A. Duwenig *et al.* (2022) "Analyzing the Weyl Construction for Dynamical Cartan Subalgebras,"
International Mathematics Research Notices, Vol. 2022, No. 20, pp. 15721–15755
Advance Access Publication July 5, 2021
https://doi.org/10.1093/imrn/rnab114

Analyzing the Weyl Construction for Dynamical Cartan Subalgebras

Anna Duwenig[1,*], Elizabeth Gillaspy[2] and Rachael Norton[3]

[1]School of Mathematics and Applied Statistics, University of Wollongong, Northfields Ave, Wollongong NSW 2522, Australia, [2]Department of Mathematical Sciences, University of Montana, 32 Campus Drive #0864, Missoula, MT 59812, USA and, [3]Mathematics Department, Fitchburg State University, 160 Pearl Street, Fitchburg, MA 01420, USA

[*]*Correspondence to be sent to: e-mail: aduwenig@uow.edu.au*

When the reduced twisted C^*-algebra $C_r^*(\mathcal{G}, c)$ of a non-principal groupoid $\mathcal{G}$ admits a Cartan subalgebra, Renault's work on Cartan subalgebras implies the existence of another groupoid description of $C_r^*(\mathcal{G}, c)$. In an earlier paper, joint with Reznikoff and Wright, we identified situations where such a Cartan subalgebra arises from a subgroupoid $\mathcal{S}$ of $\mathcal{G}$. In this paper, we study the relationship between the original groupoids $\mathcal{S}, \mathcal{G}$ and the Weyl groupoid and twist associated to the Cartan pair. We first identify the spectrum $\mathfrak{B}$ of the Cartan subalgebra $C_r^*(\mathcal{S}, c)$. We then show that the quotient groupoid $\mathcal{G}/\mathcal{S}$ acts on $\mathfrak{B}$, and that the corresponding action groupoid is exactly the Weyl groupoid of the Cartan pair. Lastly, we show that if the quotient map $\mathcal{G} \to \mathcal{G}/\mathcal{S}$ admits a continuous section, then the Weyl twist is also given by an explicit continuous 2-cocycle on $\mathcal{G}/\mathcal{S} \ltimes \mathfrak{B}$.

1 Introduction

One of the earliest theorems about C^*-algebras, the Gelfand–Naimark Theorem, establishes that any commutative C^*-algebra is of the form $C_0(X)$ for a locally compact Hausdorff space X. In addition to inspiring the "noncommutative topology" approach

Received October 13, 2020; Revised February 10, 2021; Accepted April 7, 2021
Communicated by Prof. Dan-Virgil Voiculescu

to C^*-algebras, the Gelfand–Naimark Theorem has also led researchers to search for Abelian subalgebras inside noncommutative C^*-algebras, with the goal of using topological tools to analyze the Abelian subalgebra and from there to obtain a better understanding of its noncommutative host. This program has been particularly successful when the subalgebra is *Cartan* (see Definition 2.1); it has enabled progress on Elliott's classification program for C^*-algebras [4, 15] as well as the theory of continuous orbit equivalence of topological dynamical systems [2, 16]. Even beyond the setting of Cartan subalgebras, many authors (cf. [2, 3, 12]) have successfully extended structural information from more general Abelian subalgebras to the containing C^*-algebras.

In this paper, we focus our attention on certain Cartan subalgebras, which appear in a rather unexpected context. Renault proved in [21], building on earlier work of Kumjian [14], that if a C^*-algebra A admits a Cartan subalgebra, then A is isomorphic to the reduced C^*-algebra $C_r^*(G, \Sigma)$ of a twist Σ over a groupoid G, and the Cartan subalgebra is realized as $C_0(G^{(0)})$. The groupoids G appearing in Renault's analysis must satisfy a number of structural contraints; for example, they are always topologically principal. If G is not topologically principal, then $C_0(G^{(0)})$ is not a Cartan subalgebra inside $C_r^*(G, \Sigma)$ for any twist Σ over G. Nevertheless, there are many such groupoids whose twisted C^*-algebras contain Cartan subalgebras. Examples include the rotation algebras $A_\theta \cong C_r^*(\mathbb{Z}^2, c_\theta)$ and the C^*-algebras of directed graphs, which do not satisfy Condition (L).

Together with Reznikoff and Wright, in [8, Theorem 3.1], we identified a large family of twisted groupoid C^*-algebras, associated to non-principal groupoids $\mathcal{G}$, which contain Cartan subalgebras. Moreover, these Cartan subalgebras are evident at the level of the groupoid $\mathcal{G}$: they arise from a subgroupoid $\mathcal{S}$ of $\mathcal{G}$. As mentioned above, the existence of a Cartan subalgebra in $C_r^*(\mathcal{G}, c)$ implies, by [21], the existence of a topologically principal groupoid G, the so-called *Weyl groupoid,* and a twist Σ over G such that $C_r^*(\mathcal{G}, c) \cong C_r^*(G, \Sigma)$. If $\mathcal{G}$ is a discrete group and $\mathcal{S} \leq \mathcal{G}$ satisfies the hypotheses of [8, Theorem 3.1], so that $C_r^*(\mathcal{S}, c)$ is a Cartan subalgebra of $C_r^*(\mathcal{G}, c)$, then [8, Theorem 5.2] establishes that

$$G = (\mathcal{G}/\mathcal{S}) \ltimes \widehat{\mathcal{S}}$$

as long as the action of $\mathcal{G}/\mathcal{S}$ on $\widehat{\mathcal{S}}$ is topologically free. Moreover, [8, Theorem 5.8] shows that in this case, the twist Σ arises from a 2-cocycle on G, which we described explicitly in [8, Lemma 5.6].

The prepublication version of [12] came to our attention as we were finalizing [8], and we were struck by the structural parallels between the two papers' main results. In

[12, Theorem 3.3], the authors show that if a subgroupoid $\mathcal{S}$ of a (not necessarily étale) groupoid $\mathcal{G}$ consists of a closed normal bundle of Abelian groups, then the groupoid C^*-algebra $C_r^*(\mathcal{G})$ can alternatively be computed from a twist Σ over the action groupoid $\widehat{\mathcal{S}} \rtimes (\mathcal{G}/\mathcal{S})$:

$$C_r^*(\mathcal{G}) \cong C_r^*(\widehat{\mathcal{S}} \rtimes (\mathcal{G}/\mathcal{S}), \Sigma).$$

However, they only establish that $C_r^*(\mathcal{S})$ is Cartan in $C_r^*(\mathcal{G})$ if $\mathcal{G}/\mathcal{S}$ is étale and topologically principal [12, Theorem 4.6], and they do not analyze the structure of twisted groupoid C^*-algebras $C_r^*(\mathcal{G}, c)$. As nontrivial discrete groups are never topologically principal, this excludes the setting of [8, Theorem 5.8]. Moreover, the formula given in [12] for the twist is not explicit; in particular, it is unclear when, or whether, it can be realized via a 2-cocycle on the groupoid $\widehat{\mathcal{S}} \rtimes (\mathcal{G}/\mathcal{S})$.

In this paper, we bridge the gap between [8] and [12]. Our first main result, Theorem 4.6, establishes that when a subgroupoid $\mathcal{S}$ of an étale groupoid $\mathcal{G}$ satisfies the hypotheses of [8, Theorem 3.1], so that $C_r^*(\mathcal{S}, c)$ is Cartan in $C_r^*(\mathcal{G}, c)$, then the Weyl groupoid associated to the Cartan pair $\big(C_r^*(\mathcal{G}, c), C_r^*(\mathcal{S}, c)\big)$ is an action groupoid

$$(\mathcal{G}/\mathcal{S}) \ltimes \mathfrak{B},$$

where $\mathfrak{B}$ denotes the spectrum of the commutative algebra $C_r^*(\mathcal{S}, c)$. When the 2-cocycle c is trivial, $\mathfrak{B}$ agrees with the space $\widehat{\mathcal{S}}$ of [12], and (translating our left action of $\mathcal{G}/\mathcal{S}$ into a right action) we see that our groupoid $(\mathcal{G}/\mathcal{S}) \ltimes \mathfrak{B}$ agrees with the groupoid $\widehat{\mathcal{S}} \rtimes (\mathcal{G}/\mathcal{S})$ of [12] (see Remark 8). Theorem 4.6 is a substantial improvement over [8, Theorem 5.2]. Not only do we extend [8, Theorem 5.2] from groups to groupoids, we also show that the hypothesis of topological freeness in the latter theorem is always satisfied.

Our second main result is Theorem 5.1: given a continuous section $\mathfrak{s}\colon \mathcal{G}/\mathcal{S} \to \mathcal{G}$ of the quotient map, we identify a continuous section $\psi_{\mathfrak{s}}\colon \mathcal{G}_{(A,B)} \to \Sigma_{(A,B)}$ of the Weyl groupoid extension. From this section, we follow the standard procedure to construct in Corollary 5.4 an explicit formula for a continuous 2-cocycle $c^{\mathfrak{s}}$ on the Weyl groupoid such that the Weyl twist is isomorphic to $((\mathcal{G}/\mathcal{S}) \ltimes \mathfrak{B}) \times_{c^{\mathfrak{s}}} \mathbb{T}$. Our precise description of the twist represents both an extension of [8, Theorem 5.8] to a broader setting and an improvement on [12, Theorem 3.3] in the étale case.

This paper is organized as follows. In Section 2, we recall the relevant definitions of groupoids and Cartan subalgebras, and give a detailed analysis of the Weyl construction. We expect that some of the technical results we have obtained will be

of general interest. For example, Proposition 2.2 gives a description of the equivalence relation underlying the Weyl groupoid, which, to our knowledge, has not appeared before in the literature.

The first step in providing an explicit description of the Weyl groupoid associated to a Cartan pair (A, B) is identifying the topological space $\widehat{B}$; Section 3 is devoted to describing $\widehat{B}$ in the case when $B = C_r^*(\mathcal{S}, c)$ arises from a bundle of discrete Abelian groups and c is symmetric on $\mathcal{S}$ (see Corollary 3.7). This description may be known to experts—indeed, it is similar to results such as [18, Corollary 3.4], [10, Proposition 5], and [7, Remark 5.2]—but we were unable to locate a reference in the literature for the precise result we needed.

In Section 4, we prove that if $(A, B) = (C_r^*(\mathcal{G}, c), C_r^*(\mathcal{S}, c))$ is one of the Cartan pairs identified by [8, Theorem 3.1], then its Weyl groupoid is the action groupoid $(\mathcal{G}/\mathcal{S}) \ltimes \mathfrak{B}$ alluded to above (Theorem 4.6). In Section 5, we prove that the associated Weyl twist arises from a continuous 2-cocycle on the Weyl groupoid if there exists a continuous section of the quotient map $\mathcal{G} \to \mathcal{G}/\mathcal{S}$ (Theorem 5.1).

2 Preliminaries on Cartan Subalgebras and the Weyl Construction

Intuitively, a *groupoid* $\mathcal{G}$ is a generalization of a group in which multiplication is only partially defined. More precisely, a groupoid is a set $\mathcal{G}$, together with a subset $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$; a multiplication map $(\gamma, \eta) \mapsto \gamma\eta$ from $\mathcal{G}^{(2)}$ to $\mathcal{G}$; and an inversion map $\gamma \mapsto \gamma^{-1}$ from $\mathcal{G}$ to $\mathcal{G}$, which behave like multiplication and inversion do in groups wherever they are defined. The *unit space* of $\mathcal{G}$ is $\mathcal{G}^{(0)} = \{\gamma\gamma^{-1} : \gamma \in \mathcal{G}\}$. We then have range and source maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ given by

$$r(\gamma) = \gamma\gamma^{-1}, \quad s(\gamma) := \gamma^{-1}\gamma,$$

which satisfy $r(\gamma)\gamma = \gamma = \gamma s(\gamma)$ for all $\gamma \in \mathcal{G}$. Given $u \in \mathcal{G}^{(0)}$, we write $\mathcal{G}_u := \{\gamma \in \mathcal{G} : s(\gamma) = u\}$ and $\mathcal{G}^u := \{\gamma \in \mathcal{G} : r(\gamma) = u\}$. We can also describe $\mathcal{G}^{(2)}$ using the range and source maps:

$$\mathcal{G}^{(2)} = \{(\gamma, \eta) \in \mathcal{G} \times \mathcal{G} : s(\gamma) = r(\eta)\}.$$

In this paper, we will assume that $\mathcal{G}$ is equipped with a locally compact Hausdorff topology with respect to which the multiplication and inversion maps are continuous. The groupoids considered in this paper will also be *étale* – that is, r and s will be local homeomorphisms. A subset V of $\mathcal{G}$ will be called a *bisection* if there is an

open set U containing V such that $r\colon U \to r(U)$ and $s\colon U \to s(U)$ are homeomorphisms onto open subsets of $\mathcal{G}^{(0)}$.

The link between groupoids and Cartan subalgebras was established in the seminal papers [14, 21].

Definition 2.1 ([21, Definition 5.1]). Let A be a C^*-algebra. A C^*-subalgebra B of A is a *Cartan subalgebra* if the following four conditions hold:

1. B is a maximal Abelian subalgebra of A.
2. There exists a faithful conditional expectation $\Phi\colon A \to B$.
3. B is regular; that is, the normalizer of B,

$$N(B) := \{n \in A \text{ such that } nbn^*, n^*bn \in B \text{ for all } b \in B\},$$

   generates A as a C^*-algebra.
4. B contains an approximate identity for A.

For this section, we fix a Cartan subalgebra B of some separable C^*-algebra A. Let us first recall how (A,B) gives rise to a topologically principal groupoid and twist (cf. [14, 1.6] or [21, Proposition 4.7]), and then gather a few tools to study them. Let $\widehat{B}$ be the spectrum of B, viewed as the space of one-dimensional representations of B (a subspace of B^*, the space of linear functionals on B), and let $\Omega\colon C_0(\widehat{B}) \to B$ be the Gelfand representation.

As B contains an approximate identity for A, if $n \in N(B)$, then $n^*n, nn^* \in B$. For each $n \in N(B)$, there exists a unique partial homeomorphism α_n with domain

$$\operatorname{dom}(n) := \left\{x \in \widehat{B} \,\middle|\, \Omega^{-1}(n^*n)(x) = x(n^*n) > 0\right\}$$

and with codomain $\operatorname{dom}(n^*)$ that satisfies

$$n^*\Omega(f)n = \Omega(f \circ \alpha_n)\, n^*n \tag{1}$$

for all $f \in C_0(\widehat{B})$. If $n, m \in N(B)$, then $\alpha_n \circ \alpha_m = \alpha_{nm}$ and $\alpha_{n^*} = \alpha_n^{-1}$ (cf. [14, Corollary 1.7]).

The *Weyl groupoid* $\mathcal{G}_{(A,B)}$ is the quotient of

$$D := \left\{(\alpha_n(x), n, x) \mid n \in N(B), x \in \operatorname{dom}(n)\right\}$$

by the equivalence relation

$$(\alpha_n(x), n, x) \sim (\alpha_m(y), m, y) \iff$$
$$x = y \text{ and } \alpha_n|_U = \alpha_m|_U \text{ for an open neighborhood } U \subseteq \widehat{B} \text{ of } x.$$

We will denote the equivalence class of $(\alpha_n(x), n, x)$ by $[\alpha_n(x), n, x]$. The groupoid structure on $\mathcal{G}_{(A,B)}$ is defined by

$$[\alpha_n(\alpha_m(x)), n, \alpha_m(x)] \cdot [\alpha_m(x), m, x], = [\alpha_{nm}(x), nm, x], \text{ and}$$
$$[\alpha_m(x), m, x]^{-1} = [x, m^*, \alpha_m(x)].$$

To topologize $\mathcal{G}_{(A,B)}$, we define a basic open set to be of the form $\{[\alpha_n(x), n, x] : \alpha_n(x) \in V, x \in U\}$, where $U, V \subseteq \widehat{B}$ are open and $n \in N(B)$ [21, Section 3]. It follows from the remark at the bottom of page 971 in [14] that $x \in \mathrm{dom}(n)$ if and only if $\Omega^{-1}(n^*n)$ does not vanish at x.

