ANNALS OF K-THEORY

vol. 7 no. 2 2022

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A JOURNAL OF THE K-THEORY FOUNDATION

ANNALS OF K-THEORY
Vol. 7, No. 2, 2022
https://doi.org/10.2140/akt.2022.7.395

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We initiate the study of real C^*-algebras associated to higher-rank graphs Λ, with a focus on their K-theory. Following Kasparov and Evans, we identify a spectral sequence which computes the $\mathcal{CR}$ K-theory of $C^*_{\mathbb{R}}(\Lambda, \gamma)$ for any involution γ on Λ, and show that the E^2 page of this spectral sequence can be straightforwardly computed from the combinatorial data of the k-graph Λ and the involution γ. We provide a complete description of $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$ for several examples of higher-rank graphs Λ with involution.

1. Introduction

Using the classification of simple purely infinite real C^*-algebras [Boersema 2006; Boersema et al. 2011], the first author together with Ruiz and Stacey established in [Boersema et al. 2011, Theorem 11.1] that for odd n, there are two distinct real C^*-algebras ($\mathcal{E}_n$ and $\mathcal{O}_n^{\mathbb{R}}$) whose complexification is the Cuntz algebra $\mathcal{O}_n$. While $\mathcal{O}_n^{\mathbb{R}}$ is easy to describe in terms of generators and relations, the only facts known about $\mathcal{E}_n$ (beyond its existence) are its K-theory [Boersema et al. 2011, Theorem 11.1] and that it cannot arise as the real C^*-algebra of any directed graph [Boersema 2017, Theorem 6.1]. This latter fact is quite surprising, since $\mathcal{O}_n$ is one of the most straightforward examples of a graph C^*-algebra, and every directed graph gives rise to many potentially different real C^*-algebras. Indeed, [Boersema 2017] showed that any idempotent graph automorphism γ on a graph E gives rise to the real C^*-algebra $C^*_{\mathbb{R}}(E, \gamma)$ (see (2.5) below).

To date, much of the literature on real C^*-algebras has focused on their K-theory (see [Schröder 1993; Boersema 2002; Boersema et al. 2011; Boersema and Loring 2016]), with some attention paid to other structural properties (see [Boersema 2007; Rosenberg 2016; Stacey 2003; Boersema and Ruiz 2011]). In some sense, the K-theoretic data is enough: [Boersema 2006] explains how to construct a purely infinite simple real C^*-algebra with any appropriate specified $\mathcal{CR}$ K-theory, and the $\mathcal{CR}$ K-theory is known to be a classifying invariant for simple purely infinite real C^*-algebras [Boersema et al. 2011]. However, the construction in [Boersema 2006] is quite layered and obtuse — it allows us to detect the existence of real

MSC2020: 46L80.
Keywords: higher rank graphs, real C^*-algebras, K-theory.

structures for given complex C^*-algebras, but does not otherwise shine a lot of light. We therefore wish to develop alternative constructions to help us generate more examples of real C^*-algebras in a concrete way. Specifically, as a test piece, we wish to find a concrete representation of the real C^*-algebras $\mathcal{E}_n$.

To this end, we introduce in this paper the real C^*-algebra $C^*_{\mathbb{R}}(\Lambda, \gamma)$ associated to a higher-rank graph Λ and an involution γ on Λ. Inspired by [Robertson and Steger 1999], Kumjian and Pask [2000] introduced higher-rank graphs, or k-graphs as a way to construct combinatorial examples of C^*-algebras which are more general than graph C^*-algebras. In addition to their intrinsic links with a variety of combinatorial structures, such as buildings [Robertson and Steger 1999; Konter and Vdovina 2015] and ultrametric Cantor sets [Farsi et al. 2020; 2021; Heo et al. 2021], (complex) k-graph C^*-algebras have provided important examples for Elliott's classification program [Ruiz et al. 2015] as well as for noncommutative geometry [Pask et al. 2008; Farsi et al. 2020].

The family of real C^*-algebras that arise from higher-rank graphs with involution is much larger than the family arising from graph algebras. This follows, for example, from the fact that the K_1-group of a (complex) graph C^*-algebra must be torsion-free [Raeburn and Szymański 2004], a restriction which disappears for higher-rank graphs. However, in order to answer the question of whether the exotic Cuntz algebra $\mathcal{E}_n$ arises from a higher-rank graph, we need to be able to compute the K-theory of real higher-rank graph C^*-algebras, since $\mathcal{E}_n$ can only be identified by its K-theory. In this article we develop the methods to carry out the computations of the K-theory of such algebras and demonstrate these methods with several detailed computations of interesting examples. However, we have not yet discovered an example of a higher-rank graph with involution whose associated C^*-algebra is $\mathcal{E}_n$. We discuss this, and other open questions, in Section 5.

While the K-theory of a graph C^*-algebra can be computed from the adjacency matrix of the graph using a long exact sequence [Bates et al. 2002; Raeburn and Szymański 2004], the situation is more complicated for a higher-rank graph Λ. Evans [2008] identified a spectral sequence which converges to the K-theory of $C^*(\Lambda)$, and computed the K-theory explicitly in some low-rank situations. Thus, we first confirm that given the real C^*-algebra of a higher-rank graph, there exists a spectral sequence which converges to its K-theory. This is the focus of Section 3. For a real C^*-algebra A, the K-theoretic invariant that we consider contains much more information than does the K-theory of a complex C^*-algebra. We consider the so-called $\mathcal{CR}$ K-theory of A, which includes not only the eight real K-groups $K_*(A)$, but also the two K-theory groups $K_*(A_{\mathbb{C}})$ of its complexification, as well as a number of homomorphisms between the various groups that satisfy certain compatibility conditions. The $\mathcal{CR}$ K-theory that is our focus is a variation of the $\mathcal{CRT}$ K-theory introduced in [Bousfield 1990] in the topological setting and in [Boersema 2002] in the C^*-algebraic setting. A short introduction is in Section 2B below. The spectral sequence that we develop is sufficiently functorial that it contains all of this additional structure. It can be construed as a spectral sequence in the category of $\mathcal{CR}$-modules or in the category of $\mathcal{CRT}$-modules.

Thus, the spectral sequence that we develop is simultaneously a generalization of the spectral sequence of Evans (for a complex higher-rank graph algebra) and the long exact sequence developed in [Boersema 2017] (for a real algebra from a graph with involution). Similar to the long exact sequence found in [Boersema 2017], the building blocks of our spectral sequence consist of direct sums of the K-theory of $\mathbb{C}$ and $\mathbb{R}$, viewed as real C^*-algebras. Indeed, we show in Section 3D that the E^2 page of the spectral sequence can be computed from a chain complex whose entries are the aforementioned direct sums of the K-theory of $\mathbb{C}$ and $\mathbb{R}$, and whose boundary maps are determined by the combinatorial structure of the higher-rank graph.

When it comes to computing the $\mathcal{CR}$ K-theory of specific examples of real C^*-algebras, the complicated structure of real K-theory is both boon and bane. While the intricacy of $\mathcal{CR}$ K-theory adds many additional steps to certain computations, the circumscribed relationships between the various groups (described in Section 2B) mean that often, the entire $\mathcal{CR}$ K-theory is completely determined by just a few of its constituent groups and homomorphisms. Consequently, as we see in Section 4, a small amount of information frequently enables us to completely describe the $\mathcal{CR}$ K-theory. To be precise, in Section 4, we use both the simplified description of the E^2 page of the spectral sequence from Section 3D, and the relationships between the structure maps of $\mathcal{CR}$ K-theory, to completely describe the $\mathcal{CR}$ K-theory of several examples of rank-2 graphs. In particular, for each odd n, we identify in Section 4B a 2-graph Λ and an involution γ on Λ such that $C^*(\Lambda)$ is KK-equivalent to $\mathcal{O}_n \otimes \mathcal{O}_n$, but its real structure $C^*_{\mathbb{R}}(\Lambda, \gamma)$ is not a tensor product of real Cuntz algebras. In other words, we have discovered new real structures on $\mathcal{O}_n \otimes \mathcal{O}_n$, other than $\mathcal{O}_n^{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{O}_n^{\mathbb{R}}$.

2. Preliminaries

2A. ***Higher-rank graphs and their (real) C^*-algebras.*** Higher-rank graphs were introduced in [Kumjian and Pask 2000] as a higher-dimensional generalization of directed graphs. To define them, we first specify that throughout this paper, we view $\mathbb{N}^k$ as a category with one object (namely 0), where composition of morphisms is given by addition. For consistency with the usual notation $n \in \mathbb{N}^k$ to describe a k-tuple of natural numbers (which, in the category-theoretic perspective, is a morphism in $\mathbb{N}^k$), we write $\lambda \in \Lambda$ to denote a morphism in the category Λ. We identify a category's objects with the identity morphisms, so that statements such as $0 \in \mathbb{N}^k$ are still allowed.

Definition 2.1. A *higher-rank graph* of rank k, or a *k-graph*, is a countable small category Λ equipped with a *degree functor* $d : \Lambda \to \mathbb{N}^k$ such that, whenever a morphism $\lambda \in \Lambda$ satisfies $d(\lambda) = m + n$, there exist unique morphisms $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = m$, $d(\nu) = n$.

Write e_i for the standard i-th basis vector of $\mathbb{N}^k$. We usually think of the morphisms of degree e_i as the "edges of color i" in Λ. With this perspective, if e is an edge of color i and f is an edge of color j, their composition $ef \in \Lambda$ satisfies

$$d(ef) = e_i + e_j = e_j + e_i,$$

so there must be an equivalent way of writing $ef = f'e'$, where $d(f') = e_j$ and $d(e') = e_i$.

In other words (cf. [Hazlewood et al. 2013, Theorems 4.3 and 4.4; Eckhardt et al. 2022, Theorem 2.3]) we can think of a k-graph as consisting of a directed graph G, with k colors of edges, and a *factorization rule*, or equivalence relation, $\sim$ on the multicolored paths in G^*. For each pair of colors ("red" and "blue" for this discussion), and each pair of vertices v, w, the factorization rule identifies each path from v to w which consists of a blue edge followed by a red edge (a *blue-red path*) with a unique red-blue path from v to w. The factorization rule must also satisfy certain consistency conditions which ensure that, for each path in G^*, its equivalence class under $\sim$ corresponds to a k-dimensional hyperrectangle; see [Eckhardt et al. 2022, Theorem 2.3] for more details. (For ease of readability, we omit these details here since our work in this paper does not depend on the precise details of factorization rules.)

Let Λ be a k-graph. Given objects $v, w \in \Lambda$ and $n \in \mathbb{N}^k$, we write

$$\begin{aligned} \Lambda^n &= \{\lambda \in \Lambda \mid d(\lambda) = n\}, \\ v\Lambda &= \{\lambda \in \Lambda \mid r(\lambda) = v\}, \\ \Lambda^n w &= \{\lambda \in \Lambda \mid s(\lambda) = w \text{ and } d(\lambda) = n\}, \end{aligned} \tag{2.2}$$

as well as the obvious variations. Observe that Λ^0 is the set of objects of Λ, which we also denote as *vertices* thanks to the graph-theoretic inspiration for k-graphs. We say that Λ is *row-finite* if $|v\Lambda^n| < \infty$ for all $n \in \mathbb{N}^k$ and $v \in \Lambda^0$, and that Λ is *source-free* if, for all n and v, $v\Lambda^n \neq \varnothing$.[1]

Definition 2.3 [Kumjian and Pask 2000]. Given a row-finite source-free k-graph Λ, a *Cuntz–Krieger Λ-family* is a collection $\{t_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in a C^*-algebra A which satisfy the following conditions:

(CK1) For each $v \in \Lambda^0$, t_v is a projection and $t_v t_w = \delta_{v,w} t_v$.

(CK2) For each $\lambda \in \Lambda$, $t_\lambda^* t_\lambda = t_{s(\lambda)}$.

[1] Equivalently, Λ is source-free if $v\Lambda^{e_i}$ is nonempty for all $1 \leq i \leq k$.

(CK3) For each $\lambda, \mu \in \Lambda$, $t_\lambda t_\mu = t_{\lambda\mu}$.

(CK4) For each $v \in \Lambda^0$ and each $n \in \mathbb{N}^k$,

$$t_v = \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*.$$

We define $C^*(\Lambda)$ to be the universal (complex) C^*-algebra generated by a Cuntz–Krieger family, in the sense that for any Cuntz–Krieger Λ-family $\{t_\lambda\}_{\lambda\in\Lambda}$, there is a canonical surjective $*$-homomorphism $C^*(\Lambda) \to C^*(\{t_\lambda\}_\lambda)$.

We write $\{s_\lambda\}_{\lambda\in\Lambda}$ for the generators of $C^*(\Lambda)$. One computes easily, using the Cuntz–Krieger relations, that $C^*(\Lambda) = \overline{\operatorname{span}}\,\{s_\lambda s_\mu^* \mid s(\lambda) = s(\mu)\}$.

Given a k-graph Λ, we now describe how to associate a real C^*-algebra to it. We assume that Λ is row-finite and has no sources. Observe that there is a (unique) antimultiplicative linear automorphism χ of $C^*(\Lambda)$ which satisfies $\chi(s_\lambda) = s_\lambda^*$.

Definition 2.4. An *involution* γ on a k-graph Λ is a degree-preserving functor $\gamma : \Lambda \to \Lambda$ which satisfies $\gamma \circ \gamma = \mathrm{id}_\Lambda$.

The functoriality of γ implies that $s\gamma = \gamma s$ and $r\gamma = \gamma r$ for any involution γ.

Given an involution γ on Λ, the elements $\{s_{\gamma(\lambda)} \mid \lambda \in \Lambda\}$ form a Cuntz–Krieger Λ-family, so the universal property of $C^*(\Lambda)$ implies the existence of an automorphism $C^*(\gamma)$ on $C^*(\Lambda)$, given by $C^*(\gamma)(s_\lambda) := s_{\gamma(\lambda)}$. Since χ commutes with $C^*(\gamma)$, the composition $\widetilde{\gamma} := \chi \circ C^*(\gamma)$ is an antimultiplicative involution of $C^*(\Lambda)$, which is determined uniquely by the formula

$$\widetilde{\gamma}(s_\lambda) = s_{\gamma(\lambda)}^*.$$

It follows that $(C^*(\Lambda); \widetilde{\gamma})$ is a $C^{*,\tau}$-algebra (this just means exactly that $\widetilde{\gamma}$ is an antiautomorphism of $C^*(\Lambda)$). The corresponding real C^*-algebra is given by

$$C^*_{\mathbb{R}}(\Lambda, \gamma) := \{a \in C^*(\Lambda) \mid \widetilde{\gamma}(a) = a^*\} \tag{2.5}$$

(see Definition 1.1.4 of [Schröder 1993] and the following remark).

Lemma 2.6. *Given an involution γ on a row-finite source-free k-graph Λ,*

$$C^*_{\mathbb{R}}(\Lambda, \gamma) = \overline{\operatorname{span}}_{\mathbb{R}}\,\{z s_\lambda s_\mu^* + \bar{z} s_{\gamma(\lambda)} s_{\gamma(\mu)}^* \mid z \in \mathbb{C}, \lambda, \mu \in \Lambda\}.$$

Proof. Define A to be the right-hand side. We first observe that

$$(z s_\lambda s_\mu^* + \bar{z} s_{\gamma(\lambda)} s_{\gamma(\mu)}^*)^* = \bar{z} s_\mu s_\lambda^* + z s_{\gamma(\mu)} s_{\gamma(\lambda)}^*,$$

whereas the fact that $\widetilde{\gamma}$ is antimultiplicative but linear implies that we also have

$$\widetilde{\gamma}(z s_\lambda s_\mu^* + \bar{z} s_{\gamma(\lambda)} s_{\gamma(\mu)}^*) = z s_{\gamma(\mu)} s_{\gamma(\lambda)}^* + \bar{z} s_\mu s_\lambda^*.$$

Hence $A \subseteq C^*_{\mathbb{R}}(\Lambda, \gamma)$.

To see that $A \cong C^*_{\mathbb{R}}(\Lambda, \gamma)$, we show that $A + iA = C^*(\Lambda)$. To that end, fix $\alpha \in \mathbb{C}$ and consider $\alpha s_\lambda s_\mu^* \in C^*(\Lambda)$. If we set $z = \alpha/2$, $w = -i\alpha/2$, a quick computation reveals that

$$z s_\lambda s_\mu^* + \bar{z} s_{\gamma(\lambda)} s_{\gamma(\mu)}^* + i(w s_\lambda s_\mu^* + \overline{w} s_{\gamma(\lambda)} s_{\gamma(\mu)}^*) = \alpha s_\lambda s_\mu^*.$$

As the elements $\alpha s_\lambda s_\mu^*$ densely span $C^*(\Lambda)$ as a real vector space, we conclude that $A + iA = C^*_{\mathbb{R}}(\Lambda, \gamma)$ as claimed. □

To each k-graph we can associate k commuting matrices $M_1, \dots, M_k$ in $M_{\Lambda^0}(\mathbb{N})$:

$$M_i(v, w) := |v\Lambda^{e_i} w|, \tag{2.7}$$

that is, the (v, w) entry in M_i counts the number of color-i edges in Λ with source w and range v. We call the matrices M_i the *incidence matrices* or *adjacency matrices* of the k-graph. The fact that $M_i M_j = M_j M_i$ follows from the requirement, imposed by the factorization rule, that there be an identical number of blue-red and red-blue paths between any given pair (v, w) of vertices.

Given a k-graph Λ, we can form the *skew product* $\Lambda \times_d \mathbb{Z}^k$, with $\mathrm{Obj}(\Lambda \times_d \mathbb{Z}^k) = \Lambda^0 \times \mathbb{Z}^k$ and $\mathrm{Mor}(\Lambda \times_d \mathbb{Z}^k) = \Lambda \times \mathbb{Z}^k$. We have $s(\lambda, n) = (s(\lambda), n + d(\lambda))$ and $r(\lambda, n) = (r(\lambda), n)$. Defining $d : \Lambda \times_d \mathbb{Z}^k \to \mathbb{N}^k$ by $d(\lambda, n) = d(\lambda)$ makes $\Lambda \times_d \mathbb{Z}^k$ a k-graph, which is row-finite and source-free whenever Λ is. Moreover, by [Kumjian and Pask 2000, Theorem 5.7], the universal property of $C^*(\Lambda \times_d \mathbb{Z}^k)$ implies that $C^*(\Lambda \times_d \mathbb{Z}^k)$ admits an action of $\mathbb{Z}^k$, given on the generators by $s_{\lambda,n} \cdot m := s_{\lambda,m+n}$.

2B. ***K-theory for real C*-algebras.*** The main K-theoretic invariant that we use in the category of real C^*-algebras is $\mathcal{CR}$ K-theory, denoted by $K^{CR}(A)$ for a real C^*-algebra A. In this section, we review the definition and the properties of this invariant, necessary for the rest of this article. $\mathcal{CR}$ K-theory derives from $\mathcal{CRT}$ or "united" K-theory; see [Bousfield 1990; Boersema 2002; 2004; Boersema et al. 2011].

For a real C^*-algebra A, we define $K^{CR}(A) = \{KO_*(A), KU_*(A)\}$, where $KO_*(A)$ is the usual period-8 K-theory of A and $KU_*(A) := K_*(\mathbb{C} \otimes A)$ is the K-theory of the complexification of A. In addition, $KO_*(A)$ has the structure of a graded module over the ring $KO_*(\mathbb{R})$, where the groups of this ring are given by

$$KO_*(\mathbb{R}) = \mathbb{Z} \quad \mathbb{Z}_2 \quad \mathbb{Z}_2 \quad 0 \quad \mathbb{Z} \quad 0 \quad 0 \quad 0$$

in degrees 0 through 7.

