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Entropy dissipation and propagation of chaos for the uniform reshuffling model

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We investigate the uniform reshuffling model for money exchanges: two agents picked uniformly at random redistribute their dollars between them. This stochastic dynamics is of mean-field type and eventually leads to a exponential distribution of wealth. To better understand this dynamics, we investigate its limit as the number of agents goes to infinity. We prove rigorously the so-called propagation of chaos which links the stochastic dynamics to a (limiting) nonlinear partial differential equation (PDE). This deterministic description, which is well-known in the literature, has a flavor of the classical Boltzmann equation arising from statistical mechanics of dilute gases. We prove its convergence toward its exponential equilibrium distribution in the sense of relative entropy.

Keywords: Agent-based model; uniform reshuffling; propagation of chaos; relative entropy.

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1. Introduction

Econophysics is an emerging branch of statistical physics that incorporate notions and techniques of traditional physics to economics and finance.[19, 23, 38] It has attracted considerable attention in recent years raising challenges on how various economical phenomena could be explained by universal laws in statistical physics, and we refer to Refs. 16, 17, 30 and 36, for a general review.

The primary motivation for studying such models arising from econophysics is at least two-fold: From the perspective of a policy maker, it is important to deal with the raise of income inequality[21, 22] in order to establish a more equalitarian society. From a mathematical point of view, we have to understand the fundamental mechanisms, such as money exchange resulting from individuals, which are usually agent-based models. Given an agent-based model, one is expected to identify the limit dynamics as the number of individuals tends to infinity and then its corresponding equilibrium when the model is run for a sufficiently long time (if there is one), and this guiding approach is carried out in numerous works across different fields among literature of applied mathematics, see for instance, Refs. 5, 12 and 35.

In this work, we consider the so-called uniform reshuffling model for money exchange in a closed economic system with N agents and NM total amount of dollars. The dynamics consists in choosing at random time two individuals and to redistribute their money between them. To write this dynamics mathematically, we denote by $X_i(t)$ the amount of dollar agent i has at time t for $1 \leq i \leq N$. At a random time generated by a Poisson clock with rate N, two agents (say i and j) update their purse according to the following rule:

$$\big(X_i, X_j\big) \rightsquigarrow \big(U(X_i + X_j), (1-U)(X_i + X_j)\big), \tag{1.1}$$

where U is a uniform random variable over the interval $[0,1]$ (i.e. $U \sim \text{Uniform}[0,1]$). The *uniform reshuffling model* is first studied in Ref. 23 via simulation. The agent-based numerical simulation suggests that, as the number of agents and time go to infinity, the limiting distribution of money approaches the exponential distribution as shown in Fig. 1. It is well-known (see for instance Refs. 3, 7, 24 and 33) that under the large population $N \to \infty$ limit, We can formally show that the law of the wealth of a typical agent (say X_1) satisfies the following limit PDE in a weak sense:

$$\partial_t q(t,x) = \int_0^\infty \int_0^\infty \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q(t,k)q(t,\ell)\,d\ell\,dk - q(t,x). \tag{1.2}$$

Well-posedness of the solution to (1.2) starting from a smooth initial distribution has been established in Theorem 6 in Ref. 6, thus throughout this work we will assume that $q(t,\cdot)$ is smooth for all $t \geq 0$. To our best knowledge, the rigorous derivation of the limit equation (1.2) from the particle system description is absent in most of the literature on econophysics (just like many other PDEs arising from models in econophysics[14, 26, 29]), because the propagation of chaos effect is implicitly assumed in the large N limit in most derivations. The remarkable exception

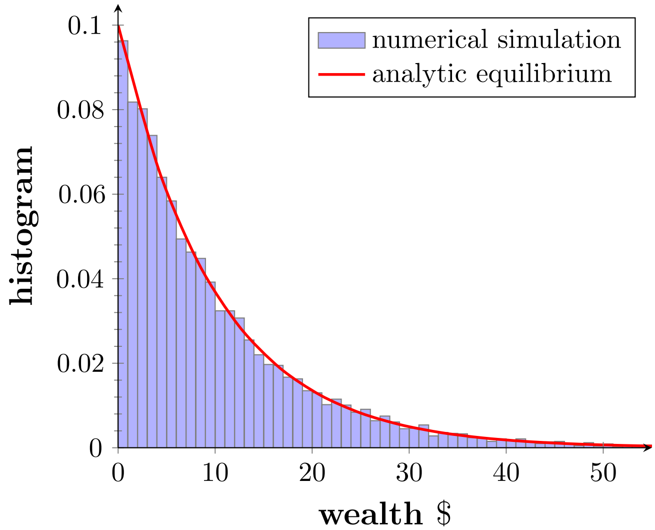

Fig. 1. (Color online) Simulation results for the uniform reshuffling model. The blue histogram shows the distribution of money after $T = 1000$ time unit. The red solid curve is the limiting exponential distribution proved in Ref. 32. We used $N = 10{,}000$ agents, each starting with \$10.

is a work of Cortez,[20] in which the author showed a uniform-in-time propagation of chaos by virtue of a delicate coupling argument based on optimal transport. In Sec. 5 of this paper, we will provide an alternative rigorous justification of Eq. (1.2) under the limit $N \to \infty$.

Once the limit PDE is identified from the interacting particle system, the natural next step is to study the problem of convergence to equilibrium of the PDE at hand, it has been shown in Refs. 24 and 33 that the unique (smooth) solution of (1.2) converges to its exponential equilibrium distribution exponentially fast in Wasserstein and Fourier metrics. In this work, we demonstrate a polynomial convergence in time using relative entropy, by establishing a entropy–entropy dissipation inequality (see Theorem 4.1) which is not available among the literature. An illustration of the general strategy used in this work (and implicitly in many of the works cited above) is shown in Fig. 2.

Although only a very specific binary exchange model is explored in the present paper, other exchange rules can also be imposed and studied, leading to different models. To name a few, the so-called immediate exchange model introduced in Ref. 26 assumes that pairs of agents are randomly and uniformly picked at each random time, and each of the agents transfer a random fraction of its money to the other agents, where these fractions are independent and uniformly distributed in $[0, 1]$. The uniform reshuffling model with saving propensity investigated in Refs. 15 and 32 suggests that the two interacting agents keep a fixed fraction λ of their fortune and only the combined remaining fortune is uniformly reshuffled between the two agents, which makes the uniform reshuffling model the particular case $\lambda = 0$.

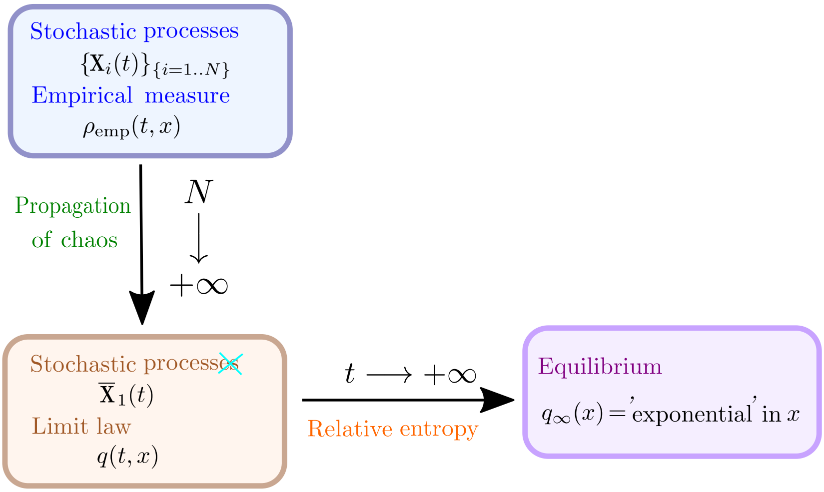

Fig. 2. Schematic illustration of the general strategy of our treatment of the uniform reshuffling dynamics.

For more variants of binary exchange models with (random) saving propensity and with debts, we refer the readers to Refs. 18 and 31. For recent work on other models from econophysics, we recommend Refs. 8–11.

This paper is organized as follows: in Sec. 2, we briefly discuss the properties of the limit equation (1.2). We show in Sec. 3 convergence results for the solution of (1.2) in Wasserstein distance and in the linearized region. We take on the most delicate analysis of the entropy–entropy dissipation relation in Sec. 4. Finally, we present a rigorous treatment of the propagation of chaos phenomenon in Sec. 5.

2. The Limit PDE and Its Properties

We present a heuristic argument behind the derivation of the limit PDE (2.8) arising from the uniform reshuffling dynamics in Sec. 2.1. Several elementary properties of the solution of (2.8) are recorded in Sec. 2.2. Section 2.3 is devoted to another formulation of the uniform reshuffling model, which can be viewed as a *lifting* of the reshuffling mechanics (1.1) and is implicitly exploited in Ref. 3. In Sec. 2.4, we highlight a key ingredient known as the *micro-reversibility*, of the collision operator determined by the right side of (2.8), which allows us to construct certain Lyapunov functions associated with (2.8) (such as entropy).

2.1. *Formal derivation of the limit PDE*

Introducing $N_t^{(i,j)}$ independent Poisson processes with intensity $1/N$, the dynamics can be written as

$$dX_i(t) = \sum_{j=1..N, j\neq i} \big(U(t-)(X_i(t-) + X_j(t-)) - X_i(t-)\big)\, dN_t^{(i,j)} \tag{2.1}$$

with $U(t) \sim \mathrm{Uniform}[0,1]$ independent of $\{X_i(t)\}_{1\le i\le N}$ and is generated as follows: whenever a Poisson clock $N_t^{(i,j)}$ rings, we generate a Uniform$[0,1]$-distributed random variable U independent of the past. As the number of players N goes to infinity, one could expect that the processes $X_i(t)$ become independent and of same law. Therefore, the limit dynamics would be of the form

$$d\overline{X}(t) = \big(U(t-)(\overline{X}(t-) + \overline{Y}(t-)) - \overline{X}(t-)\big)\, d\overline{N}_t, \tag{2.2}$$

where $\overline{Y}(t)$ is an independent copy of $\overline{X}(t)$ and $\overline{N}_t$ a Poisson process with intensity 1. Taking a test function φ, the weak formulation of the dynamics is given by

$$d\mathbb{E}[\varphi(\overline{X}(t))] = \mathbb{E}\big[\varphi\big(U(t)(\overline{X}(t) + \overline{Y}(t))\big) - \varphi(\overline{X}(t))\big]\, dt. \tag{2.3}$$

In short, the limit dynamics correspond to the jump process:

$$\overline{X} \rightsquigarrow U(\overline{X} + \overline{Y}). \tag{2.4}$$

Let us denote $q(t,x)$ the law of the process $\overline{X}(t)$. To derive the evolution equation for $q(t,x)$, we need to translate the effect of the jump of $\overline{X}(t)$ via (2.4) onto $q(t,x)$.

Lemma 2.1. (Hierarchy of probability distributions) *Suppose X and Y are two independent random variables with probability density $q(x)$ supported on $[0,\infty)$. Let $Z = U(X+Y)$ with $U \sim$ Uniform$([0,1])$ independent of X and Y. Then the density for the law of Z is given by $Q_+[q]$ with*

$$Q_+[q](x) = \int_{m=0}^{\infty} \frac{\mathbb{1}_{[0,m]}(x)}{m} \left(\int_{z=0}^{m} q(z)q(m-z)\,dz\right) dm \tag{2.5}$$

$$= \int_{\mathbb{R}_+\times\mathbb{R}_+} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q(k)q(\ell)\,d\ell\,dk. \tag{2.6}$$

Proof. Let us introduce a test function φ.

$$\begin{aligned}\mathbb{E}[\varphi(U(X+Y))] &= \int_{x\ge 0}\int_{y\ge 0}\int_{u=0}^{1} \varphi(u(x+y))q(x)q(y)\,du\,dx\,dy \\ &= \int_{m\ge 0}\int_{z=0}^{m}\int_{u=0}^{1} \varphi(um)q(z)q(m-z)\,du\,dz\,dm \\ &= \int_{m\ge 0}\int_{z=0}^{m} \frac{1}{m}\int_{s=0}^{m} \varphi(s)q(z)q(m-z)\,ds\,dz\,dm\end{aligned}$$

using the change of variables $z = x$ and $m = x+y$ followed by $s = um$. We conclude using Fubini that

$$\begin{aligned}\mathbb{E}[\varphi(U(X+Y))] &= \int_{s\ge 0} \varphi(s)\left(\int_{m\ge 0} \mathbb{1}_{[0,m]}(s)\frac{1}{m}\int_{z=0}^{m} q(z)q(m-z)\,dz\,dm\right) ds \\ &= \int_{s\ge 0} \varphi(s)Q_+[q](s)\,ds \qquad (2.7)\end{aligned}$$

with $Q_+[q]$ defined by (2.5). $\square$

We can now write the evolution equation for the law of $\overline{X}(t)$ (2.2), the density $q(t,x)$ satisfies weakly:

$$\partial_t q(t,x) = G[q](t,x) \quad \text{for } t \geq 0 \text{ and } x \geq 0 \tag{2.8}$$

with

$$G[q](x) := Q_+[q](x) - q(x) = \int_0^\infty \int_0^\infty \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q(k)q(\ell)\,d\ell\,dk - q(x). \tag{2.9}$$

Remark 2.1. We remark here that the well-posedness of the Boltzmann-type PDE (2.8) has been established in earlier works such as Ref. 33. Also, It has been shown in Ref. 33 that moments of $q(t)$ exist for all $t > 0$ as long as $q(0)$ has bounded moments.

2.2. *Evolution of moments*

Now we will establish several elementary properties of the solution of (2.8).

Proposition 2.1. *Assume that $q(t,x)$ is a classical (and global in time) solution of (2.8) and define by $m_k(t)$ the kth moment of q:*

$$m_k(t) := \int_0^\infty x^k q(t,x)\,dx. \tag{2.10}$$

Then

$$m_k'(t) = \frac{1}{k+1} \sum_{j=0}^{k} C_k^j m_j(t) m_{k-j}(t) - m_k(t), \tag{2.11}$$

where $C_k^j = \binom{k}{j} = \frac{k!}{j!(k-j)!}$ represents the binomial coefficient.

Proof. Notice that the moment can be written as $m_k(t) = \mathbb{E}[\overline{X}^k(t)]$, where $\overline{X}(t)$ satisfies (2.2). Thus, we use the weak formulation of the evolution equation of $q(t,x)$ (2.3) with $\varphi(x) = x^k$ and deduce that

$$\begin{aligned} m_k' &= \mathbb{E}\big[\big(U(\overline{X}+\overline{Y})\big)^k - \overline{X}^k\big] \\ &= \mathbb{E}[U^k]\mathbb{E}[(\overline{X}+\overline{Y}))^k] - m_k, \end{aligned}$$

since U is independent of $\overline{X}$ and $\overline{Y}$. Moreover, $\mathbb{E}[U^k] = \int_{u=0}^1 u^k\,du = \frac{1}{k+1}$. Using the independence of $\overline{X}$ and $\overline{Y}$ and expanding lead to (2.11). □

Corollary 2.1. *Let $q(t,x)$ solution of (2.8) and $m_k(t)$ its kth moment (2.10). The total mass and the mean are preserved, i.e. $m_0'(t) = m_1'(t) = 0$ and all the moments $m_k(t)$ converges in time exponentially fast.*

Proof. Writing (2.11) for $k = 2$ leads to

$$m_2' = -\frac{1}{3} m_2 + \frac{2}{3} m_1^2 \tag{2.12}$$

and thus $m_2(t) = 2\,m_1^2 + \big(m_2(0) - 2m_1^2\big)\mathrm{e}^{-\frac{1}{3}t}$. More generally, we proceed by induction to show that $m_k(t)$ converges exponentially, more precisely $m_k(t)$ is of the form

$$m_k(t) = m_k^* + \mathcal{O}(\mathrm{e}^{-\frac{k-1}{k+1}t}) \tag{2.13}$$

with m_k^* the limit value of $m_k(t)$. We first re-write the evolution equation of $m_k(t)$:

$$m_k'(t) = -\frac{k-1}{k+1} m_k(t) + P_{k-1}(t) \tag{2.14}$$

with $P_{k-1}(t) = \frac{1}{k+1}\sum_{j=1}^{k-1} C_k^j m_j(t) m_{k-j}(t)$. By induction, $P_{k-1}(t)$ has to converge in time. Using variation of constant in (2.14) gives

$$m_k(t) = m_k(0)\mathrm{e}^{-\frac{k-1}{k+1}t} + \mathrm{e}^{-\frac{k-1}{k+1}t}\int_{s=0}^{t} \mathrm{e}^{\frac{k-1}{k+1}s} P_{k-1}(s)\,ds, \tag{2.15}$$

which leads to (2.13). □

From the proposition, we observe that the second moment $m_2(t)$ converges exponentially toward the constant $2m_1^2$. This behavior could be expected as the equilibrium of the dynamics (2.8) is given by

$$q_\infty(x) := \frac{1}{m_1}\mathrm{e}^{-\frac{x}{m_1}} \mathbb{1}_{[0,\infty)}(x) \tag{2.16}$$

for which the second moment is equal $2m_1^2$.

