lambda_{\mathbf{u}+\mathbf{t}}^{-2}, 0, \cdots, 0) \in \mathbb{R}^n \end{aligned}$$

Proof. When $\mathbf{t} = n$ this follows (3.5). When $\mathbf{t} = 0$ then $\zeta_0 = \Phi_{0,0}$ and the result follows. This leaves the case $1 \le \mathbf{t} < n$. Define $F, C \in \mathrm{GL}(V)$ by $F(v_1, \cdots, v_{n-1}) = (v_{\mathbf{u}+1}, \cdots, v_{\mathbf{u}+\mathbf{t}}, v_1, \cdots, v_{\mathbf{u}})$ and $C = F \cdot \mathrm{Diag}(c_1, \cdots, c_{n-1})$, where the c_i are determined below. From (3.9)

$$\beta(\Phi_{\lambda,\kappa}^{\perp} \circ C)(v) \sim \sum_{i=1}^{n-1} c_i^2 v_i^2, \qquad \chi(\Phi_{\lambda,\kappa}^{\perp} \circ C)(v) = 2 + \mathbf{u} + \sum_{i=1}^{\mathbf{t}} \exp(\lambda_{\mathbf{u}+i} c_i v_i) \tag{15}$$

By (2.11)

$$\beta(\zeta_\psi)(v) \sim \sum_{i=1}^{\mathbf{t}} \psi_i v_i^2 + \psi_{\mathbf{t}}^{-1} \sum_{i=\mathbf{t}+1}^{n-1} v_i^2, \qquad \chi(\zeta_\psi)(v) = 2 + \mathbf{u} + \sum_{i=1}^{\mathbf{t}} \exp(\psi_{\mathbf{t}} v_i)$$

We will now show how to choose C so that ζ_ψ and $\Phi_{\lambda,\kappa}^{\perp} \circ C$ have the same complete invariant, then they are conjugate by (2.16). The characters are equal if $\lambda_{\mathbf{u}+i} c_i = \psi_{\mathbf{t}}$ for $i \le \mathbf{t}$. Now $\psi_i = \lambda_{\mathbf{u}+i}^{-2}$ when $i \le \mathbf{t}$ and $c_i = \psi_{\mathbf{t}} / \lambda_{\mathbf{u}+i}$ for $i \le \mathbf{t}$, hence $c_i^2 = \psi_{\mathbf{t}}^2 \psi_i$. Then from (15)

$$\beta(\Phi_{\lambda,\kappa}^{\perp} \circ C)(v) \sim \psi_{\mathbf{t}}^2 \sum_{i=1}^{\mathbf{t}} \psi_i v_i^2 + \sum_{i=\mathbf{t}+1}^{n-1} c_i^2 v_i^2$$

For $i > \mathbf{t}$ define $c_i = \sqrt{\psi_{\mathbf{t}}}$ then

$$\beta(\Phi_{\lambda,\kappa}^{\perp} \circ C)(v) \sim \psi_{\mathbf{t}}^2 \sum_{i=1}^{\mathbf{t}} \psi_i v_i^2 + \sum_{i=\mathbf{t}+1}^{n-1} \psi_{\mathbf{t}} v_i^2 \sim \sum_{i=1}^{\mathbf{t}} \psi_i v_i^2 + \psi_{\mathbf{t}}^{-1} \sum_{i=\mathbf{t}+1}^{n-1} v_i^2 \sim \beta(\zeta_\psi)(v)$$

□

It is messy to directly construct a conjugating matrix, since it varies continuously only when the type does not change. In general the representations $\Phi_{\lambda,\kappa}$ and $\Phi_{\lambda,\kappa'}$ are not conjugate if $\kappa \ne \kappa'$ because they have different complete invariants. However:

Corollary 3.11. *If* $\theta : V \to \mathrm{Aff}(n)$ *is a marked translation group then there are* $B, C \in \mathrm{SL}^{\pm} V$ *and* $(\lambda, \kappa) \in \widetilde{A}_n$ *such that* θ *is equivalent to* $\Phi_{\lambda,\kappa} \circ B$ *and to* $\Phi_{\lambda,\kappa}^{\perp} \circ C$.

Proof. The first claim follows from (2.5)(c) and (3.10), and the second claim from this and (3.8). □

The proof of 1.4 now follows from (3.2)(a) and (3.11).

Corollary 3.12. *If* $s > 0$ *and* $(\lambda, \kappa), (s \cdot \lambda, \kappa') \in \widetilde{A}_n$ *then* $T(\lambda, \kappa) = T(s \cdot \lambda, \kappa')$ *are conjugate subgroups of* $\mathrm{Aff}(n)$. *In particular, if* $\mathbf{t}(\lambda) < n$ *then* $T(\lambda, \kappa)$ *is conjugate to* $T(\lambda, 0)$.

Proof. By (3.10) $T(\lambda, \kappa)$ and $T(\lambda, \kappa')$ are always conjugate. Thus to prove the result we may assume that $\kappa' = \kappa$. Let $f : V \to V$ be $f(v) = sv$. By (3.5) $\chi(\Phi_{s\lambda,\kappa}) = \chi(\Phi_{\lambda,\kappa}) \circ f$ and $\beta(\Phi_{s\lambda,\kappa}) = \beta(\Phi_{\lambda,\kappa})$. Now $\beta(\Phi_{\lambda,\kappa} \circ f) \sim s^2 \beta(\Phi_{\lambda,\kappa}) \sim \beta(\Phi_{\lambda,\kappa})$. Thus $\Phi_{s\lambda,\kappa}$ and $\Phi_{\lambda,\kappa} \circ f$ are marked translation groups with the same complete invariant.

Thus they are conjugate by (2.16). The second statement follows by continuity because, if $\mathbf{t}(\lambda) < n$, then $\lambda_0 = 0$ so $(\lambda, 0) \in \widetilde{A}_n$. □

It is interesting that in the non-diagonalizable case we may choose $\kappa = 0$, and then $\phi_{\lambda,0}$ has a simple form as given in (3.4), however the diagonalizable ones are more complicated.

4. Topology of the moduli space

Recall $V = \mathbb{R}^{n-1}$ and define Rep_n to be the subspace of $\mathrm{Hom}(V, \mathrm{Aff}_n)/\mathrm{Aff}_n$ that consists of conjugacy classes of holonomy representations of marked generalized torus cusps of dimension n. First we establish that Rep_n is a quotient of $\widetilde{A}_n \times \mathrm{SL}^{\pm} V$, and that the complete invariant provides an embedding of Rep_n. We use this to prove that the holonomy map is a homeomorphism $\mathrm{hol} : \mathcal{T}_n \to \mathrm{Rep}_n$. Finally we compute the stratification of $\mathcal{T}_n$ and prove (1.5).

It follows from (2.5) and (3.10) that every marked translation group is conjugate to $\Phi^{\perp}_{\lambda,\kappa} \circ A$ for some $(\lambda, \kappa) \in \widetilde{A}_n$ and $A \in \mathrm{SL}^{\pm} V$. Moreover if $\mathbf{t}(\lambda) < n$ then it suffices to use $\kappa = 0$ so $\Phi^{\perp}_{\lambda,0} = \Phi_{\lambda,0}$.

Lemma 4.1. *The map $\widetilde{\Psi} : \widetilde{A}_n \times \mathrm{SL}^{\pm} V \to \mathrm{Rep}_n$ given by $\widetilde{\Psi}((\lambda, \kappa), B) = [\Phi^{\perp}_{\lambda,\kappa} \circ B]$ is continuous, and covers a continuous surjection $\Psi : A_n \times \mathrm{SL}^{\pm} V \to \mathrm{Rep}_n$.*

Proof. Continuity of $\widetilde{\Psi}$ follows from (3.9). To prove Ψ is well defined we must show that $\Phi^{\perp}_{\lambda,\kappa} \circ B$ is conjugate to $\Phi^{\perp}_{\lambda,\kappa'} \circ B$. To do this, it suffices to show they have the same complete invariant. Clearly it suffices to do this when $B = \mathrm{I}$. This now follows from (3.9).

Recall $p_1 : \widetilde{A}_n \to A_n$ and we have $\Psi \circ p_1 = \widetilde{\Psi}$. If $U \subset \mathrm{Rep}_n$ is open then, since $\widetilde{\Psi}$ is continuous, $\widetilde{\Psi}^{-1}(U) = p_1^{-1}(\Psi^{-1}(U))$ is open. Since p_1 is a quotient map by (3.2)(c), it follows that $\Psi^{-1}(U)$ is open, so Ψ is continuous. □

In what follows we use $\beta \in \mathcal{P}$ in place of $[\beta] \in \mathbb{P}\mathcal{P}$. Recall the compete invariant $\eta : \mathrm{Rep}_n \to \chi(V) \times \mathcal{P}$ and the codomain is homeomorphic to a subspace of Euclidean space. In particular a closed subset of the codomain is locally compact.

Lemma 4.2. *$\eta \circ \Psi : A_n \times \mathrm{SL}^{\pm} V \to \chi(V) \times \mathcal{P}$ is proper and continuous, and $X_n = \eta(\mathrm{Rep}_n)$ is a closed subset of $\chi(V) \times \mathcal{P}$, and is homeomorphic to a closed subset of Euclidean space.*

Proof. Continuity of $\eta \circ \Psi$ follows from (2.9) and (4.1). Suppose $((a_j, \kappa_j), B_j) \in \widetilde{A}_n \times \mathrm{SL}^{\pm} V$ and

$$(\chi_j, \beta_j) = \eta(\Psi(a_j, B_j)) = \eta(\Phi^{\perp}_{a_j,\kappa_j} \circ B_j)$$

is a bounded sequence in $\chi(V) \times \mathcal{P}$. Then $\beta_j = B_j^t B_j$ is bounded. The map $\theta : \mathrm{SL}^{\pm} V \to \mathrm{SL}\, V$ given by $\theta(B) = B^t B$ is proper, thus B_j is bounded. After passing to a subsequence we may assume $\lim B_j = B \in \mathrm{SL}^{\pm} V$. By (3.8)

$$\Phi^{\perp}_{a_j,\kappa_j} = \Phi_{a_j,\kappa_j} \circ \widetilde{S}_j^{-1}$$

Now $\kappa_j \in [0,1]^{n-1}$ so $\widetilde{S}_j = \widetilde{S}(I + \kappa_j \otimes \kappa_j)$ is bounded. Since the map that sends an element of $\mathrm{SL}^{\pm} V$ to its inverse is proper, B_j^{-1} is bounded. Thus $(B_j^{-1} \circ \widetilde{S}_j)$ is bounded. Also χ_j is bounded, so

$$\chi_j \circ (B_j^{-1} \circ \widetilde{S}_j) = \mathrm{trace}((\Phi_{a_j,\kappa_j} \circ \widetilde{S}_j^{-1}) \circ B_j) \circ (B_j^{-1} \circ \widetilde{S}_j) = \mathrm{trace}(\Phi_{a_j,\kappa_j})$$

is bounded. Let μ_j be the last component of a_j, then μ_j is the largest component of a_j. Referring to (3.4) we see that $\Phi_{a_j,\kappa_j}(e_{n-1})$ has an eigenvalue of $\exp\mu_j$ in the (n,n) entry and all other eigenvalues are equal to 1. Since $\operatorname{trace}(\Phi_{a_j,\kappa_j})$ is bounded, and $\mu_j > 0$, it follows that μ_j is bounded. Thus a_j is bounded. Hence $\eta\circ\Psi$ is proper. After taking a subsequence $\lim a_j = a$ and $a \in A_n$ because A_n is a closed subset of $\mathbb{R}^n$. Thus $\lim(a_j, B_j) = (a, B) \in A_n \times \mathrm{SL}^{\pm} V$, and $\lim \eta\circ\Psi(a_j, B_j) = \eta\circ\Psi(a, B) \in \operatorname{Im}\eta$. Thus $\operatorname{Im}\eta\circ\Psi$ is closed in $\chi(V)\times\mathcal{P}$. By (4.1) $\operatorname{Im}\Psi = \mathrm{Rep}_n$; thus $\operatorname{Im}\eta\circ\Psi = \eta(\mathrm{Rep}_n)$ is closed. □

By (2.16), if $B, B' \in \mathrm{SL}^{\pm} V$ then $\Phi^{\perp}_{\lambda,\kappa}\circ B$ and $\Phi^{\perp}_{\lambda,\kappa}\circ B'$ represent the same point in Rep_n if and only if they have the same complete invariant. By definition 2.8 this is equivalent to $B' \in B\cdot O(\eta(\Phi^{\perp}_{\lambda,\kappa}))$. Since $\beta(\Phi^{\perp}_{\lambda,\kappa})$ is standard, and the character does not depend on κ, the group $O(\eta(\Phi^{\perp}_{\lambda,\kappa}))$ does not depend on κ.

Let $\pi : A_n \times \mathrm{SL}^{\pm} V \to \big(A_n \times \mathrm{SL}^{\pm} V\big)/\sim$ be projection, where $(\lambda, B) \sim (\lambda', B')$ if and only if $\lambda = \lambda'$ and $B' \in B\cdot O(\eta(\Phi^{\perp}_{\lambda,\kappa}))$ for some κ with $(\lambda,\kappa) \in \widetilde{A}_n$. It follows there is an injective function

$$\Psi_* : \big(A_n \times \mathrm{SL}^{\pm} V\big)/\sim \;\longrightarrow \mathrm{Rep}_n$$

such that $\Psi = \Psi_*\circ\pi$. Equip the domain with the quotient topology, then Ψ_* is continuous by (4.1). Surjectivity of Ψ_* follows from (2.5) and (3.10). Theorem 1.1 follows from (4.5) and:

Theorem 4.3. *Ψ_* is a homeomorphism, and $\eta : \mathrm{Rep}_n \to X_n$ is a homeomorphism, and Rep_n is homeomorphic to a closed subset of Euclidean space.*

Proof. By (4.2) $\eta\circ\Psi_* : \big(A_n \times \mathrm{SL}^{\pm} V\big)/\sim \;\longrightarrow X_n$ is proper and continuous. It is bijective because η and Ψ_* are. Since X_n is homeomorphic to a closed subspace of Euclidean space, it is Hausdorff and locally compact. Given $x = (\lambda, A) \in A_n \times \mathrm{SL}^{\pm} V$ there are compact neighborhoods $L \subset A_n$ of λ and $K \subset \mathrm{SL}^{\pm} V$ of A. Then $U = L \times (O(n-1)\cdot K) \subset A_n \times \mathrm{SL}^{\pm} V$ is compact because $O(n-1)$ is compact. Since $O(\eta(\Phi^{\perp}_{\lambda,\kappa})) \subset O(n-1)$ it follows that $\pi(U)$ is a compact neighborhood of πx, thus $\big(A_n \times \mathrm{SL}^{\pm} V\big)/\sim$ is locally compact. Hence $\eta\circ\Psi_*$ is a homeomorphism by (3.1). It follows that η is an homeomorphism, and Ψ_* is a homeomorphism. The last conclusion follows from (4.2). □

In (2.5) generalized cusps were classified and shown to be equivalent to ones with holonomy in $\mathrm{Tr}(\psi)$ for some $\psi \in A_n(\Psi)$. Recall that $\psi_i = 1/\lambda_i^2$ when $\lambda_i > 0$. In [2, Theorem 0.2(v)] gives a bijection Θ that is essentially the same as Ψ_*, but the topology on the domain is different. It follows from the above that, if the *reciprocals* of the coordinates of ψ converge suitably, then the conjugacy class of $\mathrm{Tr}(\psi)$ has a limit that is another translation group.

Informally, two generalized cusps are *close* if, after shrinking them, they are nearly affine isomorphic. It turns out this is equivalent to their holonomies being close up to conjugacy. Our definition of *marked moduli space* is based on the notion of *developing maps* as is done in [9, Sec 1]. Recall $C = (V/\mathbb{Z}^{n-1})\times[0,\infty)$ so $\widetilde{C} = V\times[0,\infty)$ is the universal cover.

Definition 4.4. Let $\widetilde{\mathcal{T}}_n$ be the space of developing maps $\mathrm{dev} : \widetilde{C} \to \mathbb{A}^n$ for marked generalized cusps with underlying space C. We endow $\widetilde{\mathcal{T}}_n$ with the compact-open topology. There is an equivalence relation on $\widetilde{\mathcal{T}}_n$ that is generated by homotopy,

and composition with an affine isomorphism. The quotient space is the *marked moduli space* $\mathcal{T}_n$.

When $n \leq 3$ homotopy implies isotopy for homeomorphisms of T^n. However when $n \geq 5$, there are infinitely many isotopy classes homotopic to the identity, see [14, Theorem 4.1]. We have used *homotopy* rather than *isotopy* in the definition of $\mathcal{T}_n$ in order to obtain the following.

Theorem 4.5. *The holonomy* $\mathrm{hol} : \mathcal{T}_n \to \mathrm{Rep}_n$ *is a homeomorphism.*

Proof. First we define hol. Suppose $\mathrm{dev} : \widetilde{C} \to \mathbb{A}^n$ is the developing map of a generalized cusp. Then $g \in \mathbb{Z}^{n-1} = \pi_1 C$ acts on $\widetilde{C} = V \times [0, \infty)$ by $g \cdot (v, t) = (v + g, t)$ so the extended holonomy ρ can be recovered from dev using that for $x \in \mathrm{Im}(\mathrm{dev})$

$$(\rho g)(x) = \mathrm{dev}((g, 0) + \mathrm{dev}^{-1}(x))$$

It follows that there is a map $\widetilde{\mathrm{hol}} : \widetilde{\mathcal{T}}_n \to \mathrm{Hom}(V, \mathrm{Aff}(n))$. Moreover this formula shows ρ is determined by the restriction of dev to a compact set. Since $\widetilde{\mathcal{T}}_n$ has the compact-open topology, it follows that $\widetilde{\mathrm{hol}}$ is continuous. It is clear that ρ is the holonomy, and is therefore well defined on the equivalence class $[\mathrm{dev}] \in \mathcal{T}_n$. Thus $\widetilde{\mathrm{hol}}$ covers a continuous map $\mathrm{hol} : \mathcal{T}_n \to \mathrm{Rep}_n$.

Next, we construct an inverse to hol. By (4.3) Ψ_* is a homeomorphism so we may replace Rep_n by $\left(A_n \times \mathrm{SL}^{\pm} V\right) / \sim$. Given $((\lambda, \kappa), B) \in \widetilde{A}_n \times \mathrm{SL}^{\pm} V$, define a developing map

$$f = f_{\lambda,\kappa,B} : V \times [0, \infty) \to \mathbb{R}^n \times 1 = \mathbb{A}^n \quad \text{by} \qquad f(v, z) = (\Phi^{\perp}_{\lambda,\kappa}(Bv))(z, 0, \cdots, 0, 1)$$

Observe that $f(V, 0)$ is the orbit of the origin, so $\mathrm{Im}(f) = \Omega(\lambda, \kappa)$ defined in (3.5). More generally $f(V, z)$ is a horosphere for fixed $z > 0$. It follows that f is the developing map for a generalized cusp with holonomy $\Phi^{\perp}_{\lambda,\kappa} \circ B$, thus $f \in \widetilde{\mathcal{T}}_n$.

Define $\widetilde{F} : \widetilde{A}_n \times \mathrm{SL}^{\pm} V \to \widetilde{\mathcal{T}}_n$ by $\widetilde{F}((\lambda, \kappa), B) = f_{\lambda,\kappa,B}$. Clearly $\widetilde{F}$ is continuous. By properties of the quotient topology, $\widetilde{F}$ covers a continuous map $F : \left(A_n \times \mathrm{SL}^{\pm} V\right) / \sim \ \to \mathcal{T}_n$. Since hol has a continuous inverse $F \circ \Psi_*^{-1}$, it follows hol is a homeomorphism. □

Proof of 1.6. If C is a torus then $\widetilde{C} = C$ and the result follows from (4.5). It only remains to prove that the holonomy of $\widetilde{C}$ uniquely determines the holonomy of C. Now $\rho|\widetilde{C}$ determines the extended holonomy $\sigma : V \to \mathrm{Aff}^n$. We claim σ determines the rotational part $R : \pi_1 C \to \mathrm{O}(n)$ and therefore determines $\rho : \pi_1 C \to \mathbb{A}^n$. This follows from the observation that R is uniquely determined by the action of $\rho\pi_1 C$ on σV by conjugacy. This in turn is determined by the action by conjugacy of $\pi_1 C$ on $\pi_1 \widetilde{C}$. □

In the sequel we will use hol to identify these two spaces. If dev is the developing map for a generalized cusp with holonomy ρ then we identify $[\mathrm{dev}] \in \mathcal{T}_n$ with $[\rho] = \mathrm{hol}[\mathrm{dev}] \in \mathrm{Rep}_n$. It follows from the above that:

Theorem 4.6. $\mathrm{hol}^{-1} \circ \Psi^* : \left(A_n \times \mathrm{SL}^{\pm} V\right) / \sim \ \longrightarrow \mathcal{T}_n$ *is a homeomorphism.*

To analyze the subspaces of $\mathcal{T}_n$ consisting of cusps of fixed type it is easier to work with ζ_ψ than with $\Phi_{\lambda,\kappa}$.

Definition 4.7. Given $0 \leq \mathbf{t} \leq n$ the *stratum of type* $\mathbf{t}$ *of* $\mathcal{T}_n$ is the subspace $\mathcal{T}_n(\mathbf{t}) \subset \mathcal{T}_n$ that consists of all marked cusps with holonomy conjugate into some $\mathrm{Tr}(\psi)$ with $\mathbf{t}(\psi) = \mathbf{t}$.

