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RESEARCH

Modular Products and Modules for Finite Groups

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Abstract
Motivated by the appearance of penumbral moonshine, and by evidence that penumbral moonshine enjoys an extensive relationship to generalized monstrous moonshine via infinite products, we establish a general construction in this work which uses singular theta lifts and a concrete construction at the level of modules for a finite group to translate between moonshine in weight one-half and moonshine in weight zero. This construction serves as a foundation for a companion paper in which we explore the connection between penumbral Thompson moonshine and a special case of generalized monstrous moonshine in detail.

Keywords Moonshine · Singular theta lift · Sporadic group · Borcherds product

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1 Introduction

Penumbral moonshine [15] provides a counterpart to umbral moonshine [9–11] in which mock modularity is replaced by modularity, and Fricke genus zero groups take on the role (see [8]) played by non-Fricke genus zero groups.

Curiously, the finite groups that appear in penumbral moonshine are generally much larger than those that appear in umbral moonshine, and in many cases are identifiable as, or closely related to, centralizer subgroups of the monster. It is natural to ask if this is purely

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coincidental, or a consequence of a concrete connection. We provide evidence in support of the latter alternative in [16], wherein we explore a relationship between penumbral Thompson moonshine (which was first exposited in [24]), and a special case of generalized moonshine for the monster.

The present work provides a foundation for the analysis of [16] by establishing a "lift" of the singular theta lift of [3, 23], from modular forms to modules for a finite group. More concretely, the construction we present here associates a (virtual) graded G-module for which the McKay–Thompson series arising are weakly holomorphic modular forms of weight 0, to any (virtual) graded G-module for which the McKay–Thompson series arising are weakly holomorphic vector-valued modular forms of weight $\frac{1}{2}$ of a suitable type, for any finite group G. Moreover, the McKay–Thompson series of weight 0 that arise from this construction are defined by infinite products.

1.1 Products

The problem of obtaining automorphic forms as infinite products has received considerable attention. A famous example, which first appeared in [2] (see Example 2 in Section 13 of op. cit.), is the infinite product formula for the j-function,

$$j(\tau) = q^{-1} \prod_{n>0} (1 - q^n)^{3c_0(n^2)}. \tag{1.1.1}$$

Here $q = \mathrm{e}(\tau)$, where $\mathrm{e}(x) := e^{2\pi i x}$, and the product in (1.1.1) is constructed from the Fourier coefficients $c_0(D)$ of the unique weakly holomorphic modular form

$$f_0(\tau) = \sum_{D \geq -3} c_0(D) q^D = q^{-3} - 248q + 26752q^4 - 85995q^5 + \dots \tag{1.1.2}$$

of weight $\frac{1}{2}$ for $\Gamma_0(4)$ such that $c_0(D) = 0$ unless $D \equiv 0, 1 \bmod 4$, and such that $f_0(\tau) = q^{-3} + O(q)$ as $\Im(\tau) \to \infty$.

Building on work of Borcherds [2, 3], Zagier [30], and then Bruinier and Ono [4], established a variety of generalizations of this formula. In particular, they studied "twisted" Borcherds products (see Section 1.4), which, among other things, realize particular rational functions of the elliptic modular invariant (1.1.1) involving its values at CM points (cf. (5.1.6)).

Now the j-function is, up to an additive constant, the McKay–Thompson series

$$T_e(\tau) = j(\tau) - 744 = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \dots \tag{1.1.3}$$

attached to the identity element of the monster group, $\mathbb{M}$, by monstrous moonshine [13]. It develops that similar results involving twisted Borcherds products hold for other functions appearing in this setting. For example, infinite product formulae are obtained in [25] for rational functions of McKay–Thompson series T_g attached to elements of prime order $o(g) = p$ in the monster, for each prime p that divides $|\mathbb{M}|$.

These results on infinite products follow from the application of singular theta lifts—either similar to or the same as those considered in [3, 23]—to weakly holomorphic modular forms of weight $\frac{1}{2}$ that transform with respect to a suitably chosen Weil representation of the metaplectic double cover $\widetilde{SL}_2(\mathbb{Z})$ of the modular group $SL_2(\mathbb{Z})$. In the untwisted case,

the idea is that the infinite product constructed from the Fourier coefficients of a suitable modular form $\check{F}$ is essentially the exponential of a regularized integral

$$\int_{\mathcal{F}}^{\mathrm{reg}} \overline{\Theta_L(\tau, v^+)} \check{F}(\tau) \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2}. \tag{1.1.4}$$

Here $\mathcal{F}$ is the standard fundamental domain for the action of $SL_2(\mathbb{Z})$ on the complex upper half-plane $\mathbb{H}$, we take L to be a lattice that defines the Weil representation in question (see Section 3.1), we write Θ_L for the associated Siegel theta function (see Section 3.2), and $\tau = \tau_1 + i\tau_2$ is the decomposition of τ into its real and imaginary parts. (See Section 4.3 and especially (4.3.1) for more detail on (1.1.4).)

For an example of this framework in action, define $\check{F} = (\check{F}_0, \check{F}_1)$ by requiring that $\check{F}_r$ belongs to $q^{\frac{r^2}{4}}\mathbb{Z}[[q]][q^{-1}]$ and

$$\check{F}_0(4\tau) + \check{F}_1(4\tau) = f_0(\tau), \tag{1.1.5}$$

where f_0 is as in (1.1.2). Then $\check{F}$ transforms under the Weil representation of $\widetilde{SL}_2(\mathbb{Z})$ (see (3.1 .7–3.1 .8)) defined by the lattice $L = \sqrt{2}\mathbb{Z} \oplus \Gamma^{1,1}$ (cf. (3.1 .6)), and for this choice of L we may naturally regard the variable v^+ in $\Theta_L(\tau, v^+)$ as an element of the upper half-plane (cf. (4.3.11–4.3.12)). The integral Eq. 1.1.4 then evaluates to $-4\log|j(v^+)|$ (cf. (4.3.23)), and the method of [3, 23] produces the product formula (1.1.1) for j (cf. (4.3.24)).

1.2 Moonshine

In [24] it was observed that the cube root of (1.1.1) relates the graded dimensions of two graded modules for the sporadic simple group, *Th*, of Thompson [28, 29]. Indeed, $T_{3\mathrm{C}}(\tau) = j(3\tau)^{\frac{1}{3}}$ is the McKay–Thompson series associated to an element of the class 3C of the monster, so according to Borcherds' proof [1] of the monstrous moonshine conjecture, $j(3\tau)^{\frac{1}{3}}$ coincides with the trace of an element of the class 3C on the moonshine module $V^\natural$ of Frenkel, Lepowsky and Meurman [18–20]. The centralizer of such an element takes the form $C_{\mathbb{M}}(3\mathrm{C}) \cong \mathbb{Z}/3\mathbb{Z} \times Th$, and the group *Th* admits no non-trivial central extensions (see e.g. [12]). Applying the yoga of generalized monstrous moonshine (see e.g. [6, 7]) to these facts we obtain an action of *Th* on the 3C-twisted moonshine module $V^\natural_{3\mathrm{C}}$, whose graded dimension is given by

$$j(\tfrac{1}{3}\tau)^{\frac{1}{3}} = q^{-\frac{1}{9}} \prod_{n>0} (1 - q^{\frac{n}{3}})^{c_0(n^2)}. \tag{1.2.1}$$

The second graded *Th*-module is the one introduced in [24], which we here denote by

$$\check{W}^{(-3,1)} = \bigoplus_{D \geq -3} \check{W}_D^{(-3,1)}, \tag{1.2.2}$$

following the conventions of [15, 16]. Its (signed) graded dimension is given by

$$\check{F}^{(-3,1)}(\tau) = 2f_0(\tau) + 248\theta(\tau) = 2q^{-3} + 248 + 54000q^4 - 171990q^5 + \ldots \tag{1.2.3}$$

where f_0 is as in (1.1.2) and $\theta(\tau) := \sum_n q^{n^2}$. Significantly, the Thompson group admits a unique irreducible representation of dimension 248, which we henceforth denote **248**. Thus, at the level of Fourier coefficients we have

$$2c_0(n^2) = \mathrm{tr}(e|\check{W}_{n^2}^{(-3,1)}) - 2\mathrm{tr}(e|\mathbf{248}) \tag{1.2.4}$$

for e the identity element of Th (where we regard each $W_D^{(-3,1)}$ as a virtual Th-module, in order to handle the signs in (1.2.3)).

Comparing (1.2.1) with (1.2.4) we are motivated to ask if there is a deeper relationship between the two Th-modules $V_{3C}^\natural$ and $\check{W}^{(-3,1)}$. For example, if we twine (1.2.4) by replacing each e on the right-hand side with an arbitrary Thompson group element g, does the singular theta lift (1.1.4) of the corresponding twine of $\check{F}$ as in (1.1.5) have a meaningful relationship to the twine of the square of (1.2.1) defined by the action of g on the 3C-generalized moonshine module $V_{3C}^\natural$? This is a question we investigate in detail in [16]. One of the problems we face in determining an answer is that of appropriately defining "the corresponding twine of $\check{F}$," because if we naively replace f_0 in (1.1.5) with $\frac{1}{2}\check{F}_g^{(-3,1)} - \frac{1}{2}\mathrm{tr}(g|\mathbf{248})\theta$ (cf. (1.2.3–1.2.4)), where

$$\check{F}_g^{(-3,1)}(\tau) = \sum_{D\geq -3} \mathrm{tr}(g|\check{W}_D^{(-3,1)})q^D, \tag{1.2.5}$$

then we obtain a modular form that fails to transform appropriately under the full group $\widetilde{SL}_2(\mathbb{Z})$, and in particular fails to satisfy the hypotheses of [3].

It turns out that Borcherds himself solved an essentially similar problem earlier, in proving [1] the monstrous moonshine conjecture. In that work a key role is played by the monster Lie algebra, $\mathfrak{m}$, which is constructed in a functorial way from $V^\natural$, and has

$$\sum_{m\geq -1} c_e(m)p^m - \sum_{n\geq -1} c_e(n)q^n = p^{-1}\prod_{m>0,n\geq -1}(1-p^mq^n)^{c_e(mn)} \tag{1.2.6}$$

for its denominator formula, where $c_e(n)$ is the coefficient of q^n in the Fourier expansion of T_e (so that the left-hand side of (1.2.6) coincides with $T_e(\sigma) - T_e(\tau)$ according to (1.1.3), when $p = \mathrm{e}(\sigma)$). The identification of a McKay–Thompson series T_g, as specified by Conway and Norton [13], as the graded trace function associated to the action of $g \in \mathbb{M}$ on $V^\natural = \bigoplus_{n\geq -1} V_n^\natural$ depends upon a corresponding twine of the identity (1.2.6), whereby we identify $c_e(n) = \dim(V_n^\natural) = \mathrm{tr}(e|V_n^\natural)$, and then replace each e on the left-hand side of (1.2.6) with g. From the method of [1] it emerges that the correct generalization of (1.2.6) is

$$\sum_{m\geq -1}\mathrm{tr}(g|V_m^\natural)p^m - \sum_{n\geq -1}\mathrm{tr}(g|V_n^\natural)q^n = p^{-1}\exp\left(-\sum_{m>0}\sum_{n\geq -1}\sum_{k>0}\mathrm{tr}(g^k|V_{mn}^\natural)\frac{p^{mk}q^{nk}}{k}\right) \tag{1.2.7}$$

(see (8.3) of op. cit.). We learn from this, in particular, that the g-twine of the infinite product in (1.2.6) should involve all the powers of g.

In light of the subsequent work [3, 23] we may interpret (1.2.6) as the product formula for an $O_{2,2}$-type automorphic form resulting from an application of the singular theta lift (1.1.4) with $L = \Gamma^{1,1} \oplus \Gamma^{1,1}$ (cf. (3.1 .6)) and $\check{F} = T_e$, whereas the examples of relevance to us, such as that of (1.1.5), produce automorphic forms of type $O_{2,1}$. The interpretation of (1.2.7) as an $O_{2,2}$-type singular theta lift is included in Carnahan's work on generalized monstrous moonshine (see in particular [6]), which includes a general method for "repackaging" a family $\{T_{g^k}\}$ of modular forms of weight 0 into a vector-valued modular form T_g for $\widetilde{SL}_2(\mathbb{Z})$, for a suitably chosen Weil representation. Our solution to the problem of twining the product side of (1.2.1) will involve a directly similar strategy (see Section 4.2), but for singular theta lifts of type $O_{2,1}$ rather than $O_{2,2}$.

1.3 Modules

Beyond the technical tool of repackaging, a conceptual point that is illuminated by reflection upon Borcherds work [1] on monstrous moonshine is the analogy between $F^{(-3,1)}(\tau)$ and $j(\tau)^{\frac{2}{3}}$ on the one hand (cf. (1.2.1–1.2.4)), and $T_e(\tau)$ and $T_e(\sigma) - T_e(\tau)$ on the other (cf. (1.1.3), (1.2.6)). At the level of modules for finite groups this suggests the possibility of a relationship between $\check{W}^{(-3,1)}$ and $(V_{3C}^\natural)^{\otimes 2}$ that is analogous to that which relates $V^\natural$ to $\mathfrak{m}$. Is there an analogue of the functorial construction due to Borcherds, that produces a *Th*-invariant Lie algebra from $\check{W}^{(-3,1)}$, with root multiplicities given by the values $2c_0(n^2)$ (cf. (1.2.4))?

We face an even bigger challenge in answering this question, because the construction of $\mathfrak{m}$ due to Borcherds depends crucially upon the $\mathbb{M}$-invariant vertex operator algebra (VOA) structure on $V^\natural$, and there is—so far—no known counterpart for $\check{W}^{(-3,1)}$. For this reason we do not know yet how to proceed, if the goal is a Lie algebraic counterpart to $\mathfrak{m}$ in the context of $\check{W}^{(-3,1)}$ and $(V_{3C}^\natural)^{\otimes 2}$.

However, we do have *Th*-module structures on $\check{W}^{(-3,1)}$ and $(V_{3C}^\natural)^{\otimes 2}$, and we may ask if these structures are related, or not, by a "lift" of the singular theta lift (1.1.4) that operates at the level of G-modules, for G a finite group, rather than at the level of functions.

The construction of such a lift of the singular theta lift, in the $O_{2,1}$ setting, is the purpose and product of this work. More specifically, for G a finite group and m a non-zero integer we formulate a notion of rational weakly holomorphic G-module of weight $\frac{1}{2}$ and index m, and a notion of weakly holomorphic G-module of weight 0 (see Section 4.1), and we establish a general construction

$$W \mapsto SQ(W) \tag{1.3.1}$$

(see Theorem 4.1.3 and (4.1.10)) that produces a weakly holomorphic G-module $SQ(W)$ of weight 0 from an arbitrary rational weakly holomorphic G-module W of weight $\frac{1}{2}$ with positive index. The construction (1.3.1) depends upon the singular theta lift (1.1.4) of [3, 23], and also on $O_{2,1}$-type repackaging, and produces infinite product formulae for the McKay–Thompson series associated to the output G-module $SQ(W)$.

The cornerstone of the construction (1.3.1) is a twined counterpart to the $O_{2,1}$-type Borcherds product of (1.1.1) and (1.2.1). In the case of a rational weakly holomorphic G-module

$$W = \bigoplus_{r \bmod 2m} \bigoplus_{D \equiv r^2 \bmod 4m} W_{r,\frac{D}{4m}} \tag{1.3.2}$$

of weight $\frac{1}{2}$ and positive integer index m (cf. (4.1.1)), it may be expressed in the form

$$\Psi_g^W(\tau) := q^{-H} \exp\left(- \sum_{n>0} \sum_{k>0} C_{g^k}^W(n^2, n) \frac{q^{nk}}{k} \right), \tag{1.3.3}$$

(cf. (1.2.7)), for $g \in G$, where $H = H^W$ is a generalized class number associated to W (see (4.4.10)), and $C_g^W(D, r) := \mathrm{tr}(g | W_{r,\frac{D}{4m}})$.

A priori it is not clear that the prescription (1.3.3) makes sense, but we confirm (see Section 4.4) that the right-hand side of (1.3.3) converges for $\Im(\tau)$ sufficiently large, and extends by analytic continuation to a holomorphic function on the upper half-plane.

Note that we obtain a rational weakly holomorphic G-module $W^{(-3,1)}$ of weight $\frac{1}{2}$ and index 1, for $G = Th$, by setting

$$W^{(-3,1)}_{r,\frac{D}{4}} := \breve{W}^{(-3,1)}_{D} \tag{1.3.4}$$

(cf. (1.3.2)) for $\breve{W}^{(-3,1)}$ as in (1.2.2), where r on the left-hand side of (1.3.4) is 0 or 1 according as D is even or odd.

We denote our general construction (1.3.1) by SQ in homage to the functorial construction of Borcherds [1], that produces the monster Lie algebra $\mathfrak{m}$ from the moonshine module $V^\natural$, and is sometimes referred to as a second-quantization functor, on account of its interpretation in physics. We do not employ the formalities of categories and functors in this work, but we do provide an explicit realization of $SQ(W)$ in terms of the alternating and symmetric powers of homogeneous subspaces of W (see Section 4.5), and it follows from this that our construction is functorial at the level of G-modules.

1.4 Twists

We return now to the twisted Borcherds products of Zagier [30] and Bruinier and Ono [4] that we mentioned in Section 1.1. Inspired by their results, and by the problem of inverting our construction SQ (see Section 5.1), we suggest the consideration of twined twisted $O_{2,1}$-type Borcherds products for future work. In the case that W is a rational weakly holomorphic G-module of weight $\frac{1}{2}$ and index m for m a positive integer (cf. (1.3.2), (4.1.1)), the construction we propose takes the form

$$\Psi^{W}_{D_1,r_1,g}(\tau) := \exp\left(-\sum_{n>0}\sum_{k>0}\sum_{a \bmod D_1} C^{W}_{g^k}(D_1n^2, r_1n)\left(\frac{D_1}{a}\right)\mathrm{e}\left(\frac{ak}{D_1}\right)\frac{q^{nk}}{k}\right), \tag{1.4.1}$$

for $D_1 > 1$ a fundamental discriminant, r_1 an integer such that $D_1 = r_1^2 \bmod 4m$, and $g \in G$, where $C^{W}_{g}(D, r)$ is as in (1.3.3), and $\left(\frac{D_1}{a}\right)$ is the Kronecker symbol (see e.g. [22] for the definition). The results of [4] cover the case that g in Eq. 1.4.1 is the identity element, whereby the definition (1.4.1) we propose reduces to

$$\Psi^{W}_{D_1,r_1}(\tau) := \prod_{n>0}\prod_{a \bmod D_1}\left(1-\mathrm{e}\left(\frac{a}{D_1}\right)q^n\right)^{\left(\frac{D_1}{a}\right)C^{W}(D_1n^2,r_1n)}, \tag{1.4.2}$$

where $\Psi^{W}_{D_1,r_1} := \Psi^{W}_{D_1,r_1,e}$, and $C^{W}(D, r) := C^{W}_{e}(D, r)$ is the dimension of $W_{r,\frac{D}{4m}}$. In particular, it is shown in op. cit. that the right hand-side of (1.4.2) converges for $\Im(\tau)$ sufficiently large, and extends by analytic continuation to a meromorphic function on $\mathbb{H}$ with an explicitly determined divisor (cf. (5.1.2)).

The automorphic properties of $\Psi^{W}_{D_1,r_1}$ are established in op. cit. via an analysis of twisted Siegel theta functions $\Theta^{(m)}_{D_1,r_1}$ based on the lattices $L_m = \sqrt{2m}\mathbb{Z} \oplus \Gamma^{1,1}$ (cf. (3.1 .6)). Briefly, $\Theta^{(m)}_{D_1,r_1}$ is defined similarly to the theta function we denote $\Theta^{(m)}$ in Section 3.2, but with the generalized genus character $\chi^{(m)}_{D_1}$ (see (3.3.3)) twisting the contribution from each vector of L_m. We expect that the main step towards understanding the automorphic properties of (1.4.1) will be a counterpart analysis of similar twisted Siegel theta functions $\Theta^{(m)}_{D_1,r_1,N}$, based on the more general family $L_{m,N} = \sqrt{2m}\mathbb{Z} \oplus \Gamma^{1,1}(N)$ (see (3.1 .6)).

A main motivation for the construction (1.4.1) is the fact that the untwisted products Ψ_g^W of (1.3.3) only "see" the (virtual) G-modules $W_{r,\frac{D}{4m}}$ for which $D = n^2$ is a perfect square, whereas the twined twisted Borcherds products $\Psi^W_{D_1,r_1,g}$ of (1.4.1) incorporate all the remaining components of W. Thus it is natural to expect that twined twisted Borcherds products will play an important role in the final comparative analysis of moonshine in weight one-half and weight zero.

In fact, untwined twisted Borcherds products very similar to the $\Psi^W_{D_1,r_1}$ of (1.4.2) have already appeared in such analyses. The first examples of this were given in [26], wherein twisted Borcherds products were used to relate certain cases of umbral moonshine to conjugacy classes in the monster (see Theorem 1.1 of op. cit.). Twisted borcherds products were subsequently used to constructively relate all cases of umbral moonshine to principal moduli for genus zero groups in [8] (see Theorem 4.5.4 of op. cit.). We discuss this more in Section 5.2.

