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Full Length Article

Sign intermixing for Riesz bases and frames measured in the Kantorovich–Rubinstein norm

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Abstract

We measure a sign interlacing phenomenon for Bessel sequences (u_k) in L^2 spaces in terms of the Kantorovich–Rubinstein mass moving norm $\|u_k\|_{KR}$. Our main observation shows that, quantitatively, the rate at which $\|u_k\|_{KR} \to 0$ heavily depends on Bernstein–Kolmogorov widths of a compact set of Lipschitz functions. In particular, it depends on the dimension of the measure space. We also establish a lower bound for the rate of convergence of the norms $\|u_k\|_X \to 0$ of a basis/frame of L^2 in any larger function space $X \supset L^2$.

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1. Introduction and a summary

Let (Ω, ρ) be a metric space, and m a finite continuous (with no point masses) Borel measure on Ω. It is known [18] that for every *frame* $(u_k)_{k\geq 1}$ in $L^2 = L^2_{\mathbb{R}}(\Omega, m)$, the "$l^2$-masses" of the

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positive and negative values $u_k^{\pm}(x)$ are infinite:

$$\sum_k u_k^+(x)^2 = \sum_k u_k^-(x)^2 = \infty \text{ a.e. on } \Omega,$$

where as usual $u_k^{\pm}(x) = \max(0, \pm u_k(x))$, $x \in \Omega$. Moreover,

$$\forall f \in L^2_{\mathbb{R}}(\Omega),\ f \geq 0,\ f \neq 0 \Rightarrow \sum_k \langle f, u_k^{\pm}\rangle^2_{L^2} = \infty.$$

So, at almost every point $x \in \Omega$, there are many positive and many negative values in the sequence $(u_k(x))_{k\geq 1}$. Below, we show that for a fixed k, positive and negative values are heavily intermixed.

These statements can be considered in line with more general "positivity studies" for representing systems $u = \sum_{k\geq 1} f_k(u)u_k$, $u \in L^2$, with various possible meaning of convergence. In particular, it is known [20] that the unconditional convergence is not compatible with the non-negativity $u_k(x) \geq 0$ a.e. (it also follows from our results quoted above). On the other hand, there exists a Schauder basis $(u_k)_{k\geq 1}$ for L^2 consisting of non-negative functions, [6] by Freeman, D., Powell, A., and Taylor, M. A., Schauder basis for L^2 consisting of non-negative functions., Math. Ann., 381, 1-2, 2021, 181-208 (which solves a long standing problem). The issue was also treated for L^p and for some more general spaces, see [10,18,20], and the references quoted therein.

The point is that this is precisely the unconditional character of a representing system (u_k) (a frame/Riesz basis in our setting) which forces the functions u_k oscillate more and more. In this paper, we show that the measures $u_k^{\pm} dm$ should be closely interlaced, in the sense that the Kantorovich–Rubinstein (KR) mass moving distances $\|u_k\|_{KR} = \|u_k^+ - u_k^-\|_{KR}$ (see below) must be small enough. It is easy to see that if the supports $\mathrm{supp}(u_k^{\pm})$ are distance separated from each other then $\|u_k\|_{KR} \approx \|u_k\|_{L^1(m)}$, whereas in reality, as we will see, the norms $\|u_k\|_{KR}$ are much smaller, and so, the sets $\{x : u_k(x) > 0\}$ and $\{x : u_k(x) < 0\}$ should be increasingly mixed. For example, it follows from our results that for every frame $(u_k)_{k\geq 1}$ in $L^2(0, 1)$, we have

$$\sum \|u_k\|^2_{KR} < \infty, \text{ but } \sum \|u_k\|^2_{L^1(0,1)} = \infty.$$

Historically, one of the first results on the sign intermixing phenomenon is that of O. Kellogg [14], showing that on the unit interval $\Omega = I =: (0, 1)$, the consecutive supports $\mathrm{supp}(u_k^{\pm})$ are interlacing under quite general assumptions on an orthonormal sequence (u_k). Later on, the sign interlacing properties were intensively studied for orthogonal polynomials (starting from P. Chebyshev, and earlier, see any book on orthogonal polynomials). In particular, quite a recent survey of the field [5] counts about 780 pages and hundreds of references; many new quantitative results are also presented.

Our results are most complete for the classical case $\Omega = I^d$ $(d \geq 1)$ in $\mathbb{R}^d$, where $I = (0, 1)$, $m = m_d$ is the Lebesgue measure and ρ is the Euclidean distance on the cube. They also suggest that in general, the magnitudes of $\|u_k\|_{KR}$ are defined by certain (unknown) interrelations between m and ρ, and by a kind of the dimension of Ω (there are many in the metric geometry). This is partially confirmed by the results of a forthcoming paper [17]. In fact, all depends on and is expressed in terms of a compact subset Lip_1 of Lipschitz functions in $L^2(\Omega, m)$.

We now briefly summarize the contents of the resting sections of the paper.

2. Definitions and comments.

3. Statements on the generic behavior of $\|u_k\|_{KR}$. For every Bessel sequence (u_k) in $L^2(I^d)$, we have for $d=1$: $\sum_k \|u_k\|_{KR}^2 < \infty$, and for $d>1$: $\sum_k \|u_k\|_{KR}^{d+\epsilon} < \infty$, $\forall \epsilon > 0$.

These claims are sharp, even in a much more general setting: for every compact triple $(\Omega, \rho, \mathrm{m})$ and for every sequence $(\epsilon_k)_{k\geq 1}$, $\epsilon_k \geq 0$, such that $\sum_k \epsilon_k^2 < \infty$, there exists an orthonormal sequence $(u_k)_{k\geq 1}$ in $\mathrm{L}^2_{\mathbb{R}}(\Omega, \mathrm{m})$ such that $\|u_k\|_{KR} \geq c\epsilon_k$, $k=1,2,\dots$ $(c>0)$ (in particular, always there exists an orthonormal sequence (u_k) with $\sum_k \|u_k\|_{KR}^{2-\epsilon} = \infty$, $\forall \epsilon > 0$).

Also, as it is shown already in [18] (and recalled above), $\sum_k \|u_k\|_{L^1(\mu)}^2 = \infty$ for every frame in $L^2(\Omega, \mu)$.

For the unit cube case, there exists in $L^2(I^d)$ an orthonormal sequence (u_k) such that $\sum_k \|u_k\|_{KR}^d = \infty$.

For a generic compact triple $(\Omega, \rho, \mathrm{m})$, we can only claim $\lim_k \|u_k\|_{KR} = 0$ for every Bessel sequence in $\mathrm{L}^2_{\mathbb{R}}(\Omega, \mathrm{m})$. The property is sharp in the following sense: for every sequence $(\epsilon_k)_{k\geq 1}$, $\epsilon_k > 0$, with $\lim_k \epsilon_k = 0$, *there exists a compact triple* $(\Omega, \rho, \mathrm{m})$ (with usual properties) and an orthonormal sequence $(u_k)_{k\geq 1}$ in $\mathrm{L}^2_{\mathbb{R}}(\Omega, \mathrm{m})$ such that $\|u_k\|_{KR} = c\epsilon_k$, $k=1,2,\dots$ $(\frac{1}{2\sqrt{2}} \leq c \leq \frac{2\sqrt{2}}{\pi})$.

4. Proofs of the statements of Section 3.

5. Further examples and comments. Here we show some 1-dimensional manifolds, where the results of Section 3 still hold. Also, we give several examples to rather technical interpolation Theorem 3.2, as well as a few other comments (a direct comparisons $\|u_k\|_{KR}$ with Bernstein widths $b_k(\mathrm{Lip}_1)$; an explicit expression for $\|u\|_{KR}$).

6. The fastest rates of convergence $\|u_k\|_{KR} \to 0$ for frames/bases on $L^2(I^d)$. It is shown that (1) there exists an orthonormal basis (u_k) in $L^2(I^d)$, $d=1,2,\dots$ (namely, the Haar functions basis), such that $\sum_k \|u_k\|_{KR}^{\alpha+\epsilon} < \infty$, $\forall \epsilon > 0$, where $\alpha = \frac{2d}{d+2}$, but (2) $\sum_k \|u_k\|_{KR}^{\alpha} = \infty$, for every frame (u_k) in $L^2(I^d)$ (in particular, for every Riesz basis).

7. Here we state without proof a partial analog of Section 6 results for the limit rate of convergence $\|u_k\|_X$ in a larger function space $X \supset L^2$.

The main results of the paper are Theorems 3.1, 3.2, 6.1 and 7.1.

