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Statistical analysis of discretely sampled semilinear SPDEs: a power variation approach

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Abstract

Motivated by problems from statistical analysis for discretely sampled SPDEs, first we derive central limit theorems for higher order finite differences applied to stochastic processes with arbitrary finitely regular paths. These results are proved by using the notion of Δ-power variations, introduced herein, along with the Hölder-Zygmund norms. Consequently, we prove a new central limit theorem for Δ-power variations of the iterated integrals of a fractional Brownian motion. These abstract results, besides being of independent interest, in the second part of the paper are applied to estimation of the drift and volatility coefficients of semilinear stochastic partial differential equations in dimension one, driven by an additive Gaussian noise white in time and possibly colored in space. In particular, we solve the earlier conjecture from Cialenco et al. (Stat. Inference Stoch. Process. 23:83-103, 2020) about existence of a nontrivial bias in the estimators derived by naive approximations of derivatives by finite differences. We give an explicit formula for the bias and derive the convergence rates of the corresponding estimators. Theoretical results are illustrated by numerical examples.

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1 Introduction

The main motivation of this work comes from some open problems in statistical analysis of *discretely sampled* stochastic partial differential equations (SPDEs) of the form

$$\mathrm{d}X_t(x) = -\theta(-\Delta)^{\alpha/2}X_t(x)\mathrm{d}t + F(X_t(x))\,\mathrm{d}t + \sigma(-\Delta)^{-\gamma}\mathrm{d}W_t(x),\\ t > 0,\ x \in [0,1], \tag{1.1}$$

where $\alpha > 0, \gamma \geq 0$ are given, $\theta, \sigma > 0$ are the parameters of interest (unknown to the observer), Δ stands for the Laplace operator ∂_{xx} with Dirichlet or periodic boundary conditions, W is a cylindrical Wiener process on $L^2([0,1])$, and F is a (nonlinear) operator acting on some appropriate Hilbert space. Note that $-\Delta$ is a positive, selfadjoint, closed and densely-defined operator in $L^2([0,1])$, and hence the power operator $(-\Delta)^\beta$, $\beta \in \mathbb{R}$, is well defined. We refer to the recent survey [24] on fractional Laplacian and its applications such as modeling anomalous diffusions. Most of the existing literature on statistical inference for SPDEs is dedicated to linear SPDEs, i.e. $F = 0$, with few exceptions [1, 2, 7, 18, 29, 32]. Moreover, the majority of works were dedicated to continuous time sampling setup; cf. the survey paper [11]. The parameter estimation problem for (linear) SPDEs when the solution is discretely sampled in space and/or time component was addressed systematically only recently by quite different methods, and we refer to [4–6, 8–10, 12, 19, 21–23, 34], and to [31, 33] for earlier studies, as well as the recent work [18] on reaction-diffusion equations. The central theme in these works evolves, in one form or another, around power variations of some relevant stochastic processes, which in turn is strongly related to the regularity properties of the solution. For example, when $\alpha = 2, \gamma = 0$, and $F = 0$, one can show that for a fixed $x \in (0,1)$, the paths of the process $X_t(x)$ have continuous versions with Hölder order of continuity $1/4-\varepsilon$, for any $\varepsilon > 0$. Consequently, as proved in [8], the fourth power variation is finite and yields consistent and asymptotically normal estimators for θ and σ. Similar arguments hold true for solutions of SPDEs when the Hölder order of continuity in space or time component is smaller than one. However, this approach, as well as the existing methods from the aforementioned literature on discrete sampling, cannot be applied directly to SPDEs with regular paths (or space colored noise). One of the main goals of this work is to develop new methodologies that can treat such cases. Of course, one should not expect that the solution $X_t(x)$ as function of t will get smoother than the paths of a Brownian motion, i.e. almost 1/2 Hölder continuous. On the other hand, it is known, for example when $F = 0$, that for any fixed $t > 0$, the solution process $X_t(x)$, $x \in (0,1)$, has almost Hölder $2\gamma + \alpha/2 - 1/2$ regularity in spatial variable x, namely the solution gets smoother the more colored (correlated) in space is the driving noise. One approach is to take the maximal number of (classical) derivatives in x, say $m := \lfloor 2\gamma + \alpha/2 - 1/2 \rfloor$, and expect that $\partial_x^m X_t(x)$ behaves as a fractional Brownian motion with Hurst parameter $2\gamma + \alpha/2 - 1/2 - m$ plus a smooth process, and apply or adapt the existing results on power variations, for example, from [8, 21, 22]. However, from a statistical point of view, this assumes that the process $\partial_x^m X_t(x),\ x \in (0, 1)$ is observed, which practically speaking is an unrealistic assumption. One way to overcome this drawback, is to approximate the derivatives by using the discrete measurements of the solution itself, for example by finite differences. However, such approximations typically will yield a nontrivial and non-vanishing bias in the estimators - a phenomena noticed in [13] through numerical experiments for SPDEs driven by space-only noise and with $m = 1$, and later in [12] the bias was explicitly given and the asymptotic properties of the estimator were formally proved. We built on these line of ideas, and we focus our study on discretely sampled (in space) of semilinear SPDEs.

A key concept of this paper is to track and use the classical regularity of a continuous function in terms of conveniently chosen integro-difference operators, for which we use the Hölder-Zygmund norms and spaces rather than classical Hölder or Sobolev norms and spaces. To deal with the higher order finite differences and their power variations, we introduce the notion of Δ-power variation, and prove that the central limit theorems for Δ-power variations are invariant under smooth perturbations; see Sects. 2, 3. We note that the idea of using power variation of higher order finite differences has been used, for example, in estimation of self-similarity order of self-similar processes. We highlight [20], where quadratic variations from higher order linear filters are studied for a class of Gaussian processes. In [14], the case of p-variations is studied for fractional Brownian motion. See also [36, Sect. 5.6] and references therein. We derive a new central limit theorem for Δ-power variations of iterated integrals of a fractional Brownian motion (fBm) (see Sect. 4), where we also explicitly compute the asymptotic variance. These novel results are of independent interest, contributing to the literature on limit theorems for fractional type processes, but in addition, these results provide a method for building consistent and asymptotically normal estimators for discretely sampled process with smooth paths, such as the SPDEs mentioned earlier.

The statistical analysis of semilinear SPDEs is investigated in Sects. 5, 6. We study the estimation of the drift θ and volatility σ of (1.1), under fairly general assumptions on the nonlinear part, assuming that the solution is sampled discretely in the spatial component x at one fixed time instance $t > 0$. In particular, we do not assume that F is known to the observer. Similarly to the above cited works on nonlinear SPDEs, we first use the so-called splitting of the solution argument, where the solution is written as $X = \overline{X} + \widetilde{X}$, where $\overline{X}$ is the solution of the linear SPDE and $\widetilde{X}$ solves the corresponding nonlinear random PDE (see Eqs. 5.2 and 5.3). In typical semilinear equations (as in Example 5.3), $\widetilde{X}$ is smoother than $\overline{X}$, which allows to argue that the estimation problem can be reduced to the linear case. The latter is reduced to the results on fBm by proving that the highest order (classical) derivative of $\overline{X}$ has the same probability law as a smoothly perturbed fBm. Assuming that one of the coefficients σ or θ is known we derive an estimator for the second coefficient, prove its consistency and provide its rate of convergence. We note that, the results in [8], which is the closest in spirit to this manuscript, considers only linear equations driven by space-time white noise, i.e.

$\alpha = 2$, $\gamma = 0$, and $F = 0$. The results presented in this manuscript are the first ones on parameter estimation for SPDEs with arbitrarily regular paths that are discretely sampled in physical spatial domain. As a second application of general results of Sect. 4, in Sect. 6 we study parameter estimation problem for a version of SPDEs (1.1) on the whole space. Namely, same as in [21, 22], we consider linear equations driven by a space-time Gaussian noise with covariance structure generated by the Riesz kernel of order 4γ with $\gamma \in (0, 1/4)$. Assuming the same sampling scheme as in the bounded domain case, we derive consistent and asymptotically normal estimators for θ or σ. We remark that the obtained results hold true for any $\alpha > 0$, generalizing the results of [21, 22], where it is assumed that $\alpha \in (0, 2]$. The case of nonlinear equations on the whole space is omitted in this study due to the lack of results on fine regularity properties of the solution (the so-called L^p theory). We validate the theoretical results by numerical simulations for various sets of parameters; see Sect. 7. In particular, we compute explicitly the aforementioned bias, which indeed turns out to be a significant correction to the naively derived estimators.

Finally, we remark that extending the obtained results to more general sampling schemes, e.g. sampling on a space-time grid, generally speaking is not an easy task. See for instance [4, 5, 8, 10, 18] that deal with some particular SPDEs that admit solutions with low spatial regularity. Such schemes are also directly related to the joint estimation of θ and σ. These questions, albeit practically very important, are beyond the scope of the present work and will be addressed in the future works.

2 Preliminaries

We fix a complete probability space $\mathbb{F} = (\Omega, \mathscr{F}, \mathbb{P})$ and throughout, all equalities and inequalities are understood in $\mathbb{P}$-a.s. sense, unless otherwise stated. As usual, we will denote by $\mathbb{P}-\lim$ or $\xrightarrow{\mathbb{P}}$ the convergence in probability, and $\mathsf{w}-\lim$ or $\xrightarrow{d}$ will stand for the convergence in distribution. Correspondingly, $a_n = o_{\mathbb{P}}(b_n)$ means that $a_n/b_n \xrightarrow{\mathbb{P}} 0$. Moreover, we write $a_n \lesssim b_n$, if there exists a constant C, independent of n, such that $a_n \leq C b_n$ for all $n \in \mathbb{N}$.

Let X_t, $t \in \mathbb{R}$, be a real valued measurable function, and denote by J, and Δ_h, the integral, and respectively the difference operators of the form

$$J X_t := \int_0^t X_r \,\mathrm{d}r, \quad t \in \mathbb{R},$$
$$\Delta_h X_t := X_{t+h} - X_t, \quad t \in \mathbb{R},\ h > 0.$$

As usual, we put $J^0 X := X$, and for $m \in \mathbb{N}$, we define $J^m X := J J^{m-1} X$. Similar notations apply to Δ_h. Note that,

$$\Delta_h^M (X_t) = \sum_{k=0}^{M} (-1)^{M-k} \binom{M}{k} X_{t+kh}, \quad t \in \mathbb{R},\ h > 0.$$

We will denote by $C(\mathbb{R})$ the space of continuous and bounded functions on $\mathbb{R}$ endowed with sup-norm $\|f\|_\infty := \sup|f|$. Correspondingly, for $k \in \mathbb{N}$, we put $C^k(\mathbb{R}) := \{f \in C(\mathbb{R}) : \|f\|_{C^k(\mathbb{R})} := \sum_{j\le k} \|D^j f\|_\infty < \infty\}$, where D stands for differential operator.

