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Arnold Tongues in Area-Preserving Maps

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Abstract

In the early 60's J. B. Keller and D. Levy discovered a fundamental property: the instability tongues in Mathieu-type equations lose sharpness with the addition of higher-frequency harmonics in the Mathieu potentials. Twenty years later, V. Arnold discovered a similar phenomenon on the sharpness of Arnold tongues in circle maps (and rediscovered the result of Keller and Levy). In this paper we find a third class of object where a similar type of behavior takes place: area-preserving maps of the cylinder. loosely speaking, we show that periodic orbits of standard maps are extra fragile with respect to added drift (i.e. non-exactness) if the potential of the map is a trigonometric polynomial. That is, higher-frequency harmonics make periodic orbits more robust with respect to "drift". This observation was motivated by the study of traveling waves in the discretized sine-Gordon equation which in turn models a wide variety of physical systems.

1. Introduction

Understanding invariant sets of area-preserving maps is one of the central problems of dynamics and one of the most studied—starting with Poincaré's geometrical theorem [5,7,15], through KAM theory [2,3,14] and the Aubry-Mather theory [4,12,13]. All of the results require the exactness assumption.

Much less is known about area-preserving maps which are non-exact, such as the maps φ of the cylinder $\mathbb{S} \times \mathbb{R}$ with a "drift":

$$\int_{\varphi(\gamma)} v \; \mathrm{d}x - \int_{\gamma} v \; \mathrm{d}x = \delta \neq 0 \tag{1}$$

Here γ is a closed curve encircling the cylinder $\mathbb{R}\,\mathrm{mod}1 \times \mathbb{R}$ once; see Fig. 1.

Such maps are ubiquitous in Hamiltonian dynamics,and arise in numerous settings. We mention four examples.

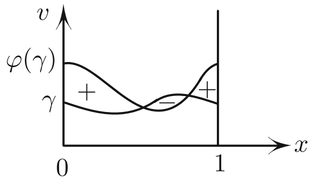

Fig. 1. Non-exact area-preserving cylinder map with $\delta > 0$

1. The Frenkel–Kontorova model of electrons in a crystal lattice [6,9]. The model consists of an infinite chain of particles on the line a periodic potential $V(x)$ and with nearest neighbor coupling. The equilibria are the critical points of the total energy

$$\sum_{i\in\mathbb{Z}} \frac{k}{2}(x_{i+1}-x_i)^2 + V(x_i);$$

although the sum is divergent, the variational equation, i.e. the equilibrium condition

$$x_{i+1} - 2x_i + x_{i-1} + k^{-1}V'(x_i), \tag{2}$$

is well defined. This discrete analog of the Euler-Lagrange equation has a Hamiltonian counterpart obtained by the introduction of the discrete momentum $v_i = x_i - x_{i-1}$:

$$\begin{cases} x_{i+1} = x_i + v_i - V'(x_i) \\ v_{i+1} = v_i - V'(x_i) \end{cases}, \tag{3}$$

Used it an area-preserving map.

If V' is periodic, then (3) defines a cylinder map. However a periodic V' leaves the possibility that V itself may have a "tilt", i.e. a linear part

$$V(x) = a\,x + V_{\text{periodic}}(x),$$

where $V_{\text{periodic}}(x+T) = V_{\text{periodic}}(x)$ for some fixed T.

The tilt causes the map (3) to be non-exact; (1) holds for this map with $\delta = -aT$ (Fig. 2).

2. Chains of coupled pendula. In the special case of $V(x) = c\sin x$, the Frenkel–Kontorova model has a mechanical interpretation as the chain of torsionally coupled pendula (Fig. 3); here x_i denotes the angle of the ith pendulum with the downward vertical. Now if each pendulum is subjected to a constant torque τ then the potential acquires a linear part: $V(x) = c\sin x + \tau x$, and the corresponding cylinder map becomes non-exact, with $\delta = -2\pi\tau$.

3. Coupled Josephson junctions. A Josephson junction consists of two superconductors separated by a narrow gap of a few angstroms [8]. The junction can behave as a superconductor or as a resistor, depending on the initial conditions

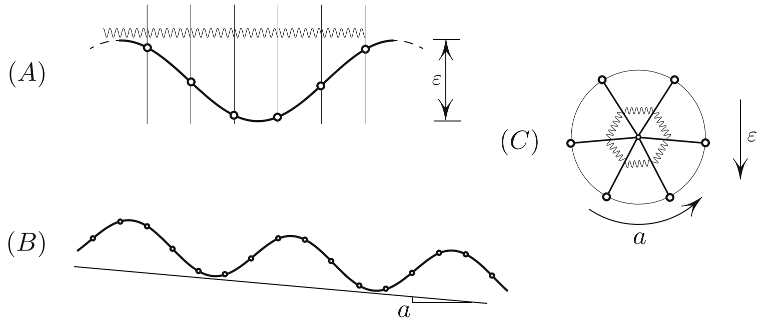

Fig. 2. A the Frenkel–Kontorova model; **B** the tilt added, leading to the non-exact cylinder map; **C** tilt interpreted as torque acting on coupled pendula

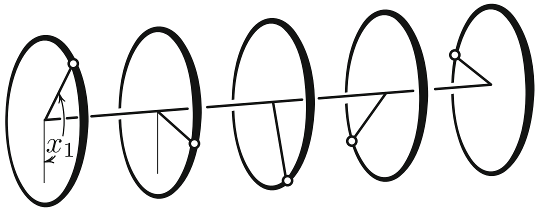

Fig. 3. Discretized sine-Gordon equation: pendula with nearest-neighbor torsional coupling

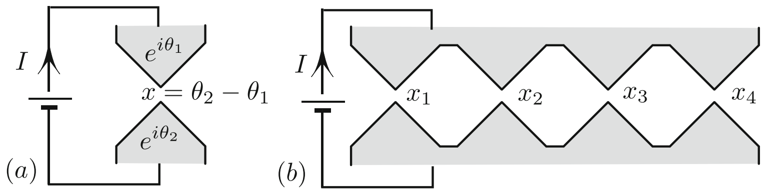

Fig. 4. Josephson junctions: single and coupled. Voltage across the junction is proportional to $\langle \dot{x} \rangle$

(there were some hopes in the 1970s to use this property as a memory device). This behavior reminds one of a pendulum with torque δ described by

$$\ddot{x} + c\dot{x} + \sin x = \delta \tag{4}$$

For the $|\delta| < 1$ there are two stable limiting regimes: either the stable equilibrium or the "running" periodic solution corresponding to the tumbling motion $x = \omega t + p(t)$ where p is periodic. In fact the same equation (4) is satisfied by the jump $x = \arg\theta_2 - \arg\theta_1$ of the phase of the electron wave function across the gap if current δ is driven across the gap, Fig. 4. The voltage across the gap is proportional to the average $\langle \dot{x} \rangle$, or the average angular velocity in the pendulum interpretation. Thus the equilibrium solution with its average $\langle \dot{x} \rangle = 0$ corresponds to zero voltage and thus to the superconducting state, while the tumbling solution with the voltage $\langle \dot{x} \rangle \neq 0$ corresponds to the resistive state.

