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ORIGINAL PAPER

Hamiltonian Paths and Cycles in Some 4-Uniform Hypergraphs

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Abstract
In 1999, Katona and Kierstead conjectured that if a k-uniform hypergraph $\mathcal{H}$ on n vertices has minimum co-degree $\lfloor \frac{n-k+3}{2} \rfloor$, i.e., each set of $k-1$ vertices is contained in at least $\lfloor \frac{n-k+3}{2} \rfloor$ edges, then it has a Hamiltonian cycle. Rödl, Ruciński and Szemerédi in 2011 proved that the conjecture is true when $k=3$ and n is large. We show that this Katona-Kierstead conjecture holds if $k=4$, n is large, and $V(\mathcal{H})$ has a partition A, B such that $|A| = \lceil n/2 \rceil$, $|\{e \in E(\mathcal{H}) : |e \cap A| = 2\}| < \epsilon n^4$ for a fixed small constant $\epsilon > 0$.

1 Introduction

A classical result of Dirac [3] states that any graph on n vertices with minimum degree at least $n/2$ contains a Hamiltonian cycle, and $K_{\lceil \frac{n}{2} \rceil -1, \lfloor \frac{n}{2} \rfloor +1}$ shows that this is best possible. However, paths and cycles may be defined in several ways for hypergraphs [1, 5, 8, 10, 11].

A hypergraph is called k-uniform if every edge of it contains k vertices. For k-uniform hypergraphs (or k-graphs, for short) with $k \geq 3$, we consider *paths* which are k-graphs with vertices $v_1, v_2, \ldots, v_l$ and edges $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\}$, $i = 1, \ldots, l-k+1$. A *cycle* is defined similarly with the additional edges $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\}$ for $i = l-k+2, \ldots, l$, where for $h \geq l$ we set $v_h = v_{h-l}$. A *Hamiltonian path (cycle)* in a k-graph $\mathcal{H}$ is a path (cycle) which is a sub-hypergraph of $\mathcal{H}$ and contains all vertices of $\mathcal{H}$.

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Given a k-graph $\mathcal{H}$ and $T \in \binom{V(\mathcal{H})}{k-1}$, the *neighbourhood* of T is denoted by $N_{\mathcal{H}}(T) := \{x : T \cup \{x\} \in E(\mathcal{H})\}$. The *collective degree* (or *co-degree*, for short) of T is $d_{k-1}(T) := |N_{\mathcal{H}}(T)|$. The *minimum co-degree* of $\mathcal{H}$ is $\delta_{k-1}(\mathcal{H}) := \min\{d_{k-1}(T) : T \in \binom{V(\mathcal{H})}{k-1}\}$.

Katona and Kierstead [10] proved that if $\mathcal{H}$ is an n-vertex k-graph with $\delta_{k-1}(\mathcal{H}) \geq (1 - \frac{1}{2k})n - k + 4$, then $\mathcal{H}$ contains a Hamiltonian cycle. In the same paper, they make the following conjecture.

Conjecture 1.1 (*Katona and Kierstead* [10]) *Let $\mathcal{H}$ be a k-graph on $n \geq k+1 \geq 4$ vertices. If $\delta_{k-1}(\mathcal{H}) \geq \lfloor \frac{n-k+3}{2} \rfloor$, then $\mathcal{H}$ has a Hamiltonian cycle.*

The bound on $\delta_{k-1}(\mathcal{H})$ is best possible due to a construction of a non-Hamiltonian k-graph on n vertices with $\delta_{k-1}(\mathcal{H}) = \lfloor \frac{n-k+3}{2} \rfloor - 1$. We describe the construction for $k = 4$. Let $\mathcal{H}_0 := \mathcal{H}_0(A, B)$ be a 4-graph with vertex set $V = A \cup B$ with $A \cap B = \emptyset$, $|A| = \lceil n/2 \rceil$ and $|B| = \lfloor n/2 \rfloor$. Its edge set consists of all $\binom{|A|}{3}|B| + |A|\binom{|B|}{3}$ quadruples of vertices having an odd intersection with A. It is easy to see that if $|A|, |B| \geq 2$ then $\delta_3(\mathcal{H}_0) = \lfloor n/2 \rfloor - 2 = \lceil \frac{n-1}{2} \rceil - 2$ and $\mathcal{H}_0$ does not have a Hamiltonian path. In [15], Rödl, Ruciński and Szemerédi prove that Conjecture 1.1 is true when $k = 3$ and n is large.

Theorem 1.2 (*Rödl, Ruciński and Szemerédi* [15]) *Let $\mathcal{H}$ be a 3-graph on n vertices, where n is sufficiently large. If $\delta_2(\mathcal{H}) \geq \lfloor n/2 \rfloor$, then $\mathcal{H}$ has a Hamiltonian cycle. Moreover, for every n there exists an n-vertex 3-graph $\mathcal{H}_n$ such that $\delta_2(\mathcal{H}_n) = \lfloor n/2 \rfloor - 1$ and $\mathcal{H}_n$ does not have a Hamiltonian cycle.*

For a 4-graph $\mathcal{H}$ on n vertices, let A, B be a partition of $V(\mathcal{H})$ and $\mathcal{H}(A, A, B, B) := \{e \in E(\mathcal{H}) : |e \cap A| = 2\}$, and let $b(\mathcal{H}) := \min |\mathcal{H}(A, A, B, B)|$, where the minimum is taken over all partitions $V(\mathcal{H}) = A \cup B$ with $|A| = \lceil n/2 \rceil$ and $|B| = \lfloor n/2 \rfloor$. We know that if $b(\mathcal{H})$ is very small, then $\mathcal{H}$ is very "close" to the $\mathcal{H}_0$, see Claim 2.1 below. We show that Conjecture 1.1 holds for these $\mathcal{H}$ with small $b(\mathcal{H})$.

Theorem 1.3 *There exists $\epsilon_0 > 0$ such that, for sufficiently large n and any 4-graph $\mathcal{H}$ on n vertices with $b(\mathcal{H}) < \epsilon_0 n^4$, the following hold:*

(i) *If $\delta_3(\mathcal{H}) \geq \lceil \frac{n-1}{2} \rceil - 1$, then $\mathcal{H}$ has a Hamiltonian path;*
(ii) *If $\delta_3(\mathcal{H}) \geq \lfloor \frac{n-1}{2} \rfloor$, then $\mathcal{H}$ has a Hamiltonian cycle.*

The bound in (i) is tight because of $\mathcal{H}_0$. The bound in (ii) is tight because of $\mathcal{H}_0'$, where $\mathcal{H}_0'$ is obtained from $\mathcal{H}_0$ by adding a new vertex v and joining it to all $\binom{n}{3}$ triples of vertices. We can see that (i) is a corollary of (ii). Indeed, for n even the thresholds in (i) and (ii) coincide. For n odd, however, they differ by 1. Suppose $\mathcal{H}$ is a 4-graph satisfying the conditions in (i). In order to see the implication in this case, consider a 4-graph $\mathcal{H}'$ obtained from $\mathcal{H}$ by adding a new vertex v and join it to all $\binom{n}{3}$ triples of vertices. Then

$$\delta_3(\mathcal{H}') \geq \delta_3(\mathcal{H}) + 1 \geq \left(\left\lceil \frac{n-1}{2} \right\rceil - 1\right) + 1 \geq \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{(n+1)-1}{2} \right\rfloor$$

and by (ii) $\mathcal{H}'$ has a Hamiltonian cycle. After removing v, $\mathcal{H}$ has a Hamiltonian path. We do not determine the optimal value of the constant ϵ_0 in the theorem. We only checked that $\epsilon_0 = 10^{-20}$ is sufficient.

For convenience, we will consider only the case when $\mathcal{H}$ has an even number of vertices. The odd case can be treated by some easy modifications and it is discussed in Sect. 5.

The rest of the paper is organized as follows. In Sect. 2, we study the typicality of vertices and edges of $\mathcal{H}$ as in [15]. The proofs of (i) and (ii) in Theorem 1.3 will be given in Sects. 3 and 4, respectively. Although (i) is a corollary of (ii), the proof of (i) given here better illustrates the proof approach of both results without involving too much technicality. Hence we also provide the proof of (i) here. In the final section, we offer some concluding remarks.

2 The Typicality of Vertices and Edges of $\mathcal{H}$

Throughout this section, unless there are special instructions, $\mathcal{H}_0$ denotes the 4-graph with $V(\mathcal{H}_0) = A \cup B$, where $A \cap B = \emptyset$ and $|A| = |B|$, and $E(\mathcal{H}_0)$ consisting of all quadruples of $V(\mathcal{H}_0)$ each of which intersects A in precisely one or three vertices. For a 4-graph $\mathcal{H}$ with $V(\mathcal{H}) = V(\mathcal{H}_0)$, we use notation $\mathcal{H}(A,B)$ and $\mathcal{H}_0(A,B)$ to indicate the partition. We will refer to the edges with exactly three vertices in A as the *AAAB* edges, the edges with exactly one vertex in A as the *ABBB* edges, etc. The *AAAB* edges and the *ABBB* edges will be referred to as the *typical edges* of $\mathcal{H}$, and the *AABB* edges will be called *atypical*. (The *AAAA* edges and *BBBB* edges remain *neutral*.)

First we show the following claim which says that if $b(\mathcal{H})$ is small and $\delta_3(\mathcal{H})$ is large, then $\mathcal{H}$ almost contains a copy of $\mathcal{H}_0$.

Claim 2.1 *Suppose $\mathcal{H}$ is a 4-graph with $V(\mathcal{H}) = A \cup B$, such that $A \cap B = \emptyset$ and $|A| = |B| = n$. For any $c, c_1 > 0$, if $|\mathcal{H}(A,A,B,B)| < cn^4$ and $\delta_3(\mathcal{H}) \geq (1 - c_1)n$, then*

$$|E(\mathcal{H}_0(A,B)) \backslash E(\mathcal{H})| \leq \frac{1}{3}(c_1 + 4c)n^4 + O(n^3).$$

Proof For convenience, let ABB and AAB denote the sets of 3-vertex subset of $V(\mathcal{H})$ with exactly one and two vertices from A respectively. Then

$$\sum_{S\in ABB} d_3(S) = 2|AABB| + 3|ABBB| \geq (1-c_1)n\cdot n\cdot \binom{n}{2}$$

and

$$\sum_{S\in AAB} d_3(S) = 2|AABB| + 3|AAAB| \geq (1-c_1)n\cdot n\cdot \binom{n}{2}.$$

Summing the above two equations, we have

$$3|ABBB| + 3|AAAB| \geq 2(1-c_1)n\cdot n\cdot \binom{n}{2} - 4|AABB|.$$

Since the number of edges of $\mathcal{H}_0(A,B)$ is $n\cdot\binom{n}{3}+\binom{n}{3}\cdot n$ and $|AABB|<cn^4$, we have

$$|E(\mathcal{H}_0(A,B))\backslash E(\mathcal{H})| \leq \frac{1}{3}(c_1+4c)n^4 + O(n^3).$$

□

From time to time, we also need to deal with hypergraphs whose vertex partitions are not balanced. Therefore, in the remainder of this section we always assume that $\mathcal{H}$ is a 4-graph on $2n$ vertices and A, B is a partition of $V(\mathcal{H})$ such that

$$\delta_3(\mathcal{H}) \geq n-1, \tag{2.1}$$

$$n - 5\epsilon_0 n \leq |A| \leq n + 5\epsilon_0 n, \tag{2.2}$$

and

$$|\mathcal{H}(A,A,B,B)| \leq \epsilon_0 n^4, \tag{2.3}$$

where $\epsilon_0 > 0$ is sufficiently small and n is sufficiently large.

2.1 Classification of Vertices

We follow the notation and the set up in [15]. The *link* of a vertex $v\in V(\mathcal{H})$ is defined as the set of triples $L_v := \{uwt : uwtv \in E(\mathcal{H})\}$; let $L_v^{V_1V_2V_3} := L_v \cap V_1V_2V_3$ and $l_v^{V_1V_2V_3} := |L_v^{V_1V_2V_3}|$, where $V_1V_2V_3 \in \{AAA, AAB, ABB, BBB\}$. Similarly, the link of a pair $u, v \in V(\mathcal{H})$ is defined as the set of pairs $L_{uv} := \{wt : uvwt \in E(\mathcal{H})\}$; let $L_{uv}^{V_1V_2} := L_{uv} \cap V_1V_2$ and $l_{uv}^{V_1V_2} := |L_{uv}^{V_1V_2}|$, where $V_1V_2 \in \{AA, AB, BB\}$.