Remark 2.2. Moment calculations can be useful in the study of classical spatially homogeneous Boltzmann equation, and we refer the readers to Ref. 2 for more information on this regard.

2.3. *Pairwise distribution*

Before studying the evolution of the entropy of the solution $q(t,x)$, we make a detour with another formulation of the reshuffling model using a two-particles distribution. Indeed, the jump process $\overline{X}(t)$ (2.4) is a "truncated version" of the following dynamics:

$$(\overline{X}, \overline{Y}) \rightsquigarrow \big(U(\overline{X} + \overline{Y}), (1-U)(\overline{X} + \overline{Y})\big), \tag{2.17}$$

where $U \sim \mathrm{Uniform}([0,1])$. Introducing a test function $\varphi(x,y)$, this dynamics lead to:

$$d\mathbb{E}[\varphi(\overline{X}, \overline{Y})] = \mathbb{E}\big[\varphi\big(U(\overline{X} + \overline{Y}), (1-U)(\overline{X} + \overline{Y})\big) - \varphi(\overline{X}, \overline{Y})\big]\,dt. \tag{2.18}$$

We now translate this evolution equation into a PDE.

Proposition 2.2. *Let $f(t,x,y)$ the density distribution of the process $(\overline{X}(t), \overline{Y}(t))$ defined via (2.17). It satisfies (weakly) the linear evolution equation*:

$$\partial_t f = L_+[f] - f \tag{2.19}$$

with

$$L_+[f](x,y) = \frac{1}{x+y}\int_{z=0}^{x+y} f(z, x+y-z)\,dz. \tag{2.20}$$

Proof. The evolution equation (2.17) gives

$$\begin{aligned}
&\frac{d}{dt}\int_{x,y\geq 0} f(t,x,y)\varphi(x,y)\,dx\,dy \\
&\quad = \int_{u=0}^{1}\int_{x,y\geq 0} f(t,x,y)\varphi\big(u(x+y),(1-u)(x+y)\big)\,dx\,dy\,du \\
&\qquad - \int_{x,y\geq 0} f(t,x,y)\varphi(x,y)\,dx\,dy.
\end{aligned} \tag{2.21}$$

To identify the operator associated with the equation, let us rewrite the "gain term" (dropping the dependency in time for simplicity) using two changes of variables:

$$\begin{aligned}
&\int_{u=0}^{1}\int_{x,y\geq 0} f(x,y)\varphi\big(u(x+y),(1-u)(x+y)\big)\,dx\,dy\,du \\
&\quad = \int_{u=0}^{1}\int_{m\geq 0}\int_{z=0}^{m} f(z,m-z)\varphi\big(um,(1-u)m\big)\,dz\,dm\,du \\
&\quad = \int_{x',y'\geq 0}\int_{z=0}^{x'+y'} f(z,x'+y'-z)\varphi(x',y')\frac{1}{x'+y'}\,dz\,dx'\,dy'
\end{aligned}$$

with $(x' = um, y' = (1-u)m)$ leading to $dx'\,dy' = m\,du\,dm$. $\square$

Remark 2.3. Notice that the operator L (2.20) "flattens" the distribution f over the diagonals $x+y = \text{constant}$ and thus minimizes its entropy over each diagonal (see Fig. 3). In particular, the equilibria for the dynamics are the distributions of the form: $f_*(x,y) = \phi(x+y)$.

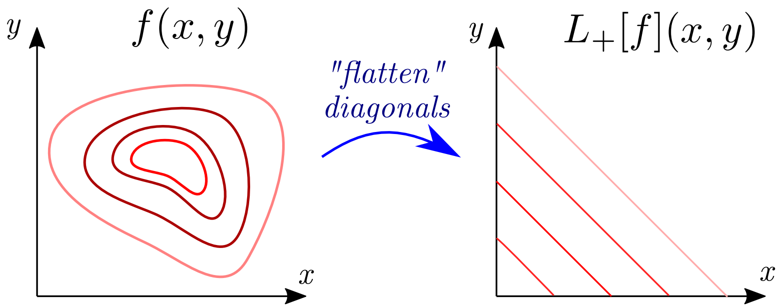

Fig. 3. The operator L_+ (2.20) flattens the distribution $f(x,y)$ over the diagonal lines $x+y = \text{constant}$.

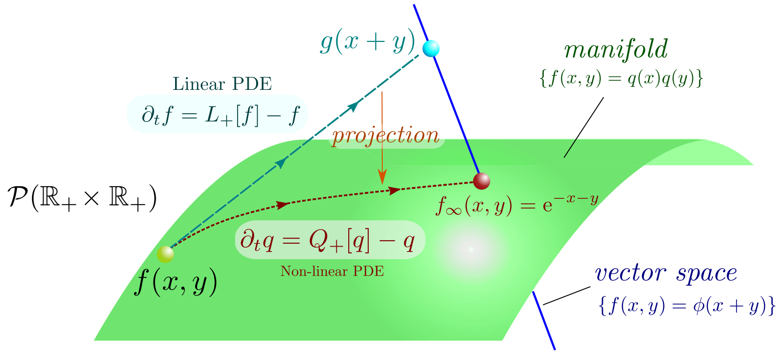

Fig. 4. Schematic representation of the evolution of $f(t,x,y)$ and $q(t,x)$. If f belongs to the manifold of independent functions, i.e. $f(t,x,y) = q(t,x)q(t,y)$, then the evolution of its marginal q satisfies *locally* the non-linear equation (2.8). Notice that the manifold of independent function is not invariant by the flow of the linear PDE. Notice that we have assumed $m_1 = 1$ so that $f_\infty(x,y) := q_\infty(x)q_\infty(y) = \mathrm{e}^{-x-y}$. Also, the definition of g appears in (4.9).

The linear operator L_+ (2.20) is linked to the non-linear operator Q_+ (2.5) as illustrated in Fig. 4. Indeed, assuming $\overline{X}$ and $\overline{Y}$ are independent, i.e. $f(x,y) = q(x)q(y)$, integrating $L_+[f]$ over the 'extra' variable y gives

$$\begin{aligned}\int_{y\geq 0} L_+[f](x,y)\,dy &= \int_{y\geq 0} \frac{1}{x+y}\int_{z=0}^{x+y} q(z)q(x+y-z)\,dz\,dy \\ &= \int_{m=x}^{+\infty} \frac{1}{m}\int_{z=0}^{m} q(z)q(m-z)\,dz\,dy = Q_+[q](x).\end{aligned}$$

2.4. *Micro-reversibility*

The evolution equation for f (2.19) corresponds to a collisional operator with the kernel:

$$K\big(x,y;x',y'\big) = \frac{1}{x+y}\delta_{x+y}(x'+y'), \tag{2.22}$$

where δ denotes the Dirac distribution. Indeed, writing $\mathbf{z} = (x,y)$, Eq. (2.19) could be written as

$$\partial_t f(\mathbf{z},t) = \int_{\tilde{\mathbf{z}}\geq 0} K\big(\tilde{\mathbf{z}};\mathbf{z}\big) f(\tilde{\mathbf{z}},t)\,d\tilde{\mathbf{z}} - \int_{\mathbf{z}'\geq 0} K\big(\mathbf{z};\mathbf{z}'\big) f(\mathbf{z},t)\,d\mathbf{z}', \tag{2.23}$$

where $\mathbf{z}' = (x',y')$ denotes the post-collision position and $\tilde{\mathbf{z}} = (\tilde{x},\tilde{y})$ the pre-collision position.

Remark 2.4. A more rigorous way to define the kernel K is through a weak formulation using a test function $\varphi(x,y)$:

$$\int_{x',y'\geq 0} K\big(x,y;x',y'\big)\varphi(x',y')\,dx'\,dy' = \frac{1}{x+y}\int_{z=0}^{x+y} \varphi(z,x+y-z)\,dz. \tag{2.24}$$

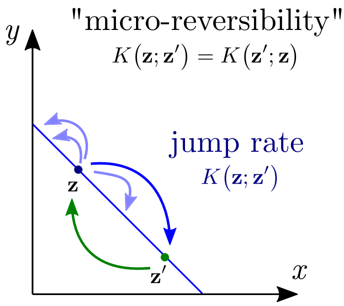

Fig. 5. The collisional kernel K (2.22) satisfies a micro-reversibility condition.

The collision kernel K satisfies a micro-reversibility condition (see Fig. 5), namely:

$$K\big(\mathbf{z};\mathbf{z}'\big) = K\big(\mathbf{z}';\mathbf{z}\big) \quad \text{for any } \mathbf{z} \text{ and } \mathbf{z}' \in \mathbb{R}_+ \times \mathbb{R}_+. \tag{2.25}$$

One has to integrate against a test function φ to make this statement rigorous. As a consequence, we deduce the lemma.

Lemma 2.2. *Let $\varphi(x,y)$ be a (smooth) test function and $f(t,x,y)$ be the solution of* (2.23)*. Then*

$$\frac{d}{dt}\int_{\mathbf{z}} f(\mathbf{z},t)\varphi(\mathbf{z})\,d\mathbf{z} = -\frac{1}{2}\int_{\mathbf{z},\mathbf{z}'} K(\mathbf{z};\mathbf{z}')\big(f(\mathbf{z}',t) - f(\mathbf{z},t)\big)\big(\varphi(\mathbf{z}') - \varphi(\mathbf{z})\big)\,d\mathbf{z}\,d\mathbf{z}'. \tag{2.26}$$

In particular, both the L^2 norm and the entropy of $f(t,x,y)$ decay in time.

Proof. We drop the dependency in time to ease the reading:

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbf{z}} f(\mathbf{z})\varphi(\mathbf{z})\,d\mathbf{z} &= \int_{\tilde{\mathbf{z}},\mathbf{z}} K(\tilde{\mathbf{z}};\mathbf{z}) f(\tilde{\mathbf{z}})\varphi(\mathbf{z})\,d\tilde{\mathbf{z}}\,d\mathbf{z} - \int_{\mathbf{z},\mathbf{z}'} K(\mathbf{z};\mathbf{z}') f(\mathbf{z})\varphi(\mathbf{z})\,d\tilde{\mathbf{z}}\,d\mathbf{z} \\
&= \int_{\mathbf{z},\mathbf{z}'} K(\mathbf{z};\mathbf{z}') f(\mathbf{z})\big(\varphi(\mathbf{z}') - \varphi(\mathbf{z})\big)\,d\mathbf{z}\,d\mathbf{z}' \\
&= \int_{\mathbf{z},\mathbf{z}'} K(\mathbf{z}';\mathbf{z}) f(\mathbf{z}')\big(\varphi(\mathbf{z}) - \varphi(\mathbf{z}')\big)\,d\mathbf{z}\,d\mathbf{z}' \\
&= \int_{\mathbf{z},\mathbf{z}'} K(\mathbf{z};\mathbf{z}') f(\mathbf{z}')\big(\varphi(\mathbf{z}) - \varphi(\mathbf{z}')\big)\,d\mathbf{z}\,d\mathbf{z}' \\
&= \frac{1}{2}\int_{\mathbf{z},\mathbf{z}'} K(\mathbf{z};\mathbf{z}')\big(f(\mathbf{z}) - f(\mathbf{z}')\big)\big(\varphi(\mathbf{z}') - \varphi(\mathbf{z})\big)\,d\mathbf{z}\,d\mathbf{z}'. \qquad \square
\end{aligned}$$

3. Convergence to Equilibrium: Wasserstein and Linearization

We carry out an linearization analysis around the exponential equilibrium distribution of the solution of (2.8) and demonstrate an explicit rate of convergence under the linearized (weighted L^2) setting in Sec. 3.1. These arguments are reinforced in Sec. 3.2 into a local convergence result for the full non-linear equation. A coupling approach is encapsulated in Sec. 3.3 in order to show that solution $q(t,x)$ of (2.8) relaxes to its equilibrium q_∞ exponentially fast in the Wasserstein distance.

3.1. *Linearization around equilibrium*

Now we perform a linearization analysis near the global exponential equilibrium q_∞, in a fashion that is similar to Ref. 4. For this purpose, we define the linear operator $\mathcal{L}$ to be

$$\mathcal{L}[h](x) := \int_0^\infty \int_0^\infty \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q_\infty(k+\ell-x)\big(h(k)+h(\ell)-h(x) - h(k+\ell-x)\big)\,dk\,d\ell.$$

Setting $q = q_\infty(1+\varepsilon h)$ in the limit $\varepsilon \to 0$, we deduce from (2.8) that

$$\partial_t h(x) = \mathcal{L}[h](x), \tag{3.1}$$

where $h \in L^2(q_\infty)$ is orthogonal to $\mathcal{N}(\mathcal{L}) := \operatorname{Span}\{1, x\}$ in $L^2(q_\infty)$ because of the conservation $\int_0^\infty q\,dx = \int_0^\infty q_\infty dx$ and $\int_0^\infty xq\,dx = \int_0^\infty xq_\infty dx$. For the linearized equation (3.1), the natural entropy is the $L^2(q_\infty)$ norm of h:

$$\mathrm{E} = \frac{1}{2}\|h\|^2_{L^2(q_\infty)} \tag{3.2}$$

and the entropy dissipation is given by

$$\begin{aligned}\frac{d}{dt}\mathrm{E} &= \int_{\mathbb{R}^3_+} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q_\infty(k)q_\infty(\ell)\big(h(k)+h(\ell) - h(k+\ell-x) - h(x)\big)h(x)\,dk\,d\ell\,dx \\ &= -\frac{1}{4}\int_{\mathbb{R}^3_+} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q_\infty(k)q_\infty(\ell)\big(h(k+\ell-x) + h(x) - h(k) - h(\ell)\big)^2\,dk\,d\ell\,dx.\end{aligned}$$

In particular, it implies that the spectrum of $\mathcal{L}$ in $L^2(q_\infty)$ is non-positive.

Remark 3.1. It is not hard to show that the linear operator $-\mathcal{L}$ enjoys a self-adjoint property on the space $L^2(q_\infty)$. Thus, the existence of a spectral gap η is equivalent to

$$\forall\, h \perp \mathcal{N}(\mathcal{L}), \quad -\langle \mathcal{L}[h],\ h\rangle_{L^2(q_\infty)} := -\int_0^\infty \mathcal{L}[h](x)h(x)q_\infty(x)\,dx \ge \eta\|h\|^2_{L^2(q_\infty)}.$$

Remark 3.2. Following Ref. 25, we give some comments on the space $L^2(q_\infty)$. If q is the unique solution of (2.8) and we set $q = q_\infty(1+\varepsilon h)$ as before for $h \perp \mathcal{N}(\mathcal{L})$, then (recall that q_∞ is the density of an exponential distribution with mean M)

$$\begin{aligned}\int_0^\infty q\log q\,dx &= \int_0^\infty q_\infty(1+\varepsilon h)\log(q_\infty(1+\varepsilon h))\,dx\\ &= \int_0^\infty q_\infty\log q_\infty\,dx + \varepsilon\int_0^\infty\left(\log\frac{1}{M}-\frac{1}{M}x\right)hq_\infty\,dx\\ &\quad+\int_0^\infty q_\infty(1+\varepsilon h)\left(\varepsilon h-\frac{(\varepsilon h)^2}{2}\pm\cdots\right)dx\\ &= \int_0^\infty q_\infty\log q_\infty\,dx+\frac{\varepsilon^2}{2}\int_0^\infty h^2q_\infty\,dx+O(\varepsilon^3),\end{aligned}$$

where we used the fact that $h\perp\mathcal{N}(\mathcal{L})$. Therefore, we can see that $\|h\|^2_{L^2(q_\infty)}=\int_0^\infty h^2q_\infty\,dx$ gives the first-order correction to the expansion of the entropy of q around q_∞.

We will prove that the linearized entropy E (3.2) decays exponentially fast in time with an explicit sharp decay rate, the essence of which lies in the following lemma.