The holonomy of a generalized cusp is conjugate to $\zeta_\psi \circ A$ where $(\psi, A) \in A_n(\Psi) \times \mathrm{SL}^{\pm} V$. The coordinates of ψ are ordered. Below we show each stratum is a manifold by showing it is the quotient of a smooth manifold by a compact group that acts freely. To do this involves enlarging the set of pairs (ψ, A) by relaxing the ordering and using $\psi \in A_n^u(\Psi)$, defined in (2.4). The equivalence relation on $A_n^u(\Psi) \times \mathrm{SL}^{\pm} V$ is then given by a free action of $\Sigma_{\mathbf{t}} \times O(u)$. This technique can only be employed with individual strata, but not all of $\mathcal{T}_n$, since the dimension of $O(u)$ changes with type. We will see that $\mathcal{T}_n$ is not a manifold with boundary when $n \geq 3$. The proof of the next result actually determines the topology of each stratum.

Proposition 4.8. *For each* $0 \leq \mathbf{t} \leq n$ *the stratum* $\mathcal{T}_n(\mathbf{t}) \subset \mathcal{T}_n$ *is a connected smooth manifold without boundary and* $\dim \mathcal{T}_n(\mathbf{t}) < \dim \mathcal{T}_n(\mathbf{t}+1)$ *when* $\mathbf{t} < n$. *Moreover* $\mathrm{cl}(\mathcal{T}_n(\mathbf{t})) = \cup_{i \leq \mathbf{t}} \mathcal{T}_n(i)$. *If* $n \geq 3$ *then the fundamental group* $\pi_1(\mathcal{T}_n(n))$ *is not trivial.*

Proof. Let $W_{\mathbf{t}} = (0,\infty)^{\mathbf{t}} \times \mathrm{SL}^{\pm} V$. By (2.5) there is a surjective map $\pi : W_{\mathbf{t}} \to \mathcal{T}_n(\mathbf{t})$ given by $\pi(\psi, A) = [\zeta_{\psi'} \circ A]$ where $A \in \mathrm{SL}^{\pm} V$, and $\psi = (\psi_1, \cdots, \psi_{\mathbf{t}}) \in W_{\mathbf{t}}$, and $\psi' = (\psi_1, \cdots, \psi_{\mathbf{t}}, 0, \cdots, 0) \in A_n^u(\Psi)$.

The first case is $\mathbf{t} < n$, so $\mathbf{t} + \mathbf{u} = n - 1$. Recall $V = D \oplus U$ from (2) where $D = \mathbb{R}^{\mathbf{t}} \oplus 0$ and $U = 0 \oplus \mathbb{R}^{\mathbf{u}}$. Let $\Sigma_{\mathbf{t}} \subset O(\mathbf{t})$ be the subgroup that permutes the coordinates axes of $\mathbb{R}^{\mathbf{t}}$, and $G_{\mathbf{t}} = \Sigma_{\mathbf{t}} \oplus O(\mathbf{u}) \subset \mathrm{SL}^{\pm} V$. There is an action of $\alpha \in G_{\mathbf{t}}$ on $(\psi, A) \in W_{\mathbf{t}}$ given by

$$\alpha(\psi, A) = (\sigma^* \psi, \alpha A), \qquad \text{where } \alpha = \begin{pmatrix} \sigma & 0 \\ 0 & B \end{pmatrix}, \quad \sigma \in \Sigma_{\mathbf{t}}, \quad B \in O(u)$$

Here we regard $\psi \in D^* = \mathrm{Hom}(D, \mathbb{R})$ and σ^* is the dual action on D^* so $\sigma^* \psi = \psi \circ \sigma^{-1}$. The marked translation groups given by (ψ, A) and $\alpha(\psi, A)$ are conjugate because they have the same complete invariant. Now B does not change η, and σ^{-1} from σ^* cancels with the σ in α, so $\pi(\alpha(\psi, A)) = \pi(\psi, A)$.

We claim that $\pi^{-1}(\psi, A) = G_{\mathbf{t}} \cdot (\psi, A)$. Suppose $\pi(\psi', A') = \pi(\psi, A)$. There is $\sigma \in \Sigma_{\mathbf{t}}$ so that the coordinates of $\sigma\psi$ are non-increasing. Thus we may assume ψ and ψ' both have this property. By the classification [2, 1.44 & 0.2(v)] it then follows that $\psi = \psi'$ and $A' \in O(\mathbf{u}) \cdot A$. The claim follows. Hence $\mathcal{T}_n(\mathbf{t})$ is homeomorphic to $W_{\mathbf{t}}/G_{\mathbf{t}}$. Moreover the subgroup $O(\mathbf{u})$ acts trivially on the first factor of $W_{\mathbf{t}}$, and by left multiplication on the second factor, so

$$\mathcal{T}_n(\mathbf{t}) \cong \left[(0,\infty)^{\mathbf{t}} \times \left(O(\mathbf{u}) \backslash \mathrm{SL}^{\pm} V\right)\right] / \Sigma_{\mathbf{t}}$$

Now $O(\mathbf{u}) \backslash \mathrm{SL}^{\pm} V$ is a symmetric space. Since $\Sigma_{\mathbf{t}}$ is finite, and acts freely on the second factor, it follows that $\mathcal{T}_n(\mathbf{t})$ is a manifold.

This leaves the case $\mathbf{t} = n$, in which case $\mathbf{u} = 0$ and the representations are diagonal. Let $\mathrm{Mono}(V, \mathrm{Aff}(n)) \subset \mathrm{Hom}(V, \mathrm{Aff}(n))$ be the subspace of injective maps. Define $f : W_n \to \mathrm{Mono}(V, \mathrm{Aff}(n))$ by $f(\psi, A) = \zeta_\psi \circ A$. Then f is injective and we use it to identify W_n with $Z = f(W_n)$. Let $\Sigma_n \subset \mathrm{Aff}(n)$ be the subgroup that permutes the standard basis of $\mathbb{R}^n$. Then Σ_n acts freely by conjugacy on $\mathrm{Mono}(V, \mathrm{Aff}(n))$.

Claim. This action preserves Z.

We identify Σ_n with the group of permutations of $\{1, \cdots, n\}$. Suppose $\sigma \in \Sigma$. If $\sigma(n) = n$ then the action of σ on W_n is as above. In particular the subgroup $\Sigma_{n-1} \subset \Sigma_n$ that fixes n preserves Z.

Let $\sigma \in \Sigma$ be the transposition $\sigma = (n-1, n)$. Since Σ_{n-1} and σ generate Σ_n, it suffices to show that σ preserves Z. Given $\psi = (\psi_1, \cdots, \psi_n) \in (0, \infty)^n$ then $\sigma \zeta_\psi \sigma^{-1} = \zeta_{\psi'} \circ B$ where

$$B = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \cdots & & \\ & & & 1 & \\ -\psi_1/\psi_n & -\psi_2/\psi_n & \cdots & & -\psi_{n-1}/\psi_n \end{pmatrix},$$

$$\psi' = (\psi_n/\psi_{n-1})(\psi_1, \cdots, \psi_{n-2}, \psi_n, \psi_{n-1})$$

Let $\delta = |\det B|^{1/(n-1)}$; then $\delta^{-1} B \in \mathrm{SL}^{\pm} V$, and $\zeta_{\delta\psi'}(v) = \zeta_{\psi'}(\delta v)$ by (2.5)(b), thus $\zeta_{\psi'} \circ B = f(\delta\psi', \delta^{-1} B) \in Z$. This proves the claim.

If two elements of Z are conjugate, then they are conjugate by an element of Σ_n. This is because both representations are diagonal, so a conjugacy must preserve the coordinate axes. Thus the conjugacy is by a signed permutation matrix. However a signed permutation matrix is the product of a permutation matrix and a diagonal matrix with ± 1 on the diagonal. But diagonal matrices centralize these representations, so they are conjugate via a coordinate permutation.

Hence $\mathcal{T}_n(n) \cong Z/\Sigma_n$. Now W_n has two components, and these are swapped by every odd element of Σ_n. Thus

$$\mathcal{T}_n(n) \cong \left[(0, \infty)^n \times \mathrm{SL}^{\pm} V\right] / \Sigma_{\mathbf{t}} \cong ((0, \infty)^n \times \mathrm{SL}\, V) \,/\, \mathrm{Alt}$$

where $\mathrm{Alt} \subset \Sigma_n$ is the alternating subgroup, and it acts freely. In particular $\pi_1 \mathcal{T}_n(n)$ surjects to Alt, and Alt is non-trivial if $n \geq 3$, the last claim in the theorem follows. Moreover $\mathrm{cl}(\mathcal{T}_n(\mathbf{t})) = \cup_{i \leq \mathbf{t}} \mathcal{T}_n(i)$ follows from the corresponding fact for the Weyl chamber A_n. Finally $\dim \mathcal{T}_{\mathbf{t}} = \dim W_{\mathbf{t}} - \dim G_{\mathbf{t}} = (\mathbf{t} + \dim \mathrm{SL}\, V) - \dim \mathrm{O}(\mathbf{u})$ and $\dim V = n - 1$. □

Proof of 1.5. There is a deformation retraction $\mathcal{T}_n \to \mathcal{T}_n(0)$ to the 0-stratum given by scaling λ, and $\mathcal{T}(0) \cong \mathcal{P}$ is homeomorphic to Euclidean space of dimension $n(n-1)/2$. Thus $\mathcal{T}_n$ is contractible. In [2, Prop 6.2] $\mathcal{T}_2$ was parameterized as $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y\}$ and is thus a manifold with boundary. Suppose $\mathcal{T}_n$ is a manifold M with boundary and $n \geq 3$. Let $\mathcal{N} \subset \mathcal{T}_n$ be the subspace of non-diagonalizable generalized cusps.

We claim that $\partial M = \mathcal{N}$. Since $Y = \mathcal{T}_n \setminus \mathcal{N}$ is the stratum of diagonalizable generalized cusps, it follow from (4.8) that Y is a manifold without boundary, and $\dim Y = \dim \mathcal{T}_n$ so $\partial M \cap Y = \emptyset$. Thus $\partial M \subset \mathcal{N}$. If $\psi \in \partial A_n(\Psi)$ and $\mathbf{t}(\psi) = n - 1$ then ψ has exactly one zero coordinate. Let $Z \subset \mathcal{N}$ be the subset of $[\rho]$ with $\rho = \zeta_\psi \circ A$ with $\mathbf{t}(\psi) = n - 1$ and all the coordinates of ψ are distinct. Then no element of Σ_{n-1} fixes $[\rho]$ because if $\sigma \in \Sigma_{n-1}$ and $\sigma\psi = \psi$ then $\sigma = \mathrm{I}$. It follows a neighborhood of $[\rho]$ in M is homeomorphic to a neighborhood U of a point in $A_n(\Psi) \times \mathrm{SL}^{\pm} V$ that projects to $[\rho]$. But ρ is in the boundary of $A_n(\Psi) \times \mathrm{SL}^{\pm} V$ so $[\rho]$ is in the boundary of the quotient. Thus $Z \subset \partial M$. But Z is dense in $\mathcal{N}$ and ∂M is closed in M so $\mathcal{N} \subset \partial M$. This proves the claim.

Since M is contractible $\pi_1 M = 0$. Also $\pi_1 M = \pi_1 Y$ because a manifold and its interior have the same fundamental group. By (4.8) $\pi_1 Y \neq 1$ when $n \geq 3$. This contradicts that $\mathcal{T}_n$ is a manifold. □

5. The weights data ν

In this section we prove (1.2). We will often identify the space $\mathcal{P}$ of unimodular positive definite quadratic forms with $\mathbb{P}\mathcal{P}$. There is an action of $A \in \mathrm{GL}\, V$ on Rep_n given by $A \cdot [\rho] = [\rho \circ A^{-1}]$. If $\nu(\rho) = ([\xi_1, \cdots, \xi_n], [\beta])$, then

$$\nu(\rho \circ A^{-1}) = ([\xi_1 \circ A^{-1}, \cdots, \xi_n \circ A^{-1}], [\beta \circ A^{-1}]) \tag{16}$$

This restricts to an action on $\operatorname{Im} \nu$ that covers a transitive action on $\mathcal{P}$.

Lemma 5.1. *If $\rho = \Phi_{\lambda,\kappa} \circ A$ with $A \in \mathrm{SL}^{\pm} V$ then $\nu(\rho) = ([\xi_0, \cdots, \xi_{n-1}], \beta)$ where*

$$\langle \xi_i, \xi_j \rangle_{\beta^*} = \varkappa \lambda_i^2 \delta_{ij} - \varpi = \varkappa \lambda_i \lambda_j \delta_{ij} - \varpi$$

and $\varkappa = (1 + \|\kappa\|^2)^{1/(n-1)}$ and $\varpi = \lambda_0^2 \varkappa^{(2-n)}$.

Proof. Let $\beta = \beta(\rho)$, and $\langle \cdot, \cdot \rangle_\beta$ is the inner product on V corresponding to $\beta \in \mathrm{S}^2 V$, and $\| \cdot \|_\beta$ the associated norm. Let $\| \cdot \|$ be the standard norm on V for which the standard basis is orthonormal and $\langle \cdot, \cdot \rangle$ the associated inner product. We may assume $\rho = \Phi_{\lambda,\kappa}$. By (3.5)(b) the matrix of β in the standard basis is given by $Q = \varkappa^{-1}(\mathrm{I} + \kappa \otimes \kappa)$. The matrix of the dual form β^* on V^* with respect to the dual basis is then Q^{-1}.

From (3.4) the Lie algebra weights for ρ are $\xi_0, \cdots, \xi_{n-1} \in V^*$ where

$$\xi_0(v) = -\lambda_0 \langle \kappa, v \rangle, \qquad \xi_i = \lambda_i e_i^* \quad \textit{for} \quad 1 \leq i \leq n-1 \tag{17}$$

For the following, refer to the discussion after (3.6). Now $Q = \varkappa^{-1}(\mathrm{I} + M)$ where $M = \kappa \otimes \kappa$, then $M^2 = \|\kappa\|^2 M$, so $Q^{-1} = \varkappa(\mathrm{I} + M)^{-1} = \varkappa(\mathrm{I} - (1 + \|\kappa\|^2)^{-1} M)$.

If $1 \leq i, j \leq n-1$ then define ϖ by

$$\langle \xi_i, \xi_j \rangle_{\beta^*} = \langle \lambda_i e_i^*, \lambda_j e_j^* \rangle_{\beta^*} = \varkappa \lambda_i \lambda_j \left(\delta_{ij} - (1 + \|\kappa\|^2)^{-1} \kappa_i \kappa_j \right) = \varkappa \lambda_i \lambda_j \delta_{ij} - \varpi \tag{18}$$

Now $\lambda_i \kappa_i = \lambda_0$ so $\varpi = \varkappa \lambda_0^2 (1 + \|\kappa\|^2)^{-1} = \lambda_0^2 \varkappa \varkappa^{1-n}$.

We claim (18) holds in all cases: $0 \leq i, j \leq n-1$. If $\lambda_0 = 0$ then $\xi_0 = 0$ and (18) holds in all cases. Otherwise $\lambda_0 > 0$ and using $\lambda_i \kappa_i = \lambda_0$ then (17) implies

$$\xi_0 = -\lambda_0^2 \sum_{i=1}^{n-1} \lambda_i^{-2} \xi_i, \qquad \text{so} \qquad \sum_{i=0}^{n-1} \lambda_i^{-2} \xi_i = 0 \tag{19}$$

To compute $\langle \xi_0, \xi_j \rangle_{\beta^*}$, replace ξ_0 by the above and then use (18) in the case $i, j \geq 1$ already established. Some algebra then shows (18) holds in all cases. □

The lemma implies the inner product of distinct weights is always $-\varpi$. This has a geometric interpretation. Consider a set of n vectors $\{v_1, \cdots, v_n\}$ in $V = \mathbb{R}^{n-1}$ equipped with the standard inner product such that for some $\varpi \geq 0$ the vectors satisfy the equation

$$\forall\ i \neq j \qquad \langle v_i, v_j \rangle = -\varpi \tag{20}$$

If $\varpi = 0$ this just says the vectors are pairwise orthogonal, and for dimension reasons at least one is zero. If $\varpi > 0$ then set $\mathbb{R}^n = V \oplus \mathbb{R} e_n$ with the standard inner product. Equation (20) is equivalent to the pairwise orthogonality of the vectors $\{u_i = v_i + \sqrt{\varpi} e_n\}$ in $\mathbb{R}^n$. In this case the $\{u_i\}$ are an orthogonal basis of

$\mathbb{R}^n$ that represent points in the hyperplane $x_n = \sqrt{\varpi}$, and the $\{v_i\}$ are the images of these vectors under orthogonal projection into V.

Proof of Theorem 1.2. We will abuse notation by identifying $\mathcal{T}_n = \mathrm{Rep}_n$ and write ρ instead of $[\rho]$ for a point in Rep_n. First we prove that ν is a homeomorphism.

Suppose $\nu(\rho) = ([\xi_0, \cdots, \xi_{n-1}], \beta)$. Then $\chi = \chi_\rho : V \to \mathbb{R}$ is given by $\chi(v) = \sum_{i=0}^{n-1} \exp \xi_i v$. Thus the complete invariant $\eta(\rho) = (\chi, \beta)$ is a continuous function of $\nu(\rho)$. By (2.16) η is injective; hence ν is injective.

Recall $\mathcal{R}_n \subset \mathrm{SP}^n(V) \times \mathcal{P}$ is the subset of all $([\xi_0, \cdots, \xi_{n-1}], \beta)$ such that

$$\exists\ \varpi \geq 0 \qquad \forall\ i \neq j \quad \langle \xi_i, \xi_j \rangle_{\beta^*} = -\varpi \tag{21}$$

We must show that $\mathrm{Im}(\nu) = \mathcal{R}_n$. By (5.1) $\mathrm{Im}\,\nu \subset \mathcal{R}_n$. It remains to show that $\mathcal{R}_n \subset \mathrm{Im}\,\nu$. In what follows we will always choose an ordering for $x = ([\xi_0, \cdots, \xi_{n-1}], \beta) \in \mathcal{R}_n$ so that $\|\xi\|_{\beta^*}$ is a non-decreasing function of i. Define k by $\xi_i \neq 0$ if and only if $i \geq k$, and define $\varpi = -\langle \xi_0, \xi_1 \rangle_{\beta^*}$.

Case 1 ($\varpi = 0$). This is the non-diagonal case. Then (21) is equivalent to requiring the ξ_i are pairwise orthogonal with respect to β^*. Since $\dim V = n - 1$ it follows that $\xi_0 = 0$. Define

$$\lambda_i = \sqrt{\beta^*(\xi_i)}, \qquad \kappa = (0, \cdots, 0) \tag{22}$$

then $\lambda_0 = 0$. Observe these values are consistent with (5.1). From (3.5)(b) the weight data is

$$\nu(\Phi_{\lambda,0}) = ([0, \cdots, 0, \lambda_k e_k^*, \cdots, \lambda_{n-1} e_{n-1}^*], \beta_0), \qquad \beta_0(v) = \|v\|^2 \tag{23}$$

Now $\beta_0^*(\lambda_i e_i^*) = \lambda_i^2 = \beta^*(\xi_i)$. Then $\lambda_i e_i^*$ are obviously pairwise β_0^*-orthogonal, and ξ_i are pairwise β^*-orthogonal since $\varpi = 0$. Thus there is an isometry $A : (V, \beta) \to (V, \beta_0)$ with

$$(\lambda_i e_i^*) \circ A^{-1} = A^*(\lambda_i e_i^*) = \xi_i, \qquad \beta = \beta_0 \circ A^{-1}$$

Then $x \in \mathrm{Im}\,\nu$ because applying (16) to (23) gives

$$\nu(\Phi_{\lambda,0} \circ A^{-1}) = ([\xi_0, \cdots, \xi_{n-1}], \beta) = x$$

This proves $\mathcal{R}_n \subset \mathrm{Im}\,\nu$ in Case 1.