In particular, multiplication by the nontrivial element η of $KO_1(\mathbb{R}) \cong \mathbb{Z}_2$ induces a natural transformation $\eta : KO_i(A) \to KO_{i+1}(A)$. We note that η satisfies the relations $2\eta = 0$ and $\eta^3 = 0$, both as an element of the ring $KO_*(\mathbb{R})$ and as a natural transformation. There is also a nontrivial element $\xi \in KO_4(\mathbb{R})$, and corresponding natural transformation, that satisfies $\xi^2 = 4\beta_O$, where β_O is the real Bott periodicity isomorphism of degree 8.

Complex K-theory $KU_*(A)$ has the structure of a module over $KU_*(\mathbb{R}) = K_*(\mathbb{C})$, but the only natural transformation which arises from this structure is the degree-2 Bott periodicity map β. There is, however, a natural transformation

$$\psi : KU_*(A) \to KU_*(A)$$

that arises from the conjugation map $\psi : \mathbb{C}\otimes A \to \mathbb{C}\otimes A$ defined by $a+ib \mapsto a-ib$.

In addition, there are natural transformations

$$c : KO_*(A) \to KU_*(A),$$
$$r : KU_*(A) \to KO_*(A)$$

which are induced by the natural inclusion maps $\mathbb{R} \hookrightarrow \mathbb{C}$ and $\mathbb{C} \hookrightarrow M_2(\mathbb{R})$, respectively.

Taken together, these natural transformations satisfy the following set of relations:

$$\begin{aligned}
rc &= 2, & cr &= 1+\psi, & 2\eta &= 0,\\
\eta r &= 0, & c\eta &= 0, & \eta^3 &= 0,\\
r\psi &= r, & \psi^2 &= \mathrm{id}, & \xi^2 &= 4\beta_O,\\
\psi c &= c, & \psi\beta &= -\beta_U\psi, & \xi &= r\beta^2 c.
\end{aligned} \tag{2.8}$$

A pair (G^O, G^U) of $\mathbb{Z}$-graded abelian groups (G^O with period 8 and G^U with period 2) together with natural transformations η, β, ζ, ψ, r, c as above, such that the equations (2.8) hold, is called a *$\mathcal{CR}$-module*, and the category of such objects is the target of the functor $K^{CR}(A)$.

We display the full structure of $K^{CR}(\mathbb{R})$ and $K^{CR}(\mathbb{C})$ in Tables 1 and 2 below. These are the only two singly generated free $\mathcal{CR}$-modules (up to suspensions) and all of the relations above are encoded in these two $\mathcal{CR}$-modules. Furthermore, these $\mathcal{CR}$-modules will be the building blocks of the spectral sequence we will use to compute $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$. (See Theorem 3.13 and Section 3D below.)

The natural transformations also combine to form a long exact sequence

$$\cdots \to KO_i(A) \xrightarrow{\eta} KO_{i+1}(A) \xrightarrow{c} KU_{i+1}(A) \xrightarrow{r\beta^{-1}} KO_{i-1}(A) \to \cdots. \tag{2.9}$$

The following theorem summarizes some of the important properties of the invariant $K^{CR}(A)$.

Theorem 2.10. (1) *If A is a real C^*-algebra, then $K^{CR}(A)$ is a $\mathcal{CR}$-module.*

(2) *If A and B are real C^*-algebras such that $\mathbb{C}\otimes A$ and $\mathbb{C}\otimes B$ are in the bootstrap category $\mathcal{N}$, then A and B are KK-equivalent if and only if $K^{CR}(A) \cong K^{CR}(B)$.*

(3) *If A and B are real C^*-algebras such that $\mathbb{C}\otimes A$ and $\mathbb{C}\otimes B$ are purely infinite simple Kirchberg algebras, then A and B are stably isomorphic if and only if $K^{CR}(A) \cong K^{CR}(B)$.*

(4) *If A and B are real C^*-algebras such that $\mathbb{C}\otimes A$ and $\mathbb{C}\otimes B$ are purely infinite simple unital Kirchberg algebras, then A and B are isomorphic if and only if $(K^{CR}(A), [1_A]) \cong (K^{CR}(B), [1_B])$.*

Proof. From [Boersema 2002, Theorem 1.12], we know that $K^{CRT}(A)$ is a $\mathcal{CRT}$-module, from which it follows immediately that $K^{CR}(A)$ is a $\mathcal{CR}$-module. By [Boersema 2004, Corollary 4.11] and [Boersema et al. 2011, Theorem 10.2], we know that statements (2), (3), and (4) are true when $K^{CR}(-)$ is replaced by $K^{CRT}(-)$ throughout. However, from [Hewitt 1996, Theorem 4.2.1], we know that $K^{CR}(A) \cong K^{CR}(B)$ if and only if $K^{CRT}(A) \cong K^{CRT}(B)$. □

From the point of view of calculations, it is often the case that once $KU_*(A)$ and a few of the $KO_*(A)$ groups are known, then the rest can be identified using the rich structure of a $\mathcal{CR}$-module, specifically the relations among the natural

n	0	1	2	3	4	5	6	7
KO_n	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
KU_n	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
η_n	1	1	0	0	0	0	0	0
c_n	1	0	0	0	2	0	0	0
r_n	2	0	1	0	1	0	0	0
ψ_n	1	0	-1	0	1	0	-1	0

Table 1. $K^{CR}(\mathbb{R})$.

n	0	1	2	3	4	5	6	7
KO_n	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
KU_n	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0
η_n	0	0	0	0	0	0	0	0
c_n	$\begin{pmatrix}1\\1\end{pmatrix}$	0	$\begin{pmatrix}-1\\1\end{pmatrix}$	0	$\begin{pmatrix}1\\1\end{pmatrix}$	0	$\begin{pmatrix}-1\\1\end{pmatrix}$	0
r_n	$(1\ 1)$	0	$(-1\ 1)$	0	$(1\ 1)$	0	$(-1\ 1)$	0
ψ_n	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	0	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	0	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	0	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	0

Table 2. $K^{CR}(\mathbb{C})$.

transformations above combined with the long exact sequence. Beatrice Hewitt [1996] found a way to boil down the information from an acyclic $\mathcal{CR}$-module into a simpler structure, called the *core* of the $\mathcal{CR}$-module. We introduce this helpful structure in Section 4 and use it to facilitate calculations of the K-theory of some specific higher-rank graph algebras.

3. The spectral sequence

This section, which is the theoretical cornerstone of the paper, takes inspiration from Evans' computations [2008] of K-theory for the complex C^*-algebras of higher-rank graphs. Our goal is to obtain a computable description of the spectral sequence which converges to $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$. The spectral sequence in question was introduced in [Kasparov 1988, 6.10] and applies to crossed product C^*-algebras. Thus, we begin by showing in Theorem 3.1 that $C^*_{\mathbb{R}}(\Lambda, \gamma)$ is stably isomorphic to $C^*_{\mathbb{R}}(\Lambda \times_d \mathbb{Z}^k, \gamma) \rtimes \mathbb{Z}^k$. Next, we establish (see Theorem 3.7) that Kasparov's spectral sequence [1988] encodes not only the real and complex K-theory groups, but also the $\mathcal{CR}$-module structure linking them. Having thus established the relevance of Kasparov's spectral sequence to our situation, in Section 3C we combine the AF structure of $C^*_{\mathbb{R}}(\Lambda \times_d \mathbb{Z}^k, \gamma)$ (Corollary 3.6) and its $\mathbb{Z}^k$-module structure to provide a more combinatorial description of the E^2 page of the spectral sequence in Theorem 3.13. Namely, we identify a chain complex whose homology computes the E^2 page of the spectral sequence. Our approach here follows the outline used in [Evans 2008] for complex C^*-algebras, although the intricate structure of real K-theory necessitates a few detours.

Thanks to the AF structure of $C^*_{\mathbb{R}}(\Lambda \times_d \mathbb{Z}^k, \gamma)$, the building blocks of this chain complex are direct sums of the K-theory of the two most basic real C^*-algebras, namely $\mathbb{R}$ and $\mathbb{C}$. In this situation, Lemma 3.14 establishes that the entire $\mathcal{CR}$-module structure is dictated by what happens at the level of the complex K-theory. Combining this insight with Evans' computations of the K-theory of complex k-graph C^*-algebras, we provide in Section 3D a more explicit description of the E^2 page of the spectral sequence for k-graphs with $k \leq 3$ and finitely many vertices. This description is fundamental to our analysis of the examples in Section 4.

3A. ***Structure of $C^*_{\mathbb{R}}(\Lambda, \gamma)$.*** Given a k-graph (Λ, γ) with involution, we can extend γ to an involution (also denoted γ) on the skew-product k-graph $\Lambda \times_d \mathbb{Z}^k$ by the formula

$$\gamma(\mu, n) = (\gamma(\mu), n).$$

The involution γ thus induces a real structure on the complex C^*-algebra $B = C^*(\Lambda \times_d \mathbb{Z}^k)$; we write

$$B_{\mathbb{R}} = C^*_{\mathbb{R}}(\Lambda \times_d \mathbb{Z}^k, \gamma)$$

for the associated real C^*-algebra. Recall that there is an action β of $\mathbb{Z}^k$ on B, given by $\beta(n)\cdot s_{\mu,m} = s_{\mu,m+n}$. Using the description of $B_\mathbb{R}$ which arises from Lemma 2.6, it is easy to see that the action β restricts to an action (also denoted β) of $\mathbb{Z}^k$ on $B_\mathbb{R}$. We also use the notation $\beta_i = \beta(e_i)$ for $i \in \{1, 2, \ldots, k\}$.

Theorem 3.1. *There is an isomorphism*

$$B_\mathbb{R} \rtimes_\beta \mathbb{Z}^k \cong C^*_\mathbb{R}(\Lambda, \gamma) \otimes_\mathbb{R} \mathcal{K}_\mathbb{R}$$

and hence

$$K^{CR}(C^*_\mathbb{R}(\Lambda, \gamma)) \cong K^{CR}(B_\mathbb{R} \rtimes_\beta \mathbb{Z}^k).$$

Proof. As in Corollary 5.3 of [Kumjian and Pask 2000], there is an isomorphism $C^*(\Lambda) \rtimes_\alpha \mathbb{T}^k \cong C^*(\Lambda \times_d \mathbb{Z}^k)$ of complex C^*-algebras, where α is the gauge action of $\mathbb{T}^k$ on $C^*(\Lambda)$. Furthermore, under this isomorphism, the dual action of $\mathbb{Z}^k$ on $C^*(\Lambda) \rtimes_\alpha \mathbb{T}^k$ corresponds to the action β on $B = C^*(\Lambda \times_d \mathbb{Z}^k)$ described above. By Takai duality (for complex C^*-algebras) we then have

$$B \rtimes_\beta \mathbb{Z}^k = C^*(\Lambda \times_d \mathbb{Z}^k) \rtimes_\beta \mathbb{Z}^k \cong (C^*(\Lambda) \rtimes_\alpha \mathbb{T}^k) \rtimes_\beta \mathbb{Z}^k \cong C^*(\Lambda) \otimes \mathcal{K}. \quad (3.2)$$

So far, all of this is exactly as indicated in [Kumjian and Pask 2000].

Now consider the involutions on each of these C^*-algebras. We show that the isomorphisms all respect the corresponding involutions. Recall from [Boersema 2014, Section 2] that a real C^*-dynamical system consists of a quintuple $(A, \bar{\cdot}, G, \bar{\cdot}, \alpha)$, where $(A, \bar{\cdot})$ is a complex C^*-algebra with conjugate-linear involution, $(G, \bar{\cdot})$ is a group with involution, and α is an action of G on A intertwining the involutions in the sense that

$$\overline{\alpha(g)(a)} = \alpha(\bar{g})(\bar{a}) \quad \text{for all } a \in A,\ g \in G.$$

If $(A, \bar{\cdot}, G, \bar{\cdot}, \alpha)$ is a real C^*-dynamical system then the crossed product $A \rtimes_\alpha G$ inherits a natural conjugate-linear involution [Boersema 2014, Theorem 2].

In our case, it is straightforward to check that the involution $\widetilde{\gamma}$ on B commutes with the action of β so that $(B, \widetilde{\gamma}, \mathbb{Z}^k, \mathrm{id}, \beta)$ is a real C^*-dynamical system. Similarly, the gauge action α intertwines with the involution $\widetilde{\gamma}$ on $C^*(\Lambda)$ so that $(C^*(\Lambda), \widetilde{\gamma}, \mathbb{T}^k, \tau, \alpha)$ is also a real C^*-dynamical system. Here τ is the involution on $\mathbb{T}^k$ given by $\tau(z_1, \ldots, z_k) = (\overline{z_1}, \ldots, \overline{z_n})$.

Furthermore, as groups with involution, $(\mathbb{T}^k, \tau)$ is dual to $(\mathbb{Z}^k, \mathrm{id})$ in the sense of [Boersema 2014, Section 3]. Therefore, by Takai duality for real C^*-algebras [Boersema 2014, Theorem 9], the isomorphisms of (3.2) are isomorphisms that respect the real structures, proving the theorem. □

As in [Evans 2008], for any $m \in \mathbb{Z}^k$ and $v \in \Lambda^0$, let

$$B_m(v) = \overline{\mathrm{span}}\,\{s_{\mu,m-d(\mu)}s^*_{\nu,m-d(\nu)} \mid s(\mu) = s(\nu) = v\},$$
$$B_m = \overline{\mathrm{span}}\,\{s_{\mu,m-d(\mu)}s^*_{\nu,m-d(\nu)} \mid s(\mu) = s(\nu) = v \text{ for some } v \in \Lambda^0\}.$$

Then, as in [Evans 2008, Lemma 3.4] or [Kumjian and Pask 2000, Lemma 5.4], we have $B = \varinjlim_{m\to\infty} B_m$ and there are isomorphisms

$$B_m(v) \cong \mathcal{K}(\ell^2(s^{-1}(v))) \quad \text{and} \quad B_m \cong \bigoplus_{v\in\Lambda^0} B_m(v),$$

which describe the structure of B as an AF-algebra. For $m \le n$, the inclusion map $\iota_{nm} : B_m \hookrightarrow B_n$ is determined on $s_{\mu,m-d(\mu)}s^*_{\nu,m-d(\nu)} \in B_m(v)$ by the fact that, by (CK4),

$$s_{\mu,m-d(\mu)}s^*_{\nu,m-d(\nu)} = \sum_{\substack{r(\alpha)=v\\ d(\alpha)=n-m}} s_{\mu\alpha,m-d(\mu)}s^*_{\nu\alpha,m-d(\nu)}. \tag{3.3}$$

Observe that the terms on the right-hand side all lie in B_n, as $d(\mu\alpha)+m-d(\mu) = d(\alpha)+m = n$; however, they generally lie in different summands $B_n(w)$.

Now, we consider the real structure on B_m and B. The involution $\widetilde{\gamma}$ on $B = C^*(\Lambda \times_d \mathbb{Z}^k)$ induced by γ satisfies $\widetilde{\gamma}(s_{\lambda,m}) = s^*_{\gamma(\lambda),m}$, so we have $\widetilde{\gamma}(B_m(v)) = B_m(\gamma(v))$ and $\widetilde{\gamma}(B_m) = B_m$. Therefore, $\widetilde{\gamma}$ gives a real structure on $B_m(v)$ (when v is a vertex fixed by γ) and on $B_m(v)\oplus B_m(\gamma(v))$ (when v is not fixed by γ). The following lemma describes the structure of the corresponding real C^*-algebras $B_m(v)_\mathbb{R}$ and $(B_m(v)\oplus B_m(\gamma(v)))_\mathbb{R}$.

Lemma 3.4. *With notation as above, if $\gamma(v) = v$, then $B_m(v)_\mathbb{R} \cong \mathcal{K}_\mathbb{R}(\ell^2(s^{-1}(v)))$. If $\gamma(v) \ne v$ then $(B_m(v)\oplus B_m(\gamma(v)))_\mathbb{R} \cong \mathcal{K}_\mathbb{C}(\ell^2(s^{-1}(v)))$.*

Proof. We first consider the case when $\gamma(v) = v$. Fix $j \in \mathbb{N}$ and decompose

$$J = J(j) := \{\lambda \in \Lambda v \mid d(\lambda) \le (j, j, \dots, j)\}$$

as $J = J_f \sqcup J_1 \sqcup J_2$, where $\gamma|_{J_f} = \mathrm{id}$ and $\gamma(J_1) = J_2$. We can view elements of $M_J(\mathbb{C})$ as lying in $B_m(v)$ under the identification $e_{\mu,\nu} \mapsto s_{\mu,m-d(\mu)}s^*_{\nu,m-d(\nu)}$. With this identification, the antimultiplicative involution $\widetilde{\gamma}$ is given on $M_J(\mathbb{C})$ by $\widetilde{\gamma}(e_{\mu,\nu}) = e_{\gamma(\nu),\gamma(\mu)}$. Furthermore, $B_m(v) = \varinjlim_{j\to\infty} M_{J(j)}(\mathbb{C})$; the connecting map $M_{J(j)} \to M_{J(j+1)}$ is determined by the inclusions $J_f(j) \subseteq J_f(j+1)$ and $J_1(j) \subseteq J_1(j+1)$.

It follows that every element in

$$M_J(\mathbb{C})_\mathbb{R} = \{a \in M_J(\mathbb{C}) \mid a^* = \widetilde{\gamma}(a)\}$$

is of the block form

$$\begin{pmatrix} A & B & \bar{B} \\ C & D & E \\ \bar{C} & \bar{E} & \bar{D} \end{pmatrix},$$

where A is real valued and B, C, D, E are complex valued matrices.

We claim that $M_J(\mathbb{C})_\mathbb{R} \cong M_J(\mathbb{R})$. Set $h = |J|$, $h_1 = |J_f|$, and $h_2 = |J_1| = |J_2|$ (so $h_1 + 2h_2 = h$). We know that (up to isomorphism) the only real C^*-algebras whose complexifications are isomorphic to $M_J(\mathbb{C}) = M_h(\mathbb{C})$ are $M_h(\mathbb{R})$ and $M_{h/2}(\mathbb{H})$, and the second possibility can only happen if h is even (see, for example, page 1 of [Schröder 1993]). We show there exists a system of h orthogonal projections in $M_J(\mathbb{C})_{\mathbb{R}}$, which precludes the existence of an isomorphism $M_J(\mathbb{C})_{\mathbb{R}} \cong M_{h/2}(\mathbb{H})$. Indeed, there are h_1 obvious orthogonal subprojections of

$$p = \begin{pmatrix} I_{h_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and similarly there are h_2 orthogonal subprojections of each of

$$q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} I_{h_2} & \frac{i}{2} I_{h_2} \\ 0 & -\frac{i}{2} I_{h_2} & \frac{1}{2} I_{h_2} \end{pmatrix} \quad \text{and} \quad q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} I_{h_2} & -\frac{i}{2} I_{h_2} \\ 0 & \frac{i}{2} I_{h_2} & \frac{1}{2} I_{h_2} \end{pmatrix}.$$

Notice that $p + q_1 + q_2 = I_h$. It follows that $M_J(\mathbb{C})_{\mathbb{R}}$ is isomorphic to $M_h(\mathbb{R})$. Moreover, it is evident that this choice of orthogonal subprojections is compatible with the inclusion maps of the inductive limit $B_m(v) \cong \varinjlim M_J(\mathbb{C})$. Hence, if $\gamma(v) = v$, $B_m(v)_{\mathbb{R}} \cong \mathcal{K}_{\mathbb{R}}(\ell^2(s^{-1}(v)))$ as claimed.

