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DERIVATION OF WEALTH DISTRIBUTIONS FROM BIASED EXCHANGE OF MONEY

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Abstract. In the manuscript, we are interested in using kinetic theory to better understand the time evolution of wealth distribution and their large scale behavior such as the evolution of inequality (e.g. Gini index). We investigate three types of dynamics denoted unbiased, poor-biased and rich-biased exchange models. At the individual level, one agent is picked randomly based on its wealth and one of its dollars is redistributed among the population. Proving the so-called propagation of chaos, we identify the limit of each dynamics as the number of individuals approaches infinity using both coupling techniques [54] and a martingale-based approach [42]. Equipped with the limit equation, we identify and prove the convergence to specific equilibrium for both the unbiased and poor-biased dynamics. In the rich-biased dynamics however, we observe a more complex behavior where a dispersive wave emerges. Although the dispersive wave is vanishing in time, it also accumulates all the wealth leading to a Gini approaching 1 (its maximum value). We characterize numerically the behavior of dispersive wave but further analytic investigation is needed to derive such dispersive wave directly from the dynamics.

1. Introduction.

Econophysics is an emerging branch of statistical physics that applies concepts and techniques of traditional physics to economics and finance [23,31,51]. It has attracted considerable attention in recent years raising challenges on how various economical phenomena could be explained by universal laws in statistical physics, and we refer to [20,21,36,47] for a general review.

The primary motivation for studying models arising from econophysics is at least two-fold: from the perspective of a policy maker, it is important to deal with the rise of income inequality [27,28] in order to establish a more egalitarian society; From a mathematical point of view, we have to understand the fundamental mechanisms, such as money exchange resulting from individuals, which are usually agent-based models. Given an agent-based model, one is expected to identify the limit dynamics

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as the number of individuals tends to infinity and then its corresponding equilibrium when you run the model for a sufficiently long time (if there is one). This guiding approach is carried out in numerous works across different fields among literatures of applied mathematics, see for instance [5, 17, 44].

Although we will only consider three distinct binary exchange models in the present work, other exchange rules can also be imposed and studied, leading to different models. To name a few, the so-called immediate exchange model introduced in [33] assumes that pairs of agents are randomly and uniformly picked at each random time, and each agent transfers a random fraction of their money to the other agent, where these fractions are independent and uniformly distributed on $[0, 1]$. The so-called uniform reshuffling model investigated in [31] and [38] suggests that the total amount of money of two randomly and uniformly picked agents possessed before interaction is uniformly redistributed among the two agents after interaction. The so-called repeated averaging model studied for instance in [14] where two randomly selected agents share half of their wealth with each other. The binomial reshuffling model proposed in a recent work [12] is a variant of the uniform reshuffling mechanism in which the agents' combined wealth is redistributed according to a binomial distribution. For models with saving propensity and with debts, we refer the readers to [16, 19, 22, 39].

1.1. **Unbiased/poor-biased/rich-biased dynamics.** In this work, we consider several dynamics for money exchange in a closed economical system, meaning that there are a fixed number of agents, denoted by N, with an (fixed) average number of dollars μ. We denote by $S_i(t)$ the amount of dollars the agent i has at time t. Since it is a closed economical system, we have:

$$S_1(t) + \cdots + S_N(t) = \text{Constant} \qquad \text{for all } t \geq 0. \tag{1}$$

As a first example of money exchange, we review the model proposed in [31]: at random time (exponential law), an agent i is picked at random (uniformly) and if it has at least one dollar (i.e. $S_i \geq 1$) it will give one dollar to another agent j picked at random (uniformly). If i does not have one dollar (i.e. $S_i = 0$), then nothing happens. From now on we will refer to this model as **unbiased exchange model** as all the agents are being picked with equal probability. We refer to this dynamics as follow:

$$\textbf{unbiased:} \qquad (S_i, S_j) \overset{\lambda}{\rightsquigarrow} (S_i - 1, S_j + 1) \quad (\text{if } S_i \geq 1). \tag{2}$$

In other words, every agent with at least one dollar gives to all of the others agents at a fixed rate. Later on, we will adjust the rate λ (more exactly $\lambda \mathbb{1}_{[1,+\infty)}(S_i)$) by normalizing by N in order to have the correct asymptotics as $N \to +\infty$ (the rate of one agent giving a dollar per unit time is of order N otherwise).

Another possible dynamics is to pick the giver agent, i.e. agent i, with higher probability if the agent is rich, i.e. S_i large. Thus *poor* agents will have a lower frequency of being picked. From now on we will call this model **poor-biased model** and is illustrated as follows:

$$\textbf{poor-biased:} \qquad (S_i, S_j) \overset{\lambda S_i}{\rightsquigarrow} (S_i - 1, S_j + 1). \tag{3}$$

Notice that since the rate of giving is S_i, an agent with no money, i.e. $S_i = 0$, will never have to give. As for the unbiased dynamics (2), we will also adjust the rate, normalizing it by N.

Our third dynamics that we would like to explore is the **rich-biased model**: we reverse the bias compared to the previous dynamics, so that rich agents are *less* likely to give:

$$\textbf{rich-biased:} \qquad (S_i, S_j) \overset{\lambda/S_i}{\rightsquigarrow} (S_i - 1, S_j + 1) \quad (\text{if } S_i \geq 1). \tag{4}$$

As a consequence of this dynamics, rich agents will tend to become even richer compared to poor agents creating a feedback that could lead to a singular behavior. The adjustment of the rate for this dynamics is more delicate since the sum of the rates λ/S_i is no longer constant. In particular, we will see that a normalization of the rates to have a constant rate of giving a dollar per agent will lead to finite time blow-up of the dynamics in the limit $N \to +\infty$.

Remark 1. The only difference between “picking simultaneously a giver i and a receiver j ” and “picking a giver i first and then pick a receiver j” lies in whether $i = j$ is allowed (i.e., whether an agent is allowed to give one dollar to himself/herself, in which case the state of the N-agent system remains unchanged). Actually in the pioneering work of Dragulescu and Yakovenko [31] it is not completely clear which rule is used in their numerical simulations. However, allowing $i = j$ only changes the probability of picking the (ordered) pair (i, j) from $\frac{1}{N(N-1)}$ to $\frac{1}{N^2}$ and thus it does not affect the asymptotic results for $N \to \infty$ nor the simulations when N is large.

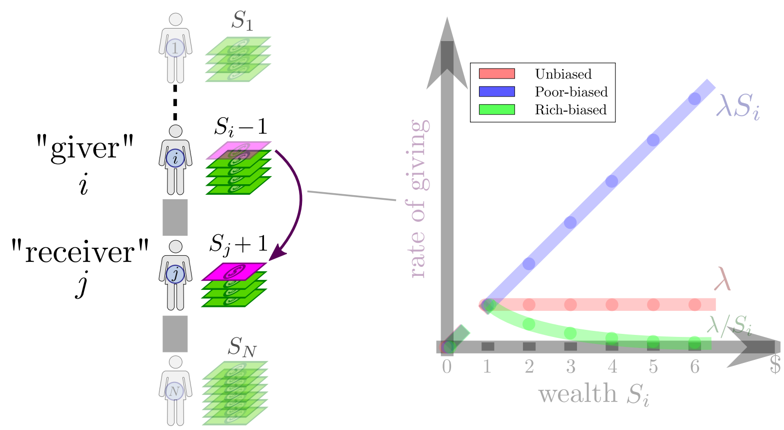


Figure 1 Left: Illustration of the 3 dynamics: at random time, one dollar is passed from a “giver” i to a “receiver” j. **Right:** The rate of picking the “giver” i depends on the wealth S_i.

We illustrate the dynamics in Figure 1-left. The key question of interest is the exploration of the limiting money distribution among the agents as the total number of agents and the number of time steps become large. We illustrate numerically (see Figure 2) the three previous dynamics using $N = 500$ agents. In the unbiased dynamics (pink), the wealth distribution is (approximately) exponential with the proportion of agent decaying as wealth increases. On the contrary, the poor-biased dynamics (blue) has the bulk of its distribution around \$10 (the average capital per agent). For the rich-biased dynamics (green), most of the agents are left with no money leaving only a few with large amounts (more than \$30). To visualize the

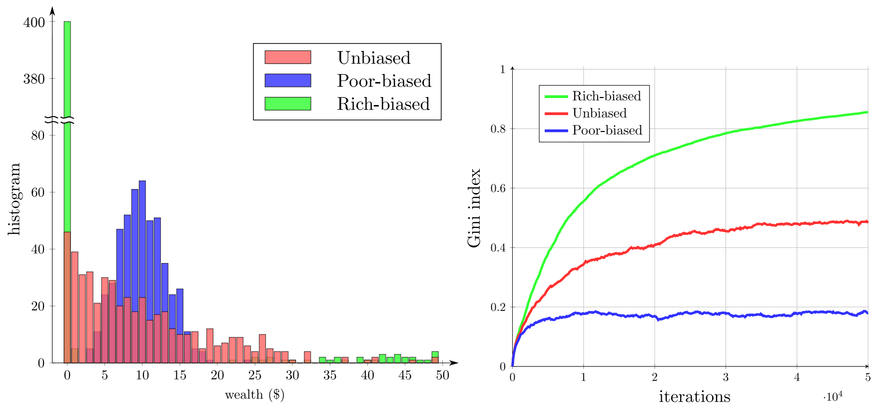


Figure 2 Left: Distribution of wealth for the three dynamics after 50,000 steps. The distribution decays for the unbiased dynamics (pink) i.e. poor agents are more frequent than rich agents, whereas in the poor-biased dynamics, the distribution (blue) is centered at the average \$10. For the rich-biased dynamics, almost all agents have zero dollars except a few with a large amount (more than \$30). **Right:** evolution of the Gini index (5) for the three dynamics. The Gini index is lower for the poor-biased dynamics (less inequality) whereas it is approaching 1 for the rich-biased dynamics.

temporal evolution of the three dynamics, we estimate the Gini index G after each iteration in Figure 1-right:

$$G = \frac{1}{2N^2\mu} \sum_{1 \le i,j \le N} |S_i - S_j|, \tag{5}$$

where μ is the average wealth ($\mu = \frac{1}{N}\sum_{i=1}^N S_i$). The widely used inequality indicator Gini index G measures the inequality in the wealth distribution and ranges from 0 (no inequality) to 1 (extreme inequality). Since all agents have the same amount of dollar initially ($S_i(t=0) = \mu$), the Gini index starts at zero (i.e. $G(t=0) = 0$). In the unbiased dynamics, the Gini index stabilizes around .5 (which corresponds to the Gini index of an exponential distribution). The Gini index is strongly reduced in the poor-biased dynamics ($G \approx .19$). On the contrary, the Gini index keeps increasing in the rich-biased dynamics and seems to approach 1 (its maximum). We study in more details this phenomena in section 5.3. We emphasize that the "rich-get-richer" phenomenon, numerically observed in the rich-biased dynamics in the present work, has also been reported in other models from econophysics, and we refer interested readers to [7,8] and references therein.

1.2. **Asymptotic dynamics: $N \to +\infty$ and $t \to +\infty$.** One of the main difficulty in any rigorous mathematical treatment lies in the general fact that models in econophysics typically consist of a large number of interacting (coupled) economic agents. Fortunately the framework of kinetic theories allows simplifications of the mathematical analysis of certain such models under some appropriate limit processes. For the unbiased model (2) and the poor-biased model (3), instead of taking the large time limit and then the large population limit as in [37], we first take the large population limit to achieve a transition from the large stochastic system of interacting agents to a deterministic system of ordinary differential equations by proving the so-called propagation of chaos [42, 43, 45, 54] through a well-designed coupling technique, see Figure 3 for a illustration of these strategies. After that, analysis of the deterministic description is then built on its (discrete) Fokker-Planck formulation and we investigate the convergence toward an equilibrium distribution by employing entropy methods [3, 34, 40]. For the rich-biased model, we prove the propagation of chaos by virtue of a novel martingale-based technique introduced in [42], and we report some interesting numerical behavior of the associated ODE system. We illustrate the various (limiting) ODE systems obtained in the present work in Figure 4.

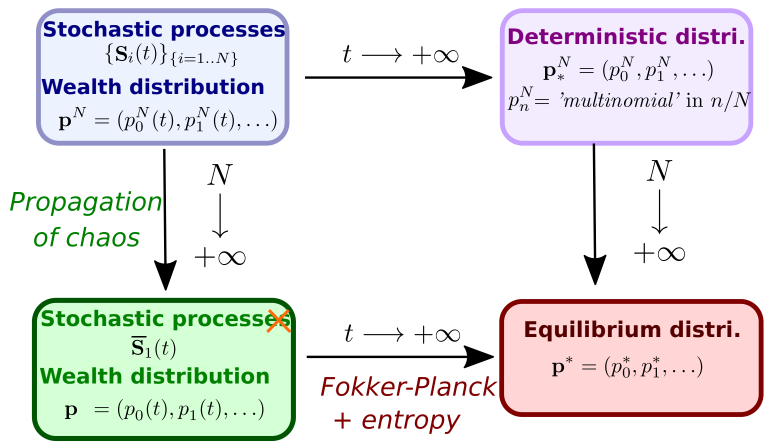
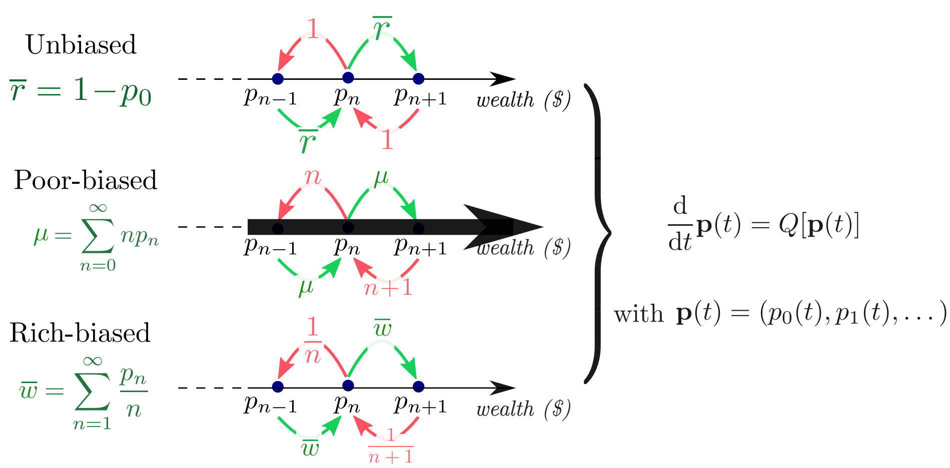


Figure 3 Schematic illustration of the strategy of proof: The approach of sending $t \to \infty$ first and then taking $N \to \infty$ is carried out in [37]. Our strategy is to perform the limit $N \to \infty$ before investigating the time asymptotic $t \to \infty$.