One of the key ideas of this paper is tracking and using the classical regularity of a continuous function in terms of conveniently chosen integral and difference operators. For this purpose, we will be using the *Hölder-Zygmund spaces* $\mathcal{C}^s(\mathbb{R})$, $s > 0$, introduced in [38] and endowed with the norm

$$\|f\|_s^{(k,M)} = \|f\|_{C^k(\mathbb{R})} + |f|_s^{(k,M)},$$

with

$$|f|_s^{(k,M)} = \sup_{h>0} h^{-(s-k)} \|\Delta_h^M D^k f\|_\infty, \tag{2.1}$$

and where $k \in \mathbb{N}_0$, $M \in \mathbb{N}$, such that $k < s$ and $M > s - k$. It can be shown (cf. [35, Section1.2.2]), that for any such k and M, and fixed s the norms $|\cdot|_s^{(k,M)}$ are equivalent. We also recall that for any $s > 0$, $\mathcal{C}^s(\mathbb{R})$ coincides with the Besov space $B^s_{\infty,\infty}(\mathbb{R})$ (see also [17]), and for $s \notin \mathbb{N}$, $\mathcal{C}^s(\mathbb{R})$ coincide with the classical Hölder spaces. Thus, the Hölder-Zygmund norms measure the regularity of a continuous function in the classical sense. In this study, we will be mainly interested in the case $k = 0$, which corresponds to statistical experiment of discrete measurements of the underlying process itself. However, if the observer evaluates discretely some derivative of f, then one should consider $k \ge 1$. Thus, we emphasize that the choice of $k = 0$ is primarily driven by practical reasons, but in principle all results can be elevated to the general case $k \in \mathbb{N}_0$. We also set $\mathcal{C}^{s-}(\mathbb{R}) := \bigcap_{r<s} \mathcal{C}^r(\mathbb{R})$.

3 Smooth perturbations of higher order power variations

In this section, we introduce the notion of a Δ-power variation for a given process X and study its stability under smooth perturbations. Let $\pi = \{t_0, \dots, t_N\}$ be the uniform partition of size N of the interval $[a, b] \subset [0, T]$, and put $h := h_N := (b-a)/N = t_{k+1} - t_k$, $k = 0, \dots, N$. For fixed $s > 0$, $q, M, N \in \mathbb{N}$, such that $N > M$, we define

$$V_{q,M,s,N}(X) := \frac{1}{b-a} \sum_{k=0}^{N-M} h \left| \frac{\Delta_h^M X_{t_k}}{h^s} \right|^q .$$

Similar to the power variation of a process, we are interested in the limiting behavior of $V_{q,M,s,N}$ as $N \to \infty$. The Δ-power variation of order (q, M, s) of a process X is defined as

$$V_{q,M,s}(X) := \mathbb{P} - \lim_{N\to\infty} V_{q,M,s,N}(X), \tag{3.1}$$

provided that the limit (in probability) exists. Note that $V_{p,1,1}$ corresponds to the (normalized) power variation of order p.

We start with a simple, but important, result that links the path continuity of the process X with its generalized power variation.

Lemma 3.1 *Let $q, M \in \mathbb{N}$, $s > 0$, such that $M > s$. If $X \in \mathcal{C}^s([a,b])$, then $V_{q,M,s,N}(X)$ is uniformly bounded in N.*

Proof This follows at once by noticing that

$$V_{q,M,s,N}(X) = \frac{1}{b-a}\sum_{k=0}^{N-M}(t_{k+1}-t_k)\left|\frac{\Delta_h^M X_{t_k}}{(t_{k+1}-t_k)^s}\right|^q \lesssim (h^{-s}\|\Delta_h^M X\|_\infty)^q$$
$$\lesssim (\|X\|_s^{(0,M)})^q.$$

□

We give the main results of this section, which in the nutshell says that the central limit theorems for Δ-power variations of a stochastic process remain invariant under smooth perturbations; see also [8, Proposition 2.1].

Theorem 3.2 *Let $q \geq 1$, $s > 0$, $M \in \mathbb{N}$ with $M > s$. Assume that $X \in \mathcal{C}^{s-}([a,b])$ and for some $\alpha > 0$, $\Sigma \geq 0$, the following limit exists*

$$h_N^{-\alpha}\left(V_{q,M,s,N}(X) - V_{q,M,s}(X)\right) \xrightarrow{d} \mathcal{N}(0,\Sigma), \quad \textit{as } N \to \infty, \tag{3.2}$$

where $\mathcal{N}(0,\Sigma)$ is a Gaussian random variable with mean zero and variance[1] *Σ. Then, for any $Y \in \mathcal{C}^{s+\eta-}([a,b])$ with $\eta > \alpha$, and $M > s+\alpha$,*

$$h_N^{-\alpha}\left(V_{q,M,s,N}(X+Y) - V_{q,M,s}(X)\right) \xrightarrow{d} \mathcal{N}(0,\Sigma), \quad \textit{as } N \to \infty. \tag{3.3}$$

Proof Without loss of generality, we assume that $M > s+\eta$, otherwise take η' instead of η with $\alpha < \eta' < \eta \wedge (M-s)$. We proceed analogously to [8, Proposition 2.1]. It suffices to show

$$\lim_{N\to\infty} h_N^{-\alpha}(V_{q,M,s,N}(X+Y) - V_{q,M,s,N}(X)) = 0, \quad \text{a.s..} \tag{3.4}$$

Let $g_N(r) = \left(V_{q,M,s,N}(X)^{1/q} + rV_{q,M,s,N}(Y)^{1/q}\right)^q$. Then, by Minkowski's inequality, $g_N(-1) \leq V_{q,M,s,N}(X+Y) \leq g_N(1)$, and there exist $\xi_1, \xi_2 \in [0,1]$ (dependent on $N \in \mathbb{N}$) such that

$$g_N'(-\xi_1) = g_N(-1) - g_N(0) \leq V_{q,M,s,N}(X+Y) - V_{q,M,s,N}(X)$$
$$\leq g_N(1) - g_N(0) = g_N'(\xi_2).$$

Thus, it remains to show $h_N^{-\alpha}\sup_{-1\leq r\leq 1} g_N'(r) \xrightarrow{a.s.} 0$, as $N \to \infty$. For $r \in [-1,1]$ and $\varepsilon > 0$,

$$\left|g_N'(r)\right| \leq q\left|V_{q,M,s,N}(X)^{1/q} + rV_{q,M,s,N}(Y)^{1/q}\right|^{q-1} V_{q,M,s,N}(Y)^{1/q}$$
$$\lesssim h_N^{\eta-2\varepsilon} V_{q,M,s+\eta-\varepsilon,N}(Y)^{1/q},$$

[1] As usual, zero variance case is interpreted as the Dirac point mass at the mean.

where in the last inequality we used that $qh_N^{\varepsilon}\,|V_{q,M,s,N}(X)^{1/q}+rV_{q,M,s,N}(Y)^{1/q}|^{q-1}$ is bounded uniformly in N and $r\in[-1,1]$ due to Lemma 3.1, and that $V_{q,M,s,N}(Y)=h_N^{q(\eta-\varepsilon)}V_{q,M,s+\eta-\varepsilon,N}(Y)$ by the definition of the Δ-power variation. Finally, since $Y\in\mathcal{C}^{s+\eta-\varepsilon}([a,b])$, and again making use of Lemma 3.1 we have that $V_{q,M,s+\eta-\varepsilon,N}(Y)$ is bounded uniformly in N. The claim follows from choosing $\varepsilon<(\eta-\alpha)/2$. □

Remark 3.3 (1) We note that the restriction $M>s+\alpha$ can be always satisfied by choosing M large enough. (2) If $\Sigma=0$, then the limits (3.2) and (3.3) can be equivalently understood as limits in probability. This in turn can be re-formulated in the terms of rates of convergence, as we do, for example, in Theorems 5.6 and 5.7. (3) The results in this section can be easily extended to Δ-power variations over arbitrary sequence of partitions, not necessarily uniform. Namely, one can replace the sequence of uniform partitions with a sequence of partitions with vanishing mesh-size in the above limits. However, generally speaking the counterpart of limit (3.1) (if exists), may depend on the choice of the sequence of partitions.

4 The case of fBM

In the Section we derive limit theorems for Δ-power variations of iteratively integrated fractional Brownian motion. We start by recalling that a two-sided fractional Brownian motion (fBm) with Hurst index $H\in(0,1)$ is a centered Gaussian process $B^H=(B_t^H)_{t\in\mathbb{R}}$ such that

$$\mathbb{E}\left(B_t^H B_r^H\right)=\frac{1}{2}\left(|t|^{2H}+|r|^{2H}-|t-r|^{2H}\right),\quad t,r\in\mathbb{R}.$$

A continuous stochastic process X is called *s-self-similar* or self-similar of index s (or self-similar for short) if the law of $(h^{-s}X_{ht})_{t\in\mathbb{R}}$ on $C(\mathbb{R})$ does not depend on $h>0$. The process X is said to be *stationary* if the law of $(X_{t+u})_{t\in\mathbb{R}}$ on $C(\mathbb{R})$ does not depend on $u\in\mathbb{R}$, and X is said to have *stationary increments* if $\Delta_h X$ is stationary for all $h>0$. A fractional Brownian motion B^H is a prominent example of a self-similar process (of index H) with stationary increments. Many core properties of fBm are directly linked to these two features. However, generally speaking, differences of integrals of fBm are not self-similar in the usual sense, but rather, one has to account for the step-width of the difference operator. Towards this end, we extend the notion of self-similarity to a parameterized family of processes, say $X^{(h)}$, $h>0$. Primarily, we will be interested in a parameterized family of process of the form $X^{(h)}=\Delta_h^M Y$, where $M\in\mathbb{N}_0$ and Y is a process that does not depend explicitly on $h>0$. We say that a parameterized family of process $X^{(h)}$ is *parameterized s-self-similar* (or just parameterized self-similar) if the law of $(h^{-s}X_{ht}^{(h)})_{t\in\mathbb{R}}$ is independent of $h>0$. We also note that in general, if X is stationary, then JX is not necessarily stationary.