Now, suppose $\gamma(v) = w \neq v$. For any fixed $j \in \mathbb{N}$, γ is a bijection from $J_v := \{\lambda \in \Lambda v \mid d(\lambda) \leq (j, j, \dots, j)\}$ to $J_w := \{\mu \in \Lambda w \mid d(\mu) \leq (j, \dots, j)\}$. Therefore, for $(a, b) \in M_{J_v}(\mathbb{C}) \oplus M_{J_w}(\mathbb{C}) \subseteq B_m(v) \oplus B_m(w)$, the involution $\widetilde{\gamma}$ satisfies

$$\widetilde{\gamma}(a, b) = (b^t, a^t),$$

and so the associated real matrix algebra is $\{M \oplus \overline{M} \mid M \in M_{J_v}(\mathbb{C})\} \cong M_{J_v}(\mathbb{C})$. As $\mathcal{K}_{\mathbb{C}}(\ell^2(s^{-1}(v))) = \varinjlim_{j \to \infty} M_{J_v}(\mathbb{C}) \cong B_m(v) \oplus B_m(w)$, the result follows. □

As a complement to the abstract reasoning above, and inspired by [Boersema 2017, Theorem 2.5], we now exhibit a choice of basis for $\mathbb{C}^J$ which gives a more concrete argument for why $M_J(\mathbb{C})_{\mathbb{R}} \cong M_J(\mathbb{R})$ when $\gamma(v) = v$. Fix an arbitrary $n \in \mathbb{Z}^k$. For $\lambda \in J_f$ we define $t_\lambda := s_{\lambda, n-d(\lambda)}$, and if $\alpha \in J_1$ set

$$t_\alpha := \tfrac{1}{\sqrt{2}}(s_{\alpha, n-d(\alpha)} + s_{\gamma(\alpha), n-d(\alpha)}).$$

If $\beta \in J_2$ we define $t_\beta := \frac{i}{\sqrt{2}}(s_{\gamma(\beta), n-d(\beta)} - s_{\beta, n-d(\beta)})$. One easily computes that

$$\widetilde{\gamma}(t_\lambda) = t_\lambda^*$$

for any $\lambda \in J$, and that for any $\alpha, \beta \in J$ we have

$$t_\beta^* t_\alpha = t_\alpha^* t_\beta = \delta_{\alpha,\beta} s_{v,n}.$$

It follows that, for any $\alpha, \beta, \lambda, \eta \in J$,

$$t_\alpha t_\beta^* t_\lambda t_\eta^* = \delta_{\beta,\lambda} t_\alpha t_\eta^*.$$

In other words, $\{t_\alpha t_\beta^* \mid \alpha, \beta \in J\}$ is a set of matrix units, which spans $M_J(\mathbb{C})$ since $\{s_{\lambda,n-d(\lambda)} s^*_{\mu,n-d(\mu)} \mid \lambda, \mu \in J\}$ does, and which satisfies $\widetilde{\gamma}(t_\alpha t_\beta^*) = t_\beta t_\alpha^*$ for all $\alpha, \beta \in J$. With this basis, it is evident that $M_J(\mathbb{C})_\mathbb{R} = M_J(\mathbb{R})$.

Remark 3.5. If Λ is a directed graph (1-graph), the operators $\{t_\alpha \mid \alpha \text{ an edge}\}$ were used in [Boersema 2017, Theorem 2.4] to show that any vertex-fixing involution γ on Λ gives rise to the same real C^*-algebra as the trivial involution. However, this proof breaks down in the k-graph case for $k > 1$, because $\{t_\alpha \mid \alpha \in \Lambda\}$ need not satisfy the Cuntz–Krieger relations, even if all vertices are fixed by γ. In particular, if $ef \sim f'e'$ we need not have $t_e t_f = t_{f'} t_{e'}$. It remains an open question whether the conclusion of [Boersema 2017, Theorem 2.4] extends to higher-rank graphs with involution.

The following corollary is immediate from Lemma 3.4.

Corollary 3.6. *For each $m \in \mathbb{Z}^k$,*

$$B_m^\mathbb{R} \cong \bigoplus_{v \in G_f} \mathcal{K}_\mathbb{R}(\ell^2(s^{-1}(v))) \oplus \bigoplus_{v \in G_1} \mathcal{K}_\mathbb{C}(\ell^2(s^{-1}(v))),$$

where G_f is the set of vertices of Λ that are fixed by γ and G_1 is a set that contains exactly one vertex from every γ-orbit of cardinality 2. Consequently, $B_\mathbb{R} = C^*_\mathbb{R}(\Lambda \times_d \mathbb{Z}^k, \gamma) = \varinjlim B_m^\mathbb{R}$ is an AF real C^*-algebra.

3B. ***The spectral sequence via group homology.*** The main result of this section is the following.

Theorem 3.7. *There exists a spectral sequence $\{E^r, d^r\}$ of $\mathcal{CR}$-modules that converges to $K^{CR}(C^*_\mathbb{R}(\Lambda, \gamma))$ and has*

$$E^2_{p,q} = H_p(\mathbb{Z}^k, K^{CR}(B_\mathbb{R})).$$

In this spectral sequence, each object $E^r_{p,q}$ is a $\mathcal{CR}$-module and each map $d^r_{p,q}$ is a $\mathcal{CR}$-module homomorphism. The spectral sequence is defined for all $p, q \in \mathbb{Z}$, but it is periodic in q. (The real part has period 8 and the complex part has period 2.) Also, $E^r_{p,q} = 0$ for $p \notin \{0, 1, \ldots, k\}$.

Proof of Theorem 3.7. Let $k_*(B_\mathbb{R})$ denote one of the graded functors $KO_*(B_\mathbb{R})$ or $KU_*(B_\mathbb{R})$. Applying [Kasparov 1988, 6.10 Theorem] to the setting where $\pi = \mathbb{Z}^k$ and $D = B_\mathbb{R}$, we obtain a spectral sequence converging to the "γ-part" of $k_*(B_\mathbb{R} \rtimes \mathbb{Z}^k)$, and whose E^1 and E^2 pages are given by

$$E^1_{p,q} \cong k_{p+q}(D_p/D_{p-1}) \cong \bigoplus_{m: 1 \le m \le \binom{k}{p}} k_q(B_\mathbb{R}) \quad \text{and} \quad E^2_{p,q} \cong H_p(\mathbb{Z}^k, k_q(B_\mathbb{R})).$$

(Here $0 \subseteq D_0 \subseteq D_1 \subseteq \cdots \subseteq D_k = D_X$ is a filtration by ideals of a certain fixed-point algebra D_X, which is Morita equivalent to $B_{\mathbb{R}}$.) Since the Baum–Connes conjecture with arbitrary coefficients is true for $\mathbb{Z}^k$ [Schick 2004], this spectral sequence in fact converges precisely to $k_*(B_{\mathbb{R}} \rtimes \mathbb{Z}^k)$, which equals $k_*(C^*_{\mathbb{R}}(\Lambda, \gamma))$ by Theorem 3.1. Taking both of these spectral sequences together, we have a spectral sequence with both a real and a complex part that converges to $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$.

Now, let $k_*(B_{\mathbb{R}})$, $\tilde{k}_*(B_{\mathbb{R}})$ each independently denote one of the groups $KO_*(B_{\mathbb{R}})$ or $KU_*(B_{\mathbb{R}})$ and let $\theta : k_*(B_{\mathbb{R}}) \to \tilde{k}_*(B_{\mathbb{R}})$ be one of the natural transformations r, c, η, β, ψ of $K^{CR}(B_{\mathbb{R}}) = K^{CR}(D_X)$. Any one of these natural transformations can actually be represented by an element in $KK_*(C_1, C_2)$, where each C_i is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Multiplying by this KK-element induces the map

$$\theta^1 : k_{p+q}(D_p/D_{p-1}) \to \tilde{k}_{p+q}(D_p/D_{p-1}).$$

Thus the E^1 page of the spectral sequence also has a natural $\mathcal{CR}$-structure. Furthermore, as observed by Schochet [1981], the spectral sequence construction is natural not only with respect to filtered homomorphisms of filtered C^*-algebras but also with respect to natural transformations of exact functors (see the comments on [Schochet 1981, page 207]). As Kasparov's spectral sequence construction follows that of Schochet, it follows that $\theta^1 : E^1_* \to E^1_*$ commutes with the differentials and converges to the map $\theta^\infty : k_*(C^*_{\mathbb{R}}(\Lambda, \gamma)) \to \tilde{k}_*(C^*_{\mathbb{R}}(\Lambda, \gamma))$ induced by the original KK-element on the E^∞ page. Therefore, we can consider the spectral sequence as a spectral sequence in the category of $\mathcal{CR}$-modules.

At the E^2 page, we also have another $\mathcal{CR}$-module structure, induced by multiplying $k_*(B_{\mathbb{R}})$ by the KK-element representing the natural transformation θ. It remains to show that the isomorphism $E^2_{p,q} \cong H_p(\mathbb{Z}^k, k_q(B_{\mathbb{R}}))$ is a $\mathcal{CR}$-module isomorphism. Recall from [Kasparov 1988, p. 199] that, under the isomorphism $E^1_{p,q} \cong \bigoplus_m k_q(B_{\mathbb{R}})$, the differential map d^1 corresponds to the boundary homomorphism of the simplicial chain complex, yielding the isomorphism $E^2_{p,q} \cong H_p(\mathbb{Z}^k, k_*(B_{\mathbb{R}}))$. It then follows immediately that under this isomorphism, the map θ^2 on E^2 which is induced from $\theta^1 : k_{p+q}(D_p/D_{p-1}) \to \tilde{k}_{p+q}(D_p/D_{p-1})$ is identical to the map on $H_p(\mathbb{Z}^k, k_*(B_{\mathbb{R}}))$ which arises from $\theta^{B_{\mathbb{R}}} : k_*(B_{\mathbb{R}}) \to \tilde{k}_*(B_{\mathbb{R}})$ and the naturality of group homology (see for example Section III.6 of [Brown 1994]). Therefore, the $\mathcal{CR}$-module structure of $H_p(\mathbb{Z}^k, k_q(B_{\mathbb{R}}))$ is the same as that of $E^2_{p,q}$. □

3C. ***A combinatorial description of $E^2_{p,q}$.*** In this section, we use the structure of $B_{\mathbb{R}}$ as an AF algebra (Corollary 3.6) to obtain (in Theorem 3.13) a more explicit formula for the E^2 page of our spectral sequence from Theorem 3.7. To be precise, we identify a chain complex $\mathcal{A}^{(0)}$ whose p-th group $\mathcal{A}^{(0)}_p$ consists of $\binom{k}{p}$ copies of a certain $\mathcal{CR}$-module, and whose homology computes $E^2_{p,q}$. In Section 3D below, we provide an explicit description of the connecting maps of this chain complex in terms of the adjacency matrices of Λ, in the situation where $k \le 3$.

Recall that $B_{\mathbb{R}} = \varinjlim(B_m^{\mathbb{R}}, \iota_{nm})$, where (for $m \in \mathbb{Z}^k$)

$$\begin{aligned} B_m^{\mathbb{R}} &= \overline{\operatorname{span}}\,\{s_{\mu,m-d(\mu)} s^*_{\nu,m-d(\nu)} \mid s(\mu) = s(\nu) = v \text{ for some } v \in \Lambda^0\} \\ &\subseteq B_{\mathbb{R}} = C^*_{\mathbb{R}}(\Lambda \times_d \mathbb{Z}^k), \end{aligned}$$

and $\iota_{nm} : B_m^{\mathbb{R}} \hookrightarrow B_n^{\mathbb{R}}$ (for $m \le n \in \mathbb{Z}^k$) are the connecting maps (3.3) of the inductive system. Let $\jmath_{nm} := (\iota_{nm})_* : K^{CR}(B_m^{\mathbb{R}}) \to K^{CR}(B_n^{\mathbb{R}})$ be the induced map on united K-theory. Partition Λ^0 into three disjoint sets, $\Lambda^0 = G_f \sqcup G_1 \sqcup G_2$, where $\gamma|_{G_f} = \mathrm{id}$ and $\gamma(G_1) = G_2$. With this notation, Corollary 3.6 implies that

$$B_m^{\mathbb{R}} \cong \bigoplus_{v \in G_f} \mathcal{K}_{\mathbb{R}}(\ell^2(s^{-1}(v))) \oplus \bigoplus_{v \in G_1} \mathcal{K}_{\mathbb{C}}(\ell^2(s^{-1}(v)))$$

and consequently,

$$A_m := K^{CR}(B_m^{\mathbb{R}}) = K^{CR}(\mathbb{R})^{G_f} \oplus K^{CR}(\mathbb{C})^{G_1}.$$

The continuity of K-theory implies that

$$A_\infty := \varinjlim(A_m, \jmath_{nm}) \cong K^{CR}(B_{\mathbb{R}}).$$

As in [Evans 2008, Section 3], we define

$$N_p = \{(\mu_1, \dots, \mu_p) \mid \mu_i \in \mathbb{N}, 1 \le \mu_1 < \cdots < \mu_p \le k\}.$$

(The authors recognize that μ is also a common notation for an element of a k-graph Λ. We have chosen to follow Evans' notation, using μ_i and μ^i in reference to elements of N_p, for ease of cross-referencing. It should always be clear from context (and the presence of sub- and super-scripts) whether λ or μ refers to an element of N_p or of Λ.)

Observe that $|N_p| = \binom{k}{p}$. If $\mu = (\mu_1, \dots, \mu_p) \in N_p$ then for any $1 \le i \le p$, we write

$$\mu^i = \begin{cases} (\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_p) \in N_{p-1} & \text{if } p > 1, \\ \star & \text{if } p = 1. \end{cases}$$

Let $\mathcal{B}$ denote the chain complex of $\mathcal{CR}$-modules,

$$\mathcal{B} : 0 \to A_\infty \to \cdots \to \bigoplus_{N_p} A_\infty \to \cdots \to A_\infty \to 0.$$

Writing $\mathcal{B}_p := \bigoplus_{N_p} A_\infty$, the differentials $\partial_p : \mathcal{B}_p \to \mathcal{B}_{p-1}$ are defined by

$$\partial_p = \bigoplus_{\lambda \in N_{p-1}} \sum_{\mu \in N_p} \sum_{i=1}^{p} (-1)^{i+1} \delta_{\lambda,\mu^i} (\mathrm{id} - (\beta_{\mu_i})_*^{-1}), \tag{3.8}$$

where β is the usual action of $\mathbb{Z}^k$ on $B_{\mathbb{R}}$ given on generators by $\beta(n)s_{\mu,m} = s_{\mu,m+n}$. We have $\beta_j = \beta(e_j)$, and we write $(\beta_j)_*$ for the induced map on A_∞.

For an element $y \in \mathcal{B}_p$, we can write $y = \bigoplus_{\mu \in N_p} y_\mu$, where $y_\mu \in A_\infty$. We find it convenient to write such an element alternatively as $y = \sum_{\mu \in N_p} y_\mu e_\mu$, where $e_\mu \in \{0, 1\}^{N_p}$ satisfies $e_\mu(\lambda) = \delta_{\mu,\lambda}$. Using this notation, we can write the differentials of the complex $\mathcal{B}$ as

$$\partial_p(y_\mu e_\mu) = \sum_{i=1}^{p} (-1)^{i+1} (\mathrm{id} - (\beta_{\mu_i})_*^{-1})(y_\mu) e_{\mu^i} \quad \text{for } \mu \in N_p \text{ and } y_\mu \in A_\infty.$$

Lemma 3.9. *There is a graded isomorphism*

$$H_*(\mathbb{Z}^k, K^{CR}(B_{\mathbb{R}})) \cong H_*(\mathcal{B}).$$

Proof. This result is proven exactly as in the proof of Lemma 3.12 of [Evans 2008], making use of the Koszul resolution for $\mathbb{Z}$ over $\mathbb{Z}G$, where $G = \mathbb{Z}^k$ and then tensoring that resolution by $k_*(B_{\mathbb{R}})$, where the functor $k_*(-)$ is any of the functors $KO_i(-)$ and $KU_i(-)$. □

We now work towards a more concrete description of $H_*(\mathcal{B})$. For each $m \in \mathbb{N}^k$, let $\mathcal{A}^{(m)}$ denote the chain complex of $\mathcal{CR}$-modules

$$\mathcal{A}^{(m)} : 0 \to A_m \to \cdots \to \bigoplus_{\mu \in N_p} A_m \to \cdots \to A_m \to 0,$$

where, we recall, $A_m = K^{CR}(B_m^{\mathbb{R}})$. The differentials $\partial_p^{(m)}$ for $\mathcal{A}^{(m)}$ are defined by

$$\partial_p^{(m)}(y_\mu e_\mu) = \sum_{i=1}^{p} (-1)^{i+1} (\mathrm{id} - \phi_{\mu_i}^m)(y_\mu) e_{\mu^i} \quad \text{for } \mu \in N_p \text{ and } y_\mu \in A_m,$$

where $\phi_j^m : K^{CR}(B_m^{\mathbb{R}}) \to K^{CR}(B_m^{\mathbb{R}})$ is the map induced on $K^{CR}(-)$ by the composition

$$B_m^{\mathbb{R}} \xrightarrow{\iota_{m+e_j,m}} B_{m+e_j}^{\mathbb{R}} \xrightarrow{\beta(-e_j)} B_m^{\mathbb{R}}.$$

Recall that $\jmath_{nm} : A_m \to A_n$ is the map induced on K-theory by the inclusion map $\iota_{nm} : B_m \to B_n$ of (3.3). For each $m \le n$, we extend the map $\jmath_{nm} : A_m \to A_n$ to a chain map $\mathfrak{J}_{nm} : \mathcal{A}^{(m)} \to \mathcal{A}^{(n)}$ defined by

$$\mathfrak{J}_{nm}^p \Big(\sum_{\mu \in N_p} y_\mu e_\mu \Big) = \sum_{\mu \in N_p} \jmath_{nm}(y_\mu) e_\mu.$$

Lemma 3.10. $\mathfrak{J}_{nm}$ *is a chain map for all* $m \le n$. *Furthermore, there is an isomorphism of chain complexes* $\mathcal{B} \cong \varinjlim(\mathcal{A}^{(m)}, \mathfrak{J}_{nm})$.