4. Particle in $\mathbb{R}$ **subject to a force that varies periodically both in time and position.** The motion of such a particle is governed by

$$\ddot{x} = \Phi(x, t),$$

where Φ is periodic in both variables of period 1 (without loss of generality). The Poincaré map $\varphi : (x, y)_{t=0} \mapsto (x, y)_{t=1}$, where $y = \dot{x}$, is generally non-exact, satisfying (1) with the drift equal to the average of force Φ:

$$\delta = \int_0^1 \int_0^1 \Phi(x, t)\, \mathrm{d}x\, \mathrm{d}t = \langle \Phi \rangle.$$

This completes our list of examples where non-exact area-preserving cylinder maps aries. In this paper we show that periodic orbits of the standard map (3) are extra sensitive to the added drift δ if the potential has harmonics of only low frequencies. There are (at least) two known phenomena with a similar flavor: (i) the sharpness of Arnold tongues in circle maps [1]

$$x \mapsto x + \omega + \varepsilon f(x),$$

where f is a trigonometric polynomial is related to the degree of f, and (ii), the sharpness of resonance zones in Hill's equations

$$\ddot{x} + (\omega^2 + \varepsilon q(t))x = 0,$$

where q is a trigonometric polynomial is related to the degree of q [11]. The present paper adds one more example to this list. According to Arnold [1], Gelfand conjectured the existence of a general theorem which covers cases (i) and (ii); to this conjecture one can add the problem studied in the present paper.

2. Results

We consider periodic potentials with a linear part added:

$$V(x) = \delta\, x + \varepsilon F(x), \quad F(x + 2\pi) = F(x).$$

Thus the standard map (3) with such V takes form

$$\begin{cases} x_{i+1} = x_i + v_i - \delta - \varepsilon f(x_i) \\ v_{i+1} = v_i - \delta - \varepsilon f(x_i) \end{cases}, \tag{5}$$

where $f(x) = F'(x)$ is periodic of period 2π.

For $\delta = 0$ the cylinder map (5) is exact, and it possesses a p/q periodic orbit for any integer $p,\ q \neq 0$, i.e. an orbit satisfying

$$x_{i+q} = x_i + 2p\pi, \quad v_{i+q} = v_i;$$

this follows from Poincaré's Last Geometric Theorem as generalized by Franks [7]. In his generalization Franks removed the requirement of invariant boundary circles

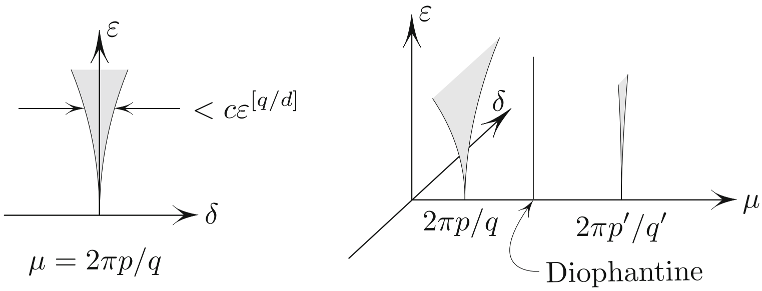

Fig. 5. Left: Arnold tongue for the Birkhoff periodic point with the rotation number $\mu = p/q$ is exponentially narrow for trigonometric polynomials. Right: for Diophantine μ one has an invariant KAM curve only for the drift $\delta = 0$, i.e. the tongue has zero width

of an annulus, replacing it with the assumption of exactness. Since the theorem no longer applies to the non-exact case $\delta \neq 0$, a natural question arises: for what range of drift δ do periodic orbits persist? We show that if V is a trigonometric polynomial, then this range becomes narrower if the degree of the trigonometric polynomial f becomes smaller; furthermore, the tightness of the range is exponentially small in terms of the period q (Fig. 5). More precisely, one has the following:

Theorem 1. *(Width of Arnold tongues) Let $f(x)$ in (5) be a trigonometric polynomial of degree d, and let $p \geq 0$, $q > 0$ be integers. There exist positive constants $\overline{\varepsilon}$ and c depending only on q and f, such that for any $0 < \varepsilon < \overline{\varepsilon}$, all p/q periodic orbits of (5) disappear if the drift $|\delta| > c\varepsilon^{[q/d]}$; here $[\cdot]$ denotes the integer part.*

The other observation of this paper is that the p/q periodic points move on special curves as δ changes.

Theorem 2. *Let $p > 0$, $q > 0$ be integers. If the perturbation term $f(x)$ in the cylinder map (5) is a 2π-periodic analytic function (*not necessarily a trigonometric polynomial*), then there exists a positive constant $\overline{\overline{\varepsilon}}$ depending only on q and f such that for any $|\varepsilon| < \overline{\overline{\varepsilon}}$ and $|\delta| < \overline{\overline{\varepsilon}}$, any p/q periodic orbit of the perturbed map (5), if it exists, lies on the graph of the function*

$$v = \mu + v_1(x)\varepsilon + v_2(x)\varepsilon^2 + \cdots$$

where $\mu = 2\pi p/q$ and $v_n(x)$ is an n^{th} degree polynomial in $f(x + k\mu)$ ($0 \leq k \leq q - 1$) and their derivatives up to order $n - 1$. In particular,

$$v_1(x) = -\frac{q+1}{2}\overline{f}(x) + \overline{\overline{f}}(x),$$

where

$$\overline{f}(x) = \frac{1}{q}\sum_{k=0}^{q-1} f\,(x + k\mu)\,,$$

$$\overline{\overline{f}}(x) = \frac{1}{q}\sum_{k=0}^{q-1}(q-k) f\,(x + k\mu)\,.$$