Lemma 4.1 *Let X and $X^{(h)}$, $h>0$, be centered Gaussian processes. Then:*

(1) $\Delta_h^2 JX=\Delta_h J\Delta_h X$.
(2) If X is s-self-similar, then JX is $(s+1)$-self-similar.

(3) *If $X^{(h)}$ is parameterized s-self-similar, then $\Delta_h X^{(h)}$ is parameterized s-self-similar and $J X^{(h)}$ is parameterized $(s+1)$-self-similar.*
(5) *If X is stationary, then $\Delta_h X$ and $\Delta_h J X$ are stationary for any $h > 0$.*

Proof First we note that if X is a centered Gaussian process, then JX and $\Delta_h X$ are also Gaussian and centered. Thus, the law of $\Delta_h J X^{(h)}$ is determined by $\mathbb{E}\left[\Delta_h J X_t^{(h)} \Delta_h J X_r^{(h)}\right]$, which is equal to $\int_t^{t+h} \int_r^{r+h} \mathbb{E}\left[X_v^{(h)} X_w^{(h)}\right] \mathrm{d}v\,\mathrm{d}w$, $t, r \in \mathbb{R}$. Using this, the above properties follow now by direct calculations. □

Next, we state some properties specific to integro-differences of the $J^m B^H$ and $\Delta_h^M J^m B^H$.

Lemma 4.2 *The following assertions hold true:*

(1) *For $m \in \mathbb{N}_0$ and $t, r \in \mathbb{R}$, we have*

$$\mathbb{E}\left(J^m B_t^H \cdot J^m B_r^H\right) = \sum_{k=0}^{m} \frac{(-1)^k \left(t^{m-k} r^{m+k+2H} + r^{m-k} t^{m+k+2H}\right)}{2(m-k)! \prod_{i=1}^{m+k} (2H+i)} + \frac{(-1)^{m+1} |t-r|^{2m+2H}}{2 \prod_{i=1}^{2m} (2H+i)}. \tag{4.1}$$

In addition, $J^m B^H$ is $(m+H)$-self-similar. By convention, $\prod_{i=1}^{0} (2H+i) = 1$.

(2) *For $M, m \in \mathbb{N}_0$ and $t, r \in \mathbb{R}$, we have*

$$\mathbb{E}\left[\Delta_h^M J^m B_t^H \Delta_h^M J^m B_r^H\right] = \sum_{k,l=0}^{M} (-1)^{2M-k-l} \binom{M}{k} \binom{M}{l} \mathbb{E}\left[J^m B_{t+kh}^H J^m B_{r+lh}^H\right]. \tag{4.2}$$

(3) *If $M \geq m$, then $\Delta_h^M J^m B^H$ is parameterized $(m+H)$-self-similar and has stationary increments.*

Proof (1) We prove (4.1) by induction in m. For $m = 0$, (4.1) is immediate. For $m = 1$, by direct computations, we have

$$\begin{aligned}
\mathbb{E}\left(J B_t^H \cdot J B_r^H\right) &= \int_0^t \int_0^r \mathbb{E}\left(B_u^H B_v^H\right) \mathrm{d}u\,\mathrm{d}v \\
&= \frac{1}{2} \int_0^t \int_0^r \left(u^{2H} + v^{2H} - |u-v|^{2H}\right) \mathrm{d}u\,\mathrm{d}v \\
&= \frac{1}{2}\left[\frac{t \cdot r^{2H+1} + r \cdot t^{2H+1}}{1!(2H+1)} + \frac{|t-r|^{2H+2} - r^{2H+2} - t^{2H+2}}{(2H+1)(2H+2)}\right],
\end{aligned}$$

and hence (4.1) is true for $m = 1$. Suppose (4.1) holds true for $m \geq 0$. Then,

$$\begin{aligned}\mathbb{E}\left(J^{m+1}B_t^H \cdot J^{m+1}B_r^H\right) &= \int_0^t \int_0^r \mathbb{E}\left(J^m B_u^H \cdot J^m B_v^H\right) \mathrm{d}u \, \mathrm{d}v \\ &= \int_0^t \int_0^r \left[\sum_{k=0}^{m} \frac{(-1)^k \left(v^{m-k}u^{m+k+2H} + u^{m-k}v^{m+k+2H}\right)}{2(m-k)! \prod_{i=1}^{m+k}(2H+i)} \right. \\ &\quad \left. + \frac{(-1)^{m+1}|v-u|^{2m+2H}}{2\prod_{i=1}^{2m}(2H+i)}\right] \mathrm{d}u \, \mathrm{d}v \\ &= \sum_{k=0}^{m+1} \frac{(-1)^k \left(t^{m+1-k}r^{m+1+k+2H} + t^{m+1-k}r^{m+1+k+2H}\right)}{2(m+1-k)! \prod_{i=1}^{m+1+k}(2H+i)} \\ &\quad + \frac{(-1)^{m+2}|t-r|^{2(m+1)+2H}}{2\prod_{i=1}^{2(m+1)}(2H+i)},\end{aligned}$$

and thus (4.1) is proved. Consequently, $(m+H)$-self-similarity of $J^m B^H$ follows from Lemma 4.1(2).

(2) Identity (4.2) is immediate.
(3) The parameterized self-similarity follows from Lemma 4.1(3). Finally, Lemma 4.1(4) yields stationarity for $\Delta_h^{M+1} J^m B^H = \Delta_h^{M-m}(\Delta_h J)^m \Delta_h B^H$, where we use Lemma 4.1(1) and the fact that $\Delta_h B^H$ is stationary for all $h > 0$. The proof is complete.

□

Let us fix $M \in \mathbb{N}$ and $s > 0$, and write $s = m + H$ with $m \in \mathbb{N}_0$ and $H \in (0, 1)$. In view of Lemma 4.2, there exists $\mu_{M,s} > 0$ such that

$$\mathbb{E}\left|\Delta_h^M J^m B_t^H\right|^2 = \mu_{M,s} h^{2s},$$

for all $t \in \mathbb{R}$ and $h > 0$, and where $\mu_{M,s}$ is given by

$$\begin{aligned}\mu_{M,s} &:= \sum_{k=0}^{M} \binom{M}{k}^2 \sum_{p=0}^{m} \frac{(-1)^p k^{2s}}{(m-p)! \prod_{i=1}^{m+p}(2H+i)} \\ &\quad + \sum_{0 \leq j < k \leq M} (-1)^{2M-k-j} \binom{M}{k}\binom{M}{j} \left[\frac{(-1)^{m+1}(k-j)^{2s}}{\prod_{i=1}^{2m}(2H+i)} \right. \\ &\quad \left. + \sum_{p=0}^{m} \frac{(-1)^p \left(k^{m-p}j^{m+p+2H} + j^{m-p}k^{m+p+2H}\right)}{(m-p)! \prod_{i=1}^{m+p}(2H+i)}\right].\end{aligned}$$

We further set

$$\rho_{M,s}(\ell) := \mu_{M,s}^{-1} h^{-2s} \mathbb{E}\left(\Delta_h^M J^m B_t^H \cdot \Delta_h^M J^m B_{t+h\ell}^H\right), \quad \ell \in \mathbb{N}_0. \tag{4.3}$$

Note that due to parametrized self-similarity and stationarity of $\Delta_h^M J^m B^H$ as in Lemma 4.1, we have that $\rho_{M,s}(\ell)$ does not depend on $t \in \mathbb{R}$ and $h > 0$.

Next, we will investigate the asymptotic behavior of the q-th (Hermite) variation of $\Delta_h^M J^m B^H$, for which we will make use of (Breuer-Major) Theorem A.1 applied to process $Y_t = \left(\mu_{M,s}^{1/2} h^s\right)^{-1} \cdot \Delta_h^M J^m B_t^H$. First we note that by Lemma 4.2 the process Y is a centered stationary Gaussian process with unit variance. Next result will be used to show that (A.1) is satisfied.

Lemma 4.3 *Assume that $M, q \in \mathbb{N}$ and $0 < s < M - \frac{1}{2q}$. Then*

$$\sum_{\ell \in \mathbb{Z}} |\rho_{M,s}(\ell)|^q < \infty. \tag{4.4}$$

Proof Without loss of generality, we assume that $\ell \geq M$. The covariance function $\rho_{M,s}(\ell)$ becomes

$$\begin{aligned}
\rho_{M,s}(\ell) &= \mu_{M,s}^{-1} \sum_{0 \leq j,k \leq M} (-1)^{2M-k-j} \binom{M}{k}\binom{M}{j} \Bigg[\frac{(-1)^{m+1}(j+\ell-k)^{2m+2H}}{\prod_{i=1}^{2m}(2H+i)} \\
&\quad + \sum_{p=0}^{m} \frac{(-1)^p \left(k^{m-p}(j+\ell)^{m+p+2H} + (j+\ell)^{m-p}k^{m+p+2H}\right)}{(m-p)! \prod_{i=1}^{m+p}(2H+i)} \Bigg] \\
&=: c_1 \Delta_1^{2M} f_1(\ell) + \sum_{p=0}^{m} \left[c_{2,p} \Delta_1^M f_{2,p}(\ell) + c_{3,p} \Delta_1^M f_{3,p}(\ell) \right],
\end{aligned} \tag{4.5}$$

where

$$f_1(x) = (x-M)^{2m+2H}, \quad f_{2,p}(x) = x^{m+p+2H}, \quad f_{3,p}(x) = x^{m-p}.$$

First note that $\Delta_1^M f_{3,p} = (\Delta_1 J)^M f_{3,p}^{(M)} \equiv 0$ for $M > m$, so

$$c_{2,p} = ((m-p)! \prod_{i=1}^{m+p} (2H+i))^{-1} \sum_{k=0}^{M} (-1)^{M-k} \binom{M}{k} k^{m-p} = 0.$$

By direct computations, one can show that $c_1 \neq 0$, and $c_{3,p} \neq 0$, for any $H \in (0,1)$. It is clear that, as $\ell \to \infty$,

$$\Delta_1^{2M} f_1(\ell) = \Delta_1^{2M} J^{2M} f_1^{(2M)}(\ell) = (\Delta_1 J)^{2M} f_1^{(2M)}(\ell) \sim f_1^{(2M)}(\ell) \sim \ell^{2m+2H-2M}.$$

If $M \leq m+p$, we similarly deduce that $\Delta_1^M f_{3,p}(\ell) \sim \ell^{m-p-M}$, and $\Delta_1^{2M} f_1$ grows faster than $\Delta_1^M f_{3,p}$, since $2m + 2H - 2M > m - p - M$. If $M > m + p$, we have $\Delta_1^M f_{3,p}(\ell) \equiv 0$. Combining the above, we have

$$\rho_{M,s}(\ell) \sim \ell^{2m+2H-2M}, \quad \ell \to \infty.$$

Thus, if $H < M - m - \dfrac{1}{2q}$, then (4.4) is true. This concludes the proof. □

As an immediate consequence of Lemma 4.3, we get that for any $0 < s < M - \dfrac{1}{2q}$, the quantity

$$\rho^2_{q,M,s} := q! \sum_{\ell \in \mathbb{Z}} \left(\rho_{M,s}(\ell)\right)^q \tag{4.6}$$

is well-defined and finite.