In the remainder of this section, vertices a and a_i (respectively, b and b_i) are contained in A (respectively, B). From (2.1), we see that

$$2l_a^{AAB} + 2l_a^{ABB} \geq |B|(|A|-1)(n-1) \quad \text{and} \quad 6l_a^{BBB} + 2l_a^{ABB} \geq |B|(|B|-1)(n-1); \tag{2.4}$$

and

$$2l_b^{AAB} + 2l_b^{ABB} \geq |A|(|B|-1)(n-1) \quad \text{and} \quad 6l_b^{AAA} + 2l_b^{AAB} \geq |A|(|A|-1)(n-1). \tag{2.5}$$

The vertices of $\mathcal{H}$ are classified according to the values of l_v^{ABB} and l_v^{AAB} as follows:

Definition 2.2 For $\epsilon > 0$ and vertex $a \in A$, a is called

- ϵ-typical if $l_a^{ABB} \leq \epsilon|A|\binom{|B|}{2}$;
- ϵ-medium if $l_a^{ABB} > \epsilon|A|\binom{|B|}{2}$ and $l_a^{AAB} > \epsilon\binom{|A|}{2}|B|$;
- an ϵ-anarchist if $l_a^{AAB} \leq \epsilon\binom{|A|}{2}|B|$.

Similarly, for vertex $b \in B$, b is called

- ϵ-typical if $l_b^{AAB} \leq \epsilon\binom{|A|}{2}|B|$;
- ϵ-medium if $l_b^{AAB} > \epsilon\binom{|A|}{2}|B|$ and $l_b^{ABB} > \epsilon|A|\binom{|B|}{2}$;
- an ϵ-anarchist if $l_b^{ABB} \leq \epsilon|A|\binom{|B|}{2}$.

We have the following observations:

Observation (i) For clarity, results and proofs below are presented in the balanced case, when $|A| = |B| = n$, but they remain valid, except for Claim 2.3, in non-balanced case with just slightly worse constants.

Observation (ii) By (2.4) and (2.5), if $a \in A$ is ϵ-typical then

$$l_a^{AAB} \geq \frac{1}{2}n(n-1)^2 - \frac{1}{2}\epsilon n^3 \quad \text{and} \quad l_a^{BBB} \geq \frac{1}{6}n(n-1)^2 - \frac{1}{6}\epsilon n^3; \tag{2.6}$$

and if $b \in B$ is ϵ-typical then

$$l_b^{ABB} \geq \frac{1}{2}n(n-1)^2 - \frac{1}{2}\epsilon n^3 \quad \text{and} \quad l_b^{AAA} \geq \frac{1}{6}n(n-1)^2 - \frac{1}{6}\epsilon n^3. \tag{2.7}$$

Hence each vertex of $\mathcal{H}$ only belongs to one of the above three types when n is sufficiently large.

Observation (iii) Assume (2.2) holds. For sufficiently large n, if $a \in A$ is an ϵ-anarchist, let $A' = A\backslash\{a\}$ and $B' = B \cup \{a\}$. If (2.2) still holds for A', B', then

$l_a^{A'A'B'} = l_a^{AAB} \le \epsilon|B|\binom{|A|}{2} \le \epsilon'|B'|\binom{|A'|}{2}$ for some $\epsilon' > \epsilon$. For any other vertex $v \neq a$, $l_v^{V_1V_2V_3}$ is changed by no more than $\max\{\binom{|A|}{2}, \binom{|B|}{2}, |A||B|\} = O(n^2)$. If v is ϵ-typical with respect to A, B, say $v \in A$, then $l_v^{ABB} \le \epsilon|A|\binom{|B|}{2}$ and $l_v^{A'B'B'} \le l_v^{ABB} + O(n^2) < \epsilon'|A'|\binom{|B'|}{2}$. So, transferring an ϵ-anarchist a in A to B makes a ϵ'-typical with respect to A', B', and other ϵ-typical vertices with respect to A, B are ϵ'-typical with respect to A', B'.

By Observation (iii), we know that an anarchist acts like a typical vertex on the other side. We claim that in the case of a balanced partition (A, B) such that $|\mathcal{H}(A, A, B, B)| = b(\mathcal{H})$, coexistence of an anarchist with an atypical vertex on the other side is impossible.

Claim 2.3 *Suppose $|A| = |B| = n$ and $b(\mathcal{H}) = |\mathcal{H}(A, A, B, B)|$. For every $\epsilon > 0$ and sufficiently large n, if there is an ϵ-anarchist in B then every vertex in A is 3ϵ-typical. Also, if there is an ϵ-anarchist in A then every vertex in B is 3ϵ-typical.*

Proof For $v \in V$, define $I_v = l_v^{AAB} - l_v^{ABB}$. Then, for $a \in A$,

$$I_a = l_a^{AAB} - l_a^{ABB} = |\mathcal{H}(A\backslash\{a\}, A\backslash\{a\}, B \cup \{a\}, B \cup \{a\})| - |\mathcal{H}(A, A, B, B)|,$$

while for $b \in B$,

$$I_b = l_b^{AAB} - l_b^{ABB} = |\mathcal{H}(A, A, B, B)| - |\mathcal{H}(A \cup \{b\}, A \cup \{b\}, B\backslash\{b\}, B\backslash\{b\})|.$$

Thus, for all $a \in A$ and $b \in B$,

$$\begin{aligned} &|\mathcal{H}(A\backslash\{a\} \cup \{b\}, A\backslash\{a\} \cup \{b\}, B\backslash\{b\} \cup \{a\}, B\backslash\{b\} \cup \{a\})| \\ &= |\mathcal{H}(A, A, B, B)| + I_a - I_b + O(n^2). \end{aligned}$$

Here the $O(n^2)$ term comes from the edges $abuv$, where $uv \in N_{\mathcal{H}}(a, b)$. Hence, by the minimality of $b(\mathcal{H})$, we must have

$$I_a \ge I_b - O(n^2).$$

Suppose that there exists $a \in A$ and $b \in B$ such that $l_b^{ABB} \le \frac{\epsilon}{2}n^3$ and $l_a^{ABB} > \frac{3}{2}\epsilon n^3$. Then by (2.5),

$$I_b = l_b^{AAB} - l_b^{ABB} = l_b^{AAB} + l_b^{ABB} - 2l_b^{ABB} \ge \frac{1}{2}n^3 - \epsilon n^3$$

and

$$I_a = l_a^{AAB} - l_a^{ABB} < \frac{1}{2}n^3 - \frac{3}{2}\epsilon n^3 \le I_b - \frac{1}{2}\epsilon n^3,$$

a contradiction.

The proof of the second statement is analogous. □

The next claim justifies the name “typical” and it shows that the number of atypical vertices is small.

Claim 2.4 *Assuming (2.1), (2.2) and (2.3), for all* $\epsilon_0, \epsilon_1 > 0$, *less than* $8(\epsilon_0/\epsilon_1)n$ *vertices in* $\mathcal{H}$ *are* ϵ_1*-atypical. Among them, less than* $5\epsilon_0 n$ *vertices in A and less than* $5\epsilon_0 n$ *vertices in B are* ϵ_1*-anarchists, provided* $\epsilon_1 < 1/5$.

Proof Let x be the number of ϵ_1-atypical vertices in $\mathcal{H}$. Then, since each of these vertices contributes more than $\frac{1}{2}\epsilon_1 n^3$ edges to $|\mathcal{H}(A,A,B,B)|$, and every such edge is counted at most four times, we have

$$\frac{1}{4}x \cdot \frac{1}{2}\epsilon_1 n^3 < \epsilon_0 n^4,$$

which implies that $x < 8(\epsilon_0/\epsilon_1)n$.

Now, let x' be the number of ϵ_1-anarchists in A. By (2.4), every ϵ_1-anarchist $a \in A$ contributes at least $l_a^{ABB} \geq \frac{1}{2}|B|(|A|-1|)(n-1) - l_a^{AAB} \geq \frac{1}{2}(1-\epsilon_1)n^3 - O(n^2)$ edges to $|\mathcal{H}(A,A,B,B)|$, and these edges are counted at most twice. Hence

$$\frac{1}{2}x' \cdot \left(\frac{1}{2}(1-\epsilon_1)n^3 - O(n^2)\right) < \epsilon_0 n^4,$$

which implies $x' < 5\epsilon_0 n$ since $\epsilon_1 < 1/5$.

The proof of the statement $b \in B$ is analogous. □

Now we classify the pair of vertices in $\mathcal{H}$ by the values l_{uv}^{AA}, l_{uv}^{AB} or l_{uv}^{BB} as follows.

Definition 2.5 Fix $\epsilon > 0$. A pair of vertices

- $\{a_1, a_2\}$ is ϵ-typical if $l_{a_1 a_2}^{BB} \leq \epsilon \binom{|B|}{2}$;
- $\{a, b\}$ is ϵ-typical if $l_{ab}^{AB} \leq \epsilon |A||B|$;
- $\{b_1, b_2\}$ is ϵ-typical if $l_{b_1 b_2}^{AA} \leq \epsilon \binom{|A|}{2}$;
- $\{u, v\} \subseteq V(\mathcal{H})$ is (ϵ_1, ϵ_2)-typical if both u and v are ϵ_1-typical and the pair $\{u, v\}$ is ϵ_2-typical.

Observation. From (2.1),

$$l_{a_1 a_2}^{AB} + 2l_{a_1 a_2}^{BB} \geq |B|(n-1),$$
$$l_{ab}^{AB} + 2l_{ab}^{AA} \geq (|A|-1)(n-1) \text{ and } l_{ab}^{AB} + 2l_{ab}^{BB} \geq (|B|-1)(n-1),$$
$$l_{b_1 b_2}^{AB} + 2l_{b_1 b_2}^{AA} \geq |A|(n-1).$$

Hence, if $\{a_1, a_2\}$, $\{a, b\}$ and $\{b_1, b_2\}$ are ϵ-typical, then by definition, we have

$$l_{a_1 a_2}^{AB} \geq n(n-1) - \epsilon n^2, \tag{2.8}$$

$$l_{ab}^{AA} + l_{ab}^{BB} \geq (n-1)^2 - \epsilon n^2, \tag{2.9}$$

$$l_{b_1 b_2}^{AB} \geq n(n-1) - \epsilon n^2. \tag{2.10}$$

Next we show that each typical vertex is contained in a small number of atypical pairs.

Claim 2.6 *Assuming* (2.1) *and* (2.2), *for all* $\epsilon_1, \epsilon_2 > 0$, *every* ϵ_1*-typical vertex in* A *belongs to at most* $(\epsilon_1/\epsilon_2)n$ ϵ_2*-atypical pairs in* AA *and at most* $(\epsilon_1/\epsilon_2)n$ ϵ_2*-atypical pairs in* AB. *Moreover, every* ϵ_1*-typical vertex in* B *belongs to at most* $(\epsilon_1/\epsilon_2)n$ ϵ_2*-atypical pairs in* BB *and at most* $(\epsilon_1/\epsilon_2)n$ ϵ_2*-atypical pairs in* AB.

Proof Let $a \in A$ be ϵ_1-typical. If a belongs to more than $(\epsilon_1/\epsilon_2)n$ ϵ_2-atypical pairs in AA, then

$$l_a^{ABB} > \frac{\epsilon_2}{2} n^2 \times \frac{\epsilon_1}{\epsilon_2} n = \frac{\epsilon_1}{2} n^3,$$

contradicting the ϵ_1-typicality of a. Similarly, if a belongs to more than $(\epsilon_1/\epsilon_2)n$ ϵ_2-atypical pairs in AB, then,

$$l_a^{ABB} > \frac{1}{2}\epsilon_2 n^2 \times \frac{\epsilon_1}{\epsilon_2} n = \frac{\epsilon_1}{2} n^3,$$

a contradiction.

The proof of the statement for ϵ_1-typical vertex in B is analogous. □

The triples of vertices in $\mathcal{H}$ are classified as follows.

Definition 2.7 Fix $\epsilon > 0$. A triple of vertices

- $\{a_1, a_2, a_3\}$ is ϵ-typical if $d_B(a_1, a_2, a_3) \geq (1-\epsilon)|B|$;
- $\{a_1, a_2, b\}$ is ϵ-typical if $d_B(a_1, a_2, b) \leq \epsilon |B|$;
- $\{a, b_1, b_2\}$ is ϵ-typical if $d_A(a, b_1, b_2) \leq \epsilon |A|$;
- $\{b_1, b_2, b_3\}$ is ϵ-typical if $d_A(b_1, b_2, b_3) \geq (1-\epsilon)|A|$;
- $\{u, v, w\} \subseteq V(\mathcal{H})$ is $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical if each of u, v and w is ϵ_1-typical, each of pairs $\{u, v\}$, $\{v, w\}$ and $\{u, w\}$ is ϵ_2-typical, and the triple $\{u, v, w\}$ is ϵ_3-typical.

Observation. From (2.1),

$$d_A(a_1, a_2, b) + d_B(a_1, a_2, b) \geq n - 1,$$
$$d_A(a, b_1, b_2) + d_B(a, b_1, b_2) \geq n - 1.$$

Hence, if $\{a_1, a_2, a_3\}, \{a, b_1, b_2\}, \{a_1, a_2, b\}$ and $\{b_1, b_2, b_3\}$ are ϵ-typical, then by definition, we have

$$d_B(a_1, a_2, a_3) \geq n - 1 - \epsilon n, \tag{2.11}$$

$$d_A(a_1,a_2,b) \geq n-1-\epsilon n, \tag{2.12}$$

$$d_B(a,b_1,b_2) \geq n-1-\epsilon n, \tag{2.13}$$

$$d_A(b_1,b_2,b_3) \geq n-1-\epsilon n. \tag{2.14}$$

The following two claims show that any typical vertex or typical pair is contained in a small number of atypical triples.