As we also discuss in Section 5.2, it develops that the untwisted Borcherds product construction, (1.3.3), cannot be applied to umbral moonshine. Thus, not only are we motivated to develop twined twisted Borcherds products as in (1.4.1) for the purpose of better understanding penumbral moonshine, and its potential monstrous aspect, but also in order to shine more light on umbral moonshine, since the lift of the untwisted Siegel theta lift that we develop in this work does not apply there. In particular, we are motivated to ask: Can the twisted Siegel theta lift,

$$\int_{\mathcal{F}}^{\mathrm{reg}} \overline{\Theta^{(m)}_{D_1,r_1}(\tau, v^+)}\check{F}(\tau)\frac{\mathrm{d}\tau_1\mathrm{d}\tau_2}{\tau_2}, \tag{1.4.3}$$

that is studied in [4], be lifted to the level of G-modules too? We offer this question as a focus for future work.

1.5 Overview

The structure of this paper is as follows. In Section 2 we give a guide to the specialized notation we use. In Section 3 we review some preliminary notions, including the Weil representations of the metaplectic double cover of the modular group that are defined by even lattices in Section 3.1, some relevant families of modular forms in Section 3.2, and the twisted Heegner divisors of [4] in Section 3.3. We recall Adams operations on modules for finite groups in Section 3.4.

The new results appear in Section 4, wherein we explain and establish our general construction *SQ*, that utilizes infinite products to translate from moonshine in weight one-half to moonshine in weight zero. We set up for and define the construction *SQ* abstractly in Section 4.1 (see Theorem 4.1.3). Then, in Section 4.2, we prove two technical lemmas (Lemmas 4.2.1 and 4.2.5). The first of these generalizes work of Carnahan [6], and constitutes the repackaging method we discussed in Section 1.2. The second lemma of Section 4.2 is used when we check that this repackaging works in the desired way. We also present some examples of repackaging in Section 4.2.

Our next step is to recall the singular theta lift from [3], but specialized to the situation specified in Section 4.1–4.2. We do this in Section 4.3, and thereby prepare for the arguments of Section 4.4, which prove (see Proposition 4.4.1) that the infinite products we propose in Section 4.1 make sense. In particular, we establish the convergence and modularity of the products of Section 4.1 in Section 4.4. The verification of the validity of the construction *SQ*

is completed in Section 4.5, wherein we offer a representation theoretic interpretation of the infinite products constructed in Section 4.1–4.4.

We put our results in perspective in Section 5. Specifically, we elucidate the relationship between our construction SQ and traces of singular moduli in Section 5.1, and comment on the import of our work for penumbral and umbral moonshine in Section 5.2.

2 Notation

$\left(\frac{\cdot}{\cdot}\right)$ The Kronecker symbol. Cf. (1.4.1).

$[A, B, C]$ Shorthand for the binary quadratic form $Ax^2 + Bxy + Cy^2$. See Section 3.3.

$\mathrm{Aut}(L, \check{F})$ The subgroup of $\mathrm{Aut}(L)$ composed of automorphisms that fix $\check{F}$. Cf. (3.1 .5).

α_Q The CM point in $\mathbb{H}$ associated to a binary quadratic form Q. Cf. (3.3.5).

$c_0(D)$ A Fourier coefficient of f_0. See (1.1.2).

C The positive cone in $K \otimes \mathbb{R}$. See (4.3.10).

$\check{C}_{i,j}(D, r)$ A Fourier coefficient of $\check{F}$. Cf. (4.3.22) and (4.3.24).

$\hat{C}_{(i,j)}(D, r)$ A Fourier coefficient of $\hat{F}_{(i,j)}$. Cf. the proof of Theorem 4.3.1.

$\check{C}_K(D, r)$ A Fourier coefficient of $\check{F}_K$. See (4.3.16).

$C^W(D, r)$ Shorthand for $C_e^W(D, r)$. See (1.4.2).

$C_g^W(D, r)$ A Fourier coefficient of $F_{g,r}^W$. See (4.1.4).

δ An element of L^* for some lattice L, or a coset of L in L^*. Cf. (3.1 .5), (3.2.3) and (4.4.7).

$\mathrm{e}(\,\cdot\,)$ We set $\mathrm{e}(x) := e^{2\pi i x}$. Cf. (1.1.1).

η_g^W An eta product defined by the action of g on $W_{0,0}$, for W as in (4.1.1). See (4.1.8).

f_0 A certain weakly holomorphic modular form of weight $\frac{1}{2}$ for $\Gamma_0(4)$. See (1.1.2).

f_g^V A McKay–Thompson series for a virtual graded G-module V as in (4.1.5). See (4.1.6).

F A certain linear operator in Section 4.5. See (4.5.4).

$\check{F}$ A weakly holomorphic vector-valued modular form of a certain type. See Lemma 4.2.1.

$\check{F}_K$ A weakly holomorphic modular form of weight $\frac{1}{2}$ and type ϱ_K. See (4.3.14).

$\check{F}_{(i,j)}$ The vector-valued function with components $\check{F}_{(i,j),r} := \check{F}_{i,j,r}$. Cf. (4.2.3).

$\hat{F}_{(i,j)}$ A function that plays a technical role in Section 4.2. See Lemma 4.2.1.

$\check{F}_{i,j,r}$ A component function of $\check{F}$. Cf. (4.2.3).

$\hat{F}_{(i,j),r}$ A component function of $\hat{F}_{(i,j)}$. See Lemma 4.2.1.

$F^{(n)}$ A weakly holomorphic modular form of weight $\frac{1}{2}$ and type $\varrho_m|_{\widetilde{\Gamma}_1(N/n)}$. See Lemma 4.2.1.

F_g^W A McKay–Thompson series for a graded G-module W as in (4.1.1). See (4.1.2).

$\check{F}_g^W$ A repackaged McKay–Thompson series. See (4.2.17).

$F_{g,r}^W$ A component function of F_g^W. See (4.1.2).

$\check{F}_g^{(-3,1)}$ A McKay–Thompson series of penumbral Thompson moonshine. See (1.2.5).

$\mathcal{F}$ A fundamental domain for the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$. Cf. (1.1.4), (4.3.1).

$\Phi_L(v^+, \check{F})$ The singular theta lift of a function $\check{F}$. See (4.3.1).

$G(L)$ The Grassmannian of positive definite subspaces of $L \otimes \mathbb{R}$. See Section 3.2, before (3.2.3).

(γ, u) An element of $\widetilde{SL}_2(\mathbb{R})$. Cf. (3.1 .1–3.1 .2).

$\Gamma^{1,1}$ The unique unimodular even lattice of signature (1, 1). Cf. (3.1 .6).

$\widetilde{\Gamma}$ The preimage of $\Gamma < SL_2(\mathbb{R})$ in $\widetilde{SL}_2(\mathbb{R})$ under the natural map $\widetilde{SL}_2(\mathbb{R}) \to SL_2(\mathbb{R})$. See Section 3.1.

$\widetilde{\Gamma}_0(N)$ The preimage of $\Gamma_0(N)$ in $\widetilde{SL}_2(\mathbb{Z})$ under the natural map $\widetilde{SL}_2(\mathbb{Z}) \to SL_2(\mathbb{Z})$. See Section 3.1.

$\widetilde{\Gamma}_1(N)$ The preimage of $\Gamma_1(N)$ in $\widetilde{SL}_2(\mathbb{Z})$ under the natural map $\widetilde{SL}_2(\mathbb{Z}) \to SL_2(\mathbb{Z})$. See Section 3.1.

$\overline{\Gamma_0(m)}_Q$ The stabilizer in $\overline{\Gamma_0(m)} := \Gamma_0(m)/\{\pm \mathrm{Id}\}$ of a binary quadratic form Q. Cf. 3.3.5.

$\Gamma_0(m) + m$ The extension of $\Gamma_0(m)$ generated by its Fricke involution. See (5.1.1).

H A shorthand for H^W. Cf. (4.1.7) and (4.4.10).

H^W The generalized class number associated to W as in (4.1.1). See (4.4.10).

$H_\ell^W(g)$ An alternative notation for $H(K, \check{F}_K)$ in a special case. See (4.3.20).

$H_m(D, r)$ A generalized class number. Cf. (4.3.16).

$H(K, \check{F}_K)$ A generalized class number. See (4.3.18).

$\mathcal{H}$ The tensor product of the $\mathcal{H}(-n)$. Cf. (4.5.6).

$\mathcal{H}_n$ A homogeneous subspace of $\mathcal{H}$. Cf. (4.5.6).

$\mathcal{H}(-n)$ A certain G-module defined in terms of U_n. See (4.5.6).

k_g^W A certain value defined for $g \in G$ and W as in (4.1.1). See (4.3.21) and cf. Theorem 4.3.1.

K A lattice defined by a choice of primitive norm zero vector ℓ in L. See (4.3.9–4.3.10).

κ The generator for K that is contained in its positive cone C. See (4.3.10).

κ' The generator for K^* that is contained in its positive cone C. See (4.3.10).

ℓ A primitive norm zero vector in L in Subsection 4.3–4.4. Cf. (4.3.9).

ℓ' A vector in L^* such that $(\ell, \ell') = 1$, for ℓ as in Subsection 4.3–4.4. Cf. (4.3.9).

L In Subsection 4.3–4.4 we use L to denote a specific realization of $L_{m,N}$. See (4.3.2).

L_m Shorthand for the lattice $L_{m,1}$. See (3.1 .6).

$L_{m,N}$ Shorthand for the lattice $\sqrt{2m}\mathbb{Z} \oplus \Gamma^{1,1}(N)$. See (3.1 .6).

$L(N)$ A lattice obtained by rescaling the quadratic form on L. Cf. (3.1 .6).

λ A vector in $L \otimes \mathbb{R}$ for some lattice L. Cf. (3.2.4).

$\lambda^\perp$ A CM point in the upper half-plane. Cf. (4.3.13) and (4.4.6).

$\lambda(c, b, a)$ A vector in the lattice $L = L_{m,N}$ in Subsection 4.3–4.4. See (4.3.2).

$\Lambda_t(U)$ A generating function for exterior powers of a virtual G-module $U \in R(G)$. See (3.4.6).

$m_\chi(U)$ The multiplicity of $\chi \in \mathrm{Irr}(G)$ in a virtual G-module U. See (3.4.4).

N_g A positive integer attached to a group element $g \in G$, in the setup of Section 4. Cf. (4.1.3).

P A certain principal $\mathbb{C}^*$-bundle over $G(L)$ for $L = L_{m,N}$. See Section 4.3, after (4.3.8).

ψ^k An Adams operation on virtual G-modules, for G a finite group. Cf. (3.4.8–3.4.9).

Ψ_g^W A twined $O_{2,1}$ Borcherds product. See (1.3.3), Eq. 4.1.7.

Ψ_{D_1,r_1}^W A shorthand for $\Psi_{D_1,r_1,e}^W$. See (1.4.2).

$\Psi_{D_1,r_1,g}^W$ A twined twisted $O_{2,1}$-type Borcherds product. See (1.4.1).

$\Psi_\ell(\tau, \check{F})$ A shorthand for $\Psi_\ell(Z_L(\tau), \check{F})$ in the special case that $\ell = \lambda(0, 1, 0)$. See (4.4.5).

$\Psi_\ell(Z, \check{F})$ A modular form on $K \otimes \mathbb{R} + iC$. See (4.3.12).

$\Psi_L(Z_L, \check{F})$ An automorphic form produced by a singular theta lift. See Theorem 4.3.1.

Q A binary quadratic form with integer coefficients. Cf. (3.3.5).

Q_L The quadratic form of an even lattice L. See Section 3.1.

$\mathcal{Q}_{D,r}^{(m)}$ A set of binary quadratic forms with integer coefficients. See Section 3.3.

$R(G)$ The Grothendieck group of the category of finitely generated $\mathbb{C}[G]$-modules. See (3.4.1).

$R^+(G)$ The subset of $R(G)$ represented by finite-dimensional $\mathbb{C}[G]$-modules. See (3.4.2).

ϱ_L The Weil representation attached to an even lattice L. See Eq. 3.1 .7.

ϱ_m A shorthand for $\varrho_{m,1}$. Cf. Eq. 3.1 .8.

$\varrho_{m,N}$ A shorthand for $\varrho_{L_{m,N}}$. See Eq. 3.1 .8.

$\rho(K, \check{F}_K)$ A shorthand for $\rho(K, C, \check{F}_K)$. See (4.3.17).

$\rho(K, C, \check{F}_K)$ A Weyl vector. Cf. (4.3.15) and (4.3.17).

$\widetilde{S}$ A specific element of $\widetilde{SL}_2(\mathbb{Z})$. See (3.1 .4).

SQ Our lift of the singular theta lift. See (1.3.1) and (4.1.10).

$S_t(U)$ A generating function for symmetric powers of a virtual G-module $U \in R(G)$. See (3.4.6).

$\widetilde{SL}_2(\mathbb{R})$ The metaplectic double cover of $SL_2(\mathbb{R})$. See (3.1 .1–3.1 .2).

$\widetilde{SL}_2(\mathbb{Z})$ The preimage of $SL_2(\mathbb{Z})$ under the natural projection $\widetilde{SL}_2(\mathbb{R}) \to SL_2(\mathbb{R})$. Cf. (3.1 .4).

$\widetilde{T}$ A specific element of $\widetilde{SL}_2(\mathbb{Z})$. See (3.1 .4).

T_g A McKay–Thompson series of monstrous moonshine. See Section 1.2.

T_g^W A McKay–Thompson series associated to $SQ(W)$. See (4.1.9) and Theorem 4.1.3.

$T^{(m+m)}$ The principal modulus associated to $\Gamma_0(m)+m$ when this group is genus zero. Cf. (5.1.3).

$\mathrm{tr}(g|U)$ The trace of $g \in G$ on some $U \in R(G)$, or on a power series $U \in R(G)[[t]]$. See Section 3.4.

θ_m^0 The vector-valued modular form $\theta_m^0 := (\theta_{m,r}^0)$. Cf. (3.2.6).

$\theta_{m,r}$ An certain 2-variable theta function. Cf. (3.2.9).

$\theta_{m,r}^0$ An index m Thetanullwert. See (3.2.6).

$\theta_{L+\delta}$ A component of the Siegel theta function Θ_L. See (3.2.4).

Θ_L The Siegel theta function attached to a lattice L. See (3.2.3–3.2.4).

$\Theta^{(m)}$ A shorthand for $\Theta_1^{(m)}$. See Section 3.2.

$\Theta_N^{(m)}$ A shorthand for Θ_L in case $L = L_{m,N}$. See Section 3.2.

τ_1 The real part of $\tau \in \mathbb{H}$. Cf. (1.1.4), 1.4.3, (4.3.1) and (4.3.15–4.3.16).

τ_2 The imaginary part of $\tau \in \mathbb{H}$. Cf. (1.1.4), 1.4.3, (4.3.1) and (4.3.15–4.3.16).

$u_d(g)$ A shorthand for $u_d(g|U)$. See the proof of Lemma 4.4.2.

$u_d(g|U)$ A positive integer defined for $U \in R(G)$ when $\mathrm{tr}(g|U)$ is a rational integer. Cf. (3.4.13).

U_n A certain virtual G-module defined in terms of W. See (4.5.1).

U_χ The class of irreducible G-modules that realizes an irreducible character χ of G. Cf. (3.4.1).

U^b, U^f We define operations $U \mapsto U^b$ and $U \mapsto U^f$ for $U \in R(G)$ in Section 3.4. See (3.4.5).

$U(-n)$ A copy of $U \in R(G)$, for n a positive integer. Cf. (4.5.4).
v^+ A positive definite subspace of $L \otimes \mathbb{R}$. Cf. (3.2.3–3.2.4).
$v_b(g)$ A shorthand for $v_b(g|U)$. See Lemma 4.4.2.
$v_b(g|U)$ An integer defined for $U \in R(G)$ when $\mathrm{tr}(g|U)$ is a rational integer. See (3.4.17–3.4.20).
V^W A realization of the virtual graded G-module $SQ(W)$. Cf. (4.1.10) and (4.5.7).
$V^{\mathrm{wh}}_{\frac{1}{2},m}(N)$ A certain space of weakly holomorphic vector-valued modular forms. See Section 3.2.
$\mathbb{V}^{\mathrm{wh}}_{\frac{1}{2},m}(N)$ A certain space of weakly holomorphic vector-valued mock modular forms. See Section 3.2.
W A weakly holomorphic G-module of weight $\frac{1}{2}$ and some index. See (1.3.2) and (4.1.1).
$W_{r,\frac{D}{4m}}$ A homogeneous component of W. See (4.1.1).
$W^{(-3,1)}$ A form of the penumbral Thompson moonshine module. Cf. (1.3.4).
$\check{W}^{(-3,1)}$ A form of the penumbral Thompson moonshine module. See (1.2.2).
$\check{W}^{(-3,1)}_D$ A homogeneous component of $\check{W}^{(-3,1)}$. See (1.2.2).
$W^{(-3,1)}_{r,\frac{D}{4}}$ A homogeneous component of $W^{(-3,1)}$. See (1.3.4).
(X_L, Y_L) An ordered basis for a given $v^+ \in G(L)$, for $L = L_{m,N}$. Cf. (4.3.7–4.3.8).
$\chi^{(m)}_{D_1}$ A generalized genus character. See (3.3.3).
Z An element of $K \otimes \mathbb{R} + iC$. Cf. (4.3.11–4.3.12).
Z_L The image of Z under a particular map from $K \otimes \mathbb{R} + iC$ to P. See (4.3.11) and cf. (4.3.8).
$Z(\tau)$ The image of $\tau \in \mathbb{H}$ under a specific isomorphism of $\mathbb{H}$ with $K \otimes \mathbb{R} + iC$. See (4.4.2).
$Z_L(\tau)$ The image of $Z(\tau)$ under the map $Z \mapsto Z_L$. See (4.4.3).
$Z^W_{D_1,r_1}$ A twisted Heegner divisor associated to W as in (4.1.1). See (5.1.2).
$Z^{(m)}_{D_1,r_1}(D,r)$ A twisted Heegner divisor. See (3.3.5).

3 Preliminaries

In this section we review some preliminary notions in preparation for the arguments of Section 4. We review the Weil representations of the metaplectic double cover of the modular group that are defined by even lattices in Section 3.1, in Section 3.2 we define families of modular forms which arise from these representations, in Section 3.3 we introduce some twisted Heegner divisors, and we finally review Adams operations in Section 3.4.

3.1 Weil Representations

Here we review some facts about Weil representations of the metaplectic group. For this recall that $SL_2(\mathbb{R})$ admits an extension

$$\widetilde{SL}_2(\mathbb{R}) := \left\{(\gamma, u) \mid \gamma \in SL_2(\mathbb{R}),\ u(\tau)^4 \mathrm{d}(\gamma\tau) = \mathrm{d}\tau\right\}, \tag{3.1 .1}$$

where the u in each pair (γ, u) in (3.1 .1) is assumed to be a holomorphic function on the complex upper half-plane, $\mathbb{H}$. Thus, using $\sqrt{\cdot}$ to denote the branch of the square root

function determined by requiring that $\sqrt{e^{2\pi it}} = e^{\pi it}$ for $-\frac{1}{2} < t \leq \frac{1}{2}$, we either have $u(\tau) = \sqrt{c\tau + d}$ or $u(\tau) = -\sqrt{c\tau + d}$ when (c, d) is the lower row of γ.

The multiplication rule in $\widetilde{SL}_2(\mathbb{R})$ is

$$(\gamma_1, u_1)(\gamma_2, u_2) = \big(\gamma_1\gamma_2, (u_1 \circ \gamma_2)u_2\big), \tag{3.1 .2}$$

and we call $\widetilde{SL}_2(\mathbb{R})$ the metaplectic double cover of $SL_2(\mathbb{R})$.

We write $\widetilde{\Gamma}_0(N)$ and $\widetilde{\Gamma}_1(N)$ for the preimages of $\Gamma_0(N)$ and $\Gamma_1(N)$ under the natural map $\widetilde{SL}_2(\mathbb{R}) \to SL_2(\mathbb{R})$, respectively, where

$$\begin{aligned} \Gamma_0(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\}, \\ \Gamma_1(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; a, d \equiv 1 \bmod N,\ c \equiv 0 \bmod N \right\}. \end{aligned} \tag{3.1 .3}$$

We also write $\widetilde{SL}_2(\mathbb{Z})$ for the preimage of the modular group $SL_2(\mathbb{Z}) = \Gamma_0(1) = \Gamma_1(1)$. This group $\widetilde{SL}_2(\mathbb{Z})$ is generated by the elements $\widetilde{S}$ and $\widetilde{T}$, where

$$\widetilde{S} := \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right), \quad \widetilde{T} := \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right). \tag{3.1 .4}$$

More generally, given $\Gamma < SL_2(\mathbb{R})$ we write $\widetilde{\Gamma}$ for its preimage in $\widetilde{SL}_2(\mathbb{R})$.

Next recall that a lattice L is a free, finitely generated Abelian group equipped with a symmetric bilinear form $(\cdot, \cdot) : L \times L \to \mathbb{R}$. We restrict our attention to even lattices (i.e. those which have $(\lambda, \lambda) \in 2\mathbb{Z}$ for each λ in L) for which $(\cdot, \cdot)$ is nondegenerate, and associate to such lattice the integer-valued quadratic form $Q_L(\lambda) := \frac{(\lambda,\lambda)}{2}$. We denote the dual lattice by $L^* := \hom(L, \mathbb{Z})$ and refer to L^*/L as the discriminant group of L, where we have implicitly made use of the inclusion $L \hookrightarrow L^*$ induced by the bilinear form. The automorphism group of L is denoted $\mathrm{Aut}(L)$; it acts on the discriminant group L^*/L, and therefore on $\mathbb{C}[L^*/L]$-valued functions as well, where

$$\mathbb{C}[L^*/L] = \mathrm{Span}_{\mathbb{C}}\{e_\delta \mid \delta \in L^*/L\} \tag{3.1 .5}$$

is the group algebra of L^*/L. For $\check{F}$ a $\mathbb{C}[L^*/L]$-valued function we use $\mathrm{Aut}(L, \check{F})$ to denote the subgroup of $\mathrm{Aut}(L)$ which fixes $\check{F}$.