Proof. The claim that $\mathfrak{J}_{nm}$ is a chain map is, by definition, the claim that the diagram

$$\begin{tikzcd} \mathcal{A}_p^{(m)} \arrow[r, "\partial_p^{(m)}"] \arrow[d, "\mathfrak{J}_{nm}"] & \mathcal{A}_{p-1}^{(m)} \arrow[d, "\mathfrak{J}_{nm}"] \\ \mathcal{A}_p^{(n)} \arrow[r, "\partial_p^{(n)}"] & \mathcal{A}_{p-1}^{(n)} \end{tikzcd}$$

commutes for all p. Focusing on each summand of $\mathcal{A}_p^{(m)} = \bigoplus_{\mu\in N_p} A_m$, this is evidently equivalent to the commuting of the diagram

$$\begin{tikzcd} A_m \arrow[r, "\phi_{\mu_i}^m"] \arrow[d, "\mathfrak{j}_{nm}"] & A_m \arrow[d, "\mathfrak{j}_{nm}"] \\ A_n \arrow[r, "\phi_{\mu_i}^n"] & A_n \end{tikzcd}$$

for all i. On the level of C^*-algebras, this follows from the relation

$$\beta(-e_j)\circ\iota_{n+e_j,m} = \iota_{n,m}\circ\beta(-e_j)\circ\iota_{m+e_j,m},$$

which holds thanks to the fact that every $\gamma\in\Lambda^{n-m+e_j}$ can be factored as $\gamma=\gamma_1\gamma_2$ for a unique $\gamma_1\in\Lambda^{e_j}$, $\gamma_2\in\Lambda^{n-m}$.

Now we prove the second statement. Using the isomorphism $\varinjlim K^{CR}(B_m^{\mathbb{R}}) = K^{CR}(B_{\mathbb{R}})$, we immediately obtain $\varinjlim \mathcal{A}_p^{(m)} = \mathcal{B}_p$ for all p. For each $m\in\mathbb{Z}^k$, let $\mathfrak{J}_m : \mathcal{A}^{(m)}\to\mathcal{B}$ be the map into the limit; this can also be described by

$$\mathfrak{J}_m^p\Big(\sum_{\mu\in N_p} y_\mu e_\mu\Big) = \sum_{\mu\in N_p} \mathfrak{j}_m(y_\mu e_\mu),$$

where $\mathfrak{j}_m : K^{CR}(B_m^{\mathbb{R}}) \to K^{CR}(B_{\mathbb{R}})$ is the map induced by the inclusion $B_m^{\mathbb{R}} \hookrightarrow B_{\mathbb{R}}$. It remains to show that the diagram

$$\begin{tikzcd} \mathcal{A}_p^{(m)} \arrow[r, "\partial_p^{(m)}"] \arrow[d, "\mathfrak{J}_m"] & \mathcal{A}_{p-1}^{(m)} \arrow[d, "\mathfrak{J}_m"] \\ \mathcal{B}_p \arrow[r, "\partial_p"] & \mathcal{B}_{p-1} \end{tikzcd}$$

commutes, for which it suffices to show that

$$\begin{tikzcd} A_m \arrow[r, "\phi_{\mu_i}^m"] \arrow[d, "\mathfrak{j}_m"] & A_m \arrow[d, "\mathfrak{j}_m"] \\ A_\infty \arrow[r, "(\beta_{\mu_i})_*^{-1}"] & A_\infty \end{tikzcd}$$

commutes for all i. This follows from the definition of $\phi^m_{\mu_i}$ and the fact that the diagram

$$\begin{tikzcd} B_m \arrow[r, "\beta(-e_i)"] \arrow[d, "\iota_m"] & B_{m-1} \arrow[d, "\iota_{m-1}"] \\ B \arrow[r, "\beta(-e_i)"] & B \end{tikzcd}$$

commutes on the level of C^*-algebras. □

The following lemma is the last key technical result that we need. The proof of the corresponding statement in the complex case is the bulk of the proof of Theorem 3.14 of [Evans 2008]. The proof there is quite technical. Our proof is too, and here we have the additional complication of working in the category of $\mathcal{CR}$-modules, rather than the category of abelian groups. We mitigate some of this technicality through the use of the e_μ notation introduced above, as well as making explicit use of the concept of a chain homotopy, which Evans did not do.

In addition to the chain map $\mathfrak{J}_{nm} : \mathcal{A}^{(m)} \to \mathcal{A}^{(n)}$ we also have the chain map $\mathfrak{B}_{nm} : \mathcal{A}^{(m)} \to \mathcal{A}^{(n)}$ for $m \neq n$, defined by the action $\beta(n-m)_* : K^{CR}(B_m) \to K^{CR}(B_n)$ extended to $\mathcal{A}^{(m)} = \bigoplus_{N_p} A_m = \bigoplus_{N_p} K^{CR}(B_m)$. It is routine to show that $\mathfrak{B}_{nm}$ is a chain map. In fact, since $\beta(n-m) : B_n \to B_m$ is an isomorphism, $\mathfrak{B}_{nm}$ is an isomorphism of chain complexes.

Lemma 3.11. *For all $m \le n$, the chain maps $\mathfrak{B}_{nm}$ and $\mathfrak{J}_{nm}$ are chain homotopic. Thus the induced map $(\mathfrak{J}_{nm})_* : H_*(\mathcal{A}^{(m)}) \to H_*(\mathcal{A}^{(n)})$ is an isomorphism.*

Proof. It suffices to prove the claim for $\mathfrak{B}_{m+e_j,m}$ and $\mathfrak{J}_{m+e_j,m}$ for arbitrary m, j. We fix m, j for the remainder of this proof and write $\mathfrak{J} = \mathfrak{J}_{m+e_j,m}$ and $\mathfrak{B} = \mathfrak{B}_{m+e_j,m}$. For $\mu \in N_p$, let $\kappa(\mu)$ denote the cardinality of $\{i \in \{1, \dots, p\} \mid \mu_i < j\}$. Now let $\sigma^p : \mathcal{A}_p^{(m)} \to \mathcal{A}_{p+1}^{(m+e_j)}$ be the map defined by

$$\sigma^p(y_\mu e_\mu) = \begin{cases} (-1)^{\kappa(\mu)}(\beta_j)_*(y_\mu)e_{\mu\cup\{j\}} & \text{if } j \notin \mu, \\ 0 & \text{if } j \in \mu. \end{cases}$$

This definition of σ is inspired by the choice of z in the proof of Theorem 3.14 in [Evans 2008].

We show that for all $y \in \mathcal{A}_p^{(m)}$ we have (suppressing the superscripts for ∂_p)

$$\partial_{p+1}\sigma^p(y) + \sigma^{p-1}\partial_p(y) = (\mathfrak{B}^p - \mathfrak{J}^p)y,$$

so that σ provides the desired chain homotopy between $\mathfrak{B}$ and $\mathfrak{J}$.

It suffices by linearity to assume that $y = y_\mu e_\mu$ for some $\mu \in N_p$ and $y_\mu \in A_m$. First we consider the case $j \in \mu$; so $\mu_{\kappa(\mu)+1} = j$. Then, writing ϕ_j for ϕ_j^m and $\jmath_j$

for $\mathfrak{j}_{m+e_j,m}$, we have

$$\begin{aligned}
&\partial_{p+1}\sigma^p(y_\mu e_\mu)+\sigma^{p-1}\partial_p(y_\mu e_\mu)\\
&\quad=0+\sigma^{p-1}\Bigg(\sum_{i=1}^{p}(-1)^{i+1}(\mathrm{id}-\phi_{\mu_i})(y_\mu)e_{\mu^i}\Bigg)\\
&\quad=(-1)^{\kappa(\mu)}\sigma^{p-1}((\mathrm{id}-\phi_j)(y_\mu)e_{\mu^{\kappa(\mu)+1}}) && \text{since } j\in\mu^i \text{ unless } i=\kappa(\mu)+1\\
&\quad=(-1)^{\kappa(\mu)}(-1)^{\kappa(\mu)}(\beta_j)_*(\mathrm{id}-\phi_j)(y_\mu)e_\mu && \text{since } \kappa(\mu^{\kappa(\mu)+1})=\kappa(\mu)\\
&\quad=((\beta_j)_*-\mathfrak{j}_j)(y_\mu)e_\mu && \text{since } \phi_j=(\beta_j)_*^{-1}(\iota_j)_*=(\beta_j)_*^{-1}(\mathfrak{j}_j)\\
&\quad=(\mathfrak{B}^p-\mathfrak{J}^p)(y_\mu)e_\mu.
\end{aligned}$$

Now, consider the case $j\notin\mu$. Then

$$\begin{aligned}
\sigma^{p-1}\partial_p(y_\mu e_\mu)&=\sigma^{p-1}\Bigg(\sum_{i=1}^{p}(-1)^{i+1}(\mathrm{id}-\phi_{\mu_i})(y_\mu)e_{\mu^i}\Bigg)\\
&=\sum_{i=1}^{p}(-1)^{i+1}(-1)^{\kappa(\mu^i)}(\beta_j)_*(\mathrm{id}-\phi_{\mu_i})(y_\mu)e_{\mu^i\cup\{j\}},\\
\partial_{p+1}\sigma^p(y_\mu e_\mu)&=\partial_{p+1}((-1)^{\kappa(\mu)}(\beta_j)_*y_\mu e_{\mu\cup\{j\}})\\
&=\sum_{i=1}^{p+1}(-1)^{\kappa(\mu)}(-1)^{i+1}(\mathrm{id}-\phi_{(\mu\cup\{j\})_i})(\beta_j)_*(y_\mu)e_{(\mu\cup\{j\})^i}.
\end{aligned}$$

In this last sum, any term with $i\le\kappa(\mu)$ is equal to

$$\begin{aligned}
&(-1)^{\kappa(\mu)}(-1)^{i+1}(\mathrm{id}-\phi_{\mu_i})(\beta_j)_*(y_\mu)e_{\mu^i\cup\{j\}}\\
&\qquad\qquad=(-1)^{\kappa(\mu^i)+1}(-1)^{i+1}(\mathrm{id}-\phi_{\mu_i})(\beta_j)_*(y_\mu)e_{\mu^i\cup\{j\}}
\end{aligned}$$

while any term with $i\ge\kappa(\mu)+2$ is equal to

$$\begin{aligned}
&(-1)^{\kappa(\mu)}(-1)^{i+1}(\mathrm{id}-\phi_{\mu_{i-1}})(\beta_j)_*(y_\mu)e_{\mu^{i-1}\cup\{j\}}\\
&\qquad\qquad=(-1)^{\kappa(\mu^{i-1})}(-1)^{i-1}(\mathrm{id}-\phi_{\mu_{i-1}})(\beta_j)_*(y_\mu)e_{\mu^{i-1}\cup\{j\}}.
\end{aligned}$$

As the maps ϕ_{μ_i} and β_j^* commute for all i, j, in the sum

$$\partial_{p+1}\sigma^p(y_\mu e_\mu)+\sigma^{p-1}\partial_p(y_\mu e_\mu),$$

all these terms cancel out, and the only term that remains is the summand of $\partial_{p+1}\sigma^p(y_\mu e_\mu)$ corresponding to $i=\kappa(\mu)+1$. Therefore,

$$\begin{aligned}
\partial_{p+1}\sigma^p(y_\mu e_\mu)+\sigma^{p-1}\partial_p(y_\mu e_\mu)&=(-1)^{\kappa(\mu)}(-1)^{\kappa(\mu)+2}(\mathrm{id}-\phi_j)(\beta_j)_*(y_\mu)e_\mu\\
&=(\mathrm{id}-\phi_j)(\beta_j)_*(y_\mu)e_\mu\\
&=(\mathfrak{B}^p-\mathfrak{J}^p)(y_\mu)e_\mu.
\end{aligned}$$

□

Lemma 3.12. $$H_*(\mathcal{B}) \cong H_*(\mathcal{A}^{(0)}).$$

Proof. From Lemma 3.10 and the continuity of the homology functor, we have $H_*(\mathcal{B}) = \lim_{m\to\infty}(H_*(\mathcal{A}^{(m)}), (\mathfrak{I}_{nm})_*)$. However, Lemma 3.11 shows that the connecting maps of the limit are all isomorphisms. Therefore $H_*(\mathcal{B}) \cong H_*(\mathcal{A}^{(0)})$. $\square$

Theorem 3.13. *Let (Λ, γ) be a k-graph with involution that is row-finite and has no sources. Then there exists a spectral sequence $\{E^r, d^r\}$ converging to $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$ such that $E^2_{p,q} \cong H_p(\mathcal{A}^{(0)})$ and $E^{k+1}_{p,q} \cong E^\infty_{p,q}$.*

Proof. Theorem 3.7 gives the existence of the spectral sequence $\{E^r, d^r\}$. Lemmas 3.9 and 3.12 combine to provide the isomorphism $E^2_{p,q} = H_p(\mathcal{A}^{(0)})$. The isomorphism $E^{k+1}_{p,q} \cong E^\infty_{p,q}$ results from the fact that $E^2_{p,q} = H_p(\mathbb{Z}^k, k_q(B_{\mathbb{R}})) = 0$ if $p \geq k+1$, so all of the differential maps $d^r_{p,q}$ are zero for $r \geq k+1$. $\square$

3D. ***Notes on computations using the spectral sequence.*** We say that a k-graph Λ is *finite* if the number of vertices is finite and the number of edges of degree e_i is finite for each i. In this subsection we articulate Theorem 3.13 more precisely in the specific cases of a finite k-graph Λ for $k = 1, 2, 3$. That is, we identify the boundary maps of the chain complex $\mathcal{A}^{(0)}$, in order to describe $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$ in terms of the purely combinatorial data coming from the k-graph and its involution.

Throughout, we assume that Λ is finite with involution γ. We partition Λ^0 into three disjoint sets, $\Lambda^0 = G_f \sqcup G_1 \sqcup G_2$, where $\gamma|_{G_f} = \mathrm{id}$ and $\gamma(G_1) = G_2$. Let A denote the $\mathcal{CR}$-module

$$A := K^{CR}(\mathbb{R})^{G_f} \oplus K^{CR}(\mathbb{C})^{G_1}.$$

Recall that $A^U_0 = \mathbb{Z}^{\Lambda^0}$. Thanks to Theorem 3.13, the E^2 page of our spectral sequence for $C^*_{\mathbb{R}}(\Lambda, \gamma)$ is given by the homology of the chain complex $\mathcal{A}^{(0)}$, all of whose component $\mathcal{CR}$-modules are direct sums of A.

We first establish a handy lemma that will facilitate our description of the boundary maps of the chain complex $\mathcal{A}^{(0)}$.

Lemma 3.14. *Let M, N be two $\mathcal{CR}$-modules, which are each isomorphic to a finite direct sum of $K^{CR}(\mathbb{R})$ and $K^{CR}(\mathbb{C})$. Then any $\mathcal{CR}$-module homomorphism $\alpha : M \to N$ is determined by the complex part α^U_0.*

Proof. It suffices to consider the cases that M is isomorphic to either $K^{CR}(\mathbb{R})$ or to $K^{CR}(\mathbb{C})$. Recall that the $\mathcal{CR}$-module $K^{CR}(\mathbb{R})$ is free with a generator in the real part in degree 0 and $K^{CR}(\mathbb{C})$ is free with a generator in the complex part in degree 0 [Bousfield 1990, Section 4.7]. Thus the result is immediate in the case $M = K^{CR}(\mathbb{C})$.

Now suppose that $M = K^{CR}(\mathbb{R})$ with generator $b \in M^O_0$. We must show that $\alpha^O_0(b)$ is uniquely determined by α^U. We have $c(\alpha^O_0(b)) = \alpha^U_0(c(b))$, where c is the complexification map from M^O to M^U, or from N^O to N^U. The complexification map c in degree 0 is injective for both $K^{CR}(\mathbb{R})$ and for $K^{CR}(\mathbb{C})$. Thus, the formula $c(\alpha_0^O(b)) = \alpha_0^U(c(b))$ determines $\alpha_0^O(b)$. $\square$

Recall from (2.7) that M_i is the adjacency matrix of Λ for the edges of degree e_i.

Definition 3.15. For $1 \le i \le k$, let $\rho^i : A \to A$ be the unique $\mathcal{CR}$-module homomorphism such that $(\rho^i)_0^U : \mathbb{Z}^{\Lambda^0} \to \mathbb{Z}^{\Lambda^0}$ is represented by the matrix $B_i = \mathrm{id} - M_i^t$.

Remark 3.16. Lemma 3.10 above combines with [Evans 2008, Lemma 3.10] to reveal that the $\mathcal{CR}$-module homomorphism $(\beta_i^{-1})_*$ used in the definition of ∂_p (see (3.8) above) agrees with ρ^i.

Lemma 3.14 tells us that $(\rho^i)_j^O$ is completely determined by $(\rho^i)_0^U$. The computation of $(\rho^i)_j^O$ from $(\rho^i)_0^U$ follows the same method as indicated in [Boersema 2017, Theorem 4.4]. In particular, if the complex part

$$(\rho^i)_0^U \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^{\Lambda^0}) = \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^{G_f} \oplus \mathbb{Z}^{G_1} \oplus \mathbb{Z}^{G_2})$$

is given by the matrix $B_i = I - M_i^t$, then the functoriality of γ implies that γ implements a bijection between the edges of color i with source in G_1 and range in G_2, and the edges of color i with source in G_2 and range in G_1. Similarly, the edges with both source and range in G_1 are in bijection with the edges with source and range in G_2. In other words,

$$B_i = \begin{pmatrix} B_{11} & B_{12} & B_{12} \\ B_{21} & B_{22} & B_{23} \\ B_{21} & B_{23} & B_{22} \end{pmatrix}.$$

It now follows that the real part $(\rho^i)_0^O \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^{G_f} \oplus \mathbb{Z}^{G_1})$ is given by the matrix

$$\begin{pmatrix} B_{11} & 2B_{12} \\ B_{21} & B_{22} + B_{23} \end{pmatrix}.$$

The other formulas for $(\rho^i)_j^O$ can be deduced from this easily; they are also given in [Boersema 2017, Theorem 4.4]. For the convenience of the reader, we reproduce the relevant table in Table 3.

Once the maps ρ^i are understood, Theorem 3.13 can be applied to develop the spectral sequence to compute $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$. The following theorems articulate exactly how this looks in the cases $k = 1, 2, 3$. We note that for the case $k = 1$ we recover Theorem 4.1 of [Boersema 2017].

Theorem 3.17 (cf. [Boersema 2017, Theorem 4.1]). *Let (Λ, γ) be a finite 1-graph with involution. Then there is a 2-column spectral sequence that converges to $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$ with $E^2_{p,q}$ equal to the homology of the chain complex $\mathcal{A}^{(0)}$,*

$$0 \to A \xrightarrow{\partial_1} A \to 0,$$

where $\partial_1 = \rho^1$.

complex part		
0	$\begin{pmatrix} B_{11} & B_{12} & B_{12} \\ B_{21} & B_{22} & B_{23} \\ B_{21} & B_{23} & B_{22} \end{pmatrix}$	$\mathbb{Z}^{\lvert G_f\rvert} \oplus \mathbb{Z}^{\lvert G_1\rvert} \oplus \mathbb{Z}^{\lvert G_2\rvert} \to \mathbb{Z}^{\lvert G_f\rvert} \oplus \mathbb{Z}^{\lvert G_1\rvert} \oplus \mathbb{Z}^{\lvert G_2\rvert}$
1	0	0
real part		
0	$\begin{pmatrix} B_{11} & 2B_{12} \\ B_{21} & B_{22}+B_{23} \end{pmatrix}$	$\mathbb{Z}^{\lvert G_f\rvert} \oplus \mathbb{Z}^{\lvert G_1\rvert} \to \mathbb{Z}^{\lvert G_f\rvert} \oplus \mathbb{Z}^{\lvert G_1\rvert}$
1	B_{11}	$\mathbb{Z}_2^{\lvert G_f\rvert} \to \mathbb{Z}_2^{\lvert G_f\rvert}$
2	$\begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22}-B_{23} \end{pmatrix}$	$\mathbb{Z}_2^{\lvert G_f\rvert} \oplus \mathbb{Z}^{\lvert G_1\rvert} \to \mathbb{Z}_2^{\lvert G_f\rvert} \oplus \mathbb{Z}^{\lvert G_1\rvert}$
3	0	0
4	$\begin{pmatrix} B_{11} & B_{12} \\ 2B_{21} & B_{22}+B_{23} \end{pmatrix}$	$\mathbb{Z}^{\lvert G_f\rvert} \oplus \mathbb{Z}^{\lvert G_1\rvert} \to \mathbb{Z}^{\lvert G_f\rvert} \oplus \mathbb{Z}^{\lvert G_1\rvert}$
5	0	0
6	$B_{22}-B_{23}$	$\mathbb{Z}^{\lvert G_1\rvert} \to \mathbb{Z}^{\lvert G_1\rvert}$
7	0	0

Table 3. Table for real K-theory.