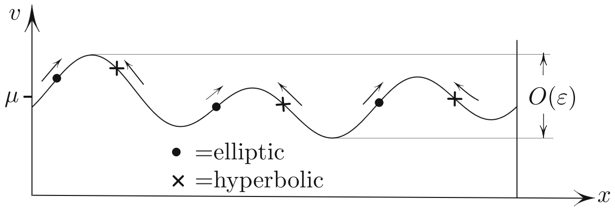

Fig. 6. Iterates of two periodic points (a center and a saddle). The arrows show the direction of motion of the iterates as δ increases. The two orbits disappear in a saddle-node bifurcation. As δ changes, these points move with large speed (at least) $O(\varepsilon^{-[q/d]})$ (at bifurcation the speed becomes infinite)

Remark 1. If f is a trigonometric polynomial, then $\overline{f}$ and $\overline{\overline{f}}$ are trigonometric polynomials as well, of a degree not exceeding the degree d of f. Moreover, if $d < q$, then $\overline{f} = 0$.

Remark 2. Birkhoff p/q periodic orbits come in saddle-center pairs. As δ increases from 0 to a critical value, a pair disappears in a saddle-node bifurcation, Fig. 6. The first theorem therefore states that critical values of δ are $O(\varepsilon^{[q/d]})$.

Remark 3. A special case of Theorem 1 for $q \leq 3$ was proven in [10] by a direct calculation. Unfortunately, as q increases, the complexity of this calculation tends to infinity. We overcome this problem by extending Arnold's approach [1] (that he used for circle maps) to the maps of the cylinder.

Remark 4. (An implication of Theorem 1 for traveling waves.) Consider an infinite periodic chain of pendula governed by the discretized sine-Gordon equation with damping:

$$\ddot{x}_k + \gamma \dot{x}_k + \varepsilon \sin x_k = (x_{k+1} - 2x_k + x_{k-1}) + \delta. \tag{6}$$

Fixing $q \in \mathbb{N}$ and the "twist" $p \in \mathbb{Z}$, consider space-periodic "twisted" solutions, i.e. the ones satisfying

$$x_{k+q}(t) = x_k(t) + 2\pi p \quad \text{for all} \quad t. \tag{7}$$

According to [10] there exists a constant $\varepsilon_0 = \varepsilon_0(\gamma)$ depending only on the damping coefficient $\gamma > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and for all δ every solution approaches asymptotically either to an equilibrium or to a traveling wave. This wave satisfies

$$x_k(t) = x_{k-1}(t + T/q),$$

(that is, each pendulum repeats what its neighbor is doing but with a delay). A q-fold application of this identity gives

$$x_k(t) = x_{k-q}(t + T) \overset{(7)}{=} x_k(t + T) - 2\pi p,$$

i.e.

$$x_k(t + T) = x_k(t) + 2\pi p,$$

so that this solution is periodic modulo rotation $x_k \mapsto x_k + 2\pi p$ $(k = 1, \dots, q)$ in $\mathbb{S} = \mathbb{R} \mod 2\pi$. Now, according to Theorem 1 the equilibria of (6) disappear if $\delta > c_0 \varepsilon^q$ (a threshhold exponentially small in the number of pendula), and thus (6) *has a globally attracting periodic traveling wave for all such* δ. *This traveling wave appears as the result of a saddle-node bifurcation of the equilibria which exist for smaller* δ.

A remarkable fragility of equilibria is illustrated in Fig. 2(C), where shows an equilibrium of $q = 6$ coupled pendula with the same torque applied to each. With the choice of "gravity" $\varepsilon = \frac{1}{2}$ the pendula sag by a comparable amount and one might expect that the equilibrium could withstand the torque of a comparable magnitude. However, the equilibrium (and hence the corresponding periodic orbit of the map) disappears for $\delta > 0.0027$, about .05% of the "gravity" ε! Sinusoidal potentials are thus quite special: they are remarkably bad at pinning down the equilibria.

The effect of non-exactness on the dynamics of the map is of interest in itself; its understanding would also shed light on physical effects, such as the fragility of Frenkel–Kontorova equilibria in crystals to imposed voltages.

3. The Preliminaries

The cylinder map (5) with $\varepsilon = \delta = 0$ has an invariant circle $v = 2\pi p/q \stackrel{\text{def}}{=} \mu$ consisting of p/q periodic points. This suggests introducing a shifted momentum

$$y := v - \mu$$

with which the cylinder map (5) takes a new form,

$$\begin{cases} x_{i+1} = x_i + y_i + \mu + g(x_i; \varepsilon, \delta) \\ y_{i+1} = y_i + g(x_i; \varepsilon, \delta), \end{cases} \tag{8}$$

where

$$g(x; \varepsilon, \delta) := -\delta - \varepsilon f(x),$$

and where $f(x)$ is a 2π-periodic analytic function as in Theorem 2.

This map (8) restricted to a neighborhood of the circle $y = 0$ can be viewed as a perturbation of the shift by μ in the x direction; the nth iterate will thus be a perturbation of the shift by $n\mu$ in the x-direction. This suggests writing the n-th iterate of (8) in the form

$$\begin{cases} x_n = x_0 + n\mu + {}_nR(x_0, y_0, \varepsilon, \delta) \\ y_n = y_0 + {}_nS(x_0, y_0, \varepsilon, \delta). \end{cases} \tag{9}$$

A quick calculation shows that

$$\begin{aligned} {}_nR(x_0, y_0, \varepsilon, \delta) &= n\,(y_0 + g(x_0)) + (n-1)g(x_1) + \cdots + 2g(x_{n-2}) + g(x_{n-1}) \\ {}_nS(x_0, y_0, \varepsilon, \delta) &= g(x_0) + g(x_1) + \cdots + g(x_{n-2}) + g(x_{n-1}), \end{aligned}$$

(10)

where $\{x_0, x_1, \ldots, x_{n-1}\}$ are the x-coordinates of the iterates of (x_0, y_0) under the cylinder map (8).
The cylinder map (8) possesses a p/q periodic orbit, $x_q = x_0 + \underbrace{2\pi p}_{q\mu}$, $y_q = y_0$, if and only if for some (x_0, y_0) the remainders of the q-th iterate vanish:

$$ {}_qR(x_0, y_0, \varepsilon, \delta) = {}_qS(x_0, y_0, \varepsilon, \delta) = 0. \tag{11}$$