The following result identifies $V_{q,M,s}(J^m B^H)$ for $s = m + H$ together with its convergence rate.

Theorem 4.4 *Let $M > m \geq 0$ and $q \geq 1$ be integers, and assume that either of the following assumptions is satisfied:*

(1) $M = m + 1$ and $0 < H < 3/4$,
(2) $M \geq m + 2$ and $0 < H < 1$.

Then, there exists $\sigma_{q,M,s} > 0$ such that

$$\sqrt{N}\left(V_{q,M,s,N}\left(J^m B^H\right) - \tau_q \mu^{q/2}_{M,s}\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2_{q,M,s}\mu^q_{M,s}\right), \quad \text{as } N \to \infty, \tag{4.7}$$

where $\tau_q := \mathbb{E}|Z|^q$ with $Z \sim \mathcal{N}(0,1)$.

Moreover, if q is an even number, then $\sigma^2_{q,M,s} = \sum_{k=1}^q \binom{q}{k}^2 \tau^2_{q-k}\rho^2_{k,M,s}$.

Proof We apply Theorem A.1, by taking $(Y_k)_{k\in\mathbb{Z}} = \left(\mu^{-1/2}_{M,s} h^{-s}\Delta^M_h J^m B^H_{t_k}\right)_{k\in\mathbb{Z}}$ and $f(x) = |x|^q - \tau_q = \sum_{k=0}^\infty a_k H_k(x)$ with $a_k = (2\pi)^{-1/2}\int(|x|^q - \tau_q)H_k(x)e^{-x^2/2}\,\mathrm{d}x$. Note that, in view of [28, Example 7.2.2] the function f has Hermite rank $d = 2$, namely $a_0 = a_1 = 0$ and $a_2 \neq 0$. It remains to show that (A.1) is satisfied, which in our case becomes $\sum_{\ell\in\mathbb{Z}} \rho^2_{M,s}(\ell) < \infty$. By Lemma 4.3, this is true if $0 < s < M - 1/4$, or equivalently if $0 < H < (M - m) - 1/4$, which is satisfied in view of assumptions (1)–(2). Thus, (4.7) is proved.

For q even, it can be shown (for example, by induction, or see [27, p.1076]) that

$$\begin{aligned} &V_{q,M,s,N}\left(J^m B^H\right) - \frac{N - M + 1}{N}\tau_q \mu^{q/2}_{M,s} \\ &\quad = \frac{1}{N}\mu^{q/2}_{M,s}\sum_{k=1}^q \binom{q}{k}\tau_{q-k}\sum_{j=0}^{N-M} H_k\left(\frac{\Delta^M_h J^m B^H_{t_j}}{\mu^{1/2}_{M,s}h^s}\right). \end{aligned} \tag{4.8}$$

□

Remark 4.5 (1) We emphasize that the limit of $V_{q,M,s,N}(J^m B^H)$ depends through $\mu_{M,s}$ on the regularity s of the process as well as the number of differences M. In particular, even for small h it is not possible to approximate the rescaled finite difference operator $h^{-1}\Delta_h$ in the definition of $V_{q,M,s,N}(J^m B^H)$ by a derivative operator without introducing a non–trivial bias, due to the change in M and s. However, as noticed in [12] for a different model with simpler computations, the bias is linked to the specific choice of the geometry of the sampling grid. If additional 'auxiliary' observation points are available that allow to evaluate $h^{-1}\Delta_h$ at a scale h smaller than the distance $1/N$ between points at which the derivative is approximated the bias term may disappear.

(2) The constant $\mu_{M,s}$ can be easily computed, and for reader's convenience we list some of its values. If $M = 1$, $m = 0$ and $0 < H < 3/4$, then $\mu_{M,s} = 1$. If $M = 2$, $m = 1$ and $H = 1/4$, then $\mu_{M,s} = (\sqrt{2} - 1)\frac{16}{15} \approx 0.44$. If $M = 2$, $m = 1$ and $H = 1/2$, then $\mu_{M,s} = 2/3$.

5 Semilinear SPDEs on a bounded domain

In this and the next section, we study the parameter estimation problem for SPDEs by using the Δ-power variations and the results on from previous section on integrated fractional Brownian motion. Specifically, we consider SPDEs on $\mathcal{D} = (0, 1)$ with zero boundary conditions. Towards this end, for $k \geq 1$, set $\Phi_k(x) = \sqrt{2}\sin(k\pi x)$ and $\lambda_k = k^2\pi^2$. The set $\{\Phi_k\}_{k\in\mathbb{N}}$ forms an orthonormal basis in $L^2(\mathcal{D})$. Further, for $s \in \mathbb{R}$, set $H^s(\mathcal{D}) := \{u \in L^2 \mid \sum_{k=1}^{\infty} \lambda_k^s (u, \Phi_k)_{L^2(\mathcal{D})}{}^2 < \infty\}$. The Laplacian $\Delta = \partial_{xx}$, acting on $C^\infty(\mathcal{D})$, can be extended to a closed, densely defined operator Δ on $L^2(\mathcal{D})$ with domain $H^2(\mathcal{D})$ and compact resolvent. The Φ_k are eigenfunctions of $-\Delta$ with eigenvalues λ_k. In this case, for $\alpha \in \mathbb{R}$, the fractional Laplacian $(-\Delta)^{\alpha/2}$ is given by $(-\Delta)^{\alpha/2} Z := \sum_{k=1}^{\infty} \lambda_k^{\alpha/2} (Z, \Phi_k)_{L^2(\mathcal{D})}\Phi_k$ whenever this term exists. Note that for $s > 1/2$, the Sobolev embedding theorem [3] $H^s(\mathcal{D}) \to C(\mathcal{D})$, and thus the point evaluations are well-defined.

We consider the following semilinear SPDE on $L^2(\mathcal{D})$:

$$\mathrm{d}X_t = \Big(-\theta(-\Delta)^{\alpha/2} X_t + F(X_t)\Big)\,\mathrm{d}t + \sigma B \mathrm{d}W_t, \quad X_0 \in L^2(\mathcal{D}), \tag{5.1}$$

where $\alpha, \theta, \sigma > 0$, W is a cylindrical Wiener process on $L^2(\mathcal{D})$, $B = (-\Delta)^{-\gamma}$ for some $\gamma > 1/4 - \alpha/4$, and F is a nonlinear operator.

We assume that (5.1) is well-posed in $L^2(\mathcal{D})$ in the analytically mild and probabilistically weak sense. We refer, for instance, to [16, 25] for sufficient conditions regarding the well-posedness, as well as Proposition 5.2 and Example 5.3 below.

As is customary in statistical inference for nonlinear SPDEs [2, 7, 32], we will use the splitting of the solution argument, by writing $X = \overline{X} + \widetilde{X}$, where

$$\mathrm{d}\overline{X}_t = -\theta(-\Delta)^{\alpha/2}\overline{X}_t \mathrm{d}t + \sigma B \mathrm{d}W_t, \quad \overline{X}_0 = 0, \tag{5.2}$$

$$\mathrm{d}\widetilde{X}_t = \Big(-\theta(-\Delta)^{\alpha/2}\widetilde{X}_t + F(\overline{X}_t + \widetilde{X}_t)\Big)\,\mathrm{d}t, \quad \widetilde{X}_0 = X_0. \tag{5.3}$$

The solution to (5.2) can be expressed either as a Fourier series, or can be given by the stochastic convolution

$$\overline{X}_t(x) = \sigma \int_0^t e^{-\theta(t-r)(-\Delta)^{\alpha/2}} B \mathrm{d}W_r = \sum_{k=1}^{\infty} \left(\sigma \lambda_k^{-\gamma} \int_0^t e^{-\theta(t-r)\lambda_k^{\alpha/2}} \mathrm{d}W_r^{(k)} \right) \Phi_k(x)$$
$$=: \sum_{k=1}^{\infty} \overline{x}_k(t) \Phi_k(x), \tag{5.4}$$

where $W^{(k)} = (W, \Phi_k)_{L^2(0,1)}, k \geq 1$, are independent one-dimensional Brownian motions, $t \mapsto e^{-\theta t(-\Delta)^{\alpha/2}}$, $t > 0$, is the C_0-semigroup on $L^2(\mathcal{D})$ generated by $-\theta(-\Delta)^{\alpha/2}$, and the convergence is understood in $L^2(\mathcal{D})$. Note that $\overline{x}_k(t) = (\overline{X}_t, \Phi_k)_{L^2}$, i.e. $\overline{x}_k(t)$ is also the Fourier coefficient of the solution $\overline{X}_t$ with respect to $\{\Phi_k\}_{k\in\mathbb{N}}$.

The next two results provide some fine continuity properties of the trajectories of $\overline{X}$ and $\widetilde{X}$.

Proposition 5.1 *For any $s < 2\gamma + \alpha/2 - 1/2$, it holds that $\overline{X} \in C(0, T; C^s(\mathcal{D}))$.*

Proof The common line of attack is to show that $\overline{X} \in C(0, T; W^{s,p}(\mathcal{D}))$ for any $p \geq 2$, and then employ the Sobolev embedding theorem. We refer, for example, to [2, Appendix B.1] for details when $\alpha = 2$, and since the proof is based on the Fourier decomposition of the solution in the base $\{\Phi_k\}$, the general case is obtained similarly. □

As a direct consequence, we note that $\overline{X}_t(\cdot)$ has up to $\lfloor 2\gamma + \alpha/2 - 1/2 \rfloor$ classical derivatives. We call $s^* = 2\gamma + \alpha/2 - 1/2$ the optimal regularity, and we make a standing assumption that $s^* > 0$ and $s^* \notin \mathbb{N}$. With this in mind, here we remark that the results from previous sections on fBm will be applied to derivatives of $\overline{X}_t(x)$ with respect to x, and thus in what follows the space variable x corresponds to t variable in the previous sections.