Claim 2.8 *Assuming (2.1) and (2.2), for all $\epsilon_1, \epsilon_3 > 0$, every ϵ_1-typical vertex in A belongs to at most $(\epsilon_1/\epsilon_3)n^2$ ϵ_3-atypical triples in each type of AAB,ABB and AAA. Moreover, every ϵ_1-typical vertex in B belongs to at most $(\epsilon_1/\epsilon_3)n^2$ ϵ_3-atypical triples in each type of AAB,ABB and BBB.*

Proof Let $a \in A$ be ϵ_1-typical. If a belongs to more than $(\epsilon_1/\epsilon_3)n^2$ ϵ_3-atypical triples in AAB or more than $(\epsilon_1/\epsilon_3)n^2$ ϵ_3-atypical triples in ABB, then

$$l_a^{ABB} > \frac{1}{2}(\epsilon_1/\epsilon_3)n^2 \times \epsilon_3 n = \frac{\epsilon_1}{2}n^3 \quad \text{or} \quad l_a^{ABB} > (\epsilon_1/\epsilon_3)n^2 \times \epsilon_3 n > \frac{\epsilon_1}{2}n^3,$$

contradicting the ϵ_1-typicality of a. Let x be the number of ϵ_3-atypical triples in AAA. Then by (2.4),

$$\left(\frac{1}{2}-\frac{\epsilon_1}{2}\right)n^3 \leq l_a^{AAB} = \sum_{a_1,a_2\neq a} d_B(a_1,a_2,a) \leq x(1-\epsilon_3)n + \left(\frac{1}{2}n^2 - x\right)n = \frac{1}{2}n^3 - x\epsilon_3 n.$$

So $x \leq (\epsilon_1/2\epsilon_3)n^2 \leq (\epsilon_1/\epsilon_3)n^2$.

The proof of the statement for ϵ_1-typical vertex in B is analogous. □

Claim 2.9 *Assuming (2.1) and (2.2), for all $\epsilon_2, \epsilon_3 > 0$, every ϵ_2-typical pair $\{a_1,a_2\}$, or $\{a,b\}$, or $\{b_1,b_2\}$ belongs to at most $(\epsilon_2/\epsilon_3)n$ ϵ_3-atypical triples in each of the four types AAA, AAB, ABB, andBBB.*

Proof Let $\{a_1,a_2\}$ be an ϵ_2-typical pair. If $\{a_1,a_2\}$ belongs to more than $(\epsilon_2/\epsilon_3)n$ ϵ_3-atypical triples in AAB, then

$$l_{a_1a_2}^{BB} > \frac{1}{2}(\epsilon_2/\epsilon_3)n \times \epsilon_3 n = \frac{\epsilon_2}{2}n^2,$$

contradicting the ϵ_2-typicality of $\{a_1,a_2\}$. Let x be the number of ϵ_3-atypical triples in AAA. Since $l_{a_1a_2}^{AB} + 2l_{a_1a_2}^{BB} \geq |B|(n-1)$, we have

$$(1-\epsilon_2)n^2 \leq l_{a_1a_2}^{AB} = \sum_{a\neq a_1,a_2} d_B(a_1,a_2,a) \leq x(1-\epsilon_3)n + (n-x)n = n^2 - x\epsilon_3 n;$$

we have $x \leq (\epsilon_2/\epsilon_3)n$.

The proof of the statement for $\{a,b\}$ and $\{b_1,b_2\}$ are analogous. □

If $|\mathcal{H}(A,A,B,B)|$ is small, then by Claim 2.4, $\mathcal{H}$ does not contain too many atypical vertices. Next, we claim that the number of atypical triples in $\mathcal{H}$ is also small.

Corollary 2.10 *Assuming (2.1), (2.2) and (2.3), for all $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3 > 0$ and for every $\epsilon_4 \geq 16(\epsilon_0/\epsilon_1) + 4(\epsilon_1/\epsilon_2) + (\epsilon_1/\epsilon_3)$, every set of at least $\epsilon_4 n^3$ triples in $\binom{V(\mathcal{H})}{3}$ contains at least one $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical triple. In particular, there are less than $\epsilon_4 n^3$ triples in $\binom{V(\mathcal{H})}{3}$ which are not $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical.*

Proof It suffices to count all triples $\{u, v, w\} \subseteq V(\mathcal{H})$, such that at least one of them is ϵ_1-atypical, or all of $\{u, v, w\}$ are ϵ_1-typical and one of the pairs from $\{u, v, w\}$ is not ϵ_2-typical, or all vertices are ϵ_1-typical and all pairs from $\{u, v, w\}$ are ϵ_2-typical, but $\{u, v, w\}$ is not ϵ_3-typical.

By Claim 2.4, the number of triples, of which at least one vertex is ϵ_1-atypical, is at most $8(\epsilon_0/\epsilon_1)n \times \binom{2n-1}{2}$. By Claim 2.6, the number of triples, of which all three vertices are ϵ_1-typical but at least one pair is ϵ_2-atypical, is at most $2n \times (2(\epsilon_1/\epsilon_2)n) \times (2n-2) \times \frac{1}{2!}$. By Claim 2.8, the number of triples, of which all vertices are ϵ_1-typical and all pairs are ϵ_2-typical but $\{u, v, w\}$ is not ϵ_3-typical, is at most $2n \times (3(\epsilon_1/\epsilon_3)n^2) \times \frac{1}{3!}$.

Hence, the number of all these atypical triples are at most $\epsilon_4 n^3$. □

2.2 Short Paths Between Typical Triples

In this section, we prove that if $|H(A, A, B, B)|$ is small and $\delta_3(\mathcal{H})$ is large then certain typical triples can be connected by a path of length at most 12. Recall that the 4-graph $\mathcal{H}_0 = \mathcal{H}_0(A, B)$ consists of all *AAAB* and *ABBB* quadruples. (Here, we allow non-balanced partitions (A, B); however, they must satisfy (2.2).) A sextuple of vertices $(v_1, v_2, v_3, w_1, w_2, w_3)$ is called *$\mathcal{H}_0$-connected* if both $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ belong to *AAV* or both $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ belong to *BBV*. We can call it an $\mathcal{H}_0$-connected sextuple formed by the triples $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$. Given a set of vertices K, a path P is *K-avoiding* if $V(P) \cap K = \emptyset$. A subset of vertices $T \subseteq V(\mathcal{H})$ is said to be *$\mathcal{H}_0$-complete* if $E(\mathcal{H}[T]) \supseteq E(\mathcal{H}_0[T])$. We show that for an $\mathcal{H}_0$-connected sextuple formed by two $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical triples, there is a path in $\mathcal{H}_0$ connecting these two triples.

Claim 2.11 *Let $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3$ be sufficiently small and assume that (2.1), (2.2) and (2.3) hold. Let $(v_1, v_2, v_3, w_1, w_2, w_3)$ be an $\mathcal{H}_0$-connected sextuple in $\mathcal{H}$, where $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ are two $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical triples. For every set of vertices $K \subseteq V(\mathcal{H}) \backslash \{v_1, v_2, v_3, w_1, w_2, w_3\}$ with $|K| \leq \frac{2}{3}n$, there exists a subset $T \subseteq V(\mathcal{H}) \backslash (K \cup \{v_1, v_2, v_3, w_1, w_2, w_3\})$ such that $|T \cap A|, |T \cap B| \geq 5$, and $T \cup \{v_1, v_2, v_3\}$ and $T \cup \{w_1, w_2, w_3\}$ are $\mathcal{H}_0$-complete. In particular, there exists a K-avoiding path P in $\mathcal{H}$ with at most 12 vertices such that the end triples of P are $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ and all edges in P are typical.*

Proof We select a set T at random, by choosing each vertex of $V(\mathcal{H})\setminus(K\cup\{v_1,v_2,v_3,w_1,w_2,w_3\})$ independently with probability $p=60/n$. We will show that it satisfies all required properties with positive probability.

Let E_v and E_w be the events that the subsets $T\cup\{v_1,v_2,v_3\}$ and $T\cup\{w_1,w_2,w_3\}$ are not $\mathcal{H}_0$-complete, and let $E=E_v\cup E_w$. We claim that

$$\mathbb{P}(E_v)\le P_0+3P_1+3P_2+P_3,$$

where P_0 is the probability that T is not $\mathcal{H}_0$-complete, P_1 is the probability that there exist $x,y,z\in T$ such that $v_ixyz\in E(\mathcal{H}_0)\setminus E(\mathcal{H}[T\cup\{v_i\}])$, P_2 is the probability that there exist $x,y\in T$ such that $v_iv_jxy\in E(\mathcal{H}_0)\setminus E(\mathcal{H}[T\cup\{v_i,v_j\}])$, and P_3 is the probability that there exist $x\in T$ such that $v_1v_2v_3x\in E(\mathcal{H}_0)\setminus E(\mathcal{H}[T\cup\{v_1,v_2,v_3\}])$.

By Claim 2.1 with $c=\epsilon_0$ and $c_1=1/n$, we know $|E(\mathcal{H}_0)\setminus E(\mathcal{H})|\le 2\epsilon_0 n^4$. (Although the partition of $V(\mathcal{H})$ might not be balanced, the result of Claim 2.1 still holds with a larger constant.) Thus, $P_0\le 2\epsilon_0 n^4p^4$. By (2.6) and (2.7), for any $1\le i\le 3$, the number of edges of $\mathcal{H}_0$ containing v_i that are not edges of $\mathcal{H}$ is at most $\epsilon_1 n^3$, since v_i is ϵ_1-typical. Thus, $P_1\le\epsilon_1 n^3p^3$. By (2.8), (2.9) and (2.10), for any $1\le i\ne j\le 3$, the number of edges in $\mathcal{H}_0$ containing the pair $\{v_i,v_j\}$ that are not edges of $\mathcal{H}$ is at most $\epsilon_2 n^2$, since $\{v_i,v_j\}$ is ϵ_2-typical. Thus, $P_2\le\epsilon_2 n^2p^2$. By (2.11), (2.12), (2.13) and (2.14), the number of edges in $\mathcal{H}_0$ containing the triple $\{v_1,v_2,v_3\}$ that are not edges of $\mathcal{H}$ is at most $\epsilon_3 n$, since $\{v_1,v_2,v_3\}$ is ϵ_3-typical. Thus, $P_3\le\epsilon_3 np$.

Hence,

$$P(E_v)\le 2\epsilon_0 n^4p^4+3\cdot\epsilon_1 n^3p^3+3\cdot\epsilon_2 n^2p^2+\epsilon_3 np<\frac{1}{4}$$

for $\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3$ sufficiently small. Similarly, $P(E_w)<\frac{1}{4}$.

Finally, recalling that $|A\setminus(K\cup\{v_1,v_2,v_3,w_1,w_2,w_3\})|\ge\frac{1}{3}n-6>\frac{1}{4}n+4$, we have

$$P(|T\cap A|\le 4)\le\left(1+np+\binom{n}{2}p^2+\binom{n}{3}p^3+\binom{n}{4}p^4\right)(1-p)^{\frac{n}{4}}<\frac{1}{4}.$$

Similarly, $P(|T\cap B|\le 4)<1/4$. Hence, the required set T does exist.

Consider the case when an H_0-connected sextuple is formed by two $(\epsilon_1,\epsilon_2,\epsilon_3)$-typical triples $\{a_1,a_2,a_3\}$ and $\{a_4,a_5,a_6\}$. By the above argument, the required set T exists. Suppose $\{b_1,a,a',a'',b_2\}\subseteq T$. Then by the properties of T, $P=a_1a_2a_3b_1aa'a''b_2a_4a_5a_6$ is a K-avoiding path with 11 vertices in $\mathcal{H}$ and all edges of P are $AAAB$ edges. For other cases, it can be checked that the two $(\epsilon_1,\epsilon_2,\epsilon_3)$-typical triples in any H_0-connected sextuple can be connected by a K-avoiding path with at most 12 vertices in which every edge is typical. Moreover, if both triples are in AAV (or BBV), all edges in this K-avoiding path connecting these two triples are $AAAB$ (or $ABBB$) edges. □

3 Hamiltonian Paths

In this section, we prove the following

Theorem 3.1 *There exists* $\epsilon_0 > 0$ *such that, for sufficiently large* n *and any* 4*-graph* $\mathcal{H}$ *on* $2n$ *vertices with* $b(\mathcal{H}) < \epsilon_0 n^4$ *the following holds. If* $\delta_3(\mathcal{H}) \geq n-1$, *then* $\mathcal{H}$ *has a Hamiltonian path.*

For typical vertices, we want to use paths similar to these in $\mathcal{H}_0$ to connect them. So we need to deal with atypical vertices, which are medium vertices or anarchists. By Claim 2.4, the number of such vertices is small. In our proof, we find a path to absorb all medium vertices. By Claim 2.3, the anarchists can only exist on one side. We may transfer all anarchists to the other side, so that all vertices will be typical in a new partition.