For the poor-biased model, we present an explicit rate of convergence of its associated system of ordinary differential equations toward its equilibrium thanks to a Poisson-Poincaré type inequality. Then, we resort to numerical simulation in the determination of the sharp rate of convergence and a heuristic argument is used in support of our numerical observation.

This paper is organized as follows: in section 2, we briefly review different approaches to tackle the propagation of chaos. Section 3 is devoted to the investigation of the unbiased exchange model, where the rigorous large population limit $N \to \infty$ is carried out via a coupling argument. We perform the analysis, for the poor-biased model in section 4 and for the rich-biased model in section 5, in a parallel fashion that resembles section 3. A subsection is dedicated in 5.3 to the emergence of a dispersive traveling wave in the rich-biased dynamics. Finally, a conclusion is drawn in section 6.

2. **Review propagation of chaos.**

2.1. **Definition.** We propose to review the method used to prove the so-called propagation of chaos. We consider a $N-$particle system denoted $\{S_i\}_{i=1..N}$ where particles are indistinguishable. In other words, the particle system is invariant by permutation, i.e. for any test function φ and permutation $\sigma \in \mathcal{S}_N$:

$$\mathbb{E}[\varphi(S_1,\ldots,S_N)] = \mathbb{E}[\varphi(S_{\sigma(1)},\ldots,S_{\sigma(N)})].$$

Denote by $\mathbf{p}^{(N)}(s_1,\ldots,s_N)$ the density distribution of the $N-$process and let $\mathbf{p}_k^{(N)}$ be the marginal density, i.e. the law of the process $(S_1,...,S_k)$ (for $1 \le k \le N$):

$$\mathbf{p}_k^{(N)}(s_1,\ldots,s_k) = \int_{s_{k+1},\ldots,s_N} \mathbf{p}^{(N)}(s_1,\ldots,s_N)\,\mathrm{d}s_{k+1}\ldots\mathrm{d}s_N.$$

Consider now a *limit* stochastic process $(\overline{S}_1,\ldots,\overline{S}_k)$ where $\{\overline{S}_i\}_{i=1,\ldots,k}$ are independent and identically distributed. Denote by $\mathbf{p}_1$ the law of a single process, thus by independence assumption the law of *all* the processes is given by:

$$\mathbf{p}_k(s_1,\ldots,s_k) = \prod_{i=1}^{k} \mathbf{p}_1(s_i).$$

Definition 2.1. We say that the stochastic process $(S_1,\ldots,S_N)$ satisfies the propagation of chaos if for any fixed k:

$$\mathbf{p}_k^{(N)} \stackrel{N\to+\infty}{\rightharpoonup} \mathbf{p}_k \tag{6}$$

which is equivalent to have for any test function φ:

$$\mathbb{E}[\varphi(S_1,\ldots,S_k)] \stackrel{N\to+\infty}{\longrightarrow} \mathbb{E}[\varphi(\overline{S}_1,\ldots,\overline{S}_k)]. \tag{7}$$

2.2. **Coupling method.** The *coupling* method [54] consists in generating the two processes $(S_1,\ldots,S_N)$ and $(\overline{S}_1,\ldots,\overline{S}_k)$ *simultaneously* in such a way that:

i) $(S_1,\ldots,S_k)$ and $(\overline{S}_1,\ldots,\overline{S}_k)$ satisfy their respective law,

ii) S_i and $\overline{S}_i$ are *closed* for all $1 \le i \le k$.

The difficulty is that $\{\overline{S}_i\}_{i=1..k}$ are independent but $\{S_i\}_{i=1..N}$ are <u>not</u>, thus the two processes cannot be *too* closed. In practice, we expect to find a bound of the form:

$$\mathbb{E}[|S_i - \overline{S}_i|] \le \frac{C}{\sqrt{N}} \stackrel{N\to+\infty}{\longrightarrow} 0 \quad , \quad \text{for all } 1 \le i \le k. \tag{8}$$

Such result is sufficient[1] to prove (7) and therefore one deduces propagation of chaos.

In a more abstract point of view, the inequality (8) gives an upper bound for the Wasserstein distance between $\mathbf{p}_k^{(N)}$ and the limit density $\mathbf{p}_k$. Since convergence in Wasserstein distance is equivalent to weak-* convergence for measures, we can conclude about the propagation of chaos (6).

2.3. **Empirical distribution - tightness of measure.** Another approach to prove propagation of chaos is to study the so-called empirical measure:

$$\mathbf{p}_{emp}^{(N)}(s) = \frac{1}{N}\sum_{i=1}^{N} \delta_{S_i}(s) \tag{9}$$

where δ is the Delta distribution. Notice that $\mathbf{p}_{emp}^{(N)}$ is a distribution of a single variable, thus the domain of $\mathbf{p}_{emp}^{(N)}$ remains the same as N increases which simplifies its study. However, $\mathbf{p}_{emp}^{(N)}$ is also a *stochastic* measure, i.e. $\mathbf{p}_{emp}^{(N)}$ is a random variable on the space of measures [6]. The link between propagation of chaos and empirical distribution relies on the following lemma.

Lemma 1. *The stochastic process $(S_1,\ldots,S_N)$ satisfies the propagation of chaos* (6) *if and only if:*

$$\mathbf{p}_{emp}^{(N)} \overset{N\to+\infty}{\rightharpoonup} \mathbf{p}_1, \tag{10}$$

i.e. for any test function φ the random variable $\langle \mathbf{p}_{emp}^{(N)}, \varphi\rangle = \frac{1}{N}\sum_{i=1}^{N}\varphi(S_i)$ converges in law to the constant value $\mathbb{E}[\varphi(\overline{S}_1)]$.

The proof can be found in [54] and we henceforth omit the detailed proof of this lemma.

3. Unbiased exchange model.

3.1. **Definition and limit equation.** We consider first the unbiased model that is briefly mentioned in the introduction above. For the three models investigated in this work, we consider a (closed) economic market consisting of N agents with μ dollars per agents for some (fixed) $\mu \in \mathbb{N}_+$, i.e. there are a total of μN dollars. We denote by $S_i(t)$ the amount of dollars that agent i has (i.e. $S_i(t) \in \{0,\ldots,\mu N\}$ and $\sum_{i=1}^{N} S_i(t) = \mu N$ for any $t \geq 0$).

Definition 3.1 (**Unbiased Exchange Model**)**.** The dynamics consist in choosing with uniform probability a "giver" i and a "receiver" j. If the receiver i has at least one dollar (i.e. $S_i \geq 1$), then it gives one dollar to the receiver j. This exchange occurs according to a Poisson process with frequency $\lambda/N > 0$.

The unbiased exchange model can be written as a stochastic differential equation [49, 52]. Introducing $\{\mathrm{N}_t^{(i,j)}\}_{1\leq i,j\leq N}$ independent Poisson processes with constant intensity $\frac{\lambda}{N}$, the evolution of each S_i is given by:

$$\mathrm{d}S_i(t) = -\sum_{j=1}^{N} \underbrace{\mathbb{1}_{[1,\infty)}\big(S_i(t-)\big)\mathrm{dN}_t^{(i,j)}}_{\text{“}i\text{ gives to }j\text{”}} + \sum_{j=1}^{N} \underbrace{\mathbb{1}_{[1,\infty)}\big(S_j(t-)\big)\mathrm{dN}_t^{(j,i)}}_{\text{“}j\text{ gives to }i\text{”}}. \tag{11}$$

[1]using as a test function $\varphi(s_1,\ldots,s_k) = \varphi_1(s_1)\ldots\varphi_k(s_k)$

To gain some insight of the dynamics, we focus on $i = 1$ and introduce some notations:

$$\mathbf{N}_t^1 = \sum_{j=1}^N \mathrm{N}_t^{(1,j)}, \quad \mathbf{M}_t^1 = \sum_{j=1}^N \mathrm{N}_t^{(j,1)}.$$

The two Poisson processes $\mathbf{N}_t^1$ and $\mathbf{M}_t^1$ are of intensity λ. The evolution of $S_1(t)$ can be written as:

$$\mathrm{d}S_1(t) = -\mathbb{1}_{[1,\infty)}\big(S_1(t-)\big)\mathrm{d}\mathbf{N}_t^1 + Y(t-)\mathrm{d}\mathbf{M}_t^1, \tag{12}$$

with $Y(t)$ Bernoulli distribution with parameter $r(t)$ (i.e. $Y(t) \sim \mathcal{B}(r(t))$) representing the proportion of "rich" people:

$$r(t) = \frac{1}{N}\sum_{j=1}^N \mathbb{1}_{[1,\infty)}\big(S_j(t)\big). \tag{13}$$

Thus, the dynamics of S_1 can be seen as a compound Poisson process.

Motivated by (12), we give the following definition of the limiting dynamics of $S_1(t)$ as $N \to \infty$ from the process point of view.

Definition 3.2 (**Asymptotic Unbiased Exchange Model**)**.** We define $\overline{S}_1(t)$ to be the (nonlinear) compound Poisson process satisfying the following SDE:

$$\mathrm{d}\overline{S}_1(t) = -\mathbb{1}_{[1,\infty)}\big(\overline{S}_1(t-)\big)\mathrm{d}\overline{\mathbf{N}}_t^1 + \overline{Y}(t-)\mathrm{d}\overline{\mathbf{M}}_t^1, \tag{14}$$

in which $\overline{\mathbf{N}}_t^1$ and $\overline{\mathbf{M}}_t^1$ are independent Poisson processes with intensity λ, and $\overline{Y}(t) \sim \mathcal{B}(\bar{r}(t))$ independent Bernoulli variable with parameter

$$\bar{r}(t) \;:=\; \mathbb{P}\big(\overline{S}_1(t) > 0\big) \;=\; 1 - \mathbb{P}\big(\overline{S}_1(t) = 0\big). \tag{15}$$

We denote by $\mathbf{p}(t) = \big(p_0(t), p_1(t), \dots\big)$ the law of the process $\overline{S}_1(t)$, i.e. $p_n(t) = \mathbb{P}\big(\overline{S}_1(t) = n\big)$. Its time evolution is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = \lambda\, Q_{unbias}[\mathbf{p}(t)] \tag{16}$$

with:

$$Q_{unbias}[\mathbf{p}]_n := \begin{cases} p_1 - \bar{r}\, p_0 & \text{if } n = 0 \\ p_{n+1} + \bar{r}\, p_{n-1} - (1+\bar{r})p_n & \text{for } n \geq 1 \end{cases} \tag{17}$$

and $\bar{r} = 1 - p_0$.

Remark 2. The coupled system (14) and (15) may look cumbersome at first glance: it is mixing a SDE for the evolution of $\overline{S}_i$ (14) with a Bernoulli random variable $\bar{Y}$ which laws depends itself on the law of $\overline{S}_i$ (through $\bar{r}$ (15)). But this expresses the non-linear nature of the SDE: the law of the process $\overline{S}_i$ has an influence on its own evolution. A classical example of such formulation is given in the seminal work by Alain-Sol Sznitman [54], and the following nonlinear SDE (of McKean-Vlasov type) appears as the limit equation of a certain interacting particle systems:

$$\mathrm{d}X_t = \mathrm{d}B_t + \int_{y\in\mathbb{R}^d} b(X_t, y)\, u(y,t)\mathrm{d}y\; \mathrm{d}t,$$

where $(B_t)_{t\geq 0}$ denotes an $\mathbb{R}^d$-valued Brownian motion and $u(.,t)$ is the law of X_t. In this SDE, the rate of change of X_t (i.e., $\mathrm{d}X_t$) depends on the law of itself, which introduces the non-linearity. This SDE is equivalent to the nonlinear PDE:

$$\partial_t u + \nabla\cdot(G[u]u) = \frac{1}{2}\Delta u \qquad \text{with} \quad G[u](x) = \int_{y\in\mathbb{R}^d} b(x,y)u(y)\,\mathrm{d}y.$$

To prove such type of nonlinear SDEs has a unique solution, one can resort to a standard fixed-point argument (see [54] for more details). Alternatively, well-posedness of the nonlinear SDE (14) follows from the well-posedness of the associated infinite system of ODEs (16). Indeed, the operator Q_{unbias} (17) is bounded and locally Lipschitz in the Banach space $\ell^1(\mathbb{N})$. See also the recent work [42].

3.2. **Coupling for the unbiased exchange model.** We now provide the coupling strategy to link the $N-$particle system $(S_1,\dots,S_N)$ with the limit dynamics $(\overline{S}_1,\dots,\overline{S}_k)$. In [54], the core of the method is to use the same "noise" in both the $N-$particle system and the limit system. Unfortunately, it is not possible in our settings: the clocks $\mathrm{N}_t^{(i,j)}$ cannot be used "as is" since they would correlate the jump of $\overline{S}_i$ with the jump of $\overline{S}_j$ which is not acceptable. Indeed, if $\overline{S}_i(t)$ and $\overline{S}_j(t)$ are independent, they cannot jump at (exactly) the same time.

For this reason, we have to introduce an intermediate dynamics, denoted by $\{\widehat{S}_i\}_{i\geq 1}$, which employs exactly the same "clocks" as our original dynamics (11), but the property of being rich or poor is decoupled.

Definition 3.3 (**Intermediate model**)**.** We define for $\{\widehat{S}_i\}_{1\leq i\leq N}$ to be a collection of identically distributed (nonlinear) compound Poisson processes satisfying the following SDEs for each $1\leq i\leq N$:

$$\begin{aligned}
\mathrm{d}\widehat{S}_i(t) \;&=\; -\sum_{j=1,j\neq i}^{N} \mathbb{1}_{[1,\infty)}\big(\widehat{S}_i(t-)\big)\mathrm{d}\mathrm{N}_t^{(i,j)} + \sum_{j=1,j\neq i}^{N} \overline{Y}(t-)\mathrm{d}\mathrm{N}_t^{(j,i)} && (18)\\
&\quad -\mathbb{1}_{[1,\infty)}\big(\widehat{S}_i(t-)\big)\mathrm{d}\overline{\mathrm{N}}_t^{(i,i)} + \overline{Y}(t-)\mathrm{d}\overline{\mathrm{M}}_t^{(i,i)} && (19)
\end{aligned}$$

in which $\overline{Y}(t)\sim\mathcal{B}(\bar{r}(t))$, the Poisson clocks $\mathrm{N}_t^{(i,j)}$ $(1\leq i\neq j\leq N)$ are the same as those used in (11), the two extra clocks $\overline{\mathrm{N}}_t^{(i,i)}$ and $\overline{\mathrm{M}}_t^{(i,i)}$ are independent with rate λ/N.