Proposition 5.2 *Assume that there exist $\eta, \epsilon > 0$, $0 \leq s_0 < s^*$, and a continuous function $g : [0, \infty) \to [0, \infty)$, such that for any $s_0 \leq s < s^*$,*

$$\|F(u)\|_{s+\eta-\alpha+\epsilon} \leq g(\|u\|_s), \tag{5.5}$$

where as before, $\|\cdot\|_s$ denotes the Hölder-Zygmund norm. Let $X \in C(0, T; C^{s_0}(\mathcal{D}))$ and $X_0 \in C^{s^*+\eta}(\mathcal{D})$. Then we have $\widetilde{X} \in C(0, T; C^{s+\eta}(\mathcal{D}))$, for any $0 \leq s < s^*$, and in particular $X \in C(0, T; C^s(\mathcal{D}))$.

Proof The case of Sobolev spaces $W^{s,p}(\mathcal{D})$ instead of Hölder spaces $C^s(\mathcal{D})$ has been treated in [2, 29] for $\alpha = 2$, the proof for general α is identical. Since for an arbitrary chosen $\epsilon > 0$, we have that $C^s(\mathcal{D}) \subset W^{s,p}(\mathcal{D}) \subset C^{s-\epsilon}(\mathcal{D})$ for large enough p, the desired result follows at once. □

Example 5.3 We present several types of semilinear SPDEs whose nonlinearity F satisfies (5.5), which in particular guarantees that all results from this section hold for the solutions to these classes of equations. For technical details see [2, 32].

(1) *(fractional) Heat equation* In the case $F = 0$, (5.1) becomes linear, sometimes called fractional heat equation, and (5.5) is trivially satisfied for any $\eta > 0$.
(2) *Reaction-diffusion equation* Let $F(u)(x) = f(u(x))$, where f is a polynomial function or $f \in C_b^\infty(\mathbb{R})$. Then (5.5) is true for any $0 < \eta < 2$.
(3) *Advection-diffusion equation* Let $F(u) = v\partial_x u$ for a given $v \in C^\infty(\mathcal{D})$. Then (5.5) holds with any $0 < \eta < 1$.
(4) If $F = F_1 + F_2$, for some F_1, F_2 that satisfy (5.5) with continuous functions g_1, g_2, then F satisfies (5.5) with $g = g_1 + g_2$.

Next, using representation (5.4), we set

$$\xi_k := \sqrt{\frac{2\theta\lambda_k^{\alpha/2+2\gamma}}{\left(1 - e^{-2\theta\lambda_k^{\alpha/2}t}\right)\sigma^2}} \cdot \overline{x}_k(t), \qquad k \geq 1.$$

Clearly, ξ_k's are independent standard Gaussian random variables, and $\overline{X}_t$ can be written as

$$\overline{X}_t(x) = \frac{\sigma}{\sqrt{2\theta}} \sum_{k\geq 1} \frac{1}{\lambda_k^{\alpha/4+\gamma}} \xi_k \Phi_k(x) + \frac{\sigma}{\sqrt{2\theta}} \sum_{k\geq 1} \frac{\left(\sqrt{1 - e^{-2\theta\lambda_k^{\alpha/2}t}} - 1\right)}{\lambda_k^{\alpha/4+\gamma}} \xi_k \Phi_k(x). \tag{5.6}$$

Recall that under our standing assumption $s^* \notin \mathbb{N}_0$, and thus us write $s^* = m + H$ for some unique $m \in \mathbb{N}_0$ and $0 < H < 1$. As one may expect, H will be linked to the Hurst parameter of a fBM. In particular, if $\alpha = 2$ and $\gamma = 0$, then $s^* = 1/2$, $m = 0$ and $H = 1/2$, and as shown in [8, p.16] the first term in (5.6) is a fBM with Hurst index H and the second term is an infinitely smooth process.

In view of (5.6), we have, for fixed $t > 0$ and $m \in \mathbb{N}_0$,

$$\partial_x^m \overline{X}_t(x) = \begin{cases} (-1)^{m/2} \frac{\sigma}{\sqrt{2\theta}} L_{\sin}^H(x) + R_{\sin,m}(x), & \text{if } m \text{ is even}, \\ (-1)^{(m-1)/2} \frac{\sigma}{\sqrt{2\theta}} L_{\cos}^H(x) + R_{\cos,m}(x), & \text{if } m \text{ is odd}, \end{cases} \tag{5.7}$$

where

$$\begin{aligned} L_{\sin}^H(x) &:= \sqrt{2} \sum_{k\geq 1} \lambda_k^{-H/2-1/4} \xi_k \sin(k\pi x), \\ L_{\cos}^H(x) &:= \sqrt{2} \sum_{k\geq 1} \lambda_k^{-H/2-1/4} \xi_k \cos(k\pi x), \end{aligned} \tag{5.8}$$

and $R_{\sin,m}, R_{\cos,m} \in C^\infty(\mathcal{D})$ almost surely. The latter follows by observing that the coefficients in the sum of the second term in (5.6) decay exponentially in k, which together with the polynomial growth of the eigenvalues λ_k, implies $||R||^2_{L^2(\Omega,H_s)} = \mathbb{E}||R||^2_{H_s} < \infty$, thus $R \in H_s$ (almost surely) for all $s \in \mathbb{R}$. The Sobolev embedding theorem yields $R \in C^\infty(\mathcal{D})$. We also note that in (5.7), we used that

$$\partial_x^m \sum_{k\geq 1} \frac{1}{\lambda_k^{\alpha/4+\gamma}} \xi_k \Phi_k(x) = \sum_{k\geq 1} \frac{1}{\lambda_k^{\alpha/4+\gamma}} \xi_k \partial_x^m \Phi_k(x), \tag{5.9}$$

which is true thanks to the uniform convergence of the last series. The latter is due to the following estimates

$$\mathbb{E}\left|\sum_{k\geq 1} \frac{1}{\lambda_k^{\alpha/4+\gamma}} \xi_k \partial_x^m \Phi_k(x)\right|^2 \lesssim \sum_{k\geq 1} k^{2m-\alpha-4\gamma} < \infty.$$

Motivated by [30, Sect. 6.4], next we show that the stochastic processes $L^H_{\sin}$ and $L^H_{\cos}$, as functions of x, are strongly related to a fBM. For $0 < H < 1$, $H \neq 1/2$, let

$$\nu_H := -\frac{2}{\pi}\Gamma(-2H)\cos(\pi H), \tag{5.10}$$

and further put $\nu_H = 1$ for $H = 1/2$. The constant ν_H corresponds to ρ_H in [30]. We emphasis that the fBm in this work, say B^H_x, $x \geq 0$, is scaled as in most of the literature, namely $\mathbb{E}[(B^H_1)^2] = 1$, in contrast to [30], where the fBm is scaled such that $\mathbb{E}[(B^H_1)^2] = \nu_H$. Respectively, some of the results from [30] used below have to be adjusted accordingly.

Lemma 5.4 *Let $0 < H < 1$, and B^H_x, $x \geq 0$, be a fBM with Hurst parameter H. There exists a stochastic process $R^H \in C^\infty(\mathbb{R})$, such that for any $0 < a < b < 1$, the following hold true:*

(1) The probability laws of $\nu_H^{1/2} B^H_\cdot$ and $(L^H_{\sin}(a+\cdot)+R^H(a+\cdot))-(L^H_{\sin}(a)+R^H(a))$ are equivalent on (canonical space) $C([0, b-a])$.
(2) The laws of $\nu_H^{1/2} B^H_\cdot$ and $L^H_{\cos}(a+\cdot) - L^H_{\cos}(a)$ are equivalent on $C([0, b-a])$.

Moreover, if $H = 1/2$, then the above laws in both items 1) and 2) are even equal.

Proof Same as in [30], we define the process

$$\widehat{B}^H_x = \xi_0 x + \sqrt{2}\sum_{k\geq 1}\left(\xi'_k \frac{\cos(2\pi kx)-1}{(2\pi k)^{H+1/2}} + \xi''_k \frac{\sin(2\pi kx)}{(2\pi k)^{H+1/2}}\right),$$

where $\xi_0, \xi'_k, \xi''_k, k \geq 1$ are i.i.d. standard normal random variables, and $x \in \mathbb{R}$. Note that in view of [30, Theorem 27], $\widehat{B}^H$ has stationary increments. Then, (i) is proved by following similar steps as in the proof of [30, Theorem 30]. First, from [30, Remark 28] it follows that the laws of B^H and the increment process $x \mapsto (B^H_{x+a} - B^H_{-x-a})/\sqrt{2} - (B^H_a - B^H_{-a})/\sqrt{2}$ are equivalent on $C([0, b-a])$. Due to self-similarity, the latter can be replaced by the increments of $x \mapsto 2^{H-1/2}(B^H_{x/2} - B^H_{-x/2})$. Now, as a consequence of [30, Theorem 27], the laws of $x \mapsto \nu_H^{1/2} B^H_{x/2}$ and $x \mapsto \widehat{B}^H_{x/2}$ are equivalent on $C([-b+a, b-a])$. Now the claim follows from noting that

$$2^{H-1/2}\left(\widehat{B}^H_{x/2} - \widehat{B}^H_{-x/2}\right) = 2^{H-1/2}\xi_0 x + L^H_{\sin}(x).$$

Consequently, taking $R^H(x) = 2^{H-1/2}\xi_0 x$ we have proved (1).
(2) We proceed similarly, and note that

$$2^{H-1/2}\left(\widehat{B}^H_{x/2} + \widehat{B}^H_{-x/2}\right) = L^H_{\cos}(x) - \frac{1}{\sqrt{2}}\sum_{k=1}^{\infty}\xi_k\lambda_k^{-H/2-1/4} =: L^H_{\cos}(x) - c^H.$$

From here, since clearly $c^H \in L^2(\Omega)$, for any $H > 0$, we conclude that the law of $L^H_{\cos} - c^H$ on $C([0, b-a])$ is equivalent to the law of $(\nu_H^{1/2}B^H_x + \nu_H^{1/2}B^H_{-x})/\sqrt{2}$. In view of [30, Remark 28], the increments of this process, and thus also the increments of $L^H_{\cos}$, have a law equivalent to the law of $\nu_H^{1/2}B^H_{\cdot-a}$ on $C([a,b])$, and the claim follows. The case $H = 1/2$ is known, and follows, for example, by Karhunen-Loève type expansions of Brownian motion. □

Proposition 5.5 *Let $t > 0$, $m \in \mathbb{N}_0$ and $0 < H < 1$ such that $m + H = s^* = 2\gamma + \alpha/2 - 1/2$. Then, there exists a stochastic process $R^{m,H} \in C^\infty(\mathcal{D})$, such that for any $0 < a < b < 1$, the laws of $(-1)^{\lfloor m/2\rfloor}\sigma^{-1}\nu_H^{-1/2}\sqrt{2\theta}\,\overline{X}_t + R^{m,H}$ and $J^m B^H_{\cdot-a}$ are equivalent on $C([a,b])$. If $H = 1/2$, the laws are even equal.*

Proof Applying J^m to (5.7), we note that it suffices to prove that for any $\bar{m} \in \mathbb{N}_0$ there exist $R^{\bar{m},H}_{\sin}$, $R^{\bar{m},H}_{\cos} \in C^\infty(\mathbb{R})$ such that the laws $P^{(\bar{m})}_{\sin}$ of $J^{\bar{m}}L^H_{\sin} + R^{\bar{m},H}_{\sin}$ and $P^{(\bar{m})}_{\cos}$ of $J^{\bar{m}}L^H_{\cos} + R^{\bar{m},H}_{\cos}$ are equivalent to the law $Q^{(\bar{m})}$ of $\nu_H^{1/2}J^{\bar{m}}B^H_{\cdot-a}$ on $C([a,b])$.