There is one particular family of lattices that will appear repeatedly in the rest of the paper. To define this family we write $\sqrt{2m}\mathbb{Z}$ for the lattice of rank 1 whose quadratic form $\mathbb{Z} \to \mathbb{Z}$ is $k \mapsto mk^2$, and $\Gamma^{1,1}$ for the unique unimodular even lattice of signature $(1, 1)$, taking the quadratic form $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ to be $(u, v) \mapsto uv$. We obtain a new lattice $L(N)$ from any given lattice L by defining it to have the same underlying $\mathbb{Z}$-module, but a quadratic form which is scaled with respect to that of L, so that $Q_{L(N)}(\lambda) := NQ_L(\lambda)$. From these building blocks we define the family of interest by setting

$$L_{m,N} := \sqrt{2m}\mathbb{Z} \oplus \Gamma^{1,1}(N), \tag{3.1 .6}$$

for arbitrary non-zero integers m and N.

To any even lattice L we can associate a representation ϱ_L of the metaplectic group $\widetilde{SL}_2(\mathbb{Z})$ (3.1 .1–3.1 .2) called the Weil representation associated to L. It acts on the group algebra $\mathbb{C}[L^*/L]$ according to the rules

$$\varrho_L(\widetilde{T})e_\delta = \mathrm{e}\big(Q_L(\delta)\big)e_\delta, \quad \varrho_L(\widetilde{S})e_\delta = \frac{\mathrm{e}(\frac{1}{8}(b^- - b^+))}{\sqrt{|L^*/L|}} \sum_{\delta' \in L^*/L} \mathrm{e}\big(-(\delta, \delta')\big)e_{\delta'} \tag{3.1 .7}$$

(cf. (3.1 .4)), where (b^+, b^-) is the signature of L. For the choice $L = L_{m,N}$ the discriminant group is $L^*_{m,N}/L_{m,N} \cong \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/2m\mathbb{Z}$ and the Weil representation becomes

$$\begin{aligned}\varrho_{m,N}(\widetilde{T})e_{i,j,r} &= \mathrm{e}\left(\frac{r^2}{4m} + \frac{ij}{N}\right) e_{i,j,r},\\ \varrho_{m,N}(\widetilde{S})e_{i,j,r} &= \frac{\mathrm{e}(-\mathrm{sgn}(m)\frac{1}{8})}{\sqrt{|2mN^2|}} \sum_{i'\in\mathbb{Z}/N\mathbb{Z}} \sum_{j'\in\mathbb{Z}/N\mathbb{Z}} \sum_{r'\in\mathbb{Z}/2m\mathbb{Z}} \mathrm{e}\left(-\frac{rr'}{2m} - \frac{(ij'+ji')}{N}\right) e_{i',j',r'}.\end{aligned} \tag{3.1 .8}$$

When $N = 1$ we make the abbreviations $L_m := L_{m,1}$ and $\varrho_m := \varrho_{m,1}$.

We are mostly concerned with the case that m and N are both positive in what follows, so that $L_{m,N}$ has signature $(2, 1)$, but we note here that the definitions (3.1 .6–3.1 .8) of $L_{m,N}$ and $\varrho_{m,N}$ make sense for arbitrary non-zero integer values of m and N. The lattices $L_{m,N}$ and $L_{m,-N}$ are isomorphic, so there is no loss of generality in restricting to positive N, but $L_{-m,N}$ and $L_{m,N}$ are different. Indeed, $\varrho_{-m,N}$ is the dual of the representation $\varrho_{m,N}$. The case that m is negative plays an important role in the discussion of Section 5.2.

3.2 Modular Forms

Here we define some spaces of modular forms. Let $k^+, k^- \in \frac{1}{2}\mathbb{Z}$, let Γ be a subgroup of $SL_2(\mathbb{R})$, let V be a complex vector space, and let $\varrho : \widetilde{\Gamma} \to GL(V)$ be a representation. Then a real analytic function $f : \mathbb{H} \to V$ is called a modular form of weight (k^+, k^-) and type ϱ for $\widetilde{\Gamma}$ if it transforms according to the equation

$$f(\gamma\tau) = u(\tau)^{2k^+} u(\bar{\tau})^{2k^-} \varrho(\gamma, u) f(\tau), \qquad (\gamma, u) \in \widetilde{\Gamma}. \tag{3.2.1}$$

We further call f a weakly holomorphic modular form of weight k^+ and type ϱ for $\widetilde{\Gamma}$ if f is holomorphic and $k^- = 0$, and if for any $(\gamma, u) \in \widetilde{SL}_2(\mathbb{Z})$ there is a $C > 0$ such that

$$u(\tau)^{-2k^+} \alpha\left(\varrho(\gamma, u)^{-1} f(\gamma\tau)\right) = O(e^{C\Im(\tau)}) \tag{3.2.2}$$

as $\Im(\tau) \to \infty$, for any linear functional $\alpha : V \to \mathbb{C}$. A holomorphic modular form of weight k^+ and type ϱ for $\widetilde{\Gamma}$ satisfies the same conditions but with the exponential growth condition (3.2.2) replaced by boundedness.

If k^+ and k^- are both integers then the transformation rule (3.2.1) is independent of the choice of u in $(\gamma, u) \in \widetilde{\Gamma}$, and so we may unambiguously speak of modular forms for $\Gamma < SL_2(\mathbb{R})$ in this case. If ϱ is trivial and Γ contains $-\mathrm{Id}$ then k^+ and k^- must both be even integers in order for non-trivial modular forms of weight (k^+, k^-) and type ϱ for Γ to exist. By a weakly holomorphic modular form of weight k^+ for Γ we mean a weakly holomorphic modular form of weight k^+ and type ϱ for Γ, where ϱ is the trivial representation.

In this work we are primarily interested in the case that ϱ is trivial, or the Weil representation associated to an even lattice L. A key example of a modular form for such a representation $\varrho = \varrho_L$ is the Siegel theta function attached to L. To define this let (b^+, b^-) be the signature of L and let $G(L)$ be the Grassmannian of positive definite subspaces v^+ of $L \otimes \mathbb{R}$. Letting λ_+ and λ_- be the projection of $\lambda \in L \otimes \mathbb{R}$ onto v^+ and its (negative definite) orthogonal complement, respectively, we define the Siegel theta function

$$\Theta_L(\tau, v^+) := \sum_{\delta \in L^*/L} e_\delta \theta_{L+\delta}(\tau, v^+), \tag{3.2.3}$$

with components given by

$$\theta_{L+\delta}(\tau, v^+) := \sum_{\lambda \in L+\delta} \mathrm{e}\left(\tau Q_L(\lambda_+) + \bar{\tau} Q_L(\lambda_-)\right). \tag{3.2.4}$$

(In (3.2.3–3.2.4) we abuse notation by using $L+\delta$ to denote the coset of L in L^* specified by $\delta \in L^*/L$. No confusion should arise from this.) Then from Theorem 4.1 of [3], for example, we have that

$$\Theta_L\left(\gamma\tau, v^+\right) = u(\tau)^{b^+} u(\bar{\tau})^{b^-} \varrho_L(\gamma, u)\, \Theta_L(\tau, v^+) \tag{3.2.5}$$

for each $(\gamma, u) \in \widetilde{SL}_2(\mathbb{Z})$. Thus the construction (3.2.3–3.2.4) furnishes an example of a modular form of weight $(\frac{b^+}{2}, \frac{b^-}{2})$ and type ϱ_L for the metaplectic double cover $\widetilde{SL}_2(\mathbb{Z})$ of the modular group. We also have the feature that $\Theta_L(\tau, v^+)$ is invariant with respect to the action of $\mathrm{Aut}(L)$ on the argument v^+.

In this work the Siegel theta functions associated to the lattices $L_{m,N}$ of (3.1 .6) play an important role. We write $\Theta_N^{(m)}$ as a shorthand for Θ_L in case $L = L_{m,N}$, and write $\Theta^{(m)}$ as a shorthand for $\Theta_1^{(m)}$.

When $b^- = 0$ the Siegel theta function (3.2.3) specializes to a holomorphic modular form of weight $\frac{b^+}{2}$ and type ϱ_L. For example, in the case that $L = \sqrt{2m}\mathbb{Z}$, the Grassmannian $G(L)$ is a point, $v^+ = \sqrt{2m}\mathbb{Z} \otimes \mathbb{R}$, and the components $\theta_{\sqrt{2m}\mathbb{Z}+\frac{r}{\sqrt{2m}}}$ of the Siegel theta function $\Theta_{\sqrt{2m}\mathbb{Z}}$ reduce to the Thetanullwerte

$$\theta^0_{m,r}(\tau) := \theta_{\sqrt{2m}\mathbb{Z}+\frac{r}{\sqrt{2m}}}(\tau) = \sum_{s \equiv r \bmod 2m} q^{\frac{s^2}{4m}}. \tag{3.2.6}$$

Here we follow [3] in dropping the variable v^+ from notation when $G(L)$ is a point. In the remainder we write $\theta^0_m = (\theta^0_{m,r})$ as a shorthand for $\Theta_{\sqrt{2m}\mathbb{Z}}$.

Recall from Section 3.1 that ϱ_m denotes the Weil representation defined by $L_m = L_{m,1}$ (cf. (3.1 .8)). We write $V^{\mathrm{wh}}_{\frac{1}{2},m}(N)$ for the space of weakly holomorphic modular forms of weight $\frac{1}{2}$ and type ϱ_m for $\widetilde{\Gamma}_0(N)$ (see (3.2.1–3.2.2)). Thus, for m positive, an element $F = (F_r)$ of $V^{\mathrm{wh}}_{\frac{1}{2},m}(N)$ is a vector-valued function whose components F_r, indexed by integers r modulo $2m$, admit Fourier expansions of the form

$$F_r(\tau) = \sum_{D \equiv r^2 \bmod 4m} C_F(D, r) q^{\frac{D}{4m}}, \tag{3.2.7}$$

with $C_F(D, r) = 0$ for $D \ll 0$. On the other hand, if m is negative then $F \in V^{\mathrm{wh}}_{\frac{1}{2},m}(N)$ still has components F_r indexed by r mod $2m$, and the Fourier expansion is still of the form (3.2.7), but with m negative the condition that F be weakly holomorphic (see (3.2.2)) requires that $C_F(D, r) = 0$ for $D \gg 0$.

We mention another simple but significant difference between the cases that m is positive and negative in $V^{\mathrm{wh}}_{\frac{1}{2},m}(N)$. Namely, using two applications of the second line of (3.1 .8) to compute (3.2.1) for $(k^+, k^-) = (\frac{1}{2}, 0)$ and $\varrho = \varrho_m$ with $(\gamma, u) = (-I, \sqrt{-1})$, we find that for $F = (F_r)$ in $V^{\mathrm{wh}}_{\frac{1}{2},m}(N)$ we have

$$F_{-r} = \begin{cases} F_r & \text{if } m > 0, \\ -F_r & \text{if } m < 0, \end{cases} \tag{3.2.8}$$

for all $r \bmod 2m$. Thus, for example, $V^{\mathrm{wh}}_{\frac{1}{2},-1}(N)$ vanishes for all N.

To help put the $V^{\mathrm{wh}}_{\frac{1}{2},m}(N)$ in perspective we point out that for m positive $V^{\mathrm{wh}}_{\frac{1}{2},-m}(N)$ is naturally isomorphic to the space of weakly holomorphic Jacobi forms of weight 1 and index m for $\Gamma_0(N)$, and $V^{\mathrm{wh}}_{\frac{1}{2},m}(N)$ is naturally isomorphic to the complex conjugate of the space of weakly skew-holomorphic Jacobi forms of weight 1 and index m for $\Gamma_0(N)$. Indeed, for $m > 0$ we obtain the weakly holomorphic Jacobi form ϕ corresponding to $F = (F_r) \in V^{\mathrm{wh}}_{\frac{1}{2},-m}(N)$ by setting

$$\phi(\tau, z) := \sum_{r \bmod 2m} F_r(\tau)\theta_{m,r}(\tau, z), \tag{3.2.9}$$

where $\theta_{m,r}(\tau, z) := \sum_{s \equiv r \bmod 2m} \mathrm{e}(sz)q^{\frac{s^2}{4m}}$ (cf. (3.2.6)), and the weakly skew-holomorphic Jacobi form φ corresponding to a given $F = (F_r) \in V^{\mathrm{wh}}_{\frac{1}{2},m}(N)$ is

$$\varphi(\tau, z) := \sum_{r \bmod 2m} \overline{F_r(\tau)}\theta_{m,r}(\tau, z). \tag{3.2.10}$$

We refer to Section 3.1 of [8] and Section 3.3 of [15] for concise introductory accounts of holomorphic and skew-holomorphic Jacobi forms, formulated with moonshine in mind.

To conclude this section we note that everything we have said about the spaces $V^{\mathrm{wh}}_{\frac{1}{2},m}(N)$ applies equally well when we replace modularity with mock modularity in the definition. In particular, the identity (3.2.8) holds when F belongs to the space $\mathbb{V}^{\mathrm{wh}}_{\frac{1}{2},m}(N)$ of mock modular forms of weight $\frac{1}{2}$ and type ϱ_m for $\widetilde{\Gamma}_0(N)$, and such an F is naturally identified with a mock Jacobi form of weight 1 for $\Gamma_0(N)$ that is either weakly holomorphic (3.2.9), or weakly skew-holomorphic (3.2.10), depending on the sign of m. We refrain from defining mock modular forms and mock Jacobi forms here, and instead refer to Section 3.2 of [8], and references cited therein, for more details.

3.3 Heegner Divisors

In this section we recall the notion of twisted Heegner divisor, as formulated in [4], specifically in preparation for the discussion in Section 5.1.

Following [22] we write $[A, B, C]$ as a shorthand for the binary quadratic form $Ax^2 + Bxy + Cy^2$. The discriminant of $[A, B, C]$ is $D = B^2 - 4AC$, and there is a discriminant-preserving right action of $SL_2(\mathbb{R})$ on binary quadratic forms with real coefficients determined by requiring that

$$[A, B, C]\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x, y) = [A, B, C](ax + by, cx + dy). \tag{3.3.1}$$

If $D = B^2 - 4AC$ is negative then there is a unique point $\alpha \in \mathbb{H}$ such that

$$[A, B, C](\alpha, 1) = 0. \tag{3.3.2}$$

The right action (3.3.1) is compatible with the usual left action of $SL_2(\mathbb{R})$ on $\mathbb{H}$ in that if α is the point in $\mathbb{H}$ associated to $[A, B, C]$ as in (3.3.2) and $\gamma \in SL_2(\mathbb{R})$, then the point in $\mathbb{H}$ associated to $[A, B, C]\gamma$ is $\gamma^{-1}\alpha$.

Henceforth we restrict to binary quadratic forms with integer coefficients and negative discriminant. Given integers m and r, and a fixed choice of $D < 0$, let $\mathcal{Q}^{(m)}_{D,r}$ denote the set

of integer-coefficient binary quadratic forms $[A, B, C]$ of discriminant D such that $A > 0$ and $A \equiv 0 \bmod m$ and $B \equiv r \bmod 2m$. Then each set $\mathcal{Q}_{D,r}^{(m)}$ is stable under the restriction of the action (3.3.1) to $\Gamma_0(m)$, and is composed of finitely many orbits for this action. Also, any point $\alpha \in \mathbb{H}$ that is associated to a quadratic form $[A, B, C]$ as in (3.3.2) with $A, B, C \in \mathbb{Z}$ is called a CM point. (At negative discriminants there is no loss of generality in restricting to quadratic forms $[A, B, C]$ with $A > 0$.)

Now suppose that D_1 is a positive fundamental discriminant that is a square modulo $4m$. Applying Proposition 1 of Section I.2 of op. cit. we may define the generalized genus character $\chi_{D_1}^{(m)}$ on $\mathcal{Q}_{DD_1}^{(m)}$, for any negative discriminant D that is a square modulo $4m$, by setting

$$\chi_{D_1}^{(m)}([Am, B, C]) := \left(\frac{D_1'}{Am'}\right)\left(\frac{D_1''}{Cm''}\right) \tag{3.3.3}$$

for $[Am, B, C] \in \mathcal{Q}_{DD_1}^{(m)}$, when there exist discriminants D_1', D_1'' and positive integers m', m'' such that $D_1 = D_1'D_1''$ and $m = m'm''$ and $\gcd(D_1', Am') = \gcd(D_1'', Cm'') = 1$, and by setting $\chi_{D_1}^{(m)}(Q) := 0$ when no such D_1', D_1'', m' and m'' exist. According to Proposition 1 in Section I.2 of [22] this generalized genus character (3.3.3) is invariant under the action of $\Gamma_0(m)$ in the sense that $\chi_{D_1}^{(m)}(Q\gamma) = \chi_{D_1}^{(m)}(Q)$ for $Q = [Am, B, C]$ as in (3.3.3) and $\gamma \in \Gamma_0(m)$, so it descends to a well-defined function on the orbit space $\mathcal{Q}_{DD_1,r}^{(m)}/\Gamma_0(m)$ for each $r \bmod 2m$.

Next let $X_0(m)$ denote the modular curve

$$X_0(m) := \Gamma_0(m)\backslash\mathbb{H} \cup \mathbb{Q} \cup \{\infty\} \tag{3.3.4}$$

associated to $\Gamma_0(m)$, and for D_1 and D as above, define the twisted Heegner divisor $Z_{D_1,r_1}^{(m)}(D, r)$ on $X_0(m)$ by setting

$$Z_{D_1,r_1}^{(m)}(D, r) := \sum_{Q \in \mathcal{Q}_{DD_1,rr_1}^{(m)}} \chi_{D_1}^{(m)}(Q) \frac{\overline{\alpha_Q}}{|\overline{\Gamma_0(m)}_Q|}, \tag{3.3.5}$$

where in (3.3.5) we write $\overline{\Gamma_0(m)}_Q$ for the stabilizer of Q in $\overline{\Gamma_0(m)} := \Gamma_0(m)/\{\pm\mathrm{Id}\}$, and in each summand take $\overline{\alpha_Q}$ to be the image under the natural map $\mathbb{H} \to X_0(m)$ of the CM point $\alpha_Q = \alpha \in \mathbb{H}$ associated to $Q = [A, B, C]$ as in (3.3.2).

Note that $Z_{D_1,r_1}^{(m)}(D, r)$ as in (3.3.5) is generally not an integral divisor, but will be in the cases of interest to us, whereby $D_1 > 1$. This is because if $D_1 > 1$ is fundamental then we have $|\overline{\Gamma_0(m)}_Q| = 1$ for every $Q \in \mathcal{Q}_{DD_1,rr_1}^{(m)}$.

3.4 Adams Operations

In this section we review Adams operations in the context of complex representations of a finite group. To set up for this suppose that G is a finite group, write $\mathbb{C}[G]$ for the complex group algebra of G (cf. (3.1 .5)), and let $R(G)$ denote the Grothendieck group of the category of finitely generated $\mathbb{C}[G]$-modules. Then, writing $\mathrm{Irr}(G)$ for the set of irreducible characters of G, and writing U_χ for the isomorphism class of irreducible modules that realize a given

$\chi \in \mathrm{Irr}(G)$, we have that $R(G)$ is naturally identified with the free $\mathbb{Z}$-module generated by the U_χ, i.e.

$$R(G) = \sum_{\chi \in \mathrm{Irr}(G)} \mathbb{Z} U_\chi. \tag{3.4.1}$$

The finitely generated (i.e. finite-dimensional) $\mathbb{C}[G]$-modules, regarded modulo isomorphisms, may be identified with the semigroup $R^+(G) < R(G)$ of non-negative integer combinations of the U_χ,

$$R^+(G) = \left\{ \sum_{\chi \in \mathrm{Irr}(G)} m_\chi U_\chi \in R(G) \;\middle|\; m_\chi \geq 0 \right\}. \tag{3.4.2}$$

Also, the tensor product operation on $\mathbb{C}[G]$-modules defines a commutative semiring structure on $R^+(G)$, and this induces a commutative ring structure on $R(G)$. We use $+$ to denote the group operation on $R(G)$, and use $\otimes$ to denote the multiplication.

By a virtual G-module we mean an element of $R(G)$. By a virtual graded G-module we mean an indexed collection $\{V_i\}_{i \in I}$ of virtual modules $V_i \in R(G)$, for some indexing set I, but we write

$$V = \bigoplus_{i \in I} V_i \tag{3.4.3}$$

for such a thing, rather than $V = \{V_i\}_{i \in I}$.

Given a virtual G-module $U \in R(G)$ we define the multiplicity $m_\chi(U)$ of χ in U, for each $\chi \in \mathrm{Irr}(G)$, by requiring that

$$U = \sum_{\chi \in \mathrm{Irr}(G)} m_\chi(U) U_\chi \tag{3.4.4}$$

(cf. (3.4.1)). We also define U^f and U^b for $U \in R(G)$ by setting

$$U^f := \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ m_\chi(U) > 0}} m_\chi(U) U_\chi, \quad U^b := \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ m_\chi(U) < 0}} (-m_\chi(U)) U_\chi, \tag{3.4.5}$$

so that $U^f, U^b \in R^+(G)$ and $U = U^f - U^b$.