We do not use the "self-giving" clocks $\mathrm{N}_t^{(i,i)}$ since we want to decouple the receiving and giving dynamics.

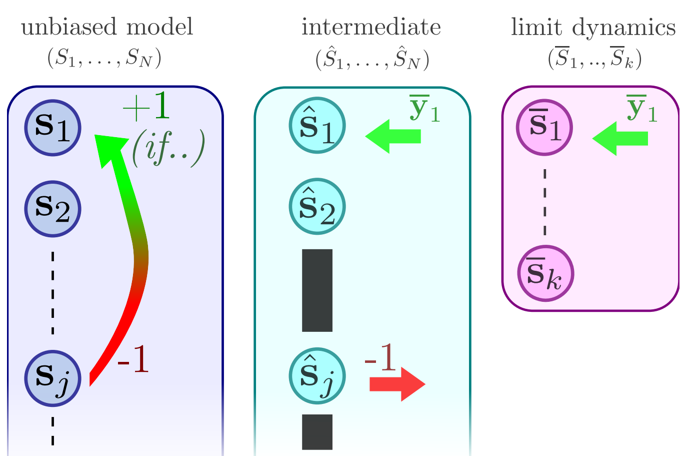


Figure 5 Schematic illustration of the coupling strategy. We use an intermediate process $(\widehat{S}_1,\dots,\widehat{S}_N)$ to *decouple* the "give" and "receive" parts of the dynamics.

A schematic illustration of the above coupling technique is shown in Fig 5. We first have to control the difference between the process $(S_1,\ldots,S_N)$ and the intermediate dynamics $(\widehat{S}_1,\ldots,\widehat{S}_N)$. The key idea is based on the following simple yet effective lemma that allows to create optimal coupling between two flipping coins [29].

Lemma 2. *For any $p,q\in(0,1)$, there exist $X\sim\mathcal{B}(p)$ and $Y\sim\mathcal{B}(q)$ such that $\mathbb{P}(X\neq Y)=|p-q|$.*

Proof. Let $U\sim\mathcal{U}[0,1]$ a uniform random variable. Define the Bernoulli random variables as $X:=\mathbb{1}_{[0,p)}(U)$ and $Y:=\mathbb{1}_{[0,q)}(U)$. It is straightforward to show that $X\sim\mathcal{B}(p)$, $Y\sim\mathcal{B}(q)$ and $\mathbb{P}(X\neq Y)=|p-q|$. □

More generally, if N_t and M_t are two inhomogeneous Poisson processes with rate $\lambda(t)$ and $\mu(t)$, respectively, then there exists a coupling such that

$$\mathrm{d}\mathbb{E}[|N_t-M_t|]\leq|\lambda(t)-\mu(t)|\mathrm{d}t.$$

This leads to the following proposition.

Proposition 1. *Let $\big(S_1,\ldots,S_N\big)$ and $\big(\widehat{S}_1,\ldots,\widehat{S}_N\big)$ be solution to (11) and (18) respectively, with the same initial condition. Then for any $1\leq i\leq N$, we have*

$$\mathrm{d}\mathbb{E}[|S_i(t)-\widehat{S}_i(t)|]\ \leq\ \lambda\,\mathbb{E}[|r(t)-\bar{r}(t)|]\,\mathrm{d}t\ +\ \lambda\frac{2}{N}\,\mathrm{d}t,\tag{20}$$

where $r(t)=\frac{1}{N}\sum_{j=1}^{N}\mathbb{1}_{[1,\infty)}\big(S_j(t)\big)$ and $\bar{r}(t)$ given by (15).

Proof. The processes $\widehat{S}_i(t)$ and $S_i(t)$ "share" the same clocks $\mathrm{N}_t^{(i,j)}$ and $\mathrm{N}_t^{(j,i)}$ for $j\neq i$. Denote the 'rich or not' random Bernoulli random variables:

$$R_i(t)=\mathbb{1}_{[1,\infty)}\big(S_i(t)\big)\qquad\text{and}\qquad\widehat{R}_i(t)=\mathbb{1}_{[1,\infty)}\big(\widehat{S}_i(t)\big).\tag{21}$$

Once a clock $\mathrm{N}_t^{(i,j)}$ rings, the processes become:

$$\begin{array}{lcl}(S_i,S_j) & \rightsquigarrow & (S_i-R_i,S_j+R_i),\\ (\widehat{S}_i,\widehat{S}_j) & \rightsquigarrow & (\widehat{S}_i-\widehat{R}_i,\widehat{S}_j+\overline{Y}).\end{array}\tag{22}$$

Notice that the difference $|S_i-\widehat{S}_i|$ can only decay after the jump from the clock $\mathrm{N}_t^{(i,j)}$ (the 'give' dynamics reduce the difference). However, the 'receive' dynamics from the clock $\mathrm{N}_t^{(j,i)}$ could increase the difference $|S_j-\widehat{S}_j|$ if $\widehat{R}_i\neq\overline{Y}$. More precisely, we find:

$$\mathrm{d}\mathbb{E}[|S_i(t)-\widehat{S}_i(t)|]\leq 0+\sum_{j=1,j\neq i}^{N}\mathbb{E}[|R_j(t-)-\overline{Y}(t-)|]\frac{\lambda}{N}\,\mathrm{d}t\ +\ \frac{2\lambda}{N}\,\mathrm{d}t\tag{23}$$

where the extra $\frac{2\lambda}{N}\,\mathrm{d}t$ is due to the extra clocks $\overline{\mathrm{N}}_t^{(i,i)}$ and $\overline{\mathrm{M}}_t^{(i,i)}$ in (19).

Now we have to couple the Bernoulli process $\overline{Y}(t-)$ with $R_j(t-)$ in a convenient way to make the difference as small as possible. Here is the strategy:

- Step 1: generate a master Poisson clock $\mathbf{N}_t$ with intensity λN which gives a collection of jumping times.
- Step 2: to select which clock $\mathrm{N}_t^{(i,j)}$ rings, calculate the proportions of "rich people" for the $N-$particle system and for the limit dynamics:

$$r(t-)=\frac{1}{N}\sum_{j=1}^{N}\mathbb{1}_{[1,\infty)}\big(S_j(t-)\big)\quad,\quad\bar{r}(t-)=1-p_0(t-).\tag{24}$$

- Step 3: let $U \sim \mathcal{U}([0,1])$ a uniform random variable.
  - if $U < r(t-)$, pick an index i uniformly among the rich people (i.e. i such that $S_i(t-) > 0$), otherwise we pick i uniformly among the poor people (i.e. i such that $S_i(t-) = 0$). Pick index j uniformly among $\{1, 2, \ldots, N\}$.
  - if $U < \bar{r}(t-)$, let $\overline{Y}(t-) = 1$, otherwise $\overline{Y}(t-) = 0$ (i.e. $\overline{Y}(t-) = \mathbb{1}_{[0,\bar{r}(t-)]}(U)$).
- Step 4: if $i \neq j$, update using (22)

Thanks to our coupling, the 'receiving' dynamics of S_i and $\widehat{S}_i$ will differ with probability $|r - \bar{r}|$:

$$\mathbb{E}[|R_j(t-) - \overline{Y}(t-)|] = \mathbb{P}\big(R_j(t-) \neq \overline{Y}(t-)\big) = \mathbb{E}\left[|r - \bar{r}|\right]. \tag{25}$$

Plug in the expression in (23) concludes the proof.

□

Remark 3. The update formula (22) for $(\widehat{S}_i, \widehat{S}_j)$ highlights that the 'give' and 'receive' dynamics are now independent in the auxiliary dynamics (i.e. $\widehat{R}_i$ and $\overline{Y}$ are independent). In contrast, we use the same process R_i to update S_i and S_j.

Now we turn our attention to the coupling between $\{\widehat{S}_i\}_{i=1..N}$ (auxiliary dynamics) and the limit dynamics $\{\overline{S}_i\}_{i=1..k}$ for a **fixed** k (while $N \to \infty$). The idea is to remove the clocks $\mathrm{N}_t^{(i,j)}$ for $1 \leq i, j \leq k$ to decouple the time of the jump in $\overline{S}_i$ and $\overline{S}_j$ as described in the Figure 6.

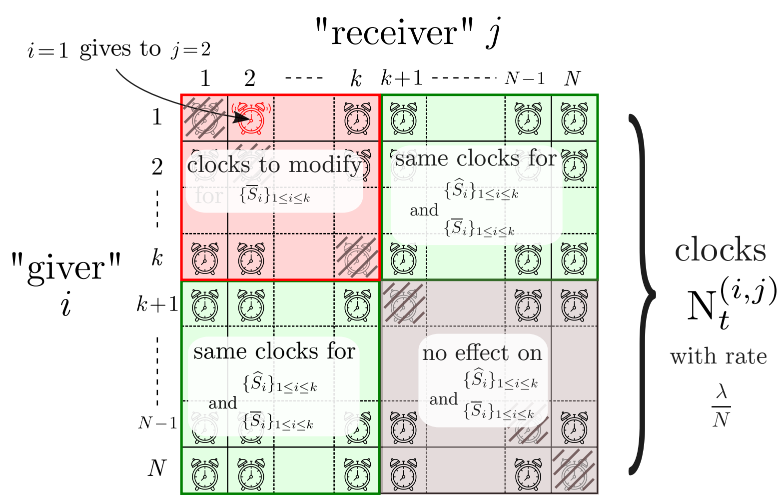


Figure 6 The clocks $\mathrm{N}_t^{(i,j)}$ used to generate the unbiased dynamics (11) have to be modified to generate the limit dynamics $(\overline{S}_1(t), \ldots, \overline{S}_k(t))$ (14). The processes $\overline{S}_i(t)$ and $\overline{S}_j(t)$ have to be independent, thus the clocks $\mathrm{N}_t^{(i,j)}$ for $1 \leq i, j \leq k$ cannot be used.

Proposition 2. *Let $(\widehat{S}_1, \ldots, \widehat{S}_N)$ be the solution to (18) and $\{\overline{S}_i\}_{1\leq i\leq k}$ be independent processes solving (14). Then for any fixed $k \in \mathbb{N}_+$, there exists a coupling such that for all $t \geq 0$:*

$$\mathrm{d}\mathbb{E}[|\widehat{S}_i(t) - \overline{S}_i(t)|] \leq \lambda \frac{4(k-1)}{N}\,\mathrm{d}t \quad , \qquad \text{for } 1 \leq i \leq k. \tag{26}$$

Proof. We assume $i=1$ to simplify the writing. To couple the two processes $\widehat{S}_1$ and $\overline{S}_1$, we use the same Bernoulli variable $\overline{Y}(t-)$ to generate both 'receive' dynamics:

$$\begin{cases} \mathrm{d}\widehat{S}_1(t) & = -\mathbb{1}_{[1,\infty)}\big(\widehat{S}_1(t-)\big)\mathrm{d}\widehat{\mathbf{N}}^1_t + \overline{Y}(t-)\mathrm{d}\widehat{\mathbf{M}}^1_t, \\ \mathrm{d}\overline{S}_1(t) & = -\mathbb{1}_{[1,\infty)}\big(\overline{S}_1(t-)\big)\mathrm{d}\overline{\mathbf{N}}^1_t + \overline{Y}(t-)\mathrm{d}\overline{\mathbf{M}}^1_t. \end{cases}$$

Meanwhile, the Poisson clocks $\widehat{\mathbf{N}}^1_t$, $\widehat{\mathbf{M}}^1_t$ are already determined in (18):

$$\widehat{\mathbf{N}}^1_t = \overline{\mathrm{N}}^{(1,1)}_t + \sum_{j=2}^{N} \mathrm{N}^{(1,j)}_t \quad \text{and} \quad \widehat{\mathbf{M}}^1_t = \overline{\mathrm{M}}^{(1,1)}_t + \sum_{j=2}^{N} \mathrm{N}^{(j,1)}_t. \tag{27}$$

Unfortunately, we cannot use the same definition for the clocks $\overline{\mathbf{N}}^1_t$ and $\overline{\mathbf{M}}^1_t$ as the clocks $\widehat{\mathbf{N}}^i_t$ and $\widehat{\mathbf{M}}^j_t$ are *not* independent (they both contain the clock $\mathrm{N}^{(i,j)}_t$). Thus, we need to remove those coupling clocks when defining $\overline{\mathbf{N}}^1$ and $\overline{\mathbf{M}}^1$. Fortunately, we only have to generate the dynamics for k process, thus we only have to replace the clocks $\mathrm{N}^{(1,i)}$ and $\mathrm{N}^{(i,1)}$ for $i = 1..k$ (see Figure 6):

$$\overline{\mathbf{N}}^1_t = \sum_{j=1}^{k} \overline{\mathrm{N}}^{(1,j)}_t + \sum_{j=k+1}^{N} \mathrm{N}^{(1,j)}_t \quad \text{and} \quad \widehat{\mathbf{M}}^1_t = \sum_{j=1}^{k} \overline{\mathrm{M}}^{(1,j)}_t + \sum_{j=k+1}^{N} \mathrm{N}^{(j,1)}_t \tag{28}$$

where $\overline{\mathrm{N}}^{(1,j)}_t$ and $\overline{\mathrm{M}}^{(1,j)}_t$ are independent Poisson clocks with rate $\frac{\lambda}{N}$.

Using this coupling strategy, the difference $|\widehat{S}_1 - \overline{S}_1|$ could only increase (by 1) if the clocks $\overline{\mathrm{N}}^{(1,j)}_t$, $\overline{\mathrm{M}}^{(1,j)}_t$, $\mathrm{N}^{(1,j)}_t$ or $\mathrm{N}^{(j,1)}_t$ ring for $2 \le j \le k$ leading to (26). $\square$

Remark 4. The explicit coupling constructed in this section is indeed the core of the manuscript. The intermediate dynamics for $\widehat{\mathbf{S}}$ is introduced to *decouple* the process of giving and receiving. In the original dynamics for $\mathbf{S}$, the total wealth is always preserved (S_i loses one dollar only if S_j receives one). This is no longer the case for the intermediate 'hat' dynamics as the total wealth is not preserved ($\widehat{S}_i$ could lose one dollar without $\widehat{S}_j$ receiving one). Moreover the limiting 'bar' dynamics decouple further the intermediate 'hat' dynamics so that components of $\overline{\mathbf{S}}$ become independent (i.e. $\overline{S}_i$ and $\overline{S}_j$ 'jump' independently from each other).