2. Definitions and comments

In order to simplify the statements, we *always assume that our sequences* $(u_k)_{k\geq 1}$ *(frames, bases, etc.) lie in the codimension one subspace*

$$L^2_0(\Omega, m) = \{f \in L^2_{\mathbb{R}}(\Omega, m) : \int_\Omega f\,dm = 0\}.$$

However, most of the results below are still true for all *Bessel sequences* $u = (u_k)_{k\geq 1}$ in L^2_0, i.e. the sequences with

$$\sum_k |\langle f, u_k\rangle|^2 \leq B(u)^2 \|f\|_2^2, \ \forall f \in L^2_0,$$

where $B(u) > 0$ stands for the best possible constant in such inequality. Recall also that *a frame* (in L^2_0) is a sequence having

$$b\|f\|_2^2 \leq \sum_k |\langle f, u_k\rangle|^2 \leq B\|f\|_2^2, \quad \forall f \in L^2_0,$$

with some constants $0 < b, B < \infty$, and *a Riesz basis* is (by definition) an isomorphic image of an orthonormal basis.

We always assume that *the space* (Ω, ρ) *is compact* (unless the contrary explicitly follows from the context) and *the measure m is finite and continuous (has no point masses)* .

Below, $\|u\|_{KR}$ *stands for the Kantorovich–Rubinstein* (also called *Wasserstein*) *norm (KR)* of a zero mean signed measure udm ($\int udm = 0$); this norm evaluates the work needed to transport the positive mass u^+dm into the negative one u^-dm. In fact, the KR distance $d(u_k^+dx, u_k^-dx)$ between measures $u_k^{\pm}dx$ (first invented by L.Kantorovich as early as 1942, [11]) is a particular case of a more general setting. Namely, given nonnegative measures μ, ν on Ω of an equal total mass, $\mu(\Omega) = \nu(\Omega)$, the KR-distance $d(\mu, \nu)$ is defined as the optimal "transfer plan" of the mass distribution μ to the mass distribution ν:

$$d(\mu, \nu) = \inf\Big\{\int_{\Omega\times\Omega} \rho(x, y)d\psi(x, y) : \psi \in \Psi(\mu, \nu)\Big\},$$

where the family $\Psi(\mu, \nu)$ consists of all "admissible transfer plans" ψ, i.e. non-negative measures on $\Omega \times \Omega$ satisfying the balance (marginal) conditions $\psi(\Omega\times\sigma) - \psi(\sigma\times\Omega) = (\mu - \nu)(\sigma)$ for every $\sigma \subset \Omega$. The value $\psi(\sigma \times \sigma')$ has the meaning of how many mass is supposed to be transferred from σ to σ'. The KR-norm of a real (signed) measure $\mu = \mu_+ - \mu_-$, $\mu(\Omega) = 0$, is defined as

$$\|\mu\|_{KR} = d(\mu_+, \mu_-).$$

It is shown in Kantorovich–Rubinstein theory [13](also see, for example [12, Ch.VIII, Section 4], or [22]) that the KR-norm of a real (signed) measure μ, $\mu(\Omega) = 0$, is the dual norm of the Lipschitz space

$$\mathrm{Lip} := \mathrm{Lip}(\Omega) = \{f : \Omega \longrightarrow \mathbb{R} : |f(x) - f(y)| \le c\rho(x, y)\}$$

modulo the constants, where the least possible constant c defines the norm $\mathrm{Lip}(f)$. Namely,

$$\|\mu\|_{KR} = d(\mu_+, \mu_-) = \sup\Big\{\int_\Omega f d\mu : \mathrm{Lip}(f) \le 1\Big\}$$

where, in fact, it suffices to test only functions $f \in \mathrm{lip}$, $\mathrm{lip} := \{f \in \mathrm{Lip} : |f(x) - f(y)| = o(\rho(x, y))\, as\, \rho(x, y) \to 0\}$. Of course, one can extend the above definition to an arbitrary real valued measure μ setting $\|\mu\| = \|\mu - \mu(\Omega)\|_{KR} + |\mu(\Omega)|$. This makes it possible to apply our results to $L^2_{\mathbb{R}}$ spaces instead of $L^2_{\mathbb{R},0}$; using the last remark in the case of Bessel sequences, we can use that the sequence $\int_\Omega u_k dm = \langle 1, u_k\rangle$ is in l^2. The KR-norm and its variations (with various cost function $h(x, y)$ instead of the distance $\rho(x, y)$) are largely used in the Monge/Kantorovich transportation problems, in ergodic theory, etc. We refer to [12] for a basic exposition and references, and to [1,2,22] for extensive and very useful surveys of the actual state of the fields.

It is clear from the above definitions that, for measuring the sign intermixing of $u_k dm$ for a Bessel sequence $(u_k) \subset L^2_0$, one can employ certain *size characteristics* of the following compact subset of $L^2(\Omega, m)$,

$$\mathrm{Lip}_1 = \Big\{f : \Omega \longrightarrow \mathbb{R} : |f(x) - f(y)| \le \rho(x, y),\ f(x_0) = 0\Big\},$$

where $x_0 \in \Omega$ stands for a fixed point of Ω; it will be easily seen that the choice of x_0 does not matter. Below, we estimate $\|u_k\|_{KR}$ making use of the known Bernstein width numbers $b_n(\mathrm{Lip}_1)$ ($n \ne k$, in general). In the case when there exists a linear Hilbert space operator T

for which Lip_1 is (or, is included into) the range of the unit ball, $b_n(\mathrm{Lip}_1)$ can be replaced by the singular numbers $s_n(T)$.

Recall that the S. Bernstein n-widths $b_n(A, X)$ of a compact, convex, centrally symmetric subset $A \subset X$ of a Banach space X are defined as follows (see [19]):

$$b_n(A, X) = \sup_{X_{n+1}} \sup\Big\{\lambda : \lambda B(X_{n+1}) \subset A,\ \lambda \geq 0\Big\}$$

where X_{n+1} runs over all linear subspaces in X of $dim X_{n+1} = n + 1$, and $B(X_{n+1})$ stands for the closed unit ball of X_{n+1}. A subspace $X_{n+1}(A)$ where $\sup_{X_{n+1}}$ is attained, is called optimal; it does not need to be unique (in general). In the case of a Hilbert space H (as everywhere below), if A is the image of the unit ball with respect to a linear (compact) operator T, $A = TB(H)$, we have $b_n(A, H) = s_n(T)$, where $s_n(T) \searrow 0$ $(n = 0, 1, \ldots)$ stands for the n-th singular number of T. In this case, optimal subspaces $H_{n+1}(T)$ are simply the linear hulls of $y_0, \ldots, y_n$ from the Schmidt decomposition of T,

$$T = \sum_{k \geq 0} s_k(T) \langle \cdot, x_k \rangle y_k,$$

(x_k) and (y_k) being orthonormal sequences in H.

3. Statements

Recall that (Ω, ρ) stands for a compact metric space, and m is a finite Borel measure on Ω *having no point masses* (for convenience normalized to 1).

Lemma 1 shows what kind of the intermixing of signs we have for free, for every Bessel sequence (u_k). Lemma 2 shows that for no triples (Ω, ρ, m), one can have an intermixing for generic sequences (u_k) better than l^2 smallness of $\|u_k\|_{KR}$. All intermediate cases can occur, following the widths properties of the compact $\mathrm{Lip}_1 \subset L^2(\Omega, m)$, see Theorems 3.1, 3.2 and the comments below.

Lemma 1. *For every Bessel sequence $(u_k)_{k\geq 1}$ in $L^2_{\mathbb{R}}(\Omega, m)$, we have*

$$\lim_k \|u_k\|_{KR} = 0.$$

Lemma 2. *For every compact measure triple (Ω, ρ, m) (with the above conditions) and every sequence $(\epsilon_k)_{k\geq 1}$, $\epsilon_k \geq 0$, such that $\sum_k \epsilon_k^2 < \infty$, there exists an orthonormal sequence $(u_k)_{k\geq 1}$ in $L^2_{\mathbb{R}}(\Omega, m)$ satisfying*

$$\|u_k\|_{KR} \geq c\epsilon_k,\ k = 1, 2, \ldots \quad c > 0.$$

In particular, there exists an orthonormal sequence $(u_k)_{k\geq 1}$ in $L^2_{\mathbb{R}}(\Omega, m)$ such that

$$\sum_k \|u_k\|_{KR}^{2-\epsilon} = \infty,\ \forall \epsilon > 0.$$

Lemma 3. *For every sequence $(\epsilon_k)_{k\geq 1}$, $\epsilon_k > 0$, with $\lim_k \epsilon_k = 0$, there exists a compact measure triple (Ω, ρ, m) (with the above conditions) and an orthonormal sequence $(u_k)_{k\geq 1}$ in $L^2_{\mathbb{R}}(\Omega, m)$ such that*

$$\|u_k\|_{KR} = c\epsilon_k,\ k = 1, 2, \ldots.$$

And $(\frac{1}{2\sqrt{2}} \leq c \leq \frac{2\sqrt{2}}{\pi})$.