Now taking t to be an indeterminate, let us write $R(G)[[t]]$ for the ring of power series in t with coefficients in $R(G)$, and for $U \in R^+(G)$ define elements $\Lambda_t(U)$ and $S_t(U)$ of $R(G)[[t]]$ by setting

$$\Lambda_t(U) := \sum_{k \geq 0} \Lambda^k(U) t^k, \quad S_t(U) := \sum_{k \geq 0} S^k(U) t^k, \tag{3.4.6}$$

where $\Lambda^k(U)$ and $S^k(U)$ denote the k-th exterior and symmetric powers of U, respectively. (Of course $\Lambda_t(U)$ actually belongs to the polynomial ring $R(G)[t]$ when $U \in R^+(G)$, but see (3.4.10) below.) Note that we have

$$\Lambda_t(U' + U'') = \Lambda_t(U') \otimes \Lambda_t(U'') \tag{3.4.7}$$

in $R(G)[[t]]$ for $U', U'' \in R^+(G)$, and similarly with S_t in place of Λ_t.

We now define the Adams operations on $R(G)$ to be the ring homomorphisms ψ^k of $R(G)$, defined for each positive integer k by requiring that

$$\Lambda_{-t}(U) = \exp\left(-\sum_{k>0} \psi^k(U)\frac{t^k}{k}\right) \tag{3.4.8}$$

for $U \in R^+(G)$ (cf. e.g. Exercise 9.3 in [27]). Note that with this definition we also have

$$S_t(U) = \exp\left(\sum_{k>0} \psi^k(U)\frac{t^k}{k}\right) \tag{3.4.9}$$

in $R(G)[[t]]$, for $U \in R^+(G)$ (cf. loc. cit.), and it follows that $\Lambda_{-t}(U) \otimes S_t(U) = 1$ in $R(G)[[t]]$ for $U \in R^+(G)$. Thus it is natural to extend Λ_t, and therefore also the Λ^k (cf. (3.4.6)), from $R^+(G)$ to $R(G)$ by requiring that

$$\Lambda_{-t}(U) = \Lambda_{-t}(U^f) \otimes S_t(U^b) \tag{3.4.10}$$

when $U = U^f - U^b$ for U^f and U^b as in (3.4.5). Then we have $\Lambda^k(-U) = (-1)^k S^k(U)$ for $U \in R^+(G)$, and (3.4.7–3.4.8) now hold for all $U, U', U'' \in R(G)$.

We now consider the trace of an element $g \in G$ on $\Lambda_{-t}(U)$ for a virtual G-module $U = U^f - U^b$. In light of (3.4.10) we may restrict to the cases that $U = U^f$ and $U = -U^b$. In the first of these cases, whereby $U = U^f$ belongs to $R^+(G)$, we find, by decomposing (a representative of) U^f into 1-dimensional representations of the cyclic group generated by g, that

$$\mathrm{tr}(g|\Lambda_{-t}(U)) = \mathrm{tr}(g|\Lambda_{-t}(U^f)) = \prod_i (1-\xi_i t) = \exp\left(-\sum_{k>0}\sum_i \xi_i^k \frac{t^k}{k}\right), \tag{3.4.11}$$

where $\{\xi_i\}$ is the multiset of eigenvalues of g on U^f. By similar considerations we find for $U = -U^b$ that

$$\mathrm{tr}(g|\Lambda_{-t}(U)) = \mathrm{tr}(g|S_t(U^b)) = \prod_i (1-\xi_i t)^{-1} = \exp\left(\sum_{k>0}\sum_i \xi_i^k \frac{t^k}{k}\right), \tag{3.4.12}$$

where now $\{\xi_i\}$ is the multiset of eigenvalues of g on U^b. In particular, we have $\mathrm{tr}(g|\psi^k(U)) = \mathrm{tr}(g^k|U)$ for all $U \in R(G)$.

We can write $\mathrm{tr}(g|\Lambda_{-t}(U))$ as a product of terms of the form $1 - t^n$ in the case that $\mathrm{tr}(g|U)$ is a rational integer, and this will be useful in Section 4.5. To explain how this works suppose that $\mathrm{tr}(g|U) \in \mathbb{Z}$, and for now let us impose the simplifying assumption that $U = U^f \in R^+(G)$ (cf. (3.4.5)). Then, since the cyclotomic fields are all Galois extensions of $\mathbb{Q}$ it must be that all the primitive d-th roots of unity appear with the same multiplicity in the multiset $\{\xi_i\}$ of eigenvalues of g on U, for each $d > 0$. Write $u_d(g|U)$ for this multiplicity. Then we have

$$\mathrm{tr}(g|\Lambda_{-t}(U)) = \prod_i (1-\xi_i t) = \prod_{d>0} c_d(t)^{u_d(g|U)} \tag{3.4.13}$$

(cf. (3.4.11)), where $c_d(t)$ denotes the d-th cyclotomic polynomial,

$$c_d(t) := \prod_{\substack{k \bmod d \\ \gcd(k,d)=1}} \left(1 - \mathrm{e}\left(-\frac{k}{d}\right)t\right) \tag{3.4.14}$$

(normalized to have constant term equal to 1).

We have $1 - t^n = \prod_{d|n} c_d(t)$ by construction (3.4.14), so we obtain

$$c_d(t) = \prod_{b|d} (1 - t^b)^{\mu(\frac{d}{b})} \tag{3.4.15}$$

by Möbius inversion, where $\mu(n)$ is the Möbius function. Now substituting (3.4.15) into (3.4.13) gives us

$$\mathrm{tr}(g|\Lambda_{-t}(U)) = \prod_{d>0} \prod_{b|d} (1 - t^b)^{\mu(\frac{d}{b}) u_d(g|U)}, \tag{3.4.16}$$

and from this we see that there exist integers $v_b(g|U)$ such that

$$\mathrm{tr}(g|\Lambda_{-t}(U)) = \prod_{b>0} (1 - t^b)^{v_b(g|U)}, \tag{3.4.17}$$

as we claimed. Indeed, rewriting (3.4.16) as a product over positive integers b, and writing the d in each factor as $d = ab$ for some $a > 0$, we obtain

$$v_b(g|U) = \sum_{a>0} \mu(a) u_{ab}(g|U). \tag{3.4.18}$$

The general case, whereby $U \in R(G)$ is such that $\mathrm{tr}(g|U) \in \mathbb{Z}$, is just the same, except that some of the $u_d(g|U)$ may be negative.

We summarize the above discussion as follows.

Lemma 3.4.1 *Let $g \in G$ and $U \in R(G)$ such that $tr(g|U) \in \mathbb{Z}$. Then we have*

$$tr(g|\Lambda_{-t}(U)) = \prod_{b>0} (1 - t^b)^{v_b(g|U)} = \exp\left(- \sum_{k>0} tr(g^k|U) \frac{t^k}{k} \right) \tag{3.4.19}$$

where the $v_b(g|U)$ are as in (3.4.17–3.4.18).

To conclude this section we mention that the Frame shape of g on U, for $g \in G$ and $U \in R(G)$ such that $\mathrm{tr}(g|U) \in \mathbb{Z}$, is the formal product

$$\pi(g|U) := \prod_{b>0} b^{v_b(g|U)} \tag{3.4.20}$$

where the $v_b(g|U)$ are as in (3.4.17–3.4.18).

4 Results

In this section we establish our construction *SQ*—a lift of the singular theta lift—which uses Borcherds products to produce G-modules of weight zero from G-modules of weight one-half. We formulate this precisely in Section 4.1. In particular, we explain what we mean by a G-module of weight zero or weight one-half, and we specify the particular infinite products that underpin the construction. The main result of this paper, Theorem 4.1.3, also appears in Section 4.1. It confirms that the construction we propose makes sense.

The proof of Theorem 4.1.3 requires preparation, which we carry out in Section 4.2–4.5. In particular, the verification that the product formulae we propose define modular forms depends upon a method we call repackaging, which we detail in Section 4.2. It also depends

upon the singular theta lift of [3, 23]. We translate this result from the formulation in [3] to the setup of Section 4.1–4.2 in Section 4.3, and use it to prove that the product formulae of Section 4.1 converge and are modular in Section 4.4. Then we explain how SQ works at the level of virtual G-modules, and use this to complete the proof of Theorem 4.1.3, in Section 4.5.

4.1 Formulation

To formulate our construction SQ precisely we let G be an arbitrary finite group, and suppose that W is a virtual graded G-module (see (3.4.3)) with a grading of the form

$$W = \bigoplus_{r \bmod 2m} \bigoplus_{D \equiv r^2 \bmod 4m} W_{r,\frac{D}{4m}} \tag{4.1.1}$$

for some positive integer m. Define the McKay–Thompson series associated to W to be the vector-valued functions $F_g^W = (F_{g,r}^W)$, for $g \in G$, that are obtained by setting

$$F_{g,r}^W(\tau) := \sum_{D \equiv r^2 \bmod 4m} \operatorname{tr}\left(g | W_{r,\frac{D}{4m}}\right) q^{\frac{D}{4m}}. \tag{4.1.2}$$

Then we say that W as in (4.1.1) is a weakly holomorphic (virtual graded) G-module of weight $\frac{1}{2}$ and index m if for each $g \in G$ there exists a positive multiple N_g of $o(g)$ such that $\frac{N_g}{o(g)}$ divides $o(g)$ and

$$F_{g^n}^W \in V_{\frac{1}{2},m}^{\mathrm{wh}}\left(\frac{N_g}{n}\right) \tag{4.1.3}$$

(see Section 3.2) for each positive divisor n of N_g. Further, we say that such a G-module W is rational if the coefficients $C_g^W(D,r)$ of the McKay–Thompson series F_g^W of (4.1.2) are rational integers,

$$C_g^W(D,r) := \operatorname{tr}\left(g | W_{r,\frac{D}{4m}}\right) \in \mathbb{Z} \tag{4.1.4}$$

for all $g \in G$ and $D, r \in \mathbb{Z}$.

Remark 4.1.1 We expect F_g^W as in (4.1.2) to transform in a natural way with respect to the action of $\widetilde{\Gamma}_0(o(g))$ for each $g \in G$. We allow N_g in (4.1.3) to be a multiple of $o(g)$ in order to allow for the possibility that this transformation is governed by a non-trivial character. We assume that any such character has order at most $o(g)$, which is why we require N_g to be a multiple of $o(g)$ that divides $o(g)^2$. Thus, in particular, we require F_e^W to belong to $V_{\frac{1}{2},m}^{\mathrm{wh}}$.

Remark 4.1.2 As we have mentioned in Section 3.2, elements of $V_{\frac{1}{2},m}^{\mathrm{wh}}(N)$ are in natural correspondence with weakly skew-holomorphic Jacobi forms of weight 1 and index m for $\Gamma_0(N)$ (see (3.2.10)), and elements of $V_{\frac{1}{2},-m}^{\mathrm{wh}}(N)$ are in natural correspondence with weakly holomorphic Jacobi forms of weight 1 and index m for $\Gamma_0(N)$ (see (3.2.9)), when m is a positive integer. Thus we may equivalently refer to a weakly holomorphic (virtual graded) G-module W of weight $\frac{1}{2}$ and index m as in (4.1.1–4.1.3) as weakly skew-holomorphic Jacobi of weight 1 and index m, when m is positive. Also, there is no obstruction to formulating the notion of weakly holomorphic (virtual graded) G-module W of weight $\frac{1}{2}$ and index $-m$, for m positive, just by replacing m with $-m$ in (4.1.1–4.1.2). If working with Jacobi forms we

should call such W weakly holomorphic Jacobi of weight 1 and index m. More generally, we obtain the notion of a weakly holomorphic mock modular (virtual graded) G-module of weight $\frac{1}{2}$ and (non-zero) index m, and an equivalent notion in terms of mock Jacobi forms, by replacing V with $\mathbb{V}$ in (4.1.3) (cf. Section 3.2).

To describe the results of our construction we consider a virtual $\mathbb{Z}$-graded G-module

$$V = \bigoplus_{n\in\mathbb{Z}} V_n \tag{4.1.5}$$

(cf. (3.4.3)), and say that such a virtual graded G-module is weakly holomorphic of weight 0 if there exists a constant h such that the graded trace function

$$f_g^V(\tau) := \sum_n \operatorname{tr}(g|V_n)\, q^{n-h} \tag{4.1.6}$$

is a weakly holomorphic modular form of weight 0 (with level depending upon g), for each $g \in G$.

Note that the h in (4.1.6) that makes f_g^V modular is unique if it exists (so there is no need to include it in the notation for V). If V is a weakly holomorphic G-module of weight 0 as in (4.1.5–4.1.6) we call the associated trace functions f_g^V the McKay–Thompson series of V.

The cornerstone of our construction is the twined $O_{2,1}$ Borcherds product

$$\Psi_g^W(\tau) := q^{-H} \exp\left(-\sum_{n>0}\sum_{k>0} C_{g^k}^W(n^2, n)\frac{q^{nk}}{k} \right), \tag{4.1.7}$$

defined for $g \in G$ with $C_g^W(D, r)$ as in (4.1.4), where $H = H^W$ is the generalized class number associated to W (defined below in (4.4.10)). A technical problem we resolve in Section 4.4 is that of showing that the right-hand side of (4.1.7) converges for $\Im(\tau)$ sufficiently large, and extends by analytic continuation to a holomorphic function on the upper half-plane. (It is this analytically continued function that we have in mind when we write Ψ_g^W.)

A less technical, but nonetheless important ingredient is the eta product

$$\eta_g^W(\tau) := \prod_{b>0} \eta(b\tau)^{2v_b(g|W_{0,0})}, \tag{4.1.8}$$

defined for $g \in G$ where $v_b(g|W_{0,0})$ is as in (3.4.17), for $W_{0,0}$ as in (4.1.1), and where $\eta(\tau) := q^{\frac{1}{24}}\prod_{n>0}(1-q^n)$ is the Dedekind eta function.

Taking the convergence of (4.1.7) on trust for now, we define

$$T_g^W := \frac{\Psi_g^W}{\eta_g^W} \tag{4.1.9}$$

for $g \in G$, when W is a rational weakly holomorphic G-module of weight $\frac{1}{2}$ and some positive integer index m. The main result of this paper is the following.

Theorem 4.1.3 *Let G be a finite group, and let W be a rational weakly holomorphic G-module of weight $\frac{1}{2}$ and index m, for some positive integer m. Then there exists a unique weakly holomorphic G-module $V = V^W$ of weight 0 such that $f_g^V = T_g^W$ for all $g \in G$.*

For W a rational weakly holomorphic G-module of weight $\frac{1}{2}$ and some positive integer index we define $SQ(W)$ to be the weakly holomorphic G-module V^W of weight 0 whose existence and uniqueness is asserted by Theorem 4.1.3,

$$SQ(W) := V^W. \tag{4.1.10}$$

The proof of Theorem 4.1.3 requires some preparation, which we carry out in the rest of Section 4. The proof itself is completed in Section 4.5. Although we do not pursue the formal language of functors in this work, we explain how to realize $SQ(W)$ explicitly in terms of tensor products of alternating and symmetric powers of subspaces of W in Section 4.5. From this we see that the construction SQ is functorial at the level of vector spaces, and in particular, maps G-modules to G-modules.

Remark 4.1.4 According to Theorem 4.1.3, the construction SQ of (4.1.10) applies to any virtual graded G-module W that is rational and weakly holomorphic of weight $\frac{1}{2}$ and index m in the sense of (4.1.1–4.1.4), for m a positive integer. It does not apply to weakly holomorphic modular or mock modular G-modules of weight $\frac{1}{2}$ and negative index, as formulated in Remark 4.1.2, because the representation ϱ_m is of the wrong type when m is negative (but see Section 5.2). It also does not apply to weakly holomorphic mock modular G-modules of weight $\frac{1}{2}$ and positive index, in general. In principle we could establish such an extension by more closely following the analysis of [4], but according to Remark 8 of op. cit. it is not expected that there are any rational weakly holomorphic mock modular G-modules of weight $\frac{1}{2}$ and positive index that are not already modular.

4.2 Repackaging

In this section we show that the McKay–Thompson series $F^W_{g^k}$, whose coefficients appear in (4.1.7), can be "repackaged" into a single modular form $\check{F}^W_g$ which transforms under the Weil representation $\varrho_{m,N}$ attached to the lattice $L_{m,N} = \sqrt{2m}\mathbb{Z} \oplus \Gamma^{1,1}(N)$. It is this modular form whose singular theta lift will ultimately yield the desired Borcherds product (4.1.7) that defines Ψ^W_g. Actually, we will provide a more general construction, which is analogous to, and heavily inspired by, Lemma 3.6 of [6].

Lemma 4.2.1 *Fix positive integers m and N. The following spaces are isomorphic:*

1. *Families $\{F^{(n)}\}_{n|N}$ of weakly holomorphic vector-valued modular forms where each $F^{(n)}$ is of weight $\frac{1}{2}$ and type ϱ_m for $\widetilde{\Gamma}_1(N/n)$.*
2. *Families $\{\hat{F}_{(i,j)}\}_{i,j\in\mathbb{Z}/N\mathbb{Z}}$ of $\mathbb{C}[L_m^*/L_m]$-valued functions on $\mathbb{H}$ that obey the transformation property $\hat{F}_{(i,j)}(\gamma\tau) = u(\tau)\varrho_m(\gamma,u)\hat{F}_{(i,j)\gamma}(\tau)$ for $(\gamma,u) \in \widetilde{SL}_2(\mathbb{Z})$.*
3. *Weakly holomorphic vector-valued modular forms $\check{F}$ of weight $\frac{1}{2}$ and type $\varrho_{m,N}$ for $\widetilde{SL}_2(\mathbb{Z})$.*

In Item 2 of the statement of Lemma 4.2.1 we write $(i,j)\gamma$ for the usual right action of matrices on row vectors, but reduce the computation modulo N since (i,j) belongs to $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$.

Proof of Lemma 4.2.1 For the course of this proof it will be convenient to use the slash operators $|_m$ and $|_{m,N}$ defined by setting

$$f|_m(\gamma,u)(\tau) := u(\tau)^{-1}\varrho_m(\gamma,u)^{-1}f(\gamma\tau), \tag{4.2.1}$$

$$\check{F}|_{m,N}(\gamma,u)(\tau) := u(\tau)^{-1}\varrho_{m,N}(\gamma,u)^{-1}\check{F}(\gamma\tau), \tag{4.2.2}$$

for $(\gamma, u) \in \widetilde{SL}_2(\mathbb{Z})$, when $f = (f_r)$ is a vector-valued function on $\mathbb{H}$ with components indexed by $\mathbb{Z}/2m\mathbb{Z}$, and $\check{F} = (\check{F}_{i,j,r})$ is a vector-valued function on $\mathbb{H}$ with components indexed by $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/2m\mathbb{Z}$, as in Item 3 above. Also, it will be convenient to use $\check{F}_{(i,j)} = (\check{F}_{(i,j),r})$ to denote the vector-valued function with components

$$\check{F}_{(i,j),r} := \check{F}_{i,j,r} \tag{4.2.3}$$

indexed by $\mathbb{Z}/2m\mathbb{Z}$, when $\check{F} = (\check{F}_{i,j,r})$ is as in (4.2.2). As an application of these conventions (4.2.1–4.2.3) we note that

$$\begin{aligned} \left(\check{F}\Big|_{m,N} \widetilde{T}\right)_{(i,j)} &= \mathrm{e}\left(-\frac{ij}{N}\right)\left(\check{F}_{(i,j)}\Big|_m \widetilde{T}\right), \\ \left(\check{F}\Big|_{m,N} \widetilde{S}\right)_{(i,j)} &= \frac{1}{N} \sum_{i',j' \bmod N} \mathrm{e}\left(\frac{ij' + ji'}{N}\right)\left(\check{F}_{(i',j')}\Big|_m \widetilde{S}\right), \end{aligned} \tag{4.2.4}$$

for $i, j \in \mathbb{Z}/N\mathbb{Z}$ (cf. (3.1 .8)), when $\check{F} = (\check{F}_{i,j,r})$ is as in (4.2.2).

(1) $\Leftrightarrow$ (2). To start we observe that the orbits of $SL_2(\mathbb{Z})$ on $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, acting on the right, are in one-to-one correspondence with the divisors of N, the bijection being given by $n \mapsto (0, n)SL_2(\mathbb{Z})$. More specifically, (i, j) belongs to the orbit of $(0, n)$ where $n = \gcd(i, j, N)$. To see this, note that there is a matrix of the form

$$\overline{\gamma_0} = \begin{pmatrix} * & * \\ i/n & j/n \end{pmatrix} \in SL_2(\mathbb{Z}/n'\mathbb{Z}), \tag{4.2.5}$$

where $n' = N/n$, and that any matrix $\gamma_0 \in SL_2(\mathbb{Z})$ which maps to this $\overline{\gamma_0}$ in (4.2.5) under the natural map $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/n'\mathbb{Z})$ will relate $(0, n)$ to (i, j) via its action. The ambiguity in this choice is given precisely by left translations by $\Gamma_1(n')$.