The overall plan of the proof of Theorem 1 (the main result) is as follows: The vanishing of the remainders (11) defines y_0 and δ as functions of x_0 and δ: $y_0 = Y(x_0, \varepsilon)$, $\delta = \Delta(x_0, \varepsilon)$ Proving prove the narrowness of the Arnold tongue (as specified in Theorem 1) amounts to showing that the range $\Delta(\mathbb{R}, \varepsilon)$ is $O(\varepsilon^r)$–small, where $r = [q/d]$. The proof of this narrow range statement goes as follows: expanding Δ in powers of ε we consider the first x_0–dependent term $\Delta_r(x_0)\varepsilon^r$. Our goal is to show that $r > [q/d]$. To that end, we prove (i) that $\Delta_r(x_0)$ is periodic of period μ, and (ii), that $\Delta_r(x_0)$ is a trigonometric polynomial of degree rd. However a non-constant trigonometric polynomial periodic of period $\mu = 2\pi p/q$ must have degree $> q$, i.e. $rd > q$, or $r > [q/d]$, as desired.

4. Structure of the Remainders

In this section we examine the n-th iterate of the cylinder map (8) for any $n \in \mathbb{N}$ and the associated remainders.

For brevity, we write $g(x) = g(x; \varepsilon, \delta)$ and $g'(x) = \frac{\partial}{\partial x} g(x; \varepsilon, \delta) = -\varepsilon f'(x)$ (recall that $g(x; \varepsilon, \delta) := -\delta - \varepsilon f(x)$), and also

$$ \begin{aligned} g_k^{(r)} &:= g^{(r)}(x_0 + k\mu), \quad g_k^{(0)} = g(x_0 + k\mu) = g_k; \\ f_k^{(r)} &:= f^{(r)}(x_0 + k\mu), \quad f_k^{(0)} = f(x_0 + k\mu) = f_k. \end{aligned} $$

The followed lemma gives the structure of the remainders for any iterate of the cylinder map (8).

Lemma 1. *The remainders ${}_nR$ and ${}_nS$ are a convergent series*

$$ \begin{aligned} {}_nR &= {}_nR_1 + {}_nR_2 + \cdots \\ {}_nS &= {}_nS_1 + {}_nS_2 + \cdots , \end{aligned} $$

where ${}_nR_m$ and ${}_nS_m$ are homogeneous polynomials of degree m in terms of the items in the list $\{y_0 + g_0,\ g_k^{(l)}\ (0 \le k \le n-1,\ 0 \le l \le m-1)\}$. The two series converge for all $x_0, y_0, \varepsilon, \delta$. Moreover, each term of degree $m \ge 2$ contains at least one derivative of g (so that, with $g = -\delta - \varepsilon f(x)$ these terms are $O(\varepsilon)$). *In particular,*

$$ \begin{aligned} {}_nR_1 &= n(y_0 + g_0) + (n-1)g_1 + (n-2)g_2 + \cdots + g_{n-1}, \\ {}_nS_1 &= g_0 + g_1 + \cdots + g_{n-1}. \end{aligned} \tag{12}$$

Proof. goes by induction on n. The statement is trivially true for $n = 1$, and we show that if the remainders ${}_nR$ and ${}_nS$ are of the form claimed in the Lemma then the same is true for $n + 1$. Indeed,

$$x_{n+1} = x_n + y_n + \mu + g(x_n) \overset{(9)}{=} x_0 + (n+1)\mu + \underbrace{{}_nR + {}_nS + g(x_n)}_{{}_{n+1}R}.$$

Thanks to the inductive assumption, it only remains to show that $g(x_n)$ is of the form claimed. Taylor expansion yields

$$g(x_n) = g(x_0 + n\mu + {}_nR) = g_n + \sum_{k\geq 1} \frac{1}{k!} g_n^{(k)} \cdot ({}_nR)^k = g_n + \sum_{k\geq 1} \frac{1}{k!} g_n^{(k)} \cdot \left(\sum_{j\geq 1} {}_nR_j \right)^k . \tag{13}$$

Now mth degree part of (13) is a linear combination (with constant coefficients) of products

$$g_n^{(k)}\, {}_nR_{j_1} \cdots {}_nR_{j_r},$$

with $j_1 + \cdots + j_r = m - 1$, and is thus a homogeneous polynomial of degree m as claimed. Furthermore, every term with $m \geq 2$ comes from the series in (13) and thus contains a derivative of g.

The claim about ${}_{n+1}S$ is proven similarly:

$$y_{n+1} = y_n + g(x_n) \overset{(9)}{=} y_0 + \underbrace{{}_nS + g(x_n)}_{{}_{n+1}S}.$$

The rest of the proof is identical to the one above. This completes the proof of Lemma 1. □

Remark 5. As an illustration of the lemma, a short computation gives an explicit form of the degree-2 terms:

$$\begin{aligned} {}_nR_2 &= (n-2)g_1' \cdot (y_0 + g_0) + (n-3)g_2' \cdot (2(y_0 + g_0) + g_1) + \cdots \\ &\quad + g_{n-1}' \cdot ((n-1)(y_0 + g_0) + (n-2)g_1 + \cdots + g_{n-2}) \\ {}_nS_2 &= g_1' \cdot (y_0 + g_0) + g_2' \cdot (2(y_0 + g_0) + g_1) + \cdots \\ &\quad + g_{n-1}' \cdot ((n-1)(y_0 + g_0) + (n-2)g_1 + \cdots + g_{n-2}). \end{aligned}$$

Each term in ${}_nR_2$, ${}_nS_2$ contains the first derivative of g at some shift $x_0 + k\mu$.

5. The Existence of Periodic Orbits

In this section we discuss the existence of the p/q periodic orbits of the cylinder map (5) using Implicit Function Theorem and then we prove Theorem 2.
We begin by showing that the equations

$$ {}_qR(x, y, \varepsilon, \delta) = {}_qS(x, y, \varepsilon, \delta) = 0 $$

uniquely determine δ and y as functions of x and ε for all $x \in \mathbb{R}$ and for all $|\varepsilon| < \bar{\bar{\varepsilon}}$ for some positive $\bar{\bar{\varepsilon}}$,

$$ \begin{cases} \delta = \Delta(x, \varepsilon) \\ y = Y(x, \varepsilon), \end{cases} \tag{14} $$

such that ${}_qR$ and ${}_qS$ vanish identically if (14) hold.