We will prove the above by induction in $\bar{m}$. First, for $z \in \mathbb{R}$, let τ_z be the shift operator, given by $\tau_z f := f_{\cdot+z}$, and we view $\tilde{J} := \tau_{-a}J\tau_a$ as a bounded operator $\tilde{J} : C([a,b]) \to C([a,b])$. For $\bar{m} = 0$, we note that by Lemma 5.4, $P^{(0)}_{\sin}$, $P^{(0)}_{\cos}$ and $Q^{(0)}$ on $C([a,b])$ are equivalent, where $R^{0,H}_{\sin}(x) = R^H(x) - (L^H_{\sin}(a) + R^H(a))$, $R^{0,H}_{\cos}(x) \equiv -L^H_{\cos}(a)$. Now assume that the claim is true for $\bar{m} \geq 0$. Then the pushforward measures $\tilde{J}^*P^{(\bar{m})}_{\sin}$, $\tilde{J}^*P^{(\bar{m})}_{\cos}$ and $\tilde{J}^*Q^{(\bar{m})}$ are equivalent measures on $C([a,b])$. As $\tilde{J} = \tau_{-a}J\tau_a$, we see that $\tilde{J}^*Q^{(\bar{m})}$ is the law of $\nu_H^{1/2}J^{\bar{m}+1}B^H_{\cdot-a}$, i.e. $\tilde{J}^*Q^{(\bar{m})} = Q^{(\bar{m}+1)}$. Likewise, $\tilde{J}^*P^{(\bar{m})}_{\sin}$ is the law of $J^{\bar{m}+1}L^H_{\sin} + R^{\bar{m}+1,H}_{\sin}$ with $R^{\bar{m}+1,H}_{\sin} = \tilde{J}R^{\bar{m},H}_{\sin} - \int_0^a J^{\bar{m}}L^H_{\sin}(y)\mathrm{d}y$. For this choice of $R^{\bar{m},H}_{\sin}$ it holds $\tilde{J}^*P^{(\bar{m})}_{\sin} = P^{(\bar{m}+1)}_{\sin}$. The case of $\tilde{J}^*P^{(\bar{m})}_{\cos}$ is treated similarly. If $H = 1/2$, one can trace the above arguments and notice that equivalent laws can be replaced with equal laws. The proof is complete. □

Now, we are in the position to prove the main result of this section. In the sequel, we fix $0 < a < b < 1$ and $t > 0$ and consider the generalized variation $V_{q,M,s,N}(X_t)$ on $[a,b]$, namely on an interval away from the boundary.

Theorem 5.6 *Let $M, q \in \mathbb{N}$, and assume that either $M = m+1$ with $H < 1/2$ or $M \geq m+2$. Suppose that (5.5) holds for some $\eta > 1/2$. Then, for any $\epsilon > 0$,*

$$V_{q,M,s^*,N}(X_t) = \tau_q\left(\frac{\sigma^2\nu_H\mu_{M,s^*}}{2\theta}\right)^{q/2} + o_{\mathbb{P}}(N^{-1/2+\epsilon}). \tag{5.11}$$

If in addition $s^* \in 1/2 + \mathbb{N}_0$, then

$$\sqrt{N}\left(V_{q,M,s^*,N}(X_t) - \tau_q\left(\frac{\sigma^2\nu_H\mu_{M,s^*}}{2\theta}\right)^{q/2}\right) \xrightarrow{d} \mathcal{N}\left(0, \left(\frac{\sigma^2\nu_H\mu_{M,s^*}}{2\theta}\right)^q \sigma^2_{q,M,s^*}\right). \tag{5.12}$$

Proof Set $Z^{m,H} := (-1)^{\lfloor m/2\rfloor}\sigma^{-1}\nu_H^{-1/2}\sqrt{2\theta}\overline{X} + R^{m,H}$, with $R^{m,H}$ as in Proposition 5.5. Since $\eta > 1/2$, Proposition 5.2 and Theorem 3.2 (with $\alpha = 1/2 - \epsilon$, $\Sigma = 0$ or $\alpha = 1/2$, $\Sigma = \sigma^{2q}\nu_H^q\mu^q_{M,s^*}\sigma^2_{q,M,s^*}/(2\theta)^q$ in the notation therein) imply that it is enough to show that (5.11), and (5.12) hold with X_t replaced by $\overline{X}_t$. Consequently, again using Theorem 3.2,[2] since $R^{m,H} \in C^\infty(\mathcal{D})$, the claims are, respectively, equivalent to

$$N^{1/2-\epsilon}\left(V_{q,M,s^*,N}(Z^{m,H}) - \tau_q\mu^{q/2}_{M,s^*}\right) \xrightarrow{\mathbb{P}} 0, \tag{5.13}$$

$$\sqrt{N}\left(V_{q,M,s^*,N}(Z^{m,H}) - \tau_q\mu^{q/2}_{M,s^*}\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2_{q,M,s^*}\mu^q_{M,s^*}\right), \tag{5.14}$$

for a general s^*, and, respectively, for $s^* \in 1/2 + \mathbb{N}_0$. From Theorem 4.4 it follows that

$$\int_{C([a,b])} \mathbb{1}\left(\left|N^{1/2-\epsilon}\left(V_{q,M,s^*,N}(f) - \tau_q\mu^{q/2}_{M,s^*}\right)\right| > \epsilon'\right) \mathrm{d}\mathcal{L}(J^m B^H_{\cdot-a})(f) \to 0, \tag{5.15}$$

for any $\epsilon, \epsilon' > 0$. By Proposition 5.5, the laws of $J^m B^H_{\cdot-a}$ and $Z^{m,H}$ are equivalent on $C([a,b])$, and thus[3]

$$\int_{C([a,b])} \mathbb{1}\left(\left|N^{1/2-\epsilon}\left(V_{q,M,s^*,N}(f) - \tau_q\mu^{q/2}_{M,s^*}\right)\right| > \epsilon'\right) \mathrm{d}\mathcal{L}(Z^{m,H})(f) \to 0 \tag{5.16}$$

for any $\epsilon, \epsilon' > 0$, which is equivalent to (5.13). Finally, if $s^* \in 1/2 + \mathbb{N}_0$, then $Z^{m,H}$ and $J^m B^H_{\cdot-a}$ are equal in law, and (5.14) becomes (4.7). This concludes the proof. □

As a direct consequence, we obtain a procedure to estimate one of the parameters σ, θ, if the other one is known, based on discrete observations on the uniform grid of $[a,b]$.

Theorem 5.7 *In the setting of Theorem 5.6, the following hold true:*

(1) If θ is known, then $\widehat{\sigma}^{q,M}_N := \tau_q^{-1}(2\theta/(\nu_H\mu_{M,s^*}))^{q/2}V_{q,M,s^*,N}(X_t)$ is a consistent estimator for σ^q, and for any $\epsilon > 0$,

$$\widehat{\sigma}^{q,M}_N = \sigma^q + o_{\mathbb{P}}(N^{-1/2+\epsilon}).$$

[2] Note that in the case $M = m+1$, the condition $M > s+\alpha$ from Theorem 3.2 imposes $H < 1/2$ instead of $H < 3/4$ as in Theorem 4.4.

[3] We recall that for two equivalent measures $P \sim Q$ on some measureable space $(M, \mathcal{M})$, it holds that $P(A_N) \to 0$ if and only if $Q(A_N) \to 0$ for any $(A_N)_{N\in\mathbb{N}} \subset \mathcal{M}$, see e.g. [37, Chapter 6].

If $s^* \in 1/2 + \mathbb{N}_0$, *then also*

$$\sqrt{N}\left(\widehat{\sigma}_N^{q,M} - \sigma^q\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^{2q}}{\tau_q^2}\sigma_{q,M,s^*}^2\right), \quad \text{as } N \to \infty.$$

(2) *If* σ *is known, then* $\widehat{\theta}_N^{q,M} := \tau_q^{2/q}\nu_H\mu_{M,s^*}\sigma^2/(2V_{q,M,s^*,N}(X_t)^{2/q})$ *is a consistent estimator for* θ, *and*

$$\widehat{\theta}_N^{q,M} = \theta + o_{\mathbb{P}}(N^{-1/2+\epsilon}),$$

for any $\epsilon > 0$. *If* $s^* \in 1/2 + \mathbb{N}_0$, *then*

$$\sqrt{N}\left(\widehat{\theta}_N^{q,M} - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{4\theta^2}{q^2\tau_q^2}\sigma_{q,M,s^*}^2\right), \quad \text{as } N \to \infty.$$

We conclude this section with several remarks:

1. The only a priori knowledge needed in order to apply our procedure is the spatial regularity s^* of X. In practice, this can be determined by examining the rate of convergence to zero (or divergence to infinity) of the Δ-power variation for some fixed (and arbitrary) $s > 0$, for example, by thinning the sampling mesh and determining the regression slope in a log-log-plot. Such an approach has been studied in a related setting in [14, 20].
2. The choice of the Dirichlet boundary conditions is not essential. By changing the role of $L_{\sin}^H$ and $L_{\cos}^H$, we immediately get an analogous result for Neumann boundary conditions. Similarly, using the representation of B^H in terms of $L_{\text{mix}}^H = \sum_{k\geq 1}(\lambda_k^M)^{-H/2-1/4}\xi_k\Phi_k^M$ (cf. [30, Theorem 6.19]) with $\Phi_k^M(x) = \sqrt{2}\sin((k-1/2)\pi x)$ and $\lambda_k^M = (k-1/2)^2\pi^2$, we get the same result for mixed boundary conditions.
3. Applying the central limit theorem from Theorem 4.4 to SPDEs, essentially depends on establishing a stronger than equivalence in law representation of the solution X_t in terms of a fractional Brownian motion. The CLT for the case $s^* \notin 1/2 + \mathbb{N}_0$ requires a different approach, for example by direct calculations of the covariance operator of $\overline{X}$, and can not be reduced to integrated fractional Brownian motion. Nevertheless, we believe that there is a CLT for $s^* \in 1/2 + \mathbb{N}_0$, and to the best of our knowledge, this remains an open problem.
4. Similar results can be derived if (5.1) is driven by an additive space-only noise (the so-called parabolic Anderson model) instead of space-time noise. The main difference in this case is that the optimal regularity is $s^* = 2\gamma + \alpha - 1/2$ (cf. [12, 13]). In particular, the results from Sect. 4 are applicable to a broad range of different models, as long as an additive dispersion operator is considered.
5. As noticed in [8, 19], under the sampling scheme of our work, but applied to linear SPDEs driven by space-time white noise, only the ratio θ/σ^2 is identifiable, or equivalently one of the two parameters assuming the second one is known. Moreover, to estimate jointly θ and σ, one has to consider spatio-temporal observations,

for example on a rectangular grid [18, 19, 23] or using infill asymptotics for one spatial and one temporal direction separately [8]. We conjecture that when the solution is observed on space-time grid, the average across time instances of the proposed estimators remains consistent and asymptotically normal under different space and/or time infill regime. A strict proof of these results relies on establishing some nontrivial asymptotics of the corresponding covariance structure and it is beyond the scope of this work.

6 Linear SPDEs on unbounded domain

We consider the (linear) counterpart of (5.1) on the whole space, namely the stochastic evolution equation of the form

$$\begin{aligned} &\partial_t X_t(x) = -\theta\,(-\Delta)^{\alpha/2}\,X_t(x) + \sigma\,\dot{W}^\gamma(t,x), \quad t>0,\ x\in\mathbb{R},\\ &X_0(x)=0, \quad x\in\mathbb{R}, \end{aligned} \tag{6.1}$$

where $\alpha,\theta,\sigma>0$ and $(-\Delta)^{\alpha/2}$ is defined via its Fourier transform: $\mathcal{F}[(-\Delta)^{\alpha/2}u](\xi) = |\xi|^\alpha\,\mathcal{F}[u](\xi)$ for $u\in L^2(\mathbb{R})$ with $|\cdot|^\alpha\,\mathcal{F}[u]\in L^2(\mathbb{R})$. $W^\gamma(t,A),\ t\geq 0, A\in\mathcal{B}(\mathbb{R})$, for some $\gamma\in(0,1/4)$, is a centered Gaussian field with covariance structure

$$\mathbb{E}[W^\gamma(t,A)W^\gamma(s,B)] = (t\wedge s)\int_A\int_B K_\gamma(x-y)\,\mathrm{d}x\,\mathrm{d}y,$$

with K_γ being the so-called Riesz kernel of order γ given by

$$K_\gamma(x) = \frac{\Gamma(1/2-2\gamma)}{2\pi^{3/2}\Gamma(2\gamma)}\cdot|x|^{4\gamma-1}, \quad \gamma\in(0,1/4).$$

We remark that traditionally in the literature the Riesz kernel has slightly different parameterization, with γ instead of 4γ above. We choose such form of Riesz kernel simply to match the spatial regularity of the solution with the one from the bounded domain case.

We recall that $G_\alpha(t,x) = \int_{\mathbb{R}} e^{ix\xi-t|\xi|^\alpha}\,\mathrm{d}\xi$ is the fundamental solution of $\partial_t G_\alpha(t,x) = -(-\Delta)^{\alpha/2}\,G_\alpha(t,x)$. Consequently, the mild solution to (6.1) is defined as

$$X_t(x) = \sigma\int_0^t\int_{\mathbb{R}} G_\alpha\left(\theta(t-s),x-z\right)W^\gamma\left(\mathrm{d}s,\mathrm{d}z\right), \tag{6.2}$$

where the above integral is a Wiener integral with respect to the Gaussian noise W^γ. For details, see for instance [26, Sect. 3] and [15, Sect. 2].

In the context of statistical inference, SPDEs similar to (6.1) were recently considered in [21] and [22].

Proposition 6.1 *For $m \in \mathbb{N} \cup \{0\}$, we have that*

$$\sup_{t\in[0,T],x\in\mathbb{R}} \mathbb{E}\left|\partial_x^m X_t(x)\right|^2 < \infty \quad \textit{for every} \quad T > 0,$$

if and only if $1 + 2m < \alpha + 4\gamma$. In particular, $\partial_x^m X_t(x)$ is well-defined for $x \in \mathbb{R}$, $0 \le t \le T$ and $1 + 2m < \alpha + 4\gamma$.

Proof Without loss of generality, we fix $\theta = \sigma = 1$ for simplicity. Note that the Fourier transforms of K_γ and $G_\alpha(t, \cdot)$ are given by

$$\mathcal{F}K_\gamma(\xi) = |\xi|^{-4\gamma} \quad \text{and} \quad \mathcal{F}G_\alpha(t,\cdot)(\xi) = e^{-t|\xi|^\alpha}, \quad \text{for } t > 0 \text{ and } \xi \in \mathbb{R}.$$

Then, for each $0 \le t \le T$ and $x \in \mathbb{R}$, we have

$$\mathbb{E}\left|\partial_x^m X_t(x)\right|^2 = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \partial_x^m G_\alpha(s, x-y) K_\gamma(y-z) \partial_x^m G_\alpha(s, x-z)\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}s.$$

From here, making the substitution $\widetilde{y} = x - y$, $\widetilde{z} = x - z$, and using that $K_\gamma(x) = K_\gamma(-x)$, we continue

$$\begin{aligned}
\mathbb{E}\left|\partial_x^m X_t(x)\right|^2 &= \int_0^t \int_{\mathbb{R}} \partial_x^m G_\alpha(s,y) \times \left(K_\gamma * \partial_x^m G_\alpha(s,\cdot)\right)(y)\, \mathrm{d}y\, \mathrm{d}s \\
&= (2\pi)^{-1} \int_0^t \int_{\mathbb{R}} \mathcal{F}\left(\partial_x^m G_\alpha(s,\cdot)\right)(\xi) \times \overline{\mathcal{F}\left(K_\gamma * \partial_x^m G_\alpha(s,\cdot)\right)(\xi)}\, \mathrm{d}\xi\, \mathrm{d}s \\
&= (2\pi)^{-1} \int_0^t \int_{\mathbb{R}} |\xi|^{2m-4\gamma} e^{-2s|\xi|^\alpha}\, \mathrm{d}\xi\, \mathrm{d}s \\
&= (2\pi)^{-1} \int_0^t s^{(4\gamma-2m-1)/\alpha}\, \mathrm{d}s \int_{\mathbb{R}} e^{-2|\xi|^\alpha} |\xi|^{2m-4\gamma}\, \mathrm{d}\xi < \infty,
\end{aligned}$$

if and only if $(4\gamma - 2m - 1)/\alpha > -1$, which is equivalent to $1 + 2m < \alpha + 4\gamma$. □

Define the following remainder term:

$$R(x) = \sigma \int_t^\infty \int_{\mathbb{R}} \partial_x^m \left(G_\alpha\left(\theta(t-s), z\right) - G_\alpha\left(\theta(t-s), x-z\right)\right) W^\gamma(\mathrm{d}s, \mathrm{d}z). \quad (6.3)$$

Note that the remainder term decays exponentially fast in Fourier space and is therefore smooth in space for each $t > 0$.

The next result is based on [22, Proposition 4.6].

Proposition 6.2 *For $t > 0$, the process $\partial_x^m X_t$ has the same distribution as a perturbed fBM of the form $c_{\alpha,\gamma,m} \frac{\sigma}{\sqrt{\theta}} B^{2\gamma+\alpha/2-1/2-m} + R$, provided that $2\gamma + \alpha/2 - 1/2 - m \in (0, 1)$, where $c_{\alpha,\gamma,m}^2 := (2\pi)^{-1} \int_{\mathbb{R}} (1 - \cos(\xi))\, |\xi|^{2m-4\gamma-\alpha}\, \mathrm{d}\xi$ and $R \in C^\infty(\mathbb{R})$ almost surely.*

Proof For every $x \in \mathbb{R}$ and a fixed $t > 0$, we set $v(x) := \partial_x^m X_t(x) - R(x)$. Then, for $x, y \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}\,|v(x) - v(y)|^2 &= \frac{\sigma^2}{\pi}\int_0^\infty \int_{\mathbb{R}} |\xi|^{2m-4\gamma}\,(1-\cos(\xi(x-y)))\,e^{-2\theta s|\xi|^\alpha}\,\mathrm{d}\xi\,\mathrm{d}s \\ &= \frac{\sigma^2}{2\pi\theta}(x-y)^{\alpha+4\gamma-1-2m}\int_{\mathbb{R}}(1-\cos(\xi))\,|\xi|^{2m-4\gamma-\alpha}\,\mathrm{d}\xi.\end{aligned}$$

We note that, by the assumption $0 < (\alpha + 4\gamma - 1)/2 - m < 1$,

$$\begin{aligned}\int_{\mathbb{R}}(1-\cos(\xi))\,|\xi|^{2m-4\gamma-\alpha}\,\mathrm{d}\xi &\lesssim \int_{|\xi|>1}|\xi|^{2m-4\gamma-\alpha}d\xi \\ &\quad + \int_{|\xi|\le 1}|\xi|^{2m-4\gamma-\alpha+2}d\xi < \infty.\end{aligned}$$

This implies that v is a fractional Brownian motion with Hurst index $\frac{\alpha+4\gamma-1}{2} - m$. The smoothness property $R \in C^\infty(\mathbb{R})$ follows from [22, Proposition 4.6]. This concludes the proof. □

The following result on estimation of drift θ or volatility σ of fractional heat equation (6.1) follows immediately from Theorem 4.4 in conjunction with Proposition 6.2.