First, we introduce a structure called *bridge*, which helps us construct a path containing all medium vertices.

Definition 3.2 Given $\epsilon_1, \epsilon_2, \epsilon_3 > 0$, an $(\epsilon_1, \epsilon_2, \epsilon_3)$-bridge is a path of at most 800 vertices whose end triples are $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical with one in AAA and the other in BBB.

For convenience, for some small ϵ we set

$$\epsilon_0 = \epsilon^4, \quad \epsilon_1 = \epsilon^3, \quad \epsilon_2 = \epsilon^2, \quad \epsilon_3 = \epsilon, \quad \epsilon_4 = 40\epsilon, \quad \epsilon_5 = 120\epsilon.$$

The proof of Theorem 3.1 can be described in four steps: Build a bridge M (cf. Lemma 3.3); arrest all medium vertices by a path Q containing M (cf. Lemma 3.5); transfer all anarchists not belonging to Q to the other side of the partition; complete the Hamiltonian path P (cf. Lemma 3.8).

3.1 Building a Bridge

Lemma 3.3 *For sufficiently small* $\epsilon > 0$, $\mathcal{H}$ *contains an* $(\epsilon_1, \epsilon_2, \epsilon_3)$*-bridge* M *with at most* 25 *vertices.*

Proof Fix two (ϵ_1, ϵ_2)-typical pairs $\{a_1, a_2\}$ and $\{b_1, b_2\}$. Suppose $a_1a_2b_1b_2 \in E(\mathcal{H})$. Since $\delta_3(\mathcal{H}) \geq n-1$ and $\{a_1, a_2\}, \{b_1, b_2\}$ are (ϵ_1, ϵ_2)-typical, it follows from Claim 2.9 that there exists $x \in N(a_1, a_2, b_1) \setminus \{b_2\}$ and $y \in N(a_1, b_1, b_2) \setminus \{a_2, x\}$, such that $\{x, a_1, a_2\}$ and $\{b_1, b_2, y\}$ are $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical. Hence, $xa_2a_1b_1b_2y$ is a path in $\mathcal{H}$. Now we show that the path $P = xa_2a_1b_1b_2y$ can be extended to an $(\epsilon_1, \epsilon_2, \epsilon_3)$-bridge by Claim 2.11. By Corollary 2.10, there exist $(\epsilon_1, \epsilon_2, \epsilon_3)$-triples $\{a_1', a_2', a_3'\}$ and $\{b_1', b_2', b_3'\}$ disjoint from $V(P)$. Since $\{a_1', a_2', a_3'\}$ and $\{x, a_1, a_2\}$ are AAV triples, there exists a $\{b_1, b_2, y, b_1', b_2', b_3'\}$-avoiding path $P_1 = a_1'a_2'a_3' \cdots xa_1a_2$ with at most 12 vertices by Claim 2.11. Similarly, there exists a $V(P_1)$-avoiding path $P_2 = b_1b_2y \cdots b_1'b_2'b_3'$ with at most 12 vertices. Hence, we extend P to $P_1 \cup P \cup P_2$ such that $P_1 \cup P \cup P_2$ is an $(\epsilon_1, \epsilon_2, \epsilon_3)$-bridge with at most 24 vertices and the end triples are $\{a_1', a_2', a_3'\}$ and $\{b_1', b_2', b_3'\}$.

So assume $a_1a_2b_1b_2 \notin E(\mathcal{H})$. Let $X = N(a_1, a_2, b_1)$ and $Y = N(a_1, b_1, b_2)$. Since

$a_1a_2b_1b_2 \notin E(\mathcal{H})$, we have $X \cup Y \subseteq V(\mathcal{H})\backslash\{a_1, a_2, b_1, b_2\}$ and $|X \cup Y| \leq 2n-4$. Since $\delta_3(\mathcal{H}) \geq n-1$, $|X| \geq n-1$ and $|Y| \geq n-1$. So $|X \cap Y| = |X| + |Y| - |X \cup Y| \geq 2$ implies that there exists a vertex $z \in X \cap Y$. Similarly, by Claim 2.11, we can find $x \in N(a_1, a_2, z)$ and $y \in N(b_1, b_2, z)$, such that $xa_2a_1zb_1b_2y$ can be extended to the desired bridge. Hence, in any case, we can always find an $(\epsilon_1, \epsilon_2, \epsilon_3)$-bridge in $\mathcal{H}$ with no moth than 25 vertices. □

The construction in the proof of Lemma 3.3 can also be used for 3-graphs, which would shorten Section 8 in [15].

3.2 Taking Care of Atypical Vertices

First, we need a simple claim from [15].

Claim 3.4 (*Rödl, Ruciński and Szemerédi* [15]) *Given $a > 0$ and $k \geq 2$, every k-graph $\mathcal{F}$ with m vertices and with at least $a\binom{m}{k}$ edges contains a path on at least am/k vertices.*

Lemma 3.5 *Let $z_1, \ldots, z_{t_1}$ be the ϵ_5-medium vertices and $K \subseteq V(\mathcal{H})$ with $|K| < \epsilon^3 n$. There exist pairwise disjoint K-avoiding paths $Q_1, \ldots, Q_{t_1}$ such that for every integer i such that $1 \leq i \leq t_1$, all edges in Q_i are typical, and Q_i contains z_i with $|V(Q_i)| = 7$ and both end triples $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical. In particular, assume that M is an $(\epsilon_1, \epsilon_2, \epsilon_3)$-bridge. Then there exists a path Q of length at most $\epsilon^3 n$, which contains M and all ϵ_5-medium vertices of $\mathcal{H}$, and whose end triples, one in AAA and one in BBB, are $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical. Moreover, all edges in $Q - M$ are typical.*

Proof By Claim 2.4, we know that $t_1 \leq 8(\epsilon_0/\epsilon_5)n$. We do an induction on the number of such paths. Suppose that we have already found paths Q_j for $j = 1, \ldots, i-1$, such that Q_j satisfies the properties in Lemma 3.5. Set $z = z_i$.

We may assume $z \in A$ as the proof is analogous for $z \in B$. Let G_z^{AAB} be the set of $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical triples in L_z^{AAB}. By Corollary 2.10, $l_z^{AAB} - |G_z^{AAB}| \leq \epsilon_4 n^3$. Then $|G_z^{AAB}| \geq l_z^{AAB} - \epsilon_4 n^3 \geq \frac{\epsilon_5}{2} n^3 - \epsilon_4 n^3$, since z is ϵ_5-medium. Further, let $F_z^{AAB} = G_z^{AAB}[(V(\mathcal{H})\backslash K)\backslash U_i]$, where $U_i = \bigcup_{j=1}^{i-1} V(Q_j)$. Note that $|F_z^{AAB}| \geq |G_z^{AAB}| - (|U_i| + |K|)n^2$ and $(|U_i| + |K|)n^2 \leq (7(i-1) + \frac{1}{2}\epsilon^3 n)n^2 \leq 7t_1 n^2 + \epsilon^3 n^3 < 2\epsilon^3 n^3$ for sufficiently large n. Thus, by the above estimates and because z is ϵ_5-medium, we have

$$|F_z^{AAB}| \geq |G_z^{AAB}| - 2\epsilon^3 n^3 \geq \left(\frac{\epsilon_5}{2} n^3 - \epsilon_4 n^3\right) - 2\epsilon^3 n^3 \ > 10\epsilon n^3;$$

so by Claim 3.4, F_z^{AAB} contains a path of length six, i.e., $a_1a_2b_1a_3a_4b_2$. Then $Q_i = a_1a_2b_1za_3a_4b_2$, disjoint from $Q_1, \ldots, Q_{i-1}$, gives the desired path, since Q_i is K-avoiding, the end triple of Q_i are $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical AAB triples and all edges in Q_i are $AAAB$ edges.

Now we have a given bridge M. Let $K = V(M)$ as $|V(M)| \leq 800 \leq \epsilon^3 n$. Set $Q_{z_i} = Q_i$ for $i = 1, \ldots, t_1$. By Claim 2.11, for all ϵ_5-medium vertices $w \in A\backslash V(M)$, since the end triples of Q_w are in AAB and $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical, we connect all paths Q_w into a $V(M)$-avoiding path, denoted by Q_{top}. Similarly, for all ϵ_5-medium vertices $w \in B \setminus V(M)$, we connect all paths Q_w into a $V(M \cup Q_{top})$-avoiding path, denoted by Q_{zig}, since the end triples of Q_w are in BBA and $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical. Now we use the given bridge M to connect Q_{top} and Q_{zig}. We connect the end triple of M in AAA with one AAB end triple of Q_{top}, and connect the end triple of M in BBB with one BBA end triple of Q_{zig}, also by Claim 2.11. Then we obtain a path P, which contains M and all ϵ_5-medium vertices of $\mathcal{H}$, whose end triples are $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical, and one of its triples is in AAB and the other is in BBA.

By Claim 2.11, $|V(P)| \leq 8(\epsilon_0/\epsilon_5)n \cdot (7 + (12 - 6)) + |V(M)| + 2 \cdot (12 - 6)$. By Corollary 2.10, there exists an $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical triple $\{a_0, a_1, a_2\}$ in AAA and an $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical triple $\{b_0, b_1, b_2\}$ in BBB, such that $\{a_0, a_1, a_2, b_0, b_1, b_2\} \cap V(P) = \emptyset$. We apply Claim 2.11 to connect $\{a_0, a_1, a_2\}$ with the AAB end triple of P, and connect $\{b_0, b_1, b_2\}$ with the BBA end triple of P. This gives a path $Q = a_2a_1a_0 \cdots b_0b_1b_2$, containing M and all ϵ_5-medium vertices of $\mathcal{H}$, such that $|V(Q)| \leq |V(P)| + 2 \cdot (12 - 3) \leq 8(\epsilon_0/\epsilon_5)n \cdot (7 + 6) + |V(M)| + 2 \cdot 6 + 2 \cdot 9 \leq \epsilon^3 n$. □

3.3 Completing the Hamiltonian Path

To complete the proof of Theorem 3.1, we need the following lemma in [12].

Lemma 3.6 (*Reiher, Rödl, Ruciński, Schacht and Szemerédi* [12]) *Every 3-graph with n vertices and minimum vertex degree at least $(\frac{5}{9} + o(1))\binom{n}{2}$ has a Hamiltonian cycle.*

Lemma 3.5 gives a path Q containing all ϵ_5-medium vertices. By Claim 2.3, we know that if there exists an ϵ_5-anarchist in one side of the vertex partition, then all vertices in the other side are $3\epsilon_5$-typical. Moreover, the number of ϵ_5-anarchists is less than $5\epsilon_0 n$. So we transfer all such vertices to the other side of the vertex partition. Then all vertices in $V(\mathcal{H}) \setminus V(Q)$ are $4\epsilon_5$-typical with respect to the new partition. We use Lemma 3.6 to derive the following.

Lemma 3.7 *Assume* (2.1), (2.2) *and* (2.3) *hold. Let X be a set of $4\epsilon_5$-typical vertices with $m = |X| \geq cn$, where c is a constant. Suppose $\{x_0, x_1, x_2\}$ and $\{x'_0, x'_1, x'_2\}$ are two disjoint $(4\epsilon_5, \epsilon_5^{3/4}, \epsilon_5^{1/2})$-typical triples disjoint from X. For sufficiently large n and sufficiently small ϵ, there exists a sequence of vertices $x_0x_1x_2x_3 \cdots x_{m+2}x_{m+3}(= x'_2)x_{m+4}(= x'_1)x_{m+5}(= x'_0)$, such that all $\{x_i, x_{i+1}, x_{i+2}\}$ are $(4\epsilon_5, \epsilon_5^{3/4}, \epsilon_5^{1/2})$-typical for $0 \leq i \leq m + 3$ and $X = \{x_3, x_4, \ldots, x_{m+2}\}$.*

Proof Construct a 3-graph G_X with vertex set $V(G_X) = X \cup \{x_2\}$ and edge set

$$E(G_X) = \{x_2uv : \{u,v\} \subseteq X \text{ such that } \{x_2,u,v\}, \{x_1,x_2,u\}, \{x'_1,x'_2,v\} \text{and} \{u,v,x'_2\} \text{ are } \epsilon_5^{1/2}\text{-typical}\} \cup \{uvw : \{u,v,w\} \subseteq X \text{ is } (4\epsilon_5, \epsilon_5^{3/4}, \epsilon_5^{1/2})\text{-typical}\}.$$

We show that $\delta_1(G_X) > \frac{2}{3}\binom{m}{2}$. Since the pairs $\{x_1,x_2\}$ and $\{x'_1,x'_2\}$ are $\epsilon_5^{3/4}$-typical, each of them belongs to at most $2\epsilon_5^{1/4}n$ $\epsilon_5^{1/2}$-atypical triples in X (by Claim 2.9); so there are at least $\binom{m-4\epsilon_5^{1/4}n}{2}$ pairs of $\{u,v\}$ such that $\{x_1,x_2,u\}, \{x'_1,x'_2,v\}$ are $\epsilon_5^{1/2}$-typical. The number of $\epsilon_5^{1/2}$-atypical triples containing x_2 or x'_2 is at most $2\cdot 3\cdot 4\epsilon_5^{1/2}n^2$ (by Claim 2.8 as they are $4\epsilon_5$-typical). Thus, $d_{G_X}(x_2) \ge \binom{m-4\epsilon_5^{1/4}n}{2} - 24\epsilon_5^{1/2}n^2 > \frac{2}{3}\binom{m}{2}$ because $m = |X| \ge cn$. Since all vertices in X are $4\epsilon_5$-typical, by Claim 2.6 and 2.8, the number of $\epsilon_5^{3/4}$-atypical pairs in X containing a fixed $4\epsilon_5$-typical vertex is at most $2\cdot 4\epsilon_5^{1/4}n$, and the number of $\epsilon_5^{1/2}$-atypical triples in X containing a fixed $4\epsilon_5$-typical vertex is at most $3\cdot 4\epsilon_5^{1/2}n^2$. Thus, for any vertex $u \in X$, we have $d_{G_X}(u) \ge \binom{m}{2} - 16\epsilon_5^{1/4}mn - 12\epsilon_5^{1/2}n^2 > \frac{2}{3}\binom{m}{2}$.