Wishing to apply the implicit function theorem, we note that ${}_qR(x_0, 0, 0, 0) = {}_qS(x_0, 0, 0, 0) = 0$ for all $x \in \mathbb{R}$ and that, using Lemma 1,

$$ \left.\frac{\partial({}_qR,\ {}_qS)}{\partial(\delta,\ y_0)}\right|_{y=\varepsilon=\delta=0} = \begin{pmatrix} -\dfrac{q(q+1)}{2} & q \\ -q & 0 \end{pmatrix} $$

for all x. Since the determinant is $q^2 \neq 0$, the implicit function theorem applies: for any x_0 there exists an open disk D_{x_0} centered at the point $(x_0, 0)$ in the (x, ε)-plane such that the implicit function $(x, \varepsilon) \mapsto (\delta, y)$ is well defined by the equations ${}_qR = {}_qS = 0$ on the disk D_{x_0}. The segment $[0, 2\pi] \times \{0\}$ in the (x, ε)-plane is covered by open disks and thus has a finite subcover; but then this finite union contains a strip nonzero width $\bar{\bar{\varepsilon}} > 0$ around the x–axis. Moreover, the functions corresponding to the overlapping disks coincide. Thus the implicit function $(x, \varepsilon) \mapsto (\Delta, Y)$ is defined for all x and for all $|\varepsilon| < \bar{\bar{\varepsilon}}$.

Proof of Theorem 2. The proof proceeds by induction. If (x_0, y_0) is a p/q-periodic point of the map (5) for some δ, then (14) holds Proving the theorem thus amounts to showing that the coefficients in the expansion of $Y(x_0, \varepsilon)$ in powers of ε are polynomials in f_k and its derivatives up to order $k-1$. We fix $\varepsilon < \bar{\bar{\varepsilon}}$ so that Δ and Y are well-defined. As the first step in induction we show that Δ_1 and Y_1 in the expansions

$$ \begin{aligned} \Delta(x_0) &= \Delta_1(x_0)\varepsilon + o(\varepsilon), \\ Y(x_0) &= Y_1(x_0)\varepsilon + o(\varepsilon) \end{aligned} $$

are polynomials of degree 1 (i.e. linear) in f_k, where $0 \leq k \leq q-1$.
Indeed, by Lemma 1,

$$ {}_qS(x_0, Y, \varepsilon, \Delta) = \sum_{m \geq 1} {}_qS_m(x_0, Y, \varepsilon, \Delta), $$

where, by (12), the linear term is

$$ {}_qS_1(x_0, Y, \varepsilon, \Delta) = -q\Delta - \varepsilon q \overline{f}(x_0) \text{ with } \overline{f}(x_0) = \frac{1}{q}\sum_{k=0}^{q-1} f_k, $$

while each higher-degree term ${}_qS_m$ ($m \geq 2$) is a homogeneous polynomial of degree m in the items from the list $\{Y - \Delta - \varepsilon f_0,\ -\Delta - \varepsilon f_k,\ \varepsilon f_k^{(l)}\}$ with $0 \leq k \leq q-1,\ 0 \leq l \leq m-1$ and thus is of order $\mathcal{O}(\varepsilon^2)$ since $\Delta, Y \sim \mathcal{O}(\varepsilon)$, so that

$$ {}_qS(x_0, Y, \varepsilon, \Delta) = -q\Delta - \varepsilon q \overline{f}(x_0) + o(\varepsilon). $$

Since the above expression vanishes by the definition of Y, Δ, we conclude that

$$ \Delta = -\varepsilon \overline{f}(x_0) + o(\varepsilon), \tag{15} $$

so that the leading coefficient of ε is

$$ \Delta_1(x_0) = -\overline{f}(x_0). $$

Similarly, by Lemma 1, we have

$$ {}_qR(x_0, Y, \varepsilon, \Delta) = \sum_{m\geq 1} {}_qR_m(x_0, Y, \varepsilon, \Delta), $$

where

$$ {}_qR_1(x_0, Y, \varepsilon, \Delta) \overset{(12)}{=} qY - \frac{q(q+1)}{2}\Delta - \varepsilon \sum_{k=0}^{q-1}(q-k) f_k, $$

while each higher-degree term ${}_qR_m$ ($m \geq 2$) is a degree-m homogeneous polynomial in the items from the list $\{Y - \Delta - \varepsilon f_0,\ \ -\Delta - \varepsilon f_k,\ \ \varepsilon f_k^{(l)}\}$, where $0 \leq k \leq q-1,\ 0 \leq l \leq m-1$; and since $Y, \Delta \sim \mathcal{O}(\varepsilon)$ all ${}_qR_m$ with $m \geq 2$ are at most $\mathcal{O}(\varepsilon^2)$, so that

$$ \underbrace{{}_qR(x_0, Y, \varepsilon, \Delta)}_{=0} = qY - \frac{q(q+1)}{2}\Delta - \varepsilon \sum_{k=0}^{q-1}(q-k) f_k + o(\varepsilon). $$

Substituting into this we obtain that $Y(x_0, \varepsilon) = Y_1(x_0)\varepsilon + o(\varepsilon)$, and (15) results in

$$ Y_1(x_0) = -\frac{q+1}{2}\overline{f}(x_0) + \overline{\overline{f}}(x_0), $$

where $\overline{\overline{f}}(x_0) = \frac{1}{q}\sum_{k=0}^{q-1}(q-k) f_k$.

This completes the first step of induction. To carry out the nth inductive step, let $n > 1$ and assume that in the expansion

$$ \begin{aligned} \Delta(x_0) &= \Delta_1(x_0)\varepsilon + \cdots + \Delta_n(x_0)\varepsilon^n + \cdots, \\ Y(x_0) &= Y_1(x_0)\varepsilon + \cdots + Y_n(x_0)\varepsilon^n + \cdots, \end{aligned} \tag{16} $$

each $\Delta_m(x_0)$ and $Y_m(x_0)$ with $m \le n$ is a polynomial of degree m in $f_k^{(l)}$ with $0 \le k \le q-1, 0 \le l \le m-1$. Our goal is show that then the coefficients Δ_{n+1} and Y_{n+1} are polynomials of degree $n+1$ in $f_k^{(l)}$ with $0 \le k \le q-1, 0 \le l \le n$.