Case 2 ($\varpi > 0$). This is the diagonal case. Identify V with the subspace of $\mathbb{R}^n$ where $x_n = 0$, and extend β to $\mathbb{R}^n$ so that $\beta(e_n) = 1$ and e_n is orthogonal to V. Let $L_\beta : (V, \beta) \to (V^*, \beta^*)$ be the natural isometry given by $(L_\beta v)w = \langle v, w \rangle_\beta$ and let $r_i = L_\beta^{-1} \xi_i$ be the vectors in V dual to the weights. Then (21) is equivalent to the pairwise orthogonality of the vectors

$$\{u_i = r_i + \sqrt{\varpi} e_n : 0 \leq i \leq n-1\} \subset \mathbb{R}^n$$

Since $\|u_i\| \geq \varpi > 0$ this is a basis of $\mathbb{R}^n$. Moreover $\langle e_n, u_i \rangle = \sqrt{\varpi}$. Writing e_n in terms of this orthogonal basis $e_n = \sum_{i=0}^{n-1} \mu_i u_i$ with $\mu_i = \sqrt{\varpi}/\|u_i\|_\beta^2 > 0$. Thus $\sum \mu_i r_i = 0$ and $\xi_i = L_\beta(r_i)$ so $\sum \mu_i \xi_i = 0$. Set $\lambda_i = \mu_i^{-2}$ and $\kappa_i = \lambda_0/\lambda_i$ then $\sum \lambda_i^{-2} \xi_i = 0$. Define

$$([\xi_0', \cdots, \xi_{n-1}'], \beta') := \nu(\Phi_{\lambda,\kappa})$$

Then $\sum \lambda_i^{-2} \xi_i' = 0$ by (19), and by (18) $\langle \xi_i', \xi_j' \rangle_{\beta'^*} = \varkappa \lambda_i^2 \delta_{ij} - \varpi$. Now $\xi_i' = \lambda_i e_i^*$ for $i > 0$, so in particular $\{\xi_i' : 1 \leq i \leq n-1\}$ is a basis of V^*. There is a unique

$A \in \mathrm{GL}\, V$ such that $A^*\xi_i' = \xi_i$ for all $i \geq 1$. By (19) $\xi_0' = -\lambda_0^2 \sum_{i=1}^{n-1} \lambda_i^{-2}\xi_i'$ and $\xi_0 = -\lambda_0^2 \sum_{i=1}^{n-1} \lambda_i^{-2}\xi_i$ it follows that $A^*\xi_0' = \xi_0$. Now

$$\langle \xi_i', \xi_j' \rangle_{\beta'^*} = \varkappa \lambda_i^2 \delta_{ij} - \varpi = \langle \xi_i, \xi_j \rangle_{\beta^*}$$

and it follows that A^* is an isometry between the metrics $(\beta')^*$ and β^* on V^*. This proves $\mathcal{R}_n \subset \mathrm{Im}\, \nu$ in Case 2.

In both cases we have shown $\mathcal{R}_n \subset \mathrm{Im}\, \nu$ so ν is surjective. Thus ν is a bijection. Let $\Upsilon = \eta \circ \nu^{-1} : \mathcal{R}_n \to X_n$. By (4.3) η is a homeomorphism, so Υ is a bijection. Above we showed that $\eta(\rho)$ is a continuous function of $\nu(\rho)$, and it follows that Υ is continuous.

We claim Υ is proper. Suppose $\nu(\rho_m) = (x_m, \beta_m)$ is unbounded, and suppose for contradiction that $\Upsilon(\nu(\rho_m)) = \eta(\rho_m) = (\chi(\rho_m), \beta_m)$ is bounded. Then there is a component $\xi_{m,i} \in V^*$ of $x_m = (\xi_{m,0}, \cdots, \xi_{m,n-1}) \in \mathrm{SP}^n V^*$ that is unbounded. Thus $\chi(\rho_m) = \sum_i \exp \xi_{m,i}$ is unbounded, a contradiction. This proves the claim.

By (4.2) X_n is locally compact, and $\mathcal{R}_n$ is a closed subset of Euclidean space and thus locally compact. By (3.1) Υ is a homeomorphism. Since η is a homeomorphism it follows that $\nu = \Upsilon^{-1} \circ \eta$ is a homeomorphism.

By definition 1 $\mathcal{R}_n$ is a semi-algebraic set. By (5.1) $\varpi = 0$ if and only if $\lambda_0 = 0$, and this is equivalent to the holonomy being non-diagonalizable by (2.4). Now $\mathcal{R}_n$ is a subspace of $\mathrm{SP}^n(V^*) \times \mathcal{P}$. There is an action of $A^{-1} \in U$ on the latter given by

$$A^{-1} \cdot ([\xi_1, \cdots, \xi_n], B) = ([\xi_1 \circ A, \cdots, \xi_n \circ A], A^t B A)$$

that preserves $\mathcal{R}_n$. By Gram-Schmidt, every $B \in \mathcal{P}$ can be written uniquely as $B = A^t A$ for some $A \in U$. Thus the action U on $\mathcal{P}$ is simply transitive. Hence the image of ν is the orbit under U of $F_n^* \times \beta_0$ where $\beta_0 \in \mathcal{P}$ is the standard form $\langle\ ,\ \rangle$. The dual of F_n^* is $F_n \subset \mathrm{SP}^n(V)$ that consists of $[v_1, \cdots, v_n]$ such that

$$\exists\ \varpi \geq 0 \qquad \forall\ i \neq j \quad \langle v_i, v_j \rangle = -\varpi$$

The solutions of this equation are preserved by multiplication by non-negative scalar, so F_n is a cone. □

6. Cubic differentials

In this section we will show that when $n \geq 3$ a generalized cusp $C \cong T^{n-1} \times [0, \infty)$ is uniquely determined up to equivalence by the projective class $[J]$, called the *shape invariant*, of a certain polynomial $J = q + c$ where $q, c : \mathbb{R}^{n-1} \to \mathbb{R}$ are homogeneous polynomials of degree 2 and 3 respectively. One may regard q as a similarity structure (Euclidean structure up to scaling) on T^{n-1}, and c as a cubic differential on T^{n-1}. When $n = 2$ then the shape invariant does not determine the cusp, but the moduli space is described in [2, Section 6].

Definition 6.1. A *calibrated vector space* is a pair (V, ϑ) where V is a vector space and $\vartheta : V \to \mathbb{R}$ is a function, called the *calibration*. A linear isomorphism $f : V \to V'$ is an *isometry* between the calibrated vector spaces (V, ϑ) and (V', ϑ') if $\vartheta = \vartheta' \circ f$. The group of self isometries of (V, ϑ) is written $\mathrm{O}(\vartheta)$. Two calibrations ϑ, ϑ' are *similar* if there is $\lambda > 0$ with $\vartheta' = \lambda\vartheta$, and this is denoted $\vartheta \sim \vartheta'$.

A calibration can be viewed as an interesting generalization of a norm. For example, there is a calibrated vector space $(\mathbb{R}^{248}, \vartheta)$ with ϑ an octic polynomial such that the compact form of the exceptional Lie group E_8 is the identity component

of $\mathrm{O}(\vartheta)$, see [12], where they use the term *stabilize* instead of *isometry*. We follow [1] in using the term *isometry*.

Definition 6.2. A *cusp-space* is a calibrated vector space, (V, ϑ), that is similar to some $(\mathbb{R}^{n-1}, \vartheta_{\lambda,\kappa})$ where $\vartheta_{\lambda,\kappa} : \mathbb{R}^{n-1} \to \mathbb{R}$ is given by

$$\vartheta_{\lambda,\kappa}(v_1, \cdots, v_{n-1}) = \left(\langle v, v\rangle + \langle v, \kappa\rangle^2\right) + \frac{1}{3}\left(-\lambda_0\langle v, \kappa\rangle^3 + \sum_{i=1}^{n-1} \lambda_i v_i^3\right)$$

and $(\lambda, \kappa) \in \widetilde{A}_n$ and $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^{n-1}$.

In the non-diagonalizable case when $\kappa = 0$, this simplifies to $\vartheta_{\lambda,0} = \langle v, v\rangle + \frac{1}{3}\sum_{i=1}^{n-1} \lambda_i v_i^3$.

Definition 6.3. The *space of cusp-space structures* on the vector space V is

$$\mathcal{J}(V) = \{[\vartheta] \ : \ (V, \vartheta) \text{ is a cusp-space}\} \subset \mathbb{P}\left(\mathrm{S}^2 V \oplus \mathrm{S}^3 V\right)$$

equipped with the subspace topology, and $\mathcal{J}_n = \mathcal{J}(\mathbb{R}^{n-1})$.

If $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, the *k-Jet* is the polynomial given by the truncated Taylor expansion of f around 0 consisting of all terms of total degree at most k.

Definition 6.4. Suppose T is a translation group, and W is a real vector space, and $\theta : W \to T$ is an isomorphism. The *shape invariant for θ is $[J]$ where h is a height function for T, and $J = J(\theta)$ is the 3-Jet of h at 0, and* $[J] \in \mathbb{P}(\mathrm{S}^2 W \oplus \mathrm{S}^3 W)$.

The height function h is unique up to multiplication by a positive real, thus the projective class $[J]$ of J is well defined. Moreover the terms of degree 0 and 1 in J vanish, so $J = q + c$ with $q \in \mathrm{S}^2 W$ and $c \in \mathrm{S}^3 W$. When $W = V$ then $\det q$ is defined using the standard basis of V, and $\beta(\theta) = \gamma q$ is unimodular where $\gamma = (\det q)^{-1/\dim V}$. We use the map $F : \mathcal{J}(V) \to \mathcal{P} \oplus \mathrm{S}^3 V$ given by $F[q + c] = \gamma(q + c)$ to identify $\mathcal{J}(V)$ with a subspace of $\mathcal{P} \oplus \mathrm{S}^3 V$. In this way we can work with polynomials instead of equivalence classes up to scaling.

It is easy to check that if $B \in \mathrm{Aff}(n)$ then $J(B\theta B^{-1}) = J(\theta)$, and that if $A \in \mathrm{GL}\, V$ then $J(\theta \circ A) = J(\theta) \circ A$. Consider the diagonal translation subgroup $G = \mathrm{Tr}(\psi)$ where $\psi = \sum_{i=1}^n \psi_i e_i^*$ with all $\psi_i > 0$ as in (2.4). Let $D(n) \subset \mathrm{GL}(n+1, \mathbb{R})$ be the subgroup of positive diagonal matrices with 1 in the bottom right corner. Then G is a codimension-1 subgroup of $D(n)$. To compute the calibration for ζ_ψ we avoid choosing a basis of the Lie algebra, $\mathfrak{g}$, of G, but instead work with the natural basis of $D(n)$.

Let $\mathbb{A} = \mathbb{R}^n$ be the $\mathbb{R}$-algebra with addition and multiplication defined componentwise, so

$$(a_1, \cdots, a_n)(b_1, \cdots, b_n) = (a_1 b_1, \cdots, a_n b_n)$$

This multiplication is called the *Hadamard product*. Observe that $p = (1, \cdots, 1)$ is the multiplicative identity in $\mathbb{A}$, and for $n > 0$ then $a^n \in \mathbb{A}$ is the element obtained by raising each component of a to power n. Let $\mathbb{A}_+ \subset \mathbb{A}$ be the subset with all coordinates strictly positive, made into a group using Hadamard multiplication. The map $\mathbb{A} \to \mathfrak{gl}(n+1, \mathbb{R})$ given by $(x_1, \cdots, x_n) \mapsto \mathrm{Diag}(x_1, \cdots, x_n, 0)$ is used to identify the Lie algebra $\mathbb{A}$ (with zero Lie bracket) to the Lie algebra of $D(n)$, and the group homomorphism $\delta : \mathbb{A}_+ \to \mathrm{GL}(n+1, \mathbb{R})$ given by $\delta(x_1, \cdots, x_n) = \mathrm{Diag}(x_1, \cdots, x_n, 1)$

identifies $\mathbb{A}_+$ (with Hadamard multiplication) to $D(n)$. Regarding $\mathbb{A}$ as the Lie algebra of $\mathbb{A}_+$ then $\exp : \mathbb{A} \to \mathbb{A}_+$ is coordinate-wise exponentiation. Define an inner product on $\mathbb{A}$ by

$$\langle x, y\rangle_\psi = \psi(xy) = \sum_{i=1}^{n} \psi_i x_i y_i$$

Then $\langle xy, z\rangle_\psi = \langle x, yz\rangle_\psi$ so $\langle x, y\rangle_\psi = \langle p, xy\rangle_\psi$, and

$$\mathfrak{g} = \ker \psi = p^\perp := \{x \in \mathbb{A} : \ \langle p, x\rangle_\psi = 0 \ \}$$

may be regarded as the Lie algebra of $\mathrm{Tr}(\psi)$.

Lemma 6.5. *If $\mathbf{t}(\psi) = n$ then $\delta \circ \exp : \mathfrak{g} \to \mathrm{Tr}(\psi)$ is a marked translation group, and the shape invariant is $[J(\delta \circ \exp)] = [J_\psi]$ where*

$$J_\psi(x) = \ (1/2)\langle p, x^2\rangle_\psi + (1/6)\langle p, x^3\rangle_\psi$$

Proof. Since $\mathfrak{g} = \ker \psi$

$$\delta \circ \exp(\mathfrak{g}) = \{\mathrm{Diag}(\exp(x_1), \cdots, \exp(x_n), 1) : \ \sum \psi_i x_i = 0 \ \} = \mathrm{Tr}(\psi)$$

Let $\partial\Omega \subset \mathbb{R}^n$ be the orbit of p under $\mathrm{Tr}(\psi)$ then the tangent space to $\partial\Omega$ at p is $p^\perp$. We use the height function $h = h_\psi \circ \exp : \mathfrak{g} \to \mathbb{R}^n$ where $h_\psi(y) = \psi(y) - \psi(p)$; then

$$h(x) = -\psi(p) + \psi(\exp(x)) = -\psi(p) + \sum_{i=0}^{\infty} \frac{1}{n!}\langle p, x^n\rangle_\psi$$

The terms of degree 0 and 1 vanish, because $\langle p, x^0\rangle_\psi = \psi(p)$, and $\langle p, x\rangle_\psi = 0$ since $x \in p^\perp$. □

The proof of the following is in Section 8.

Proposition 6.6. *If $(\lambda, \kappa) \in A_n$ then $[J(\Phi_{\lambda,\kappa})] = [\vartheta_{\lambda,\kappa}] \in \mathcal{J}(V)$. Moreover in the diagonalizable case $\lambda_0 > 0$, and $(V, J(\Phi_{\lambda,\kappa}))$ is similar to $(\mathfrak{g}, J_\psi)$ where ψ is determined in the proof.*

Lemma 6.7 shows how the weight data ν determines the calibration. The cubic term c in the 3-Jet $J = q + c$ is a weighted sum of the cubes of the weights ξ_i, see (39). Later we will see that one can recover these weights from $[J]$. In general a polynomial might be the sum of cubes in distinct way, but see [22] and Theorem (1.4) in [23] for a uniqueness statement concerning the expression of a cubic as a sum of cubes in some special cases. The proof of the following is in Section 8.

Lemma 6.7. *If $n \geq 3$ then there is a map $\mathcal{K} : \mathcal{R}_n \to \mathcal{P} \oplus \mathrm{S}^3 V$ such that $[\mathcal{K} \circ \nu] = [J]$ and $\mathcal{K}$ is continuous and proper. If ρ is a marked translation group and $x = \nu(\rho) = ([\xi_0, \cdots, \xi_{n-1}], \beta) \in \mathcal{R}_n$, then $\mathcal{K}(x) = \beta(\rho) + c(\rho)$ with*

$$c(\rho) = (1/3)\sum_{i=0}^{n-1} \xi_i^3 \left(\langle \xi_i, \xi_i\rangle_{\beta^*} + \varpi\right)^{-1}, \qquad \varpi = -\langle \xi_1, \xi_2\rangle_{\beta^*}$$

Corollary 6.8. *$J : \mathcal{T}_n \to \mathcal{P} \oplus \mathrm{S}^3 V$ is continuous and proper.*

Proof. By (4.5) $\mathrm{hol}^{-1} : \mathrm{Rep}_n \to \mathcal{T}_n$ is a homeomorphism and by (6.7) $\mathcal{K} : \mathcal{R}_n \to \mathcal{P} \oplus \mathrm{S}^3 V$ is continuous and proper, and $\nu : \mathrm{Rep}_n \to \mathcal{R}_n$ is homeomorphism by (1.2); thus $J = \mathcal{K} \circ \nu \circ \mathrm{hol}^{-1}$ is continuous and proper. □

It remains to show that the shape invariant $[J] = [q + c]$ determines a unique generalized cusp. The method used is to show that the local maxima of the cubic, c, restricted to the unit sphere of the quadratic, q, enable one to determine ψ. This follows from Lemmas 6.9 for the diagonalizable case and 6.10 in the non-diagonalizable case. The proofs are in Section 8.

Lemma 6.9 (Diagonalizable case)**.** *Assume $n \geq 3$. Given $\psi \in A_n$ let $(\mathbb{R}^n, J_\psi = q + c)$ be the calibrated vector space with $J_\psi(x) = (1/2)\langle p, x^2\rangle_\psi + (1/6)\langle p, x^3\rangle_\psi$. Let $\mathfrak{g} = \{x \in \mathbb{R}^n : \langle p, x\rangle_\psi = 0\}$, and $S = \{v \in \mathfrak{g} : \langle v, v\rangle_\psi = 1\}$, and $\mathfrak{s} = \sum \psi_i$. For $1 \leq i \leq n$ define $v_i = (\mathfrak{s}e_i - \psi_i p)/\|\mathfrak{s}e_i - \psi_i p\|_\psi$. Then*

$$K = \{x \in S : (c|S) \text{ has a local maximum at } x\} = \{v_i : 1 \leq i \leq n\}$$

Moreover $i \neq j \Rightarrow \alpha_{ij} := \langle v_i, v_j\rangle_\psi < 0$. If $1 \leq i, j, k \leq n$ and i, j, k are pairwise distinct then

$$\text{(i)} \quad \psi_i/\mathfrak{s} = \alpha_{ij}\alpha_{ik}/(\alpha_{ij}\alpha_{ik} - \alpha_{jk}), \qquad \text{(ii)} \quad 6c(v_i) = \frac{1}{\sqrt{\psi_i}} \frac{1 - 2\psi_i/\mathfrak{s}}{\sqrt{1 - \psi_i/\mathfrak{s}}}$$

Also $|K^+| \geq n - 1$ where $K^+ = \{v \in K : \ c(v) > 0 \ \}$.

Observe that the calibration determines K and thus the vectors v_i, and the inner products α_{ij}. Then (i) shows that ψ is defined up to scaling by this, and (ii) can be used to determine the scaling. For the corresponding result in the non-diagonalizable case, it is more convenient to work with $\Psi_{\lambda,0}$ instead of ζ_ψ, since the calibration is $J = \|v\|^2 + (1/3)\sum \lambda_i v_i^3$.

Lemma 6.10 (Non-diagonalizable case)**.** *Given $\lambda = (0, \lambda_1, \cdots, \lambda_{n-1}) \in A_n$, let $J(v) = \|v\|^2 + c(v)$ where $c = (1/3)\sum \lambda_i v_i^3$ and $S = \{v \in V : \sum v_i^2 = 1\}$. Then $J(\Phi_{\lambda,0}) = \vartheta_{\lambda,0} = J(v)$ and*

$$K^+ = \{v \in S : (c|S) \text{ has a local max at } v, \text{ and } c(v) > 0\} = \{e_i : \lambda_i > 0\}$$

Moreover $c(e_i) = \lambda_i/3$ for $e_i \in K^+$, and if $a \neq b \in K^+$ then $\langle a, b\rangle = 0$, and $|K^+| = \mathbf{t} - 1 \leq n - 1$.

The subgroup $\mathrm{O}(\Omega, b) \subset G(\Omega)$ that stabilizes $b \in \partial\Omega$ is conjugate to the subgroup $\mathrm{O}(\eta) \subset \mathrm{GL}\, V$ that preserves V, by (2.14). The following shows that the latter is the same as the subgroup that preserves J. These results are keys steps in showing η and $[J]$ are powerful invariants.

Lemma 6.11. *If $\theta : V \to T$ is a marked translation group then $\mathrm{O}(J(\theta)) = \mathrm{O}(\eta(\theta))$.*

Proof. This is easy when $\mathbf{t} = 0$ since the generalized cusp is standard, and the cubic term in J is 0. Thus we may assume $\mathbf{t} > 0$ and then by (2.14) $\mathrm{O}(\eta(\theta)) \subset \mathrm{GL}\, V$ is conjugate to the stabilizer of the basepoint in $\mathrm{PGL}\,\Omega$. Since $J(\theta)$ is preserved by the latter $\mathrm{O}(\eta(\theta)) \subset \mathrm{O}(J(\theta))$. To show the reverse inclusion, by (2.5)(c), every marked translation group is given by $B(\zeta_\psi \circ A)B^{-1}$ for some $A \in \mathrm{SL}^{\pm} V$ and $B \in \mathrm{Aff}(n)$. Now $\mathrm{O}(J(\theta))$ and $O(\eta(\theta))$ are both unchanged under conjugation by B. Moreover $J(\theta \circ A) = J(\theta) \circ A$ and $\eta(\theta \circ A) = \eta(\theta) \circ A$. Thus is suffices to prove the result for $\theta = \zeta_\psi$.

Suppose $J = q + c$ where $q = \beta$ is the horosphere metric on V given by ζ_ψ and $\mathrm{O}(q) \subset \mathrm{GL}(V)$ is the subgroup that preserves q. Set $\mathbf{t} = \mathbf{t}(\psi)$. Let $\mathcal{W} = \{\xi_i \in V^* : 1 \leq i \leq \mathbf{t}\}$ be the set of non-zero Lie algebra weights for ζ_ψ. Then $\mathrm{O}(\eta(\zeta_\psi))$ is the subgroup of $\mathrm{O}(q)$ that preserves the character $\chi = \chi(\zeta_\psi)$, and $\mathrm{O}(J(\zeta_\psi))$ is the subgroup of $\mathrm{Sim}(q)$ that preserves the cubic c. Arguing as in (2.14)

$\mathrm{O}(J(\zeta_\psi)) \subset \mathrm{O}(q)$ since $\mathbf{t} > 0$. The result will follow by showing that preserving χ is equivalent to preserving $\mathcal{W}$ is equivalent to preserving c.

By (2.13) preserving χ is equivalent to preserving the characteristic polynomial $G = c_{\zeta_\psi}$. Let $\mathcal{W}^+ \supset \mathcal{W}$ be the multiset of all Lie-algebra weights of the linear part of ζ_ψ. Then $|\mathcal{W}^+| = n$ and $\mathcal{W}^+$ contains the zero weight with multiplicity $n - \mathbf{t}$. The coefficients of G are the elementary symmetric functions of the elements of $\mathcal{W}^+$. Thus preserving G is equivalent to preserving $\mathcal{W}$. By (39) $c = (1/3\varkappa)\sum \lambda_i^{-2}\xi_i^3$. Thus if $\mathcal{W}$ is preserved, then c is preserved.