Remark 1. For each $x \in \mathrm{dom}(n)$, we have $\alpha_n(x)(nn^*) = x(n^*n)$. Indeed, if $b \in B$, then $x(n^*bn) = \alpha_n(x)(b)\, x(n^*n)$. In particular, for $b = nn^*$, the fact that each functional $x \in \widehat{B}$ is multiplicative implies that

$$x(n^*n)\, x(n^*n) = x(n^*(nn^*)n) = \alpha_n(x)(nn^*) \cdot x(n^*n).$$

Since $x \in \mathrm{dom}(n)$, we can divide by $x(n^*n)$ and get $0 \neq x(n^*n) = \alpha_n(x)(nn^*)$ as claimed.

The *Weyl twist* $\Sigma_{(A,B)}$ is another groupoid associated to the Cartan pair (A, B). Like the Weyl groupoid, the Weyl twist is also a quotient of D, but by the equivalence relation

$$(\alpha_n(x), n, x) \approx (\alpha_m(y), m, y) \iff$$
$$x = y \text{ and } \exists\ b, b' \in B \text{ such that } x(b), x(b') > 0 \text{ and } nb = mb'.$$

We write $[\![\alpha_n(x), n, x]\!]$ for the class of $(\alpha_n(x), n, x)$ in $\Sigma_{(A,B)}$. We point out that equivalence with respect to $\approx$ implies equivalence with respect to $\sim$.

As its name suggests, the Weyl twist is a $\mathbb{T}$-groupoid, or twist, over $\mathcal{G}_{(A,B)}$. As such, one can construct the twisted groupoid C^*-algebra $C_r^*\left(\mathcal{G}_{(A,B)}, \Sigma_{(A,B)}\right)$ (cf. [21,

Section 4]). The Weyl twist and groupoid are constructed exactly so that

$$(A,B) \cong \left(C_r^*\left(\mathcal{G}_{(A,B)}, \Sigma_{(A,B)}\right), C_0\left(\mathcal{G}_{(A,B)}^{(0)}\right)\right),$$

see [21, Theorem 5.9].

We recall the construction of a twisted groupoid C^*-algebra in the case when the twist arises from a 2-cocycle, as this is the level of generality we will need in this paper. Recall that a *($\mathbb{T}$-valued) 2-cocycle* on a groupoid $\mathcal{G}$ is a function $c\colon \mathcal{G}^{(2)} \to \mathbb{T}$, which satisfies the *cocycle condition*

$$c(x,yz)\,c(y,z) = c(x,y)\,c(xy,z) \text{ for all } (x,y),(y,z) \in \mathcal{G}^{(2)}. \tag{2}$$

Given a second countable, locally compact Hausdorff, étale groupoid $\mathcal{G}$ and a continuous 2-cocycle $c\colon \mathcal{G}^{(2)} \to \mathbb{T}$, we denote by $C_c(\mathcal{G},c)$ the collection of continuous, compactly supported $\mathbb{C}$-valued functions on $\mathcal{G}$, which we view as a *-algebra via the twisted convolution multiplication

$$fg(\gamma) := \sum_{\eta\rho=\gamma} f(\eta)g(\rho)c(\eta,\rho)$$

and the involution

$$f^*(\gamma) := \overline{f(\gamma^{-1})c(\gamma,\gamma^{-1})}.$$

For each $u \in \mathcal{G}^{(0)}$, we represent $C_c(\mathcal{G},c)$ on $\ell^2(\mathcal{G}_u)$ by left multiplication:

$$\pi_u(f)\xi := f\xi = \left(\gamma \mapsto \sum_{\eta\rho=\gamma} f(\eta)\xi(\rho)c(\eta,\rho)\right).$$

Let $\|f\|_u$ denote the operator norm of $\pi_u(f)$. The *reduced twisted groupoid C^*-algebra* $C_r^*(\mathcal{G},c)$ is then the completion of $C_c(\mathcal{G},c)$ in the norm $\|\cdot\|_r := \sup_{u\in\mathcal{G}^{(0)}} \|\cdot\|_u$.

Although the definition of the Weyl groupoid and Weyl twist seem quite different, the following very helpful proposition describes the groupoid in terms more similar to the twist. The result may be known to experts, but we were unable to locate it in the literature.

Proposition 2.2. Suppose $n_i \in N(B)$ and $x \in \mathrm{dom}(n_1) \cap \mathrm{dom}(n_2)$. Then, $[\alpha_{n_1}(x), n_1, x] = [\alpha_{n_2}(x), n_2, x]$ if and only if there exist $b_i \in B$ so that $x(b_i) \neq 0$ and $n_1 b_1 = n_2 b_2$.

For the proof, we require two lemmata.

Lemma 2.3. If $n, m \in N(B)$, then

$$\alpha_n \equiv \alpha_m \text{ on } \mathrm{dom}(n) \cap \mathrm{dom}(m) \iff nm^* \in B. \qquad \square$$

Proof. We know that $nm^* \in B$ if and only if $\alpha_{nm^*} = \mathrm{id}_{\mathrm{dom}(nm^*)}$, since B is maximal Abelian. Moreover, $\alpha_{nm^*} = \alpha_n \circ \alpha_{m^*} = \alpha_n \circ \alpha_m^{-1}$. Both things combined yield that $nm^* \in B$ if and only if $\alpha_n \circ \alpha_m^{-1}$ is the identity on $\mathrm{dom}(nm^*)$. Since $n^*n \in B$, we can use the defining property of α_{m^*} (Equation (1)) to rewrite

$$(mn^*nm^*)(x) = (n^*n \circ \alpha_{m^*})(x) mm^*(x),$$

so we have

$$\begin{aligned}\mathrm{dom}(nm^*) &= \{x \in \widehat{B} \mid x \in \mathrm{dom}(m^*) \text{ and } (n^*n \circ \alpha_{m^*})(x) \neq 0\} \\ &= \mathrm{dom}(m^*) \cap \alpha_{m^*}^{-1}\big(\mathrm{dom}(n)\big) = \alpha_m\big(\mathrm{dom}(m) \cap \mathrm{dom}(n)\big).\end{aligned}$$

Thus, $nm^* \in B$ if and only if $\alpha_n \circ \alpha_m^{-1}$ is the identity on $\alpha_m\big(\mathrm{dom}(m) \cap \mathrm{dom}(n)\big)$ if and only if $\alpha_n \equiv \alpha_m$ on $\mathrm{dom}(m) \cap \mathrm{dom}(n)$. ■

We define the *open support* of $k \in C_0(\widehat{B})$ by $\mathrm{supp}'(k) := \{x \in \widehat{B} \mid k(x) \neq 0\}$.

Lemma 2.4 (Urysohn-type Lemma). Let $f \in N(B)$ and suppose that $k \in C_0(\widehat{B})$ has $\mathrm{supp}'(k) \subseteq \mathrm{dom}(f^*)$. Then, the partial homeomorphism associated to $f_2 := \Omega(k)f$ has domain $\alpha_f^{-1}(\mathrm{supp}'(k)) = \alpha_{f^*}(\mathrm{supp}'(k))$, and $\alpha_{f_2} = \alpha_f|_{\mathrm{dom}(f_2)}$.

Proof. First note that f_2 is still a normalizer of B because f is and because $\Omega(k) \in B$; thus, it makes sense to speak of the corresponding partial homeomorphism α_{f_2} and its domain, $\mathrm{dom}(f_2)$.

By definition of f_2, we have $f_2^*f_2 = f^*\Omega(|k|^2)f$. The defining property of α_f (Equation (1) for $|k|^2 \in C_0(\widehat{B})$) yields

$$\Omega^{-1}(f_2^*f_2) = \big(|k|^2 \circ \alpha_f\big) \cdot \Omega^{-1}(f^*f). \tag{3}$$

By assumption, $V := \alpha_f^{-1}(\mathrm{supp}'(k))$ is contained in $\mathrm{dom}(f)$, so the above yields

$$\mathrm{dom}(f_2) = \mathrm{supp}'\big(\Omega^{-1}(f_2^* f_2)\big) = \mathrm{supp}'(k \circ \alpha_f) \cap \mathrm{supp}'\big(\Omega^{-1}(f^* f)\big) = V \cap \mathrm{dom}(f) = V. \tag{4}$$

By the defining property of α_{f_2}, for any $b \in B$, we have

$$\Omega^{-1}(f_2^* b f_2) = (\Omega^{-1}(b) \circ \alpha_{f_2}) \cdot \Omega^{-1}(f_2^* f_2) = (\Omega^{-1}(b) \circ \alpha_{f_2}) \cdot \big(|k|^2 \circ \alpha_f\big) \cdot \Omega^{-1}(f^* f), \tag{5}$$

where the second equality is due to Equation (3). The definition of f_2 implies that

$$f_2^* b f_2 = f^*(\Omega(k)^* b \Omega(k)) f,$$

so that the defining property of α_f (Equation (1)) yields

$$\Omega^{-1}(f_2^* b f_2) = \big((\overline{k} \cdot \Omega^{-1}(b) \cdot k) \circ \alpha_f\big) \cdot \Omega^{-1}(f^* f) = (\Omega^{-1}(b) \circ \alpha_f) \cdot (|k|^2 \circ \alpha_f) \cdot \Omega^{-1}(f^* f).$$

Combining this with Equation (5) reveals that, for any $b \in B$,

$$(\Omega^{-1}(b) \circ \alpha_{f_2}) \cdot \big(|k|^2 \circ \alpha_f\big) \cdot \Omega^{-1}(f^* f) = (\Omega^{-1}(b) \circ \alpha_f) \cdot (|k|^2 \circ \alpha_f) \cdot \Omega^{-1}(f^* f).$$

We conclude that, on $\mathrm{supp}'(\Omega^{-1}(f^* f)) \cap \mathrm{supp}'(|k|^2 \circ \alpha_f)$, we have $\alpha_{f_2} = \alpha_f$. By (4), it follows that $\alpha_{f_2} = \alpha_f$ on $\mathrm{dom}(f_2) \subseteq \mathrm{dom}(f)$. ■

Proof of Proposition 2.2. First, fix $x \in \mathrm{dom}(n_1) \cap \mathrm{dom}(n_2)$ and assume that $n_1 b_1 = n_2 b_2$ for some $b_i \in B$ such that $x(b_i) \neq 0$. In particular, there exists a neighborhood X of x in $\mathrm{dom}(n_1) \cap \mathrm{dom}(n_2)$ such that, for all $y \in X$, we have $y(b_i) \neq 0$. If g is any element of $C_0(\widehat{B})$, then by the defining property of α_{n_i} (Equation (1)) and since B is commutative, we have

$$\begin{aligned}(n_i b_i)^* \, \Omega(g) \, (n_i b_i) &= b_i^* \, (n_i^* \Omega(g) n_i) \, b_i \\ &= b_i^* \Omega(g \circ \alpha_{n_i})(n_i^* n_i) b_i = \Omega(g \circ \alpha_{n_i})(n_i^* n_i)(b_i^* b_i),\end{aligned}$$

so that the equality $n_1 b_1 = n_2 b_2$ implies that

$$\Omega(g \circ \alpha_{n_1})(n_1^* n_1)(b_1^* b_1) = \Omega(g \circ \alpha_{n_2})(n_2^* n_2)(b_2^* b_2).$$

Evaluating both sides at $y \in X$ yields

$$g\big(\alpha_{n_1}(y)\big)\, y(n_1^* n_1)\ \big|y(b_1)\big|^2 = g\big(\alpha_{n_2}(y)\big)\, y(n_2^* n_2)\ \big|y(b_2)\big|^2.$$

By our construction of X, we have $y(b_i) \neq 0$ and also $X \subseteq \mathrm{dom}(n_i)$, so that $y(n_i^* n_i) > 0$. We have shown that, for any $g \in C_0(\widehat{B})$, $g\left(\alpha_{n_1}(y)\right)$ is a positive multiple of $g\left(\alpha_{n_2}(y)\right)$. Since $C_0(\widehat{B})$ separates points (as $\widehat{B}$ is Hausdorff), this implies $\alpha_{n_1}(y) = \alpha_{n_2}(y)$ for all $y \in X$. As X is an open neighborhood of x, we arrive at the claimed equality in the Weyl groupoid.

Conversely, assume that

$$[\alpha_{n_1}(x), n_1, x] = [\alpha_{n_2}(x), n_2, x].$$

We will construct $b_1, b_2 \in B$ such that $x(b_i) \neq 0$ and $n_1 b_1 = n_2 b_2$.

By assumption, there exists a neighborhood X of x in $\mathrm{dom}(n_1) \cap \mathrm{dom}(n_2)$ on which α_{n_1} and α_{n_2} agree. Let $Y := \alpha_{n_1}(X) = \alpha_{n_2}(X)$ and note that $\alpha_{n_1^*}(Y) = \alpha_{n_2^*}(Y) = X$. As X is an open neighborhood of x, Urysohn's Lemma (see, for example, [9, 4.32]) implies the existence of $k \in C_0(\widehat{B})$ with $k(x) = 1$ and $\mathrm{supp}'(k) \subseteq X$. By our choice of X,

$$y \in \mathrm{supp}'(k) \implies \alpha_{n_1}(y) = \alpha_{n_2}(y), \text{ i.e., } y = \alpha_{n_1^*}\big(\alpha_{n_2}(y)\big). \tag{6}$$

By Lemma 2.4, we know that

$$m_i := \Omega(k) n_i^* \in N(B)$$

has $\mathrm{dom}(m_i) = \alpha_{n_i}(\mathrm{supp}'(k)) \subseteq Y \subseteq \mathrm{dom}(n_i^*)$ and that α_{m_i} is extended by $\alpha_{n_i^*}$. In particular, it follows from Implication (6) that

$$\mathrm{dom}(m_1) = \alpha_{n_1}(\mathrm{supp}'(k)) \overset{(6)}{=} \alpha_{n_2}(\mathrm{supp}'(k)) = \mathrm{dom}(m_2) \subseteq Y.$$

This means that $\alpha_{m_1} = \alpha_{n_1^*}|_{\mathrm{dom}(m_1)} = \alpha_{n_2^*}|_{\mathrm{dom}(m_2)} = \alpha_{m_2}$ on all of $\mathrm{dom}(m_1) = \mathrm{dom}(m_2)$. By Lemma 2.3, we conclude that

$$b_1 := m_1 m_2^* = \Omega(k) n_1^* n_2 \Omega(\overline{k}) \tag{7}$$

is an element of B. To see that $x(b_1) \neq 0$, use the defining property of $\alpha_{m_2^*}$ (Equation (1)) to write

$$x(b_1^* b_1) = x(m_2 m_1^* m_1 m_2^*) = x(m_2 m_2^*) \cdot \alpha_{m_2^*}(x)(m_1^* m_1).$$

Since $\mathrm{supp}'(k) \subseteq \mathrm{dom}(n_i)$ and $\alpha_{n_i^*}$ extends α_{m_i},

$$\mathrm{supp}'(k) = \alpha_{n_i^*}(\alpha_{n_i}(\mathrm{supp}'(k))) = \alpha_{n_i^*}(\mathrm{dom}(m_i)) = \alpha_{m_i}(\mathrm{dom}(m_i)) = \mathrm{dom}(m_i^*).$$

Therefore, $x \in \mathrm{supp}'(k) = \mathrm{dom}(m_2^*)$ implies $x(m_2 m_2^*) > 0$. Moreover, $\alpha_{m_2^*}(x) = \alpha_{m_1^*}(x) \in \mathrm{dom}(m_1)$, so $\alpha_{m_2^*}(x)(m_1^* m_1) > 0$. Consequently, $x(b_1^* b_1) > 0$, so $x(b_1) \neq 0$.

By the defining property of $\alpha_{n_1^*}$, we have

$$n_1 b_1 = n_1(\Omega(k) n_1^* m_2^*) = \Omega(k \circ \alpha_{n_1^*})\,(n_1 n_1^*)\, m_2^* = \Omega(k \circ \alpha_{n_1^*})(n_1 n_1^*) n_2 \Omega(\overline{k}). \tag{8}$$

Our goal is to rewrite the right-hand side of Equation (8) in the form $n_2 b_2$ for some $b_2 \in B$ such that $x(b_2) \neq 0$. Note that if

$$f := (k \circ \alpha_{n_1^*}) \cdot \Omega^{-1}(n_1 n_1^*),$$

then $f \in C_0(\widehat{B})$, Equation (8) becomes $n_1 b_1 = \Omega(f) m_2^*$, and f is supported in $\alpha_{n_1}(\mathrm{supp}'(k)) = \alpha_{n_2}(\mathrm{supp}'(k)) \subseteq Y$.