Theorems 3.1 and 3.2 describe the behavior of $\|u_k\|_{KR}$ for generic ("worst") Bessel sequences/frames/bases in $L^2_{\mathbb{R}}(I^d)$ in their dependence on the dimension d.

Theorem 3.1. *(1) Given a Bessel sequence $(u_k)_{k\geq 1}$ in $L^2_{\mathbb{R}}(I, dx)$, $I = (0, 1)$, we have*

$$\sum_k \|u_k\|^2_{KR} < \infty.$$

(2) Given a Bessel sequence $(u_k)_{k\geq 1}$ in $L^2_{\mathbb{R}}(I^d, dx)$, $d = 2, 3, \ldots$, we have

$$\sum_k \|u_k\|^{d+\epsilon}_{KR} < \infty \quad \forall \epsilon > 0.$$

(3) For the Sin orthonormal sequence $(u_n)_{n\in 2\mathbb{N}^d}$ in $L^2_{\mathbb{R}}(I^d, dx)$,

$$u_n(x) = 2^{d/2} \sin(\pi n_1 x_1) \sin(\pi n_2 x_2) \ldots \sin(\pi n_d x_d), \quad n = (n_1, \ldots, n_d) \in (2\mathbb{N})^d$$

we have

$$\sum_n \|u_n\|^d_{KR} = \infty.$$

Remark. For a generic Bessel sequence (or, an orthonormal sequence), the l^2-convergence property (1) is a best possible result (see Lemma 2). However, for certain specific sequences, (1) can be much improved . For example, let $u \in L^2_{\mathbb{R},0}(\mathbb{T})$ and

$$u_n(\zeta) = u(\zeta^n) \quad n = 1, 2, \ldots$$

Then, as it easy to see,

$$\|u_n\|_{KR} \leq \frac{1}{n}\|u\|_{KR}$$

(in fact, there is an equality), and so $\sum_n \|u_n\|^{1+\epsilon}_{KR} < \infty$ ($\forall \epsilon > 0$). Such a dilated sequence $(u_n)_n$ is Bessel if and only if the *Bohr transform* of u, $Bu(\zeta) = \sum_n \hat{u}(n)\zeta^{\alpha(n)}$, $\zeta^{\alpha(n)} = \zeta_1^{\alpha_1}\zeta_2^{\alpha_2}\ldots$, is bounded on the multitorus $\zeta = (\zeta_1, \zeta_2, \ldots) \in \mathbb{T}^\infty$; here $\alpha(n) = (\alpha_1, \alpha_2, \ldots)$ and $n = 2^{\alpha_1}3^{\alpha_2}\ldots$ stands for the Euclid prime representation of an integer $n \in \mathbb{N}$. For this claim, see [16], for example.

In fact, Theorem 3.1 is an immediate corollary of the next Theorem 3.2. We extend the property $(\|u_k\|_{KR}) \in l^2$ to any "one dimensional smooth manifold", see Proposition 5.1 for the exact statement. Lemma 2 shows that this condition describe the fastest convergence to zero of the KR-norms for a *generic* Bessel sequence. On spaces (Ω, ρ) of the dimension higher than 1, it need not be true that $(\|u_k\|_{KR}) \in l^2$ for every Bessel (or even an orthonormal) sequence.

In Theorem 3.2, we develop the approach mentioned at the end of Section 2: we compare the compact set Lip_1 with the T-range $T(B(L^2))$ of the unit ball for an appropriate compact operator T. For a direct comparison between $\|u_n\|_{KR}$ and the Bernstein numbers $b_n(\mathrm{Lip}_1)$ see Section 5.

Theorem 3.2. *Let $T : L^2_{\mathbb{R}}(\Omega, m) \to L^2_{\mathbb{R}}(\Omega, m)$ be compact linear operator, $s_k(T)$ its singular numbers, and $\varphi : [0, \infty) \to [0, \infty)$ be a continuous increasing function on $[0, \infty)$ whose inverse φ^{-1} satisfies*

$$\varphi^{-1}(x) = x^{1/2} r(1/x^{-1/2}) \quad \forall x > 0$$

with a function $x \to r(x)$ concave on $(0, \infty)$ (or is equivalent to a concave function r_0: $cr_0 \le r \le Cr_0$).

(1) If $\mathrm{Lip}_1 \subset T(B(L^2_{\mathbb{R}}(\Omega, m)))$ and $\sum_k \varphi(s_k(T)) < \infty$, then, for every Bessel sequence $(u_k) \subset L^2_{\mathbb{R}}(\Omega, m)$,

$$\sum_{k\ge 1} \varphi(a\|u_k\|_{KR}) < \infty \quad \textit{for a suitable } a > 0.$$

(2) If $\mathrm{Lip}_1 \supset T(B(L^2_{\mathbb{R}}(\Omega, m)))$, then there exists an orthonormal sequence $(u_k)_{k\ge 0}$ in $L^2_{\mathbb{R}}(\Omega, m)$, such that

$$\|u_k\|_{KR} \ge s_k(T) \quad k = 0, 1, \dots$$

In particular (in order to compare with (1)), $\sum_k h(\|u_k\|_{KR}) = \infty$ for every h for which $\sum_k h(s_k(T)) = \infty$.

Remark. See Section 5.III below for a version of Theorem 3.2, point (2), employing the Bernstein widths $b_n(\mathrm{Lip}_1)$ instead of $s_n(T)$ (T does not need to exist for the compact set Lip_1).

Corollary 3.1. *Let $\mathrm{Lip}_1 = T(B(L^2_{\mathbb{R}}(\Omega, m)))$ and*

$$p(T) := \inf\{\alpha : \sum_k s_k(T)^\alpha < \infty\}.$$

(1) If $p(T) < 2$, then $\sum_k \|u_k\|^2_{KR} < \infty$, for every Bessel sequence $(u_k) \subset L^2_{\mathbb{R}}(\Omega, m)$. On the other hand, there exists T with $p(T) = 1$ and an orthonormal sequence such that $\sum_k \|u_k\|^{2-\epsilon}_{KR} = \infty$ ($\forall \epsilon > 0$) (see Lemma 2).

(2) If $\sum_k s_k(T)^p < \infty$, $p \ge 2$, then $\sum_k \|u_k\|^p_{KR} < \infty$ for every Bessel sequence $(u_k) \subset L^2_{\mathbb{R}}(\Omega, m)$.

As we will see, Theorem 3.1, in fact, is a consequence of the last Corollary. Some concrete examples to Theorem 3.2 are presented below, in Section 5.

4. Proofs

I. Proof of Lemma 1 Since $(u_k)_{k\ge 1}$ is a Bessel sequence, it tends weakly to zero: $(u_k, f) \to 0$ as $k \to \infty$, for every $f \in L^2_{\mathbb{R}}(\Omega, m)$. On a (pre)compact set $f \in \mathrm{Lip}_1$, the limit is uniform:

$$\lim_k \|u_k\|_{KR} = \lim_k \sup\left\{\int_\Omega u_k f\, d\mu : f \in \mathrm{Lip}_1\right\} = 0. \quad \square$$

II. Proof of Lemma 2. The Borel measure m being continuous satisfies the Menger property: the values mE, $E \subset \Omega$ fill in interval $[0, m(\Omega)]$; if m is normalized, they fill in the interval $[0, 1]$. See [9], Section 41 for the history (with many retrospective references, the oldest one is to K. Menger, 1928), and [4], Prop. A1, p.645 for a complete and short proof. Below, we use that property many times.

Let $E_i \subset \Omega$ be disjoint Borel sets, $E_1 \bigcap E_2 = \emptyset$, $mE_i = 1/2$, and further, $K_i \subset E_i$ be compacts such that $mK_i = 1/3$ $(i = 1, 2)$. Denote $\delta = \mathrm{dist}(K_1, K_2) > 0$, and set

$$f(x) = (1 - \frac{2}{\delta}\mathrm{dist}(x, K_1))^+ - (1 - \frac{2}{\delta}\mathrm{dist}(x, K_2))^+, \;\; x \in \Omega.$$

Then, $f \in \mathrm{Lip}(\Omega, \rho)$, $\mathrm{Lip}(f) \le 2/\delta$ and $f(x) = 1$ for $x \in K_1$, $f(x) = -1$ for $x \in K_2$.