Proof. As $k=1$, we have $|N_0| = |N_1| = 1$. Therefore, in this case (3.8) simplifies to $\partial_1 = \mathrm{id} - (\beta_1)_*^{-1}$ By Remark 3.16, $(\beta_1^{-1})_*$ agrees with the map whose complex part is represented by the matrix M_1^t. That is, $\partial_1 = \rho^1$. □

Theorem 3.18. *Let (Λ, γ) be a finite 2-graph with involution. Then there is a 3-column spectral sequence that converges to $K^{CR}(C^*_{\mathbb{R}}(\Lambda,\gamma))$ with $E^2_{p,q}$ equal to the homology of the chain complex $\mathcal{A}^{(0)}$,*

$$0 \to A \xrightarrow{\partial_2} A^2 \xrightarrow{\partial_1} A \to 0,$$

where

$$\partial_1 = \begin{pmatrix} \rho^1 & \rho^2 \end{pmatrix}, \qquad \partial_2 = \begin{pmatrix} -\rho^2 \\ \rho^1 \end{pmatrix}.$$

Proof. When $k=2$, we have $|N_1| = 2$ and $|N_2| = |N_0| = 1$. Therefore, (3.8) and Remark 3.16 tell us that $\partial_1 : \mathcal{A}^2 \to \mathcal{A}$ and $\partial_2 : \mathcal{A} \to \mathcal{A}^2$ are given by

$$\partial_1 = \sum_{\mu \in \{1,2\}} (\mathrm{id} - (\beta_\mu)_*^{-1}) = \begin{pmatrix} \rho^1 & \rho^2 \end{pmatrix},$$

$$\partial_2 = (-1)(\mathrm{id} - (\beta_2)_*^{-1}) \oplus (\mathrm{id} - (\beta_1)_*^{-1}) = \begin{pmatrix} -\rho^2 \\ \rho^1 \end{pmatrix}. \qquad \square$$

Theorem 3.19. *Let (Λ, γ) be a finite 3-graph with involution. Then there is a 4-column spectral sequence that converges to $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$ with $E^2_{p,q}$ equal to the homology of the chain complex $\mathcal{A}^{(0)}$,*

$$0 \to A \xrightarrow{\partial_3} A^3 \xrightarrow{\partial_2} A^3 \xrightarrow{\partial_1} A \to 0,$$

where

$$\partial_1 = \begin{pmatrix} \rho^1 & \rho^2 & \rho^3 \end{pmatrix}, \qquad \partial_2 = \begin{pmatrix} -\rho^2 & -\rho^3 & 0 \\ \rho^1 & 0 & -\rho^3 \\ 0 & \rho^1 & \rho^2 \end{pmatrix}, \qquad \partial_3 = \begin{pmatrix} \rho^3 \\ -\rho^2 \\ \rho^1 \end{pmatrix}.$$

Proof. We justify the formula for ∂_3 and leave the remaining cases to the reader. As $k = 3$, we have $|N_3| = 1$ and $|N_2| = 3$. Write $N_3 = \{\{1, 2, 3\}\} = \{\mu\}$. Given $1 \le i \le 3$, there is a unique $\lambda \in N_2$ with $\lambda = \mu^i$. Ordering $N_2 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ lexicographically, (3.8) becomes

$$\partial_3 = (-1)^{3+1}(\mathrm{id} - (\beta_3)_*^{-1}) \oplus (-1)^{2+1}(\mathrm{id} - (\beta_2)_*^{-1}) \oplus (-1)^{1+1}(\mathrm{id} - (\beta_1)_*^{-1}) = \begin{pmatrix} \rho^3 \\ -\rho^2 \\ \rho^1 \end{pmatrix}. \qquad \square$$

4. Examples

In this section, we give three families of examples of real C^*-algebras that arise from rank-2 graphs with involution. These examples showcase how one can leverage the $\mathcal{CR}$-module structure of real K-theory to completely determine $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$ on the basis of a small amount of initial data. In all three examples, our strategy follows the same general outline. We begin by identifying the chain complex of Theorem 3.18 and computing its homology, which gives us the E^2 page of the spectral sequence. As $k = 2$ in all of our examples, we have $E^\infty_{pq} = E^3_{pq}$ for all p, q; thus, our next step is to identify the differential d^2, which determines the $E^3 = E^\infty$ page. However, knowing the E^∞ page does not completely describe $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$; rather, it gives a filtration (of at most 3 levels in the $k = 2$ case) of $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$.

In our chosen examples, the $\mathcal{CR}$-module structure (and in particular the concept of the *core* of a $\mathcal{CR}$-module, as introduced in [Hewitt 1996]) enable us to describe $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$, up to at most two possibilities, using only the data from the E^2 page. As the core is a key tool in all of our computations in this section, we pause to discuss it in more detail.

To that end, recall that we have an involution ψ on $KU_*(C^*_{\mathbb{R}}(\Lambda, \gamma)) = K_*(C^*(\Lambda))$ which comes from the real structure on $C^*(\Lambda)$. Moreover, since $KO_*(C^*_{\mathbb{R}}(\Lambda, \gamma))$ is a graded module over $KO_*(\mathbb{R})$, for each i we have

$$\eta_{i-1} : KO_{i-1}(C^*_{\mathbb{R}}(\Lambda, \gamma)) \to KO_i(C^*_{\mathbb{R}}(\Lambda, \gamma)),$$

which comes from multiplication by the nontrivial element in $KO_1(\mathbb{R}) = \mathbb{Z}_2$. Thus, we can define

$$\begin{aligned} MO_i &= \operatorname{im} \eta_{i-1} : KO_{i-1}(C^*_{\mathbb{R}}(\Lambda, \gamma)) \to KO_i(C^*_{\mathbb{R}}(\Lambda, \gamma)), \\ MU_i &= \frac{\ker(1-\psi_i)}{\operatorname{im}(1+\psi_i)}. \end{aligned} \tag{4.1}$$

Note that since the MO_i groups arise from the map η, which satisfies $2\eta = 0$, every MO_i group is also 2-torsion. A straightforward computation shows that the MU_i groups are also always 2-torsion.

The maps η, c, r of $K^{CR}_*(C^*(\Lambda, \gamma))$ then naturally induce maps η', c', r' on the groups MO_i and MU_i, and we obtain a long exact sequence

$$\cdots \to MO_i \xrightarrow{\eta'} MO_{i+1} \xrightarrow{c'} MU_i \xrightarrow{r'} MO_{i-2} \to \cdots \tag{4.2}$$

(see [Hewitt 1996, Section 5.1]). The core of the $\mathcal{CR}$-module $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$ is defined to consist of $KU_*(C^*_{\mathbb{R}}(\Lambda, \gamma))$, the map ψ, and the groups and maps of the long exact sequence (4.2).

Thus, $KU_*(C^*_{\mathbb{R}}(\Lambda, \gamma))$ is retained but $KO_*(C^*_{\mathbb{R}}(\Lambda, \gamma))$ itself is dropped when we pass to the core; so on the face of it, we lose information. However, it follows from Theorem 4.2.1 of [Hewitt 1996] that for two real C^*-algebras, $K^{CR}(A_1) \cong K^{CR}(A_2)$ if and only if the cores of $K^{CR}(A_1)$ and $K^{CR}(A_2)$ are isomorphic. Indeed, in our examples below, we compute some of the groups and maps in $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$ by using the spectral sequence, and then compute the core of the $\mathcal{CR}$-module to complete the identification of $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$. This saves the work of having to compute all of the groups of $KO_*(C^*_{\mathbb{R}}(\Lambda, \gamma))$ directly.

Notably, the factorization rules of the k-graph Λ are irrelevant to the computations of the E^2 page. Thus, the examples in this section support the conjecture [Barlak et al. 2018, Conjecture 5.11] that the K-theory of a k-graph C^*-algebra should be (largely) independent of the choice of factorization rules.

In cases where we have multiple possibilities for $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$, the ambiguity comes from the fact that we have multiple possibilities for the d^2 map. In more complicated examples, it is also possible that the filtration of $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$ given on the E^∞ page might not arise from a unique collection of K-theory groups. We anticipate that a careful analysis of the impact of the factorization rules on $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$ may clarify these questions.

The first family of examples we consider, in Section 4A, are 2-graphs with only one vertex but an arbitrary number of edges of each type. In the second family of examples (Section 4B) we consider 2-graphs with exactly three vertices, where the adjacency matrix is the same for both types of edges. Finally, in Section 4C, we consider a family of 2-graphs with exactly three vertices but which have two distinct adjacency matrices. Our computations result in a variety of different $\mathcal{CR}$-modules, many of which (but not all) have appeared in the literature before now or are direct sums of $\mathcal{CR}$-modules that have appeared before.

All of the examples that we present have K-theory that is not consistent with a 1-graph algebra, since they all have torsion in $KU_1(C^*(\Lambda))$ [Boersema 2017, Corollary 4.3]. In fact, in all of our examples, the complex K-theory is consistent with that of a tensor product $\mathcal{O}_m \otimes \mathcal{O}_n$ of complex Cuntz algebras. Therefore, in the case that the resulting real C^*-algebra is purely infinite and simple, [Boersema et al. 2011, Corollary 10.5] implies that they are all real forms of $\mathcal{O}_m \otimes \mathcal{O}_n$.

4A. ***A* 1*-vertex* 2*-graph.*** Let Λ be a rank-2 graph with one vertex. Since all of the edges of degree $(1, 0)$ and $(0, 1)$ are just loops based at the vertex v, an involution γ on Λ is just an involutive permutation on each of the two sets of loops, with the constraint that the permutation must be consistent with the factorization rules of Λ.

In the special case that the factorization rules for Λ are trivial and the involution γ on Λ is trivial, $C^*_{\mathbb{R}}(\Lambda, \gamma) = C^*_{\mathbb{R}}(\Lambda, \mathrm{id})$ is a tensor product of real Cuntz algebras:

$$C^*_{\mathbb{R}}(\Lambda, \mathrm{id}) = C^*_{\mathbb{R}}(\Lambda) \cong C^*_{\mathbb{R}}(\Lambda_1 \times \Lambda_2) = C^*_{\mathbb{R}}(\Lambda_1) \otimes_{\mathbb{R}} C^*_{\mathbb{R}}(\Lambda_2) \cong \mathcal{O}^{\mathbb{R}}_m \otimes_{\mathbb{R}} \mathcal{O}^{\mathbb{R}}_n$$

by [Kumjian and Pask 2000, Corollary 3.5(iv)]. The K-theory for such tensor products of real Cuntz algebras is known from [Boersema 2002]. We here compute $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$ more generally and find that essentially the same K-theory appears as in the tensor products, regardless of the factorization rules and the involution γ.

We first describe the specific $\mathcal{CR}$-modules that arise, which we denote as R_g for g odd ($g \geq 3$) and S_g, T_g for g even ($g \geq 2$). The groups in these $\mathcal{CR}$-modules are given below; in these examples, the natural transformations r, c, η, ω, ψ which complete the data of the $\mathcal{CR}$-module are completely determined by the given groups, and the relations among the homomorphisms mandated by the $\mathcal{CR}$-relations (2.8) and the long exact sequence (2.9) linking the real and complex parts of a $\mathcal{CR}$-module. The precise formulas for these natural transformations are recorded in [Boersema 2002, Section 5.2].

g odd	0	1	2	3	4	5	6	7
$(R_g)^O_i$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0	0	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0	0
$(R_g)^U_i$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$

g even	0	1	2	3	4	5	6	7
$(S_g)^O_i$	$\mathbb{Z}_g$	$\mathbb{Z}_{2g}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{2g}$	$\mathbb{Z}_g$	0	0
$(S_g)^U_i$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$

$g \equiv 0 \pmod 4$	0	1	2	3	4	5	6	7
$(T_g)_i^O$	$\mathbb{Z}_g$	$\mathbb{Z}_2 \oplus \mathbb{Z}_g$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_g$	$\mathbb{Z}_g$	0	0
$(T_g)_i^U$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$

$g \equiv 2 \pmod 4$	0	1	2	3	4	5	6	7
$(T_g)_i^O$	$\mathbb{Z}_g$	$\mathbb{Z}_2 \oplus \mathbb{Z}_g$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_g$	$\mathbb{Z}_g$	0	0
$(T_g)_i^U$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$	$\mathbb{Z}_g$

For later reference during the calculations in this section, we also record the groups MO_i and MU_i corresponding to the $\mathcal{CR}$-modules S_g and T_g in the tables below. Recall that the core of a $\mathcal{CR}$-module M consists of just the complex part of M and the groups of MO_i and MU_i (and the relevant natural transformations). For R_g, we do not make use of the core but we note for completeness that $MO_i = 0$ and $MU_i = 0$ for all i.

core of S_g for g even	0	1	2	3	4	5	6	7
MO_i	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	0	0
MU_i	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

core of T_g for g even	0	1	2	3	4	5	6	7
MO_i	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0
MU_i	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Given $m, n \in \mathbb{N}_{\geq 2}$, define $g = \gcd(m-1, n-1)$. From Section 5.2 of [Boersema 2002] we have

$$K^{CR}(\mathcal{O}_m^{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{O}_n^{\mathbb{R}}) \cong \begin{cases} R_g & \text{if } g \text{ odd,} \\ S_g & \text{if } m-1 \equiv n-1 \equiv 2 \pmod 4, \\ T_g & \text{if } m-1 \equiv 0 \text{ or } n-1 \equiv 0 \pmod 4. \end{cases} \tag{4.3}$$

In particular, there are isomorphisms $R_g \cong K^{CR}(\mathcal{O}_{g+1}^{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{O}_{g+1}^{\mathbb{R}})$ if g is odd, $T_g \cong K^{CR}(\mathcal{O}_{g+1}^{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{O}_{g+1}^{\mathbb{R}})$ if $g \equiv 0 \pmod 4$, and $S_g \cong K^{CR}(\mathcal{O}_{g+1}^{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{O}_{g+1}^{\mathbb{R}})$ if $g \equiv 2 \pmod 4$.

Proposition 4.4. *Let Λ be a rank-2 graph with one vertex. Let m be the number of edges of degree $(1, 0)$ and let n be the number of edges of degree $(0, 1)$. Assume $m, n \geq 2$. Let γ be an involution on Λ, and write $g = \gcd(m-1, n-1)$. Then*

$$K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma)) \cong \begin{cases} R_g & \text{if } g \text{ is odd,} \\ S_g \text{ or } T_g & \text{if } g \text{ is even.} \end{cases}$$

Before we begin the proof of Proposition 4.4, we pause to make a few comments. First, note that if g is odd, then $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$ depends only on the number of edges of each color, not on the choice of involution or the factorization rules defining Λ. In particular, Proposition 4.4 gives more evidence in support of [Barlak et al. 2018, Conjecture 5.11], which asserts that the K-theory of a one-vertex k-graph C^*-algebra should be independent of the factorization rules.

We also wish to remark on the uncertainty of the statement of Proposition 4.4 regarding the even case. As $\mathcal{O}_n$ is the graph C^*-algebra of the one-vertex graph E_n with n edges, [Kumjian and Pask 2000, Corollary 3.5] tells us that there exists a 2-graph $\Lambda = E_n \times E_m$ such that

$$|\Lambda^{(1,0)}| = m, \qquad |\Lambda^{(0,1)}| = n, \quad \text{and} \quad C^*(\Lambda) \cong \mathcal{O}_n \otimes \mathcal{O}_m.$$

Therefore, applying (4.3) to $C^*_{\mathbb{R}}(\Lambda, \mathrm{id})$, we see that both S_g and T_g can appear as the K-theory of a 2-graph of the type discussed in Proposition 4.4. However, we see in the calculation below that in general it is not clear how to determine which $\mathcal{CR}$-module appears from the spectral sequence.

We also have the following corollary to Proposition 4.4.

Corollary 4.5. *Fix $m, n \in \mathbb{N}_{\geq 2}$, a one-vertex 2-graph Λ with $|\Lambda^{(1,0)}| = m$, $|\Lambda^{(0,1)}| = n$, and an involution γ on Λ. If $g = \gcd(m-1, n-1)$ is odd and $C^*(\Lambda)$ is simple, then*

$$C^*_{\mathbb{R}}(\Lambda, \gamma) \cong C^*_{\mathbb{R}}(\Lambda, \mathrm{id}).$$

Proof. Recall from [Kumjian and Pask 2000, Proposition 4.8] (cf. also [Robertson and Sims 2007, Lemma 3.2]) that the factorization rules which define Λ determine whether $C^*(\Lambda)$ is simple. When $C^*(\Lambda)$ is simple, [Brown et al. 2015, Corollary 5.1] tells us that since $m, n \geq 2$, $C^*(\Lambda)$ is purely infinite. Consequently, by [Boersema et al. 2011, Theorem 10.2], $C^*_{\mathbb{R}}(\Lambda, \gamma)$ is classified by its K-theory.

If g is odd, then Proposition 4.4 tells us that this K-theory is independent of the involution γ, so

$$C^*_{\mathbb{R}}(\Lambda, \gamma) \cong C^*_{\mathbb{R}}(\Lambda, \gamma_{\mathrm{triv}}) \cong \mathcal{O}^{\mathbb{R}}_m \otimes \mathcal{O}^{\mathbb{R}}_n$$

for any involution γ on Λ. □

We now undertake the proof of Proposition 4.4.

Proof of Proposition 4.4. The incidence matrices are 1×1 matrices, so $1 - M_1^t = 1 - n$ and $1 - M_2^t = 1 - m$. As $\Lambda^0 = \{v\} = G_f$, we have

$$K^{CR}(B_{\mathbb{R}}) = A = K^{CR}(\mathbb{R}).$$

Theorem 3.18 therefore tells us that the chain complex $\mathcal{A}^{(0)}$ is

$$0 \to K^{CR}(\mathbb{R}) \xrightarrow{\begin{pmatrix} -\rho^2 \\ \rho^1 \end{pmatrix}} K^{CR}(\mathbb{R})^2 \xrightarrow{(\rho^1 \;\; \rho^2)} K^{CR}(\mathbb{R}) \to 0. \tag{4.6}$$

We now use Table 3 to compute the individual maps ρ^i_j in each degree j:

complex part	
degree 0	$0 \to \mathbb{Z} \xrightarrow{\binom{m-1}{1-n}} \mathbb{Z}^2 \xrightarrow{(1-n \;\, 1-m)} \mathbb{Z} \to 0$
real part	
degree 0	$0 \to \mathbb{Z} \xrightarrow{\binom{m-1}{1-n}} \mathbb{Z}^2 \xrightarrow{(1-n \;\, 1-m)} \mathbb{Z} \to 0$
degree 1	$0 \to \mathbb{Z}_2 \xrightarrow{\binom{m-1}{1-n}} \mathbb{Z}_2^2 \xrightarrow{(1-n \;\, 1-m)} \mathbb{Z}_2 \to 0$
degree 2	$0 \to \mathbb{Z}_2 \xrightarrow{\binom{m-1}{1-n}} \mathbb{Z}_2^2 \xrightarrow{(1-n \;\, 1-m)} \mathbb{Z}_2 \to 0$
degree 4	$0 \to \mathbb{Z} \xrightarrow{\binom{m-1}{1-n}} \mathbb{Z}^2 \xrightarrow{(1-n \;\, 1-m)} \mathbb{Z} \to 0$

(For any degree not shown, the sequence consists of all trivial groups.)