Lemma 3.1. *Let $m_1=1$ and $\mathcal{A}:=\{h\in L^2(q_\infty)\,|\,h\perp\mathcal{N}(\mathcal{L})\}$. Then*

$$\inf_{h\in\mathcal{A}}\frac{\int_0^\infty h^2(x)q_\infty(x)\,dx}{\int_0^\infty\frac{\mathrm{e}^{-z}}{z}(\int_0^z h(x)\,dx)^2\,dz}=3 \tag{3.3}$$

and the infimum in (3.3) *is attained* (*up to a non-zero multiplication constant*) *at* $h(x)=\frac{1}{2}(x^2-4x+2)$.

Proof. The key ingredient in the proof is the fact that the so-called Laguerre polynomials, defined by

$$L_n(x)=\frac{\mathrm{e}^x}{n!}\frac{d^n}{dx^n}(\mathrm{e}^{-x}x^n)=\sum_{k=0}^n\binom{n}{k}\frac{(-1)^k}{k!}x^k,\quad n\geq 0$$

form an orthonormal basis for the weighted L^2 space $L^2(q_\infty)$.[1] Thus, for any $h\in L^2(q_\infty)$ which is not identically zero, we can write $h=\sum_{n=0}^\infty\alpha_nL_n$, in which $\alpha_n\in\mathbb{R}$ for all n. Next, notice that the condition $h\in\mathcal{A}$ implies that $\alpha_0=\alpha_1=0$. Moreover, we have $\int_0^\infty h^2(x)q_\infty(x)\,dx=\sum_{n=2}^\infty\alpha_n^2$ thanks to the orthonormality of the Laguerre polynomials $\{L_n\}_{n\geq0}$. To proceed further, we recall that[37] $L_n(z)-L_{n+1}(z)=\int_0^z L_n(x)\,dx$ and $zL_n'(z)=nL_n(z)-nL_{n-1}(z)$ for all $n\geq1$, whence

$$\begin{aligned}&\int_0^\infty\frac{\mathrm{e}^{-z}}{z}\left(\int_0^z h(x)\,dx\right)^2dz\\ &\quad=\int_0^\infty\frac{\mathrm{e}^{-z}}{z}\left(\sum_{n=2}^\infty\alpha_n(L_n(z)-L_{n+1}(z))\right)^2dz\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \mathrm{e}^{-z} \left(\sum_{n,m=2}^\infty \alpha_n \alpha_m \left(\frac{L_n(z) - L_{n+1}(z)}{z} \right) (L_m(z) - L_{m+1}(z)) \right)^2 dz \\
&= - \sum_{n,m=2}^\infty \frac{\alpha_n \alpha_m}{n+1} \int_0^\infty \mathrm{e}^{-z} (L_m(z) - L_{m+1}(z))\, dL_{n+1}(z) \\
&= \sum_{n,m=2}^\infty \frac{\alpha_n \alpha_m}{n+1} \int_0^\infty L_{n+1}(z)\, d\big(\mathrm{e}^{-z}(L_m(z) - L_{m+1}(z))\big) \\
&= \sum_{n,m=2}^\infty \frac{\alpha_n \alpha_m}{n+1} \int_0^\infty L_{n+1}(z) L_{m+1}(z) \mathrm{e}^{-z}\, dz \\
&= \sum_{n=2}^\infty \frac{\alpha_n^2}{n+1} \leq \frac{1}{3} \sum_{n=2}^\infty \alpha_n^2.
\end{aligned}
$$

Finally, notice that the inequality above will become an equality if and only if $\alpha_n = 0$ for all $n \geq 3$, or in other words, if and only if $h(x) = L_2(x) = \frac{1}{2}(x^2 - 4x + 2)$ up to a non-zero multiplication constant. $\square$

We are now in a position to prove the following result.

Theorem 3.1. *Assume that $h \in L^2(q_\infty)$ solves the linearized equation* (3.1), *then we have*

$$
\|h(t)\|_{L^2(q_\infty)} \leq \|h(0)\|_{L^2(q_\infty)} \mathrm{e}^{-\frac{1}{3}t}. \tag{3.4}
$$

Proof. We will only prove the result for $m_1 = 1$, and the general case follows readily from a change of variable argument. From the discussion above, we already have that

$$
\begin{aligned}
-\frac{d}{dt}\frac{1}{2}\|h\|^2_{L^2(q_\infty)} = \int_{\mathbb{R}^3_+} &\frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q_\infty(k) q_\infty(\ell) \\
&\cdot (h(k+\ell-x) + h(x) - h(k) - h(\ell)) h(x)\, dk\, d\ell\, dx.
\end{aligned} \tag{3.5}
$$

Thanks to $h \in \mathcal{A}$, it is not hard to see through a change of variable that

$$
\int_{\mathbb{R}^3_+} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q_\infty(k) q_\infty(\ell) h(k+\ell-x) h(x)\, dk\, d\ell\, dx = 0.
$$

Also, a simple calculation yields that

$$
\int_{\mathbb{R}^3_+} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q_\infty(k) q_\infty(\ell) h^2(x)\, dk\, d\ell\, dx = \int_0^\infty h^2(x) \mathrm{e}^{-x}\, dx
$$

and

$$
\int_{\mathbb{R}^3_+} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q_\infty(k) q_\infty(\ell) h(k) h(x)\, dk\, d\ell\, dx = \int_0^\infty \frac{\mathrm{e}^{-z}}{z} \left(\int_0^z h(x)\, dx \right)^2 dz.
$$

Consequently, (3.5) reads

$$-\frac{d}{dt}\frac{1}{2}\|h\|^2_{L^2(q_\infty)} = \int_0^\infty h^2(x)\mathrm{e}^{-x}\,dx - 2\int_0^\infty \frac{\mathrm{e}^{-z}}{z}\left(\int_0^z h(x)\,dx\right)^2 dz$$
$$\geq \frac{1}{3}\int_0^\infty h^2(x)\mathrm{e}^{-x}\,dx = \frac{1}{3}\|h\|^2_{L^2(q_\infty)},$$

in which the inequality follows directly from the previous lemma. Thus, we can conclude by Gronwall's inequality since $\frac{d}{dt}\|h\|^2_{L^2(q_\infty)} \leq -\frac{2}{3}\|h\|^2_{L^2(q_\infty)}$. □

3.2. *Local convergence in L^2*

We now extend the linearization argument from the previous subsection into a local convergence result for the full non-linear equation.

Corollary 3.1. *There exists some $\varepsilon > 0$ such that if at some time $t \geq 0$,*

$$\int \frac{|q(t,x) - q_\infty(x)|^2}{q_\infty(x)}\,dx \leq \varepsilon,$$

then q converges to q_∞ and for any $\lambda < \frac{1}{3}$, there exists C_λ such that

$$\int \frac{|q(t,x) - q_\infty(x)|^2}{q_\infty(x)}\,dx \leq C_\lambda \mathrm{e}^{-\lambda t}.$$

Proof. For a solution q, we denote $h(t,x) = (q - q_\infty)/q_\infty$ and calculate

$$-\frac{d}{dt}\frac{1}{2}\|h\|^2_{L^2(q_\infty)} = -\int h\partial_t q = -\int h(Q_+[q] - q)$$
$$= -\int hq_\infty\mathcal{L}[h] - \int h(x)q_\infty(x)\frac{\mathbb{1}_{x\leq k+\ell}}{k+\ell}q_\infty(k+\ell-x)$$
$$\times h(k)h(\ell)\,dx\,dk\,d\ell.$$

Denote

$$R(x) = \int \frac{\mathbb{1}_{x\leq k+\ell}}{k+\ell}q_\infty(k+\ell-x)h(k)h(\ell)\,dk\,d\ell$$

and calculate

$$\left|\int h(x)q_\infty(x)R(x)\,dx\right| \leq \left(\int q_\infty(x)\frac{\mathbb{1}_{x\leq k+\ell}}{k+\ell}q_\infty(k+\ell-x)h^2(k)h^2(\ell)\,dx\,dk\,d\ell\right)^{1/2}$$
$$\cdot\left(\int h^2(x)q_\infty(x)\frac{\mathbb{1}_{x\leq k+\ell}}{k+\ell}q_\infty(k+\ell-x)\,dx\,dk\,d\ell\right)^{1/2}.$$

So first of all,

$$\int q_\infty(x)\frac{\mathbb{1}_{x\leq k+\ell}}{k+\ell}q_\infty(k+\ell-x)h^2(k)h^2(\ell)\,dx\,dk\,d\ell$$
$$= \int \frac{\mathbb{1}_{x\leq k+\ell}}{k+\ell}q_\infty(k)q_\infty(\ell)h^2(k)h^2(\ell)\,dx\,dk\,d\ell = \|h\|^4_{L^2(q_\infty)}.$$

On the other hand,

$$\int h^2(x) q_\infty(x) \frac{\mathbb{1}_{x \le k+\ell}}{k+\ell} q_\infty(k+\ell-x)\, dx\, dk\, d\ell = \int h^2(x) q_\infty(x)\, dx = \|h\|^2_{L^2(q_\infty)}.$$

Hence,

$$\left| \int h(x) q_\infty(x) R(x)\, dx \right| \le \|h\|^3_{L^2(q_\infty)}.$$

Coming back to the equation, we have that

$$-\frac{d}{dt}\frac{1}{2}\|h\|^2_{L^2(q_\infty)} \ge -\int h(x) q_\infty(x) \mathcal{L}[h]\, dx - \|h\|^3_{L^2(q_\infty)}.$$

Using the previous calculations on the spectral gap of $\mathcal{L}$, we can conclude that

$$-\frac{d}{dt}\frac{1}{2}\|h\|^2_{L^2(q_\infty)} \ge \frac{1}{3}\|h\|^2_{L^2(q_\infty)} - \|h\|^3_{L^2(q_\infty)},$$

which finishes the proof with a Gronwall bound. □

We can couple this with an interpolation argument to modify the smallness assumption in weighted L^2 by using the relative entropy, which leads us to Corollary 3.2, whose proof will be deferred to the appendix (as the proof of Corollary 3.2 relies on several *a priori* estimates established in Sec. 4).

Corollary 3.2. *Assume that for some* $\lambda_0 > \frac{1}{2}$, $\sup_x \mathrm{e}^{\lambda_0 x} q(0,x) < \infty$. *Then there exists some* $\delta > 0$ *such that if at some time* $t \ge 0$,

$$\int q(t,x) \log \frac{q(t,x)}{q_\infty(x)}\, dx \le \delta,$$

we have that q *converges to* q_∞ *and for any* $\lambda < \frac{1}{3}$, *there exists* C_λ *such that*

$$\int \frac{|q(t,x) - q_\infty(x)|^2}{q_\infty(x)}\, dx \le C_\lambda \mathrm{e}^{-\lambda t}.$$

3.3. *Coupling and convergence in Wasserstein distance*

In this section, we shall employ a coupling argument to demonstrate the convergence of the solution of (2.8) to the exponential probability density function given by (2.16). Before we state the main result of this section, we first collect several relevant definitions.

Definition 3.1. The Wasserstein distance with exponent 2 between two probability density functions (say f and g) is defined by

$$W_2(f,g) = \inf\left\{ \sqrt{\mathbb{E}[|X-Y|^2]};\ \mathrm{Law}(X) = f,\ \mathrm{Law}(Y) = g \right\},$$

where the infimum is taken over all pairs of random variables defined on some probability space $(\Omega, \mathbb{P})$ and distributed according to f and g, respectively.

Next, we present a stochastic representation of the evolution equation (2.8), which is interesting in its own right.

Proposition 3.1. *Assume that* $q_t(x) := q(t,x)$ *is a solution of* (2.8) *with initial condition* $q_0(x)$ *being a probability density function supported on* $\mathbb{R}_+$ *with mean* m_1*. Defining* $(X_t)_{t\geq 0}$ *to be a* $\mathbb{R}_+$*-valued continuous-time pure jump process with jumps of the form*

$$X_t \overset{\text{rate } 1}{\rightsquigarrow} U(X_t + Y_t), \tag{3.6}$$

where Y_t *is a i.i.d. copy of* X_t, $U \sim \text{Uniform}[0,1]$ *is independent of* (X_t) *and* (Y_t)*, and the jump occurs according to a Poisson clock running at the unit rate. If* $\text{Law}(X_0) = q_0$*, then* $\text{Law}(X_t) = q_t$ *for all* $t \geq 0$.

Proof. Taking φ to be an arbitrary but fixed test function, we have

$$\frac{d}{dt}\mathbb{E}[\varphi(X_t)] = \mathbb{E}[\varphi(U(X_t + Y_t))] - \mathbb{E}[\varphi(X_t)]. \tag{3.7}$$

Denoting $q(t,x)$ as the probability density function of X_t, (3.7) can be rewritten as

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}_+} q(t,x)\varphi(x)\,dx &= \int_{\mathbb{R}_+^2}\int_0^1 \varphi(u(k+\ell))q(k,t)q(\ell,t)\,du\,dk\,d\ell \\ &\quad - \int_{\mathbb{R}_+} q(t,x)\varphi(x)\,dx.\end{aligned}$$

After a simple change of variables, one arrives at

$$\frac{d}{dt}\int_{\mathbb{R}_+} q(t,x)\varphi(x)\,dx = \int_{\mathbb{R}_+} (G[q](x,t) - q(t,x))\varphi(x)\,dx. \tag{3.8}$$

Thus, q must satisfy $\partial_t q = G[q]$ and the proof is completed. □

Remark 3.3. Using a similar reasoning, we can show that if $(\overline{X}_t)_{t\geq 0}$ is a $\mathbb{R}_+$-valued continuous-time pure jump process with jumps of the form

$$\overline{X}_t \overset{\text{rate } 1}{\rightsquigarrow} U(\overline{X}_t + \overline{Y}_t), \tag{3.9}$$

where $\overline{Y}_t$ is a i.i.d. copy of $\overline{X}_t$, $U \sim \text{Uniform}[0,1]$ is independent of $(\overline{X}_t)$ and $(\overline{Y}_t)$, and the jump occurs according to a Poisson clock running at the unit rate. Then $\text{Law}(\overline{X}_0) = q_\infty$ implies $\text{Law}(\overline{X}_t) = q_\infty$ for all $t \geq 0$.

The main result of this section is recorded in the following theorem.

Theorem 3.2. *Under the setting of Proposition* 3.1, *we have*

$$W_2(q_t, q_\infty) \leq \mathrm{e}^{-\frac{1}{6}t} W_2(q_0, q_\infty), \quad \forall\, t \geq 0. \tag{3.10}$$

Proof. Fixing $t \in \mathbb{R}_+$, we need to couple the two densities q_t and q_∞. Suppose that $(X_t)_{t\geq 0}$ and $(\overline{X}_t)_{t\geq 0}$ are $\mathbb{R}_+$-valued continuous-time pure jump processes with jumps of the form (3.6) and (3.9), respectively. We can take (X_t, Y_t) and $(\overline{X}_t, \overline{Y}_t)$ as in the statement of Proposition 3.1 and Remark 3.3, respectively. Meanwhile, several independence assumptions can be imposed along the way when we introduce the coupling: the copies Y_t and $\overline{Y}_t$ are independent, respectively, to $\overline{X}_t$ and X_t, the couple $(X_t, \overline{X}_t)$ is also independent of $(Y_t, \overline{Y}_t)$. We insist that the same uniform random variable U is used in both (3.6) and (3.9) therefore X_t and $\overline{X}_t$ are not independent. Moreover, we impose that $\mathrm{Law}(X_0) = q_0$ and $\mathrm{Law}(\overline{X}_0) = q_\infty$. As a consequence of the previous proposition and remark, $q_t = \mathrm{Law}(X_t)$ and $\mathrm{Law}(\overline{X}_t) = q_\infty$ for all $t \geq 0$, whence $\mathbb{E}[\overline{X}_t] = \mathbb{E}[\overline{Y}_t] = m_1$ and $\mathbb{E}(\overline{X}_t^2) = \mathbb{E}(\overline{Y}_t^2) = 2\,m_1^2$, $\forall\, t \geq 0$. Also, we have that $\mathbb{E}[X_t] = \mathbb{E}[Y_t] = m_1$ for all $t \geq 0$. Thanks to the aforementioned coupling, we then have

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}[(X_t - \overline{X}_t)^2] &= \mathbb{E}\big[\big(U(X_t + Y_t - \overline{X}_t - \overline{Y}_t)\big)^2 - (X_t - \overline{X}_t)^2\big] \\
&= \frac{1}{3}\big(\mathbb{E}[(X_t - \overline{X}_t)^2] + \mathbb{E}[(Y_t - \overline{Y}_t)^2] + 2\mathbb{E}[(X_t - \overline{X}_t)(Y_t - \overline{Y}_t)]\big) \\
&\quad - \mathbb{E}[(X_t - \overline{X}_t)^2] \\
&= \frac{2}{3}\mathbb{E}[(X_t - \overline{X}_t)^2] + \frac{2}{3}\mathbb{E}[X_t - \overline{X}_t] \cdot \mathbb{E}[Y_t - \overline{Y}_t] - \mathbb{E}[(X_t - \overline{X}_t)^2] \\
&= -\frac{1}{3}\mathbb{E}[(X_t - \overline{X}_t)^2].
\end{aligned}$$

Now we pick $\overline{X_0}$ with law q_∞ so that $W_2^2(q, q_\infty) = \mathbb{E}[(X_0 - \overline{X}_0)^2]$, and an routine application of Gronwall's inequality yields (3.10). □

4. Entropy Dissipation

We state our main result, Theorem 4.1, in Sec. 4.1 so that readers know exactly what is at stake. We will present various expressions of the entropy and entropy dissipation associated to the solution $q(t,x)$ of (2.8), along with a discussion of the strategy of the proof of Theorem 4.1 in Sec. 4.2. A sequence of auxiliary lemmas and corollaries are recorded in Secs. 4.3 and 4.4. Finally, a full proof of Theorem 4.1, built upon all of the preparatory work from 4.1 to 4.4, is shown in 4.5.