Just as in the first step, we obtain Δ_{n+1} by extracting the coefficient of ε^{n+1} in ${}_qS$:

$$0 = {}_qS(x_0, Y, \varepsilon, \Delta) = {}_qS_1(x_0, Y, \varepsilon, \Delta) + \sum_{m \ge 2}^{n+1} {}_qS_m(x_0, Y, \varepsilon, \Delta) + \sum_{m > n+1} {}_qS_m(x_0, Y, \varepsilon, \Delta).$$

For $m > n+1$, $q S_m(x_0, Y, \varepsilon, \Delta) \sim o(\varepsilon^{n+1})$, and hence the last sum does not contribute powers of degree $n+1$. On the other hand, since

$${}_qS_1(x_0, Y, \varepsilon, \Delta) = -q\Delta - \varepsilon q \overline{f}(x_0),$$

${}_qS_1$ contributes $-q\Delta_{n+1}(x_0)\varepsilon^{n+1}$, a *constant* multiple of Δ_{n+1}. It thus suffices to show that the terms in the middle sum are polynomials of degree up to $n+1$ in terms of $f_k^{(l)}$ with $0 \le k \le q-1,\ 0 \le l \le n$.

For each $m \in [2, n+1]$, ${}_qS_m(x_0, Y, \varepsilon, \Delta)$ is a degree-m homogeneous polynomial in the items from the list $\{Y - \Delta - \varepsilon f_0,\ -\Delta - \varepsilon f_k,\ \varepsilon f_k^{(l)}\}$ with $0 \le k \le q-1,\ 0 \le l \le m-1$. Thus by the inductive assumption, the coefficient of ε^{n+1} in ${}_qS_m$ $(2 \le m \le n+1)$ is a linear combination of the terms

$$(Y_i \varepsilon^i)^{m_i} (\Delta_j \varepsilon^j)^{m_j} (f_k^{(l)} \varepsilon)^{m_s}$$

with $im_i + jm_j + m_s = n+1, 0 \le l \le m-1$.

Since, by Lemma 1, each higher-degree term ${}_qS_m$ $(m \ge 2)$ has at least one derivative of g, it follows that $m_s \ge 1$ and consequently $i, j \le n$, so that the inductive assumption applies to Y_i and Δ_j above. Thus Y_i is a polynomial of degree i in $f_k^{(l)}$ with $0 \le k \le q-1$ with $0 \le l \le i-1$, and similarly, Δ_j is a polynomial of degree j with $0 \le k \le q-1,\ 0 \le l \le j-1$. Therefore $(Y_i)^{m_i}(\Delta_j)^{m_j}(f_k^{(l)})^{m_s}$ is a polynomial of degree $n+1$ in $f_k^{(l)}$ with $0 \le k \le q-1,\ 0 \le l \le n$. This completes the inductive step for Δ_{n+1}. The step for Y_{n+1} is carried out in the same way, and we have

$$0 = {}_qR(x_0, Y, \varepsilon, \Delta) = {}_qR_1(x_0, Y, \varepsilon, \Delta) + \sum_{m \ge 2}^{n+1} {}_qR_m(x_0, Y, \varepsilon, \Delta) + \sum_{m > n+1} {}_qR_m(x_0, Y, \varepsilon, \Delta).$$

Just as before, the last sum does not contribute to the coefficient of ε^{n+1}. The first term

$${}_qR_1(x_0, Y, \varepsilon, \Delta) = qY - \frac{q(q+1)}{2}\Delta - \varepsilon q \overline{\overline{f}}(x_0)$$

contributes

$$\left(qY_{n+1} - \frac{q(q+1)}{2}\Delta_{n+1}\right)\varepsilon^{n+1}. \tag{17}$$

The coefficients of ε^{n+1} in the middle sum are polynomials of degree at most $n+1$ in terms of f, its shifts and its derivatives up to order n, precisely as we proved before when treating Δ_{n+1}. This shows that the coefficient in (17) is a polynomial of degree at most $n+1$ in terms of f, its shifts and its derivatives up to order n. Since the same is true for Δ_{n+1}, this holds for Y_{n+1} as well, thus completing the induction step and the proof of Theorem 2. □

6. The Periodicity Lemma

In this section we show that the leading *x-dependent* coefficient in the ε-expansion of $\Delta(x, \varepsilon)$ is periodic of period μ. This fact plays a key role in the proof of Theorem 1. Before proving this periodicity we show that this leading coefficient is also the leading term up to a constant factor in ${}_qR$ and ${}_qS$ as well, where is another crucial fact.

Let $r \geq 1$ be the smallest power of ε where x first appears in the coefficient of the expansion of Δ in powers of ε so that

$$\Delta(x_0, \varepsilon) = A(\varepsilon) + \Delta_r(x_0)\varepsilon^r + o(\varepsilon^r), \tag{18}$$

where $A(\varepsilon)$ is a polynomial in ε of degree at most $r-1$ with constant coefficients. We claim that replacing Δ with its constant part $A(\varepsilon)$ in ${}_qR(x_0, Y, \varepsilon, \Delta) = 0$ and ${}_qS(x_0, Y, \varepsilon, \Delta) = 0$ changes these from 0 by the amount proportional to $\Delta_r(x_0)$ in the leading order:

$${}_qR(x_0, Y, \varepsilon, A(\varepsilon)) = \frac{q(q+1)}{2}\Delta_r(x_0)\varepsilon^r + o(\varepsilon^r) \tag{19}$$

and

$${}_qS(x_0, Y, \varepsilon, A(\varepsilon)) = q\Delta_r(x_0)\varepsilon^r + o(\varepsilon^r). \tag{20}$$

Here $Y = Y(x_0, \varepsilon)$. Indeed, by Lemma 1, we have

$$\begin{aligned} &{}_qR(x_0, Y, \varepsilon, \Delta) - {}_qR(x_0, Y, \varepsilon, A(\varepsilon)) \\ &= \sum_{m\geq 1} {}_qR_m(x_0, Y, \varepsilon, \Delta) - {}_qR_m(x_0, Y, \varepsilon, A(\varepsilon)). \end{aligned}$$