For the converse, suppose c is preserved. When $\mathbf{t}(\lambda) < n$ then by (6.10) $\mathrm{O}(J(\zeta_\psi))$ preserves $K^+ = \{e_i : \psi_i > 0\}$ and since $c(e_i) = \psi_i/3$ it follows that $\mathrm{O}(J(\zeta_\psi))$ preserves $S = \{\psi_i e_i : 1 \le i \le n\}$. It follows that $\mathcal{W}$ is preserved in this case.

This leaves the case $\mathbf{t}(\lambda) = n$. By (6.9) $\mathrm{O}(J(\zeta_\psi))$ preserves K and therefore permutes the coordinates of ψ. Moreover the formula for $c(v_i)$ in (6.9) shows that $c(v_i) = c(v_j)$ if and only if $\psi_i = \psi_j$. Comparing this to (3.5) one sees that the weights are preserved. Thus $\mathrm{O}(J(\zeta_\psi))$ preserves $\mathcal{W}$. □

We now have the ingredients to show that $[J]$ determines ψ.

Lemma 6.12. *If $n \ge 3$, and $A, A' \in \mathrm{SL}^{\pm} V$, and $[J(\zeta_\psi \circ A)] = [J(\zeta_{\psi'} \circ A')]$ then $\psi = \psi'$.*

Proof. In what follows we scale $J = q + c$ so that q is unimodular, and talk about *this* calibration instead of its projective class. Let $S = \{v \in V : q(v) = 1\}$ and $K \subset S$ the set of points at which $c|S$ has a local maximum, and let $K^+ \subset K$ be the subset where $c > 0$. Observe that $|K^+|$ is an invariant of the similarity class of a cusp space.

Let $\langle \cdot, \cdot \rangle_q$ be the inner product on V determined by q. Then the set $\{\langle a, b\rangle_q : a, b \in K^+\}$ is also an invariant of the similarity class. By (6.6) the calibration on a marked translation group is similar to some $\vartheta_{\lambda,\kappa}$, and in the diagonalizable case also to some J_ψ. First suppose $|K^+| \ge 2$ and choose two distinct elements $a, b \in K^+$.

Case 1 ($\langle a, b\rangle_q = 0$). Then (6.9) implies that $\mathbf{t} < n$, and (6.10) implies the coordinates of λ are given by $c(v)$ as v ranges over K^+. Moreover $\psi_i = 1/\lambda_i^2$ so ψ is determined by $[J]$ in this case.

Case 2 ($\langle a, b\rangle_q \ne 0$). Then (6.10) implies $\mathbf{t} = n$, so $(V, J(\zeta_\psi \circ A))$ is similar to $(\mathfrak{g}, J_\psi)$. It follows from (6.9)(i) that J determines ψ up to multiplication by a positive scalar.

Thus we may assume $\psi' = s\psi$ with $s > 0$. By (2.5)(a) $\zeta_{s\psi} = \zeta_\psi \circ ((s\,\mathrm{I}_{\mathbf{r}} \oplus \mathrm{I}_{\mathbf{u}})$. If $[J(\zeta_\psi \circ A)] = [J(\zeta_{\psi'} \circ A')]$ it follows that $[J(\zeta_\psi)] = [J(\zeta_\psi \circ B)]$ where $B = ((s\,\mathrm{I}_{\mathbf{r}}) \oplus \mathrm{I}_{\mathbf{u}})A'A^{-1}$. Thus $B \in O(J(\psi))$, so $\det B = \pm 1$. Since $|\det A| = |\det A'| = 1$ it follows that $\det((s\,\mathrm{I}_{\mathbf{r}}) \oplus \mathrm{I}_{\mathbf{u}}) = s^{\mathbf{r}} = \pm 1$. Thus $s = 1$, so $\psi' = \psi$.

Case 3 ($|K^+| \le 1$). If $\mathbf{t} = n$ then $|K^+| \ge n - 1$ by (6.9). Since $n \ge 3$ it follows that $\mathbf{t} < n$ which contradicts $\mathbf{t} = n$. The result now follows from (6.10) as before.

□

Lemma 6.13. *Suppose $\rho, \rho' : V \to \mathrm{Aff}(n)$ are marked translation groups and $n \ge 3$. If $[J(\rho)] = [J(\rho')]$ then ρ and ρ' are conjugate.*

Proof. We may assume $\rho = \zeta_\psi \circ f$ and $\rho' = \zeta_{\psi'} \circ f'$ with $f, f' \in \mathrm{SL}^{\pm} V$. It follows from (6.12) that $\psi = \psi'$. Then $[J(\rho)] = [J(\rho')]$ implies $f^{-1} \circ f' \in \mathrm{O}(J(\zeta_\psi))$, thus $f^{-1} \circ f' \in \mathrm{O}(\eta(\zeta_\psi))$ by (6.11). Hence ρ and ρ' have the same complete invariant, and so are conjugate by (2.16). □

Theorem 6.14. *Suppose $n \geq 3$. Let $\mathcal{T}_n$ be the space of marked generalized cusps homeomorphic to $T^{n-1} \times [0, \infty)$. The shape invariant $J : \mathcal{T}_n \longrightarrow \mathcal{J}_n$ is a homeomorphism. Moreover $\mathcal{K} : \mathcal{R}_n \to \mathcal{J}_n$ is a homeomorphism.*

Proof. By (6.13) J is injective. By (6.8) J is continuous and proper. The image of J is contained in $\mathcal{J}_n$ by (6.6), and surjectivity follows from the proof of 6.6. Moreover $\mathcal{J}_n$ is homeomorphic to a subspace of Euclidian space and is therefore locally compact and Hausdorff. Also $\mathcal{T}_n$ is locally compact by (4.3), so J is a homeomorphism by (3.1). Now $J = \mathcal{K} \circ \nu$, and ν is a homeomorphism by (1.2), thus $\mathcal{K}$ is a homeomorphism. □

Proof of 1.3. $J = \mathcal{K} \circ \nu$ where J and $\mathcal{K}$ are both homeomorphisms by (6.14), and $\nu : \mathcal{T}_n \to \mathcal{R}_n$ is a homeomorphism by (1.1). The second component of ν is a map $\pi_\nu : \mathcal{T}_n \to \mathcal{P}$ that is a trivial bundle by (1.2). So $\pi = \pi_\nu \circ J^{-1} : \mathcal{J}_n \to \mathcal{P}$ is also a trivial bundle.

The fiber of π_ν is a closed cone $C \subset \mathrm{S}^3 V$ described in the proof of 1.2. So the fiber of π is $J(C)$ and is homeomorphic to C using the restriction of J. The cone structure on $J(C)$ can be seen directly as follows. The expression for the cubic c in (6.7) has the property that if the weights are all multiplied by $s > 0$ then the cubic is also multiplied by s. □

6.1. **The affine normal.** Refer to chapter 1 of [20], see also [19] and [16, Lemma 4.1]. Suppose $S \subset \mathbb{R}^n$ is a smooth strictly convex hypersurface and p is a point in S. Then the tangent hyperplane to S at p intersects S only at p and S lies on one side of P. An *affine normal* to S at p is vector $0 \neq \nu = \nu(p) \in \mathbb{R}^n$ with the following property. Given $\delta > 0$ let $P(\delta)$ be the hyperplane parallel to P on the side of P that contains S, and distance δ from P. Let $x(\delta)$ be the center of mass of $S \cap P(\delta)$. Then $(x(\delta) - p)/\delta$ converges to a non-zero multiple of ν. We also require that ν points to the convex side of S. Then ν is defined up to positive scalar multiples.

It follows from this that affine normals are preserved by affine maps: if A is an affine map of $\mathbb{R}^n$ then $A(\nu(p))$ is an affine normal to $A(S)$. Since affine maps are not conformal, the affine normal is not in general orthogonal to S at p. A convex hypersurface in $\mathbb{R}^n$ is an *affine sphere* if there is a point $b \in \mathbb{R}\mathrm{P}^n$ such that every affine normal passes through b.

There is a decomposition $\mathrm{S}^3(\mathbb{R}^n) = \mathscr{H}_n \oplus \mathscr{R}_n$ into the *harmonic cubics* $\mathscr{H}_n$, and the *radial cubics* $\mathscr{R}_n$ given by

$$\mathscr{H}_n = \{p \in \mathrm{S}^3(\mathbb{R}^n) : \ \Delta p = 0\}, \qquad \mathscr{R}_n = \{\|x\|^2 \langle v, x\rangle : \ v \in \mathbb{R}^n\}$$

The group $\mathrm{O}(n)$ acts on $\mathrm{S}^3(\mathbb{R}^n)$ preserving this decomposition, and by [24, Theorem 0.3] the action on each summand is irreducible.

The material from here to (6.17) is not used in this paper, so we have omitted the proofs. It is included to avert a possible misperception. The map $\pi : \mathrm{S}^3(\mathbb{R}^n) \to \mathbb{R}^n$ given by $\pi(p) = (2n+4)^{-1} \nabla(\Delta p)$ is projection onto $\mathscr{R}_n$ followed by the map $\|x\|^2 \langle v, x\rangle \mapsto v$. More generally, if β is a positive definite quadratic form on $\mathbb{R}^n$ then there is an isometry $L \in \mathrm{GL}(n, \mathbb{R})$ from $\|\cdot\|^2$ to β. Hence $L(\mathscr{H}_n)$ and $L(\mathscr{R}_n)$

are preserved by $\mathrm{O}(\beta)$ and $\pi_\beta = L \circ \pi \circ L^{-1} : \mathrm{S}^3(\mathbb{R}^n) \to \mathbb{R}^n$. The following says that the affine normal is the radial part of the cubic term in a Taylor expansion.

Proposition 6.15. *Suppose $U \subset \mathbb{R}^n$ is a neighborhood of 0 and $f : U \to \mathbb{R}$ is C^3. Let $S \subset \mathbb{R}^{n+1}$ be the graph of f and suppose $f(x) = \beta(x) + c(x) + o(\|x\|^3)$ and $\beta \in \mathrm{S}^2\,\mathbb{R}^n$ is positive definite, and $c \in \mathrm{S}^3\,\mathbb{R}^n$. Then an affine normal to S at 0 is $e_{n+1} - (2n)^{-1}\pi_\beta(c)$.*

This can be deduced from formula (3.4) on page 48 of [20]. This formula goes back at least to 1923, see Blaschke [4].

Recall that the *radial flow* $\Phi : \mathbb{R} \to \mathrm{Aff}_n$ for a generalized cusp lie group $G(\Omega)$ centralizes it, and $\Phi_t(\Omega) \subset \Omega$ whenever $t \leq 0$, see [2, (1.11)]. If $\theta = \Phi_{\lambda,\kappa}$ the radial flow is $\Phi_t(x) = x - te_1$ if $\mathbf{t} < n$, and otherwise $\mathbf{t}(\lambda) = n$ and $\Phi_t(x) = e^{-t}(x - C) + C$ where $C \in \mathbb{R}^n$ is the *center* of Φ. Refer to (4) for the definition of τ and H_b in the following. Now we may assume that $\Omega = \Omega(\lambda, \kappa)$ in (3.5) and $b = 0$ and H_b is $x_1 = 0$. Then $\tau(x_1, \cdots, x_n) = \alpha x_1$ for some $\alpha > 0$.

It is more convenient in the following to redefine the radial flow when $\mathbf{t} = n$ to be $\Phi : (-1, \infty) \to \mathrm{Aff}_n$ given by $\Phi_t(x) = (t+1)^{-1} \cdot (x - C) + C$. Then Φ_0 is always the identity and $I = \mathbb{R}$ or $(-1, \infty)$ is the domain of Φ as appropriate.

Then $F = \theta \times \Phi : V \times I \to \mathbb{R}^n$ are coordinates on a subset of $\mathbb{R}^n$ that contains Ω. *In these coordinates* the height function h_θ *describes (an open subset of) H_b as a graph over $\partial\Omega$* rather than vice versa, as one might naïvely imagine.

Lemma 6.16. *Scale τ so that if $\mathbf{t} < n$ then $\tau(x_1, \cdots, x_n) = x_1$ and if $\mathbf{t} = n$ then $\tau(C) = -1$. Then $F(V \times 0) = \partial\Omega$ and $F(\{(v, t) : \ t = h_\theta(v)\}) \subset H_b$.*

If $J(\theta) = [\beta + c]$ then (6.15) implies that β and the radial-cubic part of c determine the affine normal to $F^{-1}(H_b)$.

Proposition 6.17. *Let $\|\cdot\|$ be the standard inner product on V, and let $\theta : V \to T$ be a marked translation group, and let S be a horosphere for T, and Φ a radial flow for θ, and $J(\theta) = \beta + c$ with β unimodular. The following are equivalent*

(a) *flow lines of Φ are affine normals to S.*
(b) *S is an affine sphere.*
(c) *c is harmonic with respect to β i.e. $\pi_\beta(c) = 0$*
(d) *T is conjugate to $\mathrm{Tr}(s, \cdots, s)$ with $s \geq 0$.*

Proof. Flow lines of Φ limit on the center of the radial flow, so $(a) \Rightarrow (b)$. For the converse, assume S is an affine sphere with center $w \in \mathbb{R}\mathrm{P}^n$. Then T fixes w. If the affine normals to S are parallel, then S is an elliptic paraboloid, [5], [21]. In this case T is conjugate to $\mathrm{Tr}(0, \cdots, 0)$, and w is the center of Φ. Otherwise $w \in \mathbb{R}^n$. Thus T is diagonalizable. We may assume $T = \mathrm{Tr}(\psi)$ with all the coordinates of $\psi > 0$ and $w = 0$. Again w is the center of Φ. Thus (b)⇒(a). In this case we claim $\psi = (s, \cdots, s)$. This is because S is an affine sphere asymptotic to the sides of a simplex, and by [7] it follows that S is unique up to affine maps preserving the simplex. Thus (b)⇒(d). For (d)⇒(b) when $s = 0$ then S is an elliptic paraboloid and when $s > 0$ then S is defined by $\prod x_i = 1$. These are well known affine spheres.

It remains to show (c)⇔(d). Using (6.2) we may assume

$$J = [\beta + c], \qquad \beta(v) = \|v\|^2 + \langle v, \kappa\rangle^2, \qquad 3c(v) = -\lambda_0\langle v, \kappa\rangle^3 + \sum_{i=1}^{n-1} \lambda_i v_i^3$$

If $\lambda_0 = 0$ we may choose $\kappa = 0$ then c is harmonic with respect to $\beta(v) = \|v\|^2$ if and only if $\lambda = 0$, showing (c)⇔(d) in this case. Otherwise $\lambda_0 > 0$. First we perform a linear change of coordinates on V so that $\beta(v) = \|v\|^2$.

Let $T \in \mathrm{GL}(V)$ be defined by $T(v) = v + \alpha\langle v, \kappa\rangle\kappa$ where $\alpha = \|\kappa\|^{-2}(-1 + (1 + \|\kappa\|^2)^{-1/2})$; then $\beta(Tv) = \|v\|^2$. Now we compute the cubic $3\lambda_0^{-1}(c \circ T)$ using the Hadamard product on V, and $\kappa_i = \lambda_i/\lambda_0$.

$$\begin{aligned} 3\lambda_0^{-1}(c \circ T)v &= -\langle Tv, \kappa\rangle^3 + \langle \kappa^{-1}, (Tv)^3\rangle \\ &= \gamma\langle v, \kappa\rangle^3 + 3\alpha\langle v, \kappa\rangle\|v\|^2 + \langle \kappa^{-1}, v^3\rangle \end{aligned}$$

$$\text{where} \quad \gamma = -\left(1 + \alpha\|\kappa\|^2\right)^3 + 3\alpha^2 + \alpha^3\|\kappa\|^2 = \frac{-(2 + \|\kappa\|^2) + 2\sqrt{1 + \|\kappa\|^2}}{\|\kappa\|^4\sqrt{1 + \|\kappa\|^2}}$$

Set $m = \dim V$; then

$$\begin{aligned} 3\lambda_0^{-1}\nabla^2(c \circ T) &= \left(6\gamma\|\kappa\|^2 + 3\alpha(2m + 4)\right)\langle v, \kappa\rangle + 6\langle \kappa^{-1}, v\rangle \\ &= \langle 6u, v\rangle \end{aligned}$$

$$\text{where} \qquad u = -\left(\frac{m}{\|\kappa\|^2} + \frac{\|\kappa\|^2 - m}{\|\kappa\|^2\sqrt{1 + \|\kappa\|^2}}\right)\kappa + \kappa^{-1}$$

Then $c \circ T$ is harmonic with respect to $\|\cdot\|^2$ if and only if $u = 0$. Since u is a linear combination of κ and κ^{-1} it follows that $\kappa = s(1, \cdots, 1)$ for some $s \in [0, 1]$. Then $\|\kappa\|^2 = ms^2$ and $u = 0$ implies

$$\left(s^{-2} + m(s^2 - 1)/ms^2\sqrt{1 + ms^2}\right)s = s^{-1}$$

This implies $s^2 - 1 = 0$. Hence $s = 1$ so $\kappa_i = 1$ and $\lambda_i = \lambda_0$. Thus (c)⇔(d) when $\lambda_0 > 0$. □

7. Three dimensions

In dimension 3 every generalized cusp is equivalent to $\Omega_{\lambda,\kappa}/\Gamma$ for some lattice in $\Gamma \subset T(\lambda, \kappa)$, and $\partial\Omega_{\lambda,\kappa}$ is the orbit of 0 under $T(\lambda, \kappa)$. From (30) in the proof of 3.5 one sees that in dimension 3 $\partial\Omega_{\lambda,\kappa}$ is the graph $y = f_\lambda(x_1, x_2)$ in $\mathbb{R}^3$ shown below where for $\mathbf{t} < 3$ we have chosen $\kappa = 0$.

t	$f_\lambda(x_1, x_2)$
3	$\lambda_1^{-1}x_1 + \lambda_2^{-1}x_2 + \lambda_0^{-2}\left(-2 + (1 + \lambda_1 x_1)^{-(\lambda_0/\lambda_1)^2} + (1 + \lambda_2 x_2)^{-(\lambda_0/\lambda_2)^2}\right)$
2	$\lambda_1^{-1}x_1 - \lambda_1^{-2}\log(1 + \lambda_1 x_1) + \lambda_2^{-1}x_2 - \lambda_2^{-2}\log(1 + \lambda_2 x_2)$
1	$x_1^2/2 + \lambda_2^{-1}x_2 - \lambda_2^{-2}\log(1 + \lambda_2 x_2)$
0	$(x_1^2 + x_2^2)/2$

The function f_λ varies continuously with λ on the subspace $\lambda_0 = 0$, and is also continuous when $\lambda_1, \lambda_2 > 0$ are constant as $\lambda_0 \to 0$, but is not continuous in general. This family of surfaces only varies continuously with λ subject to these constraints.

A Euclidean torus is determined by a parallelogram in $V = \mathbb{R}^2$ of area 1. This in turn determines the quadratic form β. Using β we may identify a Lie-algebra weight in V^* with a vector in V. Thus a generalized cusp in a 3-manifold is specified by this parallelogram, together with three vectors a, b, c in V satisfying $\langle a, b\rangle = \langle b, c\rangle = \langle c, a\rangle = \varpi \leq 0$ (see Figure 1). The Lie algebra weights of the holonomy are given

by $\xi_v(x) = \langle v, x\rangle_\beta$ where $v \in \{a, b, c\}$. The cusp is diagonalizable if and only if $\varpi < 0$.

Two such collections of data define equivalent cusps if and only if there is an isometry of $\mathbb{R}^2$ taking one parallelogram to the other and that permutes the set of vectors $\{a, b, c\}$. The type of the generalized cusp is the number of non-zero vectors. The dual vectors in V^* are harmonic 1-forms that represent elements of $H^1(T^2; \mathbb{R})$ and determine the *cusp geometry* defined in (2.17).

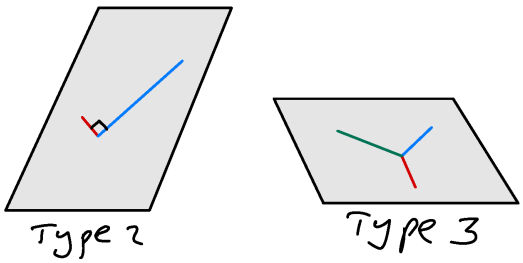


FIGURE 1. Generalized cusps in dimension 3

There is a decomposition $\mathrm{S}^3(\mathbb{R}^2) = \mathscr{H}_2 \oplus \mathscr{R}_2$ into harmonic and radial cubics is given by

$$\mathscr{H}_2 = \langle x(x^2 - 3y^2), y(y^2 - 3x^2)\rangle, \qquad \mathscr{R}_2 = \langle x(x^2 + y^2), y(x^2 + y^2)\rangle$$

with coordinate projections $\pi_{\mathscr{H}}$ and $\pi_{\mathscr{R}}$. By (6.17) the cubic is harmonic with respect to β if and only if the holonomy is conjugate into $\mathrm{Tr}(s, s, s)$ for some $s \geq 0$.