4.1. *Main result*

For the integro-differential equation (2.8), a common strategy[7, 24, 33] is to use the Laplace transform or Fourier transform of (2.8) to prove the exponential decay of solution of (2.8) to $q_\infty(x)$ in some Fourier metric. However, little analysis of (2.8) has been carried out without resorting to Laplace or Fourier transform. In particular, we would like to show the dissipation of relative entropy, i.e. $\mathrm{D_{KL}}(q(\cdot,t)\,\|\,q_\infty)$, along solution trajectories:

$$\frac{d}{dt}\int_0^\infty q \log\frac{q}{q_\infty}\,dx = \frac{d}{dt}\int_0^\infty q\log q\,dx \leq 0. \tag{4.1}$$

It is reasonable to expect the validity of (4.1) as the exponential probability density q_∞ maximizes the negative entropy $-\int_0^\infty p\log p\,dx$ among all continuous probability density functions supported on $[0,\infty)$ with prescribed mean.

The following proposition together with its proof should be a reminiscent of the calculations carried out for a standard Boltzmann equation arising from the kinetic theory of (dilute) gases.[40]

Proposition 4.1. *Let $\varphi(x)$ be a (continuous) test function on $\mathbb{R}_+$ and assume that q is a smooth solution of* (2.8), *then we have*

$$\frac{d}{dt}\int_0^\infty q(t,x)\varphi(x)\,dx = -\frac{1}{4}\int_{\mathbb{R}^3_+} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell}\big(q(k+\ell-x)q(x) - q(k)q(\ell)\big) \cdot \big(\varphi(k+\ell-x) + \varphi(x) - \varphi(k) - \varphi(\ell)\big)\,dk\,d\ell\,dx.$$

Moreover, inserting $\varphi = \log q$ and using mass conservation (i.e. $m_0'(t) = 0$ for all $t \geq 0$), we obtain the dissipation of relative entropy:

$$\frac{d}{dt}\int_0^\infty q(t,x)\log q(t,x)\,dx = -\frac{1}{4}D[q],$$

where

$$D[q] := \int_{\mathbb{R}^3_+} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell}\big(q(k+\ell-x)q(x) - q(k)q(\ell)\big)\log\frac{q(k+\ell-x)q(x)}{q(k)q(\ell)}\,dk\,d\ell\,dx \geq 0. \tag{4.2}$$

Proof. We notice that the PDE (2.8) can be rewritten as

$$\partial_t q(x) = \int_0^\infty\int_0^\infty \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell}\big(q(k)q(\ell) - q(x)q(k+\ell-x)\big)\,dk\,d\ell \tag{4.3}$$

(thanks to Proposition 2.1). Omitting the time variable for simplicity, we deduce that

$$\begin{aligned}
\frac{d}{dt}\int_0^\infty q(x)\varphi(x)\,dx &= \int_{\mathbb{R}^3_+} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell}\big(q(k)q(\ell) - q(x)q(k+\ell-x)\big)\varphi(x)\,dk\,d\ell\,dx \\
&= \int_{\mathbb{R}^3_+} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q(k)q(\ell)\big(\varphi(x) - \varphi(\ell)\big)\,dk\,d\ell\,dx \\
&= \int_{\mathbb{R}^3_+} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q(k)q(\ell)\big(\varphi(k+\ell-x) - \varphi(k)\big)\,dk\,d\ell\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}^3_+} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q(k)q(\ell) \\
&\quad \cdot (\varphi(k+\ell-x) + \varphi(x) - \varphi(k) - \varphi(\ell))\,dk\,d\ell\,dx
\end{aligned}$$

$$= -\frac{1}{4}\int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell}\big(q(k+\ell-x)q(x)-q(k)q(\ell)\big)$$
$$\cdot\big(\varphi(k+\ell-x)+\varphi(x)-\varphi(k)-\varphi(\ell)\big)\,dk\,d\ell\,dx. \qquad \square$$

Remark 4.1. The dissipation of the relative entropy can also be seen via an alternative perspective. Indeed, we fix $t \geq 0$ and assume that $X_1(t)$ and $X_2(t)$ are i.i.d. $\mathbb{R}_+$-valued random variable with its probability density function given by $q(t,x)$, and we define $(Z_1, Z_2) = (U(X_1+X_2), (1-U)(X_1+X_2))$ with $U \sim \mathrm{Uniform}[0,1]$ being independent of X_1 and X_2. Then we deduce from the PDE (2.8) and Lemma 2.1 that

$$\begin{aligned} 2\frac{d}{dt}\mathrm{D}_{\mathrm{KL}}(q \,\|\, q_\infty) &= \mathrm{H}((Z_1,Z_2),(X_1,X_2)) - \mathrm{H}((X_1,X_2)) \\ &\leq \mathrm{H}((Z_1,Z_2)) - \mathrm{H}((X_1,X_2)), \end{aligned} \tag{4.4}$$

where $\mathrm{H}(X,Y) := \int_{\mathbb{R}} \rho_X(x)\log\rho_Y(x)\,dx$ represents the cross entropy from Y to X, if the laws of X and Y are given by ρ_X and ρ_Y. It can be shown[3] that the joint entropy of (Z_1,Z_2) is always no more than the joint entropy of (X_1,X_2), whence the rightmost side of (4.4) is non-positive.

Corollary 4.1. *The exponential distribution q_∞ defined in* (2.16) *is the only* (*smooth*) *equilibrium solution of the PDE* (2.8).

Proof. By Proposition 4.1, we see that

$$q_\infty(x)q_\infty(k+\ell-x) = q_\infty(k)q_\infty(\ell) \quad \text{for all } k,\ell,x \geq 0 \text{ such that } k+\ell \geq x.$$

Since $\int_0^\infty q_\infty(x)\,dx = 1$ and $\int_0^\infty x q_\infty(x)\,dx = m_1$, q_∞ must be the exponential probability density provided by (2.16). $\square$

We will prove that $\int q\log\frac{q}{q_\infty}\,dx \xrightarrow{t\to\infty} 0$ occurs with a polynomial convergence. Without loss of generality, throughout the argument to be presented below, we will set $m_1 = 1$, i.e. $q_\infty(x) = \mathrm{e}^{-x}$ for $x \geq 0$. Our main result is stated as follows.

Theorem 4.1. *Under the assumptions of Lemma* 4.4, *there exist some constants $C > 0$ and $\mu > 0$ such that we have*

$$\int_{x=0}^{+\infty} q(x,t)\log\frac{q(x,t)}{\mathrm{e}^{-x}}\,dx \leq \frac{C}{1+t^\mu}. \tag{4.5}$$

To the best of our knowledge, Theorem 4.1 is the first entropy–entropy dissipation inequality established for the uniform reshuffling dynamics.

Our proof for Theorem 4.1 actually relies on a more technical entropy–entropy dissipation result which we state below.

Theorem 4.2. *Under the assumptions of Lemma* 4.4, *there exist some constants* $C, \tilde{C} > 0$ *and* $\theta \in (0,1)$ *such that we have either*

$$\int_{x=0}^{+\infty} q(x,t) \log \frac{q(x,t)}{\mathrm{e}^{-x}}\, dx \leq C D^{\theta} \tag{4.6}$$

or

$$\int_{x=0}^{+\infty} q(x,t) \log \frac{q(x,t)}{\mathrm{e}^{-x}} \leq \tilde{C} \mathrm{e}^{-t/\tilde{C}}. \tag{4.7}$$

The proof of Theorem 4.2 is the main goal of the rest of this section. However, we can easily check that it does imply Theorem 4.1.

Proof. (Proof of Theorem 4.1 assuming Theorem 4.2.) Observe that the relative entropy is decreasing and continuous in time and therefore we can decompose the timeline $[0, \infty)$ into intervals (s_n, t_n) and $[t_n, s_{n+1}]$ in the following manner. We have that

$$\int_{x=0}^{+\infty} q(x,t) \log \frac{q(x,t)}{\mathrm{e}^{-x}} \leq \tilde{C} \mathrm{e}^{-t/\tilde{C}}, \quad \forall t \in [t_n, s_{n+1}],$$

while

$$\int_{x=0}^{+\infty} q(x,t) \log \frac{q(x,t)}{\mathrm{e}^{-x}} > \tilde{C} \mathrm{e}^{-t/\tilde{C}}, \quad \forall t \in (s_n, t_n).$$

Applying Theorem 4.2 at any $t \in (s_n, t_n)$, we see that

$$\frac{d}{dt} \int_{x=0}^{+\infty} q(x,t) \log \frac{q(x,t)}{\mathrm{e}^{-x}} = -\frac{1}{4} D \leq -\frac{1}{C} \left(\int_{x=0}^{+\infty} q(x,t) \log \frac{q(x,t)}{\mathrm{e}^{-x}} \right)^{1/\theta},$$

as we need to satisfy the first alternative.

Denoting

$$E(t) = \int_{x=0}^{+\infty} q(x,t) \log \frac{q(x,t)}{\mathrm{e}^{-x}},$$

a straightforward Gronwall estimate then shows that for $t \in (s_n, t_n)$,

$$E(t) \leq \left(\frac{t}{C} + E(s_n)^{-(1-\theta)/\theta} \right)^{\theta/(1-\theta)}. \tag{4.8}$$

By the continuity in time of $E(t)$, we also have that

$$E(s_n) \leq \tilde{C} \mathrm{e}^{-s_n/\tilde{C}}.$$

By choosing $\mu = \theta/(1-\theta)$ and choosing $\bar{C}$ large enough s.t.

$$\tilde{C} \mathrm{e}^{-t/\tilde{C}} \leq \frac{\bar{C}}{1+t^{\mu}},$$

we automatically obtain from (4.8) that

$$E(t) \leq \frac{\bar{C}}{1+t^\mu}$$

for all $t \in (s_n, t_n)$. This also trivially applies for $t \in [t_n, s_{n+1}]$ concluding the proof. $\square$

4.2. *Basic expressions of the entropy–entropy dissipation*

Let us start by looking at the strong convergence of the pairwise distribution, which is essentially trivial. Indeed, we recall the linear PDE (2.19), which reads

$$\partial_t f = L_+[f] - f,$$

where

$$L_+[f](x,y) = \frac{1}{x+y}\int_{z=0}^{x+y} f(z, x+y-z)\,dz.$$

Then denoting

$$g(t,\lambda) = \frac{1}{\lambda}\int_0^\lambda f(t,z,\lambda-z)\,dz, \tag{4.9}$$

we can rewrite (2.19) as $\partial_t f(t,x,y) = g(t,x+y) - f(t,x,y)$, whence

$$\begin{aligned}\partial_t g(t,\lambda) &= \frac{1}{\lambda}\int_{z=0}^\lambda \partial_t f(t,z,\lambda-z)\,dz\\ &= \frac{1}{\lambda}\int_0^\lambda (g(t,\lambda) - f(t,z,\lambda-z))\,dz = 0.\end{aligned}$$

Hence, $g(t,\lambda) = g(0,\lambda)$ and trivially (by Gronwall's inequality)

$$|f(t,x,y) - g(0,x+y)| \leq \mathrm{e}^{-t}. \tag{4.10}$$

Unfortunately, this cannot be used to show the convergence on the actual equation for $q(t,x)$ because the two models are not equivalent: If $q(t,x)$ solves (2.8), which is non-linear, then in general $f(t,x,y) = q(t,x)q(t,y)$ does not solve (2.19). The one exception is when $q(t,x)$ is some exponential. Moreover, f does not necessarily converge to an exponential but to whatever $g(t=0)$ was. The rate of convergence is also too fast as the second moment of q converges much slower for example.

We will still find some of the structure above in the entropy dissipation for q but that is one reason why the entropy dissipation is not easy to handle. In particular, the entropy dissipation will vanish whenever $f(x,y) = g(x+y)$ which seems to create some degeneracy.

Next, we can rewrite the dissipation term in a manner that will make the connection with the exponential more apparent. We define for simplicity $f(x,y) = q(x)q(y)$, and as before

$$g(\lambda) = \frac{1}{\lambda}\int_0^\lambda f(z,\lambda-z)\,dz = \frac{1}{\lambda}\int_0^\lambda q(z)q(\lambda-z)\,dz.$$

Finally, we also define

$$h(x) = \int_{\mathbb{R}_+} g(x+y)\,dy.$$

We remark here that h coincides with the collision gain operator $Q_+[q]$ defined via (2.5). With these definitions, we have

Lemma 4.1. *For $D := D[q]$ in (4.2), one has that*

$$D = 2\int_{\mathbb{R}_+^2} q(x)q(y)\log\frac{q(x)q(y)}{g(x+y)}\,dx\,dy + 2\int_{\mathbb{R}_+^2} g(x+y)\log\frac{g(x+y)}{q(x)q(y)}\,dx\,dy$$

or as well that

$$D = 2\int_{\mathbb{R}_+^2} q(x)q(y)\log\frac{q(x)q(y)}{g(x+y)}\,dx\,dy + 2\int_{\mathbb{R}_+^2} g(x+y)\log\frac{g(x+y)}{h(x)h(y)}\,dx\,dy$$
$$+\,4\int_{\mathbb{R}_+} h(x)\log\frac{h(x)}{q(x)}\,dx.$$

Formally this forces $g(x+y)$ to be close to $f(x,y)$ (solution of the linear PDE (2.19)) so this is a very similar term to the one that we had found when looking at Eq. (2.19). It is some sort of degeneracy because it does not directly force f to be close to e^{-x-y} so we will have to resolve it. Of course since $f(x,y) = q(x)q(y)$, $f(x,y) = g(x+y)$ forces q to be some exponential and therefore this should be possible.