As $\mathrm{supp}'(k) \subseteq X \subseteq \mathrm{dom}(n_2)$, and $\alpha_{n_2} \circ \alpha_{n_2^*}(y) = y$ for all $y \in Y = \alpha_{n_2}(X)$, we have $n_2 \Omega(f \circ \alpha_{n_2}) = \Omega(f) n_2$ by [8, Lemma 4.2]. Equation (8) can therefore be rewritten as

$$n_1 b_1 = n_2 \Omega(f \circ \alpha_{n_2}) \Omega(\overline{k}).$$

Setting $b_2 = \Omega((f \circ \alpha_{n_2})\overline{k})$, we have $n_1 b_1 = n_2 b_2$.

We now complete the proof by showing that $x(b_2) > 0$. As $k(x) = 1$ by construction, it suffices to show that $f(\alpha_{n_2}(x)) > 0$. Our construction of $X \ni x$ implies that $k(\alpha_{n_1^*}(\alpha_{n_2}(x))) = k(x) = 1$ and that

$$\Omega^{-1}(n_1 n_1^*)(\alpha_{n_2}(x)) = \Omega^{-1}(n_1 n_1^*)(\alpha_{n_1}(x)) = x(n_1^* n_1) > 0,$$

where the last equality follows from Remark 1. Thus, $f(\alpha_{n_2}(x)) = [(k \circ \alpha_{n_1^*}) \cdot \Omega^{-1}(n_1 n_1^*)]$ $(\alpha_{n_2}(x)) > 0$, as desired. ∎

3 The Spectrum of a Twisted Bundle of Groups

Assume that $\mathcal{S}$ is a second countable, locally compact Hausdorff, étale groupoid and that $c\colon \mathcal{S}^{(2)} \to \mathbb{T}$ is a 2-cocycle on $\mathcal{S}$. We will always assume that 2-cocycles are *normalized*, i.e., $c(r(a), a) = 1 = c(a, s(a))$ for each $a \in \mathcal{S}$. In order to construct the twisted groupoid C^*-algebra $C_r^*(\mathcal{S}, c)$, we will need c to be continuous, so we will frequently impose this assumption.

In this section, we will be interested in bundles of groups, so on top of our topological assumptions above, assume that the range and source maps of $\mathcal{S}$ are equal, called $p\colon \mathcal{S} \to \mathcal{S}^{(0)}$. We write $\mathcal{S}_u := p^{-1}(\{u\})$ for u a unit. Moreover, assume that the multiplication map $\mathcal{S}^{(2)} \to \mathcal{S}$ is commutative and that the continuous 2-cocycle c is symmetric on $\mathcal{S}$, i.e., $c(a, a') = c(a', a)$ for all $a, a' \in \mathcal{S}_u$, so that its reduced twisted C^*-algebra $B := C_r^*(\mathcal{S}, c)$ is commutative by [8, Lemma 3.5].

Remark 2. Since $C_r^*(\mathcal{S}, c)$ is nuclear (being commutative) and since $\mathcal{S}$ is locally compact Hausdorff étale, it follows from [23, Theorem 5.4] that $\mathcal{S}$ is amenable. In particular, [1, Corollary 4.3] implies that $C^*(\mathcal{S}, c) \cong C_r^*(\mathcal{S}, c)$.

Definition 3.1. Given $u \in \mathcal{S}^{(0)}$ and a continuous 2-cocycle c on $\mathcal{S}$, let $\mathfrak{B}_u$ denote the set of one-dimensional c-projective representations of the Abelian group $\mathcal{S}_u$. That is, $\mathfrak{B}_u$ consists of maps $\chi : \mathcal{S}_u \to \mathbb{T}$ such that

$$\chi(a)\chi(a') = c(a, a')\chi(aa'). \tag{9}$$

Write $\mathfrak{B} = \bigsqcup_{u \in \mathcal{S}^{(0)}} \mathfrak{B}_u$ and $\rho\colon \mathfrak{B} \to \mathcal{S}^{(0)}$ for the projection map.

Remark 3. It is well known (cf. [13, Lemma 7.2], [6, Corollary 3 to Proposition 18.4]) that for a countable discrete Abelian group $\mathcal{S}_u$, every symmetric 2-cocycle on $\mathcal{S}_u$ is cohomologous to a 2-coboundary. Consequently, $\mathfrak{B}_u \cong C_r^*(\mathcal{S}_u) \cong \widehat{\mathcal{S}_u}$. However, in this paper, our main focus is the topological space $\mathfrak{B} = \bigsqcup_{u \in \mathcal{G}^{(0)}} \mathfrak{B}_u$. The proofs of the identification $\mathfrak{B}_u \cong \widehat{\mathcal{S}_u}$ that we have found in the literature are not sufficiently explicit to analyze how the fibers piece together, and so we have chosen to use the explicit description of $\mathfrak{B}_u$ given above.

Remark 4. Observe that for $\chi \in \mathfrak{B}_u$ and $a \in \mathcal{S}_u$,

$$\chi(a)\overline{c(a, a^{-1})} = \overline{\chi(a^{-1})}\chi\big(p(a)\big) = \overline{\chi(a^{-1})}, \tag{10}$$

where we used that $\chi(u) = 1$ since $c(a, u) = 1 = c(u, a)$.

Recall that the *spectrum* $\widehat{C}$ of a commutative C^*-algebra C is the set of nondegenerate one-dimensional representations of C, equipped with the weak-* topology. As $B = C_r^*(\mathcal{S}, c)$ is commutative, the Gelfand–Naimark Theorem yields $B \cong C_0(\widehat{B})$. We will show that $\widehat{B} \cong \mathfrak{B}$ for a suitable topology on $\mathfrak{B}$.

Lemma 3.2. For $\chi \in \mathfrak{B}$, let

$$\phi_\chi : C_c(\mathcal{S}, c) \to \mathbb{C}, \quad \phi_\chi(f) := \sum_{a \in \mathcal{S}_{\rho(\chi)}} \chi(a) f(a). \tag{11}$$

Then, ϕ_χ is a *-algebra homomorphism that extends to an element of $\widehat{B}$. Moreover, the map $\phi : \mathfrak{B} \to \widehat{B}$, $\chi \mapsto \phi_\chi$ is a bijection.

Remark 5. It is unclear to the authors whether the formula for ϕ_χ in Lemma 3.2 extends to elements of B when thought of as C_0-functions on $\mathcal{S}$. In particular, if $b \in B \subseteq C_0(\mathcal{S})$ with $\phi_\chi(b) \neq 0$, does it then follow that $\mathrm{supp}'(b) \cap \mathcal{S}_{\rho(\chi)} \neq \emptyset$?

Proof. Observe that B is a $C_0(\mathcal{S}^{(0)})$-algebra (cf. [7, Remark 5.1]). Then, [24, Proposition C.5] implies that $\widehat{B} = \bigsqcup_{u \in \mathcal{S}^{(0)}} \widehat{B(u)}$. When c is trivial, amenability of $\mathcal{S}$ (Remark 2) and [7, Remark 5.2] imply that $B(u) \cong C_r^*(\mathcal{S}_u, c)$. In fact, a careful examination of the proof of [7, Remark 5.2] reveals that, even if c is not trivial, the formulae used there will also give an isomorphism between the fiber algebra $B(u)$ and the twisted group C^*-algebra $C_r^*(\mathcal{S}_u, c)$.

It is a classical fact (cf. [5, Theorem 3.3(2)]) that unitary projective representations of $\mathcal{S}_u$ are in bijection with representations of the twisted group C^*-algebra $C^*(\mathcal{S}_u, c) \cong C_r^*(\mathcal{S}_u, c) \cong B(u)$. For a one-dimensional projective representation $\chi \in \mathfrak{B}_u$, the corresponding element of $\widehat{B(u)} \subseteq \widehat{B}$ is indeed given on $C_c(\mathcal{S}_u, c) \subseteq B(u)$ by the formula in Equation (11) (cf. [24, pp. 386–7]). In other words, for each u, $\phi|_{\mathfrak{B}_u}$ is a bijection $\mathfrak{B}_u \to \widehat{B(u)}$, and hence ϕ is also bijective as a map $\mathfrak{B} \to \widehat{B}$. ∎

Proposition 3.3. If we equip $\mathfrak{B}$ with the topology induced by ϕ from $\widehat{B}$, then a net $(\chi_i)_i$ converges to χ in $\mathfrak{B}$ if and only if the following two conditions hold:

1. $\rho(\chi_i) \to \rho(\chi)$ in $\mathcal{S}^{(0)}$;
2. whenever $(a_i)_i$ is a net in $\mathcal{S}$ that satisfies $p(a_i) = \rho(\chi_i)$, $p(a) = \rho(\chi)$, and $a_i \to a$ in $\mathcal{S}$, then $\chi_i(a_i) \to \chi(a)$ in $\mathbb{T}$.

Remark 6. Note that, with respect to this topology on $\mathfrak{B}$, $\rho\colon \mathfrak{B} \to \mathcal{S}^{(0)}$ is clearly continuous. Note further that, when c is trivial, this result is well known; the description of the topology on $\mathfrak{B}$ should be compared with [18, Proposition 3.3].

The proof of Proposition 3.3 proceeds through a series of lemmata.

Lemma 3.4. Suppose $(\chi_i)_{i\in I}$ and χ are elements of $\mathfrak{B}$ satisfying Condition 1 of Proposition 3.3, and suppose $f \in C_c(\mathcal{S}, c)$ is supported in a bisection. If $\mathrm{supp}'(f) \cap \mathcal{S}_{\rho(\chi)} = \emptyset$, then $\phi_{\chi_i}(f) \to 0$.

Proof. We will prove the contrapositive. Let $u_i := \rho(\chi_i)$ and assume that $\phi_{\chi_i}(f) \not\to 0$, i.e., there exists $\epsilon > 0$, so that for all $i \in I$, there exists $g(i) \in I$ with $g(i) \geq i$ such that $\left|\phi_{\chi_{g(i)}}(f)\right| > \epsilon$. This implies that, for each j in $J := \{g(i) \mid i \in I\}$, there exists a unique $a_j \in \mathrm{supp}'(f) \cap \mathcal{S}_{u_j}$ such that $\epsilon < \left|\phi_{\chi_j}(f)\right| = \left|\sum_{a\in\mathcal{S}_{u_j}} \chi_j(a)f(a)\right| = |f(a_j)|$.

Note that J is a directed set when equipped with the preorder of I: if we take $j_1, j_2 \in J$, then since I is directed, there exists $i \in I$ with $i \geq j_1, j_2$. Then, $g(i) \geq i$, so $g(i)$ is an upper bound for j_1 and j_2 in J. First, this implies that $(u_j)_{j\in J}$ is a subnet of $(u_i)_{i\in I}$ (the inclusion $J \hookrightarrow I$ is monotone and final), so that Condition 1 of Proposition 3.3 implies $\lim_{j\in J} u_j = \lim_{i\in I} u_i = \rho(\chi)$. Second, we conclude that $(a_j)_{j\in J}$ is a net in $\mathrm{supp}'(f)$. Since $\mathrm{supp}(f)$ is compact, there exists a subnet $(a_\kappa)_{\kappa\in K}$ of $(a_j)_{j\in J}$ which converges to some $a \in \mathrm{supp}(f)$. By continuity of f, we have

$$|f(a)| = \lim_\kappa \left|f(a_\kappa)\right| \geq \epsilon,$$

i.e., $a \in \mathrm{supp}'(f)$. Moreover, $p(a) = \lim_\kappa p(a_\kappa) = \lim_\kappa u_\kappa = \rho(\chi)$, i.e., $a \in \mathcal{S}_{\rho(\chi)}$, and so $\mathrm{supp}'(f) \cap \mathcal{S}_{\rho(\chi)} \neq \emptyset$. ∎

Lemma 3.5. Suppose $(\chi_i)_{i\in I}$ and χ are elements of $\mathfrak{B}$ satisfying Conditions 1 and 2 of Proposition 3.3. Then, $\phi_{\chi_i} \to \phi_\chi$ in $\widehat{B}$.

Proof. We must show that, for all $f \in C_c(\mathcal{S}, c)$ and for all $\epsilon > 0$, there exists $i_0 \in I$ such that, if $i \geq i_0$, then $|\phi_{\chi_i}(f) - \phi_\chi(f)| < \epsilon$.

We will begin by proving the claim for $f \in C_c(\mathcal{S}, c)$ such that $\mathrm{supp}(f)$ is a bisection. Let $u := \rho(\chi)$ and $u_i := \rho(\chi_i)$. If $\mathrm{supp}'(f) \cap \mathcal{S}_u = \emptyset$, then $\phi_\chi(f) = 0$ and Lemma 3.4 yields $\phi_{\chi_i}(f) \to 0 = \phi_\chi(f)$, as claimed. Otherwise, fix $\epsilon > 0$ and let $a \in \mathrm{supp}'(f)$ such that $p(a) = u$. Since $f(a) \neq 0$ and f is continuous, there exists an open neighborhood

V of a on which f is nonzero. In fact, V is a bisection around a because $V \subseteq \operatorname{supp}'(f)$, and $p(V)$ is an open neighborhood of u because $\mathcal{S}$ is étale. As $u_i \to u$, by shrinking the neighborhood V, we see that $\{a_i \in V \mid p(a_i) = u_i\}$ is a net in $\mathcal{S}$ converging to a. In this case,

$$\begin{aligned}\left|\phi_{\chi_i}(f) - \phi_\chi(f)\right| &= \left|\sum_{a_i' \in \mathcal{S}_{u_i}} \chi_i(a_i') f(a_i') - \sum_{a' \in \mathcal{S}_u} \chi(a') f(a')\right| \\ &= \left|\chi_i(a_i) f(a_i) - \chi(a) f(a)\right|.\end{aligned}$$

If we now use the fact that $f \in C_c(\mathcal{S}, c)$ is bounded in $\|\cdot\|_\infty$ and our hypothesis that $(\chi_i)_i$ and χ satisfy Condition 2 of Proposition 3.3, an easy $\epsilon/2$-argument establishes that $|\phi_{\chi_i}(f) - \phi_\chi(f)| < \epsilon$ for $i \geq i_0$ for some $i_0 \in I$.

For more general functions, recall that since $\mathcal{S}$ is a second countable, locally compact Hausdorff, étale groupoid, we have

$$C_c(\mathcal{S}, c) = \operatorname{span}\{f \in C_c(\mathcal{S}, c) \mid \operatorname{supp}(f) \text{ is a bisection}\},$$

see [22, Lemma 3.1.3]. An ϵ/k-argument now shows that, for any $g \in C_c(\mathcal{S}, c)$, there exists $i_1 \in I$ so that $i \geq i_1$ implies $|\phi_{\chi_i}(g) - \phi_\chi(g)| < \epsilon$. ■

Lemma 3.6. Let $(\chi_i)_{i \in I}$ be some net in $\mathfrak{B}$ such that $\phi_{\chi_i} \to \phi_\chi$ for some $\chi \in \mathfrak{B}$. Then, $(\chi_i)_i$ and χ satisfy Conditions 1 and 2 of Proposition 3.3.

Proof. Recall that our assumption $\phi_{\chi_i} \to \phi_\chi$ means that, for every $f \in C_c(\mathcal{S}, c)$ and $\epsilon > 0$, there exists $N_{f,\epsilon} \in I$ such that if $i \geq N_{f,\epsilon}$, then $|\phi_{\chi_i}(f) - \phi_\chi(f)| < \epsilon$.

We start by proving Condition 1 of Proposition 3.3. Let $u := \rho(\chi)$ and $u_i := \rho(\chi_i)$. Recall that the unit space in a Hausdorff étale groupoid is clopen [22, Lemmas 2.3.2 and 2.4.2], so that $u_i \to u$ in $\mathcal{S}^{(0)}$ if and only if $u_i \to u$ in $\mathcal{S}$. Let $V \subseteq \mathcal{S}^{(0)}$ be an open neighborhood of u. Since $\mathcal{S}$ is locally compact Hausdorff and V is an open neighborhood of u in $\mathcal{S}$, there exists by Urysohn's Lemma a function f in $C_c(\mathcal{S}, c)$ with $f(u) = 1$ and $f_{|\mathcal{S} \setminus V} \equiv 0$. Since $\phi_{\chi_i} \to \phi_\chi$, then for any fixed $1 > \epsilon > 0$, there exists an $M \in I$ such that if $i \geq M$, then

$$|\phi_{\chi_i}(f) - \phi_\chi(f)| < \epsilon < 1,$$

which, by definition of ϕ, implies

$$\left| \sum_{a' \in \mathcal{S}_{u_i}} \chi_i(a') f(a') - \sum_{a' \in \mathcal{S}_u} \chi(a') f(a') \right| < 1.$$

As $\mathcal{S}^{(0)} \supseteq V$ is a bisection containing $\mathrm{supp}'(f)$, and $\xi(v) = 1$ for any $\xi \in \mathfrak{B}$ and $v \in \mathcal{S}^{(0)}$, the above inequality becomes

$$\left|\chi_i(u_i) f(u_i) - \chi(u) f(u)\right| = \left|f(u_i) - 1\right| < 1,$$

for all $i \geq M$. Therefore, if $i \geq M$, then $u_i \in \mathrm{supp}'(f) \subseteq V$. This concludes the proof of Condition 1.

