# Fair Group Summarization with Graph Patterns

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***Abstract*—Given a set of node groups in a graph (*e.g.*, gender or race groups), how to succinctly summarize their neighbors, and meanwhile ensure a "fair" representation to mitigate under- or over-representation of a certain group? We propose a novel framework to compute concise summaries of node groups with fairness guarantees. (1) We introduce a pattern-correction structure called $r$-summaries. An $r$-summary uses a graph pattern set to specify representative nodes and an auxiliary edge correction set to losslessly describe their $r$-hop neighbors. (2) We formulate the fair group summarization problem, which is to compute an $r$-summary that can select and accurately describe high quality nodes and their neighbors with small edge corrections, and meanwhile guarantee a desirable coverage for each group. The need for generating such summaries is evident in social recommendation, healthcare and graph search. We show that the problem is $\Sigma_2^p$-complete with the verification problem already NP-complete. (3) We present approximation algorithms that can generate $r$-summaries with (a) guaranteed quality and coverage properties, and (b) relative approximations on optimal edge correction costs. For large groups, we introduce an efficient algorithm that interleaves node selection and localized pattern discovery to reduce unnecessary computation. In addition, we introduce an algorithm to incrementally maintain the $r$-summaries over dynamic graphs with evolving edges. Using real-world data, we experimentally verify the efficiency and effectiveness of our algorithms and verify their applications.**

***Index Terms*—attributed graph, graph summarization, fairness**

## I. Introduction

Graph summarization has been used to support large-scale graph analysis [26]. Given a graph $G$, it is to generate compact summary structures $\mathcal{S}$ that (approximately) represent $G$ that also preserves certain properties with queryable structures. A common practice is to follow Minimum Description Length (MDL) principle, which aims to minimize the size of summaries and the corresponding description length of the graph. This is often implemented by frequent pattern mining [46] in favor of subgraphs with a high compression rate, to support downstream tasks such as graph search [29], community detection [8] or influence analysis [25].

Emerging graph analysis with fairness requirements [37], [39], nevertheless, poses new challenges. A common scenario interprets fairness as group coverage constraints [44], [18], [33], [28]. Given a set of node groups, it is desirable to (1) select and concisely describe a set of representative nodes with desirable quality from each group, and (2) ensure a satisfactory "coverage" of each group to prevent under- or over-presentation of certain groups. In practice, such groups may refer to vulnerable social determined by groups *e.g.,* gender, race or professions [15], relevant yet under-represented topics [3], recommendations [17], or designated columns for query benchmarking [5]. Consider the following scenarios.

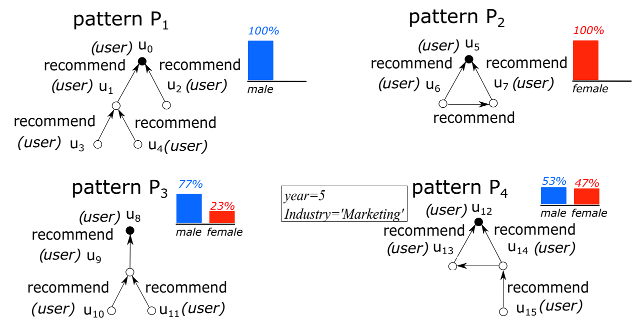


Fig. 1. Summarizing Social Connections in Talent Search.

**Example 1: [Talent Search]**. Consider a real-world social network $G$ [17] where each node in $G$ denotes a user with attributes such as *title* and *skill*. Each edge indicates a recommendation (recommend) between users. We illustrate two most frequent subgraph patterns $P_1$ and $P_2$ (illustrated in Fig. 1), which are separately mined from sub-networks of $G$ that are induced by male-only and female-only users, respectively. They interestingly demonstrate that male and female professional users in general have quite different social connectivity patterns. For example, female professionals may favor more active interactions in a small social community (dual networks or "inner circles"), while male users benefit from "high centrality" patterns, as also observed in [47].