The E^2 page of the spectral sequence has both a real part and a complex part, denoted $(E^2_{i,j})^O$ and $(E^2_{i,j})^U$, the groups of which are derived from the chain complex above. In the case that $g = \gcd(m-1, n-1)$ is odd, we use the first line of the table above to compute that

$$(E^2_{i,j})^U = H_i((\mathcal{A}^{(0)})^U_j) = \begin{cases} \mathbb{Z}_g & \text{if } i = 0, 1 \text{ and } j \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

The other lines of the table reveal that

$$(E^2_{i,j})^O = H_i((\mathcal{A}^{(0)})^O_j) = \begin{cases} \mathbb{Z}_g & \text{if } i = 0, 1 \text{ and } j \equiv 0 \pmod 4, \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

From this data, we obtain the E^2 page of the spectral sequence which converges to $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$. The left-hand diagram in Table 4 is the E^2 page for the real K-theory and the right-hand diagram is for the complex K-theory. Notice that the j index is vertical and the i index is horizontal. The spectral sequence is 0 in all nonpictured columns, and is periodic with period 8 in the vertical direction.

The d^2 map has degree $(-2, 1)$ and is hence equal to 0 everywhere. It follows that $E^2 = E^\infty$ and that $KO_q(C^*_{\mathbb{R}}(\Lambda, \gamma))$ has a filtration whose factors are the groups in the rightmost table above whose i and j coordinates sum to q. Since, for each q, there is at most one nonzero such group, we conclude that

$$KO_q(C^*_{\mathbb{R}}(\Lambda, \gamma)) = \begin{cases} \mathbb{Z}_g & \text{if } q \equiv 0, 1, 4, 5 \pmod 8, \\ 0 & \text{if } q \equiv 2, 3, 6, 7 \pmod 8. \end{cases}$$

Similarly, $KU_q(C^*_{\mathbb{R}}(\Lambda, \gamma)) = \mathbb{Z}_g$ for all q. Therefore $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma)) \cong R_g$ if g is odd.

real part			
	$\vdots$	$\vdots$	$\vdots$
7	0	0	0
6	0	0	0
5	0	0	0
4	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0
3	0	0	0
2	0	0	0
1	0	0	0
0	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0
	0	1	2

complex part			
	$\vdots$	$\vdots$	$\vdots$
7	0	0	0
6	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0
5	0	0	0
4	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0
3	0	0	0
2	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0
1	0	0	0
0	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0
	0	1	2

Table 4. $E^2_{p,q}$ when g is odd.

real part			
	$\vdots$	$\vdots$	$\vdots$
7	0	0	0
6	0	0	0
5	0	0	0
4	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0
3	0	0	0
2	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
0	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0
	0	1	2

complex part			
	$\vdots$	$\vdots$	$\vdots$
7	0	0	0
6	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0
5	0	0	0
4	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0
3	0	0	0
2	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0
1	0	0	0
0	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0
	0	1	2

Table 5. $E^2_{p,q}$ when g is even.

Now consider the case that g is even (with $m, n \geq 3$). The computations for $KU_*(C^*_{\mathbb{R}}(\Lambda, \gamma))$ are the same as in the odd case above. When computing $H_i((\mathcal{A}^{(0)})^O_j)$ for even g, we obtain nearly the same formulas as we found in (4.7) for the case that g is odd. The difference arises from the fact that all of the maps in the real part of the chain complex (4.6) in degrees 1 and 2 are zero if g is even. Hence, the E^2 page of the spectral sequence which converges to $KO_*(C^*_{\mathbb{R}}(\Lambda, \gamma))$ when g is even is as shown in Table 5.

case 1: $d^2_{2,1}=0$			
	$\vdots$	$\vdots$	$\vdots$
7	0	0	0
6	0	0	0
5	0	0	0
4	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0
3	0	0	0
2	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
0	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0
	0	1	2

case 2: $d^2_{2,1}\neq 0$			
	$\vdots$	$\vdots$	$\vdots$
7	0	0	0
6	0	0	0
5	0	0	0
4	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0
3	0	0	0
2	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	0
0	$\mathbb{Z}_g$	$\mathbb{Z}_g$	0
	0	1	2

Table 6. Real part of $E^\infty_{p,q}$.

The map d^2 again is equal to 0 everywhere except possibly $(d^2_{(2,1)})^O : \mathbb{Z}_2 \to \mathbb{Z}_2$ from degree (2, 1) to degree (0, 2) — this map may or may not be the zero map. Then the E^3 page of the spectral sequence must be as shown in Table 6. The left version corresponds to the case $(d^2_{(2,1)})^O = 0$ and the right version corresponds to the case $(d^2_{(2,1)})^O \neq 0$. For $n \geq 3$ we have $d^3 = 0$, thus $E^\infty = E^3$.

Once the E^∞ groups are settled, this determines $KO_q(C^*_{\mathbb{R}}(\Lambda, \gamma))$ only "up to extensions", meaning that there is a filtration of $KO_q(C^*_{\mathbb{R}}(\Lambda, \gamma))$ in which the successive subquotients are isomorphic to $E^\infty_{i,j}$, where $i + j = q$. However, we can deduce some specific information from the spectral sequence, namely that $KO_n(C^*_{\mathbb{R}}(\Lambda, \gamma)) = 0$ for $n = 6, 7$, and that $KO_5(C^*_{\mathbb{R}}(\Lambda, \gamma)) = KO_0(C^*_{\mathbb{R}}(\Lambda, \gamma)) = \mathbb{Z}_g$.

To complete the computation of $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$, we now consider the core. We claim that the involution ψ_* induced on $KU_*(C^*_{\mathbb{R}}(\Lambda, \gamma))$ by the real structure of $C^*_{\mathbb{R}}(\Lambda, \gamma)$ satisfies $\psi_j = 1$ for $j = 0, 1, 4, 5$ and $\psi_j = -1$ for $j = 2, 3, 6, 7$. To prove this claim, we first observe that for the $\mathcal{CR}$-module $K^{CR}(\mathbb{R})$, we have $\psi_{\mathbb{R}} = 1$ in degree 0. In the complex part

$$0 \to \mathbb{Z} \xrightarrow{\binom{m-1}{1-n}} \mathbb{Z}^2 \xrightarrow{(1-n \;\; 1-m)} \mathbb{Z} \to 0 \tag{4.8}$$

of the chain complex $\mathcal{A}^{(0)}$, each copy of $\mathbb{Z}$ represents $KU_0(\mathbb{R})$. Thus, $\psi_{\mathbb{R}}$ induces the identity map on the homology groups H_0, H_1 of the chain complex (4.8). As $H_0 = (E^2_{0,0})^U = KU_0(C^*_{\mathbb{R}}(\Lambda, \gamma))$ and $H_1 = (E^2_{1,0})^U = KU_1(C^*_{\mathbb{R}}(\Lambda, \gamma))$, we conclude that $\psi_j = 1$ for $j = 0, 1$. The relation $\psi_{j+2}\beta = -\beta\psi_j$ then implies that, as claimed,

$$\psi_j = 1 \quad \text{for } j = 0, 1, 4, 5 \qquad \text{and} \qquad \psi_j = -1 \quad \text{for } j = 2, 3, 6, 7.$$

Recall that $MU_i = (\ker(1-\psi_i))/(\operatorname{im}(1+\psi_i))$. Since g is even, one computes that $MU_0 = \mathbb{Z}_g/2\mathbb{Z}_g \cong \mathbb{Z}_2$, and $MU_2 = \{0, g/2\}/\{0\} \cong \mathbb{Z}_2$. Similar computations reveal that $MU_i \cong \mathbb{Z}_2$ for all i. The fact that $KO_i(C^*_{\mathbb{R}}(\Lambda,\gamma)) = 0$ if $i = 6, 7$ implies that $MO_i := \operatorname{im}\eta_{i-1} : KO_{i-1}(C^*_{\mathbb{R}}(\Lambda,\gamma)) \to KO_i(C^*_{\mathbb{R}}(\Lambda,\gamma))$ is zero for $i = 0, 6, 7$.

The long exact sequence (4.2) implies that r_6' and c_1' are both isomorphisms (thus $MO_1 = MO_5 = \mathbb{Z}_2$) and that $r_5' : MU_5 \to MO_3$ and $r_6' : MU_6 \to MO_4$ must be injective. From these observations, we obtain the following two segments of sequence (4.2):

$$0 \to \mathbb{Z}_2 \to MO_4 \xrightarrow{\eta_4'} \mathbb{Z}_2 \xrightarrow{c_5'} \mathbb{Z}_2 \xrightarrow{r_4'} MO_2 \to MO_3 \to \mathbb{Z}_2 \to 0, \tag{4.9}$$

$$0 \to \mathbb{Z}_2 \to MO_3 \to MO_4 \xrightarrow{c_4'} \mathbb{Z}_2 \xrightarrow{r_3'} \mathbb{Z}_2 \xrightarrow{\eta_1'} MO_2 \to \mathbb{Z}_2 \to 0. \tag{4.10}$$

Note first that η_4' must either be the zero map or be onto. In the first case, since r_6' is injective, we have $MO_4 = \mathbb{Z}_2$, and c_5' must also be injective (hence an isomorphism). We therefore have $r_4' = 0$, so (4.9) becomes

$$0 \to MO_2 \xrightarrow{\eta_2'} MO_3 \xrightarrow{c_3'} \mathbb{Z}_2 \to 0.$$

If η_4' is onto, then c_5' must be the zero map. Consequently, $MO_4 = \mathbb{Z}_2^2$ and (4.9) becomes

$$0 \to \mathbb{Z}_2 \xrightarrow{r_4'} MO_2 \to MO_3 \to \mathbb{Z}_2 \to 0. \tag{4.11}$$

Similarly, η_1' must be either injective, or the zero map. In the first case, the fact that each MO_i group is 2-torsion implies that $MO_2 = \mathbb{Z}_2^2$. Moreover, r_3' must be the zero map, so (4.10) becomes

$$0 \to \mathbb{Z}_2 \to MO_3 \xrightarrow{\eta_3'} MO_4 \xrightarrow{c_4'} \mathbb{Z}_2 \to 0. \tag{4.12}$$

If $\eta_1' = 0$ then $MO_2 = \mathbb{Z}_2$, $r_3' = 1$, and $c_4' = 0$, so (4.10) becomes

$$0 \to \mathbb{Z}_2 \to MO_3 \to MO_4 \to 0.$$

Thus, if $\eta_1' = 0$ and $\eta_4' = 0$, the fact that each MO_i group is 2-torsion implies that $MO_3 = \mathbb{Z}_2^2$. If $\eta_1' = 0$ and η_4' is onto, (4.9) becomes

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to MO_3 \to \mathbb{Z}_2 \to 0,$$

which forces $MO_3 = \mathbb{Z}_2$. However, (4.10) then implies that $MO_4 = 0$, contradicting the fact that (as we observed above) in this case we have $MO_4 = \mathbb{Z}_2^2$.

If η_1' is injective and $\eta_4' = 0$, so that $MO_4 = \mathbb{Z}_2$ and $c_4' = 1$, we must have $\eta_3' = 0$ and hence $MO_3 = \mathbb{Z}_2$. In other words, $c_3' = 1$ and $\eta_2' = 0$. This forces $MO_2 = 0$, which contradicts the fact that if η_1' is injective we have $MO_2 = \mathbb{Z}_2^2$.

Finally, suppose η_1' is injective and η_4' is onto, so that $MO_4 = \mathbb{Z}_2^2 = MO_2$. We conclude from (4.11) and (4.12) that $MO_3 = \mathbb{Z}_2^2$.

In other words, the MO_i groups are given by the first of the tables below if $\eta_4' = \eta_1' = 0$ and by the second if η_4' is onto and η_1' is injective; no other options are possible.

	0	1	2	3	4	5	6	7
MO_i	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0

	0	1	2	3	4	5	6	7
MO_i	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	0	0

As we noted at the beginning of this section, the core of the $\mathcal{CR}$-module S_g coincides with the first table above, and the core of the $\mathcal{CR}$-module T_g coincides with the second. Therefore, by [Hewitt 1996, Theorem 4.2.1], $K^{CR}(C^*(\Lambda, \gamma))$ is either isomorphic to S_g or to T_g.

Comparing the cardinality of $(S_g)_2^O$, $(T_g)_2^O$, and the two options for the E^∞ page of the spectral sequence converging to $KO_*(C^*_{\mathbb{R}}(\Lambda, \gamma))$, we see that we have $K^{CR}(C^*(\Lambda, \gamma)) \cong S_g$ when $d^2_{(2,1)} \neq 0$ and $K^{CR}(C^*(\Lambda, \gamma)) \cong T_g$ when $d^2_{(2,1)} = 0$. □

4B. ***A 3-vertex rank-2 graph.*** In this section, we consider a family of rank-2 graphs Λ with three vertices and with the adjacency matrices

$$M_1 = M_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & n-1 \\ 1 & n-1 & 0 \end{pmatrix}$$

for $n \geq 2$. We also consider an involution γ that swaps the second and third vertices. (By comparison, a rank-1 graph with involution and with the same adjacency matrix was considered in [Boersema 2017, Example 6.2].) We do not specify the factorization rules for Λ, since they do not affect our K-theory calculations. They may be any factorization rules that are consistent with the involution γ. We consider the real C^*-algebra $C^*_{\mathbb{R}}(\Lambda, \gamma)$.

Proposition 4.13. *The $\mathcal{CR}$ K-theory $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$ is isomorphic to one of two $\mathcal{CR}$-modules, P_{2n} or Q_{2n}.*

The groups of the $\mathcal{CR}$-modules P_{2n} and Q_{2n} are given by the following tables. Again, we only record the groups, not the natural transformations, as these are completely determined by the given groups. The structure of Q_{2n} differs somewhat depending on n being even or odd.

	0	1	2	3	4	5	6	7
$(P_{2n})_i^O$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{4n} \oplus \mathbb{Z}_2$	$\mathbb{Z}_{4n} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_n$	$\mathbb{Z}_n$
$(P_{2n})_i^U$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$

(n even)	0	1	2	3	4	5	6	7
$(Q_{2n})^O_i$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_n$	$\mathbb{Z}_n$
$(Q_{2n})^U_i$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$

(n odd)	0	1	2	3	4	5	6	7
$(Q_{2n})^O_i$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_n$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_n$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_n$	$\mathbb{Z}_n$
$(Q_{2n})^U_i$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$

The cores of these $\mathcal{CR}$-modules include the groups below:

core of P_{2n}	0	1	2	3	4	5	6	7
MO_i	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	0	0
MU_i	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

(4.14)

core of Q_{2n}	0	1	2	3	4	5	6	7
MO_i	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0
MU_i	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

(4.15)

We note that there is a $\mathcal{CR}$-module isomorphism

$$P_{2n} \cong \Sigma^{-2} K^{CR}(\mathcal{E}_{2n+1}) \oplus \Sigma^{-3} K^{CR}(\mathcal{E}_{2n+1}),$$

where $\mathcal{E}_{2n+1}$ is the exotic Cuntz algebra described in Section 11 of [Boersema et al. 2011]. However, to our knowledge, the $\mathcal{CR}$-modules Q_{2n} have not previously been discussed in the literature.

Proof. We develop the chain complex, and subsequent spectral sequence, as in Theorem 3.18 to compute $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$. The chain complex is

$$0 \to A \xrightarrow{\partial_2} A^2 \xrightarrow{\partial_1} A \to 0,$$

where $A = K^{CR}(\mathbb{R}) \oplus K^{CR}(\mathbb{C})$. Using Theorem 3.18 and the fact that for $i = 1, 2$ we have

$$\rho^i = B = I_3 - M_i = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 1 & 1-n \\ -1 & 1-n & 1 \end{pmatrix},$$

we can analyze the groups and maps of this chain complex in each grading, complex and real parts, as below:

complex part	
degree 0	$0 \to \mathbb{Z}^3 \xrightarrow{\left(\begin{smallmatrix} -B \\ B \end{smallmatrix}\right)} \mathbb{Z}^6 \xrightarrow{(B\ B)} \mathbb{Z}^3 \to 0$
real part	
degree 0	$0 \to \mathbb{Z}^2 \xrightarrow{\left(\begin{smallmatrix} 0 & 2 \\ 1 & n-2 \\ 0 & -2 \\ -1 & 2-n \end{smallmatrix}\right)} \mathbb{Z}^4 \xrightarrow{\left(\begin{smallmatrix} 0 & -2 & 0 & -2 \\ -1 & 2-n & -1 & 2-n \end{smallmatrix}\right)} \mathbb{Z}^2 \to 0$
degree 1	$0 \to \mathbb{Z}_2 \xrightarrow{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)} \mathbb{Z}_2^2 \xrightarrow{(0\ 0)} \mathbb{Z}_2 \to 0$
degree 2	$0 \to (\mathbb{Z}_2 \oplus \mathbb{Z}) \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ 0 & -n \\ 0 & -1 \\ 0 & n \end{smallmatrix}\right)} (\mathbb{Z}_2 \oplus \mathbb{Z})^2 \xrightarrow{\left(\begin{smallmatrix} 0 & -1 & 0 & -1 \\ 0 & n & 0 & n \end{smallmatrix}\right)} (\mathbb{Z}_2 \oplus \mathbb{Z}) \to 0$
degree 4	$0 \to \mathbb{Z}^2 \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ 2 & n-2 \\ 0 & -1 \\ -2 & 2-n \end{smallmatrix}\right)} \mathbb{Z}^4 \xrightarrow{\left(\begin{smallmatrix} 0 & -1 & 0 & -1 \\ -2 & 2-n & -2 & 2-n \end{smallmatrix}\right)} \mathbb{Z}^2 \to 0$
degree 6	$0 \to \mathbb{Z} \xrightarrow{\left(\begin{smallmatrix} -n \\ n \end{smallmatrix}\right)} \mathbb{Z}^2 \xrightarrow{(n\ n)} \mathbb{Z} \to 0$

The Smith normal form of B is

$$\mathrm{SNF}(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2n \end{pmatrix}.$$

From this, it easily follows that

$$\begin{aligned} (E^2_{0,0})^U &= \operatorname{coker}(B\ B) \cong \mathbb{Z}_{2n}, \\ (E^2_{1,0})^U &= \ker(B\ B) \big/ \operatorname{im}\begin{pmatrix} -B \\ B \end{pmatrix} \cong \mathbb{Z}_{2n}, \\ (E^2_{2,0})^U &= \ker\begin{pmatrix} -B \\ B \end{pmatrix} = 0. \end{aligned}$$

For the real part, we work out the homology of the exact sequences associated to the "real part" above to obtain $(E^2_{p,q})^O$ (or simply $E^2_{p,q}$, as we denote it when it is clear). We walk through the details of this for the first three rows and leave the rest to the reader. The Smith normal form of the matrix $\left(\begin{smallmatrix} 0 & 2 \\ 1 & n-2 \end{smallmatrix}\right)$ is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ for all n. It follows that in the real part in degree 0 (that is, when $q = 0$) we have $E^2_{0,0} = E^2_{1,0} = \mathbb{Z}_2$, and $E^2_{2,0} = 0$.