We proceed with proving Condition 2. Suppose $(a_i)_i$ is a net in $\mathcal{S}$ such that $p(a_i) = u_i$, $p(a) = u$, and $a_i \to a$. Fix $\epsilon > 0$. We must show there exists $M \in I$ such that if $i \geq M$, then $|\chi_i(a_i) - \chi(a)| < \epsilon$. By [22, Lemma 2.4.9], there exists an open bisection W in $\mathcal{S}$ that contains a. Since $\mathcal{S}$ is locally compact Hausdorff, there exists, by [9, Proposition 4.31], a precompact open set U with $a \in U \subseteq \overline{U} \subseteq W$. Since $a_i \to a$, there exists $N \in I$, such that if $i \geq N$, then $a_i \in U$.

Again by Urysohn's Lemma, there exists $f \in C_c(\mathcal{S}, c)$, which is equal to 1 on $\overline{U}$ and 0 outside of W. So for all i in I, which are larger than both N and $N_{f,\epsilon}$, we know $a_i \in U \subseteq \mathrm{supp}(f)$ and

$$\left| \sum_{a' \in \mathcal{S}_{u_i}} \chi_i(a') f(a') - \sum_{a' \in \mathcal{S}_u} \chi(a') f(a') \right| < \epsilon.$$

Note that W is a bisection, and a_i, a are elements of $U \subseteq W$ with $p(a_i) = u_i$ and $p(a) = u$. All of these facts combined yield that a_i is the unique element in $\mathcal{S}_{u_i} \cap U$ and a is the unique element in $\mathcal{S}_u \cap U$. Since f is equal to 1 on U, the inequality becomes

$$|\chi_i(a_i) - \chi(a)| < \epsilon.$$

This completes the proof of the lemma and of Proposition 3.3. ■

Corollary 3.7. The map $\phi\colon \mathfrak{B} \to \widehat{B}$ defined in Lemma 3.2 is a homeomorphism when $\mathfrak{B}$ has the topology induced by ϕ as described in Proposition 3.3. In particular, $\mathfrak{B}$ is locally compact Hausdorff and B is isomorphic to the C^*-algebra $C_0(\mathfrak{B})$.

4 Computing the Weyl Groupoid

Our standing assumptions for the remainder of this paper are the following:

1. $\mathcal{G}$ is a second countable, locally compact Hausdorff, étale groupoid;
2. c is a normalized, continuous $\mathbb{T}$-valued 2-cocycle on $\mathcal{G}$;
3. $\mathcal{S} \subseteq \mathrm{Iso}(\mathcal{G})$ is an Abelian subgroupoid, containing $\mathcal{G}^{(0)}$, on which c is symmetric;
4. $\mathcal{S}$ is clopen and normal in $\mathcal{G}$; and
5. $\mathcal{S}$ is chosen such that $B := C_r^*(\mathcal{S}, c)$ is maximal Abelian in $A := C_r^*(\mathcal{G}, c)$.

Note that Assumptions 1, 2, and 3 make sure that $(\mathcal{S}, c)$ falls into the scope of Section 3; in particular, B is a commutative algebra, and $\mathfrak{B}$ is its spectrum, which comes with the map $\rho\colon \mathfrak{B} \to \mathcal{S}^{(0)} = \mathcal{G}^{(0)}$. As $\mathcal{S}$ is clopen in $\mathcal{G}$, B is naturally a subalgebra of A, and the map $\Phi\colon C_c(\mathcal{G}, c) \to C_c(\mathcal{S}, c)$ defined by

$$\Phi(f) = f|_{\mathcal{S}} \tag{12}$$

extends to a faithful conditional expectation $A \to B$, which we will also denote by Φ, see [8, Proposition 3.13]. Normality of $\mathcal{S}$ implies that B is regular in A. Furthermore, B contains an approximate unit for A because $\mathcal{G}^{(0)} \subseteq \mathcal{S}$. Thus, Assumptions 1–5 make B a Cartan subalgebra of A.

Let us explain why our last assumption on $\mathcal{S}$ is reasonable. It was shown in [8, Theorem 3.1] that, in order to get Assumption 5, a sufficient assumption on $\mathcal{S}$ is that (1) $\mathcal{S}$ is maximal among the Abelian subgroupoids of $\mathrm{Iso}(\mathcal{G})$ on which c is symmetric, and additionally (2) $\mathcal{S}$ is *immediately centralizing* [8, Definition 2]. A careful examination of the proof of [8, Proposition 3.9] reveals that, instead of (2), we may assume that for each $\eta \in \mathrm{Iso}(\mathcal{G})$ with $u = r(\eta) = s(\eta)$, the set $\{a\eta a^{-1} \mid a \in \mathcal{S}_u\}$ is either the singleton $\{\eta\}$ or infinite.

In the sections about to come, we will use the techniques we have developed so far to compute the Weyl groupoid $\mathcal{G}_{(A,B)}$ and the Weyl twist $\Sigma_{(A,B)}$ of the Cartan pair (A, B). In particular, we will see in Theorems 4.6 and 5.1 that there is a strong connection to a certain groupoid action of $\mathcal{G}/\mathcal{S}$ on $\mathfrak{B}$. As such, it seems prudent to briefly state a few facts about the quotient groupoid $\mathcal{G}/\mathcal{S}$.

Remark 7. We let $q\colon \mathcal{G} \to \mathcal{G}/\mathcal{S} =: \mathcal{Q}, \gamma \mapsto q(\gamma) =: \dot{\gamma}$, denote the quotient map. Since $\mathcal{S}$ is a *wide* subgroupoid of $\mathcal{G}$ (i.e., $\mathcal{S} \subseteq \mathrm{Iso}(\mathcal{G})$ is closed with $\mathcal{S}^{(0)} = \mathcal{G}^{(0)}$), openness of $\mathcal{S}$ and étaleness of $\mathcal{G}$ imply that q is an open map. Since $\mathcal{G}$ is Hausdorff and $\mathcal{S}$ is closed in $\mathcal{G}$,

this implies that $\mathcal{Q}$ is Hausdorff also. Furthermore, it follows from [25, Corollary 2.13] (taking $G = \mathcal{S}$ and $X = \mathcal{G}$) that $\mathcal{Q}$ is locally compact and second countable because $\mathcal{G}$ is.

Lastly, we point out that, if one is interested in groupoids $\mathcal{S} \subseteq \mathcal{G}$ with étale quotient $\mathcal{Q}$ (as in [12, Theorem 4.6]), then one must ask for $\mathcal{S}$ to be open in $\mathcal{G}$, as we have done.

We will now construct a continuous left action $\tilde{\alpha}$ of the locally compact Hausdorff groupoid $\mathcal{Q} = \mathcal{G}/\mathcal{S}$ on the spectrum $\mathfrak{B}$ of $B = C_r^*(\mathcal{S}, c)$, with the moment map $\rho : \mathfrak{B} \to \mathcal{Q}^{(0)}$ given by $\rho|_{\mathfrak{B}_u} = u$. In the following, we will write

$$\mathcal{Q} *_\rho \mathfrak{B} := \{(\dot{\gamma}, \chi) \in \mathcal{Q} \times \mathfrak{B} \mid s_{\mathcal{Q}}(\dot{\gamma}) = \rho(\chi)\}.$$

Proposition 4.1. Let $\rho\colon \mathfrak{B} \to \mathcal{Q}^{(0)}$ be given by $\rho|_{\mathfrak{B}_u} = \mathrm{const}_u$. For $\gamma \in \mathcal{G}$, $(\dot{\gamma}, \chi) \in \mathcal{Q} *_\rho \mathfrak{B}$, and $a \in \mathcal{S}_{r(\gamma)}$, define

$$\tilde{\alpha}_{\dot{\gamma}}(\chi)(a) := \overline{c(\gamma, \gamma^{-1})}\, c(\gamma^{-1}, a)\, c(\gamma^{-1}a, \gamma)\ \chi(\gamma^{-1}a\gamma).$$

Then

(1) $\tilde{\alpha}_{\dot{\gamma}}(\chi)$ only depends on $\dot{\gamma} \in \mathcal{Q} = \mathcal{G}/\mathcal{S}$, not on $\gamma \in \mathcal{G}$.
(2) $\tilde{\alpha}_{\dot{\gamma}}(\chi) \in \mathfrak{B}_{r(\gamma)}$ and $\tilde{\alpha}_{\dot{\tau}\dot{\gamma}}(\chi) = \tilde{\alpha}_{\dot{\tau}}\big(\tilde{\alpha}_{\dot{\gamma}}(\chi)\big)$.
(3) If $u \in \mathcal{Q}^{(0)}$, then $\tilde{\alpha}_u(\chi) = \chi$ for all $\chi \in \mathfrak{B}_u$.
(4) The map $\mathcal{Q} *_\rho \mathfrak{B} \to \mathfrak{B}$, $(\dot{\gamma}, \chi) \mapsto \tilde{\alpha}_{\dot{\gamma}}(\chi)$, is continuous.

In other words, $\tilde{\alpha}$ is a continuous left action of $\mathcal{Q}$ on $\mathfrak{B}$ with moment map ρ.

Before embarking on the proof, we point out that the formula for $\tilde{\alpha}$ is not surprising. Indeed, if χ were defined on all of $\mathcal{G}$ and satisfied Equation (9) of Definition 3.1 (and, by extension, Equation (10)), then we would have

$$\begin{aligned}\tilde{\alpha}_{\dot{\gamma}}(\chi)(a) &= \overline{c(\gamma, \gamma^{-1})}\, c(\gamma^{-1}, a)\, \chi(\gamma^{-1}a)\, \chi(\gamma) \\ &= \overline{c(\gamma, \gamma^{-1})}\, \chi(\gamma^{-1})\, \chi(a)\, \chi(\gamma) = \chi(a).\end{aligned}$$

Proof. One readily verifies 1–3 using the cocycle identity (Equation (2)), the fact that c is symmetric on the Abelian subgroupoid $\mathcal{S}$, that c is normalized, and that $c(\gamma^{-1}, \gamma) = c(\gamma, \gamma^{-1})$ for any $\gamma \in \mathcal{G}$ by [8, Lemma 2.1].

For 4, suppose that $(\dot{\gamma}_i, \chi_i) \to (\dot{\gamma}, \chi)$ in $\mathcal{Q} *_\rho \mathfrak{B}$. We need to show [18, Proposition 3.3] that, if $a_i \to a$ in $\mathcal{S}$ and $s(a_i) = r(\gamma_i)$ for all i, then $\tilde{\alpha}_{\dot{\gamma}_i}(\chi_i)(a_i) \to \tilde{\alpha}_{\dot{\gamma}}(\chi)(a)$. Since the

cocycle c and both multiplication and inversion on $\mathcal{G}$ are continuous, we have

$$\overline{c(\gamma_i, \gamma_i^{-1})}c(\gamma_i^{-1}, a_i)c(\gamma_i^{-1}a_i, \gamma_i) \to \overline{c(\gamma, \gamma^{-1})}c(\gamma^{-1}, a)c(\gamma^{-1}a, \gamma).$$

Similarly, $\gamma_i^{-1}a_i\gamma_i \to \gamma^{-1}a\gamma$. Since the assumption $\chi_i \to \chi$ in $\mathfrak{B}$ implies in particular that $\chi_i(\gamma_i^{-1}a_i\gamma_i) \to \chi(\gamma^{-1}a\gamma)$, it follows that $\tilde{\alpha}_{\dot{\gamma}_i}(\chi_i)(a_i) \to \tilde{\alpha}_{\dot{\gamma}}(\chi)(a)$. ■

The action $\tilde{\alpha}$ of $\mathcal{Q}$ on $\mathfrak{B}$ allows us to endow the space $\mathcal{Q} *_\rho \mathfrak{B}$ with the structure of a topological groupoid. This so-called *left action groupoid* is denoted $\mathcal{Q} \ltimes \mathfrak{B}$, and we will show in Theorem 4.6 that it is isomorphic to the Weyl groupoid $\mathcal{G}_{(A,B)}$.

Recall [11, p. 3] that the elements $(\dot{\tau}, \chi_1)$, $(\dot{\gamma}, \chi_2) \in \mathcal{Q} \ltimes \mathfrak{B}$ are composable if $\chi_1 = \tilde{\alpha}_{\dot{\gamma}}(\chi_2)$, and their product is given by

$$(\dot{\tau}, \chi_1)(\dot{\gamma}, \chi_2) = (\dot{\tau}\dot{\gamma}, \chi_2).$$

The inverse of an element $(\dot{\gamma}, \chi)$ is $(\dot{\gamma}^{-1}, \tilde{\alpha}_{\dot{\gamma}}(\chi))$. Therefore,

$$(\mathcal{Q} \ltimes \mathfrak{B})^{(0)} = \{(\rho(\chi), \chi) | \chi \in \mathfrak{B}\}.$$

Note that $s(\dot{\gamma}, \chi) = (s(\dot{\gamma}), \chi)$ and $r(\dot{\gamma}, \chi) = (r(\dot{\gamma}), \tilde{\alpha}_{\dot{\gamma}}(\chi))$. The topology of $\mathcal{Q} \ltimes \mathfrak{B}$ is inherited from $\mathcal{Q} \times \mathfrak{B}$; since $\mathcal{Q}$ and $\mathfrak{B}$ are locally compact, so is $\mathcal{Q} \ltimes \mathfrak{B}$, and the fact that $\mathcal{Q}$ is étale and ρ is continuous implies that $\mathcal{Q} \ltimes \mathfrak{B}$ is étale. See [17, p. 5] for more details.

Our next goal will be to describe the relationship between the partial homeomorphisms α_n used to construct the Weyl groupoid $\mathcal{G}_{(A,B)}$ and the action $\tilde{\alpha}$ (see Proposition 4.4). We begin with a few preliminary results.

Lemma 4.2. The set

$$N := \left\{ f \in C_c(\mathcal{G}, c) \,|\, \mathrm{supp}(f) \text{ is a bisection} \right\} \tag{13}$$

is a subset of the normalizer $N(B)$ of B, and every element of the Weyl groupoid associated to (A, B) can be represented by some $(\alpha_n(x), n, x)$ where $n \in N$ and $x \in \mathrm{dom}(n)$.

Proof. By [8, Lemma 3.11], N is contained in the normalizer of $C_c(\mathcal{S}, c)$, which implies $N \subseteq N(B)$ since $B \subseteq A$ is closed. Since $\mathcal{G}$ is a second countable, locally compact

Hausdorff, étale groupoid, it follows from [22, Lemma 3.1.3] that $C_c(\mathcal{G}, c) = \operatorname{span}(N)$. In particular, $\operatorname{span}(N)$ is dense in A. The claim then follows from [8, Proposition 4.1]. ■

A more general variant of the following lemma was obtained in [3, Proposition 4.12]. For the convenience of the reader, we include both the precise statement of the result we need and its proof.

Lemma 4.3. Suppose $f_i \in N$ and $\chi \in \mathfrak{B}_u$. Let $\Phi\colon A \to B$ denote the faithful conditional expectation associated to the Cartan pair $(A, B) = (C_r^*(\mathcal{G}, c), C_r^*(\mathcal{S}, c))$.

1. If $[\alpha_{f_1}(\phi_\chi), f_1, \phi_\chi] = [\alpha_{f_2}(\phi_\chi), f_2, \phi_\chi]$, then $\phi_\chi(\Phi(f_2^* f_1)) \neq 0$.
2. If $[\![\alpha_{f_1}(\phi_\chi), f_1, \phi_\chi]\!] = [\![\alpha_{f_2}(\phi_\chi), f_2, \phi_\chi]\!]$, then $\phi_\chi(\Phi(f_2^* f_1)) > 0$.