A recruiter wants to explore $G$ to promote talent search with "equal opportunity" [17], for which a set of candidates with balanced gender distribution are preferred. She may also want to understand the social connections of these candidates to improve talent search. Neither $P_1$ nor $P_2$ satisfies such requirements due to their bias to a specific gender. A desirable summary structure with graph patterns should draw an almost equal number of male and female users, and describe their neighborhood via graph pattern matching as accurately as possible, to guide the talent search. □

One may consider summarization with frequent subgraphs. However, this may lead to a "skewed" distribution towards majority groups, leading to biased analysis. Another option is to "diversify" these patterns to cover different nodes. Nevertheless, it is not easy to ensure coverage for each group.

**Example 2:** Consider a graph pattern $P_3$ in Fig. 1 computed via frequent pattern mining [12], which is among the ones with the highest support. It indeed covers a large population of the groups. Nevertheless, the users that match $u_8$ come

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with a biased gender distribution of 77% male and 23% of female. This is close to the actual distribution of the gender groups in $G$. It suggests that frequent patterns are sensitive to the skewed gender distribution and over-present the majority groups. Patterns like $P_3$, if suggested as queries, may lead to both biased search results towards male candidates, but also suggest a biased understanding of how talented candidates benefit from social patterns (*e.g.*, "high centrality" only). □

**Example 3: [Pandemic Analysis]**. In a real-world pandemic spreading network [1], each node denotes a citizen with personal information such as *age groups*, *gender* and (infectious) *history*. Each edge represents routine close contact (contact) between two citizens [48]. Given a budget $k$ of vaccines, a policy maker will need to choose $n$ citizens as "seed" set to apply the vaccines to control the expected spread of the pandemic following the network. That is, she wants to select $k$ nodes that may maximize the spread of the pandemic if no vaccine is given, under a (monotonic submodular) influence maximization function [48] (group immunization). Meanwhile, she wants to investigate the impact of different age distribution to the spread by enforcing configurable coverage constraints to different age groups of the seed set, and to extract their common social connection to better understand the propagation mechanism. Existing frequent pattern discovery and graph summarization methods cannot be used to find summary structures that meanwhile satisfy the configurable coverage requirement over age groups. □

The above examples call for effective graph summary structures that can *simultaneously* support (1) *selection* of a set of high-quality nodes from groups of interests, with guaranteed group coverage that are configurable by users, and (2) *losslessly summarize* their neighbors, with small "reconstruction" effort. The problem has a general form below.

- **Input:** A graph $G$, a set of groups $\mathcal{V}$ of the same type of nodes in $G$, where each group $P_i \in \mathcal{V}$ is associated with a range $[l_i, u_i]$ (a pair of integers where $l_i \leq u_i \leq |P_i|$), denoting required coverage;
- **Output:** a summary structure $\mathcal{S}$ with graph patterns that (1) selects ("covers") $n_i$ nodes from each group $V_i \in \mathcal{V}$ via graph pattern matching, such that $n_i \in [l_i, u_i]$, and the covered nodes maximize a monotone submodular utility function $F$, and (2) provides auxiliary structure that can losslessly reconstruct the neighbors of the selected nodes.

**Example 4:** A better summary structure may present pattern $P_4$, which "integrates" high centrality and a circle social structure, with informative selection criteria on "work experience" and "industry". This pattern leads to a proper selection of 53% males and 47% females, identified by the pattern node $u_{12}$. □

Although desirable, *how to characterize and efficiently compute such summaries for configurable utility functions and coverage requirements over groups?*

**Contributions**. This paper investigates group summarization with graph patterns with fairness constraints. We characterize the group fairness as a set of coverage constraints defined on individual groups. We introduce feasible algorithms to compute and maintain summaries with guarantees on user-defined quality and coverage constraints.

(1) We introduce $r$-*summaries*, a class of "pattern-correction" structures to summarize node groups in graphs (Section II). An $r$-summary has a set of graph patterns with a designated node type, and a set of edge corrections to guide the reconstruction of neighborhood nodes and edges up to $r$-hop, for each node that is "covered" by the summary structure.