Regarding $V = \mathbb{R}^2 \cong \mathbb{C}$ via $z = x + iy$, and recalling that the real part of a holomorphic function is harmonic, it follows that

$$\mathscr{H}_2 = \{\mathrm{Re}(hz^3) : h \in \mathbb{C}\}, \qquad \mathscr{R}_2 = \{\mathrm{Re}(rz|z|^2) : r \in \mathbb{C}\}$$

This gives an isomorphism of real vector spaces $\theta : \mathbb{C}^2 \to \mathrm{S}^3\,\mathbb{R}^2$ given by $\theta(h, r) = \mathrm{Re}(hz^3 + rz|z|^2)$. The action of $\mathrm{SO}(2) \cong U(1) = \{\omega \in \mathbb{C} : |\omega| = 1\}$ on $\mathrm{S}^3\,\mathbb{R}^2$ is then $\omega.\theta(h, r) = \theta(\omega^3 h, \omega r)$. The standard Euclidean structure on $\mathbb{C}^2$ gives an inner product on $\mathrm{S}^3\,\mathbb{R}^2$ given by $\|\theta(h, r)\|^2 = |h|^2 + |r|^2$, and $\mathrm{SO}(2)$ acts by isometries. Let β_0 be the quadratic form $x^2 + y^2$ on $\mathbb{R}^2$.

Theorem 7.1. *The image of the embedding $J : \mathcal{T}_3 \to \mathcal{P}(\mathbb{R}^2) \times \mathrm{S}^3\,\mathbb{R}^2$ is*

$$J(\mathcal{T}_3) = \{(A^t A, c \circ A) \in \mathcal{P}(\mathbb{R}^2) \times \mathrm{S}^3\,\mathbb{R}^2 \ : \ |\pi_{\mathscr{R}} c| \leq 3|\pi_{\mathscr{H}} c|, \quad A \in \mathrm{SL}(2, \mathbb{R})\ \}$$

Moreover $|\pi_{\mathscr{R}} c| = 3|\pi_{\mathscr{H}} c|$ gives the subspace of non-diagonalizable generalized cusps.

Proof. In this proof we identify $\mathrm{S}^3\,\mathbb{R}^2 \equiv \mathbb{C}^2$ using θ, and $\mathcal{T}_3 \equiv \mathrm{Rep}_3$ using the holonomy. There is a surjection $\widetilde{A}_3 \times \mathrm{SL}^{\pm} V \to \mathrm{Rep}_3$. The action of $A \in \mathrm{SL}^{\pm} V$ on $\mathcal{T}_3$ defined in (16) is conjugate by J to the action on $\mathcal{P}(\mathbb{R}^2) \times \mathrm{S}^3\,\mathbb{R}$ given by $A \cdot (\beta, c) = (\beta \circ A^{-1}, c \circ A^{-1})$. This action preserves the product structure. The stabilizer of β_0 is $O(2)$.

Claim 1. Suppose $(\lambda, \kappa) \in \widetilde{A}_3$ and $\lambda = (\lambda_0 = 0, \lambda_1, \lambda_2)$ and $\kappa = 0$ and $c = c(\Phi_{\lambda,\kappa})$ is the cubic. Then $\pi_{\mathscr{H}} c = z$ and $\pi_{\mathscr{R}} c = 3\overline{z}$ where $z = (\lambda_1 + i\lambda_2)/12$.

Definition 3.4, the Lie algebra weights are $\xi_i = \lambda_i e_i^*$ for $i \in \{1,2\}$, and $\xi_0 = 0$. Using Lemma 6.7 then $\kappa = 0$ so $\varkappa = 1$ and formula (39) gives

$$3c = \lambda_1^{-2}\xi_1^3 + \lambda_2^{-2}\xi_2^3 = \lambda_1(e_1^*)^3 + \lambda_2(e_2^*)^3 = \lambda_1 x^3 + \lambda_2 y^3$$

Expressing this in terms of the generators of $\mathscr{H}_2$ and $\mathscr{R}_2$ gives

$$12c = \lambda_1[x(x^2 - 3y^2) + 3x(x^2 + y^2)] + \lambda_2[y(y^2 - 3x^2) + 3y(x^2 + y^2)]$$

So

$$\begin{aligned} 12h &= \lambda_1 x(x^2 - 3y^2) + \lambda_2 y(y^2 - 3x^2) & 12r &= 3\lambda_1 x(x^2 + y^2) + 3\lambda_2 y(x^2 + y^2) \\ &= \operatorname{Re}\left((\lambda_1 + i\lambda_2)z^3\right) & &= \operatorname{Re}\left(3(\lambda_1 - i\lambda_2)z|z|^2\right) \\ \therefore\ 12\pi_{\mathscr{H}}(c) &= \lambda_1 + i\lambda_2 & 12\pi_{\mathscr{R}}(c) &= 3(\lambda_1 - i\lambda_2) \end{aligned}$$

This proves Claim 1. □

Now $\mathcal{T}_3(3)$ is the stratum of diagonalizable cusps, so $B = \mathcal{T}_3 \setminus \mathcal{T}_3(3)$ consists of all marked generalized cusps with non-diagonalizable holonomy. Let $\pi : \mathcal{P}(\mathbb{R}^2) \times \mathrm{S}^3\,\mathbb{R} \to \mathcal{P}(\mathbb{R}^2)$ be projection and consider the subspace $N = B \cap (\pi \circ J)^{-1}\beta_0 \subset \mathcal{T}_3$ of non-diagonalizable holonomies for the standard quadratic form $\beta_0 = \|\cdot\|^2$.

Claim 2 ($J(N) = \{(\beta_0, h, r) : |r| = 3|h|\}$). If $[\rho] \in N$ then $[\rho] = [\Phi_{\lambda,0} \circ A]$ with $\mathbf{t}(\lambda) < 3$ and $A \in SO(2)$. Under the identification $V = \mathbb{C}$, the action of $\mathrm{SO}(2)$ on V is given by the action of $U(1)$ on $\mathbb{C}$. If $J(\Phi_{\lambda,0}) = (z, 3\overline{z})$, and A is rotation by θ, and $\omega = \exp(i\theta)$ then $J(\Phi_{\lambda,0} \circ A) = (\omega^3 z, 3\omega\overline{z})$.

Since $\lambda_1, \lambda_2 \geq 0$ then $z = \lambda_1 + i\lambda_2 = |z|\exp(i\phi)$ is arbitrary subject to $\phi \in [0, \pi/2]$. We need to show that given $h, r \in \mathbb{C}$ with $|r| = 3|h|$ there are $z, \omega \in \mathbb{C}$ with $|\omega| = 1$ and z as above such that $(h, r) = (\omega^3 z, 3\omega\overline{z})$.

Let $h = |h|\exp(i\alpha)$ and $r = 3|h|\exp(\beta)$ then $|z| = |h|$. We need $\alpha \equiv 3\theta + \phi$ and $\beta \equiv \theta - \phi$ where $\equiv$ means mod 2π. Thus $4\theta \equiv \alpha + \beta$. There is $\theta \in [0, \pi/2]$ with this property. The second equation implies $\phi \equiv \theta - \beta$. Rewriting the first equation gives

$$\alpha - 3\theta \equiv \alpha - 3\theta + (4\theta - \alpha - \beta) \equiv \theta - \beta \equiv \phi$$

The astute reader may notice that there are two solutions of $4\theta \equiv 0$ in $[0, \pi/2]$, and they give different values for ϕ. Thus ω and z are not always uniquely determined, even though h and r are. This proves Claim 2.

Using the action of $\mathrm{SL}\,V$ on $\mathcal{T}_3$ then B is the orbit of N so

$$J(B) = \{(\beta, c) \in \mathcal{P}(\mathbb{R}^2) \times \mathrm{S}^3\,\mathbb{R}^2 \ :\ |\pi_{\mathscr{R}} c| = 3|\pi_{\mathscr{H}} c|,\quad \beta \in \mathcal{P}\}$$

Consider $f : \mathcal{P} \times \mathbb{C}^2 \to \mathbb{R}$ given by $f(\beta, h, r) = 3|h| - |r|$, and set $P = \mathcal{T}_3(3)$. When $\lambda = (1,1,1)$ then (6.17) implies the cubic is harmonic so $r = 0$, and $h \neq 0$ thus $J(P)$ contains a point where $f > 0$. Since J is injective, $J(P) \subset \mathbb{C}^2 \setminus f^{-1}(0)$. By (4.8) P is connected, so $f \circ J(P) > 0$.

By (6.8) $J : \mathcal{T}_3 \to \mathcal{P} \times \mathbb{C}^2$ is proper, and the domain and codomain are locally compact, thus $J(\mathcal{T}_3)$ is closed. By (4.8), P is a 6-manifold without boundary. Since $J : P \to \mathcal{P} \times \mathbb{C}^2$ is an embedding, and $\mathcal{P} \times \mathbb{C}^2$ is a 6-manifold, $J(P)$ is open by invariance of domain. Hence $J(P) = f^{-1}(0, \infty)$. □

Proof of 1.7. Let $U \subset \mathrm{SL}(2, \mathbb{R})$ be the subspace of upper-triangular matrices with positive eigenvalues. Then $g : U \to \mathcal{P}$ given by $g(A) = A^t A$ is a homeomorphism. Let $G = g^{-1}$ and

$$\mathcal{C} = \{c \in \mathrm{S}^3\,\mathbb{R}^2 :\ |\pi_{\mathscr{R}} c| \leq 3|\pi_{\mathscr{H}} c|\ \} \equiv \{(r, h) \in \mathbb{C}^2 :\ |r| \leq 3|h|\ \}$$

Define $f : \mathcal{P} \times \mathcal{C} \to \mathcal{P} \times \mathrm{S}^3\,\mathbb{R}^2$ by $f(Q, c) = (Q, c \circ G(Q))$. The $\mathrm{SL}(2, \mathbb{R})$ orbit of $\mathcal{C} \times \{\beta_0\}$ is all of $\mathcal{T}(3)$, so the image of f is $J(\mathcal{T}_3)$. Hence f is an embedding, since it has inverse $f^{-1}(Q, c) = (Q, c \circ (G(Q))^{-1})$.

If $A = G(Q)$ then $f(Q, c) = (Q, c \circ A)$ and, using the action of $A \in \mathrm{SL}(2, \mathbb{R})$ on shape invariants,

$$A \cdot f(Q, c) = ((A^t)^{-1} Q A^{-1}, (c \circ A) \circ A^{-1}) = (I, c)$$

so $f^{-1} \circ J : \mathcal{T}_3 \to \mathcal{P} \times \mathcal{C}$ is a homeomorphism because J is an embedding. There is a homeomorphism $h : \mathcal{P} \to \mathcal{H} = \{z \in \mathbb{C} : \ \mathrm{Im}\, z > 0 \ \}$ given by $h(Q) = \alpha_A(i)$ where $A = G(Q)$ and α_A is the Möbius transformation corresponding to A. Then $\Theta = (h \times \mathrm{I}_{\mathcal{C}}) \circ f^{-1} \circ J : \mathcal{T}_3 \to \mathcal{H} \times \mathcal{C}$ is a homeomorphism. □

Now we describe the strata of $\mathcal{T}_3$. Let $\pi : \mathcal{T}_3 \to \mathcal{P}$ be projection. The fiber $\pi^{-1}(\beta_0)$ is the cone $F = \{(h, r) \in \mathbb{C}^2 : |r| \leq 3|h|\}$ stratified as follows. For $k \in \{0, 1, 2, 3\}$, let $T_k = \mathcal{T}_3(k) \cap \pi^{-1}(\beta_0)$. Then $T_0 = (0, 0) \in \mathbb{C}^2$ is the cone point, and $T_1 = \{(w^3 |w|^{-2}, 3w) : \ w \in \mathbb{C} \setminus 0\}$ is the open cone of a $(3, 1)$ curve in $S^1 \times S^1$. In this case c is the cube of a linear polynomial. Also $T_2 = \partial F - (T_1 \cup T_0)$, and $T_3 = \mathrm{int}(F)$. The stratification is preserved by the action of $\mathrm{SL}(2, \mathbb{R})$ which also preserves the fibering and acts transitively on the base space $\mathcal{P}$.

8. Deferred proofs

Proof of 2.11. The character and Lie-algebra weights can be read off from Definition 2.4. To compute β we use (5) with basepoint $b = (e_1 + \cdots + e_{\mathbf{t}}) + e_{n+1}$. When $\mathbf{r} = \mathbf{t} < n$ from (2.4)

$$\mu_{\theta,b}(v) - b = \sum_{i=1}^{\mathbf{t}} (\exp(\psi_{\mathbf{t}} v_i) - 1) e_i + \sum_{i=\mathbf{t}+1}^{n-1} v_i e_{i+1} + \left(- \sum_{i=1}^{\mathbf{t}} \psi_i v_i + (1/2) \sum_{i=\mathbf{t}+1}^{n-1} v_i^2 \right) e_{\mathbf{t}+1}$$

Computing $u_i = (\partial \mu_{\theta,b} / \partial v_i)_{v=0}$ gives

$$(u_1, \cdots, u_{n-1}) = (\psi_{\mathbf{t}} e_1 - \psi_1 e_{\mathbf{t}+1}, \psi_{\mathbf{t}} e_2 - \psi_2 e_{\mathbf{t}+1}, \cdots, \psi_{\mathbf{t}} e_{\mathbf{t}} - \psi_{\mathbf{t}} e_{\mathbf{t}+1}, e_{\mathbf{t}+2}, \cdots, e_n)$$

By (6) $h_\theta(v) = \pm\det(u_1, \cdots, u_{n-1}, \mu_{\theta,b}(v) - b)$ gives

$$h_\theta(v) = \pm\det\begin{pmatrix} \psi_{\mathbf{t}} & & & & & & & \exp(\psi_{\mathbf{t}}v_1) - 1 \\ & \psi_{\mathbf{t}} & & & & & & \exp(\psi_{\mathbf{t}}v_2) - 1 \\ & & \ddots & & & & & \vdots \\ & & & \psi_{\mathbf{t}} & & & & \exp(\psi_{\mathbf{t}}v_{\mathbf{t}}) - 1 \\ -\psi_1 & -\psi_2 & \cdots & -\psi_{\mathbf{t}} & 0 & \cdots & 0 & -\sum_{i=1}^{\mathbf{t}} \psi_i v_i + (1/2)\sum_{\mathbf{t}+1}^{n-1} v_i^2 \\ & & & & 1 & & & v_{\mathbf{t}+1} \\ & & & & & \ddots & & \vdots \\ & & & & & & 1 & v_{n-1} \end{pmatrix}$$

$$= \det\begin{pmatrix} \psi_{\mathbf{t}} & & & & \exp(\psi_{\mathbf{t}}v_1) - 1 \\ & \psi_{\mathbf{t}} & & & \exp(\psi_{\mathbf{t}}v_2) - 1 \\ & & \ddots & & \vdots \\ & & & \psi_{\mathbf{t}} & \exp(\psi_{\mathbf{t}}v_{\mathbf{t}}) - 1 \\ -\psi_1 & -\psi_2 & \cdots & -\psi_{\mathbf{t}} & -\sum_{i=1}^{\mathbf{t}} \psi_i v_i + (1/2)\sum_{\mathbf{t}+1}^{n-1} v_i^2 \end{pmatrix}$$

$$= {\psi_{\mathbf{t}}}^{\mathbf{t}-1}\sum_{i=1}^{\mathbf{t}} \psi_i(\exp(\psi_{\mathbf{t}}v_i) - 1 - \psi_{\mathbf{t}}v_i) + (1/2){\psi_{\mathbf{t}}}^{\mathbf{t}}\sum_{i=\mathbf{t}+1}^{n-1} v_i^2$$

Taking the second derivative of $h_\theta(v)$ at $v = 0$ yields

$$\widetilde{\beta} = \psi_{\mathbf{t}}^{\mathbf{t}+1}\sum_{i=1}^{\mathbf{t}} \psi_i dv_i^2 + {\psi_{\mathbf{t}}}^{\mathbf{t}}\sum_{i=\mathbf{t}+1}^{n-1} dv_i^2$$

Observe that the matrix of $\beta' = {\psi_{\mathbf{t}}}^{-(\mathbf{t}+1)}\widetilde{\beta}$ is diagonal in the standard basis and $\det\beta'$ is as claimed. From this it is clear that the Lie algebra weights ξ_i are pairwise β^*-orthogonal. Thus their duals are pairwise β-orthogonal.

By Definition 2.4, the non-zero Lie algebra weights are $\xi_i = \psi_{\mathbf{t}}e_i^*$ with $1 \le i \le \mathbf{t}$. Now $\langle x, e_i\rangle_{\beta'} = \psi_i e_i^*(x)$. Let $\gamma = \det(\beta')^{-1/n-1}$, then $\beta = \gamma\cdot\beta'$ so $\langle x, e_i\rangle_\beta = \gamma\psi_i e_i^*(x)$. Thus the dual of $\xi_i \in V^*$ with respect to β is

$$\xi_i^* = (\gamma\psi_i)^{-1}\psi_{\mathbf{t}}e_i \in V \tag{24}$$

and

$$\begin{aligned} \|\xi_i\|_{\beta^*} &= \beta^*(\xi_i) \\ &= \beta((\gamma\psi_i)^{-1}\psi_{\mathbf{t}}e_i) \\ &= ((\gamma\psi_i)^{-1}\psi_{\mathbf{t}})^2\beta(e_i) \\ &= (\gamma^{-2}\psi_i^{-2}\psi_{\mathbf{t}}^2)\gamma\beta'(e_i) \\ &= (\gamma^{-2}\psi_i^{-2}\psi_{\mathbf{t}}^2)\gamma\psi_i \end{aligned} \tag{25}$$

$$= \gamma^{-1}\psi_{\mathbf{t}}^2\psi_i^{-1} \tag{26}$$

If $\mathbf{t} = n$ choose basepoint $b = e_1 + \cdots + e_{n+1}$; then

$$\mu_{\theta,b}(v) - b = \sum_{i=1}^{n-1}(\exp(\psi_n v_i) - 1)e_i + \left(\exp\left(-\sum_{i=1}^{n-1}\psi_i v_i\right) - 1\right)e_n$$

thus $u_i = (\partial\mu_{\theta,b}/\partial v_i)_{v=0} = \psi_n e_i - \psi_i e_n$. Then (6) gives

$$h_\theta(v) = \det \begin{pmatrix} \psi_n & & & & \exp(\psi_n v_1) - 1 \\ & \ddots & & & \vdots \\ & & & \psi_n & \exp(\psi_n v_{n-1}) - 1 \\ -\psi_1 & -\psi_2 & \cdots & -\psi_{n-1} & \exp(-\sum_{i=1}^{n-1} \psi_i v_i) - 1 \end{pmatrix}$$
$$= \psi_n^{n-2} \sum_{i=1}^{n-1} \psi_i \left(\exp(\psi_n v_i) - 1\right) + \psi_n^{n-1} \left(\exp\left(- \sum_{i=1}^{n-1} \psi_i v_i \right) - 1 \right)$$

Taking the second derivative at $v = 0$ gives

$$\widetilde{\beta} = \psi_n^n \left(\sum_{i=1}^{n-1} \psi_i dv_i^2 + \psi_n^{-1} \left(- \sum_{i=1}^{n-1} \psi_i dv_i \right)^2 \right)$$

Then $\beta' = \psi_n^{-n} \widetilde{\beta}$ gives the form shown in the proposition. Now

$$\psi_n \beta' = \begin{pmatrix} \psi_1(\psi_n + \psi_1) & \psi_1\psi_2 & \cdots & & \psi_1\psi_{n-1} \\ \psi_2\psi_1 & \psi_2(\psi_n + \psi_2) & \psi_2\psi_3 & \cdots & \psi_2\psi_{n-1} \\ \vdots & & & & \\ \psi_{n-1}\psi_1 & \cdots & & \psi_{n-1}\psi_{n-2} & \psi_{n-1}(\psi_n + \psi_{n-1}) \end{pmatrix}$$

The determinant of this matrix is a polynomial of degree $2(n-1)$. Row i has a factor of ψ_i. The sum of the rows is a multiple of $\psi_1 + \psi_2 \cdots + \psi_n$. Setting $\psi_n = 0$ gives a matrix of rank 1 so ψ_n^{n-2} is a factor. Hence

$$\det(\psi_n \beta') = \alpha \psi_1 \cdots \psi_{n-1} \psi_n^{n-2} (1 + \psi_1 + \cdots \psi_{n-1})$$

for some constant α. Equating coefficients of ψ_n^{n-1} gives $\alpha = 1$. Thus

$$\det \beta' = \psi_1 \cdots \psi_{n-1} \psi_n^{-1} (1 + \psi_1 + \cdots \psi_{n-1})$$

□

Proof of 3.5. (a) Given $v = (v_1, \cdots, v_{n-1}) \in V$ define $v_0 \in V$ by $\lambda_0^{-1} v_0 + \cdots \lambda_{n-1}^{-1} v_{n-1} = 0$. Let

$$P = \begin{pmatrix} 1 & -\lambda_1^{-1} & \cdots & -\lambda_{n-1}^{-1} & \lambda_0^{-2} \\ 0 & 1 & 0 & 0 & \lambda_1^{-1} \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & \lambda_{n-1}^{-1} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} \lambda_0 v_0 & 0 & \cdots & & 0 \\ 0 & \lambda_1 v_1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \lambda_{n-1} v_{n-1} & 0 \\ 0 & \cdots & & & 0 \end{pmatrix}$$

then

$$P^{-1} r P = \begin{pmatrix} 0 & v_1 & \cdots & v_{n-1} & 0 \\ 0 & \lambda_1 v_1 & 0 & \cdots & v_1 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & \lambda_{n-1} v_{n-1} & v_{n-1} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} + \lambda_0 v_0 \begin{pmatrix} 1 & -\lambda_1^{-1} & \cdots & -\lambda_{n-1}^{-1} & 0 \\ 0 & \cdots & & & 0 \\ \vdots & & & & \vdots \\ & & & & \\ 0 & \cdots & & & 0 \end{pmatrix}$$