Moreover, if $\dot\gamma_i \in q(\operatorname{supp}'(f_i))$ with $s(\dot\gamma_1) = s(\dot\gamma_2) = u$, then either of the above assumptions implies $\dot\gamma_1 = \dot\gamma_2$.

Proof. We start by proving 2. By assumption, $\phi_\chi \in \operatorname{dom}(f_1) \cap \operatorname{dom}(f_2)$ and there exist $b_1, b_2 \in B$ with $\phi_\chi(b_i) > 0$ and such that $f_1 b_1 = f_2 b_2$. In particular, since $f_2 \in N \subseteq N(B)$, it follows that $f_2^* f_1 b_1 = f_2^* f_2 b_2$ is an element of B. Since $\phi_\chi \in \operatorname{dom}(f_2)$, we conclude that

$$\phi_\chi(f_2^* f_1 b_1) = \phi_\chi(f_2^* f_2 b_2) = \phi_\chi(f_2^* f_2)\,\phi_\chi(b_2) > 0.$$

As the conditional expectation Φ fixes B and is B-linear, we get the equality in the following:

$$\phi_\chi\big(\Phi(f_2^* f_1)\big)\,\phi_\chi(b_1) = \phi_\chi(f_2^* f_1 b_1) > 0.$$

It follows that $\phi_\chi\big(\Phi(f_2^* f_1)\big) > 0$, as claimed.

For 1, we use Proposition 2.2 to obtain $b_i \in B$ such that $\phi_\chi(b_i) \neq 0$ and $f_1 b_1 = f_2 b_2$. The above proof now works *mutatis mutandis*, replacing each instance of "> 0" by "$\neq 0$".

Lastly, in either of the two cases, $f_2^* f_1$ is an element of $N \subseteq C_c(\mathcal{G}, c)$. As $\Phi(g) = g|_{\mathcal{S}}$ for $g \in C_c(\mathcal{G}, c)$ (cf. [8, Proposition 3.13]), $\Phi(f_2^* f_1) \in C_c(\mathcal{S}, c)$. Thus, the definition of ϕ_χ yields

$$0 \neq \phi_\chi\big(\Phi(f_2^* f_1)\big) = \sum_{a \in \mathcal{S}_{\rho(\chi)}} \chi(a)\,\Phi(f_2^* f_1)(a) = \sum_{a \in \mathcal{S}_{\rho(\chi)}} \chi(a) f_2^* f_1(a),$$

and consequently

$$\operatorname{supp}'(f_2^* f_1) \cap \mathcal{S}_{\rho(\chi)} \neq \emptyset.$$

If $\dot{\gamma}_i \in q(\operatorname{supp}'(f_i))$ satisfy $s(\dot{\gamma}_1) = s(\dot{\gamma}_2) = u$, let γ_i denote the representative of $\dot{\gamma}_i$ in $\operatorname{supp}'(f_i)$. Since $\gamma_2^{-1}\gamma_1$ is then the unique element in $\operatorname{supp}'(f_2^* f_1)$ with source $\rho(\chi)$, it follows that $\gamma_2^{-1}\gamma_1 \in \mathcal{S}_{\rho(\chi)}$, i.e., $\dot{\gamma}_1 = \dot{\gamma}_2$. ■

Proposition 4.4. Suppose $f \in N$ and $\chi \in \mathfrak{B}$.

(1) $\rho(\chi) \in s(\operatorname{supp}'(f))$ if and only if $\phi_\chi \in \operatorname{dom}(f)$.
(2) If $\{\gamma\} = \operatorname{supp}'(f) \cap \mathcal{G}_{\rho(\chi)}$, so that $\chi \in \operatorname{dom}(\tilde{\alpha}_{\dot{\gamma}})$ and $\phi_\chi \in \operatorname{dom}(f)$, then we have $\alpha_f(\phi_\chi) = \phi_{\tilde{\alpha}_{\dot{\gamma}}(\chi)}$.

Proof. Since f is supported in a bisection, it follows from [8, Lemma 3.11] that f is a normalizer of B, so α_f exists and has domain $\operatorname{dom}(f) \subseteq \widehat{B}$. It then follows from (a twisted variant of) [22, Lemma 3.1.4] that

$$f^* f(\eta) = \begin{cases} |f(\zeta)|^2, & \eta \in \mathcal{G}^{(0)} \text{ and } \operatorname{supp}'(f) \cap \mathcal{G}_\eta = \{\zeta\}, \\ 0, & \text{otherwise.} \end{cases}$$

Using that $\chi(\rho(\chi)) = 1$ and that $f^* f \in C_c(\mathcal{S}, c) \subseteq B$, this implies

$$\phi_\chi(f^* f) = \sum_{\eta \in \mathcal{S}_{\rho(\chi)}} \chi(\eta)\,(f^* f)(\eta) = \begin{cases} |f(\gamma)|^2 & \text{if } \operatorname{supp}'(f) \cap \mathcal{G}_{\rho(\chi)} = \{\gamma\}, \\ 0 & \text{if } \operatorname{supp}'(f) \cap \mathcal{G}_{\rho(\chi)} = \emptyset. \end{cases} \tag{14}$$

This proves 1.

For 2, note first that $\rho(\chi) = s(\gamma)$, so $\chi \in \mathfrak{B}_{\rho(\chi)}$ is automatically an element of $\operatorname{dom}(\tilde{\alpha}_{\dot{\gamma}}) = \mathfrak{B}_{s(\gamma)}$. It remains to prove that $\alpha_f(\phi_\chi) = \phi_{\tilde{\alpha}_{\dot{\gamma}}(\chi)}$. Recall that α_f is uniquely determined by satisfying

$$x(f^* b f) = \alpha_f(x)(b)\, x(f^* f) \text{ for all } b \in C_r^*(\mathcal{S}, c) \text{ and } x \in \widehat{B}.$$

Observe that for $b \in C_c(\mathcal{S}, c)$ and $\eta \in \mathcal{S}$,

$$\begin{aligned} f^*bf(\eta) &= \sum_{\xi \in \mathcal{G}^{s(\eta)}} \overline{f(\xi^{-1}\eta^{-1})c(\eta\xi, \xi^{-1}\eta^{-1})}b * f(\xi^{-1})c(\eta\xi, \xi^{-1}) \\ &= \sum_{\xi \in \mathcal{G}^{s(\eta)}} \sum_{\beta \in \mathcal{G}^{s(\xi^{-1})}} \overline{f(\xi^{-1}\eta^{-1})c(\eta\xi, \xi^{-1}\eta^{-1})}b(\xi^{-1}\beta)f(\beta^{-1})c(\eta\xi, \xi^{-1})c(\xi^{-1}\beta, \beta^{-1}) \\ &= \sum_{\xi, \beta \in \mathcal{G}^{s(\eta)}} \overline{f(\xi^{-1}\eta^{-1})c(\eta\xi, \xi^{-1}\eta^{-1})}b(\xi^{-1}\beta)f(\beta^{-1})c(\eta\xi, \xi^{-1})c(\xi^{-1}\beta, \beta^{-1}). \end{aligned}$$

The factor $b(\xi^{-1}\beta)$ will only be nonzero if $r(\xi^{-1}\beta) = s(\xi^{-1}\beta)$, i.e., $r(\xi^{-1}) = s(\beta)$. In that case, since f is supported in a bisection and since $r(\beta^{-1}) = s(\beta) = r(\xi^{-1}) = r(\xi^{-1}\eta^{-1})$, the only nonzero terms in the sum occur when $\beta = \eta\xi$, so that

$$\begin{aligned} f^*bf(\eta) &= \sum_{\beta \in \mathcal{G}^{r(\eta)}} |f(\beta^{-1})|^2 b(\beta^{-1}\eta\beta)\overline{c(\beta, \beta^{-1})}c(\beta, \beta^{-1}\eta)c(\beta^{-1}\eta\beta, \beta^{-1}) \\ &= \sum_{\zeta \in \mathcal{G}_{r(\eta)}} |f(\zeta)|^2 b(\zeta\eta\zeta^{-1})\overline{c(\zeta^{-1}, \zeta)}c(\zeta^{-1}, \zeta\eta)c(\zeta\eta\zeta^{-1}, \zeta). \end{aligned}$$

In particular, if $\eta \in \mathcal{S}^{\rho(\chi)}_{\rho(\chi)}$—so that $r(\eta) = \rho(\chi)$—then the assumption $\mathrm{supp}'(f) \cap \mathcal{G}_{\rho(\chi)} = \{\gamma\}$ shows that

$$f^*bf(\eta) = |f(\gamma)|^2 b(\gamma\eta\gamma^{-1})\overline{c(\gamma^{-1}, \gamma)}c(\gamma^{-1}, \gamma\eta)c(\gamma\eta\gamma^{-1}, \gamma).$$

Therefore, for $x = \phi_\chi$ and $b \in C_c(\mathcal{S}, c) \subseteq B$,

$$\begin{aligned} \phi_\chi(f^*bf) &= \sum_{\eta \in \mathcal{S}_{\rho(\chi)}} \chi(\eta)\,(f^*bf)(\eta) \\ &= |f(\gamma)|^2 \sum_{\eta \in \mathcal{S}_{\rho(\chi)}} \chi(\eta)\, b(\gamma\eta\gamma^{-1})\overline{c(\gamma^{-1}, \gamma)}c(\gamma^{-1}, \gamma\eta)c(\gamma\eta\gamma^{-1}, \gamma). \end{aligned} \tag{15}$$

By Equation (14) and by definition of α_f, we have

$$\alpha_f(\phi_\chi)(b)\ |f(\gamma)|^2 = \alpha_f(\phi_\chi)(b)\,\phi_\chi(f^*f) = \phi_\chi(f^*bf),$$

and since $f(\gamma) \neq 0$ as $\gamma \in \operatorname{supp}'(f)$, Equation (15) allows us to conclude that

$$\begin{aligned}\alpha_f(\phi_\chi)(b) &= \sum_{\eta \in \mathcal{S}_{\rho(\chi)}} \chi(\eta)\, b(\gamma\eta\gamma^{-1})\overline{c(\gamma^{-1},\gamma)}c(\gamma^{-1},\gamma\eta)c(\gamma\eta\gamma^{-1},\gamma) \\ &= \sum_{a \in \mathcal{S}_{r(\gamma)}} \chi(\gamma^{-1}a\gamma)\,\overline{c(\gamma,\gamma^{-1})}c(\gamma^{-1},a\gamma)c(a,\gamma)b(a) \\ &= \sum_{a \in \mathcal{S}_{r(\gamma)}} \chi(\gamma^{-1}a\gamma)\,\overline{c(\gamma,\gamma^{-1})}c(\gamma^{-1}a,\gamma)c(\gamma^{-1},a)b(a) = \phi_{\tilde{\alpha}_{\dot\gamma}(\chi)}(b).\end{aligned}$$

To obtain the second equality, we invoked the fact that $c(\gamma,\gamma^{-1}) = c(\gamma^{-1},\gamma)$ for any $\gamma \in \mathcal{G}$ [8, Lemma 2.1]. It follows that, as desired, $\alpha_f(\phi_\chi)(b) = \phi_{\tilde{\alpha}_{\dot\gamma}(\chi)}(b)$ for all b in the dense subalgebra $C_c(\mathcal{S})$ and thus on all of B. ■

The fact that $\alpha_f(\phi_\chi) = \phi_{\tilde{\alpha}_{\dot\gamma}(\chi)}$ whenever $\{\gamma\} = \operatorname{supp}'(f) \cap \mathcal{G}_{\rho(\chi)}$ for $f \in N$ and $\chi \in \mathfrak{B}$ shows that there is an intimate connection between the partial action α of $N(B)$ on $\mathfrak{B}$ and the action $\tilde{\alpha}$ of $\mathcal{Q}$ on $\mathfrak{B}$. In order to describe this connection, we first need to better understand equality in the Weyl groupoid $\mathcal{G}_{(A,B)}$.

Proposition 4.5. Suppose $f_i \in N$ and $\chi \in \mathfrak{B}$. Let X^i denote $\operatorname{supp}'(f_i) \subseteq \mathcal{G}$, and for $u \in \mathcal{G}^{(0)}$, let X^i_u denote the singleton-set $X^i \cap \mathcal{G}_u$. Recall that $q\colon \mathcal{G} \to \mathcal{Q}$ is the quotient map. The following are equivalent:

1. There exists an open neighborhood U of $\rho(\chi)$ in $s(X^1) \cap s(X^2) \subseteq \mathcal{G}^{(0)}$ such that $q(X^1_u) = q(X^2_u)$ for all $u \in U$.
2. $q(X^1_{\rho(\chi)}) = q(X^2_{\rho(\chi)})$.
3. $\phi_\chi \in \operatorname{dom}(f_1) \cap \operatorname{dom}(f_2)$ and $[\alpha_{f_1}(\phi_\chi), f_1, \phi_\chi] = [\alpha_{f_2}(\phi_\chi), f_2, \phi_\chi]$.

Proof. Note that 3 $\implies$ 2 is the second assertion of Lemma 4.31, so it suffices to prove 2 $\implies$ 1 $\implies$ 3.

Assume that 2 holds. By [19, Proposition 2.2.4], $(X^1)^{-1} \cdot X^2$ is an open bisection in $\mathcal{G}$. Setting $\{\gamma_i\} = X^i_{\rho(\chi)}$, our assumption translates to $\dot\gamma_1 = \dot\gamma_2$; in particular, $\gamma_1^{-1}\gamma_2 \in [(X^1)^{-1} \cdot X^2] \cap \mathcal{S}$. Since $\mathcal{S}$ is open and $\mathcal{G}$ is étale, the set $U := s([(X^1)^{-1} \cdot X^2] \cap \mathcal{S})$ is an open subset of $\mathcal{G}^{(0)}$, which contains $s(\gamma_1^{-1}\gamma_2) = s(\gamma_2) = \rho(\chi)$. Furthermore, U is contained in $s(X^1) \cap s(X^2)$: any $u \in U$ can be written as $u = s(\gamma_u^{-1}\tau_u)$ for $\gamma_u \in X^1$ and $\tau_u \in X^2$ such that

$\gamma_u^{-1}\tau_u \in \mathcal{S}$. In particular,

$$\begin{aligned} s(\tau_u) &= s(\gamma_u^{-1}\tau_u) = u && \text{by assumption, and} \\ s(\gamma_u) &= r(\gamma_u^{-1}\tau_u) \overset{(*)}{=} s(\gamma_u^{-1}\tau_u) = u, && \text{where } (*) \text{ follows from } \mathcal{S} \subseteq \mathrm{Iso}(\mathcal{G}). \end{aligned}$$

This shows that $\{\gamma_u\} = X_u^1$, $\{\tau_u\} = X_u^2$, and $U \subseteq s(X^1) \cap s(X^2)$. Moreover, it follows from $\gamma_u^{-1}\tau_u \in \mathcal{S}$ that $q(X_u^1) = q(X_u^2)$. This proves 1.

Next, we will show 1 $\Longrightarrow$ 3. Pick any $u \in U \subseteq s(X^1) \cap s(X^2)$, and let γ_u^i denote the unique element in $X_u^i = s^{-1}(u) \cap X^i$. Note that, by our assumption on U, we have $\dot{\gamma}_u^1 = \dot{\gamma}_u^2$. Using Proposition 4.4(2) for $(*)$ in the following, we thus see that, for any $\nu \in \mathfrak{B}_u$,

$$\alpha_{f_1}(\phi_\nu) \overset{(*)}{=} \phi_{\tilde{\alpha}_{\dot{\gamma}_u^1}(\nu)} = \phi_{\tilde{\alpha}_{\dot{\gamma}_u^2}(\nu)} \overset{(*)}{=} \alpha_{f_2}(\phi_\nu).$$

Since u was arbitrary, this shows that α_{f_1} and α_{f_2} coincide on all of $\phi(\rho^{-1}(U))$. This set contains ϕ_χ by the assumption that $\rho(\chi) \in U$ and it is open in $\widehat{B}$ since ϕ is a homeomorphism, ρ is continuous, and U is open in $\mathcal{G}^{(0)}$. This proves 3. ∎

Theorem 4.6. There is an isomorphism φ of topological groupoids $\mathcal{Q} \ltimes \mathfrak{B} \to \mathcal{G}_{(A,B)}$ given by

$$\varphi(\dot{\gamma}, \chi) := [\phi_{\tilde{\alpha}_{\dot{\gamma}}(\chi)}, f, \phi_\chi] = [\alpha_f(\phi_\chi), f, \phi_\chi], \tag{16}$$

where $f \in C_c(\mathcal{G}, c)$ is any function supported on a bisection such that $\dot{\gamma} \in q(\mathrm{supp}'(f))$.