Starting with $m = 1$, the terms ${}_qR_1$ and ${}_qS_1$ are linear in δ with constant coefficients, according to (12) More precisely,

$$\begin{aligned} &{}_qR_1(x_0, Y, \varepsilon, \Delta) - {}_qR_1(x_0, Y, \varepsilon, A(\varepsilon)) \\ &= q(Y - \Delta - \varepsilon f_0) + \sum_{k=1}^{q-1}(q-k)(-\Delta - \varepsilon f_k) \end{aligned}$$

$$
\begin{aligned}
&- q(y_0 - A(\varepsilon) - \varepsilon f_0) - \sum_{k=1}^{q-1}(q-k)(-A(\varepsilon) - \varepsilon f_k) \\
&= -q\Delta_r(x_0)\varepsilon^r - \sum_{k=1}^{q-1}(q-k)\Delta_r(x_0)\varepsilon^r + o(\varepsilon^r) \\
&= -\frac{q(q+1)}{2}\Delta_r(x_0)\varepsilon^r + o(\varepsilon^r).
\end{aligned}
$$

To complete the proof of (19) it suffices to show that

$$
{}_qR_m(x_0, Y, \varepsilon, \Delta) - {}_qR_m(x_0, Y, \varepsilon, A(\varepsilon)) = \mathcal{O}(\varepsilon^{r+1}) \ \text{ for } \ m \geq 2. \tag{21}
$$

By Lemma 1, ${}_qR_m(x_0, Y, \varepsilon, \delta)$ is a homogeneous polynomial of degree m in the items from the list $\{Y + g_0, g_1, \ldots, g_{q-1}, g_k^{(l)}\}$ with $1 \leq l \leq m-1,\ 0 \leq k \leq q-1$ and it contains at least one derivative $g_k^{(l)}$ for some $1 \leq l \leq m-1,\ 0 \leq k \leq q-1$, thus contributing an extra factor of ε. Since $g(x_0; \varepsilon, \delta) = -\delta - \varepsilon f(x_0)$, replacing $\delta = \Delta = A + \Delta_r\varepsilon^r + o(\varepsilon^r)$ by $A(\varepsilon)$ changes ${}_qR_m$ by $\mathcal{O}(\varepsilon^r) \cdot \varepsilon = \mathcal{O}(\varepsilon^{r+1})$, thus completing the proof of (19). The proof of (20) is identical and therefore omitted.

Lemma 2. *(Periodicity) For all sufficiently small ε the leading x-dependent coefficient Δ_r in the expansion* (18) *of Δ is periodic in μ, and for any x we have*

$$
\Delta_r(x + \mu) = \Delta_r(x).
$$

Proof. We *fix* an initial point $(x_0, y_0 = Y(x_0, \varepsilon))$ and set $\delta = \Delta(x_0, \varepsilon)$ in the cylinder map (8) (and consequently $g(x_0; \varepsilon, \delta) = -\Delta(x_0) - \varepsilon f(x))$.

For future use we observe (dropping ε from the notation for the sake of brevity) that

$$
Y(x_1) = y_1, \quad \Delta(x_1) = \Delta(x_0) \tag{22}
$$

for all sufficiently small ε. Indeed, the orbit $(x_0, y_0 = Y(x_0)), (x_1, y_1), (x_2, y_2), \ldots$ is q-periodic under the map with $\delta = \Delta(x_0, \varepsilon)$ by the definition of Y and Δ; thus ${}_qR$ and ${}_qS$ vanish at any point of this orbit, and, in particular, at (x_1, y_1),

$$
{}_qR(x_1, y_1, \varepsilon, \Delta(x_0)) = {}_qS(x_1, y_1, \varepsilon, \Delta(x_0)) = 0. \tag{23}
$$

On the other hand, by the definition of Y and Δ,

$$
{}_qR(x_1, Y(x_1), \varepsilon, \Delta(x_1)) = {}_qS(x_1, Y(x_1), \varepsilon, \Delta(x_1)) = 0. \tag{24}
$$

Provided that the conditions of the implicit function theorem in Section 5 are satisfied, the solution is unique, and thus comparison of (23) and (24) implies (22). The conditions of the implicit function theorem are satisfied if ε is restricted to be sufficiently small, and more precisely, that $|y_1| < \bar{\bar{\varepsilon}}$. To thus end we note that

$$
|y_1| = |Y(x_0) - \Delta(x_0) - \varepsilon f(x_0)| \leq c\varepsilon,
$$

where c is a constant depending only on q and on $\max|f|$. It thus suffices to restrict ε to $\varepsilon < \bar{\varepsilon} := \min(\bar{\bar{\varepsilon}}/c, \bar{\bar{\varepsilon}})$, which we do from now on.

We now proceed with the rest of the proof. Recalling that $y_0 = Y(x_0)$ we have

$$ {}_qS(x, Y(x), \varepsilon, A(\varepsilon))\Big|_{x=x_0}^{x=x_1} \overset{20}{=} q(\Delta_r(x_1) - \Delta_r(x_0))\varepsilon^r + o(\varepsilon^r). \tag{25} $$

Since $x_1 = x_0 + \mu + y_0 - \Delta(x_0) - \varepsilon f(x_0) = x_0 + \mu + O(\varepsilon)$, this implies

$$ \begin{aligned} &{}_qS(x_1, Y(x_1), \varepsilon, A(\varepsilon)) - {}_qS(x_0, y_0, \varepsilon, A(\varepsilon)) \\ &\quad = q(\Delta_r(x_0 + \mu) - \Delta_r(x_0))\varepsilon^r + o(\varepsilon^r). \end{aligned} \tag{26} $$

We now show that the left-hand side is $o(\varepsilon^r)$ (thus completing the proof of the lemma). Consider the orbit $(\tilde{x}_k, \tilde{y}_k)$ of the same initial point (x_0, y_0) but under the map with $\delta = A(\varepsilon)$ (instead of $\delta = \Delta(x_0)$). We will show that

$$ {}_qS(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon)) - {}_qS(x_0, y_0, \varepsilon, A(\varepsilon)) = o(\varepsilon^r) \tag{27} $$

and

$$ {}_qS(x_1, Y(x_1), \varepsilon, A(\varepsilon)) - {}_qS(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon)) = o(\varepsilon^r), \tag{28} $$

thus implying that the left-hand side of 25 is $o(\varepsilon^r)$. **Proof of** (27). By (10) we have

$$ \begin{aligned} &{}_qS(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon)) - {}_qS(x_0, y_0, \varepsilon, A(\varepsilon)) \\ &= -qA(\varepsilon) - \varepsilon \sum_{k=0}^{q-1} f(\tilde{x}_k) + qA(\varepsilon) + \varepsilon \sum_{k=1}^{q} f(\tilde{x}_k) \\ &= -\varepsilon(f(\tilde{x}_q) - f(x_0)). \end{aligned} $$

However

$$ \tilde{x}_q \overset{(9)}{=} x_0 + 2p\pi + {}_qR\left(x_0, y_0, \varepsilon, A(\varepsilon)\right), $$

which together with (19), shows that the last difference in parentheses is $O(\varepsilon^r)$, thus implying 27.