Now, using the Menger property, one can find two sequences (Δ_k^1), (Δ_k^2), $k = 1, 2, \ldots$, of pairwise disjoint sets such that $\Delta_k^i \subset K_i$, $\Delta_k^i \bigcap \Delta_j^i = \emptyset$ ($i = 1, 2$, $k \neq j$), and $m\Delta_k^1 = m\Delta_k^2 = a^2\epsilon_k^2$, where $a > 0$ is chosen in such a way that $a^2 \sum_{k\geq 1} \epsilon_k^2 \leq 1/3$. Setting

$$u_k = c_k(\chi_{\Delta_k^1} - \chi_{\Delta_k^2}),\ k = 1, 2, \ldots$$

with $\|u_k\|_2^2 = 2c_k^2 m\Delta_k^1 = 1$, we obtain an orthonormal sequence $(u_k) \subset L^2(\Omega, m)$ such that

$$\|u_k\|_{KR} \geq \int_\Omega u_k(\frac{\delta}{2} f)dm = \frac{\delta}{2} 2c_k m\Delta_k^1 = \frac{\delta}{\sqrt{2}}\sqrt{m\Delta_k^1} = \frac{\delta a}{\sqrt{2}}\epsilon_k.$$

□

III. Proof of Lemma 3 Let $\Omega = \mathbb{T}^\infty$, the infinite topological product of compact abelian groups $\mathbb{T} \times \mathbb{T} \times \cdots$, endowed with its normalized Haar measure $m_\infty = m \times m \times \cdots$. The product topology on Ω is metrizable by a variety of metrics. We choose $\rho = \rho_\epsilon$, $\epsilon = (\epsilon_k)_{k\geq 1}$ defined by

$$\rho_\epsilon(\zeta, \zeta') = \max_{k\geq 1} \epsilon_k |\zeta_k - \zeta_k'|, \quad \zeta', \zeta = (\zeta_k)_{k\geq 1} \in \mathbb{T}^\infty.$$

Setting

$$u_k(\zeta) = \sqrt{2}Re(\zeta_k), \quad \zeta \in \mathbb{T}^\infty,$$

we define an orthonormal sequence in $L^2(\mathbb{T}^\infty, m_\infty)$ with $|u_k(\zeta) - u_k(\zeta')| \leq \frac{\sqrt{2}}{\epsilon_k}\rho(\zeta, \zeta')$, and so $\mathrm{Lip}(u_k) \leq \sqrt{2}/\epsilon_k$.

Further, we need the following notation: let $f \in \mathrm{Lip}_1(\mathbb{T}^\infty)$, $f(\zeta) = f(\zeta_k, \overline{\zeta})$ where $\zeta = (\zeta_k, \overline{\zeta}) \in \mathbb{T}^\infty = \mathbb{T} \times \mathbb{T}^\infty$, $\overline{\zeta}$ consists of variables different from ζ_k, and

$$\overline{u}_k(\zeta_k) = \sqrt{2}Re(\zeta_k), \quad \zeta_k \in \mathbb{T}$$

(in fact, this is one and the same function $e^{i\theta} \longmapsto \sqrt{2}\cos(\theta)$ for every k). Finally, we set $\overline{f}(\zeta_k) := \int_{\mathbb{T}^\infty} f(\zeta_k, \overline{\zeta})dm_\infty(\overline{\zeta})$ and observe that $\mathrm{Lip}(\overline{f}) \leq \epsilon_k$:

$$|\overline{f}(\zeta_k) - \overline{f}(\zeta_k')| \leq \int_{\mathbb{T}^\infty} |f(\zeta_k, \overline{\zeta}) - f(\zeta_k', \overline{\zeta})|dm_\infty(\overline{\zeta}) \leq$$
$$\leq \int_{\mathbb{T}^\infty} \epsilon_k|\zeta_k - \zeta_k'|dm_\infty(\overline{\zeta}) = \epsilon_k|\zeta_k - \zeta_k'|.$$

Now,

$$\int_{\mathbb{T}^\infty} u_k(\zeta)f(\zeta_k, \overline{\zeta})dm_\infty(\zeta) = \int_{\mathbb{T}} \overline{u}_k(\zeta_k)\int_{\mathbb{T}^\infty} f(\zeta_k, \overline{\zeta})dm_\infty(\overline{\zeta})dm(\zeta_k) =$$
$$= \int_{\mathbb{T}} \overline{u}_k(\zeta_k)\overline{f}(\zeta_k)dm(\zeta_k) \leq \epsilon_k\|\overline{u}_k\|_{KR(\mathbb{T})},$$

and hence $\|u_k\|_{KR(\mathbb{T}^\infty)} \leq \epsilon_k\|\overline{u}_k\|_{KR(\mathbb{T})}$.

Conversely, if $h \in \mathrm{Lip}_1(\mathbb{T})$ and $\underline{h}(\zeta) := h(\zeta_k)$ for $\zeta \in \mathbb{T}^\infty$, then $|\underline{h}(\zeta_k) - \underline{h}(\zeta_k')| \leq \frac{1}{\epsilon_k}\rho(\zeta, \zeta')$, and so

$$\int_{\mathbb{T}} \overline{u}_k h dm(\zeta_k) = \int_{\mathbb{T}^\infty} dm_\infty(\overline{\zeta})\int_{\mathbb{T}} \overline{u}_k(\zeta_k)h(\zeta_k)dm(\zeta_k) = \int_{\mathbb{T}^\infty} u_k(\zeta)\underline{h}(\zeta)dm_\infty(\zeta) \leq$$
$$\leq \frac{1}{\epsilon_k}\|u_k\|_{KR(\mathbb{T}^\infty)},$$

which entails $\|\overline{u}_k\|_{KR(\mathbb{T})} \leq \frac{1}{\epsilon_k}\|u_k\|_{KR(\mathbb{T}^\infty)}$. Finally, $\|u_k\|_{KR(\mathbb{T}^\infty)} = \epsilon_k\|\overline{u}_k\|_{KR(\mathbb{T})}$. Moreover, since $\mathrm{Lip}(\overline{u}_k) \leq \sqrt{2}$,

$$\frac{1}{2\sqrt{2}} = \int_{\mathbb{T}} \overline{u}_k(\overline{u}_k/\sqrt{2})dm(\zeta_k) \leq$$

$$\|\overline{u}_k\|_{KR(\mathbb{T})} \leq \|\overline{u}_k\|_{L^1(\mathbb{T})} = \frac{2\sqrt{2}}{\pi}. \quad \square$$

Remark. For the same space $L^2(\mathbb{T}^\infty, m_\infty)$, but with a *non-compact* (bounded) metric $\rho(\zeta,\zeta') = \sup_{k\geq 1}|\zeta_k - \zeta'_k|$, we have $\|u_k\|_{\mathrm{KR}} \geq 1$ for $u_k(\zeta) = \sin \pi x_k$, $\zeta = (e^{ix_1}, e^{ix_2}, \ldots, e^{ix_k}, \ldots) \in \mathbb{T}^\infty$, so that $(\|u_k\|_{\mathrm{KR}})_{k\geq 1}$ does not tend to zero.

IV. Proof of Theorem 3.1. (1) Since $u_k \in L^2_{\mathbb{R},0}(I, dx)$, $\int_I u_k dx = 0$. Taking a smooth function f with $\mathrm{Lip}(f) \leq 1$ (which are dense in the unit ball of Lip) and

$$v_k(x) = Ju_k(x) := \int_0^x u_k dx,$$

we get $v_k(0) = v_k(1) = 0$, and hence

$$\int_I f u_k dx = (f v_k)_0^1 - \int_I v_k f' dx = -\int_I v_k f' dx.$$

Making sup over all f with $|f'(x)| \leq 1$, we obtain $\|u_k\|_{KR} = \|v_k\|_{L^1}$. But the mapping

$$J : L^2(I) \longrightarrow L^2(I)$$

is a Hilbert–Schmidt operator, and hence $\sum_k \|Ju_k\|^2_{L^2} < \infty$, and so $\sum_k \|v_k\|^2_{L^1} = \sum_k \|u_k\|^2_{KR} < \infty$.