Proof. We can first simply rewrite

$$D = \int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{y+z\geq x}}{y+z}(f(y+z-x,x) - f(y,z))\log\frac{f(y+z-x,x)}{f(y,z)}\,dx\,dy\,dz.$$

Observe that by swapping x and z,

$$\int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{y+z\geq x}}{y+z}(f(y+z-x,x) - f(y,z))\log f(y+z-x,x)$$
$$= \int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{y+x\geq z}}{y+x}(f(y+x-z,z) - f(y,x))\log f(y+x-z,z).$$

Changing variable $y \to y' = y+x-z$, we get that

$$\int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{y+z\geq x}}{y+z}(f(y+z-x,x) - f(y,z))\log f(y+z-x,x)$$
$$= \int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{y'+z\geq x}}{y'+z}(f(y',z) - f(y'+z-x,x))\log f(y',z).$$

Hence,

$$D = 2\int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{y+z\geq x}}{y+z}(f(y,z) - f(y+z-x,x))\log f(y,z).$$

In other words,

$$D = 2\int_{\mathbb{R}_+^2} f(y,z)\log f(y,z)dy\,dz - 2\int_{\mathbb{R}_+^2} g(y+z)\log f(y,z)dy\,dz.$$

Now, we observe that

$$\int_{\mathbb{R}_+^2} f(y,z)\log g(y+z)dy\,dz = \int_{\mathbb{R}_+^2} g(y+z)\log g(y+z)dy\,dz.$$

Indeed, a change of variable $y = x - w$ and $z = w$ yields

$$\int_{\mathbb{R}_+^2} g(y+z)\log g(y+z)dy\,dz = \int_{\mathbb{R}_+} xg(x)\log g(x)dx.$$

By the same change of variables, we also have

$$\begin{aligned}\int_{\mathbb{R}_+^2} f(y,z)\log g(y+z)dy\,dz &= \int_{\mathbb{R}_+} \log g(x)\int_0^x f(x-w,w)dw\,dx \\ &= \int_{\mathbb{R}_+} xg(x)\log g(x)dx.\end{aligned}$$

Hence,

$$\frac{D}{2} = \int_{\mathbb{R}_+^2} f(y,z)\log\frac{f(y,z)}{g(y+z)}dy\,dz + \int_{\mathbb{R}_+^2} g(y+z)\log\frac{g(y+z)}{f(y,z)}dy\,dz.$$

Finally, as $f(y,z) = q(y)q(z)$, we may also notice that

$$\begin{aligned}\int_{\mathbb{R}_+^2} g(y+z)\log\frac{g(y+z)}{f(y,z)}dy\,dz &= \int_{\mathbb{R}_+^2} g(y+z)\log g(y+z)dy\,dz \\ &\quad - 2\int_{\mathbb{R}_+^2} g(y+z)\log q(y)dy\,dz \\ &= \int_{\mathbb{R}_+^2} g(y+z)\log g(y+z)dy\,dz \\ &\quad - 2\int_{\mathbb{R}_+} h(y)\log q(y)dy.\end{aligned}$$

So we also have that

$$\begin{aligned}\int_{\mathbb{R}_+^2} g(y+z)\log\frac{g(y+z)}{f(y,z)}dy\,dz &= \int_{\mathbb{R}_+^2} g(y+z)\log\frac{g(y+z)}{h(y)h(z)}dy\,dz \\ &\quad + 2\int_{\mathbb{R}_+} h(y)\log\frac{h(y)}{q(y)}dy,\end{aligned}$$

concluding the estimate. □

Next, we intend to collect here some various bounds stemming from the dissipation term, the essence of those bounds lies in the following lemma.

Lemma 4.2. *We have that*

$$\int q(x)\log\frac{q(x)}{H(x)}dx \le \int \varphi(y)q(x)q(y)\log\frac{q(x)q(y)}{g(x+y)}\,dx\,dy,$$

in which

$$H(x)=\int g(x+y)\varphi(y)\,dy$$

for any $\varphi \ge 0$ *such that* $\int \varphi q\,dx = 1$.

Proof. Indeed, as log is concave,

$$\begin{aligned}\int q(x)\varphi(y)q(y)\log\frac{g(x+y)}{q(x)q(y)}\,dx\,dy &\le \int q(x)\log\left(\int \frac{g(x+y)}{q(x)}\varphi(y)dy\right)dx\\ &= \int q(x)\log\frac{H(x)}{q(x)}\,dx\end{aligned}$$

and the proof is completed. □

As a consequence of this lemma, inserting $\phi(x)=1$ and then $\phi(x)=x$, we then deduce that

$$\int q(x)\log\frac{q(x)}{h(x)}dx \le \int q(x)q(y)\log\frac{q(x)q(y)}{g(x+y)}\,dx\,dy,$$

$$\int q(x)\log\frac{q(x)}{m(x)}dx \le \int xq(x)q(y)\log\frac{q(x)q(y)}{g(x+y)}\,dx\,dy,$$

where

$$\begin{aligned}m(x) &= \int_{y=0}^{\infty} g(x+y)y\,dy = \int_x^{\infty} g(z)(z-x)\,dz\\ &= \int_x^{\infty}\int_y^{\infty} g(z)dz\,dy = \int_x^{\infty} h(y)dy.\end{aligned}$$

Remark 4.2. We also note that $m(0)=1$ (since $\int h\,dx = \int q\,dx = 1$) and so

$$\int h\log m,\,dx = -\int m'\log m\,dx = -\int h\,dx = -\int x\,h(x)dx = -1,$$

by virtue of the fact that $\int xh(x)dx = \int xq(x)dx = 1$. Thus,

$$\int h\log\frac{h}{m}\,dx = \int h\log\frac{h}{\mathrm{e}^{-x}}\,dx.$$

This leads to a possible strategy: Control $\int h\log\frac{h}{m}$ in terms of $\int q\log\frac{q}{h}$, $\int h\log\frac{h}{q}$ and $\int q\log\frac{q}{m}$. Then control $\int q\log\frac{q}{\mathrm{e}^{-x}}$ by the previous quantities and $\int h\log\frac{h}{m}$. We can then estimate $\int xq(x)q(y)\log\frac{q(x)q(y)}{g(x+y)}\,dx\,dy$ via $\int q(x)q(y)\log\frac{q(x)q(y)}{g(x+y)}\,dx\,dy$ and some control on the decay of q at infinity. So in the end this would lead to some kind of bounds on $\int q\log\frac{q}{\mathrm{e}^{-x}}$ in terms of the dissipation term. We illustrate the strategy in Fig. 6.

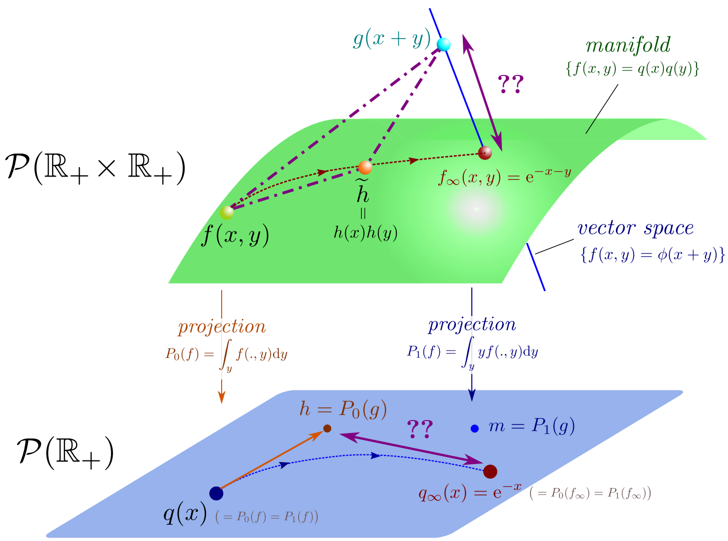

Fig. 6. (Color online) To measure the decay of the relative entropy $\int q\log\frac{q}{\mathrm{e}^{-x}}$, we have to control the term $\int h\log\frac{h}{\mathrm{e}^{-x}}$ or similarly the term $\int g\log\frac{g}{\mathrm{e}^{-x-y}}$ (represented in purple). Indeed, the dissipation term D already provides a control over the 'triangle' of relative entropies (dashed lines) $\int f\log\frac{f}{g}$, $\int g\log\frac{g}{\widetilde{h}}$ and $\int \widetilde{h}\log\frac{\widetilde{h}}{g}$ with $\widetilde{h}(x,y)=h(x)h(y)$.

However, normally it is not possible to switch relative entropy estimates. Indeed, it is not so hard to find examples of non-negative functions φ,ϕ,ψ with total mass 1 such that

$$\int\varphi\log\frac{\varphi}{\psi}=\infty,$$

while

$$\int\phi\log\frac{\phi}{\psi}+\int\phi\log\frac{\phi}{\varphi}+\int\varphi\log\frac{\varphi}{\phi}<\infty.$$

Therefore, this strategy is not obvious to implement. It should work nicely if we had a control like $\mathrm{e}^{-x}/C\le q(x)\le C\mathrm{e}^{-x}$ but the general case is certainly trickier. What saves us is the key observation that here h and m are actually very nice functions in all cases. For example, m and h are monotone decreasing so bounded from above and bounded from below on any finite interval (from the propagation of moments on q). This gives us some hope when implementing the aforementioned machinery. We emphasize here that our entropy–entropy dissipation argument draws inspiration from earlier works on Becker–Döring equations and coagulation models.[13, 28]

4.3. *Switching relative entropies*

We note that the relative entropy behaves in the following manner.

Lemma 4.3. *For any two $\mu, \nu \in \mathcal{P}(\mathbb{R}_+)$ s.t. μ has a density w.r.t. ν, denoted by $\frac{\mu}{\nu}$, and for any $C \geq 2$, then*

$$\begin{aligned} &\frac{1}{2C}\int_{\nu/C\leq\mu\leq C\nu}\frac{(\mu-\nu)^2}{\nu}+\frac{1}{8}\int_{\mu\leq\nu/C}\nu+\frac{1}{4}\int_{\mu\geq C\nu}\mu\log\frac{\mu}{\nu}\\ &\quad\leq\int\mu\log\frac{\mu}{\nu}\\ &\quad\leq\frac{C}{2}\int_{\nu/C\leq\mu\leq C\nu}\frac{(\mu-\nu)^2}{\nu}+\int_{\mu\leq\nu/C}\nu+\int_{\mu\geq C\nu}\mu\log\frac{\mu}{\nu}. \end{aligned} \tag{4.11}$$

Proof. We observe that

$$\int\mu\log\frac{\mu}{\nu}=\int\nu\left(\frac{\mu}{\nu}\log\frac{\mu}{\nu}+1-\frac{\mu}{\nu}\right).$$

On the other hand, around 1, the function $\phi(x) = x\log x + 1 - x$ satisfies that $\phi(x) \leq (x-1)^2/2$ for $x \geq 1$ and $\phi(x) \leq \frac{C}{2}(x-1)^2$ for $1/C \leq x \leq 1$. On the other hand $\phi(x) \geq (x-1)^2/2C$ for $1 \leq x \leq C$ and $\phi(x) \geq (x-1)^2/2$ for $1/C \leq x \leq 1$. Furthermore ϕ lies between $1/8$ and 1 when $x \leq 1/2$ and larger than $\frac{x}{4}\log x$ for $x \geq 2$. $\square$

Remark 4.3. One can also rewrite a little bit the statement of Lemma 4.3 so that we do not need to impose that μ and ν are probability measures.

This allows us to "switch" relative entropies between two measures that are comparable.

Corollary 4.2. *There exists a constant $C > 0$ such that if $\mu_1, \mu_2, \nu \in \mathcal{P}(\mathbb{R}_+)$ with $\lambda^{-1}\mu_1 \leq \mu_2 \leq \lambda\mu_1$ and $\lambda \geq \mathrm{e}$, then*

$$\int\mu_1\log\frac{\mu_1}{\nu}\leq C\lambda^3\int\mu_2\log\frac{\mu_2}{\nu}+\lambda^3\int\mu_2\log\frac{\mu_2}{\mu_1}.$$

Proof. Apply Lemma 4.3 with $C = 2\lambda$ first on μ_1 and ν to find

$$\int\mu_1\log\frac{\mu_1}{\nu}\leq\lambda\int_{\frac{\nu}{2\lambda}\leq\mu_1\leq2\lambda\nu}\frac{(\mu_1-\nu)^2}{\nu}+\int_{\mu_1\leq\frac{\nu}{2\lambda}}\nu+\int_{\mu_1\geq2\lambda\nu}\mu_1\log\frac{\mu_1}{\nu}.$$

Thanks to Lemma 4.3 again, we have

$$\int_{\mu_1\leq\frac{\nu}{2\lambda}}\nu\leq8\int\mu_1\log\frac{\mu_1}{\nu}.$$

Now if $\mu_1 \leq \frac{\nu}{2\lambda}$ then $\mu_2 \leq \frac{\nu}{2}$. Similarly if $\mu_1 \geq 2\lambda\nu$ then $\mu_2 \geq 2\nu$ and moreover

$$\mu_1\log\frac{\mu_1}{\nu}\leq\lambda\mu_2\log\frac{\lambda\mu_2}{\nu}\leq3\lambda\log\lambda\mu_2\log\frac{\mu_2}{\nu}.$$

Conversely if $\frac{\nu}{2\lambda} \le \mu_1 \le 2\lambda\nu$ then $\frac{\nu}{2\lambda^2} \le \mu_2 \le 2\lambda^2\nu$ and

$$\frac{(\mu_1-\nu)^2}{\nu} \le 2\left(\frac{(\mu_2-\nu)^2}{\nu} + \frac{(\mu_1-\mu_2)^2}{\nu}\right) \le 2\frac{(\mu_2-\nu)^2}{\nu} + 4\lambda\frac{(\mu_1-\mu_2)^2}{\mu_1}.$$

Hence,

$$\int_{\frac{\nu}{2\lambda}\le\mu_1\le 2\lambda\nu} \frac{(\mu_1-\nu)^2}{\nu} \le 2\int_{\frac{\nu}{2\lambda^2}\le\mu_2\le 2\lambda^2\nu} \frac{(\mu_2-\nu)^2}{\nu} + 4\lambda\int_{\frac{\mu_1}{\lambda}\le\mu_2\le\lambda\mu_1} \frac{(\mu_1-\mu_2)^2}{\mu_1}.$$

Note that by Lemma 4.3 applied with $C=\lambda$, we have that

$$\int_{\frac{\mu_1}{\lambda}\le\mu_2\le\lambda\mu_1} \frac{(\mu_1-\mu_2)^2}{\mu_1} \le 2\lambda\int \mu_2\log\frac{\mu_2}{\mu_1}.$$

Also, Lemma 4.3 applied with $C = 2\lambda^2$ gives rise to

$$\int_{\frac{\nu}{2\lambda^2}\le\mu_2\le 2\lambda^2\nu} \frac{(\mu_2-\nu)^2}{\nu} \le 4\lambda^2\int \mu_2\log\frac{\mu_2}{\nu}.$$

Assembling these estimates, the proof is completed. □

4.4. *Additional a priori estimates*

This leads us to try to compare q and h. We first observe that we can get easy upper bounds.

Lemma 4.4. *Assume that for some* $0<\lambda_0<1$, $\int \mathrm{e}^{\lambda_0 x} q(t=0,x)\,dx < \infty$. *Then we have that*

$$\sup_t \int \mathrm{e}^{\lambda_0 x} q(t,x)\,dx < \infty.$$

Proof. We use a Laplace transform by defining

$$F(t,\lambda) = \int \mathrm{e}^{\lambda x} q(t,x)\,dx$$

and note that

$$\partial_t F = \int_{\mathbb{R}_+^2} \frac{\mathrm{e}^{\lambda(y+z)}-1}{\lambda(y+z)} q(y)q(z)\,dy\,dz - F = \frac{1}{\lambda}\int_0^\lambda (F(\mu))^2\,d\mu - F.$$

It is useful to remark right away that the stationary solution to this equation satisfies that $F^2 = \partial_\lambda(\lambda F)$ which has solutions of the form $\frac{1}{1-C\lambda}$. Those do blow-up but only for λ large enough. As a matter of fact since $\partial_\lambda F|_{\lambda=0} = 1$, we can see that we should even have $C=1$. For this reason, denote now $G = (1-C\lambda)F$ with some

$C < \frac{1}{\lambda}$ such that $G(t = 0, \lambda) \leq 1$ on $[0, \lambda_0]$. We first show that $\sup_{\lambda \in [0,\lambda_0]} G(t, \lambda) \leq 1$ for all $t \geq 0$. Indeed, let $\lambda(t)$ be such that $\sup_{\lambda \in [0,\lambda_0]} G(t, \lambda) = G(t, \lambda(t))$, then

$$\partial_t \sup_{\lambda \in [0,\lambda_0]} G(t, \lambda) \leq \partial_t G(t, \lambda(t)),$$

this is because $\partial_\lambda G(t, \lambda(t)) = 0$ if $\lambda(t) < \lambda_0$, while if $\lambda(t) = \lambda_0$ then $\partial_\lambda G(t, \lambda(t)) \leq 0$ and $\lambda'(t) \leq 0$, leading to the same inequality. Now since

$$\partial_t G = (\lambda^{-1} - C) \int_0^\lambda \frac{(G(\mu))^2}{(1 - C\mu)^2} \, d\mu - G, \tag{4.12}$$

together with $\int_0^\lambda \frac{d\mu}{(1-C\mu)^2} = \frac{\lambda}{1-C\lambda}$, we deduce that

$$\partial_t \sup_{\lambda \in [0,\lambda_0]} G(t, \lambda) \leq \left(\sup_{\lambda \in [0,\lambda_0]} G(t, \lambda) \right)^2 - \sup_{\lambda \in [0,\lambda_0]} G(t, \lambda),$$

which yields via the maximum principle that $\sup_{\lambda \in [0,\lambda_0]} G(t, \lambda) \leq 1$. Now thanks to (4.12) again and the elementary observation that $\partial_t \sup_{\lambda \in [0,\lambda_0]} G(t, \lambda) \leq \sup_{\lambda \in [0,\lambda_0]} \partial_t G(t, \lambda)$, we arrive at

$$\partial_t \sup_{\lambda \in [0,\lambda_0]} G(\lambda) \leq 0,$$

which immediately proves the desired upper bound. $\square$

Remark 4.4. We believe it is possible to prove the exponential convergence of the Laplace transform $F(t, \lambda)$ to $1/(1 - \lambda)$ over $\lambda \in [0, \lambda_0)$. However, this is not strictly better than having the exponential convergence in some weak Wasserstein norm plus the control of the exponential moments that is given above, so we did not try too much in this direction.