We point out here that this result is a significant strengthening of [8, Theorem 5.2]. Not only is Theorem 4.6 true for étale groupoids (not just discrete groups), but we also do not need to assume that $\tilde{\alpha}$ is topologically free. Instead, as our theorem suggests, this simply follows from $\mathcal{G}_{(A,B)}$ being topologically principal. Moreover, Theorem 4.6 applies in the setting of [12, Section 3] if one assumes the groupoids involved to be étale, as discussed in the following remark.

Remark 8. When the 2-cocycle c on the étale groupoid $\mathcal{G}$ is trivial, $\mathfrak{B}$ is precisely the dual bundle $\widehat{\mathcal{S}}$ used in [12, Section 3]. Moreover, the right action of $\mathcal{G}/\mathcal{S}$ on $\widehat{\mathcal{S}}$ given at the top of [12, page 23],

$$\chi \cdot \dot{\gamma}(a) = \chi(\gamma a \gamma^{-1}),$$

is precisely the left action $\tilde{\alpha}_{\dot{\gamma}^{-1}}(\phi_\chi)$ of $\dot{\gamma}^{-1} \in \mathcal{G}/\mathcal{S}$ on ϕ_χ in this case. In other words, Theorem 4.6 establishes that the groupoid $\widehat{\mathcal{S}} \rtimes \mathcal{Q}$ of [12, Theorem 3.3] is indeed the Weyl groupoid, if $\mathcal{G}$ is étale and $C_r^*(\mathcal{S})$ is Cartan in $C_r^*(\mathcal{G})$.

We will use the rest of this section to prove Theorem 4.6 through a series of lemmata.

Lemma 4.7. The map $\varphi\colon \mathcal{Q} \ltimes \mathfrak{B} \to \mathcal{G}_{(A,B)}$ defined in Equation (16) is a well-defined groupoid homomorphism.

Proof. Let $(\dot{\gamma}, \chi) \in \mathcal{Q} \ltimes \mathfrak{B}$ and $f \in N$ satisfy $\dot{\gamma} \in q(\mathrm{supp}'(f))$. This assumption implies $\rho(\chi) = s(\gamma) \in s(\mathrm{supp}'(f))$, which guarantees that $\phi_\chi \in \mathrm{dom}(f)$ by Proposition 4.4(1), so that $[\phi_{\tilde{\alpha}_{\dot{\gamma}}(\chi)}, f, \phi_\chi]$ is indeed an element of $\mathcal{G}_{(A,B)}$. Moreover, this element is independent of the choice of f by Proposition 4.5, $2 \Longrightarrow 3$. In other words, φ is well defined.

Next suppose $\big((\dot{\tau}, \chi'), (\dot{\gamma}, \chi)\big)$ is a composable pair in $\mathcal{Q} \ltimes \mathfrak{B}$, i.e., $\chi' = \tilde{\alpha}_{\dot{\gamma}}(\chi)$. It follows that φ takes this composable pair to a composable pair:

$$s(\varphi(\dot{\tau}, \chi')) = \phi_{\chi'} = \phi_{\tilde{\alpha}_{\dot{\gamma}}(\chi)} = r(\varphi(\dot{\gamma}, \chi)).$$

Moreover, if $g, h \in N$ with $\dot{\tau} \in q(\mathrm{supp}'(g))$ and $\dot{\gamma} \in q(\mathrm{supp}'(h))$, then

$$\varphi(\dot{\tau}, \chi')\varphi(\dot{\gamma}, \chi) = [\phi_{\tilde{\alpha}_{\dot{\tau}}(\chi')}, g, \phi_{\chi'}][\phi_{\tilde{\alpha}_{\dot{\gamma}}(\chi)}, h, \phi_\chi] = [\phi_{\tilde{\alpha}_{\dot{\tau}}(\tilde{\alpha}_{\dot{\gamma}}(\chi))}, gh, \phi_\chi],$$

which equals $[\phi_{\tilde{\alpha}_{\dot{\tau}\dot{\gamma}}(\chi)}, gh, \phi_\chi]$ since $\tilde{\alpha}$ is an action. On the other hand,

$$\varphi\left((\dot{\tau}, \chi')(\dot{\gamma}, \chi)\right) = \varphi(q(\tau\gamma), \chi) = [\phi_{\tilde{\alpha}_{q(\tau\gamma)}(\chi)}, f, \phi_\chi],$$

where $f \in N$ with $q(\tau\gamma) \in q(\mathrm{supp}'(f))$.

In order to show that $[\phi_{\tilde{\alpha}_{q(\tau\gamma)}(\chi)}, f, \phi_\chi] = [\phi_{\tilde{\alpha}_{\dot{\tau}\dot{\gamma}}(\chi)}, gh, \phi_\chi]$, it suffices to show that, like f, gh is an element of N with $q(\tau\gamma) \in q(\mathrm{supp}'(gh))$.

Since $g, h \in N$, gh is supported on the bisection $\mathrm{supp}(g) \cdot \mathrm{supp}(h)$ (cf. [22, Lemma 3.1.4] in the untwisted case). Lastly, if $\tau \in \mathrm{supp}'(g)$ and $\gamma \in \mathrm{supp}'(h)$ are representatives of $\dot{\tau}$ and $\dot{\gamma}$, respectively, then

$$0 \neq g(\tau)h(\gamma)c(\tau, \gamma) = (gh)(\tau\gamma),$$

so $q(\tau\gamma) \in q(\mathrm{supp}'(gh))$. Therefore, the fact that φ is well defined implies that $[\phi_{\tilde{\alpha}_{q(\tau\gamma)}(\chi)}, f, \phi_\chi] = [\phi_{\tilde{\alpha}_{\dot{\tau}}(\tilde{\alpha}_{\dot{\gamma}}(\chi))}, gh, \phi_\chi]$. ■

Lemma 4.8. The groupoid homomorphism $\varphi\colon \mathcal{Q} \ltimes \mathfrak{B} \to \mathcal{G}_{(A,B)}$ is a bijection.

Proof. By Lemma 4.2, we know that every element of $\mathcal{G}_{(A,B)}$ is of the form $[\alpha_f(x), f, x]$ for $f \in N$ and $x \in \mathrm{dom}(f) \subseteq \widehat{B}$. By Lemma 3.2, we may write $x = \phi_\chi$ for a unique $\chi \in \mathfrak{B}$. Since $\phi_\chi(f^*f) > 0$ by assumption, Proposition 4.4(1) shows that $\rho(\chi) \in s(\mathrm{supp}'(f))$, i.e., $\mathrm{supp}'(f) \cap \mathcal{G}_{\rho(\chi)} = \{\gamma\}$ for some γ. In particular, $(\dot{\gamma}, \chi) \in \mathcal{Q} \ltimes \mathfrak{B}$. Moreover, the fact that φ is well defined means that

$$\varphi(\dot{\gamma}, \chi) = [\phi_{\tilde{\alpha}_{\dot{\gamma}}(\chi)}, f, \phi_\chi].$$

Proposition 4.4(2) now implies that $\varphi(\dot{\gamma}, \chi) = [\alpha_f(x), f, x]$, so φ is surjective.

For injectivity, assume that $\varphi(\dot{\gamma}_1, \chi) = \varphi(\dot{\gamma}_2, \chi')$, i.e.,

$$[\alpha_{f_1}(\phi_\chi), f_1, \phi_\chi] = [\alpha_{f_2}(\phi_{\chi'}), f_2, \phi_{\chi'}], \text{ where } f_i \in N \text{ have } \dot{\gamma}_i \in \mathrm{supp}'(f_i).$$

This immediately forces $\chi = \chi'$ by definition of the Weyl groupoid and injectivity of ϕ (Lemma 3.2). In particular, $s(\dot{\gamma}_1) = \rho(\chi) = s(\dot{\gamma}_2)$.

Proposition 4.5, 3 $\Longrightarrow$ 2, tells us that if $X^i := \mathrm{supp}'(f_i)$, then $q(X^1_{\rho(\chi)}) = q(X^2_{\rho(\chi)})$. By choice of f_i, $X^i_{\rho(\chi)} = \{\gamma_i\}$, so $\dot{\gamma}_1 = \dot{\gamma}_2$ and $(\dot{\gamma}_1, \chi) = (\dot{\gamma}_2, \chi')$. ■

Lemma 4.9. The bijective groupoid homomorphism $\varphi\colon \mathcal{Q} \ltimes \mathfrak{B} \to \mathcal{G}_{(A,B)}$ is a homeomorphism.

Proof. To see continuity of φ, suppose the net $(\dot{\gamma}_i, \chi_i)_{i\in\Lambda}$ converges to $(\dot{\gamma}, \chi)$ in $\mathcal{Q} \ltimes \mathfrak{B}$. Let $\varphi(\dot{\gamma}_i, \chi_i) = [\phi_{\tilde{\alpha}_{\dot{\gamma}_i}(\chi_i)}, f_i, \phi_{\chi_i}]$ and $\varphi(\dot{\gamma}, \chi) = [\phi_{\tilde{\alpha}_{\dot{\gamma}}(\chi)}, f, \phi_\chi]$, where $f_i, f \in C_c(\mathcal{G}, c)$ are supported on bisections such that $\dot{\gamma}_i \in q(\mathrm{supp}'(f_i))$ and $\dot{\gamma} \in q(\mathrm{supp}'(f))$.

Let U be an open neighborhood of $\phi_{\tilde{\alpha}_{\dot{\gamma}}(\chi)}$ in $\widehat{B}$ and V be an open neighborhood of ϕ_χ in $\widehat{B}$, so that

$$\mathcal{U}(U, f, V) := \{[\alpha_f(\phi_\nu), f, \phi_\nu] \in \mathcal{G}_{(A,B)} \mid \alpha_f(\phi_\nu) \in U, \phi_\nu \in V\} \tag{17}$$

is a basic open neighborhood of $[\alpha_f(\phi_\chi), f, \phi_\chi]$ [21, p. 36]. We must show that there exists $K \in \Lambda$ such that if $i \geq K$, then $[\phi_{\tilde{\alpha}_{\dot{\gamma}_i}(\chi_i)}, f_i, \phi_{\chi_i}]$ lies in $\mathcal{U}(U, f, V)$. Since $q(\mathrm{supp}'(f))$ is an open neighborhood of $\dot{\gamma}$ and $\dot{\gamma}_i$ converges to $\dot{\gamma}$, there exists $K_1 \in \Lambda$ such that if $i \geq K_1$,

then $\dot{\gamma}_i \in q(\mathrm{supp}'(f))$. Since φ is well defined (Lemma 4.7), we may assume that $f_i = f$ for $i \geq K_1$. Moreover, since χ_i converges to χ by assumption, Lemma 3.5 implies there exists $K_2 \in \Lambda$ such that $\phi_{\chi_i} \in V$ for all $i \geq K_2$. Lastly, since $(\dot{\gamma}_i, \chi_i)$ converges to $(\dot{\gamma}, \chi)$, continuity of $(\dot{\tau}, \nu) \mapsto \tilde{\alpha}_{\dot{\tau}}(\nu)$ (Proposition 4.1(4)) and of $\nu \mapsto \phi_\nu$ (Lemma 3.5) imply that there exists $K_3 \in \Lambda$ such that $\phi_{\tilde{\alpha}_{\dot{\gamma}_i}(\chi_i)} \in U$ for all $i \geq K_3$. Therefore, if i is greater than each of K_1, K_2, K_3, then $[\phi_{\tilde{\alpha}_{\dot{\gamma}_i}(\chi_i)}, f, \phi_{\chi_i}] = [\phi_{\tilde{\alpha}_{\dot{\gamma}_i}(\chi_i)}, f_i, \phi_{\chi_i}]$ lies in $\mathcal{U}(U, f, V)$.

To see that φ^{-1} is continuous, suppose $(\dot{\gamma}_i, \chi_i)_{i\in\Lambda}$ is a net in $\mathcal{Q} \ltimes \mathfrak{B}$ such that $\varphi(\dot{\gamma}_i, \chi_i)$ converges to some element Γ in $\mathcal{G}_{(A,B)}$. If we write $\Gamma = [\alpha_f(\phi_\chi), f, \phi_\chi]$ for some $f \in N$ with $\phi_\chi \in \mathrm{dom}(f)$, then a basic open neighborhood around Γ is of the form $\mathcal{U}(\widehat{B}, f, V)$ (Equation (17)), where $V \subseteq \mathrm{dom}(f)$ is some open neighborhood of ϕ_χ. Since $\varphi(\dot{\gamma}_i, \chi_i) \to \Gamma$, we know that for any fixed V, there exists $K \in \Lambda$ such that if $i \geq K$, then $\varphi(\dot{\gamma}_i, \chi_i)$ lies in $\mathcal{U}(\widehat{B}, f, V)$. This means in particular that $\phi_{\chi_i} \in V$. Since V was arbitrary, this proves that $\phi_{\chi_i} \to \phi_\chi$. The fact that the map ϕ is open (Lemma 3.6) implies that χ_i converges to χ. Since $\phi_\chi, \phi_{\chi_i} \in \mathrm{dom}(f)$, it follows from Proposition 4.4(1) that $\rho(\chi), \rho(\chi_i) \in s(\mathrm{supp}'(f))$, i.e., there exist (unique) $\tau, \tau_i \in \mathrm{supp}'(f)$ with $s(\tau) = \rho(\chi)$ and $s(\tau_i) = \rho(\chi_i)$. The definition of φ thus yields

$$\varphi(\dot{\tau}, \chi) = [\alpha_f(\phi_\chi), f, \phi_\chi] = \Gamma \quad \text{and} \quad \varphi(\dot{\tau}_i, \chi_i) = [\alpha_f(\phi_{\chi_i}), f, \phi_{\chi_i}].$$

On the other hand, since $\varphi(\dot{\gamma}_i, \chi_i) \in \mathcal{U}(\widehat{B}, f, V)$, we also know that

$$\varphi(\dot{\gamma}_i, \chi_i) = [\alpha_f(\phi_{\chi_i}), f, \phi_{\chi_i}].$$

Injectivity of φ now implies $\dot{\gamma}_i = \dot{\tau}_i$. We will show that $\tau_i \to \tau$, so that in particular $\dot{\gamma}_i = \dot{\tau}_i \to \dot{\tau}$.

Let U be any open neighborhood around τ contained in $\mathrm{supp}'(f)$. Then, since $\mathcal{G}$ is étale, $s(U)$ is an open neighborhood around $s(\tau) = \rho(\chi)$. As $\chi_i \to \chi$ and ρ is continuous, there exists i_0 so that for all $i \geq i_0$, we have $s(\tau_i) = \rho(\chi_i) \in s(U)$. Since $\tau_i \in \mathrm{supp}'(f)$ and $U = s^{-1}(s(U)) \cap \mathrm{supp}'(f)$, this means $\tau_i \in U$ for $i \geq i_0$. As U was an arbitrary neighborhood around τ, we conclude that τ_i converges to τ.

All in all, we have shown that $(\dot{\gamma}_i, \chi_i)$ converges to $(\dot{\tau}, \chi)$, the preimage of Γ under φ. This completes the proof of the lemma, and also of Theorem 4.6. ■

5 When the Weyl Twist is a 2-Cocycle

The standing assumptions for this section can be found on page 14. In this section, we identify when the Weyl twist $\Sigma_{(A,B)}$ can be described by a continuous 2-cocycle on the

Weyl groupoid $\mathcal{G}_{(A,B)}$. So let us first recall the classical construction of a twist $G\times_\sigma \mathbb{T}$ from a 2-cocycle σ on a groupoid G (cf. [20, Chapter I, Proposition 1.14] or [8]). As a set, we have $G\times_\sigma \mathbb{T} = G\times\mathbb{T}$; we set $(G\times_\sigma\mathbb{T})^{(2)} = \{((g,\lambda),(h,\mu)) : (g,h)\in G^{(2)}\}$. The multiplication is given by $(g,\lambda)(h,\mu) = (gh,\sigma(g,h)\lambda\mu)$. When σ is continuous, the product topology on $G\times_\sigma\mathbb{T}$ makes it a topological groupoid. Note that there is a continuous action of $\mathbb{T}$ on $G\times_\sigma\mathbb{T}$, given by $\lambda\cdot(g,\mu) = (g,\lambda\mu)$ and $(G\times_\sigma\mathbb{T})/\mathbb{T}\cong G$. Similarly (cf. [14, Section 3]), the Weyl twist $\Sigma_{(A,B)}$ always admits an action of $\mathbb{T}$, given by $\lambda\cdot[\![\alpha_n(\phi_\chi),n,\phi_\chi]\!] = [\![\alpha_n(\phi_\chi),\lambda n,\phi_\chi]\!]$, so that $\Sigma_{(A,B)}/\mathbb{T} = \mathcal{G}_{(A,B)}$.