(2) We introduce quality measures for an $r$-summary, in terms of conciseness, coverage properties and utility of the nodes (Section III). Based on these quality measures, we formalize the problem of *graph summarization with group fairness* (denoted as FGS) as a min-max optimization problem. Given a group $\mathcal{V}$, our goal is to compute an $r$-summary with $k$ patterns that selects $n$ nodes in $\mathcal{V}$ that satisfy the coverage constraints, minimizes correction cost, and maximizes the utility.

We establish the hardness result for FGS. We show that it is already NP-complete to verify if a summary structure is an $r$-summary for a group $\mathcal{V}$, and provide procedures for the verification problem. We further show that FGS is in general $\Sigma_2^P$-complete, by establishing a connection to graph reconstruction problem, a known $\Sigma_2^P$-complete problem. Here $\Sigma_2^P$ refers to the class of problems solvable in NP with an oracle for an NP-complete problem.

(3) We introduce an approximation scheme for FGS (Section IV). We represent the min-max form of FGS into a bi-level optimization problem, and use a "select-and-summarize" strategy to compute $r$-summaries with small accumulated cost at node level, all subject to coverage constraints. We show that this ensures a *relative optimality guarantee* in the form of $(\frac{1}{2}, \ln(n))$-approximation, which computes $r$-summaries that can (a) approximate optimal node set with $\frac{1}{2}$ ratio, and (b) simultaneously achieves $ln(n)$-approximation of optimal correction cost for the *fixed* node selection. This is a "weaker" form of global approximation guarantee, yet produces desirable summary structures given the guaranteed node quality, coverage requirements, and small accumulated cost that bounds the actual reconstruction cost.

Specifying the approximation scheme, we introduce (1) approximations for FGS with a bounded number of patterns (Section V), and (2) an efficient online algorithm that interleaves node selection and summary generation, with a matching guarantee of $(\frac{1}{4}, ln(n) - \frac{1}{k})$, when $\mathcal{V}$ is large (Section VI). These results provide flexible summarization strategies.

(4) We further develop an incremental algorithm to maintain $r$-summaries upon the arrival of new edges to the groups (Section VII). We incrementalize the computation of the node selection and summarization. Instead of rediscovering new patterns from scratch, we perform an efficient swapping strategy to control the number of $r$-summaries for conciseness.

(5) Using real-life graphs, we verify the effectiveness and efficiency of our algorithms (Section VIII). Our algorithms

can generate summaries with both desired quality and a small amount of edge corrections in covering designated groups. These algorithms are also feasible. For example, it takes up to 400 seconds to generate summaries in real-life graphs with 5 million nodes and 45 million edges. Our case analysis also verifies their applications in supporting talent search and query processing under fair constraints.

**Related Work.** We categorize the related work as follows.

*Graph summarization*. Graph summarization has been studied with various optimization goals (see [26] for a survey). Most approaches follow minimum description length (MDL) principle to discover (pre-defined) structural patterns that lead to high compression rates of large graphs [26], [23], [10], [30], [41], leveraging frequent subgraph pattern mining [12] . For example, frequent stars, bipartite graphs, cliques or chains are used as vocabularies to encode succinct descriptions of large social or knowledge graphs [23], or for visual analysis [10]. Sparse patterns are detected to understand and sample community structures [30]. $d$-summaries [41] construct computationally efficient patterns to approximately describe neighborhood information, which uses an efficient, lossy graph pattern matching process to avoid expensive subgraph isomorphism tests. To avoid information loss, Lossless graph summarization [36], [40], [21] incorporates correction structuresand extends MDL to minimize both summary sizes and the size of (edge) corrections. Unlike these work, our problem aims to compute summaries that are concise, lossless, and also ensure group coverage. This is not addressed by prior approaches.

*Subset Selection with Fairness*. Subset selection with fairness constraints has been studied [43], [35], [33]. Given a universal set and a set of groups (subsets), it computes a diverse subset that can cover each group with individual cardinality constraints. Approximation algorithms have been studied to generate subsets for max-sum and max-min diversification [35]. Submodular maximization under fairness constraints has been studied for data streams [11], where approximations with constant factors are presented. These methods study set coverage properties and cannot be directly used for graph summarization with fairness constraints. Our formal analysis verifies the hardness of graph summarization with group fairness, and shows the latter is more involved as a general counterpart of these problems. We introduce both feasible approximations and fast heuristics for fair graph summaries.