We now attach functions $\hat{F}_{(i,j)} = (\hat{F}_{(i,j),r})$ for (i, j) in $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ to the family $\{F^{(n)}\}_{n|N}$ by defining

$$\hat{F}_{(i,j)} := \left(F^{(\gcd(i,j,N))}\Big|_m (\gamma_0, u_0)\right) \tag{4.2.6}$$

(cf. (4.2.1)), where for each $(i, j) \in \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ we take (γ_0, u_0) to be an arbitrary preimage in $\widetilde{SL}_2(\mathbb{Z})$ of a matrix $\overline{\gamma_0}$ as in (4.2.5).

We require to show that the functions $\hat{F}_{(i,j)}$ satisfy the transformation property

$$\left(\hat{F}_{(i,j)}\Big|_m (\gamma, u)\right) = \hat{F}_{(i,j)\gamma} \tag{4.2.7}$$

for $(\gamma, u) \in \widetilde{SL}_2(\mathbb{Z})$. So let $(i, j) \in \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, let $(\gamma, u) \in \widetilde{SL}_2(\mathbb{Z})$, and set $n = \gcd(i, j, N)$. Also let $(\gamma_0, u_0) \in \widetilde{SL}_2(\mathbb{Z})$ be as in (4.2.6). Then by (4.2.6) we have

$$\left(\hat{F}_{(i,j)}\Big|_m (\gamma, u)\right) = \left(F^{(n)}\Big|_m (\gamma_0', u_0')\right) \tag{4.2.8}$$

where $(\gamma_0', u_0') = (\gamma_0, u_0)(\gamma, u)$. Next let $(\gamma_0'', u_0'') \in \widetilde{SL}_2(\mathbb{Z})$ be as in (4.2.6), but with (i, j) replaced by $(i, j)\gamma$. Then the bottom rows of γ_0' and γ_0'' agree modulo N/n, so there is some $(\gamma', u') \in \widetilde{\Gamma}_1(N/n)$ such that $(\gamma_0'', u_0'') = (\gamma', u')(\gamma_0', u_0')$. We have that $F^{(n)}$ is invariant under the action of $\widetilde{\Gamma}_1(N/n)$ defined by the slash operator $|_m$ of (4.2.1) by hypothesis, so

$$\hat{F}_{(i,j)\gamma} = \left(F^{(n)}\Big|_m (\gamma_0'', u_0'')\right) = \left(F^{(n)}\Big|_m (\gamma_0', u_0')\right) = \left(\hat{F}_{(i,j)}\Big|_m (\gamma, u)\right) \tag{4.2.9}$$

as required.

The inverse of the construction we have just validated is

$$\left\{\hat{F}_{(i,j)}\right\} \mapsto \left\{F^{(n)} = \hat{F}_{(0,n)}\right\}_{n|N}, \tag{4.2.10}$$

so the spaces of Items 1 and 2 are isomorphic.

(2) $\Leftrightarrow$ (3). The functions $\check{F} = (\check{F}_{i,j,r})$ we are after are obtained by taking the discrete Fourier transforms of the $\hat{F}_{(i,j)} = (\hat{F}_{(i,j),r})$ in their second index. Precisely, we take

$$\check{F}_{i,j,r} := \frac{1}{N} \sum_{j' \bmod N} \mathrm{e}\left(-\frac{jj'}{N}\right) \hat{F}_{(i,j'),r} \tag{4.2.11}$$

for $i, j \in \mathbb{Z}/N\mathbb{Z}$ and $r \in \mathbb{Z}/2m\mathbb{Z}$, and we note that the inverse transform is

$$\hat{F}_{(i,j),r} = \sum_{j' \bmod N} \mathrm{e}\left(\frac{jj'}{N}\right) \check{F}_{i,j',r}. \tag{4.2.12}$$

We require to show that $(\check{F}|_{m,N}(\gamma, u)) = \check{F}$ for $(\gamma, u) \in \widetilde{SL}_2(\mathbb{Z})$. It suffices to check that this is true on the generators $\widetilde{S}$ and $\widetilde{T}$ of $\widetilde{SL}_2(\mathbb{Z})$ (see (3.1 .4)). Employing the notation (4.2.1–4.2.3), the identities (4.2.4) and (4.2.7), and the transforms (4.2.11–4.2.12), we have

$$\begin{aligned}
\left(\check{F}\Big|_{m,N} \widetilde{T}\right)_{(i,j)} &= \mathrm{e}\left(-\frac{ij}{N}\right) \left(\check{F}_{(i,j)}\Big|_m \widetilde{T}\right) \\
&= \mathrm{e}\left(-\frac{ij}{N}\right) \frac{1}{N} \sum_{j' \bmod N} \mathrm{e}\left(-\frac{jj'}{N}\right) \left(\hat{F}_{(i,j')}\Big|_m \widetilde{T}\right) \\
&= \mathrm{e}\left(-\frac{ij}{N}\right) \frac{1}{N} \sum_{j' \bmod N} \mathrm{e}\left(-\frac{jj'}{N}\right) \hat{F}_{(i,i+j')}, \qquad (4.2.13) \\
\left(\check{F}\Big|_{m,N} \widetilde{S}\right)_{(i,j)} &= \frac{1}{N} \sum_{i',j' \bmod N} \mathrm{e}\left(\frac{ii' + jj'}{N}\right) \left(\check{F}_{(j',i')}\Big|_m \widetilde{S}\right) \\
&= \frac{1}{N} \sum_{j' \bmod N} \mathrm{e}\left(\frac{jj'}{N}\right) \left(\hat{F}_{(j',i)}\Big|_m \widetilde{S}\right) \\
&= \frac{1}{N} \sum_{j' \bmod N} \mathrm{e}\left(\frac{jj'}{N}\right) \hat{F}_{(i,-j')}. \qquad (4.2.14)
\end{aligned}$$

Now replacing j' with $j' - i$ in the last line of (4.2.13), and replacing j' with $-j'$ in the last line of (4.2.14), we find that

$$\left(\check{F}\Big|_{m,N} \widetilde{T}\right)_{(i,j)} = \left(\check{F}\Big|_{m,N} \widetilde{S}\right)_{(i,j)} = \frac{1}{N} \sum_{j' \bmod N} \mathrm{e}\left(-\frac{jj'}{N}\right) \hat{F}_{(i,j')} = \check{F}_{(i,j)} \tag{4.2.15}$$

for $i, j \in \mathbb{Z}/N\mathbb{Z}$. Thus $(\check{F}|_{m,N}\widetilde{T}) = (\check{F}|_{m,N}\widetilde{S}) = \check{F}$ as we claimed.

The statement that the spaces of Items 2 and 3 are in bijection now follows from standard facts about discrete Fourier transforms. This concludes the proof. □

Composing the transformations detailed in the proof of Lemma 4.2.1 we obtain an explicit isomorphism

$$\left\{F^{(n)}\right\}_{n|N} \mapsto \check{F} = \left(\check{F}_{i,j,r}\right) \tag{4.2.16}$$

from collections of weakly holomorphic modular forms of weight $\frac{1}{2}$ and type ϱ_m for groups $\widetilde{\Gamma}_1(N/n)$, to weakly holomorphic modular forms of weight $\frac{1}{2}$ and type $\varrho_{m,N}$ for $\widetilde{SL}_2(\mathbb{Z})$. Therefore, given a weakly holomorphic G-module W of weight $\frac{1}{2}$ and some index as in (4.1.1), we can apply this map (4.2.16) to $\{F^{(n)} = F^W_{g^n}\}_{n|N}$ (cf. (4.1.2)), and thereby encode the McKay–Thompson series $F^W_{g^n}$ associated with powers of a given group element g in a modular form $\check{F}^W_g$ transforming under $\varrho_{m,N}$, where $N = N_g$ is as in (4.1.3). In this situation we write

$$\left\{F^W_{g^n}\right\}_{n|N_g} \mapsto \check{F}^W_g = \left(\check{F}^W_{g,i,j,r}\right) \tag{4.2.17}$$

for the map (4.2.16). Extending the notational convention (4.2.3), given $i, j \in \mathbb{Z}/N\mathbb{Z}$ we write $\check{F}^W_{g,(i,j)}$ for the vector-valued function that takes the $\check{F}^W_{g,i,j,r}$ for $r \mod 2m$ as its components, so that

$$\check{F}^W_{g,(i,j)} = \left(\check{F}^W_{g,(i,j),r}\right) = \left(\check{F}^W_{g,i,j,r}\right). \tag{4.2.18}$$

Example 4.2.2 Here we give an example of a repackaging of a family of modular forms with $N = 2$ that is of relevance to the analysis of [16]. To begin we define a weakly holomorphic vector-valued modular form $F^{(1)} = (F^{(1)}_0, F^{(1)}_1)$ for the restriction to $\widetilde{\Gamma}_0(2)$ of the Weil representation $\varrho_1 = \varrho_{1,1}$ (cf. (3.1 .8)) by setting

$$F^{(1)}_0(\tau) = 128\frac{\eta(\tau)^6\eta(4\tau)^{14}}{\eta(2\tau)^{19}} = 128q - 768q^2 + 3584q^3 - 13312q^4 + 43008q^5 + \ldots,$$
$$F^{(1)}_1(\tau) = \frac{\eta(2\tau)^{23}}{\eta(\tau)^8\eta(4\tau)^{14}} = q^{-\frac{3}{4}} + 8q^{\frac{1}{4}} + 21q^{\frac{5}{4}} + 8q^{\frac{9}{4}} - 42q^{\frac{13}{4}} + 155q^{\frac{21}{4}} + \ldots, \tag{4.2.19}$$

where η is as in (4.1.8). We also let $F^{(2)} = (F^{(2)}_0, F^{(2)}_1)$ be the unique weakly holomorphic modular form of weight $\frac{1}{2}$ for the representation ϱ_1 for $\widetilde{\Gamma}_0(1) = \widetilde{SL}_2(\mathbb{Z})$ that satisfies $F^{(2)}_r(\tau) = \delta_{r,1}q^{-\frac{3}{4}} + O(q^{\frac{1}{4}})$ as $\Im(\tau) \to \infty$, so that

$$F^{(2)}_0(\tau) = 26752q + 1707264q^2 + 44330496q^3 + 708938752q^4 + \ldots,$$
$$F^{(2)}_1(\tau) = q^{-\frac{3}{4}} - 248q^{\frac{1}{4}} - 85995q^{\frac{5}{4}} - 4096248q^{\frac{9}{4}} - 91951146q^{\frac{13}{4}} + \ldots \tag{4.2.20}$$

(cf. (1.1.2)).

We can now construct the function $\hat{F}$ corresponding to $\{F^{(n)}\}_{n|2}$ in Lemma 4.2.1 by setting

$$\hat{F}_{(0,0)}(\tau) := F^{(2)}(\tau), \qquad \hat{F}_{(0,1)}(\tau) := F^{(1)}(\tau),$$
$$\hat{F}_{(1,0)}(\tau) := \tau^{-\frac{1}{2}}\varrho_1(\widetilde{S})^{-1}F^{(1)}(-\tfrac{1}{\tau}), \qquad \hat{F}_{(1,1)}(\tau) := (\tau+1)^{-\frac{1}{2}}\varrho_1(\widetilde{S}\widetilde{T})^{-1}F^{(1)}(-\tfrac{1}{\tau+1}). \tag{4.2.21}$$

Then, using (4.2.21) and the formula $\eta(-\frac{1}{\tau}) = \mathrm{e}(-\frac{1}{8})\tau^{\frac{1}{2}}\eta(\tau)$ we find that the image $\check{F}$ of $\{F^{(n)}\}_{n|2}$ under the map (4.2.16) satisfies

$$\check{F}_{(0,0)} = \tfrac{1}{2}F^{(2)} + \tfrac{1}{2}F^{(1)}, \qquad \check{F}_{(0,1)} = \tfrac{1}{2}F^{(2)} - \tfrac{1}{2}F^{(1)},$$
$$\check{F}_{(1,0)} = \tfrac{1}{2}F^{(2)} - \tfrac{1}{2}F^{(1)} + 8\theta^0_1, \qquad \check{F}_{(1,1)} = \check{F}_{(1,0)} - \tfrac{1}{2}F^{(2)} + \tfrac{1}{2}F^{(1)} - 8\theta^0_1. \tag{4.2.22}$$

To make $\check{F}_{(1,1)}$ completely explicit we note that

$$\hat{F}_{(1,0),0}(\tau) = 4\frac{\eta(\tau)^6\eta(\frac{\tau}{4})^{14}}{\eta(\frac{\tau}{2})^{19}} + 4\frac{\eta(\frac{\tau}{2})^{23}}{\eta(\tau)^8\eta(\frac{\tau}{4})^{14}} = 8+768q^{\frac{1}{2}} + 13328q+125440q^{\frac{3}{2}} + \dots, \tag{4.2.23}$$

$$\hat{F}_{(1,0),1}(\tau) = 4\frac{\eta(\tau)^6\eta(\frac{\tau}{4})^{14}}{\eta(\frac{\tau}{2})^{19}} - 4\frac{\eta(\frac{\tau}{2})^{23}}{\eta(\tau)^8\eta(\frac{\tau}{4})^{14}} = -112q^{\frac{1}{4}} - 3584q^{\frac{3}{4}} - 43008q^{\frac{5}{4}} + \dots.$$

Example 4.2.3 It is instructive to carry out the repackaging procedure in the specific case that the input family $\{F^{(n)}\}_{n|N}$ is composed of theta functions. For a basic example of this let N be a prime, and let $c^{(1)}$ and $c^{(N)}$ be integers such that $c^{(1)} \equiv c^{(N)} \bmod N$. Then for $G = \mathbb{Z}/N\mathbb{Z}$ there exists a weakly holomorphic G-module W of weight $\frac{1}{2}$ and index 1, as in (4.1.1), such that

$$F_g^W := \begin{cases} c^{(N)}\theta_1^0 & \text{if } o(g) = 1, \\ c^{(1)}\theta_1^0 & \text{if } o(g) \neq 1, \end{cases} \tag{4.2.24}$$

where $\theta_1^0 = (\theta_{1,0}^0, \theta_{1,1}^0)$ is as defined in § 3.2.

So let us take $\{F^{(n)} = F_{g^n}^W\}_{n|N}$ for F_g^W as in (4.2.24). Then, since θ_1^0 is invariant for the weight $\frac{1}{2}$ action of $\widetilde{SL}_2(\mathbb{Z})$ defined by the Weil representation ϱ_1, we have

$$\hat{F}_{(i,j)} = \begin{cases} c^{(N)}\theta_1^0 & \text{if } i \equiv j \equiv 0 \bmod N, \\ c^{(1)}\theta_1^0 & \text{else,} \end{cases} \tag{4.2.25}$$

and it follows from this that $\check{F}^W$, being the image of $\{F_{g^n}^W\}_{n|N}$ under (4.2.16), is given by

$$\check{F}_{(i,j)}^W = \begin{cases} \frac{1}{N}\left(c^{(N)} + (N-1)c^{(1)}\right)\theta_1^0 & \text{if } i \equiv 0 \bmod N \text{ and } j \equiv 0 \bmod N, \\ \frac{1}{N}\left(c^{(N)} - c^{(1)}\right)\theta_1^0 & \text{if } i \equiv 0 \bmod N \text{and} j \not\equiv 0 \bmod N, \\ c^{(1)}\theta_1^0 & \text{if } i \not\equiv 0 \bmod N \text{and} j \equiv 0 \bmod N, \\ 0 & \text{if } i \not\equiv 0 \bmod N \text{and} j \not\equiv 0 \bmod N. \end{cases} \tag{4.2.26}$$

Example 4.2.4 We now put together the previous two examples so as to demonstrate how repackaging works for the input family $\{F_{g^n}^{(-3,1)}\}_{n|2}$ of penumbral moonshine [15, 24], for g an element of order $N = 2$ in the Thompson group *Th*. (We use this example in [16].)

We first inspect Table A.6 of [21], and in so doing arrive at the identity

$$F_{g^n}^{(-3,1)} = 2F^{(n)} + \mathrm{tr}(g^n|\mathbf{248})\theta_1^0, \tag{4.2.27}$$

for $g \in Th$ with $o(g) = 2$, and $n|2$, where $F^{(n)}$ is the function so denoted in Example 4.2.2, and **248** denotes the unique irreducible representation of *Th* of dimension 248.

We have $\mathrm{tr}(g|\mathbf{248}) = -8$ (see e.g. [12]), so taking $N = 2$, $c^{(2)} = 248$ and $c^{(1)} = -8$ in Example 4.2.3 we find that (4.2.26) specializes to

$$\begin{aligned} \check{F}_{(0,0)}^2 &= 120\theta_1^0, & \check{F}_{(0,1)}^2 &= 128\theta_1^0, \\ \check{F}_{(1,0)}^2 &= -8\theta_1^0, & \check{F}_{(1,1)}^2 &= 0, \end{aligned} \tag{4.2.28}$$

where we here write $\check{F}^2$ for the function denoted simply by $\check{F}$ in Example 4.2.3. We now put (4.2.22) and (4.2.28) together according to (4.2.27), and find that

$$\begin{aligned} \check{F}_{(0,0)} &= \tfrac{1}{2}F^{(2)} + \tfrac{1}{2}F^{(1)}, & \check{F}_{(0,1)} &= \tfrac{1}{2}F^{(2)} - \tfrac{1}{2}F^{(1)}, \\ \check{F}_{(1,0)} &= \tfrac{1}{2}F^{(2)} - \tfrac{1}{2}F^{(1)} - 120\theta_1^0, & \check{F}_{(1,1)} &= \hat{F}_{(1,0)} - \tfrac{1}{2}F^{(2)} + \tfrac{1}{2}F^{(1)} + 120\theta_1^0, \end{aligned} \tag{4.2.29}$$

is the result of applying the map (4.2.16) to $\{F_{g^n}^{(-3,1)}\}_{n|2}$.

For concreteness we note that $\hat{F}_{(1,0)} = 2\hat{F}^1_{(1,0)} - 8\theta_1^0$, where $\hat{F}^1_{(1,0)}$ is the function denoted $\hat{F}_{(1,0)}$ in (4.2.21) (cf. also 4.2.23).

The top line of (4.2.21) demonstrates the general fact that $\hat{F}_{(0,n)} = F^{(n)}$ for n a divisor of N, which follows directly from the construction (4.2.6). In the situations of interest to us each $F^{(n)}$ will actually be modular for the restriction of ϱ_m to $\widetilde{\Gamma}_0(N/n)$. In this case we have the following stronger result.

Lemma 4.2.5 *Suppose that $\{F^{(n)}\}_{n|N}$ is a family of vector-valued functions such that $F^{(n)}$ is a weakly holomorphic modular form of weight $\frac{1}{2}$ and type ϱ_m for $\widetilde{\Gamma}_0(N/n)$. Then for $\{\hat{F}_{(i,j)}\}$ the corresponding family in Lemma 4.2.1 we have $\hat{F}_{(0,j)} = F^{(\gcd(j,N))}$ for all $j \in \mathbb{Z}/N\mathbb{Z}$.*

Proof Let $n = \gcd(j, N)$. Then we have $\hat{F}_{(0,j)} = (F^{(n)}|_m(\gamma_0, u_0))$ for some $(\gamma_0, u_0) \in \widetilde{SL}_2(\mathbb{Z})$ by construction (4.2.6), where the bottom row of γ_0 is congruent to $(0, j/n)$ modulo N/n. So (γ_0, u_0) belongs to $\widetilde{\Gamma}_0(N/n)$, and thus $(F^{(n)}|_m(\gamma_0, u_0)) = F^{(n)}$ by hypothesis. This proves the claim. □

4.3 Lifting

We have defined a map (4.2.17) which may be used to exchange the McKay–Thompson series $\{F_g^W\}_{g\in G}$ of a G-module W as in (4.1.1), each F_g^W (cf. (4.1.2)) being a weakly holomorphic modular form of type ϱ_m with level N_g, for an equivalent family $\{\check{F}_g^W\}_{g\in G}$, where each $\check{F}_g^W$ is a weakly holomorphic modular form of type ϱ_{m,N_g} with level 1. It is this latter collection of functions to which the machinery of [3] is most directly applied. In this section we will explain how Borcherds' computation (in op. cit.) of the singular theta lift

$$\Phi_L(v^+, \check{F}) := \int_{\mathcal{F}}^{\mathrm{reg}} \overline{\Theta_L(\tau, v^+)}\check{F}(\tau)\frac{\mathrm{d}\tau_1\mathrm{d}\tau_2}{\tau_2} \tag{4.3.1}$$

of $\check{F} = \check{F}_g^W$ reads, in the $O_{2,1}$ situation of relevance to us.

The notation in (4.3.1) is just as in (1.1.4), but here we note that complex conjugation acts on the basis elements of $\mathbb{C}[L_m^*/L_m]$ as $\overline{e_r} = e_{-r}$, so that the pairing of $\overline{\Theta_L(\tau, v^+)}$ with $F(\tau)$ is the one which multiplies the e_{-r} component of the former with the e_r component of the latter, and sums over r. Also, we will be taking $L = L_{m,N}$ for $N = N_g$ (cf. (4.1.3)), and the superscript in the integral sign indicates an application of the regularization explained in § 6 of [3] (see also [23]). We require some further preparation in order to formulate the relevant results from [3]. We perform this preparation next.