We show in Theorem 5.1 that, if $\mathfrak{s} : \mathcal{Q}\to\mathcal{G}$ is a continuous section, then we obtain a continuous section $\psi_\mathfrak{s} : \mathcal{G}_{(A,B)}\to\Sigma_{(A,B)}$. Such a continuous section $\psi_\mathfrak{s}$ implies that $\Sigma_{(A,B)}$ is given by a continuous 2-cocycle $\sigma^\mathfrak{s}$ on $\mathcal{G}_{(A,B)}$ via the formula

$$\sigma^\mathfrak{s}(Y_1,Y_2) = \psi_\mathfrak{s}(Y_1)\psi_\mathfrak{s}(Y_2)\psi_\mathfrak{s}(Y_1Y_2)^{-1},$$

see [14, Fact 4.1], [20, Proposition I.1.14], [22, Remark 5.1.6]. In Corollary 5.4, we give an explicit formula for the induced 2-cocycle $C^\mathfrak{s}\colon(\mathcal{Q}\ltimes\mathfrak{B})^{(2)}\to\mathbb{T}$.

Theorem 5.1. Let $\mathfrak{s}\colon\mathcal{Q}\to\mathcal{G}$ be a section for $q\colon\mathcal{G}\to\mathcal{Q}$ and for each $\dot\gamma\in\mathcal{Q}$, choose one $f_{\dot\gamma}\in N$ such that $f_{\dot\gamma}(\mathfrak{s}(\dot\gamma))>0$. Define $\psi_\mathfrak{s}\colon\mathcal{G}_{(A,B)}\to\Sigma_{(A,B)}$ for $m\in N$ and $\phi_\chi\in\mathrm{dom}(m)$ by

$$\psi_\mathfrak{s}([\alpha_m(\phi_\chi),m,\phi_\chi]) = [\![\phi(\tilde\alpha_{\dot\gamma}(\chi)),f_{\dot\gamma},\phi_\chi]\!] = [\![\alpha_m(\phi_\chi),f_{\dot\gamma},\phi_\chi]\!],$$

where $(\dot\gamma,\chi) = \varphi^{-1}([\alpha_m(\phi_\chi),m,\phi_\chi])$. If $\mathfrak{s}$ is continuous, then $\psi_\mathfrak{s}$ is a continuous section of the groupoid extension $\Sigma_{(A,B)}$.

Remark 9. By Lemma 4.2, every element of $\mathcal{G}_{(A,B)}$ indeed has a representative of the form $[\alpha_m(\phi_\chi),m,\phi_\chi]$ with $m\in N$. Moreover, surjectivity of φ (see Theorem 4.6) implies the existence of the specified $(\dot\gamma,\chi)\in\mathcal{Q}\ltimes\mathfrak{B}$, and Proposition 4.4(2) shows that $\alpha_m(\phi_\chi) = \phi(\tilde\alpha_{\dot\gamma}(\chi)) = \alpha_{f_{\dot\gamma}}(\phi_\chi)$.

Proving continuity of $\psi_\mathfrak{s}$ requires a few preliminary results. The following analogue of [8, Lemma 5.4] states that every element of the Weyl twist can be represented by a function supported in a bisection and scaled by an explicitly computed factor.

Proposition 5.2. Let $[\![\alpha_n(\phi_\chi),n,\phi_\chi]\!]$, $n\in N(B)$, be an arbitrary element of the Weyl twist $\Sigma_{(A,B)}$, and let $\dot\gamma\in\mathcal{Q}$ be such that $\varphi(\dot\gamma,\chi) = [\alpha_n(\phi_\chi),n,\phi_\chi]$. If $f\in N$ satisfies

$\dot{\gamma} \in q(\mathrm{supp}'(f))$, then

$$[\![\alpha_n(\phi_\chi), n, \phi_\chi]\!] = [\![\alpha_f(\phi_\chi), \lambda f, \phi_\chi]\!] \quad \text{where} \quad \lambda = \frac{\phi_\chi\big(\Phi(f^* n)\big)}{\Big|\phi_\chi\big(\Phi(f^* n)\big)\Big|} \in \mathbb{T}. \qquad \square$$

Proof. Write $x := \phi_\chi \in \mathrm{dom}(n)$. We have to find two elements $b, b' \in B$ such that $x(b), x(b') > 0$ and $nb = (\lambda f)b'$. By assumption on f and definition of φ, we have

$$[\alpha_f(x), f, x] = \varphi(\dot{\gamma}, \chi) = [\alpha_n(x), n, x].$$

Proposition 2.2 therefore tells us that there exist $b_1, b_2 \in B$ such that $fb_1 = nb_2$ and $x(b_1), x(b_2) \neq 0$. In fact, the construction of b_1, b_2 in the proof of that proposition gives $x(b_2) > 0$. Thus, if we set

$$b' = \frac{|x(b_1)|}{x(b_1)} b_1, \qquad b = b_2, \qquad \lambda = \frac{x(b_1)}{|x(b_1)|},$$

then $x(b), x(b') > 0$ and $\lambda f b' = fb_1 = nb_2$ as desired.

To see that λ can be equivalently written as in the statement of the proposition, recall from Equation (7) that

$$b_1 = \Omega(k) f^* n \, \Omega(\overline{k}),$$

where $k \in C_0(\widehat{B})$ is supported on $\mathrm{dom}(f) \cap \mathrm{dom}(n)$ and satisfies $k(x) = 1$. As the conditional expectation $\Phi : A \to B$ is B-linear, it follows that

$$x(b_1) = \phi_\chi(\Phi(b_1)) = \phi_\chi(\Omega(k)\Phi(f^* n)\Omega(\overline{k})) = \phi_\chi(\Phi(f^* n)).$$

This yields the asserted expression for λ. ■

As we will frequently need to explicitly compute the constant λ appearing in Proposition 5.2, the following corollary will be helpful.

Corollary 5.3. In the setting of Proposition 5.2, suppose further that $n \in N$. If the unique element γ of $\mathrm{supp}'(n) \cap \mathcal{G}_{\rho(\chi)}$ is also an element of $\mathrm{supp}'(f)$, then

$$[\![\alpha_n(\phi_\chi), n, \phi_\chi]\!] = [\![\alpha_f(\phi_\chi), \lambda f, \phi_\chi]\!] \quad \text{where} \quad \lambda = \frac{\overline{f(\gamma)}}{|f(\gamma)|} \frac{n(\gamma)}{|n(\gamma)|}. \qquad \square$$

Remark 10. The above corollary shows, in particular, that the definition of $\psi_{\mathfrak{s}}$ in Theorem 5.1 does not depend on the choice of $f_{\dot\gamma}$ but only on $\mathfrak{s}$. If f is another element of N with $f(\mathfrak{s}(\dot\gamma)) > 0$, then Corollary 5.3 applied to $n := f_{\dot\gamma}$ and $\gamma := \mathfrak{s}(\dot\gamma)$ yields

$$\psi_{\mathfrak{s}}([\alpha_m(\phi_\chi), m, \phi_\chi]) = [\![\alpha_{f_{\dot\gamma}}(\phi_\chi), f_{\dot\gamma}, \phi_\chi]\!] = [\![\alpha_f(\phi_\chi), f, \phi_\chi]\!] \quad \text{since} \quad 1 = \frac{\overline{f(\gamma)}}{|f(\gamma)|}\frac{f_{\dot\gamma}(\gamma)}{\left|f_{\dot\gamma}(\gamma)\right|}. \qquad \square$$

Proof of Corollary 5.3. Since $f^*n \in N$, the definition of the conditional expectation (Equation (12)) implies that $\Phi(f^*n)$ is given by restriction to $\mathcal{S}$. In particular, $\Phi(f^*n) \in C_c(\mathcal{S}, c)$. The definition of ϕ_χ therefore yields

$$\phi_\chi(\Phi(f^*n)) = \sum_{a\in\mathcal{S}_{\rho(\chi)}} \chi(a) f^*n(a) = \sum_{a\in\mathcal{S}_{\rho(\chi)}} \chi(a) \sum_{r(\eta)=\rho(\chi)} f^*(a\eta)n(\eta^{-1})c(a\eta, \eta^{-1}).$$

Since f and n are supported in bisections, and both have the same element of $\mathcal{G}_{\rho(\chi)}$ in their support, only the summand where $\eta := \gamma^{-1}$ and $a := \rho(\chi)$ does not vanish, i.e., the sum above simplifies to a single term:

$$\phi_\chi(\Phi(f^*n)) = \chi(\rho(\chi))\overline{f(\gamma\rho(\chi)^{-1})c(\gamma\rho(\chi)^{-1}, \rho(\chi)\gamma^{-1})}n(\gamma)c(\rho(\chi)\gamma^{-1}, \gamma).$$

As $\chi(\rho(\chi)) = 1$ and $c(\gamma^{-1}, \gamma) = c(\gamma, \gamma^{-1})$, we conclude that $\phi_\chi(\Phi(f^*n)) = \overline{f(\gamma)}n(\gamma)$. This yields the desired formula for λ. ∎

We are now prepared to prove Theorem 5.1.

Proof of Theorem 5.1. The fact that $\psi_{\mathfrak{s}}$ is a section follows from Proposition 4.5, as $\dot\gamma \in q(\mathrm{supp}'(m)) \cap q(\mathrm{supp}'(f_{\dot\gamma}))$ by construction and hence

$$[\alpha_m(\phi_\chi), f_{\dot\gamma}, \phi_\chi] = [\alpha_m(\phi_\chi), m, \phi_\chi].$$

To see that $\psi := \psi_{\mathfrak{s}}$ is continuous when $\mathfrak{s}$ is continuous, assume that $(\dot\gamma_i, \chi_i)_i$ is a net in $\mathcal{Q} \ltimes \mathfrak{B}$ that converges to $(\dot\gamma, \chi)$. Since φ is a homeomorphism, it now suffices to show that $\psi(y_i) \to \psi(y)$, where $y_i := \varphi(\dot\gamma_i, \chi_i)$ and $y := \varphi(\dot\gamma, \chi)$. If $f := f_{\dot\gamma}$, then a basic open set around $\psi(y) = [\![\alpha_f(\phi_\chi), f, \phi_\chi]\!]$ (according to [21, Lemma 4.16]) is of the form

$$\mathfrak{U}(U, V, f) := \left\{ [\![\alpha_f(\phi_\nu), \kappa f, \phi_\nu]\!] \mid \phi_\nu \in U, \kappa \in V \right\}, \tag{18}$$

where $U \subseteq \widehat{B}$ is an open neighborhood around ϕ_χ and $V \subseteq \mathbb{T}$ an open neighborhood around 1. We need to show that $\psi(y_i) \in \mathfrak{U}(U,V,f)$ for large enough i. By definition of ψ, we have

$$\psi(y_i) = [\![\alpha_{f_i}(\phi_{\chi_i}), f_i, \phi_{\chi_i}]\!], \quad \text{where } f_i := f_{\dot{\gamma}_i}.$$

Let $\gamma_i := \mathfrak{s}(\dot{\gamma}_i)$ and $\gamma := \mathfrak{s}(\dot{\gamma})$. It follows from $\dot{\gamma}_i \to \dot{\gamma}$ and continuity of $\mathfrak{s}$ that $\gamma_i \to \gamma$. By choice of f, we have that $\mathrm{supp}'(f)$ is an open neighborhood of γ, so $\gamma_i \in \mathrm{supp}'(f)$ for all i larger than some i_1. Since we also have $f_i(\gamma_i) > 0$, Corollary 5.3 implies that for $i \geq i_1$,

$$\psi(y_i) = [\![\alpha_f(\phi_{\chi_i}), \lambda_i f, \phi_{\chi_i}]\!], \quad \text{where } \lambda_i := \frac{\overline{f(\gamma_i)}}{|f(\gamma_i)|}\frac{f_i(\gamma_i)}{|f_i(\gamma_i)|} = \frac{\overline{f(\gamma_i)}}{|f(\gamma_i)|}.$$

To show that $\psi(y_i)$ is an element of $\mathfrak{U}(U,V,f)$ for all large enough i, we must show that $\lambda_i \in V$ and $\phi_{\chi_i} \in U$. For the latter, note that since $\chi_i \to \chi$ by hypothesis, and since the map $\nu \mapsto \phi_\nu$ is continuous by Lemma 3.5, we must have $\phi_{\chi_i} \in U$ for all i larger than some i_2. For the former, note that since $\gamma_i \to \gamma$ and f is a continuous function with $f(\gamma) = \overline{f(\gamma)} > 0$, $\frac{\overline{f(\gamma_i)}}{|f(\gamma_i)|} = \lambda_i$ converges to 1. We conclude that λ_i is in the neighborhood V of 1 for all large enough i. ∎

Corollary 5.4. If $\mathfrak{s}\colon \mathcal{Q} \to \mathcal{G}$ is a continuous section for $q\colon \mathcal{G} \to \mathcal{Q}$, then the function $C^{\mathfrak{s}}\colon (\mathcal{Q} \ltimes \mathfrak{B})^{(2)} \to \mathbb{T}$ defined by

$$C^{\mathfrak{s}}((\dot{\tau}, \tilde{\alpha}_{\dot{\gamma}}(\chi)), (\dot{\gamma}, \chi)) = c(\mathfrak{s}(\dot{\tau}), \mathfrak{s}(\dot{\gamma}))\, c(\mathfrak{s}(\dot{\tau}\dot{\gamma})^{-1}, \mathfrak{s}(\dot{\tau})\mathfrak{s}(\dot{\gamma}))\, \overline{c(\mathfrak{s}(\dot{\tau}\dot{\gamma})^{-1}, \mathfrak{s}(\dot{\tau}\dot{\gamma}))}\, \chi(\mathfrak{s}(\dot{\tau}\dot{\gamma})^{-1}\mathfrak{s}(\dot{\tau})\mathfrak{s}(\dot{\gamma}))$$

is a continuous 2-cocycle on $\mathcal{Q} \ltimes \mathfrak{B}$, and the Weyl twist $\Sigma_{(A,B)}$ and the twist $(\mathcal{Q} \ltimes \mathfrak{B}) \times_{C^{\mathfrak{s}}} \mathbb{T}$ are isomorphic as topological groupoids.

Proof. If $\mathfrak{s}$ is continuous, then $\psi_{\mathfrak{s}} : \mathcal{G}_{(A,B)} \to \Sigma_{(A,B)}$ is a continuous section by Theorem 5.1. Given this continuous section, it is well known (cf. [14, Fact 4.1], [20, Proposition I.1.14], [22, Remark 5.1.6]) that the function $\sigma^{\mathfrak{s}} : \mathcal{G}^{(2)}_{(A,B)} \to \mathbb{T}$ defined by

$$\sigma^{\mathfrak{s}}(y_1, y_2) = \psi_{\mathfrak{s}}(y_1)\psi_{\mathfrak{s}}(y_2)\psi_{\mathfrak{s}}(y_1y_2)^{-1} \tag{19}$$

is a continuous 2-cocycle on $\mathcal{G}_{(A,B)}$ and $\Sigma_{(A,B)} \cong \mathcal{G}_{(A,B)} \times_{\sigma^{\mathfrak{s}}} \mathbb{T}$. Since $\mathcal{G}_{(A,B)} \cong \mathcal{Q} \ltimes \mathfrak{B}$ by Theorem 4.6, it remains to prove that the definition of $\sigma^{\mathfrak{s}}$ in Equation (19) induces the asserted formula for $C^{\mathfrak{s}}$ on $(\mathcal{Q} \ltimes \mathfrak{B})^{(2)}$.