Now $\lambda_0^{-1}v_0 = -\lambda_1^{-1}v_1 \cdots - \lambda_{n-1}^{-1}v_{n-1}$ so $v_0 = -(\kappa_1 v_1 + \cdots + \kappa_{n-1}v_{n-1}) = -\langle v, \kappa\rangle$ where $\kappa_i = \lambda_0/\lambda_i$. Then

$$P^{-1}rP = \begin{pmatrix} 0 & v_1 & \cdots & v_{n-1} & 0 \\ 0 & \lambda_1 v_1 & 0 & \cdots & v_1 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & \lambda_{n-1}v_{n-1} & v_{n-1} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} + \langle v, \kappa\rangle \begin{pmatrix} -\lambda_0 & \kappa_1 & \cdots & \kappa_{n-1} & 0 \\ 0 & \cdots & & & 0 \\ \vdots & & & & \vdots \\ & & & & \\ 0 & \cdots & & & 0 \end{pmatrix}$$
$$= \phi_{\lambda,\kappa}(v)$$

Set $R = \exp r$ then $\Phi_{\lambda,\kappa} = P^{-1}RP$. From (2.4) using

$$\mathfrak{f}(v_1, \cdots, v_{n-1}) = \lambda_0^2(\lambda_1 v_1, \cdots, \lambda_{n-1}v_{n-1}),$$

$$\zeta_\psi(\mathfrak{f}v) = \exp \begin{pmatrix} \psi_n\lambda_0^2\lambda_1 v_1 & 0 & \cdots & & & 0 \\ 0 & \psi_n\lambda_0^2\lambda_2 v_2 & 0 & \cdots & & 0 \\ 0 & 0 & \ddots & 0 & \vdots & \\ \vdots & \vdots & 0 & \psi_n\lambda_0^2\lambda_{n-1}v_{n-1} & & 0 \\ & & & & -\sum_{i=1}^{n-1}\psi_i\lambda_0^2\lambda_i v_i & 0 \\ 0 & \cdots & & & & 0 \end{pmatrix}$$

Using $\psi_n\lambda_0^2 = 1$ and $\psi_i\lambda_i = \lambda_i^{-1}$ gives

$$\zeta_\psi(\mathfrak{f}v) = \exp \begin{pmatrix} \lambda_1 v_1 & 0 & \cdots & & & 0 \\ 0 & \lambda_2 v_2 & 0 & \cdots & & 0 \\ 0 & 0 & \ddots & & & \vdots \\ \vdots & \vdots & 0 & \lambda_{n-1}v_{n-1} & & 0 \\ & & & & -\lambda_0^2\sum_{i=1}^{n-1}\lambda_i^{-1}v_i & 0 \\ 0 & \cdots & & & & 0 \end{pmatrix}$$

By the definition of v_0 above we have $-\lambda_0^2\sum_{i=1}^{n-1}\lambda_i^{-1}v_i = \lambda_0 v_0$. Let $M \in \mathrm{GL}(n+1, \mathbb{R})$ be defined by

$$M(x_1, \cdots, x_{n+1}) = (x_n, x_1, \cdots, x_{n-1}, x_{n+1})$$

Then

$$M^{-1}\zeta_\psi(\mathfrak{f}v)M = \exp \begin{pmatrix} \lambda_0 v_0 & 0 & \cdots & & 0 \\ 0 & \lambda_1 v_1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & 0 & \lambda_{n-1}v_{n-1} & 0 \\ 0 & \cdots & & & 0 \end{pmatrix} = R$$

so

$$M^{-1}(\zeta_\psi \circ \mathfrak{f})M = R = P\Phi_{\lambda,\kappa}P^{-1} \tag{27}$$

Set $Q = MP$; then $Q\Phi_{\lambda,\kappa}Q^{-1} = \zeta_\psi \circ \mathfrak{f}$ as asserted.

To prove (b) we exploit the fact that every $\Phi_{\lambda,\kappa}$ is a limit of the diagonalizable ones above. Given an integer $k \geq 0$ define $f_k : \mathbb{R}^2 \to \mathbb{R}$ by

$$f_k(s,t) = \sum_{j=k}^{\infty} s^{j-k}t^j/j!$$

This is analytic and $f_0(s,t) = \exp(st)$, and for $s \neq 0$

$$f_1(s,t) = s^{-1}(e^{st} - 1), \qquad f_2(s,t) = s^{-2}(e^{st} - 1 - st)$$

Also $f_1(0,t) = t$ and $f_2(0,t) = t^2/2$. For $s \geq 0$ the map $f_1(s,-) : \mathbb{R} \to (-s^{-1}, \infty)$ is a diffeomorphism when we interpret $-0^{-1} = -\infty$, and $f_2(s,-) : \mathbb{R} \to \mathbb{R}$ is convex and proper. Then

$$P^{-1}RP = \begin{pmatrix} e^{\lambda_0 v_0} & * & \cdots & * & \sum_{i=0}^{n-1} \lambda_i^{-1} f_1(\lambda_i, v_i) \\ 0 & e^{\lambda_1 v_1} & 0 \cdots & 0 & f_1(\lambda_1, v_1) \\ 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & e^{\lambda_{n-1} v_{n-1}} & f_1(\lambda_{n-1}, v_{n-1}) \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

Set $x_i = f_1(\lambda_i, v_i)$ and $y = \sum_{i=0}^{n-1} \lambda_i^{-1} f_1(\lambda_i, v_i)$. Write the last column of $P^{-1}RP$ as $(y, x_1, \cdots, x_{n-1}, 1)^T$. Now

$$\lambda_0^{-1} v_0 = -(\lambda_1^{-1} v_1 + \cdots + \lambda_{n-1}^{-1} v_{n-1}) \tag{28}$$

Observe that $s^{-1} f_1(s,t) = f_2(s,t) + s^{-1} t$. Thus

$$\lambda_0^{-1} f_1(\lambda_0, v_0) = f_2(\lambda_0, v_0) + \lambda_0^{-1} v_0 = f_2(\lambda_0, v_0) - \sum_{i=1}^{n-1} \lambda_i^{-1} v_i \tag{29}$$

Then

$$\begin{aligned} y &= \sum_{i=0}^{n-1} \lambda_i^{-1} f_1(\lambda_i, v_i) \\ &= \lambda_0^{-1} f_1(\lambda_0, v_0) + \sum_{i=1}^{n-1} \lambda_i^{-1} f_1(\lambda_i, v_i) \\ &= \left(f_2(\lambda_0, v_0) - \sum_{i=1}^{n-1} \lambda_i^{-1} v_i \right) + \sum_{i=1}^{n-1} \left(f_2(\lambda_i) + \lambda_i^{-1} v_i \right) \qquad \textit{using (29)} \\ &= \sum_{i=0}^{n-1} f_2(\lambda_i, v_i) \end{aligned} \tag{30}$$

The orbit of the origin under $T(\lambda, \kappa)$ is a hypersurface $S = S(\lambda, \kappa)$ in $\mathbb{R}^n$ that is the locus of the points $(y, x_1, \cdots, x_{n-1})$ as v varies in V. Solving $x = f_1(\ell, v)$ for v gives

$$v = h(\ell, x) := \ell^{-1} \log(1 + \ell x) \tag{31}$$

This defines $h(\ell, x)$ whenever $1 + \ell x > 0$ and $\ell \neq 0$. Observe that $h(\ell, x) = x + \ell \cdot \mathrm{O}(x^2)$, so if we define $h(0, x) = x$ then h is analytic on the subset of $\mathbb{R}^2$ where $1 + \ell x > 0$. Define $g(\ell, x) = \ell^{-2}(\ell x - \log(1 + \ell x))$ for $1 + \ell x > 0$ and $\ell \neq 0$. Observe that $g(\ell, x) = x^2/2 + \mathrm{O}(x^3)$, thus if we define $g(0, x) = x^2/2$, then g is analytic for

$1 + \ell x > 0$. Then

$$\begin{aligned} f_2(\ell, v) &= f_2(\ell, \ell^{-1}\log(1+\ell x)) \\ &= \ell^{-2}(e^{\ell\ell^{-1}\log(1+\ell x)} - 1 - \log(1+\ell x)) \\ &= \ell^{-2}(\ell x - \log(1+\ell x)) \\ &= g(\ell, x) \end{aligned} \tag{32}$$

The hypersurface $S = S(\lambda, \kappa)$ is given by

$$\begin{aligned} y &= \sum_{i=0}^{n-1} f_2(\lambda_i, v_i) && \text{by (30)} \\ &= f_2\left(\lambda_0, -\lambda_0 \sum_{i=1}^{n-1} \lambda_i^{-1} v_i\right) + \sum_{i=1}^{n-1} g(\lambda_i, x_i) \\ &\qquad \because\ v_0 = -\lambda_0 \sum_{i=1}^{n-1} \lambda_i^{-1} v_i \ \&\ f_2(\lambda_i, v_i) = g(\lambda_i, x_i) \\ &= f_2\left(\lambda_0, -\sum_{i=1}^{n-1} \kappa_i h(\lambda_i, x_i)\right) + \sum_{i=1}^{n-1} g(\lambda_i, x_i) \\ &\qquad \because\ v_i = h(\lambda_i, x_i) \ \&\ \kappa_i = \lambda_0/\lambda_i \\ &=: F(\lambda, \kappa, x) && \text{definition} \end{aligned}$$

Here $x = (x_1, \cdots, x_{n-1})$. Up to this point we have assumed $(\lambda, \kappa) \in D_n$ so every $\lambda_i > 0$. However the function F is defined and analytic whenever $(\lambda, \kappa) \in \widetilde{A}_n \cup D_n$ and $1 + \lambda_i x_i > 0$ for all i. It follows that $y = F(\lambda, \kappa, x)$ defines a hypersurface $S(\lambda, \kappa)$ for each $(\lambda, \kappa) \in \widetilde{A}_n$.

Also $S(\lambda, \kappa)$ is the orbit of 0 under $T(\lambda, \kappa)$ whenever $(\lambda, \kappa) \in D_n$. Since $\widetilde{A}_n \subset \operatorname{cl} D_n$ and $\Phi_{\lambda,\kappa}$ is a continuous function of (λ, κ) it follows that $S(\lambda, \kappa)$ is the orbit of 0 under $T(\lambda, \kappa)$ whenever $(\lambda, \kappa) \in \widetilde{A}_n$. For fixed (λ, κ)

$$\begin{aligned} h(\lambda_i, x_i) &= x_i + \mathrm{O}(x^2) && \text{by (31)} \\ \therefore \quad \sum_{i=1}^{n-1} \kappa_i h(\lambda_i, x_i) &= \sum_{i=1}^{n-1} \kappa_i (x_i + \mathrm{O}(x_i^2)) \\ &= \langle \kappa, x \rangle + \mathrm{O}(\|x\|^2) \end{aligned}$$

Using this and $f_2(\lambda, x) = x^2/2 + \mathrm{O}(x^3)$ gives

$$f_2\left(\lambda_0, -\sum_{i=1}^{n-1} \kappa_i h(\lambda_i, x_i)\right) = (1/2)\langle \kappa, x \rangle^2 + \mathrm{O}(\|x\|^3)$$

Also

$$\begin{aligned} g(\lambda_i, x_i) &= x_i^2/2 + \mathrm{O}(x_i^3) && \text{by (32)} \\ \therefore \quad F(\lambda, \kappa, x) &= f_2\left(\lambda_0, -\sum_{i=1}^{n-1} \kappa_i h(\lambda_i, x_i)\right) + \sum_{i=1}^{n-1} g(\lambda_i, x_i) && \text{by definition above} \\ &= (1/2)\left(\langle \kappa, x \rangle^2 + \|x\|^2\right) + \mathrm{O}(\|x\|^3) \end{aligned}$$

It follows that $S(\lambda,\kappa)$ is strictly convex at 0. Since $T(\lambda,\kappa)$ acts transitively by affine maps $S(\lambda,\kappa)$ is strictly convex everywhere. One checks that $F(\lambda,\kappa,x)$ is a proper function of $x \in \{(x_1,\cdots,x_{n-1}) \ : \ 1+\lambda_i x_i > 0\}$ for fixed λ,κ. Hence $S(\lambda,\kappa)$ is properly embedded, and therefore bounds a convex domain $\Omega(\lambda,\kappa) \subset \mathbb{R}^n$ that is preserved by $T(\lambda,\kappa)$.

By (31) $v_i = x_i + \mathrm{O}(x_i^2)$; thus $\Phi_{\lambda,\kappa}(v) = (y, x_1, \cdots, x_{n-1})$ where

$$y = (1/2)\left(\langle\kappa, v\rangle^2 + \|v\|^2\right) + \mathrm{O}(\|v\|^3)$$

which gives the formula for the metric β'. The formula for the character $\chi_{\lambda,\kappa}$ follows immediately from Definition 3.4 as the sum of the exponentials of the diagonal terms. It only remains to compute $\det\beta'$. Now

$$\beta'(v) = \langle\kappa, v\rangle^2 + \|v\|^2$$

Choose an orthonormal basis with respect to $\|\cdot\|^2$ of V that contains $\kappa/\|\kappa\|$. In this basis β' is diagonal, and the only diagonal entry that is not 1 is $1 + \langle\kappa, \kappa/\|\kappa\|\rangle^2 = 1 + \|\kappa\|^2$. Hence $\det\beta' = 1 + \|\kappa\|^2$. □

Proof of 6.6. Suppose $(\lambda,\kappa) \in A_n$. First consider the diagonalizable case. By (3.5), $\Phi_{\lambda,\kappa}$ is conjugate to $\zeta_\psi \circ \mathfrak{f}$, where $\psi_n = \lambda_0^{-2}$ and $\psi_i = \lambda_i^{-2}$ for $1 \le i \le n-1$. This defines a linear map $\psi : \mathbb{A} \to \mathbb{R}$ and we have $\mathfrak{g} = \ker\psi$. Since J is an invariant of conjugacy classes, we may replace $\Phi_{\lambda,\kappa}$ by $\zeta_\psi \circ \mathfrak{f}$. In this proof summation is over the integers from 1 to $n-1$. Consider the linear map $f : \mathbb{R}^{n-1} \to \mathbb{A}$ given by

$$x := f(v_1,\cdots,v_{n-1}) = \left(\psi_n v_1, \cdots, \psi_n v_{n-1}, -\sum \psi_i v_i\right)$$

Then $\mathfrak{g} = \mathrm{Im}\, f = \ker\psi$. Recall $\mathfrak{f}(v_1,\cdots,v_{n-1}) = \lambda_0^2(\lambda_1 v_1, \cdots, \lambda_{n-1} v_{n-1})$. Thus

$$\begin{aligned} f\circ\mathfrak{f}(v) &= \lambda_0^2(\psi_n\lambda_1 v_1, \cdots, \psi_n\lambda_{n-1}v_{n-1}, -\sum \psi_i\lambda_i v_i) \\ &= (\lambda_1 v_1, \cdots, \lambda_{n-1}v_{n-1}, -\lambda_0^2 \sum \lambda_i^{-1} v_i) && \because \lambda_0^2\psi_n = 1 \ \text{ by (3.5)(a)} \\ &= (\lambda_1 v_1, \cdots, \lambda_{n-1}v_{n-1}, -\lambda_0 \sum \kappa_i v_i) && \because \lambda_0\lambda_i^{-1} = \kappa_i \ \text{ by (14)} \end{aligned} \tag{33}$$

It follows from Definition 2.4 that $\zeta_\psi = \delta \circ \exp \circ f$. The calibration, J_ψ, on $\mathfrak{g}$ is given by (6.5)

$$J_\psi(x) = \ (1/2)\langle p, x^2\rangle_\psi + (1/6)\langle p, x^3\rangle_\psi \tag{34}$$

The calibration $J = J(\zeta_\psi \circ \mathfrak{f}) = J(\delta \circ \exp \circ f \circ \mathfrak{f})$. By (6.5) $J_\psi = J(\delta \circ \exp)$, so $J = J_\psi \circ f \circ \mathfrak{f}$. The calibration J on V is obtained from this factorization by using (33) to substitute $x = f(\mathfrak{f}v)$ into (34).

$$\begin{aligned} \langle p, x^2\rangle_\psi &= \sum \psi_i(\lambda_i v_i)^2 + \psi_n\left(-\lambda_0 \sum \kappa_i v_i\right)^2 \\ &= \sum v_i^2 + \left(\sum \kappa_i v_i\right)^2 && \because \psi_i\lambda_i^2 = 1 \ \&\ \psi_n\lambda_0^2 = 1 \end{aligned}$$

Let $\langle\cdot,\cdot\rangle$ denote the standard inner product on $\mathbb{R}^{n-1}$; then

$$\langle p, x^2\rangle_\psi \ = \ \langle v, v\rangle + \langle v, \kappa\rangle^2 \tag{35}$$

and

$$\begin{aligned} \langle p, x^3\rangle_\psi &= \sum \psi_i(\lambda_i v_i)^3 + \psi_n\left(-\lambda_0 \sum \kappa_i v_i\right)^3 \\ &= \sum \lambda_i v_i^3 - \lambda_0\langle\kappa, v\rangle^3 && \because \ \ \psi_i\lambda_i^2 = 1 \ \&\ \psi_n\lambda_0^2 = 1 \end{aligned} \tag{36}$$

Then (35) and (36) give

$$\begin{aligned} J(v) &= (1/2)\langle p, x^2\rangle_\psi + (1/6)\langle p, x^3\rangle_\psi \\ &= (1/2)\left(\langle v, v\rangle + \langle v, \kappa\rangle^2\right) + (1/6)\left(-\lambda_0\langle \kappa, v\rangle^3 + \sum \lambda_i v_i^3\right) \end{aligned}$$

This gives the result in the diagonalizable case.

By (3.12) in the non-diagonalizable case we may assume $\theta = \Phi_{\lambda,\kappa} = \exp\circ\phi_{\lambda,\kappa}$ with $\lambda_0 = 0$ and $\kappa = 0$. For $1 \le i \le n-1$ define $v_i' = v_i + \langle v, \kappa\rangle\kappa_i$. By (3.4) $\phi_{\lambda,\kappa}(v_1, \cdots, v_{n-1}) = D + N$ where

$$D = \begin{pmatrix} 0 & 0 & 0 & \cdots & & 0 \\ 0 & \lambda_1 v_1 & 0 & \cdots & & \\ \vdots & & \ddots & & & \vdots \\ & & & & \lambda_{n-1}v_{n-1} & 0 \\ 0 & & & & \cdots & 0 \end{pmatrix} \quad N = \begin{pmatrix} 0 & v_1' & v_2' & \cdots & v_{n-1}' & 0 \\ 0 & & 0 & \cdots & 0 & v_1 \\ \vdots & & \ddots & & & \vdots \\ & & & & & v_{n-1} \\ 0 & & & & \cdots & 0 \end{pmatrix}$$

Relabel the standard basis of $\mathbb{R}^{n+1}$ as $e_0, \cdots, e_n$. Then $\partial\Omega$ is the orbit in affine space $\mathbb{R}^n \oplus e_n \subset \mathbb{R}^{n+1}$ of $0 \oplus e_n$ under this group. We compute the series expansion for $\exp(D+N)e_n$ to degree 3.

$$\exp(D+N) = I + (D+N) + (1/2)(D+N)^2 + (1/6)(D+N)^3 + O(\|v\|^4)$$

Using that $De_n = 0$ and $N^3 = 0$ and $DN^2e_n = 0$ gives

(37)

$$\exp(D+N)e_n = \left(I + N + (1/2)(DN + N^2) + (1/6)(D^2N + NDN)\right)e_n + O(\|v\|^4)$$

In the following summation is over integers from 1 to $n-1$

$$\begin{aligned} Ne_n &= \sum v_i e_i, & N^2e_n &= \left(\|v\|^2 + \langle v, \kappa\rangle^2\right)e_0 \\ DNe_n &= \sum \lambda_i v_i^2 e_i, & NDNe_n &= \left(\sum \lambda_i v_i^3\right)e_0, & D^2Ne_n &= \sum \lambda_i^2 v_i^3 e_i \end{aligned}$$

The only term linear in v_i is Ne_n, so the supporting hyperplane to $\partial\Omega$ at 0 is the coordinate hyperplane $v_0 = 0$ in $\mathbb{R}^n$. Thus in the definition of J we may take the height function τ to be the v_0-coordinate, and it follows that J is the coefficient of e_0 in (37)

$$Je_0 = (1/2)N^2e_n + (1/6)NDNe_n = \left((1/2)\left(\|v\|^2 + \langle v, \kappa\rangle^2\right) + (1/6)\sum \lambda_i v_i^3\right)e_0$$

Since $\lambda_0 = 0$ in the non-diagonalizable case, this is the calibration $\vartheta_{\lambda,\kappa}$ in (6.2) as claimed. □

Proof of 6.7. We claim the formula for $c(\rho)$ holds when $\rho = \Phi_{\lambda,\kappa}$. By (6.6)

$$\varkappa J(\Phi_{\lambda,\kappa})(v) = \left(\langle v, v\rangle + \langle v, \kappa\rangle^2\right) + \frac{1}{3}\left(-\lambda_0\langle v, \kappa\rangle^3 + \sum_{i=1}^{n-1} \lambda_i v_i^3\right)$$

Then $J = q + c$, and the term $\varkappa$ defined in (5.1) ensures that q is unimodular. Thus

$$c := c(\rho) = \frac{1}{3\varkappa}\left(-\lambda_0\langle v, \kappa\rangle^3 + \sum_{i=1}^{n-1} \lambda_i v_i^3\right) \tag{38}$$

By (3.4) the weights of $\Phi_{\lambda,\kappa}$ are $\xi_i = \lambda_i e_i^*$ for $1 \le i \le n-1$ and ξ_0 given by $\xi_0(v) = -\lambda_0\langle v, \kappa\rangle$. Then (38) becomes:

$$3\varkappa c = \sum_{i=0}^{n-1} \lambda_i^{-2}\xi_i^3 \tag{39}$$

By (5.1) $\varkappa\lambda_i^2 = \langle \xi_i, \xi_i\rangle_{\beta^*} + \varpi$ and $\varpi = -\langle \xi_1, \xi_2\rangle_{\beta^*}$, so $\varkappa^{-1}\lambda_i^{-2} = (\langle \xi_i, \xi_i\rangle_{\beta^*} + \varpi)^{-1}$. This proves the claim.