Several fairness measurements have been proposed [31], [42]. Group fairness measures the equality of proportions of each group from algorithm outcomes. [9] proposes a KL-Divergence based fairness that computes the similarity between the distribution of group members and a user-defined target distribution. [43] ensures fairness by satisfying cardinality constraints for protected groups. Unlike [9], our approach evaluates the fairness of node selection with cardinality constraints, to help users explicitly define the desired coverage. Compared with [43], our approach allows soft range constraints with upper and lower bounds, which are easier to set, and more practical to discover meaningful summaries.

*Diversified Pattern Mining*. Diversified subgraph pattern discovery [34] aims to discover subgraph patterns that maximize the coverage of a node set and the pairwise diversity of individual nodes that are covered. A greedy approximation is introduced given the submodular quality measure. The problem is relevant to a special case of our problem when group coverage is consistently defined on a single set. On the other hand, it has been observed that group fairness and diversity may come with conflict [43], [11]. We study a more involved setting and introduce feasible algorithms that compute the summaries under monotone submodular quality measures and explicit group coverage constraints.

## II. Graph Patterns and Summaries

**Graphs.** We consider directed, attributed graphs $G = (V, E, L, T)$, where $V$ is a node set, and $E \subseteq V \times V$ is a set of edges. Each node $v \in V$ (resp. edge $e \in E$) has a label $L(v)$ (resp. $L(e)$). Each node $v$ carries a tuple $T(v) = <(A_1, a_1),\ldots,(A_n, a_n)>$, where $A_i$ ($i \in [1, n]$) from a finite set $\mathcal{A}$ is a node attribute with value $a_i$.

We use the following notations. The $r$-hop neighbors (resp. edges) of $v$, denoted as $N_v^r$ (resp. $E_v^r$), refers to the nodes (resp. edges) that can be reached from or reach $v$ in $r$ hops. The $r$-hop neighbors of a node set $X$, denoted as $N_X^r$, refers to the set $\bigcup_{v\in X} N_v^r$. The $r$-hop edge set $E_X^r$ is defined similarly.

**Graph patterns**. A graph pattern $P(u_o)$ is a connected graph $(V_P, E_P, L_P, T_P)$, where $V_P$ (resp. $E_P \subseteq V_P \times V_P$) is a set of pattern nodes (resp. pattern edges). Each node $u \in V_P$ (resp. edge $e \in E_P$) has a label $L_P(u)$ (resp. $L_P(e)$). Each pattern node $u$ has a set of equality literals $T_P(u)$ in the form of $u.A = a$ ($A \in \mathcal{A}$), where $a$ is a constant.

The node $u_o$ is a designated *focus* of $P$. In practice, a pattern with a focus captures a "center" of interests and its egocentric structures, as seen in *e.g.,* social network analysis [4], [25].

*Coverage*. We extend graph pattern matching with *induced subgraph isomorphism* to characterize the coverage of a pattern. Given a pattern $P$ and a graph $G$, a matching from $P$ to $G$ is a function $h : V_P \to V$, where (a) for each node $u \in V_P$, $L_P(u) = L(h(u))$, and for each literal $u.A = a$ in $T_P$, $h(u).A = a$; and (b) for each edge $e = (u, u')$ in $P$, $h(e) = (h(u), h(u'))$ is an edge in $G$ where $L_P(e) = L(h(e))$.

A graph pattern $P(u_o)$ *covers* a node $v$ (resp. edges $e$) if there exists a matching $h$ such that $v = h(u)$ (resp. $e = h(e_p)$). The set of all the nodes (resp. edges) covered by $P(u_o)$ at the *focus* is denoted as $P_V$ (resp. $P_E$). Given a set of graph patterns $\mathcal{P}(u_o) = \{P_1(u_o), \ldots P_n(u_o)\}$ with a common focus $u_o$, the nodes (resp. edges) covered by $\mathcal{P}(u_o)$ in $G$ at $u_o$, denoted as $\mathcal{P}_V$ (resp. $\mathcal{P}_E$), refers to the set $\bigcup_{P\in\mathcal{P}} P_V$ (resp. $\bigcup_{P\in\mathcal{P}} P_E$), *i.e.,* the union of nodes (resp. edges) covered by the graph patterns in $\mathcal{P}(u_o)$ at $u_o$.