Since $|V(G_X)| = m+1$ and $m \ge cn$, by Lemma 3.6, G_X has a Hamiltonian cycle. So we can find a Hamiltonian path in G_X, say $P_X = x_2x_3\cdots x_{m+2}$, such that $\{x_1,x_2,x_3\}, \{x_{m+1},x_{m+2},x'_2\}$ and $\{x_{m+2},x'_2,x'_1\}$ are $\epsilon_5^{1/2}$-typical. Hence, we obtain a sequence of vertices

$$x_0x_1x_2x_3x_4\cdots x_{m+2}x_{m+3}(=x'_2)x_{m+4}(=x'_1)x_{x_1+5}(=x'_0),$$

where $\{x_i,x_{i+1},x_{i+2}\}$ is $\epsilon_5^{1/2}$-typical for $0 \le i \le m+3$ and $X = \{x_3,x_4,\ldots,x_{m+2}\}$. □

Now we are ready to prove the following lemma, which implies Theorem 3.1.

Lemma 3.8 *Suppose that $\mathcal{H}$ contains a path $Q = a_2a_1a_0\cdots b_0b_1b_2$ of length at most $\epsilon^3 n$ such that*

- *$\{a_0,a_1,a_2\} \in AAA$ and $\{b_0,b_1,b_2\} \in BBB$, and both are $(2\epsilon_1, 2\epsilon_2, 2\epsilon_3)$-typical;*
- *every vertex of $V(\mathcal{H})\backslash V(Q)$ is $4\epsilon_5$-typical.*

Then Q can be extended to a Hamiltonian path in $\mathcal{H}$.

Proof We use typical edges to connect all remaining vertices in $\mathcal{H}$. Note that all vertices in $V(\mathcal{H})\backslash V(Q)$ are $4\epsilon_5$-typical, but with respect to a (possibly) slightly modified partition, still denoted by (A, B), in which the two sides may differ in size by at most $10\epsilon_0 n$. Let $A' = A\backslash V(Q)$ and $B' = B\backslash V(Q)$ and let $m_1 = |A'|$ and $m_2 = |B'|$. Without loss of generality, suppose $m_1 \le m_2$. Then $m_1 \ge n - 5\epsilon_0 n - \epsilon^3 n \ge n - 2\epsilon^3 n$ and $m_2 - m_1 \le 2\epsilon^3 n$.

First, we label the vertices in B'. Since all vertices in B' are $4\epsilon_5$-typical and

$|B'| = m_2 \geq n - 2\epsilon^3 n$, there exists a $(4\epsilon_5, \epsilon_5^{3/4}, \epsilon_5^{1/2})$-typical triple $\{b_{m_2}, b_{m_2+1}, b_{m_2+2}\}$ such that $b_{m_2}, b_{m_2+1}, b_{m_2+2} \in B'$. Applying Lemma 3.7 to $\{b_0, b_1, b_2\}$, $\{b_{m_2}, b_{m_2+1}, b_{m_2+2}\}$ and $B' \setminus \{b_{m_2}, b_{m_2+1}, b_{m_2+2}\}$, we have a sequence, denoted by $P_B = b_0 b_1 \cdots b_{m_2+2}$ such that $\{b_i, b_{i+1}, b_{i+2}\}$ is $\epsilon_5^{1/2}$-typical for $0 \leq i \leq m_2$ and $B' = \{b_3, b_4, \ldots, b_{m_2+2}\}$. Construct an auxiliary bipartite graph Γ_B between A' and B_1, where $B_1 = \{b_2, b_5, \ldots, b_{3p_1-1}\}$ with $p_1 = \lfloor \frac{m_2+3}{3} \rfloor$ such that for any $a \in A'$ and $b_i \in B_1$, $ab_i \in E(\Gamma_B)$ if and only if

$$a \in N(b_{i-2}, b_{i-1}, b_i) \cap N(b_{i-1}, b_i, b_{i+1}) \cap N(b_i, b_{i+1}, b_{i+2}) \cap N(b_{i+1}, b_{i+2}, b_{i+3}).$$

Observe that if we find a matching in Γ_B, then we find a path in $\mathcal{H}$ similar to paths in $\mathcal{H}_0$. Since $\{b_i, b_{i+1}, b_{i+2}\}$ $(0 \leq i \leq m_2)$ is typical, $d_{\Gamma_B}(b) \geq 0.99 m_1$ for all $b \in B'$; so at least $0.9 m_1$ vertices $a \in A'$ have degree $d_{\Gamma_B}(a) \geq 0.9 p_1$. We need to deal with the vertices in A' of small degree since vertices of larger degree can be included in a matching of Γ_B. Let $A_{big} = \{a \in A' : d_{\Gamma_B}(a) \geq 0.9 p_1\}$; so $|A_{big}| \geq 0.9 m_1$. Let $A_{small} = A' \setminus A_{big}$. We claim the following.

(1) There exists a path P_{top} in $\mathcal{H}$ such that $A_{small} \subseteq V(P_{top}) \cap A \subseteq \{a_0, a_1, a_2\} \cup A'$, $V(P_{top}) \cap B \subseteq B'$ and one end triple of P_{top} is $a_0 a_1 a_2$.

Let $t := \lceil \frac{3m_1 - m_2}{8} \rceil$; then $\frac{3m_1 - m_2 - 3}{8} \leq t \leq \frac{3m_1 - m_2 + 9}{8}$. Since $(1 - 2\epsilon^3) n \leq m_1 \leq m_2 \leq (1 + \epsilon^3) n$, we have $0.24 n \leq t \leq 0.25 n$. Let A_S be a subset of A' such that $A_{small} \subseteq A_S$ and $|A_S| = 3t - 3 \geq 0.7n$. Note that $|A_{small}| \leq 0.1 m_1 \leq 3t - 3$, so we can find such A_S. Since $|A' \setminus A_S| = m_1 - (3t - 3) \geq 0.2n$, there exist $a_{3t}, a_{3t+1}, a_{3t+2} \in A' \setminus A_S$ such that $\{a_{3t}, a_{3t+1}, a_{3t+2}\}$ is $(4\epsilon_5, \epsilon_5^{3/4}, \epsilon_5^{1/2})$-typical. We apply Lemma 3.7 to $\{a_0, a_1, a_2\}$, $\{a_{3t}, a_{3t+1}, a_{3t+2}\}$ and A_S. Then there exists a sequence of vertices $a_0 a_1 a_2 a_3 \ldots a_{3t-1} a_{3t} a_{3t+1} a_{3t+2}$, such that $\{a_i, a_{i+1}, a_{i+2}\}$ is $\epsilon_5^{1/2}$-typical and $A_{small} \subseteq A_S = \{a_3, a_4, \ldots, a_{3t-1}\}$. Construct another auxiliary bipartite graph Γ_A between A_2 and B'', where $A_2 = \{a_2, a_5, \ldots, a_{3t-1}\}$ and B'' is the set of the last $p_2 = \lceil 0.3 m_1 \rceil$ vertices of P_B, i.e., $B'' = \{b_{m_2+2}, b_{m_2+1}, \ldots, b_{m_2+3-p_2}\}$. For any $a_i \in A_2$ and $b \in B''$, $a_i b \in E(\Gamma_A)$ if and only if

$$b \in N(a_{i-2}, a_{i-1}, a_i) \cap N(a_{i-1}, a_i, a_{i+1}) \cap N(a_i, a_{i+1}, a_{i+2}) \cap N(a_{i+1}, a_{i+2}, a_{i+3}).$$

Again, since $\{a_i, a_{i+1}, a_{i+2}\} (0 \leq i \leq t)$ is $\epsilon_5^{1/2}$-typical, $d_{\Gamma_A}(a) \geq 0.99 p_2$ for all $a \in A_2$; so at least $0.9 p_2$ vertices $b \in B''$ have degree $d_{\Gamma_A}(b) \geq 0.9t$. Let $B_{big} = \{b \in B'' : d_{\Gamma_A}(b) \geq 0.9t\}$. Then $|B_{big}| \geq 0.9 p_2 > t$ (since $t \leq \frac{3m_1 - m_2 + 9}{8} < 0.26 m_1 < 0.9 \cdot \lceil 0.3 m_1 \rceil = 0.9 p_2$ for sufficiently large n). We choose a set $\overline{B} \subseteq B_{big}$ of size $|\overline{B}| = t$. Consider the subgraph $\Gamma'_A = \Gamma_A[A_2 \cup \overline{B}]$. Since $|A_2| = |\overline{B}| = t$ and all vertices in A_2 and $\overline{B}$ have degree at least $0.9t$ in Γ'_A, there exists a perfect matching in Γ'_A by Dirac's theorem. The perfect matching forms a path in $\mathcal{H}$, denoted by

$$P_{top} = a_0a_1a_2\overline{b_1}a_3a_4a_5\overline{b_2}\cdots a_{3t-1}\overline{b_t}\ldots,$$

where $a_{3i-1}\overline{b_i} \in E(\Gamma_A)$. Note that the end of P_{top}, possibly $a_{3t-3}a_{3t-2}a_{3t-1}\overline{b_t}$ or $a_{3t-2}a_{3t-1}\overline{b_t}a_{3t}$ or $a_{3t-1}\overline{b_t}a_{3t}a_{3t+1}$ or $\overline{b_t}a_{3t}a_{3t+1}a_{3t+2}$, is determined by the numbers m_1 and m_2. This proves (1).

We now show the following claim.

(2) There exists a path P_{zig} in $\mathcal{H}$ such that $V(P_{zig}) \cap A = A' \backslash V(P_{top}), V(P_{zig}) \cap B = \{b_0, b_1, b_2\} \cup B' \backslash V(P_{top})$, and one end triple of P_{zig} is $b_0b_1b_2$.

Consider $P_{B_1} = b_0b_1b_2\ldots b_{m_2+2-p_2}$ and $V(P_{B_1}) = V(P_B)\backslash B''$. Since all vertices in $B_2 := B''\backslash\overline{B}$ are $4\epsilon_5$-typical and $|B_2| = p_2 - t \geq 0.04n$, there exist vertices $b'_{m_2-t}, b'_{m_2+1-t}, b'_{m_2+2-t} \in B_2$ such that $\{b'_{m_2-t}, b'_{m_2+1-t}, b'_{m_2+2-t}\}$ is $(4\epsilon_5, \epsilon_5^{3/4}, \epsilon_5^{1/2})$-typical. We extend P_{B_1} by applying Lemma 3.7 to $\{b_{m_2-p_2}, b_{m_2+1-p_2}, b_{m_2+2-p_2}\}$, $\{b'_{m_2-t}, b'_{m_2+1-t}, b'_{m_2+2-t}\}$, and $B_2\backslash\{b'_{m_2-t}, b'_{m_2+1-t}, b'_{m_2+2-t}\}$. Hence, we obtain $b'_0b'_1b'_2\ldots b'_{m_2+2-t}$ such that $b'_j = b_j$ for all $0 \leq j \leq m_2 + 2 - p_2$, $B_2 = \{b'_{m_2+3-p_2}, \ldots, b'_{m_2+2-t}\}$, and $\{b'_i, b'_{i+1}, b'_{i+2}\}$ is $\epsilon_5^{1/2}$-typical for any $0 \leq i \leq m_2 - t$.