The penultimate inequality is obvious if (u_k) is an orthonormal (or only Riesz) sequence, but it is still true for every Bessel sequence $(u_k)_{k\geq 1}$. Indeed, taking an auxiliary orthonormal basis $(e_j)_{j\geq 1}$ in $L^2_{\mathbb{R}}(I, dx)$, we can write

$$\sum_k \|Ju_k\|^2_{L^2} = \sum_k \sum_j |\langle Ju_k, e_j\rangle|^2 = \sum_j \sum_k |(u_k, J^*e_j)|^2 \leq$$

$$\sum_j \mathrm{const} \cdot \|J^*e_j\|^2 < \infty,$$

since the adjoint J^* is a Hilbert–Schmidt operator. $\square$

(2) This is a d-dimensional version of the previous reasoning. Anew, we use the dual formula for the KR norm,

$$\|u_k\|_{KR} = \sup\{\int_{I^d} f u_k dx : f \in C^\infty, \mathrm{Lip}(f) \leq 1, \int f dx = 0\},$$

the last requirement does not matter since $\mathrm{Lip}(f) = \mathrm{Lip}(f + const)$. Notice that for $f \in C^\infty(I^d)$, $\mathrm{Lip}(f) \leq 1 \Leftrightarrow |\nabla f(x)|_{\mathbb{R}^d} \leq 1$ $(x \in I^d)$, where ∇f stands for the gradient vector $\nabla f = (\frac{\partial f}{\partial x_j})_{1\leq j\leq d}$. Now, define a linear mapping on the set $\mathfrak{P}_0$ of vector valued trigonometric polynomials of the form $\sum_{n\in\mathbb{Z}^d} c_n \nabla e^{2\pi i(n,\cdot)} \in L^2(I^d, \mathbb{C}^d)$ with the zero mean $(c_0 = 0)$ by the formula

$$A(\nabla e^{2\pi i(n,x)}) = \left|n\right|_{\mathbb{R}^d} e^{2\pi i(n,x)}, \quad n \in \mathbb{Z}^d \backslash \{0\}.$$

It is clear that A extends to a unitary operator

$$A : \mathrm{clos}_{L^2(I^d,\mathbb{C}^d)}(\nabla \mathfrak{P}_0) \longrightarrow L^2_0(I^d).$$

Further, let $M : L^2_0(I^d) \longrightarrow L^2_0(I^d)$ be a (bounded) multiplier,

$$M(e^{2\pi i(n,x)}) = \frac{1}{\left|n\right|_{\mathbb{R}^d}} e^{2\pi i(n,x)}, \quad n \in \mathbb{Z}^d \backslash \{0\},$$

and finally, $T(\nabla f) = f$, $f \in C_0^\infty(I^d)$. Then,

$$\int_{I^d} f u_k dx = \int_{I^d} (T(\nabla f)) u_k dx = \int_{I^d} \nabla f \cdot (T^* u_k) dx,$$

T^* being the adjoint between L^2 Hilbert spaces. It follows

$$\left\|u_k\right\|_{KR} \le \sup\left\{\int_{I^d} \nabla f (T^* u_k) dx : \left|\nabla f(x)\right|_{\mathbb{R}^d} \le 1, x \in \mathbb{R}^d\right\} \le \left\|T^* u_k\right\|_{L^1(I^d,\mathbb{C}^d)} \le$$

$$\le \left\|T^* u_k\right\|_{L^2(I^d,\mathbb{C}^d)}.$$

Moreover, $T = MA$, where A is unitary up to numerical multiple (between the corresponding spaces) and M in a Schatten–von Neumann class $\mathfrak{S}_p$ for every p, $p > d$, since M is diagonal and $\sum_{n \in \mathbb{Z}^d \backslash \{0\}} \frac{1}{\left|n\right|^p_{\mathbb{R}^d}} < \infty \Leftrightarrow p > d$. Using the dual definition of the Bessel sequence as $\left\| \sum a_k u_k \right\|^2 \le c(\sum a_k^2)$ for every real finite sequence (a_k), we can write (u_k) as the image $u_k = Be_k$ of an orthonormal sequence (e_k) under a linear bounded map B. This gives

$$\left\|u_k\right\|_{KR} \le \left\|T^* B e_k\right\|_{L^2}.$$

For every $p > d$, this implies $\sum_k \left\|u_k\right\|^p_{KR} \le \sum_k \left\|T^* B e_k\right\|^p_{L^2} < \infty$ since $T^* B \in \mathfrak{S}_p$ and $d \ge 2$ (see Remark below). $\square$

Remark. For the last argument in the proof, we refer for example to [7]. Here is a brief explanation: given a linear bounded operator $S : H \longrightarrow K$ between two Hilbert spaces and an orthonormal sequence (e_k) in H, define a mapping $j : S \longrightarrow (Se_k)$; then, j is bounded as a map $\mathfrak{S}_2 \longmapsto l^2(K)$ and as a map $\mathfrak{S}_\infty \longmapsto c_0(K)$ (compact operators); by operator interpolation, $j : \mathfrak{S}_p \longmapsto l^p(K)$ is also bounded for $2 < p < \infty$. $\square$

For $1 \le p \le 2$, the things go differently: the best summability exponent α, $\sum_k \left\|Se_k\right\|^\alpha < \infty$, which one can generally guarantee for $S \in \mathfrak{S}_p$, is $\alpha = 2$ (look at rank one operators $S = (\cdot, x)y$). This observation explains the jump behavior of the summability exponent for KR-norms from $\alpha = d + \epsilon$ in the dimension $d \ge 2$ to exactly $\alpha = 2$ in the dimension 1 (and not $\alpha = 1 + \epsilon$ as one can expect).

(3) We use anew the duality formula

$$\left\|u_n\right\|_{KR} = \sup\left\{\int_{I^d} f u_n d\mu : \mathrm{Lip}(f) \le 1\right\}.$$

Taking $f = u_n/\mathrm{Lip}(u_n)$ we get $\|u_n\|_{KR} \geq 1/\mathrm{Lip}(u_n)$ where $\mathrm{Lip}(u_n) \leq \max \|\nabla u_n(x)\| \leq 2^{d/2}|n|$, and so

$$\sum_n \|u_n\|_{KR}^d \geq 2^{-d^2/2} \sum_{n\in(2\mathbb{N})^d} |n|_{\mathbb{R}^d}^{-d} = \infty. \quad \square$$

V. Proof of Theorem 3.2. Let $T = \sum_{k\geq 0} s_k(T)\langle\cdot, x_k\rangle y_k$ be the Schmidt decomposition of a compact operator T acting on a Hilbert space H, $s_k(T) \searrow 0$ being the singular numbers. Let further, $A : H \longrightarrow H$ be a bounded operator, and $(e_k)_{k\geq 0}$ an arbitrary (fixed) orthonormal basis. Given a sequence $\alpha = (\alpha_j)_{j\geq 0}$ of real numbers, $\alpha \in l^\infty$, define a bounded operator

$$T_\alpha = \sum_{k\geq 0} \alpha_k \langle\cdot, x_k\rangle y_k,$$

and then a mapping

$$j : \alpha \longmapsto \langle T_\alpha^* A e_k\rangle_{k\geq 0},$$

a H-vector valued sequence in $l^\infty(H)$.

We are using a J. Gustavsson–J. Peetre interpolation theorem [8] for Orlicz spaces. Recall that, in the case of sequence spaces, an *Orlicz space* l^φ, where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ = (0, \infty)$ is increasing and continuous, and meets the so-called Δ_2-condition $\varphi(2x) \leq C\varphi(x)$, $x \in \mathbb{R}_+$, is the vector space of real sequences $c = (c_k)$ satisfying

$$\sum_k \varphi(a|c_k|) < \infty$$

for a suitable $a > 0$. Similarly, a vector valued Orlicz space consists of sequences $c = (c_k)$, $c_k \in H$, having $\sum_k \varphi(a|c_k|) < \infty$ for a suitable $a > 0$. We need the Hilbert space valued spaces only. The Gustavsson–Peetre interpolation theorem (Theorem 9.1 in [8]) implies that if the mappings $j : l^\infty \longrightarrow l^\infty(H)$ and $j : l^2 \longrightarrow l^2(H)$ are bounded, then

$$j : l^\varphi \longrightarrow l^\varphi(H)$$

is bounded whenever the measuring function φ satisfies the conditions given in Theorem 3.2.

(1) Now, in the notation and the assumptions of statement (1), the Bessel sequence (u_k) is of the form $u_k = Ae_k$, where A is a bounded operator and (e_k) an orthonormal sequence. It follows

$$\|u_k\|_{KR} = \sup_{f\in\mathrm{Lip}_1} |\langle Ae_k, f\rangle| \leq \sup_{f\in T(B(L^2))} |\langle Ae_k, f\rangle_{L^2}| = \|T^* Ae_k\|_{L^2}.$$

For every $\alpha \in l^2$, $T_\alpha \in \mathfrak{S}_2$ (Hilbert–Schmidt), and then $T_\alpha^* A \in \mathfrak{S}_2$, and hence $j(\alpha) \in l^2(H)$. By Gustavsson–Peetre, $\alpha \in l^\varphi \Rightarrow j(\alpha) \in l^\varphi(H)$. Applying this with $\alpha = (s_k(T))$, we get $\sum_k \varphi(a\|u_k\|_{KR}) \leq \sum_k \varphi(a\|T^* Ae_k\|) < \infty$ for a suitable $a > 0$. $\square$

(2) In the assumptions of (2), and with the Schmidt decomposition

$$T = \sum_{k\geq 0} s_k(T)\langle\cdot, x_k\rangle y_k, \quad k \geq 0$$

st $u_k = y_k$. Then $\|u_k\|_{KR} = \sup_{f\in\mathrm{Lip}_1} |\langle y_k, f\rangle| \geq \sup_{f\in T(B(L^2))} |\langle y_k, f\rangle| = \|T^* y_k\|_2 = s_k(T)$. $\square$

5. Examples and comments to Sections 3–4

I. Fastest and slowest rates of convergence $\|u_k\|_{KR} \to 0$. Lemma 2 shows that, the KR-norms of a *generic* Bessel sequence do not have to be smaller than required by the condition $\sum_k \|u_k\|_{KR}^2 < \infty$. On the other hand, point (1) of Theorem 3.1 gives an example of (Ω, ρ, dx), where every Bessel sequence meets this property.