Theorem 6.3 *Let $m \in \mathbb{N}_0$ and $0 < H < 1$ such that $m + H = s^* = 2\gamma + \alpha/2 - 1/2$. Let $M, q \in \mathbb{N}$, and assume that either $M = m + 1$ with $H < 1/2$ or $M \ge m + 2$. Then, we have,*

$$\begin{aligned} & w - \lim_{N\to\infty}\sqrt{N}\left(V_{q,M,s^*,N}(X_t) - c^q_{\alpha,\gamma,m}\tau_q\mu^{q/2}_{M,s^*}\left(\frac{\sigma}{\sqrt{\theta}}\right)^q\right) \\ &\quad = \mathcal{N}\left(0,\, c^{2q}_{\alpha,\gamma,m}\sigma^2_{q,M,s^*}\mu^q_{M,s^*}\left(\frac{\sigma}{\sqrt{\theta}}\right)^{2q}\right). \end{aligned} \tag{6.4}$$

Moreover,

(1) If θ is known, then $\widetilde{\sigma}^{q,M}_N := c^{-1}_{\alpha,\gamma,m}\tau_q^{-1/q}\mu^{-1/2}_{M,s^*}\sqrt{\theta}\,V_{q,M,s^*,N}(X_t)^{1/q}$ is an asymptotically normal estimator for σ;
(2) If σ is known, $\widetilde{\theta}^{q,M}_N := c^2_{\alpha,\gamma,m}\tau_q^{2/q}\mu_{M,s^*}\sigma^2 V_{q,M,s^*,N}(X_t)^{-2/q}$ is an asymptotically normal estimator for θ.

We conclude this section with several clarifying remarks on the class of considered SPDEs in this section. The choice of Riesz kernel was primarily prompted by [22] that considers same equations. In particular this allows to have a direct compassion of the results obtained in this paper and those from [21, 22]. A careful reader will also notice that working with Riesz kernel, which is characterized by its Fourier transform $\mathcal{F}K_\gamma(\xi) = |\xi|^{-4\gamma}$, is technically convenient. On the other hand, such correlation structure of the noise limits $\gamma \in (0, 1/4)$, thus limiting the range of regularity of the solution in spatial component (as described above). To overcome this, but also to be on par with SPDEs from Sect. 5, one can replace the Riesz kernel with Bessel kernel with Fourier transform $\mathcal{F}K_\gamma^B(\xi) = (1+|\xi|^2)^{-2\gamma}$, for any $\gamma > 0$. This case indeed can be addressed, and results similar to those from Sect. 5 can be obtained. For the sake of brevity, we shortly sketch the main arguments of the proof. For simplicity, let us also assume that the drift operator $-\theta(-\Delta)^{\alpha/2}$ is substituted by $-\theta(I-\Delta)^{\alpha/2}$ in (6.1). First, we note that for $0 < \gamma < 1/4$, there exists a positive definite kernel $K_\gamma^{R/B}$ such that $\mathcal{F}K_\gamma^{R/B} = \mathcal{F}K_\gamma - \mathcal{F}K_\gamma^B$. Let $m \in \mathbb{R}$ such that $\gamma' := 2\gamma + \alpha/2 - 1/2 - m \in (0,1)$, let $\widetilde{W}^{\gamma'}$ a centered Gaussian field with covariance kernel $K_{\gamma'}^{R/B}$, independent of W^γ. Then, similarly to [22, Proposition 4.6], one can prove that the increments of $(I-\Delta)^{m/2}X - R^{(1)} - R^{(2)}$ are the increments of a fractional Brownian motion, where

$$R^{(1)}(x) = \sigma \int_t^\infty \int_{\mathbb{R}} (I-\Delta)^{m/2}(G_\alpha(\theta(t-s),z) - G_\alpha(\theta(t-s),x-z))W^\gamma(\mathrm{d}z,\mathrm{d}s),$$
$$R^{(2)}(x) = \sigma \int_0^\infty \int_{\mathbb{R}} (G_\alpha(\theta(t-s),z) - G_\alpha(\theta(t-s),x-z))\widetilde{W}^{\gamma''}(\mathrm{d}z,\mathrm{d}s).$$

Then, for $m \in 2\mathbb{N}$, we have that $(I-\Delta)^{m/2}X$ is a linear combination of $\partial_x^{2m'}X$, where $0 \le m' \le m/2$, so $V_{q,M,s,N}(X) = V_{q,M,s,N}(J^m\partial_x^m X) = V_{q,M,s,N}(J^m(I-\Delta)^{m/2}X)$, up to negligible terms. Furthermore, $J^m(I-\Delta)^{m/2}X$ behaves like $J^m B^H$ with $H = \gamma'$, up to a perturbation by $R^{(1)}$ and $R^{(2)}$. Consequently, similar statements concerning consistency and rate of convergence of the Δ-power variation can be made. However, the central limit theorem does not transfer since $R^{(2)}$ is not arbitrarily smooth. Similar to the bounded domain, the asymptotic normality property of the corresponding estimators for θ and σ remains an open question.

The emphasis that the extension of the results from linear to nonlinear equations of the form (1.1) via a splitting argument depends on spatial regularity properties of the solution X to (6.1). In contrast to the case of bounded domains, the covariance operator as given by the Riesz (or Bessel) kernel is not of trace class, so X will not belong to $L^2(\mathbb{R})$ or any higher-order Sobolev space derived from $L^2(\mathbb{R})$. Instead, we believe that suitably chosen weighted Sobolev spaces can help to mitigate this issue. To the best of our knowledge, this has not been investigated systematically in the literature.

7 Numerical example

In this section we illustrate the theoretical results of Sect. 5 via numerical simulations, by considering the stochastic heat equation

$$\mathrm{d}X_t = \theta\Delta X_t \mathrm{d}t + \sigma(-\Delta)^{-\gamma}\mathrm{d}W_t, \tag{7.1}$$

with initial condition $X_0 = 0$ on $\mathcal{D} = [0,1]$ with Dirichlet boundary conditions. We take the true values of the parameters $\theta, \sigma = 1$. As far as the smoothing parameter γ, we consider the following representative cases $\gamma \in \{0.0, 0.375, 0.5, 0.625\}$, which correspond to the regularity level $s^* = 2\gamma + 1/2 \in \{0.5, 1.25, 1.5, 1.75\}$. To numerically simulate a path, we use the Fourier series decomposition of the solution (5.4)

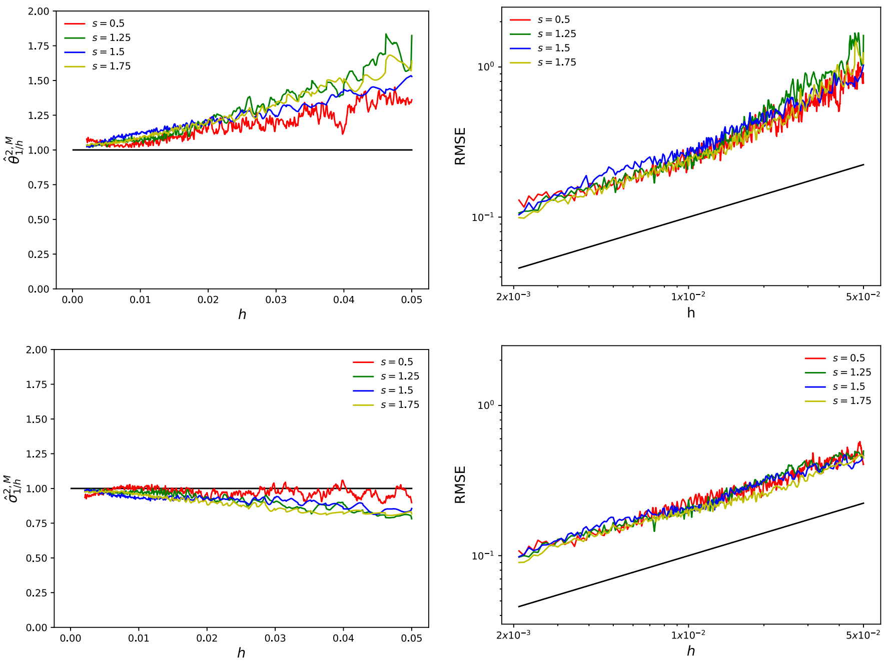

Fig. 1 Estimation of θ (top row) and σ^2 (bottom row). Left panel: the average of 100 Monte Carlo estimates as function of spatial sampling resolution h. The solid black line corresponds to the true value 1.0. Right panel: The RMSE (root mean square error) as function of h. The black line corresponds to the theoretical convergence rate $h^{1/2}$

by taking $N_0 = 1 \times 10^4$ eigenmodes, and each eigenmode is numerically simulated by the Euler implicit scheme with temporal stepsize $\delta t = 1 \times 10^{-8}$. Correspondingly, the solution is computed at $N_0 + 1$ uniformly spaced spatial grid points with step size $h = 1 \times 10^{-4}$.

Next, we assume that the solution X is observed at time $T = 1$ on spatial grid points belonging to the interval $[a, b]$, with $a = 0.2$, $b = 0.8$. We apply Theorem 5.7, with $q = 2$ and $M = \lceil s^* \rceil + 2$, to estimate one of the parameters μ or σ^2 assuming that the second one is known. For each set of the parameters, we perform these evaluations on 100 Monte Carlo sample paths of the solution. The average values of the estimates as function of step size h are displayed in Fig. 1, left panel. Clearly, the estimators converge to the true value (horizontal solid line $\theta = 1$ and $\sigma^2 = 1$), as the mesh size gets smaller. Moreover, as shown in Fig. 1, right panel, the root mean square error of the estimators behaves as $h^{1/2}$, confirming the theoretical rate of converges of the proposed estimators, regardless of the order of regularity s^* of the solution. Similar results were obtained for various sets of the parameters. Finally, while not shown here, we remark that the results from Sect. 4 were also confirmed via numerical simulations.

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Data availability The numerical computations were performed using programming language Python. The source code and the simulated data are available from the corresponding author upon request.

Appendix

For reader's convenience, we recall a useful asymptotic result of Hermite polynomials of a stationary Gaussian sequence.

Theorem A.1 [28, Theorem 7.2.4 Breuer-Major Theorem] *Let* $Y = \{Y_k\}_{k\in\mathbb{Z}}$ *be a centered stationary Gaussian sequence with unit variance, and* $f(x) = \sum_{q=d}^{\infty} a_q H_q(x),\ a_q \in \mathbb{R}$, *where* H_q *is the* q*-th Hermite polynomial. Assume that*

$$\sum_{\ell\in\mathbb{Z}} |\rho(\ell)|^d < \infty, \tag{A.1}$$

where $\rho(\ell) = \mathbb{E}(Y_0 Y_\ell),\ \ell \in \mathbb{Z}$. *Then,*

$$w - \lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(Y_k) = \mathcal{N}\left(0, \sum_{q=d}^{\infty} q! a_q^2 \sum_{\ell\in\mathbb{Z}} \rho(\ell)^q\right).$$

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