Proof of (28). By Lemma 1,

$$ {}_qS(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon)) = \sum_{m\geq 1} {}_qS_m(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon)) $$

and

$$ {}_qS(x_1, y_1, \varepsilon, A(\varepsilon)) = \sum_{m\geq 1} {}_qS_m(x_1, y_1, \varepsilon, A(\varepsilon)), $$

where ${}_qS_m(x, y, \varepsilon, A(\varepsilon))$ is a degree-m homogeneous polynomial in the items from the list $\{y - A(\varepsilon) - \varepsilon f(x),\ -A(\varepsilon) - \varepsilon f(x + k\mu),\ \varepsilon f^{(l)}(\ x + k\mu)\ \}$ with $0 \leq k \leq q-1,\ 1 \leq l \leq m-1$. We will show that the corresponding terms in each sum differ by $o(\varepsilon^r)$. We have

$$ \begin{aligned} x_1 &= x_0 + y_0 + \mu - \Delta(x_0) - \varepsilon f(x_0) \\ y_1 &= y_0 - \Delta(x_0) - \varepsilon f(x_0); \end{aligned} $$

and

$$\begin{aligned}\tilde{x}_1 &= x_0 + y_0 + \mu - A(\varepsilon) - \varepsilon f(x_0)\\ \tilde{y}_1 &= y_0 - A(\varepsilon) - \varepsilon f(x_0).\end{aligned}$$

Since $\Delta(x_0) = A(\varepsilon) + \Delta_r(x_0)\varepsilon^r + o(\varepsilon^r)$, this implies

$$\begin{aligned}\tilde{x}_1 - x_1 &= \Delta_r(x_0)\varepsilon^r + o(\varepsilon^r)\\ \tilde{y}_1 - y_1 &= \Delta_r(x_0)\varepsilon^r + o(\varepsilon^r)\\ f^{(l)}(\tilde{x}_1 + k\mu) - f^{(l)}(x_1 + k\mu) &= f^{(l+1)}(x_1 + k\mu)\Delta_r(x_0)\varepsilon^r + o(\varepsilon^r).\end{aligned} \tag{29}$$

This $O(\varepsilon^r)$ difference in x and y results in the $o(\varepsilon^r)$ difference in the terms ${}_qS_m$ as we now show. Starting with $m = 1$ we have

$$\begin{aligned}&{}_qS_1(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon)) - {}_qS_1(x_1, y_1, \varepsilon, A(\varepsilon))\\ &\overset{(12)}{=} -\varepsilon \sum_{k=0}^{q-1} (f(\tilde{x}_1 + k\mu) - f(x_1 + k\mu))\\ &= -\varepsilon \sum_{k=0}^{q-1} f'(x_1 + k\mu)\Delta_r(x_0)\varepsilon^r + o(\varepsilon^r) = o(\varepsilon^r).\end{aligned}$$

For $m \geq 2$, according to Lemma 1, each term in ${}_qS_m$ contains at least one derivative of $g(x; \varepsilon, A(\varepsilon)) = -A(\varepsilon) - \varepsilon f(x)$ for both ${}_qS_m(x_1, y_1, \varepsilon, A(\varepsilon))$ and ${}_qS_m(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon))$, which contributes a factor of ε. This, together with (29), implies

$${}_qS_m(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon)) - {}_qS_m(x_1, y_1, \varepsilon, A(\varepsilon)) = o(\varepsilon^r).$$

We showed that ${}_qS(x_1, y_1, \varepsilon, A(\varepsilon)) - {}_qS(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon)) = o(\varepsilon^r)$. Since $y_1 = Y(x_1)$ this proves (28), thus completing the proof of the lemma. □

7. End of Proof of the Main Theorem

In this last section we complete the proof of Theorem 1 using the results of the previous sections. The main idea, similar to [1], is to observe that if f is a trigonometric polynomial of degree d then Δ_r is a trigonometric polynomial of degree rd. Since Δ_r is nonconstant (by the definition) and periodic of period $2\pi p/q$, one must have $rd > q$, so that $r > [q/d]$. This would complete the proof of the theorem, since $\Delta(x, \varepsilon) = A(\varepsilon) + \Delta_r(x)\varepsilon^r + o(\varepsilon^r)$ implies that the range of δ for which p/q-periodic points exist is at most $O(\varepsilon^r)$ with $r > [q/d]$.

It remains therefore to show that Δ_r is indeed a trigonometric polynomial of degree at most rd. According to (20),

$$\varepsilon^r \Delta_r(x) = q^{-1} \sum_{m \geq 1} {}_qS_m(x, Y(x, \varepsilon), \varepsilon, A(\varepsilon)) + o(\varepsilon^r),$$

and we must show that the coefficient of ε^r in the above sum is a trigonometric polynomial of degree at most rd. According to Lemma 1, ${}_qS_m(x, Y(x), \varepsilon, A(\varepsilon))$ is homogeneous polynomial of degree m in the items from the list

$$\{Y(x) - A(\varepsilon) - \varepsilon f_0,\ -A(\varepsilon) - \varepsilon f_k,\ \varepsilon f_k^{(l)}\}, \tag{30}$$

with $0 \le k \le q-1$, $1 \le l \le m-1$, and thus only finitely many terms - namely the ones with $m \le r$ - contribute to the coefficient of ε^r. According to Theorem 2 the coefficients Y_n in the expansion

$$Y(x) = Y_1(x)\varepsilon + Y_2(x)\varepsilon^2 + \cdots$$

are nth degree polynomials in f, its shifts by μ, and its derivatives. The key point here is that the power of ε in $Y(x)$ is also the degree of the polynomial Y_k in f, its derivatives and shifts. In addition, ε enters with power 1 in the list (30) as a factor of every f and its derivatives and shifts (while $A(\varepsilon)$ has constant coefficients). This shows that the coefficient of ε^r is a polynomial of degree at most r in f, its derivatives and shifts. Moreover, this coefficient is a finite sum, since only finitely many terms ($m \le r$) contribute to it. Finally, since f is a trigonometric polynomial of degree d, it follows that the coefficient of ε^r is a trigonometric polynomial of degree at most rd, as claimed.

This completes the proof of Theorem 1.

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