Out of Lemma 4.4, we may deduce pointwise bounds on q and h, for this purpose, we need the following preparatory result.

Lemma 4.5. *We have that*

$$\sup_{t \geq 0} h(t, 0) < \infty,$$

i.e. $h(t, 0)$ *is uniformly bounded in time.*

Proof. To show $h(t, 0)$ is uniformly bounded in time, we write

$$\begin{aligned} h(t, 0) &= \int_{\mathbb{R}_+^2} \frac{q(y)q(z)}{y + z} \, dy \, dz = 2 \iint_{y \leq z} \frac{q(y)q(z)}{y + z} \, dy \, dz \\ &\leq 2 \iint_{y \leq z} \frac{q(y)q(z)}{z} \, dy \, dz \\ &\leq 2 \sup_{y \leq r} q(y) \int_{z \geq r} \frac{rq(z)}{z} \, dz + 2 \iint_{y \leq z, z \leq r} \frac{q(z)}{z} \, dy \, dz + \frac{2}{r}. \end{aligned}$$

We know that there exists some r uniformly in time such that

$$\iint_{y\leq z, z\leq r} \frac{q(z)}{z}\,dy\,dz = \int_{z\leq r} q(z)\,dz \leq \frac{1}{8}.$$

Moreover, for this r we also have $\int_{z\geq r} \frac{rq(z)}{z}\,dz \leq \frac{1}{8}$. Thus,

$$h(t,0) \leq \frac{1}{2}\sup_{x\leq r} q(x) + \frac{2}{r}.$$

Now we recall the equation for q to find that for any $x \leq r$,

$$\partial_t q(t,x) \leq h(t,0) - q(t,x) \leq \frac{1}{2}\sup_{x\leq r} q(t,x) + \frac{2}{r} - q(t,x),$$

so if x_* is such that $q(t,x_*) = \sup_{x\leq r} q(t,x)$, then

$$\partial_t q(t,x_*) \leq \frac{2}{r} - \frac{1}{2} q(t,x_*).$$

By Grownwall's inequality, we deduce that $\sup_{x\leq r} q(t,x) \leq \frac{4}{r}$, which allows us to finish the proof. $\square$

Corollary 4.3. *Assume that for some* $0 < \lambda_0 < 1$, $\int \mathrm{e}^{\lambda_0 x} q(0,x)\,dx < \infty$, *then we have that*

$$\sup_t \int \mathrm{e}^{\lambda_0 x} h(t,x)\,dx < \infty, \qquad \sup_{t,x} \mathrm{e}^{\lambda_0 x} h(t,x) < \infty,$$
$$q(t,x) \leq C\,\mathrm{e}^{-\lambda_0 x} + q(0,x)\mathrm{e}^{-t} \quad \textit{for some } C > 0.$$

Proof. The first bound follows from the definition of h. Indeed, as $h = Q_+[q]$, we have

$$\begin{aligned}\int \mathrm{e}^{\lambda_0 x} h(t,x)\,dx &= \int \frac{\mathrm{e}^{\lambda_0 (y+z)} - 1}{\lambda_0 (y+z)} q(y)q(z)\,dy\,dz \\ &\leq \int \mathrm{e}^{\lambda_0 (y+z)} q(y)q(z)\,dy\,dz < \infty.\end{aligned}$$

Next, we observe that h is decreasing in x, so for any $x \geq 0$,

$$\begin{aligned}\int_0^\infty \mathrm{e}^{\lambda_0 y} h(t,y)\,dy &\geq \int_0^x \mathrm{e}^{\lambda_0 y} h(t,y)\,dy \\ &\geq h(t,x)\int_0^x \mathrm{e}^{\lambda_0 y}\,dy \\ &= h(t,x)\frac{\mathrm{e}^{\lambda_0 x} - 1}{\lambda_0}.\end{aligned}$$

Since $h(t,x) \leq h(t,0)$ is uniformly bounded in time, this shows the second point.

Finally, we recall the equation for q, which reads $\partial_t q = h - q$, so we may rewrite (2.8) as

$$q(t,x) = q(0,x)\mathrm{e}^{-t} + \int_0^t h(s,x)\mathrm{e}^{-(t-s)}\,ds. \tag{4.13}$$

Moreover, notice that

$$\mathrm{e}^{\lambda_0 x}\int_0^t h(s,x)\mathrm{e}^{-(t-s)}\,ds \leq \sup_s(\mathrm{e}^{\lambda_0 x}h(s,x))\int_0^t \mathrm{e}^{-(t-s)}\,ds \leq \sup_s(\mathrm{e}^{\lambda_0 x}h(s,x)).$$

Combining these estimates with (4.13) ends the proof. □

We now turn to lower bounds on q and hence h. We start with a lower bound on q in terms of h.

Lemma 4.6. *There exists C such that for any $t \geq 1$,*

$$q(t,x) \geq \frac{1}{C}h(t-1,x). \tag{4.14}$$

Proof. We note from Eq. (2.8) that

$$\partial_t h(t,x) = 2\int_x^\infty \frac{1}{\lambda}\int_0^\lambda h(t,z)q(t,\lambda - z)\,dz\,d\lambda - 2h(t,x).$$

Therefore,

$$\partial_t h(t,x) \geq -2h(t,x)$$

and we have that for any $s \leq t$ that

$$\partial_t q(t,x) \geq \mathrm{e}^{-(t-s)}h(s,x) - q(t,x),$$

leading for example to the claimed result

$$q(t,x) \geq \frac{h(t-1,x)}{C}$$

with $C = \frac{\mathrm{e}^2}{\mathrm{e}-1}$, thereby completing the proof. □

Unfortunately, this is not enough to give us a bound between q and h which would solve everything. Instead, we can first deduce a bound near the origin.

Lemma 4.7. *There exists a constant C such that*

$$\inf_{t\geq 1}\inf_{x\in[0,2]} h(t,x) \geq \frac{1}{C}, \quad \inf_{t\geq 2}\inf_{x\in[0,2]} q(t,x) \geq \frac{1}{C}. \tag{4.15}$$

Proof. For any $x \leq 2$, we have that

$$\begin{aligned} h(t,x) &= \int \frac{\mathbb{1}_{x\leq y+z}}{y+z}q(t,y)q(t,z)\,dy\,dz \\ &\geq \int_{y,z\geq 1} \frac{1}{(y+1)(z+1)}q(y)q(z)\,dy\,dz = \left(\int_1^\infty \frac{q(y)}{1+y}\,dy\right)^2. \end{aligned}$$

By Cauchy–Schwartz, we have that

$$\begin{aligned}\int_1^\infty q(y)\,dy &\le \left(\int_1^\infty \frac{q(y)}{1+y}\,dy\right)^{1/2}\left(\int_1^\infty (1+y)q(y)\,dy\right)^{1/2}\\ &\le \left(\int_1^\infty \frac{q(y)}{1+y}\,dy\right)^{1/2}\left(\int_0^\infty (1+y)q(y)\,dy\right)^{1/2}\\ &= \sqrt{2}\left(\int_1^\infty \frac{q(y)}{1+y}\,dy\right)^{1/2}.\end{aligned}$$

On the other hand the convergence of all moments of q shows that there exists C such that for all $t \ge 1$,

$$\int_1^\infty q(y)dy \ge \frac{1}{C}.$$

Therefore, there exists C such that $h(t,x) \ge \frac{1}{C}$ whenever $x \le 2$ and $t \ge 1$. Finally, we deduce the second result from Lemma 4.6. □

We combine the previous result with the following doubling type of argument.

Lemma 4.8. *There exists a constant C such that for any x and $t \ge 1$, there holds*

$$q(t,x) \ge \frac{x}{C}\left(\inf_{s\in[t-1,t]}\ \inf_{y\in[x/2,3x/4]} q(s,y)\right)^2.$$

Proof. This is a simple consequence of a lower bound on h. Indeed, we have

$$h(t,x) = \int \frac{\mathbb{1}_{x\le y+z}}{y+z} q(t,y)q(t,z)\,dy\,dz \ge \frac{2}{3x}\int_{y,z\in[x/2,3x/4]} q(y)q(z)\,dy\,dz.$$

Therefore,

$$h(t,x) \ge \frac{x}{24}\left(\inf_{y\in[x/2,3x/4]} q(t,y)\right)^2.$$

We can again conclude by virtue of Lemma 4.6. □

Lemma 4.9. *There exists a constant C such that for any $t \ge 2$ and $x \ge 2$, we have*

$$q(t,x) \ge \int_{y\ge x} \frac{q(t-1,y)}{Cy}\,dy.$$

Proof. This is again a consequence of a lower bound on h. Indeed,

$$h(t,x) = \int_{\mathbb{R}_+^2} \frac{\mathbb{1}_{x\le y+z}}{y+z} q(t,y)q(t,z)\,dy\,dz \ge \int_{y\le x}\int_{z\ge x} q(y)\frac{q(z)}{2z}\,dy\,dz.$$

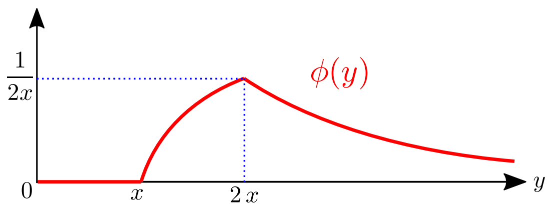

Fig. 7. The function ϕ used in the proof of Corollary 4.4. Notice that $\phi(y) \le \frac{1}{y}$ for all $y > 0$.

Thus, by the lower bound on q on $[0, 2]$ (thanks to Lemma 4.7), we arrive at

$$h(t,x) \ge \int_{y\ge x} \frac{q(t,y)}{Cy}\,dy.$$

Using Lemma 4.6, we can again conclude. □

Owing to Lemma 4.9, we immediately deduce that

Corollary 4.4. *There exists some* $C > 0$ *such that for any* $x \ge 2$ *and any* $t \ge \max(Cx, 1)$

$$h(t,x) \ge \frac{\mathrm{e}^{-Cx}}{C}, \quad q(t,x) \ge \frac{\mathrm{e}^{-Cx}}{C}.$$

Proof. Define $\phi(y) = \frac{(y/x-1)_+}{y}$ for $y \le 2x$ and $\phi = 1/y$ if $y \ge 2x$ (see Fig. 7).

Note that ϕ is Lipschitz with

$$\|\nabla\phi\|_{L^\infty} \le \frac{1}{x^2}.$$

Hence,

$$x^2 \int \phi(y)q(y)\,dy \ge x^2 \int \phi(y)\mathrm{e}^{-y}\,dy - W_1(q, \mathrm{e}^{-x}),$$

in which $W_1(q, \mathrm{e}^{-x})$ represents the Wasserstein distance (with exponent 1) between q and e^{-x}. Thanks to the exponentially fast in time of the convergence $W_1(q, \mathrm{e}^{-x}) \to 0$, which is a simple consequence of Theorem 3.2, we deduce that

$$\int_{y\ge x} \frac{q(y)}{y}\,dy \ge \int \phi(y)\mathrm{e}^{-y}\,dy - \frac{C}{x^2}\mathrm{e}^{-t/6}.$$

Note that

$$\int \phi(y)\mathrm{e}^{-y}\,dy \ge \int_{y\ge 2x} \frac{\mathrm{e}^{-y}}{y}\,dy \ge \frac{\mathrm{e}^{-3x}}{3x}\int_{2x\le y\le 3x} dy = \frac{\mathrm{e}^{-3x}}{3}.$$

Therefore, from Lemma 4.9, we can conclude provided that $\frac{C}{x^2}\mathrm{e}^{-t/6} \le \frac{\mathrm{e}^{-3x}}{6}$. □

4.5. *Proof of the main technical result*

Armed with all the previous estimates, we can finally present the proof of Theorem 4.2.

Proof. (Theorem 4.2) We start from the estimates derived in Sec. 4.2 and in particular from Lemma 4.2 for $\phi(x) = x$, yielding

$$\int q \log \frac{q}{m}\, dx \leq \int x q(x) q(y) \log \frac{q(x)q(y)}{g(x+y)}\, dx\, dy,$$

where we recall that

$$g(x) = \frac{1}{x} \int_0^x q(z) q(x-z) dz, \quad m(x) = \int_{y=0}^{\infty} g(x+y) y dy = \int_{y=x}^{\infty} h(y) dy.$$

In particular

$$\int x g(x+y) dx dy = \int x q(x) q(y) dx\, dy,$$

so that

$$\int q \log \frac{q}{m}\, dx \leq \int \left(x q(x) q(y) \log \frac{q(x)q(y)}{g(x+y)} + x g(x+y) - x q(x) q(y) \right) dx\, dy.$$

For some $K > 0$ to be chosen later, we can decompose the integral into the domain $x \in [0, K]$ and $x \in [K, +\infty)$. For the first part, we can simply bound

$$\begin{aligned} &\int_{x \leq K} \left(x q(x) q(y) \log \frac{q(x)q(y)}{g(x+y)} + x g(x+y) - x q(x) q(y) \right) dx\, dy \\ &\quad \leq K \int_{x \leq K} \left(q(x) q(y) \log \frac{q(x)q(y)}{g(x+y)} + g(x+y) - q(x) q(y) \right) dx\, dy, \end{aligned}$$

since $a \log \frac{a}{b} + b - a \geq 0$ for any $a, b \geq 0$.

By Lemma 4.1, this immediately implies that

$$\int_{x \leq K} \left(x q(x) q(y) \log \frac{q(x)q(y)}{g(x+y)} + x g(x+y) - x q(x) q(y) \right) dx\, dy \leq K D.$$

On the other hand, denoting $\phi(x) = x \log x + 1 - x$, which is a non-negative convex function on $\mathbb{R}_+$ and satisfies $\phi(x) \leq C x$ for some constant C if x is bounded, we can first write that for any $\lambda > 0$,

$$\begin{aligned} &\int_{x \geq K} \left(x q(x) q(y) \log \frac{q(x)q(y)}{g(x+y)} + x g(x+y) - x q(x) q(y) \right) dx\, dy \\ &\quad = \int_{x \geq K} g(x+y) x \phi\left(\frac{q(x)q(y)}{g(x+y)} \right) dx\, dy \\ &\quad = \frac{1}{\lambda} \int_{x \geq K} g(x+y) \lambda x \phi\left(\frac{q(x)q(y)}{g(x+y)} \right) dx\, dy. \end{aligned}$$

We now perform a classical Fenchel duality estimate on ϕ, namely $xy \leq \phi(x) + \phi^*(y)$, in which ϕ^* denotes the Legendre convex conjugate of ϕ. One may readily check that $\phi^*(y) = \mathrm{e}^y - 1 \le \mathrm{e}^y$. This lets us deduce for any $\lambda \in (0, \lambda_0)$ that

$$\begin{aligned}\int_{x\ge K} &\left(xq(x)q(y)\log\frac{q(x)q(y)}{g(x+y)} + xg(x+y) - xq(x)q(y)\right) dx\,dy\\ &\le \frac{1}{\lambda}\int_{x\ge K} g(x+y)\phi\circ\phi\left(\frac{q(x)q(y)}{g(x+y)}\right) dx\,dy + \frac{1}{\lambda}\int_{x\ge K} \mathrm{e}^{\lambda x} g(x+y)dx\,dy.\end{aligned}$$

We can immediately note that $\phi\circ\phi \le x\log x$ for large x. Thus from Corollary 4.3, we have that

$$\begin{aligned}\int_{x\ge K} &\left(xq(x)q(y)\log\frac{q(x)q(y)}{g(x+y)} + xg(x+y) - xq(x)q(y)\right) dx\,dy\\ &\le \frac{D}{\lambda} + \frac{C}{\lambda}\mathrm{e}^{-(\lambda_0-\lambda)K}.\end{aligned}$$

Combining both estimates gives rise to

$$\int q\log\frac{q}{m} \le (K+1)\frac{D}{\lambda} + \frac{C}{\lambda}\mathrm{e}^{-(\lambda_0-\lambda)K}$$

and optimizing in K leads to

$$\int q\log\frac{q}{m} \le CD\log\frac{1}{D}. \tag{4.16}$$

The next step is to change this to $\int h\log\frac{h}{m}$. We decompose again

$$\int h\log\frac{h}{m} = \int_{x\le K}\left(h\log\frac{h}{m} + m - h\right) + \int_{x\ge K}\left(h\log\frac{h}{m} + m - h\right).$$

We note that since $h = -\partial_x m$,

$$\int_{x\ge K} h\log m = -\int_{x\ge K}\partial_x m\log m = m(K)\log m(K) - m(K).$$

Applying Corollary 4.3 again, this shows that for some constant C, we have that

$$\int_{x\ge K}\left(h\log\frac{h}{m} + m - h\right) \le C\mathrm{e}^{-K/C}. \tag{4.17}$$

From Corollaries 4.3 and 4.4, we note that on $x \le K$ there holds $\mathrm{e}^{-CK} \le \frac{q}{h} \le \mathrm{e}^{CK}$, at least provided that $t \ge Cx$.