Below, in Proposition 5.1, we extend the latter result to measure spaces over (almost) arbitrary 1-dimensional "smooth manifolds" of finite length.

As to the fastest possible convergence $\|u_k\|_{KR} \to 0$ for frames/bases, we treat the question in Section 6 for spaces $L^2(I^d)$ over the cubes.

Proposition 5.1. *Let $\varphi : I \longrightarrow X$ be a continuous injection of $I = [0, 1]$ in a normed space X differentiable a.e. (with respect to Lebesgue measure dx), and the distance on I be defined by*

$$\rho(x, y) = \|\varphi(x) - \varphi(y)\|_X, \quad x, y \in I.$$

Let further, μ be a continuous (without point masses) probability measure on I, satisfying

$$\int_I d\mu(y) \int_y^1 \|\varphi'(x)\|_X dx =: C^2(\mu, \varphi) < \infty.$$

Then, every Bessel sequence $u = (u_k)$ in $L^2(\mu) =: L_0^2(I, \mu)$ fulfills

$$\sum_k \|u_k\|_{KR}^2 \le B^2 C(\mu, \varphi)^2 < \infty,$$

where $B(u) > 0$ comes from the Bessel condition.

Proof. Following the proof of Theorem 3.1 (1) and using that for $f \in C^\infty$,

$$\mathrm{Lip}(f) \le 1 \Leftrightarrow |f(x) - f(y)| \le \|\varphi(x) - \varphi(y)\| \Leftrightarrow |f'(x)| \le \|\varphi'(x)\|_X \quad x \in I,$$

we obtain, for every $h \in L_0^2(\mu)$ and $J_\mu(h)(x) := \int_0^x h d\mu$,

$$\|h\|_{KR} = \sup\left\{\int_I f h d\mu : f \in C^\infty, \mathrm{Lip}(f) \le 1\right\} =$$

$$\sup\left\{\int_I f' J_\mu(h) dx : |f'(x)| \le \|\varphi'(x)\|_X\right\} = \int_I |J_\mu(h)(x)| \cdot \|\varphi'(x)\|_X dx \le$$

$$\le \|J_\mu(h)\|_{L^2(I, v dx)},$$

where $v(x) = \|\varphi'(x)\|_X$. Mapping $Th := J_\mu(h)$, $Th(x) = \int_I k(x, y) h(y) d\mu$ acting as $T : L^2(\mu) \longrightarrow L^2(I, v dx)$ is in the Hilbert–Schmidt class $\mathfrak{S}_2$ if and only if

$$\|T\|_2^2 = \int\int_{I\times I} |k(x, y)|^2 d\mu(y) v(x) dx = \int_0^1 d\mu(y) \int_y^1 v(x) dx =: C^2(\mu, \varphi) < \infty.$$

If $u = (u_k)$ is a Bessel sequence (with $\sum_k |(h, u_k)|^2 \le B(u)^2 \|h\|^2$, $\forall h \in L^2(\mu)$), and the last condition is fulfilled, then $u_k = A e_k$ where (e_k) is orthonormal and $\|A\| \le B(u)$, and hence

$$\sum_k \|u_k\|_{KR}^2 \le \sum_k \|(TA) e_k\|_2^2 \le \|TA\|_2^2 \le \|T\|_2^2 \|A\|^2 \le B^2(u) C^2(\mu, \varphi). \quad \square$$

Remark. In particular, the following formula appeared in the proof:

$$\|h\|_{KR} = \int_I \left|J_\mu(h)(x)\right| \cdot \|\varphi'(x)\|_X dx,$$

see also Remark V below.

II. Examples of interpolation spaces appearing conspicuously in Theorem 3.2. Lemma 3 suggests that all decreasing rates of $\|u_k\|_{KR}$ can really occur, and so all cases of convergence/divergence of $\sum_k \varphi(\|u_k\|_{KR})$ are different and non empty. The following examples make explicit the links between some Orlicz functions φ and the corresponding singular numbers $s_k(T)$.

(**1**) The most well-known interpolation space between l^2 and l^∞ is l^p, $2 < p < \infty$, which is included in Theorem 3.2 with

$$r(t) = t^{1-\frac{2}{p}},$$

it serves for the case of power-like decreasing of $b_n(\mathrm{Lip}_1)$, or $s_n(T)$ (if $\mathrm{Lip}_1 = T(B(L^2))$), and consequently of $\|u_n\|_{KR}$:

$$\log \frac{1}{s_n} \approx \log(n), \quad n \longrightarrow \infty.$$

In particular, point (2) of Theorem 3.1 (where $\Omega = I^d$, $d \geq 2$) can be seen now as a partial case of Theorem 3.2 since, in the hypotheses of 3.1 (2), $\mathrm{Lip}_1 = TB(L^\infty) \subset TB(L^2)$ and $T \in \bigcap_{p>d} \mathfrak{S}_p(L^2 \longrightarrow L^2)$ (which was already observed in the proof of Theorem 3.1).

(**2**) The following spaces l^φ of slowly decreasing sequences (s_n) can appear as s-numbers (or Bernstein n-widths) of Lip_1 for the triples $\Omega = \mathbb{T}^\infty$, $\rho = \rho_\epsilon$, m_∞, described in the proof of Lemma 3, for convenient choices of the sequence $\epsilon = (\epsilon_n)_{n\geq 1}$.

$$\sum_n s_n^{C \log\log \frac{1}{s_n}} < \infty \quad \text{corresponding to } \log \frac{1}{s_n} \approx \frac{\log(n)}{\log\log(n)}$$

the case is included in Theorem 3.2 with

$$r(t) = t \cdot \exp\left\{ -\frac{1}{C} \cdot \frac{\log(t^2)}{\log\log(t^2)}(1 + o(1)) \right\}, \quad t \longrightarrow \infty$$

(follows from the known $b^{-1}(y) = \frac{y}{\log(y)}(1 + o(1))$ for $b(x) = x \cdot \log(x)$), which is eventually concave (since $t \longmapsto r(t) = o(t)$ for $t \longrightarrow \infty$ and lies in the Hardy fields, see [3], L'Appendice du Ch.V);

$$\sum_n s_n^{C(log\frac{1}{s_n})^\alpha} < \infty, \quad \alpha > 1 \text{ corresponding to } \log \frac{1}{s_n} \approx (\log(n))^{1/\alpha};$$

the case is included in Theorem 3.2 with

$$r(t) = t \cdot \exp\left\{ -(\frac{1}{C} \cdot \log(t^2))^{1/\alpha} \right\},$$

which is eventually concave as $t \to \infty$ (by the same argument as above);

$$\sum_n e^{-\frac{C}{s_n^\beta}} < \infty, \quad \beta > 0 \text{ corresponding to } \log \frac{1}{s_n} \approx (c + \frac{1}{\beta} \log\log(n));$$

the case is included in Theorem 3.2 with

$$r(t) = Ct/(\log(t^2))^{1/\beta},$$

which is eventually concave as $t \to \infty$ (by the same argument as above).

III. In terms of the Bernstein n-widths. It is quite easy to see that a part of Theorem 3.2, namely point (2), is still true with a relaxed hypothesis: we replace the assumption that Lip_1 is of the form $\mathrm{Lip}_1 \supset T(B(L^2))$ for a compact T with a hypothesis that the optimal subspaces for Bernstein widths $b_n(\mathrm{Lip}_1)$ are ordered by inclusion (see Section 2 for the definitions): $H_n(\mathrm{Lip}_1) \subset H_{n+1}(\mathrm{Lip}_1)$, $n = 1, 2, \dots$ The latter is always true if Lip_1 is of the form $T(B(L^2))$. Namely, the following property holds.

Proposition 5.2. *Let (Ω, ρ, m) be a compact probability triple for which there exist Bernstein optimal subspaces $H_n(\mathrm{Lip}_1) \subset L^2(\Omega, m)$ such that*

$$H_n(\mathrm{Lip}_1) \subset H_{n+1}(\mathrm{Lip}_1), \quad n = 1, 2, \dots$$

Then there exists an orthonormal sequence $(u_k)_{k\geq 0} \subset \mathrm{Lip}(\Omega) \subset L^2_{\mathbb{R}}(\Omega, m)$, such that

$$\|u_n\|_{KR} \geq b_n(\mathrm{Lip}_1), \quad n = 1, 2, \dots$$

Proof.