Suppose $((\dot{\tau},\chi_1),(\dot{\gamma},\chi_2)) \in (\mathcal{Q} \ltimes \mathfrak{B})^{(2)}$. In particular, this means that $\chi_1 = \tilde{\alpha}_{\dot{\gamma}}(\chi_2)$ and that $\varphi(\dot{\tau},\chi_1),\varphi(\dot{\gamma},\chi_2)$ are composable elements of $\mathcal{G}_{(A,B)}$. With $f_{\dot{\gamma}} \in N$ for $\dot{\gamma} \in \mathcal{Q}$ as specified in Theorem 5.1, Equation (19) and the definition of $\psi_{\mathfrak{s}}$ (see Theorem 5.1) yield

$$\begin{aligned}
\sigma^{\mathfrak{s}}(\varphi(\dot{\tau},\chi_1),\varphi(\dot{\gamma},\chi_2)) &= \psi_{\mathfrak{s}}(\varphi(\dot{\tau},\chi_1))\psi_{\mathfrak{s}}(\varphi(\dot{\gamma},\chi_2))\psi_{\mathfrak{s}}(\varphi(\dot{\tau},\chi_1)\varphi(\dot{\gamma},\chi_2))^{-1} \\
&= [\![\phi_{\tilde{\alpha}_{\dot{\tau}}(\chi_1)}, f_{\dot{\tau}}, \phi_{\chi_1}]\!][\![\phi_{\tilde{\alpha}_{\dot{\gamma}}(\chi_2)}, f_{\dot{\gamma}}, \phi_{\chi_2}]\!][\![\phi_{\tilde{\alpha}_{\dot{\tau}\dot{\gamma}}(\chi_2)}, f_{\dot{\tau}\dot{\gamma}}, \phi_{\chi_2}]\!]^{-1} \\
&= [\![\phi_{\tilde{\alpha}_{\dot{\tau}}(\tilde{\alpha}_{\dot{\gamma}}(\chi_2))}, f_{\dot{\tau}} f_{\dot{\gamma}}, \phi_{\chi_2}]\!][\![\phi_{\chi_2}, f^*_{\dot{\tau}\dot{\gamma}}, \phi_{\tilde{\alpha}_{\dot{\tau}\dot{\gamma}}(\chi_2)}]\!] \\
&= [\![\phi_{\tilde{\alpha}_{\dot{\tau}\dot{\gamma}}(\chi_2)}, f_{\dot{\tau}} f_{\dot{\gamma}} f^*_{\dot{\tau}\dot{\gamma}}, \phi_{\tilde{\alpha}_{\dot{\tau}\dot{\gamma}}(\chi_2)}]\!].
\end{aligned}$$

To interpret this as an element of $\mathbb{T}$, recall [21, p. 47] that an element $[\![x,b,x]\!] = [\![x, \frac{1}{|x(b)|}b, x]\!]$ in $\Sigma_{(A,B)}$ with $b \in B$ corresponds to $\left(x, \frac{x(b)}{|x(b)|}\right)$ in $\widehat{B} \times \mathbb{T}$. Denote $f_{\dot{\tau}} f_{\dot{\gamma}} f^*_{\dot{\tau}\dot{\gamma}} \in N$ by n and denote $\mathfrak{s}(\dot{\tau})\mathfrak{s}(\dot{\gamma})\mathfrak{s}(\dot{\tau}\dot{\gamma})^{-1} \in \mathcal{S}$ by a, the unique element in $\mathrm{supp}'(n) \cap \mathcal{G}_{r(\dot{\tau})}$. Let $f \in C_c(\mathcal{S},c)$ be supported in a bisection and such that $f(a) = 1$. Then, since $r(\dot{\tau}) = \rho(\tilde{\alpha}_{\dot{\tau}\dot{\gamma}}(\chi_2))$ (Proposition 4.1(2)), it follows from Corollary 5.3 that

$$\sigma^{\mathfrak{s}}(\varphi(\dot{\tau},\chi_1),\varphi(\dot{\gamma},\chi_2)) = [\![\phi_{\tilde{\alpha}_{\dot{\tau}\dot{\gamma}}(\chi_2)}, n, \phi_{\tilde{\alpha}_{\dot{\tau}\dot{\gamma}}(\chi_2)}]\!] = \left[\!\!\left[\phi_{\tilde{\alpha}_{\dot{\tau}\dot{\gamma}}(\chi_2)}, \frac{n(a)}{|n(a)|}f, \phi_{\tilde{\alpha}_{\dot{\tau}\dot{\gamma}}(\chi_2)}\right]\!\!\right].$$

With $x = \phi_{\tilde{\alpha}_{\dot{\tau}\dot{\gamma}}(\chi_2)}$ and $b = \frac{n(a)}{|n(a)|}f$, the element of $\mathbb{T}$ that we are after is therefore

$$\begin{aligned}
C^{\mathfrak{s}}((\dot{\tau},\chi_1),(\dot{\gamma},\chi_2)) :&= \frac{x(b)}{|x(b)|} = \phi_{\tilde{\alpha}_{\dot{\tau}\dot{\gamma}}(\chi_2)}\left(\frac{n(a)}{|n(a)|}f\right) \\
&= \frac{n(a)}{|n(a)|}\sum_{a' \in \mathcal{S}_{r(\dot{\tau})}} f(a')\,\tilde{\alpha}_{\dot{\tau}\dot{\gamma}}(\chi_2)(a') = \frac{n(a)}{|n(a)|}\tilde{\alpha}_{\dot{\tau}\dot{\gamma}}(\chi_2)(a).
\end{aligned}$$

If we write $\tau := \mathfrak{s}(\dot{\tau})$, $\gamma := \mathfrak{s}(\dot{\gamma})$, and $\varepsilon := \mathfrak{s}(\dot{\tau}\dot{\gamma})$, then

$$\begin{aligned}
n(a) &= \left(f_{\dot{\tau}} f_{\dot{\gamma}} f^*_{\dot{\tau}\dot{\gamma}}\right)\left(\tau\gamma\varepsilon^{-1}\right)\left(f_{\dot{\tau}} f_{\dot{\gamma}}\right)(\tau\gamma)\; f^*_{\dot{\tau}\dot{\gamma}}\left(\varepsilon^{-1}\right)\; c\left(\tau\gamma, \varepsilon^{-1}\right) \\
&= f_{\dot{\tau}}(\tau)\; f_{\dot{\gamma}}(\gamma)\; c(\tau,\gamma)\; \overline{f_{\dot{\tau}\dot{\gamma}}(\varepsilon)}\; \overline{c\left(\varepsilon^{-1},\varepsilon\right)}\; c\left(\tau\gamma,\varepsilon^{-1}\right)
\end{aligned}$$

and, using the definition of $\tilde{\alpha}$ (see Proposition 4.1) and of a,

$$\begin{aligned}
\tilde{\alpha}_{\dot{\tau}\dot{\gamma}}(\chi_2)(a) &= \overline{c(\varepsilon,\varepsilon^{-1})}\, c(\varepsilon^{-1},a)\, c(\varepsilon^{-1}a,\varepsilon)\; \chi_2(\varepsilon^{-1}a\varepsilon) \\
&= \overline{c(\varepsilon,\varepsilon^{-1})}\, c(\varepsilon^{-1},\tau\gamma\varepsilon^{-1})\, c(\varepsilon^{-1}\tau\gamma\varepsilon^{-1},\varepsilon)\; \chi_2(\varepsilon^{-1}\tau\gamma).
\end{aligned}$$

Since $f_{\dot\tau}(\tau)$, $f_{\dot\gamma}(\gamma)$, and $f_{\dot\tau\dot\gamma}(\varepsilon)$ are positive by assumption, we conclude

$$\begin{aligned}C^{\mathfrak{s}}((\dot\tau,\chi_1),(\dot\gamma,\chi_2)) &= \frac{n(a)}{|n(a)|}\tilde\alpha_{\dot\tau\dot\gamma}(\chi_2)(a)\\ &= c(\tau,\gamma)\,\overline{c\left(\varepsilon^{-1},\varepsilon\right)}\,c(\tau\gamma,\varepsilon^{-1})\,\overline{c(\varepsilon,\varepsilon^{-1})}\,c(\varepsilon^{-1},\tau\gamma\varepsilon^{-1})\,c(\varepsilon^{-1}\tau\gamma\varepsilon^{-1},\varepsilon)\,\chi_2(\varepsilon^{-1}\tau\gamma)\\ &= c(\tau,\gamma)\,\overline{c(\varepsilon^{-1},\varepsilon)}\,c(\varepsilon^{-1},\tau\gamma)\,\chi_2(\varepsilon^{-1}\tau\gamma),\end{aligned}$$

where the last equality follows from the fact that

$$\begin{aligned}&c(\tau\gamma,\varepsilon^{-1})\,\overline{c(\varepsilon,\varepsilon^{-1})}\left(c(\varepsilon^{-1},\tau\gamma\varepsilon^{-1})\,c(\varepsilon^{-1}\tau\gamma\varepsilon^{-1},\varepsilon)\right)\\ &\quad= c(\tau\gamma,\varepsilon^{-1})\,\overline{c(\varepsilon,\varepsilon^{-1})}\left(c(\varepsilon^{-1},\tau\gamma)\,c(\tau\gamma\varepsilon^{-1},\varepsilon)\right)\\ &\quad= \left(c(\tau\gamma,\varepsilon^{-1})\,c(\tau\gamma\varepsilon^{-1},\varepsilon)\right)\overline{c(\varepsilon,\varepsilon^{-1})}\,c(\varepsilon^{-1},\tau\gamma)\\ &\quad= \left(c(\varepsilon^{-1},\varepsilon)\right)\overline{c(\varepsilon,\varepsilon^{-1})}\,c(\varepsilon^{-1},\tau\gamma) = c(\varepsilon^{-1},\tau\gamma)\end{aligned}$$

since $c(g,g^{-1}) = c(g^{-1},g)$ for any $g\in\mathcal{G}$ by [8, Lemma 2.1]. ■

Remark 11. For each $\dot\gamma\in\mathcal{Q}$, choose as before one $f_{\dot\gamma}\in N$ such that $f_{\dot\gamma}(\mathfrak{s}(\dot\gamma))>0$. Then, a straightforward computation yields an alternative formula for $C^{\mathfrak{s}}$:

$$C^{\mathfrak{s}}((\dot\tau,\tilde\alpha_{\dot\gamma}(\chi)),(\dot\gamma,\chi)) = \frac{\phi_\chi(\Phi(f^*_{\dot\tau\dot\gamma}f_{\dot\tau}f_{\dot\gamma}))}{|\phi_\chi(\Phi(f^*_{\dot\tau\dot\gamma}f_{\dot\tau}f_{\dot\gamma}))|}, \tag{20}$$

where ϕ_χ and Φ are defined in Equations (11) and (12), respectively.

Corollary 5.5. If $\mathfrak{s}$ is continuous, then the Cartan pair $(C^*_r(\mathcal{G},c),C^*_r(\mathcal{S},c))$ is isomorphic to the pair $\left(C^*_r(\mathcal{Q}\ltimes\mathfrak{B},C^{\mathfrak{s}}),C_0(\mathfrak{B})\right)$.

Proof. Since the Weyl groupoid is isomorphic, as a topological groupoid, to $\mathcal{Q}\ltimes\mathfrak{B}$ by Theorem 4.6, and the Weyl twist is also topologically isomorphic to $(\mathcal{Q}\ltimes\mathfrak{B})\times_{C^{\mathfrak{s}}}\mathbb{T}$ by Corollary 5.4, [21, Theorem 5.9] implies that $C^*_r(\mathcal{G},c)$ is isomorphic to $C^*_r(\mathcal{Q}\ltimes\mathfrak{B},C^{\mathfrak{s}})$ and that isomorphism carries $C^*_r(\mathcal{S},c)$ onto $C_0((\mathcal{Q}\ltimes\mathfrak{B})^{(0)})\cong C_0(\mathfrak{B})$. ■

Acknowledgments

The authors thank the referees for their very helpful comments, which have substantially streamlined and improved the paper.

A.D. would further like to thank Aidan Sims and Chris Bruce for helpful discussions and the Department of Mathematics and Statistics at the University of Victoria for its hospitality.

Funding

The research presented here grew out of the work [8], which was begun during the authors' participation in the 2018 BIRS workshop "Women in Operator Algebras" and continued during a 2019 stay at the Mathematical Sciences Research Institute. The latter stay was supported by the National Security Agency [H98230-19-1-0119], The Lyda Hill Foundation, The McGovern Foundation, and Microsoft Research. E.G. was partially supported by the National Science Foundation [DMS-1800749].

References

[1] Brown, J. H. and A. an Huef. "Decomposing the C*-algebras of groupoid extensions." *Proceedings of the American Mathematical Society* 142 (2014): 1261–74.

[2] Brownlowe, N., T. M. Carlsen, and M. F. Whittaker. "Graph algebras and orbit equivalence." *Ergodic Theory and Dynamical Systems* 37, no. 2 (2017): 389–417.

[3] Brown, J. H., A. Fuller, D. Pitts, and S. Reznikoff. "Graded C*-algebras and twisted groupoid C*-algebras." *New York Journal of Mathematics* 27 (2021): 205–52.

[4] Barlak, S. and X. Li. "Cartan subalgebras and the UCT problem." *Advances in Mathematics* 316 (2017): 748–69.

[5] Busby, R. C. and H. A. Smith. "Representations of twisted group algebras." *Transactions of the American Mathematical Society* 149 (1970): 503–37.

[6] Calabi, L. "Sur les extensions des groupes topologiques." *Annali di Matematica Pura ed Applicata* 32, no. 4 (1951): 295–370.

[7] Crytser, D. and G. Nagy. "Simplicity criteria for étale groupoid C*-algebras." *Journal of Operator Theory* 83, no. 1 (2020): 95–138.

[8] Duwenig, A., E. Gillaspy, R. Norton, S. Reznikoff, and S. Wright. "Cartan subalgebras for non-principal twisted groupoid C*-algebras." *Journal of Functional Analysis* 279, no. 6 (2020): 108611.

[9] Folland, G. B. *Real Analysis: Modern Techniques and Their Applications*. New York: Wiley, 1999.

[10] Goehle, G. "Group bundle duality." *Illinois Journal of Mathematics* 52, no. 3 (2008): 951–6.

[11] Ionescu, M. and A. Kumjian. "Groupoid actions on fractafolds." *Symmetry Integrability and Geometry: Methods and Applications (SIGMA)* 10 (2014): 14 Paper 068.

[12] Ionescu, M., A. Kumjian, J. N. Renault, A. Sims, and D. P. Williams. "C* -algebras of extensions of groupoids by group bundles." *Journal of Functional Analysis* 280, no. 5 (2021): 108892.

[13] Kleppner, A. "Multipliers on abelian groups." *Mathematische Annalen* 158 (1965): 11–34.

[14] Kumjian, A. "On C*-diagonals." *Canadian Journal of Mathematics* 38, no. 4 (1986): 969–1008.

[15] Li, X. "Every classifiable simple C*-algebra has a Cartan subalgebra." *Inventiones Mathematicae* 219 (2020): 653–99.

[16] Matsumoto, K. and H. Matui. "Continuous orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras." *Kyoto Journal of Mathematics* 54, no. 4 (2014): 863–77.

[17] Muhly, P. S., J. N. Renault, and D. P. Williams. "Equivalence and isomorphism for groupoid C*-algebras." *Journal of Operator Theory* 17 (1987): 3–22.

[18] Muhly, P. S., J. N. Renault, and D. P. Williams. "Continuous-trace groupoid C*-algebras. III." *Transactions of the American Mathematical Society* 348, no. 9 (1996): 3621–41.

[19] Paterson, A. L. T. "Groupoids, inverse semigroups, and their operator algebras." *Progress in Mathematics* 170 (1999).

[20] Renault, J. "A groupoid approach to C*-algebras." *Lecture Notes in Mathematics* 793 (1980).

[21] Renault, J. "Cartan subalgebras in C*-algebras." *Irish Mathematical Society Bulletin* 61 (2008): 29–63.

[22] Sims, A. "Hausdorff étale groupoids and their C*-algebras." *Advanced Courses in Mathematics. CRM Barcelona* (2017) arXiv:1710.10897v2.

[23] Takeishi, T. "On nuclearity of C*-algebras of Fell bundles over étale groupoids." *Publications of the Research Institute for Mathematical Sciences* 50 (2014): 251–68.

[24] Williams, D. P. "Crossed products of C*-algebras." *Mathematical Surveys and Monographs* 134 (2007).

[25] Williams, D. P. "A tool kit for groupoid C*-algebras." *Mathematical Surveys and Monographs* 241 (2019).