Similarly, construct an auxiliary bipartite graph Γ'_B with partition classes $\overline{A}$, B_3, where $\overline{A} = A\backslash V(Q \cup P_{top})$ and $B_3 = \{b'_2, b'_5, \ldots, b'_{3p_3-1}\}$ with $p_3 = \lfloor\frac{m_2+3-t}{3}\rfloor$. We choose the end triple for P_{top} to make $|\overline{A}| = |B_3|$. For any $a \in \overline{A}$ and $b'_i \in B_3$, $ab'_i \in E(\Gamma'_B)$ if and only if

$$a \in N(b'_{i-2}, b'_{i-1}, b'_i) \cap N(b'_{i-1}, b'_i, b'_{i+1}) \cap N(b'_i, b'_{i+1}, b'_{i+2}) \cap N(b'_{i+1}, b'_{i+2}, b'_{i+3}).$$

We know $d_{\Gamma'_B}(b) \geq 0.99p_3$ for all $b \in B_3$. Since $A_{small} \subseteq V(P_{top})$, $\overline{A} \subseteq A_{big}$ and $d_{\Gamma_B}(a) \geq 0.9p_1$ for each $a \in \overline{A}$. Hence, $d_{\Gamma'_B}(a) \geq 0.9p_1 - \lceil\frac{p_2}{3}\rceil \geq 0.3m_2 - 0.1m_1 \geq 0.8p_3$. Therefore, $\delta(\Gamma'_B) \geq 0.8p_3$. By Dirac's theorem, Γ'_B has a perfect matching, which forms a path in $\mathcal{H}$, denoted by

$$P_{zig} = b'_0b'_1b'_2\overline{a_1}b'_3b'_4b'_5\overline{a_2}\cdots b'_{3p_3-1}\overline{a_{p_3}}\ldots,$$

where $\overline{a_i}b'_{3i-1} \in E(\Gamma'_B)$.

Now, $Q \cup P_{top} \cup P_{zig}$ is a Hamiltonian path in $\mathcal{H}$. □

4 Hamiltonian Cycles

In this section, we prove the following

Theorem 4.1 *There exists $\epsilon_0 > 0$ such that, for sufficiently large n and any 4-graph $\mathcal{H}$ on $2n$ vertices with $b(\mathcal{H}) < \epsilon_0 n^4$ the following holds: If $\delta_3(\mathcal{H}) \geq n - 1$, then $\mathcal{H}$ has a Hamiltonian cycle.*

The proof of Theorem 4.1 proceeds along the lines of the proof of Theorem 3.1, except that to get a Hamiltonian cycle we will need a second bridge. Suppose we have two disjoint bridges M_1 and M_2. The arguments for taking care of medium and anarchist vertices are essentially the same as in the proof of Theorem 3.1. So we have a path Q which contains M_1 and all remaining medium vertices, and is disjoint from M_2. Let $n_1 = |A\backslash V(Q \cup M_2)|$, $n_2 = |B\backslash V(Q\cup M_2)|$, and let $m = \frac{3n_1 - n_2 + 6}{8}$. When m is an integer, we may apply the following variation of Lemma 3.8, whose proof basically repeats the proof of Lemma 3.8 with just minor modifications.

Lemma 4.2 *Suppose that $\mathcal{H}$ contains a path $Q = a_2a_1a_0\cdots b_0b_1b_2$ of length at most $\epsilon^3 n$ and a bridge $M_2 = a_2'a_1'a_0'\cdots b_0'b_1'b_2'$ such that*

- *$V(Q)\cap V(M_2) = \emptyset$;*
- *$m = \frac{3n_1-n_2+6}{8}$ is an integer;*
- *all end triples $\{a_0,a_1,a_2\}$, $\{b_0,b_1,b_2\}$, $\{a_0',a_1',a_2'\}$ and $\{b_0',b_1',b_2'\}$ are $(2\epsilon_1, 2\epsilon_2, 2\epsilon_3)$-typical;*
- *every vertex of $V(\mathcal{H})\backslash V(Q\cup M_2)$ is $4\epsilon_5$-typical.*

Then $Q \cup M_2$ can be extended to a Hamiltonian cycle in $\mathcal{H}$.

Proof Let $A' = A\backslash V(Q\cup M_2)$ and $B' = B\backslash V(Q\cup M_2)$. It follows that $|A'| = n_1, |B'| = n_2$. We build a top path P_{top} with $V(P_{top})\cap A \subseteq \{a_0, a_1, a_2, a_2', a_1', a_0'\}\cup A'$ and $V(P_{top})\cap B\subseteq B'$ such that $|V(P_{top})\cap A| = 3(m+1), |V(P_{top})\cap B| = m$, and P_{top} connects two AAA end triples of Q and M_2, i.e., $a_0a_1a_2$ and $a_2'a_1'a_0'$. Then, we use P_{zig} with $V(P_{zig})\cap A = A'\backslash V(P_{top})$ and $V(P_{zig})\cap B = \{b_0, b_1, b_2, b_2', b_1', b_0'\}\cup B'\backslash V(P_{top})$ to connect $b_0b_1b_2$ with $b_2'b_1'b_0'$. Note that $|V(P_{zig})\cap A| = n_1 - (3(m+1) - 6) = m$ and $|V(P_{zig})\cap B| = 6 + n_2 - m = 3(m+1)$. Therefore, $Q\cup M_2\cup P_{top}\cup P_{zig}$ is a Hamiltonian cycle in $\mathcal{H}$.

To construct P_{top}, we apply Lemma 3.7 to $\{b_0, b_1, b_2\}, \{b_2', b_1', b_0'\}$, and B'. We obtain a sequence of vertices $b_0b_1b_2b_3\cdots b_{n_2+2}b_{n_2+3}(=b_2')b_{n_2+4}(=b_1')b_{n_2+5}(=b_0')$, where $\{b_i, b_{i+1}, b_{i+2}\}$ is $(4\epsilon_5, \epsilon_5^{3/4}, \epsilon_5^{1/2})$-typical for $0\le i\le n_2+3$ and $B' = \{b_3, b_4, \ldots, b_{n_2+2}\}$. Consider a bipartite graph Γ_B between A' and B_1, where $B_1 = \{b_2, b_5, \ldots, b_{3p_1-1}\}$ with $p_1 = \lfloor\frac{n_2+3}{3}\rfloor$. For any $a\in A'$ and $b_i\in B_1$, $ab_i\in E(\Gamma_B)$ if and only if

$$a\in N(b_{i-2}, b_{i-1}, b_i)\cap N(b_{i-1}, b_i, b_{i+1})\cap N(b_i, b_{i+1}, b_{i+2})\cap N(b_{i+1}, b_{i+2}, b_{i+3}).$$

By the typicality of all triples, $d_{\Gamma_B}(b)\ge 0.99n_1$ for any $b\in B_1$, and there are a lot of vertices in A' having large degree in Γ_B. We partition $A' = A_{big}\cup A_{small}$, where $A_{big} = \{a\in A' : d_{\Gamma_B}(a)\ge 0.9p_1\}$. Then $|A_{small}|\le 0.1n_1 < 3(m-1)$. By Lemma 3.7 again, there exists a sequence of vertices $a_0a_1a_2a_3\ldots a_{3m-2}a_{3m-1}a_{3m}(=a_2')a_{3m+1}(=a_1')$ $a_{3m+2}(=a_0')$, such that$\{a_i, a_{i+1}, a_{i+2}\}$ is $\epsilon_5^{1/2}$-typical for $0\le i\le 3m$ and $A_{small}\subseteq\{a_3, a_4, \ldots, a_{3m-1}\}$.

Consider a bipartite graph Γ_A with partition classes A_2, B'', where $A_2 =$

$\{a_2, a_5, \ldots, a_{3m-1}\}$ and B'' is the set containing the last $p_2 = \lceil 0.3n_1 \rceil$ vertices of B', i. e., $B'' = \{b_{n_2+2}, b_{n_2+1}, \ldots, b_{n_2+3-p_2}\}$. Then we can find a perfect matching in $\Gamma_A[A_2, \overline{B}]$, where $\overline{B}$ is a subset of B'' and $|\overline{B}| = |A_2|$. Therefore,

$$P_{top} = a_0 a_1 a_2 \overline{b_1} a_3 a_4 a_5 \overline{b_2} a_6 \cdots a_{3m-1} \overline{b_m} a_{3m}(= a'_2) a_{3m+1}(= a'_1) a_{3m+2}(= a'_0).$$

For the remaining vertices, we use a similar step in the proof of Lemma 3.8 to find P_{zig}, and $Q \cup M_2 \cup P_{top} \cup P_{zig}$ is a Hamiltonian cycle in $\mathcal{H}$. □

Since Lemma 4.2 has some requirements on the number of vertices in $A \backslash V(Q \cup M_2)$ and $B \backslash V(Q \cup M_2)$, the above proof works only if $m \in \mathbb{N}$. When m is not an integer, we use a *good set* defined below. Here, we will consider the order of sets in $V_1 V_2 V_3$. For example, AAB and ABA are different.

Definition 4.3 For a 4-graph $\mathcal{H}$, let A, B be a partition of $V(\mathcal{H})$.

- The difference of a path P in $\mathcal{H}$ with respect to A,B is the number $p^* \equiv 3|V(P) \cap A| - |V(P) \cap B| \pmod 8$.
- An $(\epsilon_1, \epsilon_2, \epsilon_3)$-switcher is a path S, which contains no ϵ_5-anarchists, has two $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical end triples type of BAA and AAA or type ABB and BBB, and has nonzero difference.
- A set X of vertices in $V(\mathcal{H})$ is called good if $|X| < 1600$ and X does not contain any ϵ_5-anarchists of $\mathcal{H}$, and, for any number $a \in \{0, 1, 2, \ldots, 7\}$, there exists two disjoint bridges M_1 and M_2 such that $V(M_1), V(M_2) \subseteq X$ and $m_1^* + m_2^* \equiv a \pmod 8$ where m_i^* is the difference of M_i for $i = 1, 2$.

Note. Given two disjoint $(\epsilon_1, \epsilon_2, \epsilon_3)$-bridges R_1 and R_2, let r_i^* be the difference of R_i for $i = 1, 2$. By Claim 2.11, we can connected any two $(\epsilon_1, \epsilon_2, \epsilon_3)$-bridges to obtain a path with both end triples in AAA or in BBB. For example, the BBB triples of the two bridges can be connected by two vertices in A and three vertices in B (these vertices are from the vertex set T by Claim 2.11), and we have a path with both end triples in AAA. Adding some vertex from B to one end of this path to make it have an $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical end triple of type BAA, we can obtain a path P with difference $r_1^* + r_2^* + 2$ and both end triples are $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical. Therefore, if $r_1^* + r_2^* \not\equiv 6 \ (mod\ 8)$, R_1 and R_2 can form an $(\epsilon_1, \epsilon_2, \epsilon_3)$-switcher with difference $r_1^* + r_2^* + 2$.

If there exists a good set X in $\mathcal{H}$, then firstly, make a small modification of the partition of $V(\mathcal{H})$ by transferring all ϵ_5-anarchists to the other side, denoted by (A', B'). Since X is good, we can find two disjoint bridges M_1 and M_2 in X to make $\frac{3n'_1 - n'_2 + 6}{8}$ an integer, where $n'_1 = |A' \backslash V(M_1 \cup M_2)|$ and $n'_2 = |B' \backslash V(M_1 \cup M_2)|$ (We can do it since X does not contain any ϵ_5-anarchists of $\mathcal{H}$). Next, by the proof of Lemma 3.5, there is a path Q_{top} connecting all ϵ_5-medium vertices in $A' \backslash V(M_1 \cup M_2)$ and a path Q_{zig} connecting all ϵ_5-medium vertices in $B' \backslash V(M_1 \cup M_2)$. By Claim 2.11, using M_1 connects Q_{top} and Q_{zig} to get a path Q whose end triples are AAA and BBB, and both of them are $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical. It can be checked that $\frac{3n_1 - n_2 + 6}{8}$ is an integer when $\frac{3n_1'-n_2'+6}{8}$ an integer, where $n_1 = |A' \setminus V(Q \cup M_2)|$ and $n_2 = |B' \setminus V(Q \cup M_2)|$. (This is because $Q - M_1$ contains two paths Q_1 and Q_2 such that all edges of Q_1 are $AAAB$ edges and Q_1 contains $3x$ vertices in A' and x vertices in B' while Q_2 of which all edges are $ABBB$ edges contains $3y$ vertices in B' and y vertices in A' for some integers x and y.) Finally, by Lemma 4.2, we can find a Hamiltonian cycle in $\mathcal{H}$.

So the key is to prove the following lemma.

Lemma 4.4 *For any 4-graph $\mathcal{H}$, let A, B be a partition of $V(\mathcal{H})$ and assume that (2.1), (2.2), (2.3) hold. Then $\mathcal{H}$ contains a good set X.*

Next, we introduce a special type of edges, called *seed*. It is used to find a good set X.

Definition 4.5 A quadruple of vertices (a, a', b, w) is called a seed if

- $aa'bw \in E(\mathcal{H})$,
- $\{a, a', b\}$ is $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical, and
- $w \in B$ is ϵ_5-typical.

Similarly, a quadruple of vertices (b, b', a, w) is called a seed if

- $bb'aw \in E(\mathcal{H})$,
- $\{b, b', a\}$ is $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical, and
- $w \in A$ is ϵ_5-typical.