Let $e_1 \in H_1$, $\|e_1\|_2 = b_1$, and assume that e_k, $k \leq n$ are chosen in a way that $e_k \in H_n$, $e_k \perp e_j$ $(k \neq j)$ and $\|e_k\|_2 = b_k$. Since $b_{n+1}B(H_{n+1}) \subset \mathrm{Lip}_1$, there exists a vector $e_{n+1} \in H_{n+1} \ominus H_n \subset \mathrm{Lip}(\Omega)$ with $\|e_{n+1}\|_2 = b_{n+1}$ (and hence, $e_{n+1} \in \mathrm{Lip}_1$). For the constructed sequence (e_n), we set

$$u_n = e_n/b_n$$

and obtain an orthonormal sequence $(u_n) \subset \mathrm{Lip}(\Omega)$ such that $\mathrm{Lip}(u_n) \leq 1/b_n$, and hence $\|u_n\|_{KR} \geq \int_\Omega u_n e_n dm = b_n(\mathrm{Lip}_1)$. □

IV. Remark: an "uncertainty inequality" for $\|u\|_{KR}$. The following simple inequality implicitly appeared several times (in the proofs of points II–IV of Section 4, or just above, in the proof of 5.2). It merits to be stated separately: for a function $u \in \mathrm{Lip}(\Omega)$ the following "uncertainty principle" holds

$$\|u\|_{KR}\,\mathrm{Lip}(u) \geq \|u\|_2^2.$$

(Indeed, $\|u\|_{KR} \geq \int_\Omega u(u/\,\mathrm{Lip}(u))dm$ □).

As a consequence, one can observe that for every normalized Bessel sequence (u_k), its Lip norms must be sufficiently large, so that $\sum_k \varphi(\frac{1}{\mathrm{Lip}(u_k)}) < \infty$ for any monotone increasing function $\varphi \geq 0$ for which $\sum_k \varphi(\|u_k\|_{KR}) < \infty$ (compare with the statements of Section 3).

V. Remark: an explicit formula for $\|u\|_{KR}$. The definitions of the KR-norm are rather implicit, and the question on a simpler formula was discussed, for example in [21,22]. In our setting, there are some cases where the norm $\|\cdot\|_{KR}$ can be explicitly expressed in terms of the triple (Ω, ρ, m). In particular, if $\mathrm{Lip}_1 = T(B(L^\infty(\Omega, m)))$ then

$$\|u\|_{KR} = \|T^*u\|_{L^1(\Omega,m)}, \quad \forall u \in L^1(\Omega, m).$$

Indeed, $\|u\|_{KR} = \sup\left\{\int_\Omega uf\,dm : f \in \mathrm{Lip}_1\right\} = \|T^*u\|_{L^1(\Omega,m)}$. □

In particular, such a formula holds for $(\Omega, m) = (I^d, m_d)$, as it was mentioned in the proof of Theorem 3.1. The corresponding operator $T(\sum_{k\neq 0} c_k e^{2\pi i(k,x)}) = \sum_{k\neq 0} |k|_{\mathbb{R}^d}^{-1} c_k e^{2\pi i(k,x)}$ is a multiplier on L_0^p; for $d = 1$, the formula is mentioned in [21]. See also Remark after the proof of Proposition 5.1.

6. The fastest rate of convergence $\|u_k\|_{KR} \to 0$ for frames/bases in $L^2(I^d)$

As before, we measure the rate mentioned in the title with the convergence/divergence exponents. The following theorem shows that the best possible sign intermixing for bases/frames $(u_k) \subset L^2(I^d)$ gives the much smaller convergence exponents α than for generic sequences treated in Sections 3–4. In particular, always $\alpha < 2$, and for $d = 1$ it is simply $\alpha = 2/3+\epsilon < 1$.

Theorem 6.1. *Let $d = 1, 2, \ldots$ and $\alpha = \frac{2d}{d+2}$ ($\alpha < 2$). Then,*

(1) there exists an orthonormal basis (u_k) in $L^2(I^d)$ such that $\sum_k \|u_k\|_{KR}^{\alpha+\epsilon} < \infty$, for all $\epsilon > 0$, but

(2) $\sum_k \|u_k\|_{KR}^{\alpha} = \infty$, for every frame (u_k) in $L^2(I^d)$ (in particular, for every Riesz basis).

Proof. (1) Let (u_n) be the Haar basis in $L_0^2(I^d)$ enumerated with the following notation:

$$h = \chi_{(0,1/2)} - \chi_{(1/2,1)}$$

stands for the *Haar basic wavelet* on $I \subset \mathbb{R}$; taking a subset $\sigma \subset D := \{1, 2, \ldots, d\}$, $\sigma \neq \emptyset$, and a multiinteger $k = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}_+^d$, where $0 \le k_s < 2^j$ for every s and $j \in \mathbb{Z}_+$, define *the Haar functions* $(u_n) := (h_{j,k,\sigma})$ as

$$h_{j,k,\sigma}(x) = 2^{dj/2} \prod_{s\in\sigma} h(2^j x_s - k_s) \prod_{s\in D\setminus\sigma} \chi_{(0,1)}(2^j x_s - k_s)$$

where $x = (x_1, x_2, \ldots, x_d) \in I^d$. Then (see for example, [15, Section 3.9]), (u_n) forms an orthonormal basis in $L_0^2(I^d)$ (j and k run over all mentioned above values, σ runs a finite set of $2^d - 1$ elements). Obviously,

$$\mathrm{supp}(h_{j,k,\sigma}) = Q_{j,k} := \{x \in \mathbb{R}^d : 2^j x - k \in I^d\} = \prod_{s=1}^{d} [k_s 2^{-j}, (k_s + 1)2^{-j}].$$

Lemma 4. *Let $u \in L^\infty(I^d)$, $\mathrm{supp}(u) \subset Q_{j,k}$ and $\int_{I^d} u\,dx = 0$. Then,*

$$\|u\|_{KR} \le \frac{d}{2}\|u\|_\infty 2^{-(d+1)j}.$$

Proof of the Lemma. Since $\int_{I^d} u\,dx = 0$, we can restrict ourselves in the formula

$$\|u\|_{KR} = \sup\left\{\int_{I^d} uf\,dx : \mathrm{Lip}(f) \le 1\right\}$$

to the functions f with $f(l) = 0$, $\mathrm{Lip}(f) \le 1$ where $l = (k_s 2^{-j})_{s=1}^d$, and so $\left|f(x)\right| \le \left|l - x\right|$, $x \in Q_{j,k}$. Changing variables, we have

$$\|u\|_{KR} \le \int_{Q_{j,0}} \|u\|_\infty |x|_{\mathbb{R}^d} dx \le \int_{Q_{j,0}} \|u\|_\infty \sum_{s=1}^d x_s dx =$$

$$= \|u\|_\infty \frac{d}{2} 2^{-2j} 2^{-j(d-1)} = \|u\|_\infty \frac{d}{2} 2^{-j(d+1)}. \quad \square$$

Applying Lemma to $u = h_{j,k,\sigma}$,

$$\|h_{j,k,\sigma}\|_{KR} \le 2^{jd/2} \frac{d}{2} 2^{-j(d+1)}.$$

Summing up (with a $\gamma > \alpha$, $\alpha = \frac{2d}{d+2}$), we get

$$\sum_n \|u_n\|_{KR}^\gamma \le \sum_\sigma \sum_{j\ge 0} \sum_k \|h_{j,k,\sigma}\|_{KR}^\gamma \le \sum_\sigma \sum_{j\ge 0} 2^{jd} \left(2^{jd/2} \frac{d}{2} 2^{-j(d+1)}\right)^\gamma < \infty. \quad \square$$

(2) Recall that the space $L_0^1(I^d)$ endowed with the KR-norm is isometrically embedded into the dual space $(\mathrm{Lip}_0)^*$ (with respect to the standard duality $\langle u, f\rangle = \int_{I^d} uf\,dm$).

The plan of the proof (suggested by E. Gluskin) is the following: consider some metric properties of the embedding $E^* : L_0^2(I^d) \longrightarrow (\mathrm{Lip}_0)^*$ and its predual embedding $E : \mathrm{Lip}_0 \longrightarrow L_0^2(I^d)$ from two different point of view. Namely, assuming that there exists a frame (u_k) in $L_0^2(I^d)$ such that $\sum_k \|u_k\|_{KR}^\alpha < \infty$, we show that

(I) embeddings E, E^* are 2-nuclear operators (see below) and the 2-nuclear approximation numbers $a_N^{(2)}(E^*)$ decrease as $o(1/N^{1/d})$ when $N \to \infty$;

(II) on the other hand, one can see that – at least for $N = 2^{jd}$, $j = 1, 2, \ldots$ – the numbers $a_N^{(2)}(E)$ (which coincide with $a_N^{(2)}(E^*)$) cannot be less than $cN^{-1/d}$.