Finally, combining propositions 1 and 2 gives rise to the following theorem.

Theorem 3.4. *Let $\big(S_1,\ldots,S_N\big)$ to be a solution to (11). Then for any fixed $k \in \mathbb{N}_+$ and $t \ge 0$, there exists a coupling between $\big(S_1,\ldots,S_k\big)$ and $\big(\overline{S}_1,\ldots,\overline{S}_k\big)$ (with the same initial conditions) such that:*

$$\mathbb{E}[|S_i(t) - \overline{S}_i(t)|] \le \frac{C(t)}{\sqrt{N}}\frac{(\mathrm{e}^{\lambda t}-1)}{\lambda} + \lambda\frac{4(k-1)t}{N} \tag{29}$$

with $C(t) = \big(\frac{1}{4} + \lambda 4t\big)^{1/2} + \lambda\frac{2}{\sqrt{N}}$ holding for each $1 \le i \le k$.

Proof. We assume without loss of generality that $i=1$. First, we show that the processes S_1 and $\widehat{S}_1$ remain closed. We denote:

$$R_i = \mathbb{1}_{[1,\infty)}\big(S_i\big) \quad , \quad \widehat{R}_i = \mathbb{1}_{[1,\infty)}\big(\widehat{S}_i\big) \quad , \quad \overline{R}_i = \mathbb{1}_{[1,\infty)}\big(\overline{S}_i\big).$$

We have:

$$\begin{aligned}\mathbb{E}[|r-\bar{r}|] &= \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N} R_i - \bar{r}\right|\right] = \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}(R_i - \widehat{R}_i) \;+\; \frac{1}{N}\sum_{i=1}^{N}(\widehat{R}_i - \bar{r})\right|\right] \\ &\leq \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[|R_i - \widehat{R}_i|] + \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}(\widehat{R}_i - \bar{r})\right|\right] \\ &\leq \mathbb{E}[|S_1 - \widehat{S}_1|] + \mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N}(\widehat{R}_i - \bar{r})\right)^2\right]^{1/2},\end{aligned}$$

where we use $|R_i - \widehat{R}_i| \leq |S_i - \widehat{S}_i|$. To control the variance, we expand:

$$\begin{aligned}\mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N}(\widehat{R}_i - \bar{r})\right)^2\right] &= \frac{1}{N}\mathrm{Var}[\widehat{R}_1] + \frac{N(N-1)}{N^2}\mathrm{Cov}(\widehat{R}_1, \widehat{R}_2) \\ &\leq \frac{1}{4N} + \mathrm{Cov}(\widehat{R}_1, \widehat{R}_2),\end{aligned}$$

since $\widehat{R}_1$ is a Bernoulli variable its variance is bounded by $1/4$. Notice that we have used $\mathbb{E}\left[\widehat{R}_i\right] = \bar{r}$. Indeed, for a given i, the SDE satisfied by the intermediate dynamics $\widehat{S}_i$ is *exactly* the same SDE satisfied by $\overline{S}_i$. Thus the law of $\widehat{S}_i$ is the same as the law of $\overline{S}_i$ which implies $\mathbb{E}\left[\widehat{R}_i\right] = \bar{r}$. But as a system, $(\widehat{S}_1, \ldots, \widehat{S}_N)$ does not have the same law as $(\overline{S}_1, \ldots, \overline{S}_N)$ because the processes $\widehat{S}_i$ and $\widehat{S}_j$ are not independent due to the clock $\mathbf{N}^{(i,j)}$ (clocks that we get rid of in the following steps of the proof).

Controlling the covariance of $\widehat{R}_1$ and $\widehat{R}_2$ is more delicate since the two processes are not independent due to the clocks $\mathrm{N}_t^{(1,2)}$ and $\mathrm{N}_t^{(2,1)}$. Fortunately, these clocks have a rate of only λ/N and thus the covariance has to remain small for a given time interval. To prove it, let's use the independent processes $\overline{R}_1$ and $\overline{R}_2$:

$$\mathrm{Cov}(\widehat{R}_1, \widehat{R}_2) = \mathrm{Cov}(\widehat{R}_1 - \overline{R}_1, \widehat{R}_2 - \overline{R}_2) \leq \left(\mathbb{E}[|\widehat{R}_1 - \overline{R}_1|^2] \cdot \mathbb{E}[|\widehat{R}_2 - \overline{R}_2|^2]\right)^{1/2}$$

using Cauchy–Schwarz. Since the two processes $\widehat{S}_i$ and $\overline{S}_i$ remain close, we deduce:

$$\mathbb{E}[|\widehat{R}_1(t) - \overline{R}_1(t)|^2] = \mathbb{E}[|\widehat{R}_1(t) - \overline{R}_1(t)|] \leq \mathbb{E}[|\widehat{S}_1(t) - \overline{S}_1(t)|] \leq \lambda\frac{4t}{N},$$

using proposition 2 (with $k = 2$). We conclude that:

$$\mathbb{E}[|r(t) - \bar{r}(t)|] \leq \mathbb{E}[|S_1(t) - \widehat{S}_1(t)|] + \left(\frac{1}{4N} + \lambda\frac{4t}{N}\right)^{1/2}.$$

Going back to proposition 1, we find:

$$\begin{aligned}\mathrm{d}\mathbb{E}[|S_i(t) - \widehat{S}_i(t)|] &\leq \lambda\,\mathbb{E}[|S_1(t) - \widehat{S}_1(t)|]\,\mathrm{d}t + \left(\frac{1}{4N} + \lambda\frac{4t}{N}\right)^{1/2}\mathrm{d}t \;+\; \lambda\frac{2}{N}\,\mathrm{d}t \\ &\leq \lambda\,\mathbb{E}[|S_1(t) - \widehat{S}_1(t)|]\,\mathrm{d}t + \frac{C(t)}{\sqrt{N}}\,\mathrm{d}t\end{aligned}$$

with $C(t) = \left(\frac{1}{4} + \lambda 4t\right)^{1/2} + \lambda \frac{2}{\sqrt{N}} = \mathcal{O}(1)$. Using Gronwall's lemma, since $|S_i(0) - \widehat{S}_i(0)| = 0$, we obtain:

$$\mathbb{E}[|S_i(t) - \widehat{S}_i(t)|] \leq \frac{C(t)}{\sqrt{N}} \frac{(\mathrm{e}^{\lambda t} - 1)}{\lambda} + \lambda \frac{4(k-1)t}{N}. \tag{30}$$

We finally conclude by using proposition 2 and triangular inequality. □

Remark 5. After we achieved the transition from the SDEs (3.1) to the deterministic system of nonlinear ODEs (16), the natural follow-up step is to analyze (16) with the intention of proving convergence of the solution of (16) to its (unique) equilibrium solution, which turns out to be a geometric distribution defined by

$$p_n^* = p_0^*(1 - p_0^*)^n, \quad n \geq 0, \tag{31}$$

where $p_0^* = \frac{1}{1+\mu}$ if we put initially that $\sum_{n=0}^{\infty} n\, p_n(0) = \mu$ for some $\mu \in \mathbb{N}_+$. The main ingredient underlying the proof lies in the reformulation of (16) into a (discrete) Fokker-Planck type equation, combined with the standard entropy method [3, 34, 40]. We emphasize that the convergence of the solution of (16) to (31) has already been established in [15, 32, 42] so we refer the interested readers to the aforementioned references for further details and results on this model.

4. **Poor-biased exchange model.**

We now investigate our second model where the 'given' dynamics is biased toward richer agent: the wealthier an agent becomes, the more likely it will give a dollar. As for the previous model, we first investigate the limit dynamics as the number of agents N goes to infinity, then we study the large time behavior and show rigorously the convergence of the wealth distribution to a Poisson distribution.

4.1. **Definition and limit equation.**

We use the same setting as the unbiased model: there are N agents with initially the same amount of money $S_i(0) = \mu$.

Definition 4.1 (**Poor-biased exchanged model**). The dynamics consists in choosing a "giver" i with a probability proportional to its wealth (the wealthier an agent is, the more likely it will be a "giver"). Then it gives one dollar to a "receiver" j chosen uniformly at random.

From another point of view, the dynamics consist in taking one dollar from the common pot (tax system) and re-distribute the dollar uniformly among the individuals [24]. Thus instead of 'taxing the agents' in the unbiased exchange model, the poor-biased model is 'taxing the dollar'.

The poor-biased model can be written in term of stochastic differential equations, the wealth S_i of agent i evolves according to:

$$\mathrm{d}S_i(t) = -\sum_{j=1}^{N} \mathrm{dN}_t^{(i,j)} + \sum_{j=1}^{N} \mathrm{dN}_t^{(j,i)}, \tag{32}$$

with $\mathrm{N}_t^{(i,j)}$ Poisson process with intensity $\lambda_{i,j}(t) = \frac{\lambda S_i(t)}{N}$.

Since the clocks $\{\mathrm{N}_t^{i,j}\}_{1\leq i,j\leq N}$ are now time dependent (in contrast to the unbiased model), the dynamics might appear more difficult to analyze. But it turns out to be simpler, since the rate of receiving a dollar is constant:

$$\sum_{j=1}^{N} \lambda_{j,i}(t) = \sum_{j=1}^{N} \frac{\lambda S_j(t)}{N} = \lambda\mu,$$

where μ is the (conserved) initial mean. In contrast, in the unbias dynamics, the rate of receiving a dollar is equal to the proportion of rich people $r(t)$ which fluctuates in time. Let's focus on $i=1$ and sum up the clocks introducing:

$$\mathbf{N}_t^1 = \sum_{j=1}^{N} \mathrm{N}_t^{(1,j)}, \quad \mathbf{M}_t^1 = \sum_{j=1}^{N} \mathrm{N}_t^{(j,1)}, \tag{33}$$

where the two Poisson processes $\mathbf{N}_t^1$ and $\mathbf{M}_t^1$ have intensity λS_1 and $\lambda\mu$ (respectively). Thus, the poor-biased model leads to the equation:

$$\mathrm{d}S_1(t) = -\mathrm{d}\mathbf{N}_t^1 + \mathrm{d}\mathbf{M}_t^1. \tag{34}$$

Notice that $S_1(t)$ is not independent of $S_j(t)$ as both processes can jump at the same time due to the two clocks $\mathrm{N}_t^{(1,j)}$ and $\mathrm{N}_t^{(j,1)}$.

Motivated by the equation above, we give the following definition of the limiting dynamics as $N \to \infty$.

Definition 4.2. (**Asymptotic Poor-biased model**) We define $\overline{S}_1$ to be the compound Poisson process satisfying the following SDE:

$$\mathrm{d}\overline{S}_1(t) = -\mathrm{d}\overline{\mathbf{N}}_t^1 + \mathrm{d}\overline{\mathbf{M}}_t^1, \tag{35}$$

in which $\overline{\mathbf{N}}_t^1$ and $\overline{\mathbf{M}}_t^1$ are independent Poisson processes with intensity $\lambda\overline{S}_1(t)$ and λm (respectively) where μ is the mean of $\overline{S}_1(0)$ (i.e. $\mu = \mathbb{E}[\overline{S}_1(0)]$).

If we denote by $\mathbf{p}(t) = \big(p_0(t), p_1(t), \ldots\big)$ the law of the process $\overline{S}_1(t)$, its time evolution is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = \lambda\, Q_{poor}[\mathbf{p}(t)] \tag{36}$$

$$\text{with} \qquad Q_{poor}[\mathbf{p}]_n := \begin{cases} p_1 - \mu\, p_0 & \text{if } n = 0 \\ (n+1)p_{n+1} + \mu\, p_{n-1} - (n+\mu)\, p_n & \text{for } n \geq 1 \end{cases} \tag{37}$$

and $\mu = \sum_{n=0}^{+\infty} n\, p_n(t) = \sum_{n=0}^{+\infty} n\, p_n(0)$.

4.2. **Proof of propagation of chaos.** The aim of this subsection is to prove the propagation of chaos, i.e. that the process $\big(S_1, \ldots, S_k\big)$ converges to $\big(\overline{S}_1, \ldots, \overline{S}_k\big)$ as N goes to infinity. As for the unbiased exchange model, the key is to define the Poisson clocks for the limit dynamics $\overline{\mathbf{N}}_t^i$ and $\overline{\mathbf{M}}_t^i$ *close to* the clocks of the $N-$particle system $\mathbf{N}_t^i$ and $\mathbf{M}_t^i$ for $1 \leq i \leq k$, but at the same time making the clocks independent. With this aim, we have to 'remove' the clocks $\mathbf{N}_t^{(i,j)}$ and $\mathbf{M}_t^{(i,j)}$ for $1 \leq i, j \leq k$.

Theorem 4.3. *Let $\big(S_1, \ldots, S_N\big)$ to be a solution to (32) and $\big(\overline{S}_1, \ldots, \overline{S}_k\big)$ a solution to (35). Then for any fixed $k \in \mathbb{N}_+$, there exists a coupling between $\big(S_1, \ldots, S_k\big)$ and $\big(\overline{S}_1, \ldots, \overline{S}_k\big)$ (with the same initial conditions) such that:*

$$\mathbb{E}[|\overline{S}_i(t) - S_i(t)|] \leq \frac{4k\lambda\mu}{N}(\mathrm{e}^{\lambda t} - 1), \tag{38}$$

holding for each $1 \leq i \leq k$.