If $A \in \mathrm{SL}^{\pm} V$ then $c(\rho \circ A) = c(\rho) \circ A$ and $\xi_i(\rho \circ A) = \xi_i(\rho) \circ A$. It follows that the formula holds for $\rho = \Phi_{\lambda,\kappa} \circ A$. Every marked translation group is conjugate to such ρ, and both sides are conjugacy invariants, so the formula for $c(\rho)$ holds in general.

The formula shows $\mathcal{K}$ is continuous. It only remains to show $\mathcal{K}$ is proper. Suppose $J(\rho_m)$ is a bounded sequence, then we must show

$$\kappa(\rho_m) = ([\xi_1(\rho_m), \cdots, \xi_n(\rho_m)], \beta(\rho_m))$$

is bounded. Now $\beta(\rho_m)$ and $c(\rho_m)$ are both bounded, so it only remains to prove the weights $\xi_i(\rho_m)$ are bounded. Suppose for a contradiction that some $\xi_i(\rho_m)$ is unbounded. We show that this implies $c(\rho_m)$ is not bounded, which gives a contradiction.

We may assume $\rho_m = \Phi^{\perp}_{\lambda(m),\kappa(m)} \circ B_m$ with $B_m \in \mathrm{SL}^{\pm} V$. The matrix of $\beta(\rho_m)$ in the standard basis of V is $B_m^t B_m$. Since this is bounded, B_m is bounded, and we may subsequence so B_m converges to $B_\infty \in \mathrm{SL}^{\pm} V$. It follows that the weights of $\Phi^{\perp}_{\lambda(m),\kappa(m)}$ are unbounded. Hence $\lambda(m) = (\lambda_{m,0}, \cdots, \lambda_{m,n-1})$ is unbounded. Since $\lambda_{m,i}$ increases with i it follows that $\lambda_{m,n-1} \to \infty$.

The matrix of $\beta(\Phi_{\lambda(m),\kappa(m)})$ is $\mathrm{I} + \kappa \otimes \kappa$ and $\kappa \in [0,1]^{n-1}$ is bounded. Thus the weights of $\Phi_{\lambda(m),\kappa(m)}$ are unbounded. The weights determine the cubic via (39) and $\xi_i = \lambda_i e_i^*$ for $1 \le i \le n-1$, and $\xi_0(v) = -\lambda_0\langle v, \kappa\rangle = -\lambda_0 \sum_{i=1}^{n-1} \lambda_0\lambda_i^{-1} v_i$. We now evaluate this cubic at the point $v_m = e_{n-1} - t_m(e_1 + \cdots + e_{n-2}) \in V$ where t_m is chosen so that $\langle \kappa, v\rangle = 0$. This simplifies the first summand in (39) to $\xi_0(v_m) = 0$. If $\mathbf{t} < n$ then we may choose $\kappa = 0$ and $t_m = 0$.

If $\mathbf{t} = n$ since $\lambda_{m,n-1} \ge \lambda_{m,i}$ for all i and $\kappa_i = \lambda_0\lambda_i^{-1}$ it follows that $\kappa_{n-1} \le \kappa_i$ for all i. Now t_m is determined by

$$0 = \langle \kappa, v\rangle = \kappa_{n-1} - t_m \sum_{i=1}^{n-2} \kappa_i \qquad \& \ \kappa_i \ge 0$$

implies $0 < t_m \le 1/(n-2)$. In what follows we omit the subscript m from $\lambda_{m,i}$. Setting $v = v_m$ in (39) and recalling that $\xi_i = \lambda_i e_i^*$ gives

$$3\varkappa c(\Phi_{\lambda,\kappa})(v_m) = \left(\sum_{i=1}^{n-2} \lambda_i^{-2}(-\lambda_i t_m)^3\right) + \lambda_{n-1}^{-2}(\lambda_{n-1})^3 = \lambda_{n-1} - t_m^3 \sum_{i=1}^{n-2} \lambda_i \tag{40}$$

Since $\lambda_i \le \lambda_{n-1}$, and $t_m \le 1/(n-2)$,

$$3\varkappa c(\Phi_{\lambda,\kappa})(v_m) \ge \lambda_{n-1}(1 - (n-2)/(n-2)^3) \tag{41}$$

Since $0 \le \kappa_i \le 1$ it follows that $\|\kappa\| \le n-1$; thus $\varkappa = (1 + \|\kappa\|^2)^{1/(n+1)} \le 1 + n^2$. If $n > 3$ then $(1 - (n-2)/(n-2)^3) > 0$. Using $\lambda_{n-1} \to \infty$ as $m \to \infty$ and $\varkappa$ is bounded, it follows that $c(\Phi_{\lambda,\kappa})$ is unbounded, a contradiction.

This leaves the case that $n = 3$ then (38) gives

$$3\varkappa c(\Phi_{\lambda,\kappa})(x,y) = \lambda_1 x^3 + \lambda_2 y^3 + \lambda_0^{-2}(-\lambda_0((\lambda_0/\lambda_1)x + (\lambda_0/\lambda_2)y))^3$$

The coefficients of this cubic are

$$(\lambda_2^4 - \lambda_0^4)\lambda_2^{-3}, \quad -3\lambda_0^4/(\lambda_1\lambda_2^2), \quad -3\lambda_0^4/(\lambda_1^2\lambda_2), \quad (\lambda_1^4 - \lambda_0^4)\lambda_1^{-3}$$

Since c is bounded we may assume each coefficient has absolute value at most b. The second term gives $\lambda_0^4 \le b \cdot \lambda_1\lambda_2^2$. Then $\lambda_2^4 - \lambda_0^4 \ge \lambda_2^4 - b \cdot \lambda_1\lambda_2^2$. But $\lambda_2 \ge \lambda_1$ so

$$\lambda_2^4 - \lambda_0^4 \ge \lambda_2^4 - b \cdot \lambda_1\lambda_2^2 \ge \lambda_2^4 - b \cdot \lambda_2^3 = \lambda_2^3(\lambda_2 - b)$$

Hence $\lambda_2 - b \le (\lambda_2^4 - \lambda_0^4)\lambda_2^{-3} \le b$, so $\lambda_2 \le 2b$. Since $\lambda_i \le \lambda_2$ for $i = 0, 1$ it follows that all the $\lambda_i \le 2b$. This is a contradiction. Hence $\mathcal{K}$ is proper. □

Proof of 6.9. Let $L = \{s \cdot e_i : 1 \le i \le n, \quad s > 0\}$ be the set of positive coordinate axes in $\mathbb{R}^n$ and $\pi : \mathbb{R}^n \to \mathfrak{g}$ orthogonal projection with respect to $\langle \cdot, \cdot \rangle_\psi$. We will show that the local maxima of $(c|S)$ are the points $(\pi L) \cap S$.

Write $J = J_\psi$. Since $q|S = 1$ it follows that $J|S = 1 + c|S$. First we find the critical points of $J|S$. The derivative of J at $v \in \mathbb{R}^n$ is

$$dJ_v(w) = \langle v, w \rangle_\psi + (1/2)\langle v^2, w \rangle_\psi \tag{42}$$

If $v \in S$ then $w \in T_v S$ if and only if $\langle v, w \rangle_\psi = 0$ and $\langle p, w \rangle_\psi = 0$. Thus v is a critical point of $J|S$ if and only if

$$\forall w \in \mathbb{R}^n \qquad (\langle v, w \rangle_\psi = 0 \quad \text{and} \quad \langle p, w \rangle_\psi = 0) \quad \Longrightarrow \quad \langle v^2, w \rangle_\psi = 0$$

This is equivalent to

$$\exists\ \alpha, \beta \in \mathbb{R} \qquad v^2 = \alpha v + \beta p$$

Writing $v = (x_1, \cdots, x_n)$ then each x_i is a solution of $t^2 = \alpha t + \beta$. Let $\mathfrak{s}_\pm$ be the two solutions of this quadratic and set

$$A_\pm = \{i : \ 1 \le i \le n \quad v_i = \mathfrak{s}_\pm\}$$

Thus $\{A_+, A_-\}$ is a partition of $\{1, \cdots, n\}$ and $i \in A_+$ if and only if $v_i = \mathfrak{s}_+$. Let $e_1, \cdots, e_n$ be the standard basis of $\mathbb{R}^n$ and define

$$e_\pm = \sum_{i \in A_\pm} e_i \qquad \text{so} \qquad p = e_+ + e_- \tag{43}$$

then

$$v = v(A_+) = \mathfrak{s}_+ e_+ + \mathfrak{s}_- e_- \tag{44}$$

The standard basis is orthogonal so $\langle e_+, e_- \rangle_\psi = 0$. Now $v \in p^\perp$ implies

$$0 = \langle p, v \rangle_\psi = \langle e_+ + e_-, \mathfrak{s}_+ e_+ + \mathfrak{s}_- e_- \rangle_\psi = \mathfrak{s}_+ \langle e_+, e_+ \rangle_\psi + \mathfrak{s}_- \langle e_-, e_- \rangle_\psi$$

Since $\langle e_+, e_+ \rangle_\psi, \langle e_-, e_- \rangle_\psi > 0$ it follows that $\mathfrak{s}_+\mathfrak{s}_- < 0$. We choose the labelling so that

$$\mathfrak{s}_+ > 0 \qquad \text{and} \qquad \mathfrak{s}_- < 0 \tag{45}$$

Then there is $t > 0$ so that

$$\mathfrak{s}_+ = t \cdot \langle e_-, e_- \rangle_\psi \qquad \mathfrak{s}_- = -t \cdot \langle e_+, e_+ \rangle_\psi$$

Hence

$$t^{-1} \cdot v = \langle e_-, e_- \rangle_\psi e_+ - \langle e_+, e_+ \rangle_\psi e_-$$

We will ignore the t factor in what follows. This is justified by observing that the critical points of J restricted to $t \cdot S$ are the critical points of $J|S$ multiplied by t. Then

$$\mathfrak{s}_+ = \langle e_-, e_- \rangle_\psi \qquad \mathfrak{s}_- = -\langle e_+, e_+ \rangle_\psi \qquad 0 = -\langle e_+, e_- \rangle_\psi \tag{46}$$

$$\mathfrak{s}_+ = \sum_{i \in A_-} \psi_i, \qquad \mathfrak{s}_- = -\sum_{i \in A_+} \psi_i, \qquad \text{and} \qquad \mathfrak{s} = \mathfrak{s}_+ - \mathfrak{s}_- = \sum_{i=1}^{n} \psi_i = \langle p, p \rangle_\psi$$

We have shown the critical points of $J|S$ are in one to one correspondence with the non-empty subsets $A_+ \subset \{1, \cdots, n\}$ with non-empty complement. Given a quadratic form Q define $\mu(Q)$ to be the dimension of the positive eigenspace. This is the *Morse index* of $-Q$. Thus a non-degenerate critical point is a local maximum if and only if the Hessian has $\mu = 0$.

Claim 1. The critical point of $f = J|S$ at $v = v(A_+)$ is non-degenerate, and $\mu(d^2 f_v) = |A_+| - 1$.

Assuming this we prove the lemma. The claim implies the local maxima occur when $|A_+| = 1$ so $A_+ = \{i\}$ for some $1 \leq i \leq n$. When $A_+ = \{i\}$ by (43)

$$e_+ = e_i \qquad e_- = p - e_i$$

By (46)

$$\mathfrak{s}_+ = \mathfrak{s} - \langle e_i, e_i \rangle_\psi \qquad \mathfrak{s}_- = -\langle e_i, e_i \rangle_\psi$$

Using $\langle e_i, e_i \rangle_\psi = \psi_i$ and (44)

$$v(A_+) = (\mathfrak{s} - \psi_i) e_i - \psi_i (p - e_i) = \mathfrak{s} e_i - \psi_i p$$

Now e_i and $p - e_i$ are idempotents, and $e_i(p - e_i) = 0$ so

$$\left(v(A_+)\right)^3 = (\mathfrak{s} - \psi_i)^3 e_i - \psi_i^3 (p - e_i) \tag{47}$$

Using $\langle p, e_i \rangle_\psi = \psi_i$ and $\langle p, p - e_i \rangle_\psi = \mathfrak{s} - \psi_i$ gives

$$\begin{aligned} 6c(v(A_+)) &= \langle p, (v(A_+))^3 \rangle_\psi \\ &= (\mathfrak{s} - \psi_i)^3 \langle p, e_i \rangle_\psi - \psi_i^3 \langle p, p - e_i \rangle_\psi \quad \textit{by (47)} \\ &= \left(\psi_i (\mathfrak{s} - \psi_i)^3 - \psi_i^3 (\mathfrak{s} - \psi_i)\right) \\ &= \psi_i (\mathfrak{s} - \psi_i) \left((\mathfrak{s} - \psi_i)^2 - \psi_i^2\right) \\ &= \psi_i (\mathfrak{s} - \psi_i) \mathfrak{s} (\mathfrak{s} - 2\psi_i) \end{aligned} \tag{48}$$

Now

$$\|v(A_+)\|_\psi^2 = (\mathfrak{s} - \psi_i)^2 \psi_i + \psi_i^2 (\mathfrak{s} - \psi_i) = \mathfrak{s} \psi_i (\mathfrak{s} - \psi_i) \tag{49}$$

It follows that the critical point, v_i, on S for $A = \{i\}$ is

$$v_i := \frac{v(A_+)}{\|v(A_+)\|_\psi} = \frac{(\mathfrak{s} - \psi_i) e_i - \psi_i (p - e_i)}{\sqrt{\mathfrak{s} \psi_i (\mathfrak{s} - \psi_i)}} = \frac{\mathfrak{s} e_i - \psi_i p}{\|\mathfrak{s} e_i - \psi_i p\|_\psi}$$

Thus

$$\begin{aligned} 6c(v_i) &= 6c(v(A_+))/\|v(A_+)\|_\psi^3 \\ &= \psi_i(\mathfrak{s}-\psi_i)\mathfrak{s}(\mathfrak{s}-2\psi_i)/(\mathfrak{s}\psi_i(\mathfrak{s}-\psi_i))^{3/2} \qquad \text{using (48), (49)} \\ &= (\mathfrak{s}-2\psi_i)/\sqrt{\mathfrak{s}\psi_i(\mathfrak{s}-\psi_i)} \\ &= \frac{1}{\sqrt{\psi_i}}(1-2\psi_i/\mathfrak{s})/\sqrt{1-\psi_i/\mathfrak{s}} \end{aligned}$$

If $c(v_i) < 0$ then $\psi_i > \mathfrak{s}/2$. Since $\mathfrak{s} = \sum \psi_i$, and all $\psi_i > 0$, it follows that $c(v_i) < 0$ for at most one value of i. Thus $|K^+| \geq n-1$. To compute α_{ij} start with

$$\begin{aligned} \langle \mathfrak{s}e_i - \psi_i p, \mathfrak{s}e_j - \psi_j p\rangle_\psi &= \mathfrak{s}^2\langle e_i, e_j\rangle_\psi + \psi_i\psi_j\langle p,p\rangle_\psi - \mathfrak{s}\left(\psi_j\langle e_i,p\rangle_\psi + \psi_i\langle p,e_j\rangle_\psi\right) \\ &= \delta_{ij}\mathfrak{s}^2\psi_i + \psi_i\psi_j\mathfrak{s} - 2\mathfrak{s}\psi_i\psi_j \\ &= \mathfrak{s}\psi_i(\delta_{ij}\mathfrak{s} - \psi_j) \end{aligned}$$

Using this for $i \neq j$ gives

$$\begin{aligned} \alpha_{ij} = \langle v_i, v_j\rangle_\psi &= \langle \mathfrak{s}e_i - \psi_i p, \mathfrak{s}e_j - \psi_j p\rangle_\psi / (\|\mathfrak{s}e_i - \psi_i p\|_\psi \|\mathfrak{s}e_j - \psi_j p\|_\psi) \\ &= (-\mathfrak{s}\psi_i\psi_j)/\sqrt{\mathfrak{s}\psi_i(\mathfrak{s}-\psi_i)\mathfrak{s}\psi_j(\mathfrak{s}-\psi_j)} \\ &= -\sqrt{\psi_i\psi_j/(\mathfrak{s}-\psi_i)(\mathfrak{s}-\psi_j)} \\ &< 0 \end{aligned}$$

When i, j, k are all distinct

$$\begin{aligned} 1 - \alpha_{jk}/\alpha_{ij}\alpha_{ik} &= 1 + \sqrt{\frac{\psi_j\psi_k(\mathfrak{s}-\psi_i)(\mathfrak{s}-\psi_j)(\mathfrak{s}-\psi_i)(\mathfrak{s}-\psi_k)}{(\mathfrak{s}-\psi_j)(\mathfrak{s}-\psi_k)\psi_i\psi_j\psi_i\psi_k}} \\ &= 1 + (\mathfrak{s}-\psi_i)/\psi_i \\ &= \mathfrak{s}/\psi_i \end{aligned}$$

Hence $\alpha_{ij}\alpha_{ik}/(\alpha_{ij}\alpha_{ik} - \alpha_{jk}) = \psi_i/\mathfrak{s}$.

Let $\pi : \mathbb{R}^n \to \mathfrak{g}$ be orthogonal projection. Since $\mathfrak{g} = p^\perp$ it follows that

$$\pi(x) = x - \frac{\langle p, x\rangle_\psi}{\langle p,p\rangle_\psi} p$$

Using that $p = e_1 + \cdots e_n$, and that the standard basis $\{e_1, \cdots, e_n\}$ is orthogonal, gives

$$\langle p, e_i\rangle_\psi = \langle e_i, e_i\rangle_\psi$$

and $\langle p,p\rangle_\psi = \mathfrak{s}$ so

$$\pi(e_i) = e_i - (\langle e_i, e_i\rangle_\psi/\mathfrak{s})p$$

Thus the local maxima are on the projections of the coordinate axes:

$$v(A_+) = \mathfrak{s}\cdot\pi(e_i)$$

This proves the lemma, modulo Claim 1.

Claim 2. At $v = v(A_+)$ then

$$d^2(J|S)_v(w,w) = (\mathfrak{s}/2)\langle e_+ - e_-, w^2\rangle_\psi \quad \text{for } w \in T_vS.$$

Assuming this, the quadratic form

$$Q(w,w) = \langle e_+ - e_-, w^2\rangle_\psi \tag{50}$$

is defined and non-singular on all of $T_v\mathbb{R}^n \equiv \mathbb{R}^n$ and $\mu(Q|T_vS) = \mu(d^2(J|S)_v)$. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map defined by

$$L|A_\pm = \pm \mathrm{I}$$

Then

$$Q(x,y) = \langle Lx, y\rangle_\psi \tag{51}$$

Now $p = e_+ + e_-$ so $Lp = e_+ - e_-$ and

$$Lv = L(\mathfrak{s}_+ e_+ + \mathfrak{s}_- e_-) = \mathfrak{s}_+ e_+ - \mathfrak{s}_- e_-$$

Now T_vS is the orthogonal complement with respect to the inner product $\langle\cdot,\cdot\rangle_\psi$ of the subspace spanned by $\{p, v\}$, because S is a sphere in the orthogonal complement of p. Using (51) and $L = L^{-1}$ shows that T_vS is also the orthogonal complement with respect to Q of the subspace W spanned by $\{Lp, Lv\}$.

Claim 3. $Q|W$ is non-singular and $\mu(Q|W) = 1$.

Assuming this, since W and T_vV are orthogonal with respect to Q, it follows that

$$\mu(Q) = \mu(Q|W) + \mu(Q|T_sV) = 1 + \mu(Q|T_vS)$$

From (50) $\mu(Q) = |A_+|$ so $\mu(Q|T_vS) = |A_+| - 1$ which proves Claim 1.