**Groups**. A group set $\mathcal{V} = \{V_1, \ldots, V_n\}$ is a set of disjoint node sets in $G$ with a same type, where each group $V_i \in \mathcal{V}$ is a subset of $V$, and carries a *coverage constraint* $[l_i, u_i]$, where

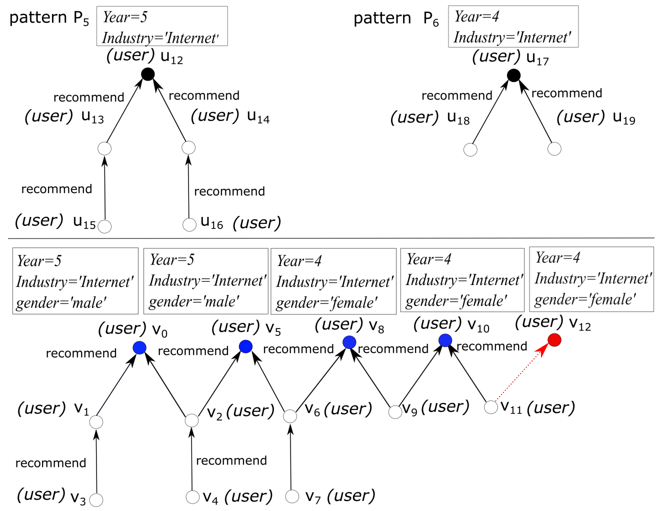



Fig. 2. Graph patterns, groups and $r$-summaries

$0 \leq l_i \leq u_i \leq |V_i|$. In practice, users may specify the group set $\mathcal{V}$ as vulnerable social groups (*e.g.,* gender, age, or race groups)for *e.g.,* social search and healthcare [17], [48]; and defines coverage constraints $[l_i, u_i]$ to express fairness constraints such as equal opportunity [17] or disparity constraints [14].

We use the following simplified conventions. (1) We assume a fixed designated focus $u_o$ and denote a graph pattern $P(u_o)$ (or simply a "pattern") and pattern set $\mathcal{P}(u_o)$ as $P$ and $\mathcal{P}$, respectively. (2) Given a set of sets $\mathcal{X}$, we denote the set $\bigcup_{X \in \mathcal{X}} X$ as $\bigcup \mathcal{X}$. (3) We use $P_X$ (resp. $\mathcal{P}_X$) to denote the nodes or edges in a node or edge set $X$ that are also covered by a pattern $P$ (resp. pattern set $\mathcal{P}$). The symbol $E_X^r$ refers to the $r$-hop edges of a node set $X$.

We next introduce $r$-*summaries*, a class of summary structures for group summarization.

**$r$-Summaries**. Given a graph $G$ with node set $V$, and a group set $\mathcal{V}$ of $n$ sets duin $G$, an $r$-*summary* of $\mathcal{V}$ is a two-part "pattern-correction" structure $\mathcal{S}$ = $(\mathcal{P}, \mathcal{C})$, where

- $\mathcal{P}$ is a pattern set with a common focus $u_o$, such that $|P_{\mathcal{V}} \cap V_i| \in [l_i, h_i]$; here $\mathcal{P}_{\mathcal{V}}$ = $\mathcal{P}_V \cap \bigcup \mathcal{V}$, *i.e.,* the *group nodes* covered by $\mathcal{P}$; and
- $\mathcal{C}$ refers to a set of *edge corrections*, and is defined as $\mathcal{C}$ = $E^r_{\mathcal{P}_{\mathcal{V}}} \setminus \mathcal{P}_E$, *i.e.,* the edges in $r$-hop neighbors of $\mathcal{P}_{\mathcal{V}}$ that are not covered by $\mathcal{P}_E$.