Proof. To simplify the writing, we suppose $i=1$. We define for $1 \leq i \leq k$ the clocks for the limit dynamics as follow:

$$\overline{\mathbf{N}}_t^1 = \overline{G}_1 \cdot \left(\sum_{j=k+1}^{N} \mathrm{N}_t^{(1,j)} \right) + \widehat{\mathrm{N}}_t^1 \quad , \quad \overline{\mathbf{M}}_t^1 = \left(\sum_{j=k+1}^{N} \mathrm{N}_t^{(j,1)} \right) + \widehat{\mathrm{M}}_t^1. \tag{39}$$

Here, $\overline{G}_1$ is a Bernoulli random variable that prevents the clocks to ring for $\overline{S}_1$ if the rates of the clocks $\mathrm{N}_t^{(1,j)}$ from $k+1 \leq j \leq N$ are too large compare to $\overline{S}_1$. The parameter of this Bernoulli random variable is given by:

$$\overline{G}_1(t) \sim \mathcal{B}\Big(1 \wedge \frac{N\overline{S}_1(t)}{(N-k)S_1(t)}\Big), \tag{40}$$

with $a \wedge b = \min\{a,b\}$ for any $a,b \in \mathbb{R}$. On the contrary, the two processes $\widehat{\mathrm{N}}_t^1$ and $\widehat{\mathrm{M}}_t^1$ are used to compensate if the rates of the clocks $\mathrm{N}_t^{(1,j)}$ and $\mathrm{N}_t^{(j,1)}$ from $k+1 \leq j \leq N$ are not large enough. Both processes $\widehat{\mathrm{N}}_t^1$ and $\widehat{\mathrm{M}}_t^1$ are independent (inhomogeneous) Poisson processes with rates respectively:

$$\widehat{\mu}(t) = \lambda\left(\overline{S}_1(t) - \frac{(N-k)S_1(t)}{N}\right)_+ \quad \text{and} \quad \widehat{\nu}(t) = \lambda\left(\mu - \sum_{j=k+1}^{N} \frac{S_j(t)}{N}\right) \tag{41}$$

where $a_+ = \max\{a, 0\}$ for any $a \in \mathbb{R}$. One can check that under the aforementioned setup (coupling of Poisson clocks), $\overline{\mathbf{N}}_t^1$ and $\overline{\mathbf{M}}_t^1$ are indeed independent counting processes with intensity $\lambda\,\overline{S}_i(t)$ and $\lambda\,\mu$, respectively.
The difference $|\overline{S}_1(t) - S_1(t)|$ could increase due to 3 types of events:

i) $\mathrm{N}_t^{(1,j)}$ and $\mathrm{N}_t^{(j,1)}$ ring for $1 \leq j \leq k$,
ii) $\widehat{\mathrm{N}}_t^1$ and $\widehat{\mathrm{M}}_t^1$ ring
iii) $\mathrm{N}_t^{(1,j)}$ ring for $j \geq k+1$ and $\overline{G}_1 = 0$.

Notice that the third type of event leads to:

$$S_1(t) = S_1(t-) - 1 \quad , \quad \overline{S}_1(t) = \overline{S}_1(t-) \tag{42}$$

i.e. only S_1 gives. However, the event $\{\overline{G}_1 = 0\}$ only occurs if $S_1(t-) > \overline{S}_1(t-)$. Therefore, the event iii) could only make $|\overline{S}_1(t) - S_1(t)|$ to decay.

Therefore, we deduce:

$$\begin{aligned} \mathrm{d}\mathbb{E}[|\overline{S}_1(t) - S_1(t)|] \;&\leq\; \sum_{j=1}^{k} \frac{\lambda}{N}\mathbb{E}[S_1(t)]\mathrm{d}t \;+\; \sum_{j=1}^{k} \frac{\lambda}{N}\mathbb{E}[S_j(t)]\mathrm{d}t \\ &\quad +\mathbb{E}[\widehat{\mu}(t)]\mathrm{d}t \;+\; \mathbb{E}[\widehat{\nu}(t)]\mathrm{d}t \\ &\leq\; \frac{2k\lambda\mu}{N}\mathrm{d}t \;+\; \mathbb{E}[\widehat{\mu}(t)]\mathrm{d}t \;+\; \mathbb{E}[\widehat{\nu}(t)]\mathrm{d}t \end{aligned} \tag{43}$$

using $\mathbb{E}\left[S_j(t)\right] = \mu$ for any j. Let's bound the rates $\widehat{\mu}$ and $\widehat{\nu}$:

$$\begin{aligned} \mathbb{E}[\widehat{\mu}] \;&=\; \mathbb{E}\left[\lambda\Big(\overline{S}_1 - \frac{(N-k)S_1}{N}\Big)_+\right] \;\leq\; \lambda\mathbb{E}\left[\Big(\overline{S}_i - S_i\Big)_+ + \frac{kS_1}{N}\right] \\ &\leq\; \lambda\mathbb{E}\left[|\overline{S}_1 - S_1|\right] + \frac{\lambda k\mu}{N} \\ \mathbb{E}[\widehat{\nu}] \;&=\; \mathbb{E}\left[\lambda\Big(\mu - \sum_{j=k+1}^{N}\frac{S_j}{N}\Big)\right] = \frac{\lambda k\mu}{N}. \end{aligned}$$

We deduce from (43):

$$\mathrm{d}\mathbb{E}[|\overline{S}_1(t) - S_1(t)|] \leq \lambda\mathbb{E}\left[|\overline{S}_1(t) - S_1(t)|\right]\mathrm{d}t + \frac{4k\lambda\mu}{N}\mathrm{d}t. \tag{44}$$

Applying the Gronwall's lemma to (44) yields the result.

□

4.3. **Large time behavior.** After we achieved the transition from the interacting system of SDEs (32) to the deterministic system of linear ODEs (36), we now analyze the long time behavior of the distribution $\mathbf{p}(t)$ and its convergence to an equilibrium. The main tool behind the proof relies again on the reformulation of (36) into a (discrete) Fokker-Planck type equation, in conjunction with the standard entropy method [3, 34, 40].

Let's introduce a function space to study $\mathbf{p}(t)$:

$$\begin{aligned} V_\mu &:= \{\mathbf{p} \in \ell^2(\mathbb{N}) \mid \sum_{n=0}^{\infty} p_n = 1,\ p_n \geq 0,\ \sum_{n=0}^{\infty} n\, p_n = \mu\}, && (45) \\ \mathcal{D}(Q_{poor}) &:= \{\mathbf{p} \in \ell^2(\mathbb{N}) \mid Q_{poor}[\mathbf{p}] \in \ell^2(\mathbb{N})\}, && (46) \end{aligned}$$

where ℓ^2 denote the vector space of square-summable sequences. In contrast to the unbias model with the dynamics (36), the operator Q_{poor} is an unbounded operator (i.e. $\mathcal{D}(Q_{poor}) \not\subset \ell^2(\mathbb{N})$). For any $\mathbf{p} \in V_\mu \cap \mathcal{D}(Q_{poor})$, it is straightforward to show that:

$$\sum_{n=0}^{\infty} Q_{poor}[\mathbf{p}]_n = 0 \quad , \quad \sum_{n=0}^{\infty} n\, Q_{poor}[\mathbf{p}]_n = 0, \tag{47}$$

which express that the total mass and the mean value is conserved. Moreover, there exists a unique equilibrium $\mathbf{p}^*$ for Q_{poor} in V_μ given by a Poisson distribution:

$$p_n^* = \frac{\mu^n}{n!} \mathrm{e}^{-\mu}, \quad n \geq 0. \tag{48}$$

To investigate the convergence of $\mathbf{p}(t)$ solution to (36) to the equilibrium $\mathbf{p}^*$ (48), we introduce two function spaces.

Definition 4.4. We define the sub-vector spaces of ℓ^2:

$$\begin{aligned} \mathcal{H}^0 &= \{\mathbf{p} \in \ell^2(\mathbb{N}) \mid \sum_{n=0}^{\infty} \frac{p_n^2}{p_n^*} < +\infty\}, && (49) \\ \mathcal{H}^1 &= \{\mathbf{p} \in \ell^2(\mathbb{N}) \mid \sum_{n=0}^{\infty} p_n^* \left(\frac{p_{n+1}}{p_{n+1}^*} - \frac{p_n}{p_n^*}\right)^2 < +\infty\}, && (50) \end{aligned}$$

and define corresponding scalar products:

$$\langle \mathbf{p}, \mathbf{q} \rangle_{\mathcal{H}^0} := \sum_{n=0}^{\infty} \frac{p_n q_n}{p_n^*} \quad , \quad \langle \mathbf{p}, \mathbf{q} \rangle_{\mathcal{H}^1} := \sum_{n=0}^{\infty} p_n^* \left(\frac{p_{n+1}}{p_{n+1}^*} - \frac{p_n}{p_n^*}\right)\left(\frac{q_{n+1}}{p_{n+1}^*} - \frac{q_n}{p_n^*}\right). \tag{51}$$

The advantage of using the scalar product $\langle ., . \rangle_{\mathcal{H}^0}$ is that the operator Q_{poor} becomes symmetric. To prove it, we rewrite the operator *a la* Fokker-Planck.

Lemma 3. *For any* $\mathbf{p} \in \mathcal{H}^0$*, we have:*

$$Q_{poor}[\mathbf{p}]_n = \mu\, D^- \left(p_n^* D^+ \left(\frac{p_n}{p_n^*}\right)\right) \tag{52}$$

with $D^+(p_n) = p_{n+1} - p_n$*,* $D^-(p_n) = p_n - p_{n-1}$ *and the convention* $p_{-1} = p_{-1}^* = 0$.

Proof. Since $p_n^*/p_{n+1}^* = (n+1)/\mu$, we find

$$\begin{aligned} \frac{1}{\mu} Q_{poor}[\mathbf{p}]_n &= \frac{p_n^*}{p_{n+1}^*} p_{n+1} - \frac{p_{n-1}^*}{p_n^*} p_n - \left(\frac{p_n^*}{p_n^*} p_n - \frac{p_{n-1}^*}{p_{n-1}^*} p_{n-1}\right) \\ &= \mu p_n^* u_{n+1} - p_{n-1}^* u_n \ - \ \left(p_n^* u_n - p_{n-1}^* u_{n-1}\right) \end{aligned}$$

with $u_n = p_n/p_n^*$. Using the notation D^+ and D^-, we write:

$$\frac{1}{\mu} Q_{poor}[\mathbf{p}]_n \quad = \quad p_n^* D^+ u_n - p_{n-1}^* D^+ u_{n-1} = D^-(p_n^* D^+ u_n).$$

□

Remark 6. Equation (52) has a flavor of a Fokker-Planck equation of the form

$$\partial_t \rho = \nabla \cdot \left(\rho_\infty \nabla \left(\frac{\rho}{\rho_\infty} \right) \right), \tag{53}$$

where ρ_∞ is an equilibrium distribution to which ρ converges (and ρ_∞ may also depend on ρ, making the equation nonlinear).

As a consequence, we deduce that the operator Q_{poor} is symmetric on $\mathcal{H}^0$.

Proposition 3. *For any $\mathbf{p}, \mathbf{q} \in \mathcal{H}^0$, the operator Q_{poor} (37) satisfies:*

$$\langle Q_{poor}[\mathbf{p}], \mathbf{q} \rangle_{\mathcal{H}^0} = \langle \mathbf{p}, Q_{poor}[\mathbf{q}] \rangle_{\mathcal{H}^0} \qquad \text{for any } \mathbf{p}, \mathbf{q} \in \mathcal{H}^0. \tag{54}$$

Moreover,

$$\langle Q_{poor}[\mathbf{p}], \mathbf{p} \rangle_{\mathcal{H}^0} = -\mu \sum_{n=0}^{\infty} p_n^* \left(D^+ \left(\frac{p_n}{p_n^*} \right) \right)^2 = -\mu \|\mathbf{p}\|_{\mathcal{H}^1}^2. \tag{55}$$

Proof. We simply use integration by parts:

$$\begin{aligned} \frac{1}{\mu} \langle Q_{poor}[\mathbf{p}], \mathbf{q} \rangle_{\mathcal{H}^0} &= \sum_{n=0}^{\infty} D^- \left(p_n^* D^+ \frac{p_n}{p_n^*} \right) \frac{q_n}{p_n^*} = -\sum_{n=0}^{\infty} p_n^* \left(D^+ \frac{p_n}{p_n^*} \right) \left(D^+ \frac{q_n}{p_n^*} \right) \\ &= \sum_{n=0}^{\infty} \frac{p_n}{p_n^*} D^- \left(p_n^* D^+ \frac{q_n}{p_n^*} \right) = \frac{1}{\mu} \langle \mathbf{p}, Q_{poor}[\mathbf{q}] \rangle_{\mathcal{H}^0}. \end{aligned}$$

□

Furthermore, the operator $-Q_{poor}$ would have a so-called spectral gap if one can show that the norm $\|.\|_{\mathcal{H}^1}$ controls the norm $\|.\|_{\mathcal{H}^0}$. To prove it, we establish a Poincaré inequality. We use for that the following Poisson-Poincaré inequality taken from the monograph [9].

Proposition 4. *Let f be a real-valued function defined on the set of non-negative integers. Suppose that X obeys a Poisson distribution with parameter μ, then*

$$\mathrm{Var}\left(f(X)\right) \leq \mu \, \mathbb{E}\left[\left(f(X+1) - f(X) \right)^2 \right]. \tag{56}$$

For the sake of completeness, we give a proof of proposition 4 in Appendix A. The proof is based on the Efron-Stein inequality as well as the infinite divisibility of the Poisson distribution. The result of the previous proposition reads:

$$\sum_{n=0}^{+\infty} (f_n - m)^2 p_n^* \leq \mu \sum_{n=0}^{+\infty} (f_{n+1} - f_n)^2 p_n^* \tag{57}$$

with $m = \sum_{n=0}^{+\infty} f_n p_n^*$. Thus, using $f_n = p_n/p_n^*$ with $\mathbf{p} = (p_0, p_1, \ldots)$, we deduce the following Poincaré inequality:

Corollary 1. *For any $\mathbf{p} \in \mathcal{H}^1$ satisfying $\sum_n p_n = 1$, we have:*

$$\|\mathbf{p} - \mathbf{p}^*\|_{\mathcal{H}^0}^2 \leq \mu \|\mathbf{p}\|_{\mathcal{H}^1}^2 \tag{58}$$

where $\|.\|_{\mathcal{H}^0}$ and $\|.\|_{\mathcal{H}^1}$ are defined in (51) and $\mathbf{p}^*$ is the equilibrium (48).

As a result of the corollary, the operator $-Q_{poor}$ has a spectral gap of at least $1/\mu$ since:

$$\langle -Q_{poor}[\mathbf{p}-\mathbf{p}_\infty]\,,\,\mathbf{p}-\mathbf{p}_\infty\rangle_{\mathcal{H}^0} = \langle -Q_{poor}[\mathbf{p}]\,,\,\mathbf{p}\rangle_{\mathcal{H}^0} = \|\mathbf{p}\|^2_{\mathcal{H}^1} \geq \frac{1}{\mu}\|\mathbf{p}-\mathbf{p}^*\|^2_{\mathcal{H}^0}. \quad (59)$$

We shall establish the existence of a unique global solution to the linear ODE system (36). The key ingredient in our proof relies heavily on standard theory of maximal monotone operators (see for instance Chapter 7 of [10]).