To prove Claim 3 we first evaluate $Q(Lp, Lp)$, and $Q(Lp, Lv)$, and $Q(Lv, Lv)$ to obtain the matrix of Q in the basis $\{Lp, Lv\}$.

$$\begin{aligned} Q(Lp,Lp) &= \langle L^2p, Lp\rangle_\psi && \text{by (51)}\\ &= \langle p, Lp\rangle_\psi \\ &= \langle e_+ + e_-, e_+ - e_-\rangle_\psi \\ &= -\mathfrak{s}_- - \mathfrak{s}_+ && \text{by (46)} \end{aligned}$$

$$\begin{aligned} Q(Lv,Lv) &= \langle v, Lv\rangle_\psi \\ &= \langle \mathfrak{s}_+ e_+ + \mathfrak{s}_- e_-, \mathfrak{s}_+ e_+ - \mathfrak{s}_- e_-\rangle_\psi \\ &= \mathfrak{s}_+^2\langle e_+, e_+\rangle_\psi - \mathfrak{s}_-^2\langle e_-, e_-\rangle_\psi \\ &= \mathfrak{s}_+^2(-\mathfrak{s}_-) - \mathfrak{s}_-^2(\mathfrak{s}_+) \\ &= -\mathfrak{s}_-\mathfrak{s}_+(\mathfrak{s}_+ + \mathfrak{s}_-) \end{aligned}$$

$$\begin{aligned} Q(Lp,Lv) &= \langle p, Lv\rangle_\psi \\ &= \langle e_+ + e_-, \mathfrak{s}_+ e_+ - \mathfrak{s}_- e_-\rangle_\psi \\ &= \mathfrak{s}_+\langle e_+, e_+\rangle_\psi - \mathfrak{s}_-\langle e_-, e_-\rangle_\psi \\ &= \mathfrak{s}_+(-\mathfrak{s}_-) - \mathfrak{s}_-\mathfrak{s}_+ \\ &= -2\mathfrak{s}_-\mathfrak{s}_+ \end{aligned}$$

$$\begin{aligned}
\therefore\ \det(Q|W) &= \det\begin{bmatrix} Q(Lp,Lp) & Q(Lp,Lv) \\ Q(Lp,Lv) & Q(Lv,Lv) \end{bmatrix} \\
&= \det\begin{bmatrix} -(\mathfrak{s}_+ + \mathfrak{s}_-) & -2\mathfrak{s}_+\mathfrak{s}_- \\ -2\mathfrak{s}_+\mathfrak{s}_- & -\mathfrak{s}_-\mathfrak{s}_+(\mathfrak{s}_+ + \mathfrak{s}_-) \end{bmatrix} \\
&= \mathfrak{s}_+\mathfrak{s}_- \left[(\mathfrak{s}_+ + \mathfrak{s}_-)^2 - 4\mathfrak{s}_+\mathfrak{s}_-\right] \\
&= \mathfrak{s}_+\mathfrak{s}_- (\mathfrak{s}_+ - \mathfrak{s}_-)^2 \\
&= \mathfrak{s}_+\mathfrak{s}_-\mathfrak{s}^2 \\
&< 0
\end{aligned}$$

Thus $Q|W$ has one eigenvalue of each sign, which proves Claim 3.

It only remains to prove Claim 2. Give a critical point $v = v(A_+)$ of $f := J|S$ we compute $d^2(J|S)_v$. Let $\gamma : (-\epsilon, \epsilon) \to S$ be a smooth curve with $\gamma(0) = v$ and $\gamma'(0) = w \in T_vS$. Then

$$\begin{aligned}
(f \circ \gamma)'(t) &= \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x_i}\right|_{x=\gamma(t)} \gamma_i'(t) \\
\therefore\ (f \circ \gamma)''(0) &= \sum_{i,j=1}^{n} \left.\frac{\partial^2 f}{\partial x_i^2}\right|_{x=v} \gamma_i'(0)\gamma_j'(0) + \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x_i}\right|_{x=v} \gamma''(0) \\
&= d^2J_v(w,w) + dJ_v(\gamma''(0))
\end{aligned}$$

In the following everything is evaluated at $t = 0$

$$\begin{aligned}
&\langle \gamma, \gamma \rangle_\psi = 1 && \because \gamma = \gamma(0) \in S \\
&\Rightarrow \langle \gamma', \gamma \rangle_\psi = 0 \\
&\Rightarrow \langle \gamma'', \gamma \rangle_\psi + \langle \gamma', \gamma' \rangle_\psi = 0 \\
&\Rightarrow \gamma'' \in (-\langle \gamma', \gamma' \rangle_\psi / \langle \gamma, \gamma \rangle_\psi)\, \gamma + T_vS
\end{aligned}$$

Using $\gamma(0) = v$ and $\gamma'(0) = w$

$$\gamma''(0) = -\left(\langle w, w \rangle_\psi / \langle v, v \rangle_\psi\right) v + t$$

for some $t \in T_vS$. Since dJ_v vanishes on T_vS we get

$$d^2 f_v(w,w) = (f \circ \gamma)''(0) = d^2J_v(w,w) - \left(\langle w, w \rangle_\psi / \langle v, v \rangle_\psi\right) dJ_v(v) \tag{52}$$

Now we compute these two terms

$$\begin{aligned}
d^2J_v(w,w) &= \left.\frac{d^2}{dt^2}\right|_{t=0} \left(\frac{1}{2}\langle (v+tw)^2, p \rangle_\psi + \frac{1}{6}\langle (v+tw)^3, p \rangle_\psi\right) \\
&= \langle w^2, p \rangle_\psi + \langle vw^2, p \rangle_\psi \\
&= \langle w, w \rangle_\psi + \langle v, w^2 \rangle_\psi \qquad (53)
\end{aligned}$$

By (42) $dJ_v(v) = \langle v, v \rangle_\psi + (1/2)\langle v^2, v \rangle_\psi$, so

$$\left(\langle w, w \rangle_\psi / \langle v, v \rangle_\psi\right) dJ_v(v) = \langle w, w \rangle_\psi + (1/2)\left(\langle v, v^2 \rangle_\psi / \langle v, v \rangle_\psi\right) \langle w, w \rangle_\psi \tag{54}$$

At the critical point $v = v(A_+) = \mathfrak{s}_+ e_+ + \mathfrak{s}_- e_-$ so

$$\begin{aligned}
\langle v, v\rangle_\psi &= \langle \mathfrak{s}_+ e_+ + \mathfrak{s}_- e_-, \mathfrak{s}_+ e_+ + \mathfrak{s}_- e_-\rangle_\psi \\
&= \mathfrak{s}_+^2 \langle e_+, e_+\rangle_\psi + \mathfrak{s}_-^2 \langle e_-, e_-\rangle_\psi \\
&= \mathfrak{s}_+^2 (-\mathfrak{s}_-) + \mathfrak{s}_-^2 \mathfrak{s}_+ \\
&= \mathfrak{s}_+ \mathfrak{s}_- (\mathfrak{s}_- - \mathfrak{s}_+) \\
&= -\mathfrak{s}_+ \mathfrak{s}_- \mathfrak{s} \\
\langle v, v^2\rangle_\psi &= \langle \mathfrak{s}_+ e_+ + \mathfrak{s}_- e_-, \mathfrak{s}_+^2 e_+ + \mathfrak{s}_-^2 e_-\rangle_\psi \\
&= \mathfrak{s}_+^3 \langle e_+, e_+\rangle_\psi + \mathfrak{s}_-^3 \langle e_-, e_-\rangle_\psi \\
&= \mathfrak{s}_+^3 \mathfrak{s}_- + \mathfrak{s}_-^3 \mathfrak{s}_+ \\
&= \mathfrak{s}_+ \mathfrak{s}_- (\mathfrak{s}_- - \mathfrak{s}_+)(\mathfrak{s}_- + \mathfrak{s}_+) \\
&= -\mathfrak{s}_+ \mathfrak{s}_- \mathfrak{s} (\mathfrak{s}_+ + \mathfrak{s}_-)
\end{aligned}$$

Thus $\langle v, v^2\rangle_\psi / \langle v, v\rangle_\psi = \mathfrak{s}_+ + \mathfrak{s}_-$. Using this with (54) gives

$$(\langle w, w\rangle_\psi / \langle v, v\rangle_\psi)\, dJ_v(v) = \langle w, w\rangle_\psi + (1/2)\left(\mathfrak{s}_+ + \mathfrak{s}_-\right) \langle w, w\rangle_\psi$$

Using this and (53) to substitute into (52) gives

$$\begin{aligned}
d^2 f_v(w, w) &= \langle w, w\rangle_\psi + \langle v, w^2\rangle_\psi - (1 + (1/2)(\mathfrak{s}_+ + \mathfrak{s}_-))\langle w, w\rangle_\psi \\
&= \langle v, w^2\rangle_\psi - (1/2)\langle (\mathfrak{s}_+ + \mathfrak{s}_-) w, w\rangle_\psi \\
&= \langle v, w^2\rangle_\psi - (1/2)\langle (\mathfrak{s}_+ + \mathfrak{s}_-) p, w^2\rangle_\psi
\end{aligned}$$

From (43) $p = e_+ + e_-$ and $v = \mathfrak{s}_+ e_+ + \mathfrak{s}_- e_-$ from (44). Then

$$\begin{aligned}
d^2 f_v(w, w) &= \langle \mathfrak{s}_+ e_+ + \mathfrak{s}_- e_-, w^2\rangle_\psi - (1/2)\langle (\mathfrak{s}_+ + \mathfrak{s}_-)(e_+ + e_-), w^2\rangle_\psi \\
&= \langle \mathfrak{s}_+ e_+ + \mathfrak{s}_- e_- - (1/2)(\mathfrak{s}_+ + \mathfrak{s}_-)(e_+ + e_-), w^2\rangle_\psi \\
&= (1/2)\mathfrak{s}\langle e_+ - e_-, w^2\rangle_\psi
\end{aligned}$$

where we used $\mathfrak{s} = \mathfrak{s}_+ - \mathfrak{s}_-$. This proves Claim 2. □

Proof of 6.10. This proof uses different coordinates to the proof of 6.9. In this lemma summation is over the set of integers in $[1, n-1]$ unless otherwise indicated, and $\langle \cdot, \cdot \rangle$ is the standard inner product on $V = \mathbb{R}^{n-1}$. A point $v = \sum v_i e_i \in S$ is a critical point of $c|S$ if and only if there is some $\alpha \in V$ such that for all $w \in V$ we have $dc_v(w) = \alpha \cdot \langle v, w\rangle$ because $T_v S = v^\perp$. Thus

$$dc_v(w) = \sum \lambda_i v_i^2 w_i = \alpha \sum v_i w_i \tag{55}$$

This equation is satisfied if and only if $\forall i\ \lambda_i v_i^2 = \alpha v_i$. Since $\lambda_i \geq 0$ the requirement that $c(v) = (1/3)\psi v^3 > 0$ implies $\psi v^2 > 0$; thus $\alpha \neq 0$. Thus the set of positive critical points of $c|S$ is

$$W = \{v \in S :\ \exists\, \alpha \neq 0\ \ \psi v^2 = \alpha v\}$$

Given $v \in W$, let $A = \{i : v_i \neq 0\}$, then A is not empty and $i \in A \Rightarrow \lambda_i \neq 0$ and

$$v = v(A) = \alpha \sum_{i \in A} \lambda_i^{-1} e_i$$

Claim. $d^2(c|S)$ at $v = v(A)$ is the restriction to T_vS of the quadratic form on V

$$Q(w) = \alpha \left(\sum_A w_i^2 - \sum_{A^c} w_i^2 \right)$$

where $A^c = \{1, \cdots, n-1\} \setminus A$.

Observe that Q is non-degenerate and $v = v(A) \in \langle e_i : i \in A \rangle$ and $T_vS = v^\perp$ so $Q|T_vS$ is also non-degenerate. If follows that $v(A)$ is a local maximum of $(c|S)$ if $\alpha > 0$ and $|A| = 1$ or $\alpha < 0$ and $A^c = \emptyset$. In the first case $A = \{i\}$ and $v(A) = e_i$ and $c(e_i) = \lambda_i/3 > 0$. In the second case $A = \{1, \cdots, n-1\}$ and $v(A) = -(n-1)^{-1/2} \sum e_i$ so $c(v(A)) < 0$. This proves the lemma modulo the claim.

To prove the claim, adapting the derivation of (52) gives

$$d^2(c|S)_v(w,w) = d^2c_v(w,w) - (\langle w,w\rangle/\langle v,v\rangle) dc_v(v) \tag{56}$$

Using $\lambda_i v_i = \alpha$ for $i \in A$ and $\lambda_i v_i = 0$ for $i \notin A$ gives

$$d^2c_v(w,w) = 2\sum \lambda_i v_i w_i^2 = 2\sum_{i\in A} \alpha w_i^2 \tag{57}$$

Using (55) gives

$$d_c(v) = \sum \alpha v_i v_i = \alpha \langle v,v\rangle = \alpha \qquad \because v \in S \quad \therefore \langle v,v\rangle = 1 \tag{58}$$

Hence

$$(\langle w,w\rangle/\langle v,v\rangle) dc_v(v) = \alpha \sum w_i^2 \tag{59}$$

Substituting into (56)

$$d^2(c|S)_v(w,w) = 2\sum_{i\in A} \alpha w_i^2 - \alpha \sum w_i^2 = \alpha \left(\sum_{i\in A} w_i^2 - \sum_{i\in A^c} w_i^2 \right) \tag{60}$$

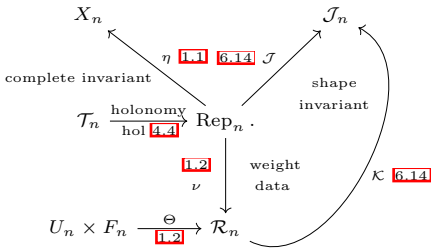
which proves the claim. □

9. Notation/Cheat sheet

$V = \mathbb{R}^{n-1} \qquad T^{n-1} = V/\mathbb{Z}^{n-1}$
$\mathcal{T}_n$ = space of generalized cusps for $C \cong T^{n-1} \times [0,1)$
$\mathcal{T}_n = \{\text{developing maps}\}/\{\text{homotopy, affine maps}\}$
$\mathrm{Rep}_n \subset \mathrm{Hom}(V, \mathrm{Aff}_n)/\mathrm{Aff}_n \qquad \mathrm{Rep}_n \cong A_n \times \mathrm{SL}^\pm V/\sim_\psi \;\cong \tilde{A}_n \times \mathrm{SL}^\pm V/\sim_{(\lambda,\kappa)}$
$[\xi_\psi \circ A]$ (2.4) $[\Phi_{(\lambda,\kappa)} \circ A]$ (3.4)
$U_n \subset \mathrm{SL}\, V$ is upper-triangular unipotent matrices
$F_n = \{(v_1, \cdots, v_n) \in V^n \;:\; \exists\, \varpi \geq 0 \;\; \forall\, i \neq j \;\; \langle v_i, v_i\rangle = -\varpi\}/\Sigma_n$
$\mathcal{T}_n \cong F_n \times U_n$
$\mathrm{S}^2_+ V =$ all positive definite quadratic forms on V
$\mathcal{P} = \{\beta \in \mathrm{S}^2_+ V \,|\, \det \beta = 1\} \equiv \mathbb{P}\,\mathrm{S}^2_+ V$
$U_n \cong \mathcal{P}$ using $A^{-1} \mapsto \beta = A^t A$ is metric on T^{n-1}
F_n parameterizes $\sigma \in \mathrm{Rep}_n$ with $\beta(\sigma) = \beta_0$ and $\beta_0(v) = \sum v_i^2$, so $A = \mathrm{I}$
General $\rho \in \mathrm{Rep}_n$ is $\rho = \sigma \circ A$ with $A^{-1} \in U_n$ and $\sigma \in F_n$
Lie algebra weights of *linear part of* ρ are $\xi_1, \ldots, \xi_n \in V^*$ st for all $i \neq j$ we have $\langle \xi_i, \xi_j \rangle_\beta = -\varpi$

	in terms of Lie alg. weights	invariant		$\in$ space
cubic	$c(\rho) = (1/3)\sum(\xi_i)^3/(\|\xi_i\|^2_\beta + \varpi)$	shape	$J(\rho) = (c(\rho), \beta(\rho))$	$\mathcal{J}_n$
character	$\chi(\rho) = \sum \exp \xi_i$	complete	$\eta(\rho) = (\chi(\rho), \beta(\rho))$	X_n
	$[\xi(\rho)] = [\xi_1, \dots \xi_n] \in V^{*n}/\Sigma_n$	weight data	$\nu(\rho) \in ([\xi(\rho)], \beta(\rho))$	$\mathcal{R}_n$

All arrows are homeomorphisms



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References

[1] Michael Aschbacher, *Chevalley groups of type G_2 as the group of a trilinear form*, J. Algebra **109** (1987), no. 1, 193–259, DOI 10.1016/0021-8693(87)90173-6. MR898346

[2] Samuel A. Ballas, Daryl Cooper, and Arielle Leitner, *Generalized cusps in real projective manifolds: classification*, J. Topol. **13** (2020), no. 4, 1455–1496, DOI 10.1112/topo.12161. MR4125754

[3] Ludwig Bieberbach, *Über die Bewegungsgruppen der Euklidischen Räume* (German), Math. Ann. **70** (1911), no. 3, 297–336, DOI 10.1007/BF01564500. MR1511623

[4] Wilhelm Blaschke, *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie. Band I. Elementare Differentialgeometrie* (German), Dover Publications, New York, N.Y., 1945. 3d ed. MR0015247

[5] Eugenio Calabi, *Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens*, Michigan Math. J. **5** (1958), 105–126. MR106487

[6] Leonard S. Charlap, *Bieberbach groups and flat manifolds*, Universitext, Springer-Verlag, New York, 1986, DOI 10.1007/978-1-4613-8687-2. MR862114

[7] Shiu Yuen Cheng and Shing Tung Yau, *On the regularity of the Monge-Ampère equation* $\det(\partial^2 u/\partial x_i \partial s x_j) = F(x, u)$, Comm. Pure Appl. Math. **30** (1977), no. 1, 41–68, DOI 10.1002/cpa.3160300104. MR437805

[8] Suhyoung Choi, *The convex real projective orbifolds with radial or totally geodesic ends: a survey of some partial results*, In the tradition of Ahlfors-Bers. VII, Contemp. Math., vol. 696, Amer. Math. Soc., Providence, RI, 2017, pp. 51–85, DOI 10.1090/conm/696/14016. MR3715441

[9] Daryl Cooper, Darren Long, and Stephan Tillmann, *Deforming convex projective manifolds*, Geom. Topol. **22** (2018), no. 3, 1349–1404, DOI 10.2140/gt.2018.22.1349. MR3780436

[10] D. Cooper, D. D. Long, and S. Tillmann, *On convex projective manifolds and cusps*, Adv. Math. **277** (2015), 181–251, DOI 10.1016/j.aim.2015.02.009. MR3336086

[11] Albrecht Dold, *Homology of symmetric products and other functors of complexes*, Ann. of Math. (2) **68** (1958), 54–80, DOI 10.2307/1970043. MR97057

[12] Skip Garibaldi and Robert M. Guralnick, *Simple groups stabilizing polynomials*, Forum Math. Pi **3** (2015), e3, 41, DOI 10.1017/fmp.2015.3. MR3406824

[13] Ross Geoghegan, *Topological methods in group theory*, Graduate Texts in Mathematics, vol. 243, Springer, New York, 2008, DOI 10.1007/978-0-387-74614-2. MR2365352

[14] A. E. Hatcher, *Concordance spaces, higher simple-homotopy theory, and applications*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978, pp. 3–21. MR520490

[15] N. J. Hitchin, *Lie groups and Teichmüller space*, Topology **31** (1992), no. 3, 449–473, DOI 10.1016/0040-9383(92)90044-I. MR1174252

[16] B. Klartag, *Affine hemispheres of elliptic type*, Algebra i Analiz **29** (2017), no. 1, 145–188, DOI 10.1090/spmj/1484; English transl., St. Petersburg Math. J. **29** (2018), no. 1, 107–138. MR3660687

[17] François Labourie, *Flat projective structures on surfaces and cubic holomorphic differentials*, Pure Appl. Math. Q. **3** (2007), no. 4, Special Issue: In honor of Grigory Margulis., 1057–1099, DOI 10.4310/PAMQ.2007.v3.n4.a10. MR2402597

[18] Serge Lang, *Algebra*, 3rd ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002, DOI 10.1007/978-1-4613-0041-0. MR1878556

[19] John C. Loftin, *Affine spheres and convex $\mathbb{RP}^n$-manifolds*, Amer. J. Math. **123** (2001), no. 2, 255–274. MR1828223

[20] Katsumi Nomizu and Takeshi Sasaki, *Affine differential geometry*, Cambridge Tracts in Mathematics, vol. 111, Cambridge University Press, Cambridge, 1994. Geometry of affine immersions. MR1311248

[21] A. V. Pogorelov, *On the improper convex affine hyperspheres*, Geometriae Dedicata **1** (1972), no. 1, 33–46, DOI 10.1007/BF00147379. MR319126

[22] Boris Reichstein, *On expressing a cubic form as a sum of cubes of linear forms*, Linear Algebra Appl. **86** (1987), 91–122, DOI 10.1016/0024-3795(87)90289-8. MR870934

[23] Bruce Reznick, *Some new canonical forms for polynomials*, Pacific J. Math. **266** (2013), no. 1, 185–220, DOI 10.2140/pjm.2013.266.185. MR3105781

[24] Tuong Ton That, *Lie group representations and harmonic polynomials of a matrix variable*, Trans. Amer. Math. Soc. **216** (1976), 1–46, DOI 10.2307/1997683. MR399366

Florida State University
Email address: ballas@math.fsu.edu

University of California, Santa Barbara
Email address: cooper@math.ucsb.edu

Afeka College of Engineering
Email address: ariellel@afeka.ac.il