Given positive integers m and N define traceless 2×2 matrices $\lambda(c, b, a)$, and a set L of such things, by taking

$$\lambda(c, b, a) := \begin{pmatrix} a & b/m \\ Nc & -a \end{pmatrix}, \quad L := \{\lambda(c, b, a) \mid a, b, c \in \mathbb{Z}\}. \tag{4.3.2}$$

Then L becomes a copy of the lattice $L_{m,N}$ of (3.1 .6) once we equip it with the quadratic form Q_L given by $Q_L(\lambda) := -m\det(\lambda)$. In terms of the $\lambda(c, b, a)$, the quadratic form and associated bilinear form on $L \cong L_{m,N}$ are given explicitly by

$$\begin{aligned} Q_L(\lambda(c, b, a)) &= ma^2 + Nbc, \\ (\lambda(c, b, a), \lambda(c', b', a')) &= 2maa' + N(b'c + bc'). \end{aligned} \tag{4.3.3}$$

Using this description we find that the dual of $L \cong L_{m,N}$ is

$$L^* = \left\{ \lambda \left(\frac{c}{N}, \frac{b}{N}, \frac{a}{2m} \right) \,\middle|\, a, b, c \in \mathbb{Z} \right\}. \tag{4.3.4}$$

One convenience of this realization (4.3.2–4.3.3) of $L_{m,N}$ is that it makes the automorphisms of the lattice easier to see. For this we define a left action of $SL_2(\mathbb{R})$ on $L \otimes \mathbb{R}$ by isometries by setting

$$\gamma \cdot \lambda := \gamma \lambda \gamma^{-1} \tag{4.3.5}$$

for $\gamma \in SL_2(\mathbb{R})$ and $\lambda \in L \otimes \mathbb{R}$. Then the automorphism group $\mathrm{Aut}(L_{m,N})$ is the subgroup of $SL_2(\mathbb{R})$ composed of the elements that preserve the lattice, when acting as in (4.3.5). In general it contains at least $\Gamma_0(mN)$ as a subgroup.

The computation of (4.3.1) in [3] (see also [23]) produces an automorphic form on a Grassmannian. We now explain what such an object is in the setup of interest to us. Since L has signature $(2, 1)$ we may consider the Grassmannian $G(L)$ of 2-dimensional positive definite subspaces of $L \otimes \mathbb{R}$. Following op. cit. we equip this space with a complex structure as follows. First consider the particular 2-dimensional positive definite subspace

$$v^+ = \left\{ \lambda(\tfrac{1}{N}x, mx, y) \mid x, y \in \mathbb{R} \right\} \in G(L), \tag{4.3.6}$$

and give it an orientation by declaring that the ordered basis (X_L, Y_L) for v^+ is positively oriented, for the particular choice

$$X_L = \lambda(\frac{1}{N}, m, 0), \quad Y_L = \lambda(0, 0, 1). \tag{4.3.7}$$

Then we obtain an orientation on every element of $G(L)$ by requiring that the choice varies continuously, as we move away from v^+. Next, for an arbitrary oriented 2-dimensional positive definite subspace $v^+ \in G(L)$ we consider the points

$$Z_L = X_L + iY_L \in L \otimes \mathbb{C}, \tag{4.3.8}$$

such that (X_L, Y_L) is a positively oriented ordered orthogonal basis of v^+ with $Q_L(X_L) = Q_L(Y_L)$. Each such Z_L has norm zero, and all define the same point $\mathbb{C}Z_L$ in the complex projective space $\mathbb{P}(L \otimes \mathbb{C})$. So finally we use the association $v^+ \mapsto \mathbb{C}Z_L$ to identify $G(L)$ with an open subset of $\mathbb{P}(L \otimes \mathbb{C})$, and thereby obtain a complex structure on $G(L)$.

Now to define automorphic forms on $G(L)$ we observe that the norm zero points Z_L of (4.3.8) form a principal $\mathbb{C}^*$-bundle P over $G(L)$, so realized as a subset of $\mathbb{P}(L \otimes \mathbb{C})$. An automorphic form of weight k on $G(L)$, in the sense of op. cit., is a function on P which is homogeneous of degree $-k$ and invariant under some subgroup Γ of finite index in $\mathrm{Aut}(L)^+$, the orientation preserving automorphisms of L.

Borcherds provides product formulae for the automorphic forms produced by the singular theta lift; one for each cusp of the invariance group. To formulate this concretely let ℓ be a primitive norm zero vector in L, and let $\ell' \in L^*$ be such that $(\ell, \ell') = 1$. Then for any finite-index subgroup $\Gamma < \mathrm{Aut}(L)^+$ as in the last paragraph, ℓ represents a cusp of Γ. Define a corresponding lattice K by setting

$$K := (L \cap \ell^\perp)/\mathbb{Z}\ell. \tag{4.3.9}$$

Identify $K \otimes \mathbb{R}$ with the orthogonal complement of ℓ' in $L \otimes \mathbb{R} \cap \ell^\perp$ in the natural way, and in so doing identify K with a subgroup of $L \otimes \mathbb{R}$.

In our situation K is even, positive definite, and has rank 1 by construction, so $K \cong \sqrt{2M}\mathbb{Z}$ for some positive integer M. In particular, there are two choices of positive cone in $K \otimes \mathbb{R}$. Following op. cit. we take $C \subset K \otimes \mathbb{R}$ to be the one that contains a vector Y_L, where $Z_L = X_L + iY_L$ is a point of P as in (4.3.8), and furthermore $(X_L, \ell) > 0$. (We necessarily have $(Y_L, \ell) = 0$ for $Y_L \in K \otimes \mathbb{R}$.) In the remainder of this section we use κ to denote the generator of K that lies in C, and set $\kappa' := \frac{1}{2M}\kappa$, so that

$$K = \mathbb{Z}\kappa, \quad K^* = \mathbb{Z}\kappa', \quad C = \mathbb{R}^+\kappa = \mathbb{R}^+\kappa'. \tag{4.3.10}$$

Now consider the set of points $Z = X + iY \in K \otimes \mathbb{C}$ with $Y \in C$. Since the rank of K is 1 this is actually a copy of the upper half-plane $\mathbb{H}$. We may map it into P, and thereby interpret the automorphic form produced by (4.3.1) as a modular form, by sending such a $Z = X + iY$ to the unique norm zero vector $Z_L \in P$ as in (4.3.8) that satisfies $(Z_L, \ell) = 1$, and whose projection onto K recovers Z. Explicitly, the map $Z \mapsto Z_L$ is given by setting

$$Z_L := Z - Q_L(Z)\ell - Q_L(\ell')\ell + \ell'. \tag{4.3.11}$$

With Z_L as in (4.3.11) the association $Z \mapsto \mathbb{C}Z_L$ defines an isomorphism of $K \otimes \mathbb{R} + iC \simeq \mathbb{H}$ with $G(L)$. In particular, automorphic forms on $G(L)$ may naturally be regarded as modular forms on the upper half-plane. (Note that the weight picks up a factor of two when the automorphic form produced by (4.3.1) is regarded as a modular form. Cf. Example 14.4 in [3].) We write

$$\Psi_\ell(Z, \check{F}) := \Psi_L\left(Z - Q_L(Z)\ell - Q_L(\ell')\ell + \ell', \check{F}\right) \tag{4.3.12}$$

for the modular form defined by (4.3.11).

A further key feature of the singular theta lift of op. cit. is that it gives us direct knowledge of the divisors of the automorphic forms that it produces. In our situation the zeros and poles on the upper half-plane occur at points $\mathbb{C}Z_L$, for Z_L as in (4.3.11) that satisfy

$$(Z_L, \lambda) = 0 \tag{4.3.13}$$

for some $\lambda \in L$ with $Q_L(\lambda) < 0$. Following Borcherds we denote such a point $\mathbb{C}Z_L$ by $\lambda^\perp$.

We compute the zeros and poles at cusps by computing the corresponding Weyl vectors, $\rho(K, W, \check{F}_K)$, as defined after the proof of Theorem 10.3 in op. cit. Here K is as in (4.3.9), for a given cusp representative ℓ, and the symbol W denotes a Weyl chamber, as defined after the proof of Theorem 6.2 in op. cit. In our situation the Weyl chamber is simply a choice of positive cone in $K \otimes \mathbb{R}$, so we may and do take $W = C$. The function $\check{F}_K = (\check{F}_{K,r})$ is the weakly holomorphic modular form of weight $\frac{1}{2}$ and type $\varrho_K = \varrho_M$ (cf. (4.3.10)) for $\widetilde{SL}_2(\mathbb{Z})$ that we obtain by setting

$$\check{F}_{K,r} := \sum_{\substack{i,j \bmod N,\, s \bmod 2m \\ (i,j,s)|(L\cap\ell^\perp)=r}} \check{F}_{i,j,s} \tag{4.3.14}$$

for $r \bmod 2M$, where the summation in (4.3.14) is over the (i, j, s), for $i, j \bmod N$ and $s \bmod 2m$, such that the vectors in the coset of L represented by $\lambda(\frac{i}{N}, \frac{j}{N}, \frac{s}{2m})$ define the same subset of $\hom(L \cap \ell^\perp, \mathbb{Z})$ as those in the coset of K represented by $r\kappa'$ (cf. (4.3.10)). (This is a specialization of the definition preceding the proof of Theorem 5.3 in [3].)

With the above understanding the Weyl vector $\rho(K, C, \check{F}_K)$ may now be defined by requiring that

$$(\rho(K, C, \check{F}_K), \kappa) = \frac{\sqrt{Q_L(\kappa)}}{8\pi} \int_{\mathcal{F}}^{\mathrm{reg}} \overline{\theta_M^0(\tau)} \check{F}_K(\tau) \tau_2^{-\frac{3}{2}} \mathrm{d}\tau_1 \mathrm{d}\tau_2 \tag{4.3.15}$$

(cf. (4.8) in [4]), where $\theta_M^0 = (\theta_{M,r}^0)$ is the theta function associated to K (cf. (3.2.6)), and the notation in (4.3.15) is otherwise as in (4.3.1). We henceforth write $\rho(K, \check{F}_K)$ for $\rho(K, C, \check{F}_K)$, since there is a natural choice $W = C$ of Weyl chamber in the situation at hand.

Note that the integral in (4.3.15) is computed in Corollary 9.6 in [3], with the result being that

$$\int_{\mathcal{F}}^{\mathrm{reg}} \overline{\theta_M^0(\tau)} \check{F}_K(\tau) \tau_2^{-\frac{3}{2}} \mathrm{d}\tau_1 \mathrm{d}\tau_2 = -\frac{8\pi}{\sqrt{M}} \sum_{r \bmod 2M} \sum_{\substack{D \equiv r^2 \bmod 4M \\ D \leq 0}} \check{C}_K(D, r) H_M(D, r), \tag{4.3.16}$$

where $\check{C}_K(D, r)$ is the coefficient of $q^{\frac{D}{4M}}$ in the Fourier expansion of $\check{F}_{K,r}$, and $H_M(D, r)$ is the coefficient of $q^{-\frac{D}{4M}}$ in the Fourier expansion of $G_{M,r}$, for $G_M = (G_{M,r})$ as defined in loc. cit. We have $Q_L(\kappa) = M$ according to (4.3.10), so from (4.3.15–4.3.16) we obtain that

$$\rho(K, \check{F}_K) := \rho(K, C, \check{F}_K) = -H(K, \check{F}_K)\kappa', \tag{4.3.17}$$

where

$$H(K, \check{F}_K) := \sum_{r \bmod 2M} \sum_{\substack{D \equiv r^2 \bmod 4M \\ D \leq 0}} \check{C}_K(D, r) H_M(D, r). \tag{4.3.18}$$

Note that the $H_M(D, r)$ are generalized class numbers. For example, when $M = 1$ and D is a negative discriminant $H_1(D, D)$ is the Hurwitz class number of D (commonly denoted $H(D)$, cf. e.g. [30]). Taking $\check{G}_1(\tau) := G_{1,0}(4\tau) + G_{1,1}(4\tau)$ we have

$$\check{G}_1(\tau) = \sum_{D \leq 0} H_1(D, D) q^D = -\frac{1}{12} + \frac{1}{3}q^3 + \frac{1}{2}q^4 + q^7 + q^8 + \ldots, \tag{4.3.19}$$

and $H_1(D, D)$ is the usual class number of the quadratic imaginary number field $\mathbb{Q}(\sqrt{D})$ when $D < -4$ is fundamental. In light of this we call $H(K, \check{F}_K)$ the *generalized class number* associated to $\check{F}$ and the cusp ℓ, and also use the notation

$$H_\ell^W(g) := H(K, \check{F}_K), \tag{4.3.20}$$

when $K = (L \cap \ell^\perp)/\mathbb{Z}\ell$, in the specific situation that $\check{F} = \check{F}_g^W$ is the image under the map (4.2.16) of a family $\{F_{g^n}^W\}_{n|N}$, for $g \in G$ and $N = N_g$, for some W as in the statement of Theorem 4.1.3.

We are now ready to state the result of [3] that will allow us to validate and interpret the singular theta lift (4.3.1). To ease notation in the statement we define

$$k_g^W := \frac{1}{N} \sum_{n|N} \phi\left(\frac{N}{n}\right) C_{g^n}^W(0, 0), \tag{4.3.21}$$

where ϕ denotes the Euler totient function, and $C_g^W(D, r)$ is as in (4.1.4).

Theorem 4.3.1 *[Borcherds] Let G and W be as in the statement of Theorem 4.1.3, choose $g \in G$, and let $\check{F} = \check{F}_g^W$ be obtained as the image under the map (4.2.17) of $\{F_{g^n}^W\}_{n|N}$ where $N = N_g$ and $F_g^W = (F_{g,r}^W)$ is as in (4.1.2). Also, let L be as in (4.3.2–4.3.3) and let P be as above. Then there is a meromorphic function $\Psi_L(Z_L, \check{F})$ for $Z_L \in P$ with the following properties.*

1. *$\Psi_L(Z_L, \check{F})$ is an automorphic form of weight $\frac{1}{2}k_g^W$ for the group $Aut(L, \check{F})$, with respect to some unitary character of finite order.*
2. *The zeros and poles of $\Psi_L(Z_L, \check{F})$ in $\mathbb{H}$ occur at points $\lambda^\perp$ for $\lambda \in L$, and when $\lambda = \lambda(c, b, a)$ the contribution of $\lambda^\perp$ to the divisor of $\Psi_L(Z_L, \check{F})$ is*

$$\sum_{\substack{x \in \mathbb{R}^+ \\ Nxc, Nxb, 2mxa \in \mathbb{Z}}} \check{C}_{xc,xb}(x^2(ma^2 + Nbc), xa). \tag{4.3.22}$$

3. *The function Ψ_L is related to the singular theta lift Φ_L of (4.3.1) by*

$$\log|\Psi_L(Z_L, \check{F})| = -\frac{\Phi_L(Z_L, \check{F})}{4} - \frac{k_g^W}{2}\left(\log|Y_L| + \frac{\Gamma'(1)}{2} + \log\sqrt{2\pi}\right). \tag{4.3.23}$$

4. *For each primitive norm zero vector $\ell \in L$ the infinite product*

$$e(-(H(K, \check{F}_K)\kappa', Z)) \prod_{n>0} \prod_{\substack{i,j \bmod N, r \bmod 2m \\ (i,j,r)|(L\cap\ell^\perp)=n}} \left(1 - e((n\kappa', Z) + (\lambda, \ell'))\right)^{\check{C}_{i,j}(n^2,r)}, \tag{4.3.24}$$

where in each factor $\lambda = \lambda(\frac{i}{N}, \frac{j}{N}, \frac{r}{2m})$, converges for Z in some neighborhood of the cusp of $Aut(L, \check{F})$ represented by ℓ, and is proportional to $\Psi_\ell(Z, \check{F})$ (cf. (4.3.12)) in the domain of its convergence.

In (4.3.22) and (4.3.24) we write $\check{C}_{i,j}(D, r)$ for the coefficient of $q^{\frac{D}{4m}}$ in the Fourier expansion of $\check{F}_{(i,j),r} = \check{F}_{i,j,r}$, and we interpret the condition $(i, j, r)|(L\cap\ell^\perp) = n$ as we did for (4.3.14). In (4.3.23) we write $\Gamma'(1)$ for the value at $s = 1$ of the derivative of the Gamma function. In the proof of Theorem 4.3.1 we will write $\hat{C}_{(i,j)}(D, r)$ for the coefficient of $q^{\frac{D}{4m}}$ in the Fourier expansion of $\hat{F}_{(i,j),r}$.

Proof of Theorem 4.3.1 The claimed result is a specialization of Theorem 13.3 of [3] to the situation of interest in this work. More specifically, Items 1–3 above correspond to Items 1–3 in loc. cit., and Item 4 above corresponds to Item 5 in loc. cit. (Item 4 of loc. cit. does not apply to our choice of L.) The only part that requires some translation is the computation of the weight $\frac{1}{2}k_g^W$, which is $\frac{1}{2}\check{C}_{0,0}(0, 0)$ according to the statement in loc. cit. To perform this translation recall that $\check{C}_{0,0}(0, 0) = \frac{1}{N}\sum_{j' \bmod N} \hat{C}_{(0,j')}(0, 0)$ according to (4.2.11), and $\hat{C}_{(0,j')}(0, 0) = C_{g^n}^W(0, 0)$, where $n = \gcd(j', N)$, according to Lemma 4.2.5 and our hypothesis that $F^{(n)} = F_{g^n}^W$. So

$$\check{C}_{0,0}(0, 0) = \frac{1}{N}\sum_{j' \bmod N} C_{g^n}^W(0, 0), \tag{4.3.25}$$

where in each summand $n = \gcd(j', N)$. The coincidence of $\check{C}_{0,0}(0, 0)$ with k_g^W as defined in (4.3.21) now follows. □

4.4 Convergence

The purpose of this section is to verify that the functions T_g^W, defined by (4.1.7–4.1.9), are weakly holomorphic modular forms of weight 0. (Cf. Theorem 4.1.3.)

The main challenge at hand is to check the convergence and modularity of the product formula (4.1.7). We use Theorem 4.3.1 to do this. More specifically, we apply (4.3.24) in the special case that ℓ represents the infinite cusp, so we begin this section with a closer look at the constructions of (4.3.9–4.3.20) for such ℓ. We then carry out the application of Theorem 4.3.1 in Proposition 4.4.1. The modularity of η_g^W follows directly from the definition (4.1.8), but it is not immediate that it has the same weight as Ψ_g^W. For this we require to understand how the $v_b(g|U)$ of (3.4.17) transform under power maps $g \mapsto g^p$. We present the necessary analysis in Lemmas 4.4.2 and 4.4.3.

For the rest of this section we take $\ell = \lambda(0, 1, 0)$, whereby ℓ represents the infinite cusp of $\mathrm{Aut}(L)^+$ (and any finite-index subgroup thereof). Then we may take $\ell' = \lambda(\frac{1}{N}, 0, 0)$ (cf. (4.3.3)), in which case $Q_L(\ell') = 0$. With these choices

$$L \cap \ell^\perp = \{\lambda(0, b, a) \mid a, b \in \mathbb{Z}\}, \quad K = \{\lambda(0, 0, a) \mid a \in \mathbb{Z}\} \tag{4.4.1}$$

(cf. (4.3.9)), and in particular $M = m$ in (4.3.10).

Now taking X_L and Y_L as in (4.3.7) we see that $Y_L \in K \otimes \mathbb{R}$ and $(X_L, \ell) = 1 > 0$, so the positive cone C is composed of the positive real multiples of $\lambda(0, 0, 1)$. That is, $\kappa = \lambda(0, 0, 1)$ and $\kappa' = \lambda(0, 0, \frac{1}{2m})$, and $C = \{\lambda(0, 0, y) \mid y > 0\}$, in the notation of (4.3.10).

Identify the upper half-plane $\mathbb{H}$ with $K \otimes \mathbb{R} + iC$ by mapping $\tau \in \mathbb{H}$ to

$$Z(\tau) := \lambda(0, 0, \tau) = \tau\kappa. \tag{4.4.2}$$

Then, writing $Z_L(\tau)$ for the point Z_L corresponding to $Z = Z(\tau)$ as in (4.3.11), we have $Z_L(\tau) = Z(\tau) - Q_L(Z(\tau))\ell + \ell'$ since $Q_L(\ell') = 0$, so that

$$Z_L(\tau) = \lambda(\tfrac{1}{N}, -m\tau^2, \tau) = \begin{pmatrix} \tau & -\tau^2 \\ 1 & -\tau \end{pmatrix} \tag{4.4.3}$$

(cf. (4.3.2)).

It is a pleasant exercise to check now that the action (4.3.5) of $SL_2(\mathbb{R})$ induces the usual action by Möbius transformations on $G(L) \simeq \mathbb{H}$. Concretely, we have

$$\mathbb{C}(\gamma \cdot Z_L(\tau)) = \mathbb{C}Z_L(\gamma\tau) \tag{4.4.4}$$

for $\gamma \in SL_2(\mathbb{R})$ and $\tau \in \mathbb{H}$. We set

$$\Psi_\ell(\tau, \check{F}) := \Psi_\ell(Z(\tau), \check{F}) = \Psi_L(Z_L(\tau), \check{F}) \tag{4.4.5}$$

(cf. (4.3.12)) when $\ell = \lambda(0, 1, 0)$.

To explain the computation of the points $\lambda^\perp$ (cf. (4.3.13)) suppose that $\lambda = \lambda(c, b, a) \in L$ satisfies $Q_L(\lambda) = ma^2 + Nbc < 0$. Then we have

$$\begin{aligned} (Z_L(\tau), \lambda) &= (\lambda(\tfrac{1}{N}, -m\tau^2, \tau), \lambda(c, b, a)) \\ &= 2ma\tau - mNc\tau^2 + b, \end{aligned} \tag{4.4.6}$$

so $(Z_L(\tau), \lambda) = 0$ just when τ is a solution of the quadratic polynomial $mNcX^2 - 2maX - b$. This polynomial has discriminant $4m(ma^2 + Nbc)$, which is negative by our choice of λ. So there is indeed a solution in $\mathbb{H}$, and it is unique. This is the point we call $\lambda^\perp$.