Proposition 5. *Given any $\mathbf{p}_0 \in \mathcal{D}(Q_{poor})$, there exists a unique function*

$$\mathbf{p}(t) \in C^1\big([0,\infty);\mathcal{H}^0\big) \cap C\big([0,\infty);\mathcal{D}(Q_{poor})\big)$$

satisfying (36).

Proof. We use the Hille-Yosida theorem and show that the (unbounded) linear operator $-Q_{poor}$ on $\mathcal{H}^0$ is a maximal monotone operator. The monotonicity of $-Q_{poor}$ follows from its symmetric property on $\mathcal{H}^0$:

$$\langle -Q_{poor}[\mathbf{v}],\mathbf{v}\rangle_{\mathcal{H}^0} = \mu\sum_{n=0}^{\infty} p_n^*\left(D^+\left(\frac{v_n}{p_n^*}\right)\right)^2 \geq 0 \quad \text{for all } \mathbf{v}\in\mathcal{D}(Q_{poor}).$$

To show the maximality of $-Q_{poor}$, it suffices to show $R(I-Q_{poor}) = \mathcal{H}^0$, i.e., for each $\mathbf{f}\in\mathcal{H}^0$, the equation $\mathbf{p}-Q_{poor}[\mathbf{p}] = \mathbf{f}$ admits at least one solution $\mathbf{p}\in\mathcal{D}(-Q_{poor})$. To this end, the weak formulation of $\mathbf{p}-Q_{poor}[\mathbf{p}] = \mathbf{f}$ reads

$$\langle \mathbf{p},\mathbf{q}\rangle_{\mathcal{H}^0} + \langle -Q_{poor}[\mathbf{p}],\mathbf{q}\rangle_{\mathcal{H}^0} = \langle \mathbf{f},\mathbf{q}\rangle_{\mathcal{H}^0} \quad \text{for all } \ \mathbf{q}\in\mathcal{H}^0, \quad (60)$$

whence the Lax-Milgram theorem yields a unique $\mathbf{p}\in\mathcal{H}^1$.

□

We can now prove the convergence of $\mathbf{p}(t)$ solution of (36) to its equilibrium solution (48).

Theorem 4.5. *Let $\mathbf{p}(t)$ be the solution of (36) and $\mathbf{p}^*$ the corresponding equilibrium. Then:*

$$\|\mathbf{p}(t)-\mathbf{p}^*\|_{\mathcal{H}^0} \leq \|\mathbf{p}_0-\mathbf{p}^*\|_{\mathcal{H}^0}\mathrm{e}^{-\lambda t} \quad (61)$$

where $\mathbf{p}_0$ is the initial condition, i.e., $\mathbf{p}(t=0)=\mathbf{p}_0$.

Proof. Taking the derivative of the square norm gives:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{p}(t)-\mathbf{p}^*\|^2_{\mathcal{H}^0} &= \langle \mathbf{p}'(t)\,,\,\mathbf{p}(t)-\mathbf{p}^*\rangle_{\mathcal{H}^0} = \lambda\langle Q_{poor}[\mathbf{p}(t)]\,,\,\mathbf{p}(t)-\mathbf{p}^*\rangle_{\mathcal{H}^0} \\ &= \lambda\langle \mathbf{p}(t)\,,\,Q_{poor}[\mathbf{p}(t)]\rangle_{\mathcal{H}^0} = -\lambda\mu\|\mathbf{p}(t)\|^2_{\mathcal{H}^1}, \quad (62)\end{aligned}$$

using the symmetry of Q_{poor} and the relation (55). Using the Poincaré constant from corollary (1), we deduce:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{p}(t)-\mathbf{p}^*\|^2_{\mathcal{H}^0} \quad \leq \quad -\lambda\|\mathbf{p}(t)-\mathbf{p}^*\|^2_{\mathcal{H}^0}.$$

Applying the Gronwall's lemma leads to the result.

□

4.4. **Numerical illustration poor-biased model.** We investigate numerically the convergence of $\mathbf{p}(t)$ solution to the poor-biased model (36) to the equilibrium distribution $\mathbf{p}^*$ (48). We use $\mu = 5$ (average money) and $\lambda = 1$ (rate of jumps) for the model. To discretize the model, we use $1,001$ components to describe the distribution $\mathbf{p}(t)$ (i.e. $(p_0(t), \ldots, p_{1000}(t))$). As initial conditions, we use $p_\mu(0) = 1$ and $p_i(0) = 0$ for $i \neq \mu$. The standard Runge-Kutta fourth-order method (e.g. RK4) is used to discretize the ODE system (36) with the time step $\Delta t = 0.01$.

We plot in Figure (7)-left the numerical solution $\mathbf{p}$ at $t = 12$ unit time and compare it to the equilibrium distribution $\mathbf{p}^*$. The two distributions are indistinguishable. Indeed, plotting the evolution of the difference $\|\mathbf{p}(t) - \mathbf{p}^*\|_{\mathcal{H}^0}$ (Figure (7)-right) shows that the difference is already below 10^{-10}. Moreover, the decay is clearly exponential as we use semi-logarithmic scale.

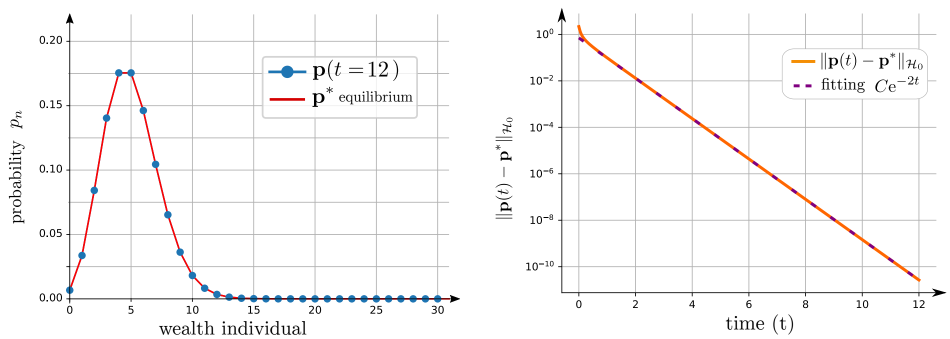


Figure 7 **Left**: comparison between the numerical solution $\mathbf{p}(t)$ (36) of the poor-bias model and the equilibrium $\mathbf{p}^*$ (48). The two distributions are indistinguishable. **Right**: decay of the difference $\|\mathbf{p}(t) - \mathbf{p}^*\|_{\mathcal{H}^0}$ in semilog scale. The decay is exponential as predicted by the theorem 4.5.

Notice that the numerical simulation suggests that the optimal decay rate of $\|\mathbf{p}(t) - \mathbf{p}^*\|_{\mathcal{H}^0}$ is 2λ, which is twice the analytical decay rate λ proved in proposition 4. The reason for this discrepancy is that the solution of $\mathbf{p}(t)$ remains in the subspace $V_\mu \cap \mathcal{D}(Q_{poor})$, i.e. the mean of $\mathbf{p}(t)$ is preserved. The analysis of the spectral gap of Q_{poor} in the proposition 4 does not take account this constraint.

We numerically investigate the spectrum of $-Q_{poor}$ denoted $\{\alpha_n\}_{n=1}^\infty$. The first eigenvalue satisfies $\alpha_1 = 0$ due to the equilibrium $\mathbf{p}^*$ (i.e. $Q_{poor}[\mathbf{p}^*] = 0$). The other eigenvalues are $\alpha_n = n - 1$ and in particular the spectral gap is $\alpha_2 = 1$. One can find explicitly a corresponding eigenfunction given by:

$$\mathbf{p}^{(2)} = D^-(\mathbf{p}^*) = (p_0^*, p_1^* - p_0^*, \ldots, p_n^* - p_{n-1}^*, \ldots). \tag{63}$$

Thus, for any $\mathbf{p} \in V_\mu \cap \mathcal{D}(Q_{poor})$, we find:

$$\langle \mathbf{p} \, , \, \mathbf{p}^{(2)} \rangle_{\mathcal{H}^0} = \sum_{n=0}^{\infty} p_n (p_n^* - p_{n-1}^*) \frac{1}{p_n^*} = \sum_{n=0}^{\infty} p_n (1 - n/\mu) = 1 - \mu/\mu = 0.$$

This explains why the effective spectral gap for the dynamics is given by α_3 and not α_2: the solution $\mathbf{p}(t)$ (36) lives in $V_\mu \cap \mathcal{D}(Q_{poor})$ and therefore it is orthogonal to $\mathbf{p}^{(2)}$.

Remark 7. We can find explicitly the exact formulation of the eigenfunction $\mathbf{p}^{(k)}$ of $-Q_{poor}$ for all $k \in \mathbb{N}_+$. We find by induction:

$$\mathbf{p}^{(k)} = \left(p_0^*, p_1^* - (k-1)p_0^*, \cdots, p_n^* + \sum_{j=0}^{n-1} (-1)^{n-j} \frac{\prod_{\ell=1}^{n-j}(k-\ell)}{(n-j)!} p_j^*, \cdots \right) \tag{64}$$

leading to:

$$p_n^{(k)} = \sum_{j=0}^{n} \binom{k-1}{j} (-1)^j \frac{\mu^{n-j}}{(n-j)!} \mathrm{e}^{-\mu}, \quad n \geq 0, \tag{65}$$

with $\binom{k}{j}$ binomial coefficient (i.e. $\binom{k}{j} = \frac{k!}{(k-j)!\,j!}$). Moreover, through an induction argument and some combinatorial identities, we can verify that $\langle \mathbf{p}^{(m)}, \mathbf{p}^{(k)} \rangle_{\mathcal{H}^0} = 0$ for $m \neq k$. We speculate that $\{\mathbf{p}^{(k)}\}_{k=1}^{\infty}$ spans the entire space $\mathcal{H}^0$, but we do not have a proof for this conjecture.

5. Rich-biased exchange model.

In our third model, the selection of the 'giver' is biased toward the poor instead of the rich, i.e. the more money an individual has the less likely it will be chosen.

5.1. Definition and limit equation.

As before, the definition of the model is given first.

Definition 5.1. (**Rich-biased exchange model**) A "giver" i is chosen with inverse proportionality of its wealth. The "receiver" j is chosen uniformly.

The rich-biased model leads to the following stochastic differential equation:

$$\mathrm{d}S_i(t) = -\sum_{j=1}^{N} \mathrm{dN}_t^{(i,j)} + \sum_{j=1}^{N} \mathrm{dN}_t^{(j,i)}, \tag{66}$$

with $\mathrm{N}_t^{(i,j)}$ Poisson process with intensity λ_{ij} given by:

$$\lambda_{ij} = \left\{ \begin{array}{cl} 0 & \text{if } S_i = 0 \\ \frac{\lambda}{N} \cdot \frac{1}{S_i} & \text{if } S_i > 0 \end{array} \right. \tag{67}$$

An agent i receives a dollar at rate λw where w is the inverse of the harmonic mean:

$$w = \frac{1}{N} \sum_{S_k > 0} \frac{1}{S_k}. \tag{68}$$

Definition 5.2. (**Asymptotic Rich-biased model**)

$$\mathrm{d}\overline{S}_1(t) = -\mathrm{d}\overline{\mathbf{N}}_t^1 + \mathrm{d}\overline{\mathbf{M}}_t^1, \tag{69}$$

in which $\overline{\mathbf{N}}_t^1$ and $\overline{\mathbf{M}}_t^1$ are independent Poisson processes with intensity $\lambda/\overline{S}_1(t)$ (if $\overline{S}_1(t) > 0$) and $\lambda \overline{w}(t)$ respectively. The inverse mean $\overline{w}(t)$ is given by:

$$\overline{w}[\mathbf{p}(t)] := \sum_{n=1}^{\infty} \frac{p_n(t)}{n} \tag{70}$$

where $\mathbf{p}(t) = \big(p_0(t), p_1(t), \dots\big)$ the law of the process $\overline{S}_1(t)$. The time evolution of $\mathbf{p}(t)$ is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = \lambda\, Q_{rich}[\mathbf{p}(t)] \tag{71}$$

with:

$$Q_{rich}[\mathbf{p}]_n := \begin{cases} p_1 - \overline{w}\, p_0 & \text{if } n = 0 \\ \frac{p_{n+1}}{n+1} + \overline{w} p_{n-1} - \left(\frac{1}{n} + \overline{w}\right) p_n & \text{for } n \geq 1 \end{cases} \tag{72}$$

We will also need the weak form of the operator: for any test function φ:

$$\langle Q_{rich}[\mathbf{p}]\,,\,\varphi\rangle = \sum_{n\geq 0} p_n \left(\overline{w}\varphi(n+1) + \frac{\mathbb{1}_{\{n\geq 1\}}}{n}\varphi(n-1) - \left(\overline{w} + \frac{\mathbb{1}_{\{n\geq 1\}}}{n}\right)\varphi(n)\right) \tag{73}$$

5.2. **Propagation of chaos using empirical measure.** We investigate the propagation of chaos for the rich-biased dynamics using the empirical measure (see subsection 2.3). We consider $\{S_i(t)\}_{1\leq i\leq N}$ the solution to (66) and use the empirical measure $\mathbf{p}_{emp}(t)$ (9). The goal is to show that the stochastic measure $\mathbf{p}_{emp}(t)$ converges to the deterministic density $\mathbf{p}(t)$ solution of (71). The main difficulty is that the empirical measure is a stochastic process on a Banach space $\ell^1(\mathbb{N})$ and thus of infinite dimension. Fortunately, the space is a discreet (i.e. $\mathbb{N}$) and therefore we do not have to consider stochastic partial differential equations which are famously difficult. Moreover, we only have to consider a finite number of possible jumps.

When agent i gives a dollar to j (i.e. $(S_i, S_j) \rightsquigarrow (S_i - 1, S_j + 1)$), the empirical measure is transformed as

$$\mathbf{p}_{emp} \;\rightsquigarrow\; \mathbf{p}_{emp} + \frac{1}{N}\Big(\delta_{S_i-1} + \delta_{S_j+1} - \delta_{S_i} - \delta_{S_j}\Big). \tag{74}$$

To write down the evolution equation satisfied by $\mathbf{p}_{emp}$, we regroup the agents with the same number of dollars (i.e. we project the dynamics on a subspace).