Now in the region $x \le K$, we can use Lemma 4.3 in exactly the same manner as what we did in Corollary 4.2, which yields that

$$\begin{aligned}\int h\log\frac{h}{m} &= \int\left(h\log\frac{h}{m} + m - h\right)\\ &\le C\mathrm{e}^{CK}\int q\log\frac{q}{m} + C\mathrm{e}^{-K/C}\\ &\le C\mathrm{e}^{CK} D\log\frac{1}{D} + C\mathrm{e}^{-K/C}.\end{aligned} \tag{4.18}$$

Now we recall that, as a simple consequence of Lemma 4.2, we have

$$\int h \log \frac{h}{m} = \int h \log \frac{h}{\mathrm{e}^{-x}}. \tag{4.19}$$

Therefore, we now want to change back from h to q. This is the same process and taking a second different $\tilde{K}$ and inserting (4.18), it leads to

$$\begin{aligned}\int q \log \frac{q}{\mathrm{e}^{-x}} &\leq C\mathrm{e}^{C\tilde{K}} \int h \log \frac{h}{\mathrm{e}^{-x}} + C\mathrm{e}^{-\tilde{K}/C} \\ &\leq C^2 \mathrm{e}^{C(K+\tilde{K})} D \log \frac{1}{D} + C\mathrm{e}^{-K/C+C\tilde{K}} + C\mathrm{e}^{-\tilde{K}/C},\end{aligned}$$

provided that $t \geq CK$ and $t \geq C\tilde{K}$. Just take now $\tilde{K} = K/2C^2$ so that we have automatically that $t \geq C\tilde{K}$ if $t \geq CK$ and

$$\int q \log \frac{q}{\mathrm{e}^{-x}} \leq C^2 \mathrm{e}^{2CK} D \log \frac{1}{D} + C\mathrm{e}^{-K/2C^2}. \tag{4.20}$$

We now have two distinct situation: First of all, consider the case where $t \geq C \log \frac{1}{D(t)}$. In that situation, we can take

$$K = s \log \frac{1}{D}$$

for any exponent $s \leq 1$ as this ensures $t \geq CK$. By optimizing the choice of s in (4.20), we find some exponent $\theta > 0$ (which depends only on C) such that, as

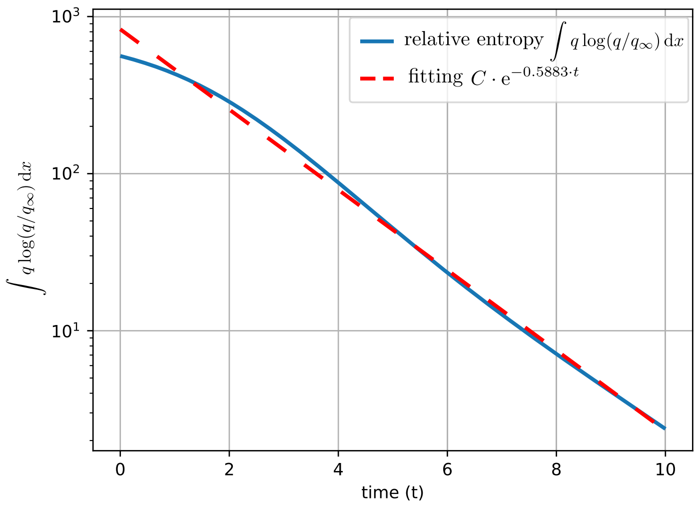

Fig. 8. Simulation of the relative entropy from q to q_∞ after $t = 10$ in the semilogy scale. We employed forward Euler method with time step-size $\Delta t = 0.05$, space step-size $\Delta x = 0.01$, and a “random” initial condition $q(t = 0, x)$ having mean value $m_1 = 5$ for the numerical simulation of (2.8). This experiment suggests that the relaxation of $\int q \log(q/q_\infty) dx$ might be exponentially fast in time, instead of polynomial convergence in time as guaranteed by Theorem 4.1.

claimed,

$$\int q \log \frac{q}{e^{-x}} \le C D^{\theta}. \tag{4.21}$$

In the alternative, we have that $t \le C \log \frac{1}{D(t)}$ or $D(t) \le e^{-t/C}$. Inserting this into (4.20) yields that

$$\int q \log \frac{q}{e^{-x}} \le C^2 e^{2CK} e^{-t/C} + C e^{-K/2C^2}.$$

We now choose $K = t/4C^2$, again ensuring $t \ge CK$ and giving

$$\int q \log \frac{q}{e^{-x}} \le \tilde{C} e^{-t/\tilde{C}} \tag{4.22}$$

for some different constant $\tilde{C}$. □

We end this section with a numerical experiment demonstrating the entropic convergence of q to q_∞, see Fig. 8.

5. Propagation of Chaos

We give the statement of the propagation of chaos, Theorem 5.1 in Sec. 5.1. A technical lemma that will be employed in the proof of Theorem 5.1 is displayed in Sec. 5.2. We reveal the full proof of Theorem 5.1 in Sec. 5.3.

5.1. *Statement of propagation of chaos*

In this section, we try to adapt the martingale-based techniques developed in Refs. 27 and 34 to justify the propagation of chaos.[39] For this purpose, we equip the space $\mathcal{P}(\mathbb{R}_+)$ with the Wasserstein distance with exponent 1, which is defined via

$$W_1(\mu, \nu) = \sup_{\|\nabla\varphi\|_\infty \le 1} \langle \mu - \nu, \varphi \rangle$$

for $\mu, \nu \in \mathcal{P}(\mathbb{R}_+)$ having finite first moment. We will also need the following version of Itô's formula.

Lemma 5.1. *Consider an inhomogeneous Poisson process* N_t *with intensity* $\lambda(t)$, *and a random variable* $Y(t)$ *left-continuous and adapted to the filtration* $\mathcal{F}_t$ *generated by* N_t. *We define the compound jump process* $Z(t)$ *and* $M(t)$ *its associated compensated martingale by*

$$dZ(t) = Y(t)\, d\mathrm{N}_t, \quad M(t) = Z(t) - Z(0) - \int_0^t \tilde{Y}(s)\lambda(s)\, ds, \tag{5.1}$$

where $\tilde{Y}$ *is any other left-continuous and adapted process. Itô's lemma then implies that for any* C^1 *function* Φ,

$$\begin{aligned} d\mathbb{E}[\Phi(M(t))] = {}& \mathbb{E}[\Phi(M(t-) + Y(t)) - \Phi(M(t-))]\lambda(t)\, dt \\ & - \mathbb{E}[\nabla\Phi(M(t)) \cdot \tilde{Y}(t)\lambda(t)]\, dt. \end{aligned} \tag{5.2}$$

Our main result in this section is stated as follows.

Theorem 5.1. *Denote the empirical distribution of the uniform reshuffling stochastic system* (1.1) *at time t as*

$$\rho_{\mathrm{emp}}(t) := \frac{1}{N}\sum_{i=1}^{N}\delta_{X_i(t)},$$

and let $q(t)$ be the solution of (2.8) *with initial condition $q(0)$. If*

$$\mathbb{E}[W_1(\rho_{\mathrm{emp}}(0), q(0))] \to 0 \quad \text{as } N \to \infty, \tag{5.3}$$

then we have that

$$\mathbb{E}[W_1(\rho_{\mathrm{emp}}(t), q(t))] \to 0 \quad \text{as } N \to \infty,$$

holding for all $0 \le t \le T$ with any prefixed $T > 0$.

5.2. *Switching supremum and expectation*

We will also make use of the following result, which allows us to interchange the operation of supremum and of expectation.

Lemma 5.2. *Consider a random Radon measure Z on $\mathbb{R}$ with $\int Z(dx) = 0$ and with uniformly bounded second moment $\int(1+|x|^2)|Z|(dx) \le m_2$ almost surely for some constant m_2. Then there exists a fixed constant $C > 0$ such that*

$$\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty \le 1}\int \varphi dZ\right] \le C m_2 \left(\sup_{\|\nabla\varphi\|_\infty \le 1}\mathbb{E}\left[\int \varphi dZ\right]^2\right)^{\frac{1}{8}}.$$

Proof. This is essentially an interpolation argument. First of all, we can always assume that $\varphi(0) = 0$ by subtracting a constant. Introduce a classical convolution kernel K_ε. We have that $\|K_\varepsilon \star \varphi - \varphi\|_{L^\infty} \le C\varepsilon$ which implies that

$$\int \varphi Z(dx) \le \int K_\varepsilon \star \varphi Z(dx) + C\varepsilon.$$

Then we reduce ourselves to a compact support: since $\|\nabla\varphi\|_\infty \le 1$ then $|\varphi(x)| \le |x|$ and

$$\begin{aligned}\int K_\varepsilon \star \varphi Z(dx) &\le \int_{|x|\le R} K_\varepsilon \star \varphi Z(dx) + 2\int_{|x|\ge R}|x||Z|(dx)\\ &\le \int_{|x|\le R} K_\varepsilon \star \varphi Z(dx) + 2\frac{m_2}{R}.\end{aligned}$$

On $[-R, R]$, we have on the other hand that $\|K_\varepsilon \star \varphi\|_{H^2} \le \frac{C}{\varepsilon}\|\varphi\|_{W^{1,\infty}} \le C\frac{R}{\varepsilon}$. Hence,

$$\sup_{\|\nabla\varphi\|_\infty \le 1}\int \varphi Z(dx) \le C\frac{R}{\varepsilon}\sup_{\|\varphi\|_{H^2}\le 1}\int_{|x|\le R}\varphi Z(dx) + Cm_2\left(\varepsilon + \frac{1}{R}\right).$$

Of course

$$\sup_{\|\varphi\|_{H^2}\le 1}\int_{|x|\le R}\varphi Z(dx)=\|Z\|_{H^{-2}([-R,R])}$$

and by using Fourier series

$$\|Z\|^2_{H^{-2}([-R,R])}=\sum_k\frac{R^2}{1+k^4}\left(\int_{-R}^{R}\mathrm{e}^{-ik\pi x/R}dZ\right)^2.$$

Hence by Cauchy–Schwartz,

$$\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty\le 1}\int\varphi Z(dx)\right]\le Cm_2\left(\varepsilon+\frac{1}{R}\right)$$
$$+C\frac{R^2}{\varepsilon}\left(\sum_k\frac{1}{1+k^4}\mathbb{E}\left[\left(\int_{-R}^{R}\mathrm{e}^{-ik\pi x/R}dZ\right)^2\right]\right)^{1/2}.$$

Finally, we have that

$$\|\nabla\mathrm{e}^{-ik\pi x/R}\|_\infty\le Ck,$$

so that

$$\mathbb{E}\left[\left(\int_{-R}^{R}\mathrm{e}^{-ik\pi x/R}dZ\right)^2\right]\le Ck^2\sup_{\|\nabla\varphi\|_\infty\le 1}\mathbb{E}\left[\left(\int\varphi dZ\right)^2\right].$$

This allows us to conclude that

$$\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty\le 1}\int\varphi Z(dx)\right]\le Cm_2\left(\varepsilon+\frac{1}{R}\right)$$
$$+C\frac{R^2}{\varepsilon}\left(\sum_k\frac{k^2}{1+k^4}\sup_{\|\nabla\varphi\|_\infty\le 1}\mathbb{E}\left[\left(\int\varphi dZ\right)^2\right]\right)^{1/2}$$

or

$$\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty\le 1}\int\varphi Z(dx)\right]\le Cm_2\left(\varepsilon+\frac{1}{R}\right)+C\frac{R^2}{\varepsilon}\left(\sup_{\|\nabla\varphi\|_\infty\le 1}\mathbb{E}\left[\left(\int\varphi dZ\right)^2\right]\right)^{1/2},$$

which finishes the proof by setting $R=\mathbb{E}[(\int\varphi dZ)^2]^{-1/8}$ and $\varepsilon=\frac{1}{R}$. □

5.3. *Proof of propagation of chaos*

The proof of Theorem 5.1 occupies the rest of the section.

Proof. We recall that the map $Q_+[\cdot]:\mathcal{P}(\mathbb{R}_+)\to\mathcal{P}(\mathbb{R}_+)$ is defined via

$$Q_+[q](x)=\int_0^\infty\int_0^\infty\frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell}q(k)q(\ell)\,dk\,d\ell$$

and that a classical solution $q(t,x)$ of

$$q(t,x) = q(0,x) + \int_0^t G[q](s,x)\,ds \tag{5.4}$$

exists for $0 \le t < \infty$, where $G = Q_+ - \mathrm{Id}$ and $q(0,x)$ is an continuous probability density function with mean m_1 whose support is contained in $\mathbb{R}_+$. The map Q_+ is Lipschitz continuous in the sense that

$$W_1(Q_+[f], Q_+[g]) \le W_1(f,g) \tag{5.5}$$

for any $f, g \in \mathcal{P}(\mathbb{R}_+)$. Indeed, we have

$$W_1(Q_+[f], Q_+[g]) = \sup_{\|\nabla\varphi\|_\infty \le 1} \mathbb{E}[\varphi(U(X_1+Y_1)) - \varphi(U(X_2+Y_2)],$$

where X_1, Y_1 are i.i.d. with law f, X_2, Y_2 are i.i.d. with law g and $U \sim \mathrm{Uniform}[0,1]$ is independent of X_i and Y_i for $i = 1, 2$. By Lipschitz continuity of the test function φ, we obtain

$$W_1(Q_+[f], Q_+[g]) \le \mathbb{E}[2U|X_1 - X_2|] = \mathbb{E}[|X_1 - X_2|].$$

We now recall an alternative formulation of $W_1(f,g)$, given by

$$W_1(f,g) = \inf\{\mathbb{E}[|X-Y|];\ \mathrm{Law}(X) = f,\ \mathrm{Law}(Y) = g\},$$

so in particular, we may take a coupling of X_1 and X_2 so that $W_1(f,g) = \mathbb{E}[|X_1 - X_2|]$. Assembling these pieces together, we arrive at (5.5).

We are going to prove a more precise control than (5.5), by working directly on $Q_+[f]$. Consider now two random probability measures f and g with bounded second moment and a deterministic test function φ. We have that

$$\begin{aligned}
&\int \varphi(x)(Q_+[f] - Q_+[g])\,dx \\
&\quad = \int \frac{\mathbb{1}_{x \le k+\ell}}{k+\ell}\varphi(x)(f(dk) - g(dk))(f(d\ell) + g(d\ell))\,dx \\
&\quad = \int (f(d\ell) + g(d\ell)) \int \Phi_\ell(k)(f(dk) - g(dk)),
\end{aligned}$$

where we denote

$$\Phi_\ell(k) = \frac{1}{k+\ell}\int_0^{k+\ell} \varphi(x)\,dx.$$

Since $\int Q_+[f] = \int Q_+[g]$, we can always assume without loss of generality that $\varphi(0) = 0$, whence $|\varphi(x)| \le \|\nabla\varphi\|_\infty |x| \le |x|$. Now we observe that Φ_ℓ is deterministic with

$$\begin{aligned}
|\partial_k \Phi_\ell(k)| &\le \frac{|\varphi(k+\ell)|}{k+\ell} + \frac{1}{(k+\ell)^2}\int_0^{k+\ell} |\varphi(x)|\,dx \\
&\le 1 + \frac{1}{(k+\ell)^2}\int_0^{k+\ell} x\,dx \le \frac{3}{2}.
\end{aligned} \tag{5.6}$$

By (5.6) and recalling that again Φ_ℓ is deterministically obtained from φ,

$$\mathbb{E}\left[\int \Phi_\ell(k)(f(dk) - g(dk))\right] \le \frac{3}{2}\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty \le 1} \int \varphi(x)(f(dx) - g(dx))\right].$$

Therefore, we conclude that

$$\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty \le 1} \int \varphi(x)(Q_+[f] - Q_+[g])\right] \le 3\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty \le 1} \int \varphi(x)(f(dx) - g(dx))\right]. \tag{5.7}$$

We now observe that the empirical measure is a compound jump process: Define N_t a homogeneous Poisson process with constant intensity $\lambda = (N-1)/2$. Given $\tau_1, \ldots, \tau_k$ the times when N_t jumps, we take the Y_{τ_k} independent: At each τ_k, with uniform probability $\frac{2}{N\,(N-1)}$ we choose a pair $i < j$ and take

$$\begin{aligned} Y_{\tau_k} = \frac{1}{N}\big(&\delta(x - U_k(X_i(\tau_k-) + X_j(\tau_k-)) + \delta(x - (1-U_k)(X_i(\tau_k-) + X_j(\tau_k-)) \\ &- \delta(x - X_i(\tau_k-)) - \delta(x - X_j(\tau_k-))\big), \end{aligned}$$

where the U_k are i.i.d. in $[0,1]$.