To compute $\check{F}_K = (\check{F}_{K,r})$ explicitly in terms of $\check{F} = (\check{F}_{i,j,r})$ (cf. (4.3.14)) let $\lambda \in L^*$ and let $\delta \in K^*$ belong to the coset defined by $r\kappa' = \lambda(0, 0, \frac{r}{2m})$. Then $\lambda = \lambda(\frac{c}{N}, \frac{b}{N}, \frac{a}{2m})$ for

some $a, b, c \in \mathbb{Z}$, and $\delta = n\kappa' = \lambda(0, 0, \frac{n}{2m})$ for some $n \equiv r \bmod 2m$. Letting $\lambda(0, b', a')$ be an arbitrary element of $L \cap \ell^\perp$ (cf. (4.4.1)) we have

$$(\lambda, \lambda(0, b', a')) = aa' + b'c, \quad (\delta, \lambda(0', b', a')) = na', \tag{4.4.7}$$

according to (4.3.3), so λ and δ coincide as homomorphisms on $L \cap \ell^\perp$ just when $a = n$ and $c = 0$. We conclude from this that $\check{F}_{K,r}$ is the sum over the components of $\check{F}$ indexed by $\lambda(0, \frac{j}{N}, \frac{r}{2m})$ for arbitrary $j \bmod N$. That is, the definition (4.3.14) works out to

$$\check{F}_{K,r} = \sum_{j \bmod N} \check{F}_{0,j,r}, \tag{4.4.8}$$

or more simply $\check{F}_K = \sum_{j \bmod N} \check{F}_{0,j}$ in this case.

Observe now using (4.2.12) that $\check{F}_K = \hat{F}_{(0,0)} = F^{(N)} = F_e^W$ under our hypotheses on $\check{F}$. Thus we have

$$H_\ell^W(g) = H(K, \check{F}_K) = \sum_{r \bmod 2m} \sum_{\substack{D \equiv r^2 \bmod 4m \\ D \le 0}} C^W(D, r) H_m(D, r) \tag{4.4.9}$$

(cf. (4.3.18–4.3.20)) for the generalized class number associated to ℓ and $\check{F} = \check{F}_g^W$, when ℓ is the infinite cusp, where $C^W(D, r)$ denotes the coefficient of $q^{\frac{D}{4m}}$ in $F_{e,r}^W$, for $F_e^W = (F_{e,r}^W)$. In particular, the generalized class number $H_\ell^W(g)$ is independent of g when ℓ represents the infinite cusp. For this reason we drop g from notation in $H_\ell^W(g)$ when $\ell = \lambda(0, 1, 0)$. Also dropping ℓ we obtain the definition of $H = H^W$ in (4.1.7) that we promised in § 4.1. Specifically, the definition is

$$H^W := \sum_{r \bmod 2m} \sum_{\substack{D \equiv r^2 \bmod 4m \\ D \le 0}} C^W(D, r) H_m(D, r), \tag{4.4.10}$$

where $C^W(D, r)$ and $H_m(D, r)$ are as in (4.4.9).

With the definition (4.4.10) of H^W in place we are ready to confirm the validity of the Borcherds product construction (4.1.7).

Proposition 4.4.1 *Suppose that G and W are as in the statement of Theorem 4.1.3, and set $H = H^W$. Then for each $g \in G$ the expression*

$$q^{-H} \exp\left(- \sum_{n>0} \sum_{k>0} C_{g^k}^W(n^2, n) \frac{q^{nk}}{k} \right) \tag{4.4.11}$$

converges for $\Im(\tau)$ sufficiently large, and extends by analytic continuation to a modular form Ψ_g^W of weight k_g^W (see (4.3.21)) for $\Gamma_0(mN_g)$.

Proof We will prove the claimed result by showing that the infinite product expression (4.4.11) coincides with the product (4.3.24) that we get from Theorem 4.3.1 when $\ell = \lambda(0, 1, 0)$.

To begin we take $Z = Z(\tau) = \tau\kappa$ (cf. (4.4.2)) in (4.3.24), interpret the second product in (4.3.24) as in (4.4.7–4.4.8), apply the computation (4.4.9–4.4.10) of $H^W = H_\ell^W(g)$, and recall the choice $\ell' = \lambda(\frac{1}{N}, 0, 0)$ so as to obtain the expression

$$q^{-H} \prod_{n>0} \prod_{j \bmod N} \left(1 - \mathrm{e}\left(\frac{j}{N}\right) q^n\right)^{\check{C}_{0,j}(n^2,n)}, \tag{4.4.12}$$

for (4.3.24). According to Item 4 of Theorem 4.3.1 this product (4.4.12) converges for $\Im(\tau)$ sufficiently large, and defines a modular form of weight k_g^W on $\mathbb{H}$ where k_g^W is as in (4.3.21).

Next we remove the factor q^{-H} from (4.4.12) for a moment, take a logarithm, and expand so as to arrive at

$$\log \prod_{n>0} \prod_{j \bmod N} \left(1 - \mathrm{e}\left(\frac{j}{N}\right) q^n\right)^{\check{C}_{0,j}(n^2,n)} = -\sum_{n>0} \sum_{k>0} \sum_{j \bmod N} \check{C}_{0,j}\left(n^2, n\right) \frac{(\mathrm{e}(\frac{j}{N}) q^n)^k}{k} \tag{4.4.13}$$

for $\Im(\tau)$ sufficiently large. Then we apply the inverse discrete Fourier transform (4.2.12) to the sum over j in the right-hand side of (4.4.13), and thus obtain the simpler expression

$$\log \prod_{n>0} \prod_{j \bmod N} \left(1 - \mathrm{e}\left(\frac{j}{N}\right) q^n\right)^{\check{C}_{0,j}(n^2,n)} = -\sum_{n>0} \sum_{k>0} \hat{C}_{(0,k)}\left(n^2, n\right) \frac{q^{nk}}{k}, \tag{4.4.14}$$

where $\hat{C}_{(i,j)}(D, r)$ denotes the coefficient of $q^{\frac{D}{4m}}$ in the Fourier expansion of $\hat{F}_{(i,j),r}$, and the relationship between the $\hat{F}_{(i,j)} = (\hat{F}_{(i,j),r})$ and $\check{F} = (\check{F}_{i,j,r})$ is as in Lemma 4.2.1.

The next step is to exponentiate, and replace the factor q^{-H}, and thus find that (4.4.12) coincides with

$$q^{-H} \exp\left(-\sum_{n>0} \sum_{k>0} \hat{C}_{(0,k)}(n^2, n) \frac{q^{nk}}{k}\right). \tag{4.4.15}$$

Finally we apply Lemma 4.2.5, which tells us that $\hat{F}^W_{(0,k)} = F^W_{g^k}$ when $F^{(n)} = F^W_{g^n}$. Thus we have $\hat{C}_{(0,k)}(n^2, n) = C^W_{g^k}(n^2, n)$, and the coincidence between (4.4.11) and (4.4.12) follows. This completes the proof. □

Now we know that the T_g^W are weakly holomorphic modular forms, we can move on to computing their weight. For this we need two lemmas. The first tells us how the $v_b(g|U)$ of (3.4.17) change under power maps $g \mapsto g^p$. The second uses the result of the first to identify the weight k_g^W (4.3.21) of Ψ_g^W with the weight of η_g^W, and thus confirms that the T_g^W all have weight 0.

Note that if $\mathrm{tr}(g|U)$ is a rational integer then $\mathrm{tr}(g^n|U)$ is a rational integer too, for every $n \in \mathbb{Z}$, because the eigenvalues defined by the action of g^n on U are just the n-th powers ξ_i^n of the eigenvalues ξ_i defined by g. Thus the question of how the $v_b(g|U)$ of (3.4.17) change under power maps makes sense.

Lemma 4.4.2 *Suppose that G is a finite group and U is a finite-dimensional G-module, and suppose that $g \in G$ is such that $tr(g|U)$ is a rational integer. Then for p a prime and $v_b(g) = v_b(g|U)$ as in (3.4.17) we have*

$$v_b(g^p) = \begin{cases} p v_{bp}(g) & \text{if } p \text{ divides } b, \\ p v_{bp}(g) + v_b(g) & \text{if } p \text{ does not divide } b. \end{cases} \tag{4.4.16}$$

Proof To begin we consider the behavior of the multiplicities $u_d(g) = u_d(g|U)$ of (3.4.13) under power maps $g \mapsto g^p$ for p prime. Thus we let p be a prime, and note first that if $\gcd(d, p) = 1$ then the p-th power of a primitive d-th root of unity is again a primitive d-th root of unity. So $u_d(g^p) = u_d(g)$ if d is not divisible by p. Next we observe that if d is divisible by p then the p-th power of a primitive d-th root of unity is a primitive $\frac{d}{p}$-th root of unity, but the multiplicity of the map $\xi \mapsto \xi^p$ in this case depends upon whether

d is divisible by p^2 or not. If so, then every primitive $\frac{d}{p}$-th root has p primitive d-th root preimages under $\xi \mapsto \xi^p$, while if p exactly divides d then there are just $p-1$ primitive preimages. Replacing d with dp in the previous sentence we obtain that the p-th powers of the primitive dp-th roots of unity constitute p copies of each primitive d-th root of unity if p divides d, and constitute $p-1$ copies of each primitive d-th root of unity if p does not divide d. So we have

$$u_d(g^p) = \begin{cases} pu_{dp}(g) & \text{if } p \text{ divides } d, \\ (p-1)u_{dp}(g) + u_d(g) & \text{if } p \text{ does not divide } d. \end{cases} \tag{4.4.17}$$

To finish we combine (3.4.18) and (4.4.17) in order to calculate $v_b(g^p)$. Taking b to be a multiple of p we find that

$$\begin{aligned} v_b(g^p) &= \sum_{a>0} \mu(a) v_{ab}(g^p) \\ &= \sum_{a>0} \mu(a) p u_{abp}(g) \\ &= p v_{bp}(g) \end{aligned} \tag{4.4.18}$$

(cf. (3.4.18)), in agreement with the first part of (4.4.16). Taking b now to be coprime to p we compute that

$$\begin{aligned} v_b(g^p) &= \sum_{a>0} \mu(a) v_{ab}(g^p) \\ &= \sum_{\substack{a>0 \\ \gcd(a,p)=1}} \mu(a)((p-1)u_{abp}(g) + u_{ab}(g)) + \sum_{\substack{a>0 \\ p|a}} \mu(a) p u_{abp}(g) \\ &= S_1 + S_0, \end{aligned} \tag{4.4.19}$$

where $S_1 = p \sum_{a>0} \mu(a) u_{abp}(g)$, and

$$\begin{aligned} S_0 &= \sum_{\substack{a>0 \\ \gcd(a,p)=1}} \mu(a) u_{ab}(g) - \sum_{\substack{a>0 \\ \gcd(a,p)=1}} \mu(a) u_{abp}(g) \\ &= \sum_{\substack{a>0 \\ \gcd(a,p)=1}} \mu(a) u_{ab}(g) + \sum_{\substack{a>0 \\ \gcd(a,p)=1}} \mu(ap) u_{abp}(g) \\ &= \sum_{\substack{a>0 \\ \gcd(a,p)=1}} \mu(a) u_{ab}(g) + \sum_{a>0} \mu(ap) u_{abp}(g) \\ &= \sum_{a>0} \mu(a) u_{ab}(g). \end{aligned} \tag{4.4.20}$$

Inspecting (3.4.18) we obtain that $S_1 = p v_{bp}(g)$ and $S_0 = v_b(g)$, so we have verified the second part of (4.4.16), and the proof of the lemma is complete. □

By repeated application of Lemma 4.4.2 we can compute $v_b(g^n)$ in terms of $v_b(g)$ for any $n \in \mathbb{Z}$. For example, it follows from an inductive argument using Lemma 4.4.2 that if g is as in the lemma and p is a prime, then

$$v_b(g^{p^k}) = \begin{cases} p^k v_{bp^k}(g) & \text{if } p \text{ divides } b, \\ \sum_{j=0}^{k} p^j v_{bp^j}(g) & \text{if } p \text{ does not divide } b, \end{cases} \tag{4.4.21}$$

for all $k \geq 0$.

Lemma 4.4.3 *Suppose that G is a finite group and U is a finite-dimensional G-module, and suppose that $g \in G$ is such that $tr(g|U)$ is a rational integer. Then for N an arbitrary positive integer and for $v_b(g) = v_b(g|U)$ as in (3.4.17–3.4.18) we have*

$$\frac{1}{N}\sum_{n|N}\phi\left(\frac{N}{n}\right)tr(g^n|U) = \sum_{n|N} v_n(g). \tag{4.4.22}$$

Proof To begin we note that $\mathrm{tr}(g^n|U) = v_1(g^n)$ for all $n \in \mathbb{Z}$, under our hypothesis that $\mathrm{tr}(g|U) \in \mathbb{Z}$. This is because we have

$$\begin{aligned}\mathrm{tr}(g|\Lambda_{-t}(U)) = \prod_i (1-\xi_i t) &= 1 - \left(\sum_i \xi_i\right) t + O(t^2) \\ &= 1 - \mathrm{tr}(g|U)t + O(t^2)\end{aligned} \tag{4.4.23}$$

for the left-hand side of the defining identity (3.4.17) (cf. (3.4.13)), where the ξ_i are the eigenvalues defined by the action of g on U, while the right-hand side satisfies

$$\prod_{b>0}(1-t^b)^{v_b(g)} = 1 - v_1(g)t + O(t^2). \tag{4.4.24}$$

The same argument applies with any power of g in place of g, so we require to show that

$$\sum_{n|N}\phi\left(\frac{N}{n}\right)v_1(g^n) = \sum_{n|N} N v_n(g) \tag{4.4.25}$$

for arbitrary $N > 0$. To obtain (4.4.25) we will actually prove the more general statement that

$$\sum_{n|N}\phi\left(\frac{N}{n}\right)v_m(g^n) = \sum_{n|N} N v_{mn}(g) \tag{4.4.26}$$

for $m, N > 0$, when $\gcd(m, N) = 1$.

To establish (4.4.26) we begin with the case that N is a prime power. Using (4.4.21) and the fact that $\sum_{n|N}\phi(n) = N$ we find that for $N = p^k$ with p a prime such that $\gcd(m, p) = 1$, the left-hand side of (4.4.26) works out to be

$$\begin{aligned}\sum_{j=0}^{k}\phi\left(p^{k-j}\right)v_m(g^{p^j}) &= \sum_{j=0}^{k}\phi(p^{k-j})\sum_{i=0}^{j} p^i v_{mp^i}(g) \\ &= \sum_{i=0}^{k} p^i \sum_{j=i}^{k}\phi(p^{k-j})v_{mp^i}(g) \\ &= \sum_{i=0}^{k} p^i \sum_{j=0}^{k-i}\phi(p^j)v_{mp^i}(g) \\ &= \sum_{i=0}^{k} p^k v_{mp^i}(g),\end{aligned} \tag{4.4.27}$$

which agrees with the right-hand side of (4.4.26).

Now suppose that we have proven (4.4.26) for $N = N'$ and for $N = N''$, where m, N' and N'' are pairwise coprime. Then for $N = N'N''$ we have

$$\begin{aligned}\sum_{n|N}\phi\left(\frac{N}{n}\right)v_m(g^n) &= \sum_{n'|N'}\sum_{n''|N''}\phi\left(\frac{N'N''}{n'n''}\right)v_m((g^{n'})^{n''})\\ &= \sum_{n'|N'}\phi\left(\frac{N'}{n'}\right)\sum_{n''|N''}\phi\left(\frac{N''}{n''}\right)v_m((g^{n'})^{n''})\\ &= \sum_{n'|N'}\phi\left(\frac{N'}{n'}\right)\sum_{n''|N''}N''v_{mn''}(g^{n'})\\ &= \sum_{n''|N''}N''\sum_{n'|N'}\phi\left(\frac{N'}{n'}\right)v_{mn''}(g^{n'})\\ &= \sum_{n''|N''}N''\sum_{n'|N'}N'v_{mn'n''}(g),\end{aligned} \tag{4.4.28}$$

which is $\sum_{n|N} Nv_{mn}(g)$. The identity (4.4.26) now follows for arbitrary $N > 0$ with $\gcd(m, N) = 1$ by induction on the number of prime divisors of N. Specializing to $m = 1$ we have thus proven (4.4.25), as required.

4.5 Construction

In this section we explain how to realize the infinite products that define T_g^W (cf. (4.1.7–4.1.9)) in terms of alternating and symmetric powers of subspaces of W (cf. (4.1.1)), and finally complete the proof of Theorem 4.1.3.

Taking G and W to be as in the statement of Theorem 4.1.3, the task at hand is to define a virtual graded G-module $V = V^W$ as in (4.1.5) with the property that $f_g^V = T_g^W$ for each $g \in G$, where f_g^V is as in (4.1.6) and T_g^W is as in (4.1.9). To ease notation as we do this let us define

$$U_n := \begin{cases} W_{n,\frac{n^2}{4m}} & \text{for } n > 0,\\ -2W_{0,0} & \text{for } n = 0,\end{cases} \tag{4.5.1}$$

and for each $n \geq 0$ write $U_n = U_n^f - U_n^b$, where U_n^f and U_n^b are as in (3.4.5) (with U_n in place of U). Then by applying Lemma 3.4.1 to the definitions (4.1.7–4.1.9) we obtain

$$\begin{aligned}T_g^W(\tau) &= q^{-h}\prod_{n>0}\mathrm{tr}(g|\Lambda_{-q^n}(U_n))\mathrm{tr}(g|\Lambda_{-q^n}(U_0))\\ &= q^{-h}\prod_{n>0}\mathrm{tr}(g|\Lambda_{-q^n}(U_n^f)\otimes S_{q^n}(U_n^b)\otimes\Lambda_{-q^n}(U_0^f)\otimes S_{q^n}(U_0^b)),\end{aligned} \tag{4.5.2}$$

where

$$h = H + \frac{1}{24}(\dim(U_0^b) - \dim(U_0^f)). \tag{4.5.3}$$

Next we make the convention that $U(-n)$ denotes a copy of U, for $U \in R(G)$ and n a positive integer, and define operators F and deg on such $U(-n)$ by requiring that

$$F = \begin{cases}\mathrm{Id} \text{ on } U^f(-n),\\ 0 \text{ on } U^b(-n),\end{cases} \qquad \deg = n\,\mathrm{Id} \text{ on } U(-n). \tag{4.5.4}$$

Then, we extend these operators to alternating, symmetric and tensor products of such spaces $U(-n)$ by applying the Lie-like coproduct, $X \mapsto X\otimes 1+1\otimes X$, so that deg acts as $(n'+n'')\mathrm{Id}$ on $U'(-n')\otimes U'(-n'')$, &c.

With F and deg so defined the right-hand side of (4.5.2) may be rewritten as

$$q^{-h}\prod_{n>0}\mathrm{tr}(g(-1)^F q^{\mathrm{deg}}|\Lambda(U_n^f(-n))\otimes S(U_n^b(-n))\otimes\Lambda(U_0^f(-n))\otimes S(U_0^b(-n))) \tag{4.5.5}$$

where h is as in (4.5.3). Thus, if we define $\mathcal{H}(-n)$ for $n>0$ by setting

$$\mathcal{H}(-n) := \Lambda(U_n^f(-n))\otimes S(U_n^b(-n))\otimes\Lambda(U_0^f(-n))\otimes S(U_0^b(-n)), \tag{4.5.6}$$

and take $\mathcal{H} := \bigotimes_{n>0}\mathcal{H}(-n)$, then deg defines a $\mathbb{Z}$-grading $\mathcal{H}=\bigoplus_{n\geq 0}\mathcal{H}_n$ on $\mathcal{H}$, and F defines a superspace structure $\mathcal{H}_n=\mathcal{H}_n^0\oplus\mathcal{H}_n^1$ on each $\mathcal{H}_n$ via the requirement that $(-1)^F=(-1)^j$ on $\mathcal{H}_n^j$.

We now define $V^W=\bigoplus_{n\geq 0}V_n^W$ to be the virtual graded G-module that is determined by requiring that

$$m_\chi(V_n^W)=m_\chi(\mathcal{H}_n^0)-m_\chi(\mathcal{H}_n^1) \tag{4.5.7}$$

(cf. (3.4.4)) for each $\chi\in\mathrm{Irr}(G)$ and $n\geq 0$. Then taking $V=V^W$ we have that

$$f_g^V(\tau)=\sum_{n\geq 0}\mathrm{tr}(g|V_n)q^{n-h}=\mathrm{tr}\left(g(-1)^F q^{\mathrm{deg}-h}\middle|\mathcal{H}\right) \tag{4.5.8}$$

coincides with (4.5.5) for h as in (4.5.3). That is, $f_g^V=T_g^W$ for all $g\in G$, and we have realized the T_g^W as graded traces on a virtual graded G-module constructed (4.5.6–4.5.7) from tensor products of alternating and symmetric powers of subspaces of W, as we promised we would. In particular, since we have shown in § 4.4 that the T_g^W are weakly holomorphic modular forms of weight 0, and since a virtual graded G-module is determined by its graded traces, we have completed the proof of Theorem 4.1.3.