An $r$-summary $(\mathcal{P}, \mathcal{C})$ of group set $\mathcal{V}$ in $G$ ensures to (1) *select* a set of group nodes $\mathcal{P}_{\mathcal{V}}$ from group set $\mathcal{V}$ as the matches of the focus of the patterns $\mathcal{P}$, which satisfies the coverage constraints enforced by each group, and (2) *losslessly summarizes* $r$-hop neighbors of the selected group nodes $\mathcal{P}_{\mathcal{V}}$, by reconstructing $E^r_{\mathcal{P}_{\mathcal{V}}}$ with $\mathcal{P}_E \cup \mathcal{C}$.

**Example 5:** Fig. 2 illustrates a fraction of a profession network $G$, induced by the 2-hop neighbors of a candidate set $\{v_0, v_5, v_8, v_{10}, v_{12}\}$. Consider a gender group set $\mathcal{V}$, which contains a male group $\{v_0, v_5\}$ with a coverage constraint $[1, 2]$ and female group $\{v_8, v_{10}, v_{12}\}$ with coverage constraint $[2, 3]$. A 2-summary $(\mathcal{P}, \mathcal{C})$ for $\mathcal{V}$ is illustrated in Fig. 2, where $\mathcal{P}$ contains two patterns $P_5$ and $P_6$. (1) $P_5$ selects candidates in "Internet" industry with 5 years of experience and are recommended by other two users, each further recommended by another user. It only covers the male group $\{v_0, v_5\}$, and misses 5 edges of their 2-hop neighbors. (2) $P_6$ selects candidates with 4 years' experience in "Internet" and are recommended by two users. It covers females $\{v_8, v_{10}\}$, and leaves 2 edges $(v_6, v_7)$ and $(v_{11}, v_{12})$ in their 2-hop neighbors not covered. Note that the edge $(v_6, v_7)$ is covered by $P_5$. (3) Putting these together, $\mathcal{P}$ covers all the group nodes except $v_{12}$ ($\mathcal{P}_{\mathcal{V}}$ = $\{v_0, v_5, v_8, v_{10}\}$) and satisfies the coverage constraints, and losslessly describes their 2-hop neighbors with a single edge correction $\mathcal{C}$ = $\{(v_{11}, v_{12})\}$.

The missing edge $(v_{11}, v_{12})$ may be used to interpret why $v_{12}$ is not selected, and can be recommended to her as a new social link. □

TABLE I
MAJOR NOTATIONS AND SYMBOLS.

| Symbol | Description |
|---|---|
| $G$, $P(u_o)$ (or $P$) | graph, (graph) pattern |
| $N_V^r$ (resp. $E_V^r$) | $r$-hop neighbors (resp. edges) of nodes $V$ |
| $\mathcal{P}(u_o)$ (or $\mathcal{P}$) | a set of patterns with a common focus $u_o$ |
| $\mathcal{V}$, $V_i$ | a set of groups, and a group in $\mathcal{V}$ |
| $[l, u]$ | coverage constraint with lower/upper bounds $l/u$ |
| $P_V$, $P_E$ | nodes in $V$, edges in $E$ that are covered by $P$ |
| $\mathcal{P}_{\mathcal{V}}$, $\mathcal{P}_E$ | group nodes and edges covered by $\mathcal{P}$ |
| $\mathcal{S}$ = $(\mathcal{P}, \mathcal{C})$ | an $r$-summary, with edge corrections $\mathcal{C}$ |
| $C$ | a configuration $r, k, n$ |
| $F$ | monotone submodular utility function |
| $\mathcal{C}_l$ (resp. $\mathcal{C}_P$) | edge correction loss of $\mathcal{S}$ (resp. single pattern $P$) |

We summarize the major notations in Table I.

## III. Group Summaries with Coverage Constraint

### A. Quality Measurement

Given a set of node groups $\mathcal{V}$ in $G$, we are interested in finding $r$-summaries that can select high-quality group nodes from $\mathcal{V}$, and meanwhile describe their neighbors with small correction. These can be characterized by the following quality measures.