Proposition 6. *The empirical measure $\mathbf{p}_{emp}(t)$ (9) satisfies:*

$$\mathrm{d}\mathbf{p}_{emp}(t) = \frac{1}{N}\sum_{k=1,l=0}^{+\infty}\Big(\delta_{k-1} + \delta_{l+1} - \delta_k - \delta_l\Big)\mathrm{dN}_t^{(k,l)} \tag{75}$$

where $\mathrm{N}_t^{(k,l)}$ independent Poisson clock with intensity:

$$\lambda_{k,l} = N \cdot p_{emp,k} \cdot (N \cdot p_{emp,l} - \mathbb{1}_{\{k=l\}}) \cdot \frac{\lambda}{k \cdot N} \tag{76}$$

where $p_{emp,k}$ is the $k-th$ coordinate of $\mathbf{p}_{emp}$.

Proof. Following the jump process given in (74), the empirical measure satisfies:

$$\mathrm{d}\mathbf{p}_{emp}(t) = \frac{1}{N}\sum_{i,j=1,i\neq j}^{N}\Big(\delta_{S_i-1} + \delta_{S_j+1} - \delta_{S_i} - \delta_{S_j}\Big)\mathrm{dN}_t^{(i,j)} \tag{77}$$

Introducing $\mathrm{N}_t^{(k,l)}$ the Poisson process regrouping all the clocks corresponding to a giver with k dollars giving to a receiver with l dollars:

$$\mathrm{N}_t^{(k,l)} = \sum_{\{i\neq j\,|\,S_i=k,S_j=l\}} \mathrm{N}_t^{(i,j)}, \tag{78}$$

In this sum, each clock $\mathrm{N}_t^{(i,j)}$ has the same intensity $\lambda/(S_i{\cdot}N) = \lambda/(k{\cdot}N)$. Moreover, counting the number of clocks involved in the sum (78) leads to (76). The indicator $\mathbb{1}_{\{k=l\}}$ is here to remove the self-giving clocks $\mathrm{N}_t^{(i,i)}$: when an agent gives to itself, nothing happens. $\square$

Corollary 2. *For any test function φ, the empirical measure $\mathbf{p}_{emp}(t)$ (9) satisfies:*

$$\mathrm{d}\mathbb{E}[\langle \mathbf{p}_{emp}(t), \varphi\rangle] = \lambda \mathbb{E}[\langle Q_{rich}[\mathbf{p}_{emp}(t)], \varphi\rangle]\mathrm{d}t \;-\; \frac{\lambda}{N}\mathbb{E}[\langle R[\mathbf{p}_{emp}(t)], \varphi\rangle]\mathrm{d}t \qquad (79)$$

where Q_{rich} is the operator defined in (72) and R defined by:

$$R[\mathbf{p}]_n := \frac{p_{n+1}}{n+1} + \frac{p_{n-1}}{n-1}\mathbb{1}_{\{n\geq 2\}} - \frac{2}{n}p_n \mathbb{1}_{\{n\geq 1\}}. \qquad (80)$$

Proof. From the proposition 6, we find:

$$\begin{aligned}
\mathrm{d}\mathbb{E}[\langle \mathbf{p}_{emp}(t), \varphi\rangle] \;&=\; \mathbb{E}\left[\sum_{k=1,l=0}^{+\infty} \Big(\varphi(k-1)+\varphi(l+1)-\varphi(k)-\varphi(l)\Big) p_{emp,k}\cdot p_{emp,l}\cdot \frac{\lambda}{k}\right]\mathrm{d}t \\
&\quad -\frac{1}{N}\mathbb{E}\left[\sum_{k=1}^{+\infty}\Big(\varphi(k-1)+\varphi(k+1)-2\varphi(k)\Big)p_{emp,k}\cdot\frac{\lambda}{k}\right]\mathrm{d}t \\
&=\; \lambda\mathbb{E}\left[\sum_{k=1}^{+\infty}\Big(\varphi(k-1)-\varphi(k)\Big)\frac{p_{emp,k}}{k}\right]\mathrm{d}t \\
&\quad +\lambda\mathbb{E}\left[\sum_{l=0}^{+\infty}\Big(\varphi(l+1)-\varphi(l)\Big)\overline{w}[\mathbf{p}_{emp}]\cdot p_{emp,l}\right]\mathrm{d}t \\
&\quad -\frac{\lambda}{N}\mathbb{E}\left[\sum_{k=1}^{+\infty}\Big(\varphi(k-1)+\varphi(k+1)-2\varphi(k)\Big)p_{emp,k}\cdot\frac{1}{k}\right]\mathrm{d}t
\end{aligned}$$

where $\overline{w}[\mathbf{p}_{emp}]$ is defined in (70). We recognize the weak formulation of Q_{rich} (73) leading to (79). □

The operator R (80) corresponds to the *bias* in the evolution of the empirical measure $\mathbf{p}_{emp}(t)$ compared to the evolution of $\mathbf{p}(t)$ solution to the limit equation (71). This bias vanishes as λ/N goes to zero when the number of agents N becomes large. The other source of discrepancy between $\mathbf{p}_{emp}(t)$ and $\mathbf{p}(t)$ is the *variance* of $\mathbf{p}_{emp}(t)$ (as it is a stochastic measure). Let's review an elementary result on compensated Poisson process.

Remark 8. Denote $Z(t)$ a compound jump process and $M(t)$ its compensated version:

$$\mathrm{d}Z(t) = Y(t)\,\mathrm{dN}_t \quad , \quad M(t) = Z(t) - \int_0^t \mu(s)\lambda(s)\,\mathrm{d}s \qquad (81)$$

where $Y(t)$ denotes the (independent) jumps and N_t Poisson process with intensity $\lambda(t)$ and $\mu(t) = \mathbb{E}[Y(t)]$. The Ito's formula is given by:

$$\mathrm{d}\mathbb{E}[\varphi(M(t))] = \mathbb{E}\Big[\varphi\Big(M(t-)+Y(t-)\Big) - \varphi(M(t-)\Big]\lambda(t)\mathrm{d}t \;-\; \mathbb{E}[\varphi'(M(t))\mu(t)\lambda(t)]\,\mathrm{d}t.$$

In particular, for $\varphi(x) = x^2$, we obtain:

$$\begin{aligned}
\mathrm{d}\mathbb{E}[M^2(t)] \;&=\; \mathbb{E}[2M(t-)Y(t-) + Y^2(t-)]\,\lambda(t)\mathrm{d}t - \mathbb{E}[2M(t)\mu(t)\lambda(t)]\,\mathrm{d}t \\
&=\; \mathbb{E}[Y^2(t)]\,\lambda(t)\mathrm{d}t. \qquad (82)
\end{aligned}$$

Here, we assume that the jump $Y(t)$ is independent of the value $Z(t)$. To generalize the formula, one has to replace $\mu(t) = \mathbb{E}[Y(t)]$ by $\mathbb{E}[Y(t)|Z(t)]$.

Motivated by this remark, we obtain the following result.

Proposition 7. *Denote $M(t)$ the compensated process of the empirical measure $\mathbf{p}_{emp}(t)$:*

$$M(t) = \mathbf{p}_{emp}(t) - \left(\mathbf{p}_{emp}(0) + \lambda \int_0^t \left(Q_{rich}[\mathbf{p}_{emp}(s)] + \frac{1}{N} R[\mathbf{p}_{emp}(s)]\right) \mathrm{d}s\right) \quad (83)$$

then $M(t)$ is a ℓ^1-value martingale and satisfies:

$$\mathbb{E}[\|M(t)\|_{\ell^1}] \leq \sqrt{\frac{4\lambda}{N}}\, t. \quad (84)$$

Proof. The key observation is that the jump (74) for the empirical measure are of order $\mathcal{O}(1/N)$. Indeed:

$$E\left[\left\|\frac{1}{N}(\delta_{k-1} + \delta_{l+1} - \delta_k - \delta_l)\right\|_{\ell^1}^2\right] \leq \frac{4}{N^2}. \quad (85)$$

Applying the formula (82) we obtain::

$$\mathrm{d}\mathbb{E}[\|M(t)\|_{\ell^1}^2] \leq \sum_{k=1,l=0}^{+\infty} \mathbb{E}\left[\frac{4}{N^2} \cdot N p_{emp,k} \cdot N p_{emp,l}\right] \frac{\lambda}{k \cdot N} \mathrm{d}t \;\leq\; \frac{4\lambda}{N} \mathrm{d}t. \quad (86)$$

Integrating in time gives (84). □

We are now ready to prove the propagation of chaos for the rich-biased dynamics by showing that the empirical measure $\mathbf{p}_{emp}(t)$ converges to $\mathbf{p}(t)$ as $N \to +\infty$. The key observation is the following

Lemma 4. *The operator Q_{rich} (72) is globally Lipschitz on $\ell^1(\mathbb{N}) \cap \mathcal{P}(\mathbb{N})$ and R is an bounded on $\ell^1(\mathbb{N})$.*

$$\|Q_{rich}[\mathbf{p}] - Q_{rich}[\mathbf{q}]\|_{\ell^1} \leq 4\|\mathbf{p} - \mathbf{q}\|_{\ell^1} \qquad \text{for any } \mathbf{p}, \mathbf{q} \in \ell^1(\mathbb{N}) \cap \mathcal{P}(\mathbb{N}) \quad (87)$$

$$\|R[\mathbf{p}]\|_{\ell^1} \leq 4\|\mathbf{p}\|_{\ell^1} \qquad \text{for any } \mathbf{p} \in \ell^1(\mathbb{N}) \quad (88)$$

Proof. Since $\mathbf{p} \in \ell^1(\mathbb{N}) \cap \mathcal{P}(\mathbb{N})$, the rate of receiving $w[\mathbf{p}]$ (68) satisfies $0 \leq w[\mathbf{p}] \leq 1$. Thus,

$$|Q_{rich}[\mathbf{p}]_n - Q_{rich}[\mathbf{q}]_n| \leq |p_{n+1} - q_{n+1}| + |p_{n-1} - q_{n-1}| + 2|p_n - q_n|.$$

Summing over n gives the result. We proceed similarly for the operator R. □

Theorem 5.3. *Consider $\mathbf{p}(t)$ solution to the limit equation (71) and $\mathbf{p}_{emp}(t)$ empirical measure (9). Then:*

$$\mathbb{E}[\|\mathbf{p}_{emp}(t) - \mathbf{p}(t)\|_{\ell^1}] \leq \mathcal{O}\left(\frac{t e^{4\lambda t}}{\sqrt{N}}\right), \quad (89)$$

in particular $\mathbf{p}_{emp}(t) \overset{N \to +\infty}{\rightharpoonup} \mathbf{p}(t)$ for any $t \geq 0$.

Proof. First we write down the integral form of the equation satisfied by both $\mathbf{p}(t)$ and $\mathbf{p}_{emp}(t)$:

$$\begin{aligned}
\mathbf{p}(t) &= \mathbf{p}_0 + \int_0^t Q_{rich}[\mathbf{p}(s)]\,\mathrm{d}s \\
\mathbf{p}_{emp}(t) &= \mathbf{p}_0 + \int_0^t Q_{rich}[\mathbf{p}_{emp}(s)]\,\mathrm{d}s + \frac{1}{N}\int_0^t R[\mathbf{p}_{emp}(s)]\,\mathrm{d}s + M(t)
\end{aligned}$$

Combining the two equations give:

$$\begin{aligned}\|\mathbf{p}_{emp}(t) - \mathbf{p}(t)\|_{\ell^1} &\leq \lambda \int_0^t \|Q_{rich}[\mathbf{p}_{emp}(s)] - Q_{rich}[\mathbf{p}(s)]\|_{\ell^1}\,\mathrm{d}s \\ &\quad + \frac{\lambda}{N}\int_0^t \|R[\mathbf{p}_{emp}(s)]\|_{\ell^1}\,\mathrm{d}s + \|M(t)\|_{\ell^1} \\ &\leq 4\lambda \int_0^t \|\mathbf{p}_{emp}(s) - \mathbf{p}(s)\|_{\ell^1}\,\mathrm{d}s + \frac{\lambda 4t}{N} + \|M(t)\|_{\ell^1}\end{aligned}$$

using lemma 4. Denoting $\phi(t) = \mathbb{E}[\|\mathbf{p}_{emp}(t) - \mathbf{p}(t)\|_{\ell^1}]$, we deduce from the bound (84) of $M(t)$:

$$\phi(t) \leq 4\lambda \int_0^t \phi(s)\,\mathrm{d}s + \frac{\lambda 4t}{N} + \sqrt{\frac{4\lambda}{N}}t.$$

Applying Gronwall's lemma gives rise to:

$$\phi(t) \leq \left(\frac{\lambda 4t}{N} + \sqrt{\frac{4\lambda}{N}}t\right)\mathrm{e}^{4\lambda t}$$

leading to the result. □

Remark 9. The martingale-based technique, developed in [42] and employed here for justifying the propagation of chaos, is remarkable since it does not require us to study the N-particle process $(S_1, \ldots, S_N)$ but solely its generator. One drawback is that this method might not work if the generator Q of the limit process is unbounded, which is the case for the generator Q_{poor} of the (limit) poor-biased dynamics (36).

5.3. **Dispersive wave leading to vanishing wealth.** As illustrated in the introduction (Figure 2), the rich-biased dynamics tend to accentuate inequality, i.e. the Gini index $G(t)$ was approaching 1 (its maximum value) for the agent-based model (4) (66). We would like to investigate numerically the behavior of the solution to the rich-biased dynamics using the limit equation (71) and the distribution $\mathbf{p}(t) = (p_0(t), p_1(t), \ldots)$.

In Figure 8, we plot the evolution of the distribution $\mathbf{p}(t)$ starting from a Dirac distribution with mean $\mu = 5$ (i.e. $p_5 = 1$ and $p_i = 0$ for $i \neq 5$). We observe that the distribution spreads in two parts: the bulk of the distribution moves toward zero whereas a smaller proportion is moving to the right. One can identify the two pieces as the "poor" and the "rich". Thus, the dynamics could be interpreted as the poor getting poorer and the rich getting richer. Notice that the proportion of poor is increasing (e.g. $p_0(t)$ is increasing) whereas the "rich" distribution resembles a dispersive traveling wave. Since both the total mass and the total amount of dollar are preserved (i.e. $\sum_n n \cdot p_n(t) = \mu$ for any t), the dispersive traveling wave contains the bulk of the money but it is also vanishing in time.