We immediately note that

$$\begin{aligned} \lambda\mathbb{E}[Y_t] = \frac{1}{N^2}\sum_{i<j}\mathbb{E}\big[&\delta(x - U\,(X_i(t-) + X_j(t-)) + \delta(x - (1-U) \\ &\times (X_i(t-) + X_j(t-)) - \delta(x - X_i(t-)) - \delta(x - X_j(t-))\big], \end{aligned} \tag{5.8}$$

where U is uniformly distributed in $[0,1]$ and independent of all $X_i(t-)$.

We also remark that we can easily mimic the propagation of moments shown in Sec. 2 with in particular Proposition 2.1 to prove that

$$\mathbb{E}\left[\int x^2 \rho_{\text{emp}}(t, dx)\right] \le m_2 = 2 + \mathbb{E}\left[\int x^2 \rho_{\text{emp}}(0, dx)\right]. \tag{5.9}$$

We now show that the empirical measure of the stochastic system satisfies an approximate version of (5.4). Fix a deterministic test function φ with $\|\nabla\varphi\|_\infty \le 1$, and consider the time evolution of $\langle\rho_{\text{emp}}, \varphi\rangle$, where for some probability measure ν, we denote by the duality bracket $\langle\nu, \varphi\rangle = \int \varphi\, d\nu$. We emphasize here that φ can also be random and will indeed be chosen according to ρ_{emp} to estimate Wasserstein distances involving ρ_{emp}. Then

$$d\mathbb{E}[\langle\rho_{\text{emp}}, \varphi\rangle] = d\mathbb{E}[\langle Y_t\, dN_t, \varphi\rangle] = \lambda\langle\mathbb{E}[Y_t], \varphi\rangle\, dt.$$

Hence by (5.8),

$$\begin{aligned} d\mathbb{E}[\langle\rho_{\text{emp}}, \varphi\rangle] = \frac{1}{N^2}\sum_{i<j}\mathbb{E}\big[&\varphi\big(U(X_i + X_j)\big) + \varphi\big((1-U)(X_i + X_j)\big) \\ &- \varphi(X_i) - \varphi(X_j)\big]\, dt \end{aligned}$$

$$= \frac{1}{N^2} \sum_{i,j=1\ldots N, i\neq j} \mathbb{E}\big[\varphi\big(U(X_i+X_j)\big) - \varphi(X_i)\big]\, dt$$

$$= \frac{1}{N^2} \sum_{i,j=1}^{N} \mathbb{E}\big[\varphi\big(U(X_i+X_j)\big) - \varphi(X_i)\big]\, dt + R\, dt,$$

where all X_i, X_j are taken at time $t-$ and where

$$R = -\frac{1}{N^2} \sum_i \mathbb{E}\big[\varphi\big(2UX_i\big) - \varphi(X_i)\big].$$

Hence, $|R| \le \mathcal{O}(\frac{1}{N})$ uniformly over φ and $t \ge 0$. On the other hand, we may calculate

$$\langle Q_+[\rho_{\mathrm{emp}}], \varphi\rangle = \frac{1}{N^2} \sum_{i,j} \int \varphi(x) \frac{\mathbb{1}_{x \le X_i+X_j}}{X_i+X_j}\, dx = \frac{1}{N^2} \sum_{i,j} \int_0^1 \varphi(u(X_i+X_j))\, du,$$

by the change of variables $x = u\,(X_i+X_j)$. Therefore,

$$d\mathbb{E}[\langle \rho_{\mathrm{emp}}, \varphi\rangle] = \mathbb{E}[\langle G[\rho_{\mathrm{emp}}], \varphi\rangle]\, dt + R\, dt. \tag{5.10}$$

By Dynkin's formula, the compensated process

$$M_\varphi(t) := \langle \rho_{\mathrm{emp}}(t), \varphi\rangle - \langle \rho_{\mathrm{emp}}(0), \varphi\rangle - \int_0^t (\mathbb{E}[\langle G[\rho_{\mathrm{emp}}(s)], \varphi\rangle] + R(s))\, ds \tag{5.11}$$

is a martingale. Furthermore, comparing with (5.4), we easily obtain that

$$\begin{aligned}\langle \rho_{\mathrm{emp}}(t) - q(t), \varphi\rangle &= M_\varphi(t) + \langle \rho_{\mathrm{emp}}(0) - q(0), \varphi\rangle \\ &\quad + \mathbb{E}\int_0^t \langle G[\rho_{\mathrm{emp}}(s)] - G[q(s)], \varphi\rangle\, ds + \mathcal{O}\left(\frac{t}{N}\right).\end{aligned}$$

Taking the supremum over φ, we therefore have that

$$\begin{aligned}&\mathbb{E} \sup_{\|\nabla\varphi\|_\infty \le 1} \langle \rho_{\mathrm{emp}}(t) - q(t), \varphi\rangle \\ &\quad \le \mathbb{E} \sup_{\|\nabla\varphi\|_\infty \le 1} (|M_\varphi(t)| + \langle \rho_{\mathrm{emp}}(0) - q(0), \varphi\rangle) \\ &\qquad + \int_0^t \mathbb{E} \sup_{\|\nabla\varphi\|_\infty \le 1} \langle G[\rho_{\mathrm{emp}}(s)] - G[q(s)], \varphi\rangle\, ds + \mathcal{O}\left(\frac{t}{N}\right).\end{aligned}$$

By the definition of the W_1 distance, we deduce from (5.7) that

$$\mathbb{E}W_1(\rho_{\mathrm{emp}}(t), q(t)) \le \eta(t) + C \int_0^t \mathbb{E}W_1(\rho_{\mathrm{emp}}(t), q(t))\, ds + \frac{Ct}{N},$$

in which we have set

$$\eta(t) := \mathbb{E} \sup_{\|\nabla\varphi\|_\infty \le 1} |M_\varphi(t)| + \mathbb{E}W_1(\rho_{\mathrm{emp}}(0), q(0)). \tag{5.12}$$

Thus, Gronwall's inequality gives rise to

$$\mathbb{E}W_1(\rho_{\text{emp}}(t), q(t)) \leq \left(\sup_{t\in[0,T]} \eta(t) + \frac{CT}{N}\right) \mathrm{e}^{CT}. \tag{5.13}$$

In order to establish propagation of chaos for $t \leq T$, it therefore suffices to show that

$$\sup_{t\in[0,T]} \eta(t) \xrightarrow{N\to\infty} 0. \tag{5.14}$$

To prove (5.14), we treat each term appearing in the definition of $\eta(t)$ separately. The second term in (5.12) approaches to 0 as $N \to \infty$ by our assumption.

To handle the first term, let us write $Z(t) = \langle \rho_{\text{emp}}(t), \varphi\rangle$ and $M(t) = M_\varphi(t)$ to simplify the notations. Of course $Z(t)$ is a compound jump process itself and by combining (5.10) and (5.11)

$$M_\varphi(t) = Z(t) - Z(0) - \int_0^t \tilde{Y}(s)\,ds, \quad \tilde{Y}(t) = \langle G[\rho_{\text{emp}}(t)], \varphi\rangle + R.$$

We may hence use Itô's lemma as stated in Lemma 5.1, which yields

$$\begin{aligned} d\mathbb{E}[M^2(t)] = &\sum_{i<j} \mathbb{E}\big[M_{ij}^2(t) - M^2(t)\big]\frac{dt}{N} - \mathbb{E}\big[2M(t)\langle G[\rho_{\text{emp}}(t)], \varphi\rangle\big]\,dt \\ &+ \mathcal{O}\left(\frac{1}{N}\right) dt, \end{aligned}$$

where $M_{ij} = M + Y_{ij}$ and we define

$$Y_{ij} := \left\langle \frac{1}{N}(\delta_{U_k(X_i+X_j)} + \delta_{(1-U_k)(X_i+X_j)} - \delta_{X_i} - \delta_{X_j}), \varphi \right\rangle.$$

Therefore, we have

$$d\mathbb{E}[M^2(t)] = \sum_{i<j} \mathbb{E}\big[2M(t)Y_{ij} + Y_{ij}^2\big]\frac{dt}{N} - \mathbb{E}\big[2M(t)\langle G[\rho_{\text{emp}}(t)], \varphi\rangle\big]\,dt + \mathcal{O}\left(\frac{1}{N}\right) dt.$$

By our previous calculations

$$\begin{aligned} &\frac{1}{N}\sum_{i<j} \mathbb{E}[M(t)Y_{ij}] \\ &\quad = \frac{1}{N^2}\sum_{i<j} \mathbb{E}[M(t)(\varphi(U(X_i + X_j)) + \varphi((1-U)(X_i+X_j) - \varphi(X_i) - \varphi(X_j))] \\ &\quad = \frac{1}{N^2}\sum_{i\neq j} \mathbb{E}[M(t)(\varphi(U(X_i + X_j)) - \varphi(X_i))] \\ &\quad = \frac{1}{N^2}\sum_{i,j} \mathbb{E}[M(t)(\varphi(U(X_i + X_j)) - \varphi(X_i))] + O\left(\frac{1}{N}\right), \end{aligned}$$

as U is random variable independent of $M(t)$ and $\rho_{\text{emp}}(t)$.

Therefore,

$$\frac{1}{N}\sum_{i<j}\mathbb{E}[M(t)Y_{ij}] = \mathbb{E}[M(t)\langle G[\rho_{\mathrm{emp}}(t)],\varphi\rangle] + O\left(\frac{1}{N}\right)$$

and consequently

$$d\mathbb{E}[M^2(t)] = \sum_{i<j}\mathbb{E}\left[Y_{ij}^2\right]\frac{dt}{N} + \mathcal{O}\left(\frac{1}{N}\right)dt \le \frac{C}{N}dt,$$

for a constant C that depends only on $\|\nabla\varphi\|_\infty$. This lets us deduce that

$$\sup_{\|\nabla\varphi\|_\infty\le 1}\mathbb{E}\left[M_\varphi^2(t)\right] \le \frac{Ct}{N}.$$

Recalling the definition of $M_\varphi(t)$, we have that

$$M_\varphi(t) = \int \varphi(x)\mu(t,dx)$$

for some random Radon measure μ with uniformly bounded second moment. Furthermore $\int\mu(t,dx) = 0$ since $\int\rho_{\mathrm{emp}}(t,dx) = 1 = \int\rho_{\mathrm{emp}}(0,dx)$ and $\int G[\rho_{\mathrm{emp}}(t)]\,dx = 0$.

We may hence apply Lemma 5.2 to obtain that

$$\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty\le 1} M_\varphi(t)\right] \le C\frac{t^{1/8}}{N^{1/8}},$$

which allows to conclude that $\sup_{t\in[0,T]}\eta(t) \xrightarrow{N\to\infty} 0$. □

Remark 5.1. One can readily check that

$$\|Q_+[f] - Q_+[g]\|_{L^1(\mathbb{R}_+)} \le 2\|f-g\|_{L^1(\mathbb{R}_+)}$$

for all probability densities f, g whose support are contained in $\mathbb{R}_+$, but as we are working on $\mathcal{P}(\mathbb{R}_+)$, we cannot use any strong distances. Hence, equipping $\mathcal{P}(\mathbb{R}_+)$ with an appropriate distance so that the operator Q_+ has enjoys a Lipschitz continuity with respect to the chosen distance is an indispensable step to make the argument above work.

Appendix A. Proof of Corollary 3.2

Proof. The whole strategy is of course to find some δ such that if

$$\int q(t,x)\log\frac{q(t,x)}{q_\infty(x)}\,dx \le \delta, \tag{A.1}$$

then we have for the ε of Corollary 3.1

$$\int \frac{|q(t,x)-q_\infty(x)|^2}{q_\infty(x)}\,dx \le \varepsilon. \tag{A.2}$$

We start with using Lemma 4.3 for $C = 2$ and note that

$$\begin{aligned}\frac{1}{4}\int_{q_\infty/2\le q\le 2q_\infty}&\frac{|q(t,x)-q_\infty(x)|^2}{q_\infty(x)}\,dx+\frac{1}{8}\int_{q\le q_\infty/2}q_\infty(x)\,dx\\&+\frac{\log 2}{4}\int_{q\ge 2q_\infty}q(t,x)\,dx\\&\le\int q(t,x)\log\frac{q(t,x)}{q_\infty(x)}\,dx.\end{aligned}\tag{A.3}$$

Observe that if $q \le q_\infty/2$ then

$$\frac{|q(t,x)-q_\infty(x)|^2}{q_\infty(x)}\le q_\infty(x),$$

so the first two terms already provides the straightforward bound

$$\int_{q\le 2q_\infty}\frac{|q(t,x)-q_\infty(x)|^2}{q_\infty(x)}\,dx\le 8\int q(t,x)\log\frac{q(t,x)}{q_\infty(x)}\,dx.\tag{A.4}$$

Now if $q \ge q_\infty$ then

$$\frac{|q(t,x)-q_\infty(x)|^2}{q_\infty(x)}\le\frac{(q(t,x))^2}{q_\infty(x)}.$$

Therefore, for any $p > 1$,

$$\begin{aligned}\int_{q\ge 2q_\infty}&\frac{|q(t,x)-q_\infty(x)|^2}{q_\infty(x)}\,dx\\&\le\int_{q\ge 2q_\infty}\frac{|q(t,x)|^2}{q_\infty(x)}\,dx\\&\le\left(\int_{q\ge 2q_\infty}q(t,x)\,dx\right)^{1-1/p}\left(\int_{q\ge 2q_\infty}\frac{|q(t,x)|^{p+1}}{(q_\infty(x))^p}\,dx\right)^{1/p}.\end{aligned}$$

We now use Corollary 4.3 to find that

$$\begin{aligned}\int_{q\ge 2q_\infty}\frac{|q(t,x)|^{p+1}}{(q_\infty(x))^p}\,dx&\le C_p\int\left(\mathrm{e}^{(-(p+1)\lambda_0+p)x}+\mathrm{e}^{px}(q(0,x))^{p+1}\right)dx\\&\le C_p'\int\mathrm{e}^{(-(p+1)\lambda+p)x}\,dx,\end{aligned}$$

in which $\lambda \in (\frac{1}{2}, \lambda_0)$. Now we take p close enough to 1 such that $p - (p+1)\lambda < 0$ which is always possible if $\lambda_0 > \frac{1}{2}$. For this choice of p, we hence obtain that

$$\int_{q\ge 2q_\infty}\frac{|q(t,x)-q_\infty(x)|^2}{q_\infty(x)}\,dx\le C_p\left(\int_{q\ge 2q_\infty}q(x)\,dx\right)^{1-1/p}.$$

Going back to (A.3), we can conclude that

$$\int_{q \geq 2q_\infty} \frac{|q(t,x) - q_\infty(x)|^2}{q_\infty(x)} dx \leq C_p \left(\int q(t,x) \log \frac{q(t,x)}{q_\infty(x)} dx \right)^{1-1/p}$$

and combining this with (A.4), we deduce that for some C and $\theta \in (0,1)$,

$$\int \frac{|q(t,x) - q_\infty(x)|^2}{q_\infty(x)} dx \leq C \left(\int q(t,x) \log \frac{q(t,x)}{q_\infty(x)} dx \right)^\theta \leq C\delta^\theta.$$

It is enough to choose δ being small enough to conclude the proof. □

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