Monotone Submodular Utility. An $r$-summary $\mathcal{S}$ = $(\mathcal{P}, \mathcal{C})$ should be able to identify a set of high quality nodes $\mathcal{P}(u_o, G)$ that maximizes a utility.

This is often determined by a user-specified function $F$, which typically captures submodular properties. A utility function $F$ is submodular, if for any two node sets $V_1 \subseteq V_2 \subseteq V$, and any node $v \in V \setminus V_2$, $F(V_1 \cup \{v\})$ - $F(V_1) \geq F(V_2 \cup \{v\})$ - $F(V_2)$.

For example, (1) talent and social recommendation favors candidates with high social influence as a submodular function [20]; (2) data systems use submodular informativeness function to select training examples [32], or to diversify solution space [38]; (3) budgeted sensor network design sample sensors that maximize submodular utilities determined by benefits that are weighted by the distance from a root sensor [6]; (4) Social media analysis detects blogs that are likely to cause information outbreaks, modeled with the sum of submodular reward functions [24]. Our summarization framework generally applies to the need for choosing group nodes with submodular utility functions.

Conciseness. One also wants to inspect a small number of representative nodes (*e.g.,* candidates in talent search) from a large group $\mathcal{V}$. While the summary structures are often small, it is desirable to find $r$-summaries that can cover a bounded number of nodes over all the groups $\mathcal{V}$. Moreover, one often wants to inspect a bounded number of patterns.

Edge coverage loss. It is also desirable to ensure a small reconstruction cost to restore the $r$-hop neighbors of $\mathcal{P}_\mathcal{V}$, the group nodes covered by $\mathcal{S}$. This can be determined by the accumulated number of the $r$-hop edges surrounding $\mathcal{P}_\mathcal{V}$ that each pattern $P$ in $\mathcal{P}$ "missess". Let $\mathcal{C}_P = E^r_{P_\mathcal{V}} \setminus P_E$, where $P_\mathcal{V} \subseteq \bigcup \mathcal{V}$ refers to the group nodes $P$ covers. We define an *accumulated edge coverage loss* as $\mathcal{C}_l = \sum_{P\in\mathcal{P}} |\mathcal{C}_P|$. The smaller $\mathcal{C}_l$ is, the better. Note that $|\mathcal{C}| \leq \mathcal{C}_l$, and smaller $\mathcal{C}_l$ indicates less uncovered edges by $\mathcal{S}$.

**Remarks**. Another option is to simply define the cost as $|\mathcal{C}|$. We consider accumulated loss as a reasonable upperbound for $|\mathcal{C}|$, and closer to the actual algorithmic reconstruction cost, given the need of pattern-wise inspection in practice.

**Problem statement**. We now formalize our problem as a min-max optimization problem. Given a graph $G$, a set of disjoint groups $\mathcal{V}$ with associated coverage constraints, a monotone submodular utility function $F$, and a user-specified *configuration* $C = \{r, k, n\}$, the *fair group summarization* problem, denoted as FGS, is to compute an $r$-summary $\mathcal{S}$ = $(\mathcal{P}, \mathcal{C})$ of the group $\mathcal{V}$ with the following general form:

$$(\mathcal{P}, \mathcal{C}) = \min_{|\mathcal{P}|\leq k, \mathcal{C}_l} \max_{|\mathcal{P}_\mathcal{V}|\leq n} F(\mathcal{P}_\mathcal{V})$$

The solution of FGS leads to desirable summary structures $\mathcal{S}$ = $(\mathcal{P}, \mathcal{C})$ as justified by the following properties (also see "Case study" in Section VIII). (1) The group nodes $\mathcal{P}_\mathcal{V}$ covered by $\mathcal{P}$ can be readily suggested as high-quality answers for *e.g.,* talent search and recommendation with fairness constraints [17]. (2) The patterns $\mathcal{P}$ can be directly suggested as meaningful graph queries, to guide query and graph generation with cardinality constraints [7], for *e.g.,* benchmarking. (3) The "pattern-correction" structure $\mathcal{S}$ is *queryable*, where $\mathcal{P}$ naturally serve as (virtual) views to support *e.g.,* view-based query processing [13] with small reconstruction effort. (4) The edge corrections $\mathcal{C}$ also facilitate the interpretation between selected and unselected nodes, by explicitly suggesting their difference via edge corrections.