When $q = 1$ we have $\partial_1 = \partial_2 = 0$, so it immediately follows that $E^2_{0,1} = \mathbb{Z}_2$, $E^2_{1,1} = \mathbb{Z}_2^2$, and $E^2_{1,2} = \mathbb{Z}_2$.

For the next row (when $q = 2$), we first observe that

$$\operatorname{im}\begin{pmatrix} 0 & -1 & 0 & -1 \\ 0 & n & 0 & n \end{pmatrix} = \big\{([1], (2k+1)n) \mid k \in \mathbb{Z}\big\} \cup \big\{([0], 2kn) \mid k \in \mathbb{Z}\big\} \subseteq \mathbb{Z}_2 \oplus \mathbb{Z}.$$

In particular, the sum $([1], 1)$ of the two generators $([1], 0)$ and $([0], 1)$ of $\mathbb{Z}_2 \oplus \mathbb{Z}$ lies in $\operatorname{im}\left(\begin{smallmatrix} 0 & -1 & 0 & -1 \\ 0 & n & 0 & n \end{smallmatrix}\right)$. Consequently,

$$E^2_{0,2} = \operatorname{coker}\begin{pmatrix} 0 & -1 & 0 & -1 \\ 0 & n & 0 & n \end{pmatrix} = \big\langle [([0], 1)] \big\rangle = \mathbb{Z}_{2n}.$$

To show that $E^2_{1,2} = \mathbb{Z}_2 \oplus \mathbb{Z}_{2n}$, we note that

$$\ker \partial_1 = \ker\begin{pmatrix} 0 & -1 & 0 & -1 \\ 0 & n & 0 & n \end{pmatrix} = \{(x, y, z, -y) \mid x, z \in \mathbb{Z}_2, y \in \mathbb{Z}\}$$

while

$$\operatorname{im} \partial_2 = \big\{([x], -nx, [x], nx) \mid x \in \mathbb{Z}\big\}.$$

Consequently, $E^2_{1,2} = \ker \partial_1 / \operatorname{im} \partial_2 = \big\langle [(0, 0, 1, 0)], [(0, 1, 0, -1)] \big\rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2n}$ because $(0, n, 0, -n)$ is not in $\operatorname{im} \partial_2$ but $(0, 2n, 0, -2n)$ is.

Finally, note that

$$E^2_{0,2} = \ker \partial_2 = \ker\begin{pmatrix} 0 & 1 \\ 0 & -n \\ 0 & 1 \\ 0 & n \end{pmatrix} = \{(x, 0) \mid x \in \mathbb{Z}_2\} = \mathbb{Z}_2.$$

The $E^2_{p,q}$ groups of the spectral sequence converging to $KO_*(C^*_{\mathbb{R}}(\Lambda, \gamma))$ are shown on the left in Table 7. From this and similar calculations for $3 \le q \le 7$ we obtain the E^2 page of the spectral sequence as shown.

The map d^2 is forced to be 0 everywhere except possibly the map $d^2_{(2,1)} : \mathbb{Z}_2 \to \mathbb{Z}_{2n}$ from degree $(2, 1)$ to degree $(0, 2)$ in the real case.

The complex spectral sequence (right side of Table 7) gives $KU_i(C^*_{\mathbb{R}}(\Lambda, \gamma)) \cong \mathbb{Z}_{2n}$ for all i. Thus $KU_0(C^*_{\mathbb{R}}(\Lambda, \gamma)) = KU_1(C^*_{\mathbb{R}}(\Lambda, \gamma)) \cong \mathbb{Z}_{2n}$. It follows that the complex C^*-algebra $C^*(\Lambda)$ is KK-equivalent to $\mathcal{O}_{2n+1} \otimes \mathcal{O}_{2n+1}$.

With a bit more work, we can identify the maps ψ, by tracing the elements $KU_*(C^*_{\mathbb{R}}(\Lambda, \gamma))$ as they arise from the chain complex through the spectral sequence. In the $i = 0$ case we have that $KU_0(C^*_{\mathbb{R}}(\Lambda, \gamma))$ is isomorphic to

$$A^U_0 / \operatorname{im} B = \mathbb{Z}^3 / \operatorname{im} B,$$

and the generator of $KU_0(C^*_{\mathbb{R}}(\Lambda, \gamma)) \cong \mathbb{Z}_{2n}$ is represented by the element $(0, 1, 0)$, which is equivalent to the element $(0, 0, -1)$ (since $(0, -1, -1)$ is in the image of B). Similarly, one computes that

$$\ker(B \;\; B) = \{(x, y, z, -x, -y, -z) \in \mathbb{Z}^6\},$$

real part			
	$\vdots$	$\vdots$	$\vdots$
7	0	0	0
6	$\mathbb{Z}_n$	$\mathbb{Z}_n$	0
5	0	0	0
4	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
3	0	0	0
2	$\mathbb{Z}_{2n}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}$	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
	0	1	2

complex part			
	$\vdots$	$\vdots$	$\vdots$
7	0	0	0
6	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	0
5	0	0	0
4	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	0
3	0	0	0
2	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	0
1	0	0	0
0	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	0
	0	1	2

Table 7. $E^2_{p,q}$.

so $[(0, 1, 0, 0, -1, 0)] = [(0, 0, -1, 0, 0, 1)]$ generates

$$KU_1(C^*_{\mathbb{R}}(\Lambda, \gamma)) = \ker(B\ \ B) \,\big/\, \mathrm{im}\begin{pmatrix} -B \\ B \end{pmatrix}.$$

As $A^U_0 = KU_0(\mathbb{R}) \oplus KU_0(\mathbb{C})$, the fact that $(\psi_{\mathbb{C}})_0 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on $KU_0(\mathbb{C}) = \mathbb{Z}^2$ implies that $(\psi_A)_0(x, y, z) = (x, z, y)$. Thus, the involution ψ_0 on $KU_0(C^*_{\mathbb{R}}(\Lambda, \gamma)) \cong A^U_0 / \mathrm{im}\, B \cong \mathbb{Z}^3 / \mathrm{im}\, B$ induced by ψ_A satisfies $\psi_0([0, 1, 0]) = [0, 0, 1]$. It follows that ψ_0 is given by multiplication by -1 in $KU_0(C^*_{\mathbb{R}}(\Lambda, \gamma)) = \mathbb{Z}_{2n}$. A similar analysis also shows that $\psi = -1$ in $KU_1(C^*_{\mathbb{R}}(\Lambda, \gamma)) = \mathbb{Z}_{2n}$. Using the fact that ψ anticommutes with the Bott isomorphism (that is, $\psi\beta = -\beta\psi$), we find that $\psi_i = -1$ for $i = 0, 1, 4, 5$ and $\psi_i = 1$ for $i = 2, 3, 6, 7$.

In addition to ψ, it would be possible to compute the action of most of the natural transformations r, c, η in this way, based on the corresponding actions in A. Alternatively, once we have computed a few of these natural transformations, we can complete the calculation of $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$ using the long exact sequence (2.9) and the core exact sequence (4.2).

From the E^2 page of the spectral sequence for $KO_*(C^*_{\mathbb{R}}(\Lambda, \gamma))$ we see immediately that $KO_0(C^*_{\mathbb{R}}(\Lambda, \gamma)) \cong \mathbb{Z}_2$ and $KO_1(C^*_{\mathbb{R}}(\Lambda, \gamma))$ is either isomorphic to $\mathbb{Z}_4$ or to $\mathbb{Z}_2^2$. Less immediately, we also find that η_0 and η_1 are nontrivial.

To see that η_0 is nontrivial, observe that

$$KO_0(C^*_{\mathbb{R}}(\Lambda, \gamma)) \cong (KO_0(\mathbb{R}) \oplus KO_0(\mathbb{C})) \,\big/\, \mathrm{im}\begin{pmatrix} 0 & -2 & 0 & -2 \\ -1 & 2-n & -1 & 2-n \end{pmatrix}$$

is generated by $[(1, 0)]$. Since $(\eta_{\mathbb{R}})_0([1]) \in KO_1(\mathbb{R})$ is the nontrivial element of $(A_1)^O = \mathbb{Z}_2$, which is the $E^2_{0,1}$ group of the spectral sequence converging to $KO_*(C^*_{\mathbb{R}}(\Lambda, \gamma))$, we find that $\eta_0([1, 0])$ is a nontrivial element of $KO_1(C^*_{\mathbb{R}}(\Lambda, \gamma))$. Therefore, $\eta_0 \neq 0$. In addition, since $(\eta_{\mathbb{R}})_1 : \mathbb{Z}_2 \to \mathbb{Z}_2$ is nontrivial and the image in $E^2_{2,0} = \operatorname{coker}\left(\begin{smallmatrix} 0 & -1 & 0 & -1 \\ 0 & n & 0 & n \end{smallmatrix}\right)$ of the generator $([1], 0)$ of $\mathbb{Z}_2 \subseteq \mathbb{Z}_2 \oplus \mathbb{Z}$ is nontrivial, we conclude that $\eta_1 : KO_1(C^*_{\mathbb{R}}(\Lambda, \gamma)) \to KO_2(C^*_{\mathbb{R}}(\Lambda, \gamma))$ is also nontrivial.

Now we claim that $r_i : KU_i(C^*_{\mathbb{R}}(\Lambda, \gamma)) \to KO_i(C^*_{\mathbb{R}}(\Lambda, \gamma))$ is surjective for $i = 5, 6, 7$. In degree 6, since $A = K^{CR}(\mathbb{R}) \oplus K^{CR}(\mathbb{C})$ and $(r_{\mathbb{C}})_6 = \begin{pmatrix} -1 & 1 \end{pmatrix}$, we conclude that $(r_A)_6 : \mathbb{Z}^3 \to \mathbb{Z}$ on $KO_*(A)$ is given by $(x, y, z) \mapsto z - y$. Now, recall that the generator of $KU_6(C^*_{\mathbb{R}}(\Lambda, \gamma)) = \operatorname{coker}(B \ B) = \mathbb{Z}_{2n}$ is represented by $(0, 1, 0)$. Thus, $r_6 : KU_6(C^*_{\mathbb{R}}(\Lambda, \gamma)) \to KO_6(C^*_{\mathbb{R}}(\Lambda, \gamma))$ satisfies

$$r_6([0, 1, 0]) = [-1] \in \mathbb{Z}_n \cong KO_6(C^*_{\mathbb{R}}(\Lambda, \gamma)).$$

Thus r_6 is onto.

To see that r_5 is onto, recall that for i odd

$$KU_i(C^*_{\mathbb{R}}(\Lambda, \gamma)) = \ker(B \ B) \Big/ \operatorname{im}\begin{pmatrix} -B \\ B \end{pmatrix} = \mathbb{Z}_{2n}$$

is generated by $g = [(0, 1, 0, 0, -1, 0)]$. Also, for $i = 5$ we find (referring to the degree 4 part of the chain complex) that

$$KO_5(C^*_{\mathbb{R}}(\Lambda, \gamma)) = \ker(\partial_1)_4 / \operatorname{im}(\partial_2)_4 = \mathbb{Z}_2$$

and the nontrivial element can be determined to be represented by $h = [(0, 1, 0, -1)]$. The map $r_5 : KU_5(C^*_{\mathbb{R}}(\Lambda, \gamma)) \to KO_5(C^*_{\mathbb{R}}(\Lambda, \gamma))$ is therefore induced by the map $(r_A)_4 : \mathbb{Z}^6 \to \mathbb{Z}^4$, which is given by the formula

$$(x, y, z, u, v, w) \mapsto (x, y + z, u, v + w)$$

since $(r_{\mathbb{C}})_4 = (1 \ 1) : \mathbb{Z}^2 \to \mathbb{Z}$. Thus $r_5(g) = [(0, 1, 0, -1)]$, so r_5 is surjective.

With a similar argument, we can show that $r_7 : KU_7(C^*_{\mathbb{R}}(\Lambda, \gamma)) \to KO_7(C^*_{\mathbb{R}}(\Lambda, \gamma))$ is onto. As

$$KU_7(C^*_{\mathbb{R}}(\Lambda, \gamma)) = \ker(B \ B) \Big/ \operatorname{im}\begin{pmatrix} -B \\ B \end{pmatrix},$$

r_7 is induced from $(r_{A^2})_6 : \mathbb{Z}^6 \to \mathbb{Z}^2$, and $(r_{A^2})_6(x, y, z, u, v, w) = (z - y, w - v)$. Therefore, using the generator g of $KU_7(C^*_{\mathbb{R}}(\Lambda, \gamma))$ identified above,

$$r_6(g) = [(1, -1)] \in \ker\begin{pmatrix} n & n \end{pmatrix} \Big/ \operatorname{im}\begin{pmatrix} -n \\ n \end{pmatrix} = KO_6(C^*_{\mathbb{R}}(\Lambda, \gamma)).$$

As $[(1, -1)]$ generates $KO_6(C^*_{\mathbb{R}}(\Lambda, \gamma))$, we conclude that r_7 is also surjective.

Since r_i is surjective for $i = 5, 6, 7$, it immediately follows from (2.8) that $\eta_i = 0$ for $i = 5, 6, 7$.

Now we turn to the core of the exact sequence. Given that we've already computed $KU_i(C^*_{\mathbb{R}}(\Lambda,\gamma)) = \mathbb{Z}_{2n}$ and $\psi_i = \pm 1$, (4.1) implies that $MU_i(C^*_{\mathbb{R}}(\Lambda,\gamma)) = \mathbb{Z}_2$ for all i. The fact that $\eta_i = 0$ for $5 \le i \le 7$ implies that $MO_i(C^*_{\mathbb{R}}(\Lambda,\gamma)) = 0$ for $i = 0, 6, 7$. From this information, the long exact sequence (4.2) relating MO_i and MU_i can be used to determine that MO_* must be one of the following (using the same argument as used in the previous section):

	0	1	2	3	4	5	6	7
MO_i	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	0	0

	0	1	2	3	4	5	6	7
MO_i	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0

The former possibility coincides with the core of P_{2n}. Comparing $(P_{2n})^O_3$ with the groups $(E^O)^3_{p,q}$ of the spectral sequence converging to $KO_*(C^*_{\mathbb{R}}(\Lambda,\gamma))$ for $p+q=3$, we see that this possibility coincides with the case when $d^2_{2,1} = 0$. Therefore, in that case $K^{CR}(C^*_{\mathbb{R}}(\Lambda,\gamma)) = P_{2n}$. The latter case occurs when $d^2_{2,1} \neq 0$ and yields $K^{CR}(C^*_{\mathbb{R}}(\Lambda,\gamma)) = Q_{2n}$. See the tables (4.14) and (4.15). $\square$

If the factorization rules of Λ are such that $C^*(\Lambda)$ is simple and purely infinite, then the complex C^*-algebra $C^*(\Lambda)$ is isomorphic to a matrix algebra over $\mathcal{O}_{2n+1} \otimes \mathcal{O}_{2n+1}$. A little more work is necessary to track the class of [1] in $KU_0(C^*_{\mathbb{R}}(\Lambda,\gamma))$ to determine the value of k in the isomorphism $C^*(\Lambda) \cong M_k(\mathcal{O}_{2n+1} \otimes \mathcal{O}_{2n+1})$. The real C^*-algebra $C^*_{\mathbb{R}}(\Lambda,\gamma)$ is then a real structure of $M_k(\mathcal{O}_{2n+1} \otimes \mathcal{O}_{2n+1})$ but is not isomorphic (nor stably isomorphic) to $\mathcal{O}^{\mathbb{R}}_{2n+1} \otimes_{\mathbb{R}} \mathcal{O}^{\mathbb{R}}_{2n+1}$ or any other tensor product of real Cuntz algebras. This follows, for example, from the fact that $KO_7(C^*_{\mathbb{R}}(\Lambda,\gamma)) \cong \mathbb{Z}_2$, but $KO_7(-)$ is trivial for any tensor product of real Cuntz algebras. Furthermore, $C^*_{\mathbb{R}}(\Lambda,\gamma)$ is not isomorphic to any real C^*-algebra arising from a rank-1 graph with involution, as $KO_7(-)$ is torsion-free for such a C^*-algebra by Corollary 4.3 of [Boersema 2017].

4C. ***Another 3-vertex 2-graph.*** In this section we examine another 2-graph for which the two adjacency matrices are not the same. Fix an integer $n \ge 2$. Let Λ be a rank-2 graph with three vertices with adjacency matrices

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & n-1 \\ 1 & n-1 & 0 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & n-1 & 0 \\ 1 & 0 & n-1 \end{pmatrix}$$

and with an involution γ that swaps the second and third vertices.

Proposition 4.16. *Fix an integer $n \geq 2$. If n is odd, then*

$$K^{CR}(C^*(\Lambda,\gamma)) \cong S_2 \quad or \quad K^{CR}(C^*(\Lambda,\gamma)) \cong T_2.$$

If n is even, then

$$\begin{aligned} K^{CR}(C^*(\Lambda,\gamma)) &\cong \Sigma K^{CR}(\mathcal{O}_3^{\mathbb{R}}) \oplus \Sigma^{-2} K^{CR}(\mathcal{O}_3^{\mathbb{R}}) \\ &\cong \Sigma^{-1} K^{CR}(\mathcal{E}_3) \oplus \Sigma^{4} K^{CR}(\mathcal{E}_3). \end{aligned}$$

We find it intriguing that for n even, the choice of n has no impact on the $\mathcal{CR}$ K-theory groups of the real C^*-algebra. For all odd integers n, there are only two possible $\mathcal{CR}$-modules that can be realized by $K^{CR}(C^*(\Lambda,\gamma))$.