Claim 4.6 *Let $K \subseteq V(\mathcal{H})$ with $|K| \leq \epsilon n$. Given two disjoint seeds not intersecting with K, (a_i, a_i', b_i, w_i), $i = 1, 2$, we can build a K-avoiding $(\epsilon_1, \epsilon_2, \epsilon_3)$-switcher of odd difference and at most 100 vertices. Analogically, two disjoint seeds not intersecting with K, (b_i, b_i', a_i, w_i), $i = 1, 2$, give a K-avoiding $(\epsilon_1, \epsilon_2, \epsilon_3)$-switcher of odd difference and at most 100 vertices.*

Proof For the simplicity of the proof, we do not involve K in our proof. But all vertices we need to choose in the following paragraphs can be chosen from the vertex set not intersecting with K as the size of K is small.

Since w_1 is ϵ_5-typical and $\delta_3(\mathcal{H}) \geq n - 1$, we can extend the edge $a_1 a_1' b_1 w_1$ to a path $P = a_1 a_1' b_1 w_1 u_1 v_1$, such that $\{u_1, v_1\}$ is an (ϵ_1, ϵ_2)-typical pair, $\{w_1, u_1, v_1\}$ is an $\epsilon_5^{1/2}$-typical triple and $\{b_1, u_1, v_1\}$ is $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical. If $u_1 v_1 \in AA$, then there exist three ϵ_1-typical vertices $\overline{a_1}, a_1'', b_1'$ such that all triples $\{\overline{a_1}, a_1, a_1'\}$, $\{u_1, v_1, a_1''\}$ and $\{v_1, a_1'', b_1'\}$ are $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical. Hence, $S = \overline{a_1} a_1 a_1' b_1 w_1 u_1 v_1 a_1'' b_1'$ is a path with both ends in AAA and AAB, and then S is a switcher with $s^* \equiv 7$.

Otherwise, $u_1 v_1 \in BA$, or $u_1 v_1 \in AB$, or $u_1 v_1 \in BB$. Similarly, extending the seed $a_1 a_1' b_1 w_1$ to a path $P = a_1 a_1' b_1 w_1 u_1 v_1$, we can find an $(\epsilon_1, \epsilon_2, \epsilon_3)$-bridge R_1, R_2 *or* R_3 with $r_1^* \equiv 6$, $r_2^* \equiv 7$ or $r_3^* \equiv 5$, respectively. We repeat the same construction on the second seed to get $P' = a_2 a_2' b_2 w_2 u_2 v_2$. If we cannot get a switcher with odd difference, then there exists an $(\epsilon_1, \epsilon_2, \epsilon_3)$-bridge obtained from the second seed, R_1', or R_2', or R_3' with $(r_1')^* \equiv 6$, or $(r_2')^* \equiv 7$, or $(r_3')^* \equiv 5$, respectively. By applying

Claim 2.11, we can use these two $(\epsilon_1, \epsilon_2, \epsilon_3)$-bridges R_i and R'_j, $i,j \in \{1,2,3\}$, to form a switcher with odd difference.

If the differences of these two bridges have different parity, connecting these two bridges results in a switcher with odd difference. Now assume the difference of these two bridges have the same parity.

Suppose there are two bridges R_1, R_2 with even difference $r_1^* = r_2^* \equiv 6$. Then $u_1v_1 \in BA$ and $u_2v_2 \in BA$. If $l^B_{a_1a'_1w_1} \geq \frac{n}{2}$, then we have a path $b'_1a_1a'_1w_1b_1u_1v_1$, where $b'_1 \in B$ and $\{b'_1, a_1, a'_1\}$ is $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical; this path can be extended to a bridge R with difference $r^* \equiv 3$. If $l^A_{a_1a'_1w_1} \geq \frac{n}{2}$, consider the path $b_1a_1a'_1w_1a''_1$ where $a''_1 \in A$ is ϵ_1-typical. Extending this path, we can get either a switcher with difference 5, or a bridge with difference 5.

Now assume there are two bridges R_1 and R_2 with odd differences. Then $u_iv_i \in AB$ *or* BB for $i = 1,2$. If $l^A_{a'_1b_1w_1} \geq \frac{n}{2}$, we may assume $u_1v_1 \in AB$; otherwise we obtain a switcher with odd difference. If $l^B_{a_1a'_1w_1} \geq \frac{n}{3}$, then $a_1a'_1w_1b_1u_1v_1$ can be extended to a bridge with difference 4. Connecting this bridge with R_2 gives a switcher with odd difference. We may assume $l^A_{a_1a'_1w_1} \geq \frac{2n}{3}$. Then $|L^A_{a_1a'_1w_1} \cap L^A_{a'_1b_1w_1}| \geq \frac{n}{6}$. In this case, there exists $a \in A$ such that $a \in N(a_1, a'_1, w_1) \cap N(a'_1, b_1, w_1)$ and the pair $\{a, w_1\}$ is $\epsilon_5^{3/4}$-typical. If $l^A_{a'_1w_1a} \geq \frac{n}{2}$, we have a switcher with difference 5 by the path $b_1a_1a'_1w_1a$. If $l^B_{a'_1w_1a} \geq \frac{n}{2}$, the path $a_1b_1a'_1w_1a$ gives a bridge with even difference 2, which also gives a switcher with odd difference by connecting it and R_2.

If $l^B_{a'_1b_1w_1} \geq \frac{n}{2}$, we may assume $u_1v_1 \in BB$ for all possible choices of u_1v_1; otherwise we can obtain a bridge with even difference. Consider the path $P_1 = a_1b_1a'_1w_1u_1$. If $l^B_{a'_1w_1u_1} \geq \frac{n}{6}$, we extend P_1 to a bridge with difference 0. We may assume $l^A_{a'_1w_1u_1} \geq \frac{5n}{6}$ and hence $l^{AB}_{a'_1w_1} \geq \frac{n^2}{3}$, as there are at least $\frac{2n}{5}$ possible choices of $u_1 \in B$. Let $F := \{ab \in L^{AB}_{a'_1w_1} : \{a'_1, a, b\}$ is $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical $\}$. Since a'_1 is ϵ_1-typical and $l^{AB}_{a'_1w_1} \geq \frac{n^2}{3}$, $|F| \geq \frac{n^2}{4}$. We know that (a'_1, a, b, w_1) is a seed for any $ab \in F$. Then for all possible $a \in A$ with $ab \in F$ (the number of such vertices is at least $\frac{n}{4}$), we may assume $l^{AB}_{aw_1} \geq \frac{n^2}{3}$. Otherwise we have a switcher with odd difference by the above analysis. Hence, $l^{AAB}_{w_1} \geq \frac{n^2}{3} \cdot \frac{n}{4} \cdot \frac{1}{2} = \frac{n^3}{24}$, contradicting the fact that w_1 is ϵ_5-typical. □

We know that connecting a bridge with a switcher forms a new bridge with different difference. If there are two given disjoint bridges with small lengths, switchers can help construct a good set. By Claim 4.6, a lot of pairwise disjoint seeds give many switchers with odd differences. In the proof of Lemma 4.4, we explore when $\mathcal{H}$ contains many seeds or not and this completes the proof of Theorem 4.1.

Proof of Lemma 4.4 First, we claim that for sufficiently small $\epsilon > 0$, $\mathcal{H}$ contains two disjoint $(\epsilon_1, \epsilon_2, \epsilon_3)$-bridges M_1 and M_2 with $|V(M_i)| \leq 25$ for $i = 1,2$.

We repeat the proof of Lemma 3.3 to build the first $(\epsilon_1, \epsilon_2, \epsilon_3)$-bridge M_1, and find two (ϵ_1, ϵ_2)-typical pairs $\{a_1, a_2\}$ and $\{b_1, b_2\}$. If $a_1a_2b_1b_2 \in E(\mathcal{H})$, we can extend this edge to a bridge. If $a_1a_2b_1b_2 \notin E(\mathcal{H})$, then there exists a vertex $z \in$

$N(a_1,a_2,b_1)\cap N(a_1,b_1,b_2)$ such that we can extend $a_2a_1zb_1b_2$ to an $(\epsilon_1,\epsilon_2,\epsilon_3)$-typical bridge.

To build the second bridge, find two (ϵ_1,ϵ_2)-typical pairs $\{a'_1,a'_2\}$ and $\{b'_1,b'_2\}$, such that if $a'_1a'_2b'_1b'_2\notin E(\mathcal{H})$ then a_1,a_2,b_1,b_2 are not contained in the common neighbors of $\{a'_1,a'_2,b'_1\}$ and $\{a'_1,b'_1,b'_2\}$, in order to get disjoint bridges. We can do it since these vertices a_1,a_2,b_1 and b_2 are ϵ_1-typical, by Claim 2.8, there exists an (ϵ_1,ϵ_2)-typical pair $\{a'_1,b'_1\}$, such that all triples $\{a'_1,b'_1,a_1\}$, $\{a'_1,b'_1,a_2\}$, $\{a'_1,b'_1,b_1\}$ and $\{a'_1,b'_1,b_2\}$ are ϵ_3-typical. Then we know $d_A(a'_1,b'_1,b_i)\leq\epsilon_3 n$ and $d_B(a'_1,b'_1,a_i)\leq\epsilon_3 n$ for $i=1,2$. By Claim 2.6, we find two vertices a'_2, b'_2, such that $\{a'_1,a'_2\}$ and $\{b'_1,b'_2\}$ are (ϵ_1,ϵ_2)-typical pairs and for $i=1,2$, $a'_2a'_1b'_1b_i\notin E(\mathcal{H})$, and $b'_2b'_1a'_1a_i\notin E(\mathcal{H})$. Similarly, if $a'_2a'_1b'_1b'_2\in E(\mathcal{H})$, $b'_2b'_1a'_1a'_2$ can be extended to an $(\epsilon_1,\epsilon_2,\epsilon_3)$-typical bridge M_2 and $M_1\cap M_2=\emptyset$, since $\delta(\mathcal{H})\geq n-1$. Otherwise, there exist two vertices z',z'_1 different from a_1,a_2,b_1,b_2 satisfying $z',z'_1\in N(a'_2,a'_1,b'_1)\cap N(b'_2,b'_1,a'_1)$. Without loss of generality, suppose $z'\neq z$, then $a'_2a'_1z'b'_1b'_2$ can be extended to a $(\epsilon_1,\epsilon_2,\epsilon_3)$-bridge M_2 such that $M_1\cap M_2=\emptyset$. So in any case, there are two disjoint $(\epsilon_1,\epsilon_2,\epsilon_3)$-bridges M_1 and M_2 in $\mathcal{H}$ with $|V(M_i)|\leq 25$ for $i=1,2$.

Case 1. All vertices in B are ϵ_5-typical (or all vertices in A are ϵ_5-typical).

We may consider the case when all vertices in B are ϵ_5-typical. Let $V':=V(M_1\cup M_2)$. It suffices to show that $\mathcal{H}$ has fourteen pairwise disjoint seeds of type (a,a',b,w) that are also disjoint from V'. Then by Claim 4.6, every two such seeds can form a switcher with odd difference. Hence, we can obtain seven pairwise disjoint switchers and each has odd difference.

Since all $b\in B$ are ϵ_5-typical, we have $l_B^{AAB}\leq\epsilon_5 n^3$. Consider the set of triples $E=\bigcup_{b\in V'\cap B}L_B^{AAB}$. Since $|V'|\leq 50$, we have $|E|\leq 50\epsilon_5 n^3$ and, thus, by Corollary 2.10, there exists an $(\epsilon_1,\epsilon_2,\epsilon_3)$-typical triple $\{a_1,a'_1,b_1\}$ such that $a_1a'_1b_1\notin E$ and $a_1,a'_1,b_1\notin V'$. Let $w_1\in N_B(a_1,a'_1,b_1)$; the existence of w_1 follows from (2.1). By the definition of E, $w_1\notin V'$. So we get a seed (a_1,a'_1,b_1,w_1). Assume that we have produced $i-1$ seeds, (a_j,a'_j,b_j,w_j), for $j=1,\ldots,i-1$. Set

$$E_{i-1}=E\cup(L_{b_1}^{AAB}\cup L_{w_1}^{AAB})\cdots\cup(L_{b_{i-1}}^{AAB}\cup L_{w_{i-1}}^{AAB})$$

and note that $|E_{i-1}|\leq 100\epsilon_5 n^3$ if $i\leq 15$. Similarly as before, we can find an $(\epsilon_1,\epsilon_2,\epsilon_3)$-typical triple $\{a_i,a'_i,b'_i\}$ such that $a_ia'_ib_i\notin E_{i-1}$ and $a_i,a'_i,b_i\notin V'\cup\{a_1,a'_1,b_1,w_1\}\cup\cdots\cup\{a_{i-1},a'_{i-1},b_{i-1},w_{i-1}\}$. We can also find $w_i\in N_B(a_i,a'_i,b_i)$ such that a_i,a'_i,b'_i,w_i is a seed. So there are at least fourteen pairwise disjoint seeds and, applying Claim 4.6, we can form seven pairwise disjoint $(\epsilon_1,\epsilon_2,\epsilon_3)$-switchers with odd differences. For any $a\in\{0,1,2,\ldots,7\}$, we can find some numbers from those seven odd differences such that the summation of them is a (mod 8). In particular, for the case $a=0$, we do not use any switchers. Let V'' denote the set of all vertices of these seven switchers. Then $V'\cup V''$ and a small number of ϵ_5-typical vertices, which are used to connect bridges and switchers, form a good set in $\mathcal{H}$.