The above contradiction shows property (2) of Theorem 6.1.

Proof of point (I). A linear operator $T : X \longrightarrow Y$ between Banach spaces X and Y is said *p-nuclear* if $Tx = \sum_k T_k x$, $x \in X$ (norm convergence), $\mathrm{rank}(T_k) \le 1$ and $\sum_k \|T_k\|^p < \infty$;

$$\inf\left\{\left(\sum_k \|T_k\|^p\right)^{1/p} : \text{over all such representations}\right\} =: \|T\|_{N(p)}$$

is called its p-norm. *N-th p-nuclear approximation number of T* ($N = 1, 2, \ldots$) is

$$a_N^{(p)}(T) := \inf\left\{\|T - A\|_{N(p)} : A : X \longrightarrow Y,\ \mathrm{rank}(A) < N\right\}.$$

Assume now that there exists a frame (u_k) in $L_0^2(I^d)$ such that $\sum_k \|u_k\|_{KR}^\alpha < \infty$ where $\alpha = \frac{2d}{d+2}$. Let $Sf = \sum_k \langle f, u_k\rangle u_k$ be the frame operator on $L_0^2(I^d)$; S is an isomorphism $S : L_0^2(I^d) \longrightarrow L_0^2(I^d)$, and $E^*S : L_0^2(I^d) \longrightarrow (\mathrm{Lip}_0)^*$ is a 2-nuclear operator,

$$E^*Sf = \sum_{k\ge 1} \langle f, u_k\rangle E^* u_k,$$

since $\|E^* u_k\| = \|u_k\|_{KR}$ and $\alpha < 2$. Moreover, letting (u_k) in the decreasing order of $\|u_k\|_{KR}$, we get $\|u_k\|_{KR}^{\alpha} = o(1/k)$ (as $k \to \infty$), and hence

$$a_N^{(2)}(E^*S)^2 \le \sum_{k\ge N} \|u_k\|_{KR}^2 \le \|u_N\|_{KR}^{2-\alpha} \sum_{k\ge N} \|u_k\|_{KR}^{\alpha} = o(\frac{1}{N^{2/\alpha-1}}),$$

and $a_N^{(2)}(E^*S) = o(\frac{1}{N^{1/\alpha-1/2}}) = o(\frac{1}{N^{1/d}})$, as $N \to \infty$. Since S is invertible, and $\|UTV\|_{N(p)} \le \|U\| \cdot \|T\|_{N(p)} \cdot \|V\|$ for every T, U, V, we have

$$a_N^{(2)}(E^*) = o(\frac{1}{N^{1/d}}), \quad N \longrightarrow \infty. \quad \square$$

Proof of point (II). We need to show that there exists a constant $c > 0$ such that for every operator $A_N : \mathrm{Lip}_0 \longrightarrow L_0^2(I^d)$, $\mathrm{rank}(A_N) < N = 2^{jd}$ $(j = 1, 2, \ldots)$, one has $\|E - A_N\|_{N(2)} \ge cN^{-1/d}$. To this end, we construct two linear mappings $V = V_N : \mathbb{R}^N \longrightarrow \mathrm{Lip}_0$ and $U = U_N : L_0^2(I^d) \longrightarrow \mathbb{R}^N$ such that

$$UEV = id_{\mathbb{R}^N}, \ \|V : \mathbb{R}^N \longrightarrow \mathrm{Lip}_0\| \le CN^{\frac{1}{2}+\frac{1}{d}}, \ \|U : L_0^2(I^d) \longrightarrow \mathbb{R}^N\| = 1,$$

where $C > 0$ does not depend on N.

Having these mappings at hand, we get $U_{2N}(E - A_N)V_{2N} = id_{\mathbb{R}^{2N}} - B_N$, where $\mathrm{rank}(B_N) < N$ and so,

$$\|U_{2N}(E - A_N)V_{2N}\|_{N(2)} = \|id_{\mathbb{R}^{2N}} - B_N\|_{N(2)} \ge N^{1/2},$$

and on the other hand,

$$\|U_{2N}(E - A_N)V_{2N}\|_{N(2)} \le \|U_{2N}\| \cdot \|E - A_N\|_{N(2)} \|V_{2N}\| \le$$

$$\le C(2N)^{\frac{1}{2}+\frac{1}{d}} \|E - A_N\|_{N(2)},$$

which gives $\|E - A_N\|_{N(2)} \ge cN^{-1/d}$.

Construction of the mappings $V = V_N : \mathbb{R}^N \longrightarrow \mathrm{Lip}_0$ and $U = U_N : L_0^2(I^d) \longrightarrow \mathbb{R}^N$, $N = 2^{jd}$, $j = 1, 2, \ldots$. We use the similar scaling procedure as in the above proof of part (1) of Theorem 6.1: let ψ be a smooth function on $\mathbb{R}^d$ such that $\mathrm{supp}(\psi) \subset Q_0 = I^d$, $\|\psi\|_{L^2(I^d)} = 1$, $\int_{I^d} \psi\, dm = 0$, and, for every $j \in \mathbb{Z}_+$,

$$\psi_k = \psi_{j,k}(x) := 2^{jd/2}\psi(2^j x - k), \quad k \in K_j,$$

where $K_j = \{k = (k_1, \ldots, k_d) \in \mathbb{Z}_+^d : 0 \le k_s < 2^j \ (1 \le s \le d)\}$. Then, ψ_k $(k \in K_j)$ have pairwise disjoint supports and form an orthonormal family in $L_0^2(I^d)$, $\mathrm{card}(K_j) = 2^{jd} := N$. Now, setting

$$Va = \sum_{k\in K_j} a_k \psi_k, \quad a \in \mathbb{R}^N,$$

we obtain

$$\|Va\|_{\mathrm{Lip}} \le c \cdot \sup_{x\in I^d} |\nabla(Va)(x)|_{\mathbb{R}^d} = c \cdot \max_{k\in K_j} \sup_{x\in I^d} |a_k \nabla\psi_k(x)|_{\mathbb{R}^d} \le C2^{jd/2}2^j |a|_{\mathbb{R}^N},$$

where $c > 0$, $C > 0$ depend only on d (and the choice of ψ), which gives the needed $\|V : \mathbb{R}^N \longrightarrow \mathrm{Lip}_0\| \le CN^{\frac{1}{2}+\frac{1}{d}}$.

For $U = U_N : L_0^2(I^d) \longrightarrow \mathbb{R}^N$, we let $Uf = (\langle f, \psi_k\rangle)_{k\in K_j}$, and obviously get $UEV = id_{\mathbb{R}^N}$ and $\|U : L_0^2(I^d) \longrightarrow \mathbb{R}^N\| = 1$. $\square$

7. Limiting the rate of convergence of $\|u_k\|_X$ in a function space $X \supset L^2$

Here, we briefly explain an application of the techniques of Section 6 to a lower estimate for the convergence exponents $\sum_k \|u_k\|_X^\alpha < \infty$ for a Riesz basis (u_k) in L^2 embedded in a larger Banach function space $X \supset L^2$. The following theorem has a similar proof to that of Theorem 6.1, point (2).

Theorem 7.1. *Let X be a normed space of measurable functions on (Ω, μ) in which the space $L^2 = L^2(\Omega, \mu)$ is continuously embedded as a dense subset, $E : L^2(\Omega, \mu) \longrightarrow X$, and let $\gamma, \delta \geq 0$, $\gamma + \delta > 1/2$. Assume that, for every $N \in \mathbb{N}$, the identity map $id_N : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ can be factored through the embedding $E^* : X^* \longrightarrow L^2$ so that*

$$U_N E^* V_N = id_N,\ V_N : \mathbb{R}^N \longrightarrow X^*,\ U_N : L^2 \longrightarrow \mathbb{R}^N,$$

and

$$\|V_N\| = O(N^\gamma),\ \|U_N\| = O(N^\delta) \quad N \longrightarrow \infty.$$

Then, for every Riesz basis (u_k) in L^2, $\displaystyle\sum_k \|u_k\|_X^{\frac{1}{\gamma+\delta}} = \infty.$

The details will be given elsewhere. Theorem 6.1 follows from Theorem 7.1 with $X = (L^1, \|\cdot\|_{KR})$, $\gamma = \frac{1}{2} + \frac{1}{d}$, $\delta = 0$. For $X = L^1$ and $\gamma = 1/2$, $\delta = 0$, we get (only) $\displaystyle\sum_k \|u_k\|_{L^1}^2 = \infty$ (which fact we know already, [18], see the beginning of Section 1).

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