To investigate more carefully the dispersive wave, we try to fit numerically its profile. After numerically examination, we choose to approximate it by a Gaussian distribution. Meanwhile we approximate the "poor" distribution by a Dirac centered at zero δ_0. Thus, we approximate the distribution $\mathbf{p}(t)$ by the following Ansatz:

$$p_n(t) \;\approx\; (1 - r(t)) \cdot \delta_0 \quad + \quad r(t) \cdot \frac{1}{\sigma(t)}\phi\left(\frac{n - c(t)}{\sigma(t)}\right), \tag{90}$$

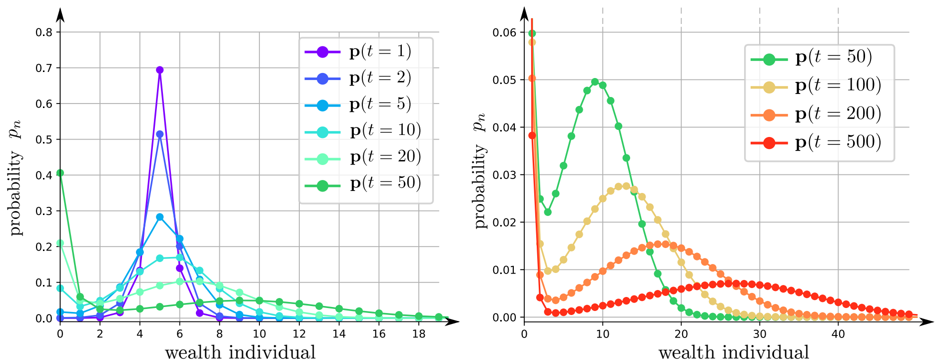


Figure 8 Evolution of the wealth distribution $\mathbf{p}(t)$ for the rich-biased dynamics (71). The distribution spreads in two parts: a large proportion starts to concentrate at zero ("poor distribution") and while the other part forms a dispersive traveling wave. Parameters: $\Delta t = 5 \cdot 10^{-3}$, $\mathbf{p}(t) \approx (p_0(t), p_1(t), \ldots, p_{1,000}(t))$. A standard Runge-Kutta of order 4 has been used to discretize the system.

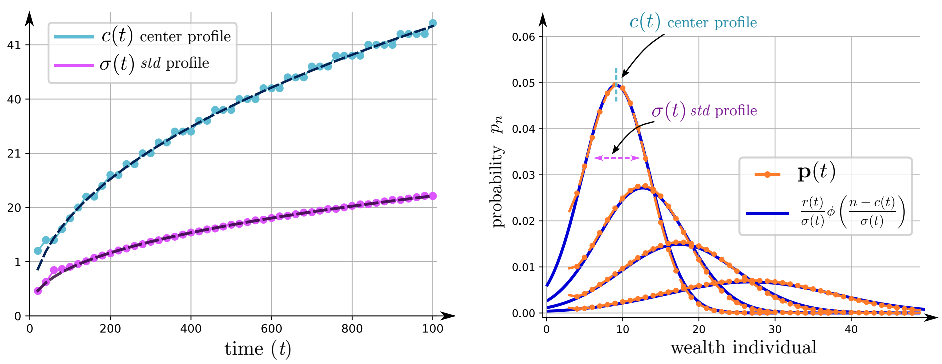


Figure 9 Left: Estimation of the center $c(t)$ and standard deviation $\sigma(t)$ of the dispersive wave along with their parametric (power-law) estimation (91). **Right**: Comparison of the distribution $\mathbf{p}(t)$ (see Figure 8) with the dispersive wave using the standard normal distribution ϕ.

where ϕ is the standard normal distribution (i.e. $\phi(x) = \mathrm{e}^{-x^2/2}/\sqrt{2\pi}$), $c(t)$ is the center of the profile, $\sigma(t)$ its standard deviation and $r(t)$ the proportion of rich. The speed of the wave $c(t)$ and its standard deviation $\sigma(t)$ are estimated numerically and plotted in Figure 9. Their growth is well-approximated by a power-law of the form:

$$c(t) = 1.4748 \cdot t^{.466}, \quad \sigma(t) = 0.9261 \cdot t^{.399}. \tag{91}$$

Since the total amount of money is preserved, the proportion of rich $r(t)$ can be easily deduced from $c(t)$ since we must have $\mu = r(t) \cdot c(t)$. Such approximation leads to the fitting in Figure 8-right (dotted-black curves). We notice that the proportion of rich in our Ansatz is vanishing:

$$r(t) = \frac{\mu}{c(t)} \stackrel{t\to+\infty}{\longrightarrow} 0. \tag{92}$$

Thus, we make the conjecture that $\mathbf{p}(t)$ converges weakly toward δ_0, i.e. all the money will asymptotically disappear.

To further assess our conjecture, we measure the evolution of the Gini index for the distribution $\mathbf{p}(t)$:

$$G[\mathbf{p}] = \frac{1}{2\mu} \sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} |i-j| p_i p_j \tag{93}$$

with μ the standard mean. Using the Ansatz (90), we can approximate the value of the Gini index given (see Appendix B):

$$G(t) \approx 1 - \frac{\mu}{c(t)} + \frac{\mu \cdot \sigma(t)}{\sqrt{\pi}\, c^2(t)}. \tag{94}$$

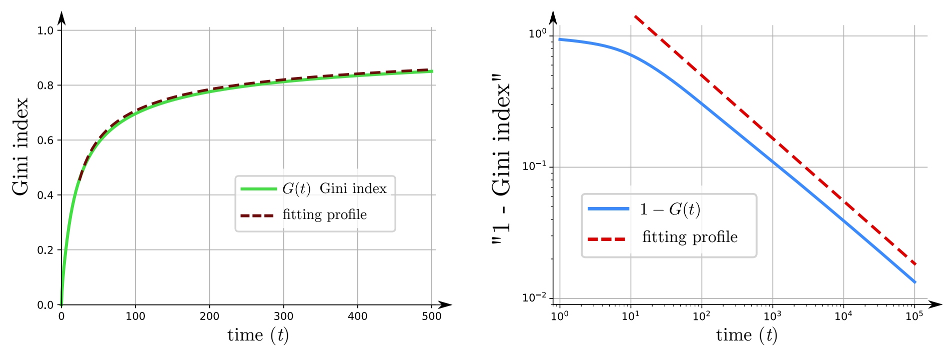


Figure 10 Left: Evolution of the corresponding Gini index (93) along with the analytical approximation using the dispersive wave profile (94). **Right** The Gini index converges to 1 due to the vanishing dispersive wave transporting all the wealth to infinity.

Remark 10. In the approximation (91), the coefficient $c(t)$ grows faster than $\sigma(t)$, thus the Gini index has no risk of exceeding one in the approximation (94). More generally, as long as $c(t)$ is of the same order as $\sigma(t)$, the approximated Gini index given by (94) will not become larger than one.

We plot in Figure 10-left the evolution of the Gini index $G(t)$ along with its approximation (94). We observe a good agreement between the two curves. To examine closely the long time behavior of the curves, we plot the evolution of $1 - G(t)$ in log-scales (Figure 10-right) over a longer time interval (up to $t = 10^5$). Both curves seem to converges similarly toward 0 (indicating that $G(t) \xrightarrow{t \to +\infty} 1$) with a slight overshoot for the Ansatz. This overshoot might be due to our approximation that the "poor distribution" of $\mathbf{p}(t)$ is concentrated exactly at zero (i.e. $(1 - r(t))\delta_0$). This approximation amplifies the inequality between the "poor" and "rich" parts of the distribution and hence increases slightly the Gini index. But overall the asymptotic behavior of the Gini index for $\mathbf{p}(t)$ matches with the formula (94) and thus strengthens our assumption that $\mathbf{p}(t)$ will converge (weakly) to a Dirac δ_0. However, further analytical studies are needed to derive the asymptotic behavior of $\mathbf{p}(t)$ directly from the rich-biased evolution equation (71).

6. **Conclusion.** In this manuscript, we have investigated three related models for money exchange originated from econophysics. For the unbiased and poor biased dynamics, we rigorously proved the so-called *propagation of chaos* by virtue of a coupling technique, and we found an explicit rate of convergence of the limit dynamics for the poor biased model thanks to the Bakry-Emery approach. We have also introduced a more challenging dynamics referred to as the rich biased model, and a propagation of chaos result was established via a powerful martingale-based argument presented in [42]. In contrast to the two other dynamics, the rich-biased dynamics do not converge (strongly) to an equilibrium. Instead, we have found numerical evidence of the emergence of a (vanishing) dispersive wave. Such wave of extreme wealthy individual increases the inequality in the wealth distribution making the corresponding Gini index converging to its maximum 1.

Although we have shown numerically strong evidence of a dispersive wave, it is desirable to derive such emerging behavior directly from the evolution equation. One direction of future work would be to derive space continuous dynamics of evolution equations in order to investigate analytically the profile of traveling waves. However, space continuous description such as the uniform reshuffling model could lead to additional challenges. For instance, proving propagation of chaos using the martingale technique for the uniform reshuffling model was more involved [13].

From a modeling perspective, one should explore how selecting the "receiver" as well as the "giver" could impact the dynamics. Indeed, in the three dynamics studied in the manuscript, the re-distribution process (how the one-dollar is re-distributed) is uniform among all the agent. It would be reasonable to have the redistribution of the dollar based on the individual wealth (e.g. poor individual being more likely to receive a dollar). The interplay between receiver and giver selection could lead to novel emerging behaviors.

Appendix A. Poisson-Poincaré inequality.

Proof Proposition 4. We use the notations provided in [9].

Let $S_n = \sum_{i=1}^n Y_i$, where $\{Y_i\}_{i=1}^n$ are independent and identically distributed with $Y_i \sim \text{Bernoulli}(\mu/n)$, so that $S_n \to X$ in distribution (as $n \to \infty$). Using $\mathbb{E}^{(i)}$ the conditional expectation with respect to $Y^{(i)} = (Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n)$, we find:

$$\mathbb{E}^{(i)}[f(S_n)] = \left(1 - \frac{\mu}{n}\right) f(S_n - Y_i) \; + \; \frac{\mu}{n} f(S_n - Y_i + 1).$$

Notice that $S_n - Y_i$ is independent of Y_i. After computations, we deduce the following formula for the conditional variance:

$$\begin{aligned}\text{Var}^{(i)}\left(f(S_n)\right) &= \mathbb{E}^{(i)}\left[\left(f(S_n) - \mathbb{E}^{(i)}[f(S_n)]\right)^2\right] \\ &= \left(1 - \frac{\mu}{n}\right)\frac{\mu}{n}\,\left(f(S_n - Y_i + 1) - f(S_n - Y_i)\right)^2.\end{aligned}$$

As $\{Y_i\}_{i=1}^n$ are independent, the standard Efron-Stein inequality yields that

$$\begin{aligned}\text{Var}\left(f(S_n)\right) &\leq \left(1 - \frac{\mu}{n}\right)\frac{\mu}{n}\sum_{i=1}^n \mathbb{E}\left[\left(f(S_{n-1}+1) - f(S_{n-1})\right)^2\right] \\ &\leq \left(1 - \frac{\mu}{n}\right)\mu\,\mathbb{E}\left[\left(f(S_{n-1}+1) - f(S_{n-1})\right)^2\right] \\ &= \left(1 - \frac{\mu}{n}\right)\mu\,\mathbb{E}\left[\left(Df(S_{n-1})\right)^2\right],\end{aligned}$$

whence the advertised inequality (56) follows by sending $n \to \infty$. $\square$

Appendix B. **Gini index dispersive wave.** We estimate the Gini coefficient for a (continuous) distribution of the form:

$$\rho(x) = (1-r)\cdot\delta_0(x) + r\cdot\frac{1}{\sigma}\phi\left(\frac{x-c}{\sigma}\right) \tag{95}$$

where ϕ is the standard normal distribution, r, c, σ some positive constant with $r \in [0,1]$. The law ρ can be represented by a random variable:

$$X = (1-Y)\cdot 0 + Y\cdot(c+\sigma Z) \tag{96}$$

with Y random Bernoulli variable with probability r (i.e. $Y \sim B(r)$), Z a random variable with normal law (i.e. $Z \sim \mathcal{N}(0,1)$), Y and Z being independent. To estimate the Gini index of ρ, we take two independent random variables X_1 and X_2 with such law and estimate the expectation of their difference:

$$\begin{aligned} G &= \frac{1}{2\mu}\mathbb{E}[|X_1 - X_2|] = \frac{1}{2\mu}\mathbb{E}[|Y_1\cdot(c+\sigma Z_1) \ - \ Y_2\cdot(c+\sigma Z_2)|] \\ &= \frac{1}{2\mu}\mathbb{E}[|c(Y_1-Y_2) + \sigma(Y_1Z_1 - Y_2Z_2)|] \end{aligned} \tag{97}$$

We then take the conditional expectation with respect to Y_1 and Y_2:

$$\begin{aligned} 2\mu G &= 0 + \mathbb{E}[|c+\sigma Z_1|]\mathbb{P}[Y_1=1, Y_2=0] \\ &\quad + \mathbb{E}[|-c-\sigma Z_2|]\mathbb{P}[Y_1=0, Y_2=1] \\ &\quad + \mathbb{E}[|\sigma(Z_1-Z_2)|]\mathbb{P}[Y_1=1, Y_2=1] \\ &= 2\cdot\mathbb{E}[|c+\sigma Z_1|]r(1-r) + \mathbb{E}[|\sigma(Z_1-Z_2)|]r^2 \end{aligned} \tag{98}$$

For large c, we made the approximation $\mathbb{E}[|c+\sigma Z_1|] \approx \mathbb{E}[c+\sigma Z_1] = c$. Moreover, the expectation of the difference between two standard Gaussian random variables is known explicitly: $\mathbb{E}[|Z_1 - Z_2|] = 2/\sqrt{\pi}$. We deduce:

$$2\mu G \approx 2c\cdot r(1-r) + \sigma\frac{2}{\sqrt{\pi}}r^2. \tag{99}$$

Furthermore, if $r = \mu/c$, we obtain:

$$G \approx 1 - \frac{\mu}{c} + \frac{\sigma\mu}{\sqrt{\pi}c^2}. \tag{100}$$

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