Proof. Again, we use Theorem 3.18. The chain complex $\mathcal{A}^{(0)}$ used to build the spectral sequence to compute $K^{CR}(C^*(\Lambda,\gamma))$ has the components shown in the table below, based on the matrices

$$\rho^1 = B_1 = I_3 - M_1 = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 1 & 1-n \\ -1 & 1-n & 1 \end{pmatrix},$$

$$\rho^2 = B_2 = I_3 - M_2 = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 2-n & 0 \\ -1 & 0 & 2-n \end{pmatrix}.$$

complex part	
degree 0	$0 \to \mathbb{Z}^3 \xrightarrow{\left(\begin{smallmatrix} -B_2 \\ B_1 \end{smallmatrix}\right)} \mathbb{Z}^6 \xrightarrow{(B_1\ B_2)} \mathbb{Z}^3 \to 0$
real part	
degree 0	$0 \to \mathbb{Z}^2 \xrightarrow{\left(\begin{smallmatrix} 0 & 2 \\ 1 & n-2 \\ 0 & -2 \\ -1 & 2-n \end{smallmatrix}\right)} \mathbb{Z}^4 \xrightarrow{\left(\begin{smallmatrix} 0 & -2 & 0 & -2 \\ -1 & 2-n & -1 & 2-n \end{smallmatrix}\right)} \mathbb{Z}^2 \to 0$
degree 1	$0 \to \mathbb{Z}_2 \xrightarrow{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)} \mathbb{Z}_2^2 \xrightarrow{(0\ 0)} \mathbb{Z}_2 \to 0$
degree 2	$0 \to (\mathbb{Z}_2 \oplus \mathbb{Z}) \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ 0 & n-2 \\ 0 & -1 \\ 0 & n \end{smallmatrix}\right)} (\mathbb{Z}_2 \oplus \mathbb{Z})^2 \xrightarrow{\left(\begin{smallmatrix} 0 & -1 & 0 & -1 \\ 0 & n & 0 & 2-n \end{smallmatrix}\right)} (\mathbb{Z}_2 \oplus \mathbb{Z}) \to 0$
degree 4	$0 \to \mathbb{Z}^2 \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ 2 & n-2 \\ 0 & -1 \\ -2 & 2-n \end{smallmatrix}\right)} \mathbb{Z}^4 \xrightarrow{\left(\begin{smallmatrix} 0 & -1 & 0 & -1 \\ -2 & 2-n & -2 & 2-n \end{smallmatrix}\right)} \mathbb{Z}^2 \to 0$
degree 6	$0 \to \mathbb{Z} \xrightarrow{\left(\begin{smallmatrix} n-2 \\ n \end{smallmatrix}\right)} \mathbb{Z}^2 \xrightarrow{(n\ 2-n)} \mathbb{Z} \to 0$

The Smith normal forms of the matrices in the complex part of the chain complex are

$$\mathrm{SNF}(B_1, B_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{SNF}\begin{pmatrix} -B_2 \\ B_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then we have

$$\operatorname{coker}(B_1\ B_2) \cong \mathbb{Z}_2, \qquad \ker(B_1\ B_2) \big/ \operatorname{im}\begin{pmatrix} -B_2 \\ B_1 \end{pmatrix} \cong \mathbb{Z}_2, \quad \text{and} \quad \ker\begin{pmatrix} -B_2 \\ B_1 \end{pmatrix} = 0.$$

Thus, the E^2 groups of the chain complex computing $KU_*(C^*_{\mathbb{R}}(\Lambda, \gamma))$ satisfy

$$(E^2_{p,q})^U = \begin{cases} \mathbb{Z}_2 & \text{for } p \in \{0, 1\}, q \text{ even}, \\ 0 & \text{otherwise}. \end{cases}$$

Consequently, $KU_0(C^*_{\mathbb{R}}(\Lambda, \gamma)) \cong KU_1(C^*_{\mathbb{R}}(\Lambda, \gamma)) \cong \mathbb{Z}_2$. It follows that the complex C^*-algebra $C^*(\Lambda)$ has the same K-theory as $\mathcal{O}_3 \otimes \mathcal{O}_3$.

From the chain complexes exhibited above, we can compute that the E^2 page of the spectral sequence computing $KO_*(C^*_{\mathbb{R}}(\Lambda, \gamma))$ is given as in Table 8. The left-hand table corresponds to the case where n is odd and the right-hand table corresponds to the case where n is even. It is a remarkable fact that, even though the matrices ρ^i look very different when $n = 2$, the E^2 page in this case is the same as for any other even n.

case 1: n is odd			
	$\vdots$	$\vdots$	$\vdots$
7	0	0	0
6	0	0	0
5	0	0	0
4	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
3	0	0	0
2	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
	0	1	2

case 2: n is even			
	$\vdots$	$\vdots$	$\vdots$
7	0	0	0
6	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
5	0	0	0
4	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
3	0	0	0
2	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
	0	1	2

Table 8. Real part of $E^2_{p,q}$.

As an illustration of how this spectral sequence was obtained, we explain the computations of $E^2_{p,2}$ for $p = 0, 1, 2$. We have

$$E^2_{0,2} = \operatorname{coker}(\partial_1)_2 = \operatorname{coker}\begin{pmatrix} 0 & -1 & 0 & -1 \\ 0 & n & 0 & 2-n \end{pmatrix} = (\mathbb{Z}_2 \oplus \mathbb{Z})/G,$$

where G is the subgroup of $\mathbb{Z}_2 \oplus \mathbb{Z}$ generated by $([1], n)$ and by $([0], 2)$. This in turn is isomorphic to $(\mathbb{Z}_2 \oplus \mathbb{Z}_2)/G'$, where G' is the subgroup of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ generated by $([1], [n])$. Note that in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $([1], [n])$ is equal to $([1], [0])$ or $([1], [1])$, depending on whether n is even or odd. In either case, the quotient is isomorphic to $\mathbb{Z}_2$.

To compute $E^2_{1,2} = \ker(\partial_1)_2 / \operatorname{im}(\partial_2)_2$, we first compute that

$$\begin{aligned} \ker\begin{pmatrix} 0 & -1 & 0 & -1 \\ 0 & n & 0 & 2-n \end{pmatrix} &= \{([x], y, [z], w) \mid y + w \equiv 0 \pmod 2,\, ny = (n-2)w\} \\ &= \{([x], k(n-2), [z], kn) \mid k \in \mathbb{Z}\}. \end{aligned}$$

Now, observe that

$$\operatorname{im}(\partial_2)_2 = \operatorname{im}\begin{pmatrix} 0 & 1 \\ 0 & n-2 \\ 0 & -1 \\ 0 & n \end{pmatrix} = \{([x], (n-2)x, [x], nx) \mid x \in \mathbb{Z}\} \subseteq (\mathbb{Z}_2 \oplus \mathbb{Z})^2.$$

Thus a generic element in $E^2_{1,2}$ can be written as

$$[([x], y, [z], w)] = [([x], k(n-2), [z], kn)] = [([k+x], 0, [k+z], 0)],$$

since $([1], n-2, [1], n) \in \operatorname{im}(\partial_2)_2$ and hence $[(0, n-2, 0, n)] = [([1], 0, [1], 0)]$. It now follows easily that $E^2_{1,2} \cong \mathbb{Z}_2^2$.

Finally,

$$E^2_{2,2} = \ker(\partial_2)_2 = \ker\begin{pmatrix} 0 & 1 \\ 0 & n-2 \\ 0 & -1 \\ 0 & n \end{pmatrix} = \{([x], y) \mid y = 0\} \cong \mathbb{Z}_2.$$

Leaving it to the reader to calculate $E^2_{p,q}$ for the remaining values of p, q, we turn now to analyzing the spectral sequence from the E^2 stage. In the case where n is odd, the spectral sequence is the same as one that we saw in Section 4A, so, as there, we can conclude that either $K^{CR}(C^*(\Lambda, \gamma)) \cong S_2$ (if $d^2_{(2,1)} \neq 0$) or $K^{CR}(C^*(\Lambda, \gamma)) \cong T_2$ (if $d^2_{(2,1)} = 0$).

In the case where n is even, there is some work to do. Observe first that once again, we have $d^2_{(i,j)} = 0$ for all (i, j), with the possible exception of $d^2_{(2,1)}$. Thus, for all $(i, j) \notin \{(0,2),(2,1)\}$ we have $E^2_{ij} = E^\infty_{ij}$. It follows that $KO_i(C^*(\Lambda,\gamma)) \cong \mathbb{Z}_2$ for $i = 5, 6, 7, 0$, and that $|KO_i(C^*(\Lambda,\gamma))| = 4$ for $i = 1, 4$. If $d^2_{(2,1)} = 0$ then

$$|KO_2(C^*_{\mathbb{R}}(\Lambda,\gamma))| = |KO_3(C^*_{\mathbb{R}}(\Lambda,\gamma))| = 8,$$

and if $d^2_{(2,1)} \neq 0$ then

$$KO_2(C^*_{\mathbb{R}}(\Lambda,\gamma)) \cong KO_3(C^*_{\mathbb{R}}(\Lambda,\gamma)) \cong \mathbb{Z}_2^2.$$

Now we determine the maps η, r, and c. We also show that $|KO_2(C^*_{\mathbb{R}}(\Lambda,\gamma))| = |KO_3(C^*_{\mathbb{R}}(\Lambda,\gamma))| = 4$ (and hence that $d^2_{(2,1)} \neq 0$). We start with the following segments of the long exact sequence (2.9) relating $KO_*(C^*_{\mathbb{R}}(\Lambda,\gamma))$ and $KU_*(C^*_{\mathbb{R}}(\Lambda,\gamma))$:

$$KO_0(C^*_{\mathbb{R}}(\Lambda,\gamma)) \xrightarrow{\eta_0} KO_1(C^*_{\mathbb{R}}(\Lambda,\gamma)) \xrightarrow{c_1} KU_1(C^*_{\mathbb{R}}(\Lambda,\gamma)),$$
$$KU_4(C^*_{\mathbb{R}}(\Lambda,\gamma)) \xrightarrow{r_4} KO_4(C^*_{\mathbb{R}}(\Lambda,\gamma)) \xrightarrow{\eta_4} KO_5(C^*_{\mathbb{R}}(\Lambda,\gamma)).$$

Since $|KO_1(C^*_{\mathbb{R}}(\Lambda,\gamma))| = 4$ and $KO_0(C^*_{\mathbb{R}}(\Lambda,\gamma)) \cong KU_2(C^*(\Lambda,\gamma)) \cong \mathbb{Z}_2$, it follows that η_0 must be injective and c_1 must be surjective. Similarly, r_4 must be injective and η_4 must be surjective.

Since η_0 is injective, $r_0 = 0$ and c_2 is surjective. Since c_1 is surjective, r_7 must be 0. Continuing to use the long exact sequence (2.9), we find that

$$r_7 = 0, \quad \eta_7 = 1, \quad c_0 = 0, \quad r_6 = 1, \quad \eta_6 = 0, \quad c_7 = 1, \quad r_5 = 0, \quad \eta_5 = 1, \quad c_6 = 0.$$

Also, since η_4 is surjective, we know that $c_5 = 0$.

Now η_7 and η_0 are both injective. It follows that

$$\eta_1 : KO_1(C^*_{\mathbb{R}}(\Lambda,\gamma)) \to KO_2(C^*_{\mathbb{R}}(\Lambda,\gamma))$$

cannot be injective also, due to the relation $\eta^3 = 0$. So $|\ker \eta_1|$ is either equal to 2 or to 4. But $|\ker \eta_1| = |\operatorname{im} r_1|$ and the latter cannot be equal to 4, since $KU_1(C^*_{\mathbb{R}}(\Lambda,\gamma) = \mathbb{Z}_2$. Thus $|\ker \eta_1| = 2$. Then the exact sequence

$$0 \to KO_1(C^*_{\mathbb{R}}(\Lambda,\gamma))/\ker \eta_1 \xrightarrow{\eta_1} KO_2(C^*_{\mathbb{R}}(\Lambda,\gamma)) \xrightarrow{c_2} KU_2(C^*_{\mathbb{R}}(\Lambda,\gamma)) \to 0$$

implies that $|KO_2(C^*_{\mathbb{R}}(\Lambda,\gamma))| = 4$ (since the groups on the left and the right each have order 2). Thus, r_1 is an injection and $c_3 = 0$. Moreover, the fact that $|KO_2(C^*_{\mathbb{R}}(\Lambda,\gamma))| \neq 8$ implies that $d^2_{(2,1)} \neq 0$ and consequently

$$KO_2(C^*_{\mathbb{R}}(\Lambda,\gamma)) \cong \mathbb{Z}_2^2 \cong KO_3(C^*_{\mathbb{R}}(\Lambda,\gamma)).$$

As $c_3 = 0$, we must have $\eta_2 : KO_2(C^*_{\mathbb{R}}(\Lambda,\gamma)) \to KO_3(C^*_{\mathbb{R}}(\Lambda,\gamma))$ onto, which implies that η_2 is an isomorphism and consequently $r_2 = 0$ and c_4 is onto.

Now, because η_2 and η_4 are both surjective, the relation $\eta^3 = 0$ implies that $\eta_3 : \mathbb{Z}_2^2 \to KO_4(C^*_{\mathbb{R}}(\Lambda,\gamma))$ cannot be surjective. So $|\operatorname{im} \eta_3|$ is equal to 0 or to 2.

But if $|\operatorname{im}\eta_3| = 0$, then $c_4 : KO_4(C^*_{\mathbb{R}}(\Lambda,\gamma)) \to KU_4(C^*_{\mathbb{R}}(\Lambda,\gamma)) \cong \mathbb{Z}_2$ would be injective, which is not possible. Therefore $|\operatorname{im}\eta_3| = |\ker c_4| = 2$.

We have now calculated all of the groups $KO_*(C^*_{\mathbb{R}}(\Lambda,\gamma))$, at least up to order, and the action of the maps η, r, c. This enables us to compute the core of the $\mathcal{CR}$-module $K^{CR}(C^*_{\mathbb{R}}(\Lambda,\gamma))$.

Recall from (2.8) that $\psi : KU_*(C^*_{\mathbb{R}}(\Lambda,\gamma)) \to KU_*(C^*_{\mathbb{R}}(\Lambda,\gamma))$ satisfies $\psi^2 = 1$. As $KU_i(C^*_{\mathbb{R}}(\Lambda,\gamma)) \cong \mathbb{Z}_2$ for each i, we may conclude that $\psi = 1$ for all i in this case. It follows that $MU_i = (\ker(1-\psi_i))/(\operatorname{im}(1+\psi_i)) = KU_i(C^*_{\mathbb{R}}(\Lambda,\gamma)) \cong \mathbb{Z}_2$ for all i.

Furthermore, our descriptions of the maps η_i above reveal that the groups $MO_i = \operatorname{im}\eta_{i-1}$ are as follows:

	0	1	2	3	4	5	6	7
MO_i	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0

Now from the K-theory calculations in [Boersema 2002, Section 5.1, Table 5] and [Boersema et al. 2011, Section 11, Table 2], we find the core of $K^{CR}(\mathcal{O}_3^{\mathbb{R}}) \cong K^{CR}(\mathcal{E}_3)$ is given by

	0	1	2	3	4	5	6	7
MO_i	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0
MU_i	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0

By comparing cores, we conclude that

$$\begin{aligned} K^{CR}(C^*_{\mathbb{R}}(\Lambda,\gamma)) &\cong \Sigma K^{CR}(\mathcal{O}_3^{\mathbb{R}}) \oplus \Sigma^{-2} K^{CR}(\mathcal{O}_3^{\mathbb{R}}) \\ &\cong \Sigma^{-1} K^{CR}(\mathcal{E}_3) \oplus \Sigma^{4} K^{CR}(\mathcal{E}_3). \end{aligned}$$

That is,

	0	1	2	3	4	5	6	7
$KO_i(C^*_{\mathbb{R}}(\Lambda,\gamma))$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$KU_i(C^*_{\mathbb{R}}(\Lambda,\gamma))$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

□

5. Questions

Our investigation has highlighted many unanswered questions about higher-rank graph C^*-algebras and about the spectral sequence of Theorem 3.7 that computes $K^{CR}_*(C^*_{\mathbb{R}}(\Lambda,\gamma))$. First of all, Theorem 3.7 gives no information about the differential d^r of the spectral sequence. How is this map determined by the higher-rank graph with involution (Λ,γ)? Can we compute d^r from the combinatorial data — the adjacency matrices, the factorization rule, the involution — of (Λ,γ)?

A related question is to better understand the role that γ plays in these constructions. In setting up the spectral sequence, it is important to know which vertices are fixed and which are not fixed by γ. But beyond that, the action of γ on the edges does not seem to play a role (unless it plays a role in determining the differential maps d^r in a way that we are not aware of — see the previous paragraph).

In fact, we know from [Boersema 2017, Theorem 2.4] that in the case of a 1-graph the isomorphism class of the real C^*-algebra $C^*_{\mathbb{R}}(\Lambda, \gamma)$ may depend on the action of γ on the vertices of Λ but not on the way γ acts on the edges of Λ. We found that the proof of this theorem does not extend in an obvious way to the case of k-graphs with $k \geq 2$, but on the other hand we have no counterexamples to the analogous statement. How does γ affect the K-theory of $C^*_{\mathbb{R}}(\Lambda, \gamma)$ and indeed how does γ affect the isomorphism class of $C^*_{\mathbb{R}}(\Lambda, \gamma)$?

Another question concerns the functoriality of the spectral sequence. Specifically, suppose that (Λ, γ) is a rank-k graph with involution. Then for any $0 \leq \ell \leq k$, there is an obvious rank-ℓ graph (Λ', γ') with involution:

$$\Lambda' = \{\lambda \in \Lambda \mid d(\lambda) \in \mathbb{N}^\ell = \{(x_1, \dots, x_\ell, 0, \dots, 0) \mid x_i \in \mathbb{N}\} \subseteq \mathbb{N}^k\}.$$

We define γ' to be the restriction of γ. There is an obvious corresponding map $i : C^*_{\mathbb{R}}(\Lambda', \gamma') \to C^*_{\mathbb{R}}(\Lambda, \gamma)$, which induces a map on K-theory:

$$i_* : K^{CR}(C^*_{\mathbb{R}}(\Lambda', \gamma')) \to K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma)).$$

On the purely algebraic level, there is consequently a homomorphism from the chain complex associated to $C^*_{\mathbb{R}}(\Lambda', \gamma')$ to that associated to $C^*_{\mathbb{R}}(\Lambda, \gamma)$ that commutes with the differentials ∂. We conjecture that this map on the level of the chain complexes induces a map on the level of spectral sequences which commutes with the differentials d^r, and that it converges in the appropriate sense to the map i_* on K-theory.

In particular, taking $\ell = 0$, this conjecture would provide a way to identify the class of any projection $[p_v]$ in $KO_0(C^*_{\mathbb{R}}(\Lambda, \gamma))$ when v is a vertex in Λ fixed by γ, or the class of $[p_v + p_{\gamma(v)}]$ when v is not fixed by γ. It would also provide a way to identify the class of the identity in $KO_0(C^*_{\mathbb{R}}(\Lambda, \gamma))$ when Λ is finite, which is part of the Elliot invariant when $C^*_{\mathbb{R}}(\Lambda, \gamma))$ is simple and purely infinite. Such a result would be a direct generalization of Theorem 4.5 of [Boersema 2017] and Theorem 3.2 of [Raeburn and Szymański 2004].

Finally, we wonder if our spectral sequence can be used to characterize the $\mathcal{CR}$-modules that can arise as $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$, where (Λ, γ) is a rank k-graph. Corollary 4.3 of [Boersema 2017] gives a necessary condition for a given $\mathcal{CR}$-module to be isomorphic to $K^{CR}(C^*_{\mathbb{R}}(\Lambda, \gamma))$, but we do not have a complete characterization, even when Λ is a rank-1 graph. Which real Kirchberg algebras can be realized as $C^*_{\mathbb{R}}(\Lambda, \gamma)$ for some directed graph with involution (Λ, γ)? More generally, which real Kirchberg algebras can be realized as $C^*_{\mathbb{R}}(\Lambda, \gamma)$ for some higher-rank graph with involution (Λ, γ)? In particular, the original question that motivated this work is still unanswered: can we find concrete representations of the exotic real Cuntz algebras $\mathcal{E}_n$ using a family of higher-rank graphs with involution?

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Received 25 Aug 2021. Revised 8 Feb 2022. Accepted 24 Feb 2022.

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Annals of K-Theory is a journal of the K-Theory Foundation (ktheoryfoundation.org). The K-Theory Foundation acknowledges the precious support of Foundation Compositio Mathematica, whose help has been instrumental in the launch of the Annals of K-Theory.

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Annals of K-Theory (ISSN 2379-1681 electronic, 2379-1683 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

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