Remark 4.5.1 Each alternating algebra factor $\Lambda(U_n^f(-n))$ in (4.5.6) may be realized as a Fock space built out of the repeated application of fermionic creation operators, and similarly for the $\Lambda(U_0^f(-n))$, while each symmetric algebra factor $S(U_n^b(-n))$ may be realized as a Fock space built out of bosonic creation operators, and similarly for the $S(U_0^b(-n))$. Thinking along these lines, it is tempting to interpret deg in (4.5.4–4.5.5) and (4.5.8) as the grading operator L_0 defined by an action of the Virasoro algebra with central charge $c=24h$ on V, for h as in (4.5.3). Indeed, such an action exists in the case that $W=W_{3\mathrm{C}}$, in the notation of [16], as V^W is isomorphic to a tensor power of the 3C-twisted module $V_{3\mathrm{C}}^\natural$ for the moonshine module VOA for this choice of W, according to Theorem 4.5 of op. cit. (See also Example 5.1.1 below.) However, the construction we have given does not (obviously) entail any Virasoro action, or VOA or CFT structure on V^W in general.

5 Outlook

We provide further perspective on the results of this paper in this section. To do this we first explain an approach for applying traces of singular moduli to the problem of constructing an inverse to SQ in Section 5.1. Then, with this approach in place, we are in a better position to

appreciate the import of our results for penumbral and umbral moonshine. We comment on this in Section 5.2.

5.1 Inversion

Here we consider the problem of inverting the construction SQ. That is, given a weakly holomorphic G-module V of weight 0 as in (4.1.5), and given the hypothesis that $V = V^W = SQ(W)$ (cf. (4.1.10)), for some weakly holomorphic G-module W of weight $\frac{1}{2}$ and positive integer index, we here pursue a strategy for determining the values $C_g^W(D, r)$ as in (4.1.4), that characterize the G-module structure on W, given just the functions $f_g^V = T_g^W$ (cf. (4.1.9)), that characterize the G-module structure on $V = SQ(W)$. The challenge in this is that the T_g^W depend only on the $C_g^W(D, r)$ for perfect-square values of D, according to (4.1.7–4.1.8). In particular, we can read off the $C_g^W(n^2, n)$ for $n \in \mathbb{Z}$ from these formulas (4.1.7–4.1.8), if given the functions $f_g^V = T_g^W$. Thus we are tasked with determining the values $C_g^W(D_1 n^2, r_1 n)$ for $D_1 > 1$ fundamental, for r_1 such that $D_1 \equiv r_1^2 \bmod 4m$, and for integers n, given just the $C_g^W(n^2, n)$. As we will see, it develops that we can solve this problem for $g = e$ the identity element of G in terms of traces of singular moduli, as introduced in [30], by applying results of [4]. The problem of extending our approach to non-identity values of g depends upon a natural extension of the methods of op. cit. (cf. Section 1.4).

To explain our approach let G be a finite group, let V be a weakly holomorphic G-module of weight 0, and let us assume, as above, that a rational weakly holomorphic G-module W of weight $\frac{1}{2}$ and index m such that $V = V^W = SQ(W)$ exists. Next let $D_1 > 1$ be a fundamental discriminant that is a square modulo $4m$, and let r_1 be a witness to this fact, so that $D_1 \equiv r_1^2 \bmod 4m$. Also let W_m denote the coset of $\Gamma_0(m)$ represented by the Fricke involution, $\frac{1}{\sqrt{m}}\left(\begin{smallmatrix} 0 & -1 \\ m & 0 \end{smallmatrix}\right)$, and write $\Gamma_0(m) + m$ for the extension of $\Gamma_0(m)$ it defines,

$$\Gamma_0(m) + m := \Gamma_0(m) \cup W_m. \tag{5.1.1}$$

Then according to Theorem 6.1 of [4] we have that the twisted Borcherds product Ψ_{D_1,r_1}^W, defined by (1.4.2), is a meromorphic function on $\mathbb{H}$ that is invariant for the action of $\Gamma_0(m)+m$. Moreover, the induced meromorphic function on $X_0(m)$ has divisor given by

$$Z_{D_1,r_1}^W := \sum_{r \bmod 2m} \sum_{\substack{D<0 \\ D \equiv r^2 \bmod 4m}} C^W(D, r) Z_{D_1,r_1}^{(m)}(D, r), \tag{5.1.2}$$

where $Z_{D_1,r_1}^{(m)}(D, r)$ is as in (3.3.5).

Now suppose that m is chosen so that the compact Riemann surface $X_0(m)+m$ associated to $\Gamma_0(m) + m$ (cf. (3.3.4)) has genus zero. Then $\Gamma_0(m) + m$ admits a unique normalized principal modulus (a.k.a. Hauptmodul), $T^{(m+m)}$. From the statement just made about Ψ_{D_1,r_1}^W, and in particular from the explicit description (5.1.2) of the divisor Z_{D_1,r_1}^W (cf. (3.3.5)), we may write Ψ_{D_1,r_1}^W as a product of (finitely many) integer powers of terms of the form $T^{(m+m)}(\tau) - T^{(m+m)}(\alpha_Q)$, for various CM points α_Q (cf. (3.3.5)). This is useful because $T^{(m+m)}$ is replicable in the sense of [13] (see also [14]). According to § 1 of [14] (see

also § 5 of [5]) this means that there exist holomorphic functions $T_{(a)}^{(m+m)}$ for $a > 0$ such that

$$T^{(m+m)}(\tau) - T^{(m+m)}(\alpha_Q) = q^{-1} \exp\left(- \sum_{n>0} \sum_{ad=n} \sum_{b \bmod d} T_{(a)}^{(m+n)}\left(\frac{a\alpha_Q + b}{d}\right) \frac{q^n}{n} \right). \tag{5.1.3}$$

(In fact, according to [13] we have $T_{(a)}^{(m+m)} = T^{(m+m)}$ if $\gcd(a, m) = 1$, and $T_{(a)}^{(m+m)} = T^{(m')}$ for $m' = \frac{m}{\gcd(a,m)}$ otherwise, where $T^{(m)}$ denotes the normalized principal modulus associated to $\Gamma_0(m)$. In particular, $X_0(m')$ is genus zero whenever $X_0(m) + m$ is genus zero and m' is a proper divisor of m.)

Now by comparing powers of q in $\log \Psi_{D_1,r_1}^W(\tau)$ and $\log(T^{(m+m)}(\tau) - T^{(m+m)}(\alpha_Q))$, where the latter is computed using (5.1.3), we can deduce explicit expressions for the $C^W(D_1 n^2, r_1 n)$ in terms of the values $T_{(a)}^{(m+m)}(\frac{a\alpha_Q+b}{d})$, for explicitly determined CM points α_Q.

Example 5.1.1 Suppose that V is such that $f^V(\tau) = f_e^V(\tau)$ is the elliptic modular invariant j, that appears in (1.1.1). (Such a G-module $V = (V_{3C}^\natural)^\otimes$ is considered in [16], for $G = Th$ the sporadic simple Thompson group. Cf. the discussion in § 1.2, and see also Remark 4.5.1). Noting that j is invariant for the action of $SL_2(\mathbb{Z}) = \Gamma_0(1)$ we guess that the corresponding weakly holomorphic G-module W of weight $\frac{1}{2}$ has index $m = 1$. The relevant normalized principal modulus in this case is $T^{(1)} = j - 744$, and we have $T_{(a)}^{(1)} = T^{(1)}$ for all $a > 0$ in (5.1.3). (We write $T^{(1)}$ instead of $T^{(1+1)}$ because $\Gamma_0(1) + 1 = \Gamma_0(1)$.) Thus, given $D_1 > 1$ fundamental, and taking $r_1 \equiv D_1 \bmod 2$, we expect Ψ_{D_1,r_1}^W as in (1.4.2) to coincide with a product of expressions of the form

$$J(\tau) - J(\alpha_Q) = q^{-1} \exp\left(- \sum_{n>0} \sum_{ad=n} \sum_{b \bmod d} J\left(\frac{a\alpha_Q + b}{d}\right) \frac{q^n}{n} \right), \tag{5.1.4}$$

for CM points α_Q, where we follow tradition in writing J for $T^{(1)} = j - 744$.

The particular CM points arising are determined by the divisor Z_{D_1,r_1}^W, which according to (5.1.2) depends only on the singular part of W. The singular part of W also determines the generalized class number $H = H^W$ that appears in the definition (1.3.3) of $\Psi^W = \Psi_e^W$, according to (4.4.10), so we can use knowledge of the latter to constrain the possibilities for the former. Let us suppose that $W_{0,0}$ is vanishing, so that $f^V = T_e^W = \Psi_e^W$ (cf. (4.1.7–4.1.9)). Then from (1.1.1) we have that $H = 1$. Comparing with (4.3.19) we conclude that the simplest possibility is that $C^W(D, r)$ vanishes for $D \leq 0$ unless $D = -3$ and $r = 1$, in which case $C^W(-3, 1) = 3$. Supposing that this possibility holds we can compute Z_{D_1,r_1}^W concretely. Indeed, using (3.3.5) and (5.1.2) we obtain that

$$Z_{D_1,r_1}^W = 3Z_{D_1,r_1}^{(1)}(-3, 1) = \sum_{Q \in \mathcal{Q}_{-3D_1,r_1}^{(1)}} 3\chi_{D_1}^{(1)}(Q)\overline{\alpha_Q}, \tag{5.1.5}$$

where $\chi_{D_1}^{(1)}$ is as in (3.3.3). Thus, applying (5.1.4), and the fact that $\sum_{Q\in\mathcal{Q}_{-3D_1,r_1}^{(1)}} \chi_{D_1}^{(1)}(Q) = 0$, we have

$$\Psi_{D_1,r_1}^W(\tau) = \prod_{Q\in\mathcal{Q}_{-3D_1,r_1}^{(1)}} \big(J(\tau) - J(\alpha_Q)\big)^{3\chi_{D_1}^{(1)}(Q)} \tag{5.1.6}$$

$$= \exp\left(-\sum_{n>0}\sum_{Q\in\mathcal{Q}_{-3D_1,r_1}^{(1)}} 3\chi_{D_1}^{(1)}(Q)\sum_{ad=n}\sum_{b \bmod d} J\left(\frac{a\alpha_Q+b}{d}\right)\frac{q^n}{n}\right).$$

To obtain concrete expressions for the coefficients $C^W(D_1n^2, r_1n)$ we apply the Gauss identity

$$\sum_{a \bmod D}\left(\frac{D}{a}\right)\mathrm{e}\left(\frac{ab}{D}\right) = \sqrt{D}\left(\frac{D}{b}\right) \tag{5.1.7}$$

to the $g = e$ case of (1.4.1) so as to obtain

$$\Psi_{D_1,r_1}^W(\tau) = \exp\left(-\sum_{n>0}\sqrt{D_1}\sum_{ad=n} a\left(\frac{D_1}{d}\right)C^W(D_1a^2, r_1a)\frac{q^n}{n}\right). \tag{5.1.8}$$

Then, identifying the coefficients of q^n in the logarithms of (5.1.6) and (5.1.8) we obtain

$$\sqrt{D_1}\sum_{ad=n} a\left(\frac{D_1}{d}\right)C^W(D_1a^2, r_1a) = \sum_{Q\in\mathcal{Q}_{-3D_1,r_1}^{(1)}} 3\chi_{D_1}^{(1)}(Q)\sum_{ad=n}\sum_{b \bmod d} J\left(\frac{a\alpha_Q+b}{d}\right) \tag{5.1.9}$$

for $n > 0$. Thus we determine the $C^W(D_1n^2, r_1n)$ recursively in terms of the $C^W(D_1a^2, r_1a)$ for $a < n$, and the values $J(\frac{a\alpha_Q+b}{d})$ for $Q \in \mathcal{Q}_{-3D_1,r_1}^{(1)}$. In particular, taking $n = 1$ we obtain the formula

$$C^W(D_1, r_1) = \frac{1}{\sqrt{D_1}}\sum_{Q\in\mathcal{Q}_{-3D_1,r_1}^{(1)}} 3\chi_{D_1}^{(1)}(Q)J\left(\alpha_Q\right) \tag{5.1.10}$$

for $D_1 > 1$ fundamental. We refer to the right hand-side of (5.1.10) as a (twisted) trace of singular moduli, following [30].

From the above discussion we conclude that we can solve the problem of recovering W from $V = SQ(W)$ in terms of traces of singular moduli, at least in the case that $G = \{e\}$ is trivial, so long as the graded dimension function $f^V = f_e^V$ is invariant for a group $\Gamma_0(m)+m$ that has genus zero, and so long as we have some knowledge of the singular terms in the graded dimension function $F^W = F_e^W$ of W. (We can actually weaken the condition that $\Gamma_0(m)+m$ have genus zero, and handle non-trivial G in certain circumstances. See Remarks 5.1.2 and 5.1.2 below.)

In principle the problem of recovering the $C_g^W(D_1n^2, r_1n)$ for non-trivial $g \in G$ can be handled in a directly similar way, except that the results of [4] do not extend to the twined twisted Borcherds products $\Psi_{D_1,r_1,g}^W$ of (1.4.1), for $g \neq e$ (but see Remark 5.1.2 below). As we have mentioned in § 1.4, we expect that the main step in determining such an extension of the results of op. cit. will be an analysis of twisted Siegel theta functions $\Theta_{D_1,r_1,N}^{(m)}$, based on the lattices $L_{m,N}$ (see (3.1 .6)) for $N > 1$. For concrete hints as to what to expect from

such an analysis we refer the reader to § 4.3 of [16], wherein we present (mostly conjectural) expressions in terms of traces of singular moduli for (most of) the McKay–Thompson series associated to a weakly holomorphic *Th*-module $W_{3\mathrm{C}}$ of weight $\frac{1}{2}$. This weakly holomorphic *Th*-module $W_{3\mathrm{C}}$ has index 1, and is mapped by *SQ* to a *Th*-module $(V_{3\mathrm{C}}^{\natural})^{\otimes 3}$ with graded dimension given (up to rescaling) by j, just as in Example 5.1.1.

To better appreciate the generality, and specificity, of the method we have described for inverting *SQ* via traces of singular moduli, we offer the following remarks.

Remark 5.1.2 It is not necessary that m be such that $\Gamma_0(m) + m$ has genus zero in order for the approach we have sketched to apply. This is because the invariance group of $V = V^W = SQ(W)$ will be an extension

$$\Gamma_0(m) + n, n', \ldots, m \tag{5.1.11}$$

of $\Gamma_0(m) + m$ by further Atkin–Lehner involutions, W_n, $W_{n'}$, ..., in addition to W_m (cf. (5.1.1)), under suitable conditions on W. If this extension (5.1.11) has genus zero then the above approach goes through, with the associated normalized principal modulus taking on the role of $T^{(m+m)}$.

Remark 5.1.2 The fact that the right hand-side of (5.1.10) reduces to a formula in terms of quadratic forms of a single discriminant depends upon our assumption in Example 5.1.1 that there is a unique negative discriminant $D = D_0 < 0$ such that $C^W(D, r)$ does not vanish. As we see from (5.1.2), the divisor of $\Psi^W_{D_1,r_1}$ generally involves CM points of every discriminant DD_1 such that $D < 0$ and $C^W(D, r)$ is not zero. Thus the same is true for the sums in general counterparts to (5.1.5) and (5.1.10).

Remark 5.1.2 It develops that the untwined twisted Borcherds products $\Psi^W_{D_1,r_1}$ of (1.4.2) can sometimes be used to recover expressions in terms of traces of singular moduli for Fourier coefficients $C^W_g(D, r)$, for non-trivial $g \in G$. This is because the assignment $F(\tau) \mapsto F(M\tau)$, for M a positive integer, defines a level-raising map

$$\mathrm{V}^{\mathrm{wh}}_{\frac{1}{2},mM}(N) \to \mathrm{V}^{\mathrm{wh}}_{\frac{1}{2},m}(MN) \tag{5.1.12}$$

(and similarly with $\mathbb{V}$ in place of V), which in special situations can be used to relate coefficients $C^W_g(D, r)$ and $C^{W'}_{g'}(D, r)$, for weakly holomorphic modules W and W' of weight $\frac{1}{2}$, and respective indexes m and m', when it holds that $mo(g) = m'o(g')$. That is, in certain circumstances we can circumvent the problem that F^W_g has level $N_g > 1$ (cf. (4.1.3)), by identifying its coefficients as those of $F^{W'}_e$, for some weakly holomorphic G'-module W' of weight $\frac{1}{2}$ and index $m' = mo(g)$, for some auxiliary group G'.

5.2 Import

We conclude with some comments on the import of this work for moonshine.

To begin we point out that for each lambdency $\lambda = (D_0, \ell)$ of penumbral moonshine, as described in [15], the $G^{(\lambda)}$-module $W^{(\lambda)}$ is rational weakly holomorphic of weight $\frac{1}{2}$ with (positive integer) index m, in the sense of Section 4.1, where m is the level of $\ell = m + n, n', \ldots, m$. By a similar token, for each lambency $\ell = m + n, n', \ldots$ of umbral moonshine, as described in [9–11], the $G^{(\ell)}$-module $K^{(\ell)}$ is rational weakly holomorphic mock modular of weight $\frac{1}{2}$ with index $-m$, in the sense of Remark 4.1.2. Thus, according to Remark 4.1.4, the construction *SQ* of (4.1.10) applies to all cases of penumbral moonshine,

but to no cases of umbral moonshine. However, the notion of twisted Borcherds product, as formulated in Section 1.4, allows us to put this in a broader perspective, as we will presently see.

With respect to twisted Borcherds products, it is significant that the special circumstances mentioned in Remarks 5.1.2–5.1.2 all manifest in penumbral and umbral moonshine. To explain this write $F_g^{(\lambda)}$ in place of F_g^W when $W = W^{(\lambda)}$ and $g \in G^{(\lambda)}$, for $\lambda = (D_0, \ell)$ a lambdency of penumbral moonshine. Then for such $W = W^{(\lambda)}$ the D_0 arising in Remark 5.1.2 is the D_0 in $\lambda = (D_0, \ell)$, and the genus zero group (5.1.11) arising in Remark 5.1.2 is specified by the lambency symbol $\ell = m + n, n', \ldots, m$. In umbral moonshine the situation is similar, except that D_0 is 1 in every case, so it is suppressed from notation, and the lambency symbol $\ell = m + n, n', \ldots$ specifies a genus zero group that does not include the Fricke involution W_m. (It develops that the identity (3.2.8) is responsible for the presence, or absence, of W_m, and there are analogous identities for the $n, n', \ldots$ in (5.1.11). See [8] or [15] for more detail.) Also, concrete examples of the multiplicative relations of Remark 5.1.2, wherein coefficients of $F_g^{(\lambda)}$ are written in terms of those of $F^{(\lambda')} = F_e^{(\lambda')}$, for suitable lambdencies λ and λ', and $g \in G^{(\lambda)}$, can be found in Tables 4–5 of [15], and multiplicative relations for umbral moonshine can be found in Tables 8–9 of [10] (see also Table 2 of [8]).

The primary motivation for the data D_0 and ℓ that constitute a penumbral lambdency $\lambda = (D_0, \ell)$ is that they allow us to formulate the sense in which $F^{(\lambda)} = F_e^{(\lambda)}$ is genus zero and optimal, and thereby serves as a natural weight $\frac{1}{2}$ counterpart to a principal modulus. We refer to the introduction of [15], and especially § 1.6 of op. cit., for a detailed discussion of this. (See also § 4.4 of [16], and the forthcoming work [17].) We emphasize here that the multiplicative relations of penumbral and umbral moonshine, which we have sketched in a general way in Remark 5.1.2, depend upon the genus zero and optimality properties that we have just described, in relation to Remarks 5.1.2–5.1.2. We refer to § 4.2 of [15], and also the latter part of Section 4.1 of op. cit., for more detail on the multiplicative relations of penumbral moonshine. A counterpart discussion for umbral moonshine can be found in [10].

We have indicated that the special circumstances of Remarks 5.1.2–5.1.2 hold in umbral moonshine, but our lift of the singular theta lift (4.1.10) does not apply to umbral moonshine, according to Remark 4.1.4. This brings us back to the results of [8, 26] that we referred to in Section 1.4, wherein untwined twisted Borcherds products very similar to the $\Psi^W_{D_1,r_1}$ of (1.4.2) are used to constructively connect cases of umbral moonshine with principal moduli for genus zero groups. In fact the essential difference in the umbral case is that D_1 should be a negative fundamental discriminant rather than a positive one, because the modules of umbral moonshine are weakly holomorphic (mock) modular of weight $\frac{1}{2}$ with negative index, rather than positive index (cf. Remark 4.1.2). The untwisted Borcherds product construction (1.3.3) that underpins SQ should be regarded as the $D_1 = r_1 = 1$ case of the more general twisted construction (1.4.1) that we specified in Section 1.4, so we may say that our main result in this paper, Theorem 4.1.3, fails to apply to umbral moonshine, for the simple reason that $D_1 = 1$ is not negative.

Our final comment is that, apart from the fact that D_1 in (1.4.2) should be negative in the umbral setting, the aforementioned results of [8, 26] are very similar to what we have described in this section, in that they produce formulae very similar to (5.1.10), for the coefficients of the graded dimension functions of umbral moonshine in terms of traces of singular moduli. We conclude that the apparent prejudice of our Borcherds product construction SQ (4.1.10), for penumbral moonshine over umbral moonshine, is appeased if we can positively answer the question posed at the end of Section 1.4, on lifting the twisted Siegel theta lift (1.4.3), for arbitrary fundamental D_1, to the level of G-modules too.

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