Case 2. There exists an ϵ_5-anarchist in $\mathcal{H}$.

By Claim 2.3, all vertices in one side are $3\epsilon_5$-typical. Then a similar proof argument as in Case 1 completes this case.

Case 3. There exists an ϵ_5-medium vertices in $\mathcal{H}$ and $\mathcal{H}$ doesn't contain any ϵ_5-anarchist.

We may assume there are at least two ϵ_5-medium vertices, otherwise we get back to Case 1. If there are at least 28 pairwise disjoint seeds in $\mathcal{H}$ that are also disjoint from $V' = V(M_1 \cup M_2)$, then there exist at least 14 pairwise disjoint seeds of the same type. Then we can find a good set X by Claim 4.6 and the argument in Case 1. So we may assume that the number of pairwise disjoint seeds is less than 28. Let V_s denote a maximal set of vertices containing pairwise disjoint seeds in $\mathcal{H} - V'$ and let V_m denote the set of ϵ_5-typical vertices in V'. Then all vertices in $V := V_s \cup V_m$ are ϵ_5-typical. Let $V_A = V \cap A$ and $V_B = V \cap B$. Then $|V_A| \leq 2 \cdot 25 + 2 \cdot 28 = 106$ and $|V_B| \leq 2 \cdot 25 + 2 \cdot 28 = 106$ (by $|M_i| \leq 25$ for $i = 1, 2$ and the fact that each seed forms an $AABB$ edge).

Let $E_A = \bigcup_{a \in V_A} L_A^{ABB}$, $E_B = \bigcup_{b \in V_B} L_B^{AAB}$. Then $|E_A| \leq 106\epsilon_5 n^3$ and $|E_B| \leq 106\epsilon_5 n^3$ since all vertices in V are ϵ_5-typical. Let T be a set of quadruples (a_i, a_j, b_k, b_l) such that both $\{a_i, a_j, b_k\}$ and $\{a_j, b_k, b_l\}$ are $(\epsilon_1, \epsilon_2, \epsilon_3)$-typical triples, $a_i a_j b_k \notin E_B$, $a_j b_k b_l \notin E_A$ and $a_i, a_j, b_k, b_l \notin V$, where $a_i \neq a_j \in A$ and $b_k \neq b_l \in B$. By Corollary 2.10, $\mathcal{H}$ contains at most $\epsilon_4 n^3$ $(\epsilon_1, \epsilon_2, \epsilon_3)$-atypical triples. So $|T| \geq n^2(n-1)^2 - 2 \cdot \epsilon_4 n^3 \cdot (n-1) - 2 \cdot 2 \cdot 106\epsilon_5 n^3 \cdot (n-1) - 2 \cdot 106 \cdot n(n-1)^2 > \frac{n^4}{2}$.

For any $(a_i, a_j, b_k, b_l) \in T$, $a_i, a_j, b_k, b_l \notin V' \cup V_s$ (by the definition of T) and $a_i a_j b_k b_l \notin \mathcal{H}$ (by the maximality of V_s). Since $a_i a_j b_k b_l \notin \mathcal{H}$, it follows from the proof of Lemma 3.3 that $|N(a_j, b_k, b_l) \cap N(a_i, a_j, b_k)| \geq 2$. Now we claim that for each vertex $v \in N(a_j, b_k, b_l) \cap N(a_i, a_j, b_k)$, either $v \in V' \cup V_s$ or v is ϵ_5-medium. Suppose $v \notin V' \cup V_s$ and v is not ϵ_5-medium. Then v is ϵ_5-typical, and hence, if $v \in B$ then (a_i, a_j, b_k, v) is a seed disjoint from $V' \cup V_s$, and if $v \in A$ then (b_k, b_l, a_j, v) is a seed disjoint from $V' \cup V_s$. This contradicts the maximality of V_s. Since the number of ϵ_5-medium vertices in $\mathcal{H}$ is at most $8\epsilon_0/\epsilon_5 n$ and $|V' \cup V_s| \leq 2 \cdot 25 + 28 \cdot 4 = 162$, the number of all possible vertices in $N(a_j, b_k, b_l) \cap N(a_i, a_j, b_k)$ for all $(a_i, a_j, b_k, b_l) \in T$ is at most $8\epsilon_0/\epsilon_5 n + 162 < \frac{\epsilon^3 n}{10}$. Therefore, we can find a vertex u, such that at least $\frac{n^4}{2} / \frac{\epsilon^3 n}{10} = \frac{5n^3}{\epsilon^3}$ quadruples (a_i, a_j, b_k, b_l) satisfy $u \in N(a_i, a_j, b_k) \cap N(a_j, b_k, b_l)$. Since $|N(a_j, b_k, b_l) \cap N(a_i, a_j, b_k)| \geq 2$, we can find two such vertices u, v by applying Pigeonhole principle twice.

Next, we claim that for each integer $i \in I = \{0, 3, 6, 7\}$, u is contained in a bridge U_i with difference i, and v is contained in a bridge V_i with difference i. Moreover, $(\cup_{i \in I} V(U_i)) \cap (\cup_{i \in I} V(V_i)) = \emptyset$.

Without loss of generality, we may assume $u \in B$. (The proof for the case when $u \in A$ is analogous.) Construct an auxiliary bipartite graph G with partition classes Y, Z, where $Y = \{(a_i, a_j) : a_i, a_j \in A, a_i \neq a_j\}$ and $Z = \{(b_k, b_l) : b_k, b_l \in B, b_k \neq b_l\}$, and $(a_i, a_j) \sim (b_k, b_l)$ if and only if $(a_i, a_j, b_k, b_l) \in T$ and $u \in N(a_i, a_j, b_k) \cap N(a_j, b_k, b_l)$. Then $|E(G)| \geq \frac{5n^3}{\epsilon^3} > 8n^3$ and the average degree of G is at least $8n$.

Note that every graph H contains a subgraph D, of which the minimum degree is at least half of the average degree of H. Hence there exists $G' \subseteq G$ such that $\delta(G') \geq 4n$. In G', $d_{G'}((a_1, a_2)) \geq 4n$ for $(a_1, a_2) \in V(G') \cap Y$. There exist

$(b_1,b_3),(b_2,b_3) \in V(G') \cap Z$ such that $(a_1,a_2)(b_1,b_3),(a_1,a_2)(b_2,b_3) \in E(G')$ with $b_1 \neq b_2$; otherwise $d_{G'}((a_1,a_2)) \leq n$. Since $d_{G'}((b_2,b_3)) \geq 4n$, there exists $(a_3,a_4) \in V(G') \cap Y$ such that $(a_3,a_4) \in N_{G'}((b_2,b_3))$ and $a_3,a_4 \notin \{a_1,a_2\}$. Hence, we have

$$u \in N(a_1,a_2,b_1) \cap N(a_2,b_1,b_3) \cap N(a_1,a_2,b_2) \cap N(a_2,b_2,b_3) \cap N(a_3,a_4,b_2) \cap N(a_4,b_2,b_3).$$

Now the path $a_1b_1a_2ub_3b_2a_4$ can be extended to a bridge with difference 0 by Claim 2.11. Similarly, the path $b_1a_1a_2ub_2b_3a_4$ gives a bridge with difference 3 and the path $a_3a_4b_2ub_3a_2b_1$ gives a bridge with difference 6.

To obtain a bridge with difference 7, we consider another bipartite graph H with partition classes U, W, where $U = \{(a_i,b_k) : a_i \in A, b_k \in B\}$ and $W = \{(a_j,b_l) : a_j \in A, b_l \in B\}$, and $(a_i,b_k) \sim (a_j,b_l)$ if and only if $(a_i,a_j,b_k,b_l) \in T$ and $u \in N(a_i,a_j,b_k) \cap N(a_j,b_k,b_l)$. Then $|E(H)| \geq \frac{5n^3}{\epsilon^3} > 8n^3$ and the average degree of H is at least $8n$. Similarly, for some $(a_1,b_1) \in U$, there exists $(a_2,b_2),(a_3,b_3) \in W$ such that $a_2 \neq a_3, d_H((a_2,b_2)) \geq 4n$ and $(a_1,b_1)(a_2,b_2),(a_1,b_1)(a_3,b_3) \in E(H)$. Since $d_H((a_2,b_2)) \geq 4n$, there exists $(a_4,b_4) \in U$ such that $(a_4,b_4) \in N_H((a_2,b_2))$ and $b_4 \neq b_1$. Hence, we have

$$u \in N(a_1,a_2,b_1) \cap N(a_2,b_1,b_2) \cap N(a_1,a_3,b_1) \cap N(a_2,b_4,b_2).$$

The path $a_3a_1b_1ua_2b_2b_4$ results in a bridge with difference 7.

To summarize, we found four bridges with difference 0, 3, 6, 7 respectively, and all contain u. Let V_1' be the set of vertices of these four bridges. Since each bridge is obtained by extending a path with 7 vertices and both end triples $(\epsilon_1,\epsilon_2,\epsilon_3)$-typical by the application of Claim 2.11, it has at most $7 + 2 \cdot (12-3) = 25$ vertices. Then $|V_1'| \leq 4 \cdot 25 = 100$. Repeat the same argument for v, we find four bridges with difference 0, 3, 6, 7 respectively, and all are disjoint from V_1' and contain v. We complete the proof of this claim.

Now we find a good set X. Let V_2' be the set of vertices of such four bridges containing v. We can choose one bridge M_1 containing u and one bridge M_2 containing v to make $\frac{3n_1'-n_2'+6}{8}$ an integer where $n_1' = |A \backslash V(M_1 \cup M_2)|$ and $n_2' = |B \backslash V(M_1 \cup M_2)|$, since

$$0+0 \equiv 0 (\text{mod } 8);\ 3+6 \equiv 1 (\text{mod } 8);\ 3+7 \equiv 2 (\text{mod } 8);\ 0+3 \equiv 3 (\text{mod } 8);$$
$$6+6 \equiv 4 (\text{mod } 8);\ 6+7 \equiv 5 (\text{mod } 8);\ 0+6 \equiv 6 (\text{mod } 8);\ 0+7 \equiv 7 (\text{mod } 8).$$

Therefore, $X = V_1' \cup V_2'$ is a good set in $\mathcal{H}$. □

5 Concluding Remarks

For the case when $|V(\mathcal{H})| = 2n+1$, choose a partition A, B of $V(\mathcal{H})$ with $|A| = n+1$ and $|B| = n$ and, subject to this, $|\mathcal{H}(A,A,B,B)|$ is minimal. The proof is almost exactly the same as that of Theorems 3.1 and 4.1, since one extra vertex almost does not make any difference here. Theorem 1.3 will become the following

Theorem 5.1 *There exists $\epsilon_0 > 0$ such that, for sufficiently large n and any 4-graph $\mathcal{H}$ on $2n+1$ vertices with $b(\mathcal{H}) < \epsilon_0 n^4$, the following hold.*

(i) *If $\delta_3(\mathcal{H}) \geq n-1$, then $\mathcal{H}$ has a Hamiltonian path;*
(ii) *If $\delta_3(\mathcal{H}) \geq n$, then $\mathcal{H}$ has a Hamiltonian cycle.*

Thus, if a 4-graph $\mathcal{H}$ with n vertices is close to extremal graph $\mathcal{H}_0$ and its minimum co-degree is at least $\lfloor \frac{n-1}{2} \rfloor$, then $\mathcal{H}$ must contain a Hamiltonian cycle. It remains to consider the other case, that is, when $\mathcal{H}$ is far from $\mathcal{H}_0$.

Conjecture 5.2 *For all $c > 0$ there exists $c_1 > 0$ such that, for sufficiently large n and a 4-graph on n vertices, if $b(\mathcal{H}) \geq cn^4$ and $\delta_3(\mathcal{H}) \geq (1-c_1)\frac{n}{2}$ then $\mathcal{H}$ has a Hamiltonian cycle.*

Conjecture 5.2 is equivalent to Conjecture 1.1 for $k=4$. It is likely that this case requires the use of absorption technique that Rödl, Ruciński and Szemerédi [15] used to prove the case of 3-graphs.

On the other hand, using the tools in this paper, one might ask if Theorem 1.3 holds for k-graphs with $k \geq 5$.

Conjecture 5.3 *There exists $\epsilon_0 > 0$ such that, for sufficiently large n and any k-graph $\mathcal{H}$ on n vertices with $b(\mathcal{H}) < \epsilon_0 n^k$ the following holds: If $\delta_{k-1}(\mathcal{H}) \geq \lfloor \frac{n-k+3}{2} \rfloor$, then $\mathcal{H}$ has a Hamiltonian cycle.*

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Declarations

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