**Example 6:** Continuing the example in Fig. 2. Assuming the utility function $F$ quantifies the social influence as the the number of neighbors of the nodes covered by $r$-summary. Given $r = 2$, $n = 4$ and equal cardinality constraints which is $[2, 2]$ for both male and female groups. The 2-summary $\mathcal{S}$ in Fig. 2 thus covers $\{v_0, v_8, v_5$ and $v_{10}\}$ with $P_5, P_6$ (marked as blue) that achieves a total influence 8. As node $v_{12}$ is not a match for either $P_5$ or $P_6$, the edge $(v_{11}, v_{12})$ (marked as red dashed line) can not be covered by $\mathcal{S}$. While $|\mathcal{C}|$ = 1 with one missing edge $(v_{11}, v_{12})$, one can verify that $\mathcal{C}_{P_5}$ = 0, $\mathcal{C}_{P_6}$ = 2 and it takes a cost $\mathcal{C}_l$ = 2 to reconstruct the 2-hop neighbor of all the covered group nodes. □

### B. Verification and Hardness

To understand the hardness of FGS, we first study a *verification problem.* Given $G$, $\mathcal{V}$, a configuration $C = \{r, k, n\}$, and constants $b_c$ and $b_f$, it is to determine if a summary structure $\mathcal{S} = (\mathcal{P}, \mathcal{C})$ (1) is *feasible*, *i.e.,* an $r$-summary of $\mathcal{V}$ that covers at most $n$ nodes $|\mathcal{P}_\mathcal{V}| \leq n$ and also satisfies the coverage constraints for every group, and (2) the covered group nodes at $u_o$ have utility at least $b_f$, and edge coverage loss $|\mathcal{C}_l| \leq b_c$.

**Lemma 1:** *The verification problem alone is* NP*-complete.* □

The hardness follows from the reduction from the subgraph isomorphism problem between a single pattern and a graph. Below we outline a procedure to show it's in NP.

**Verification**. The procedure, denoted as rverify, first checks if $|\mathcal{P}| \leq k$, and performs subgraph isomorphism tests to decide if $\mathcal{P}(u_o, G) \cap \bigcup \mathcal{V} = \emptyset$ (in NP) for at most $k$ patterns. It then verifies if $|\mathcal{P}(u_o, G) \cap \bigcup \mathcal{V}| \leq n$, and $|\mathcal{P}(u_o, G) \cap \mathcal{V}_i| \leq [l_i, h_i]$ for each group $V_i \in \mathcal{V}$. It finally verifies if $\mathcal{C}_l \leq b_c$, and the utility is at least $b_f$. The above verification process takes $O(k \cdot |\bigcup \mathcal{V}| \cdot T_I + |\bigcup \mathcal{V}|)$ time. Here $T_I$ is the cost of verifying if a single pattern $P \in \mathcal{P}$ covers a group node at $u_o$, which is typically small in practice. Note that the verification does not require to compute the complete set $\mathcal{P}(u_o, G)$.

We next investigate the hardness of FGS.

**Theorem 2:** *The* FGS *problem is* $\Sigma^p_2$*-complete.* □

**Proof sketch:** Given $G$, $\mathcal{V}$, a configuration $\mathcal{C}$=$(r, k, n)$ and two constants $b_c$ and $b_f$, the decision problem of FGS is to decide if there exists a feasible $r$-summary $\mathcal{S}$ of $\mathcal{V}$ with a utility no less than $b_f$ and edge correction size no more than $b_c$. The problem can be solved in $\Sigma^p_2$. As the verification can be done in NP (Lemma 1), FGS can be solved in $\Sigma^p_2$ by guessing an $r$-summary $\mathcal{S}$ and verify its properties with rverify.

To show it's $\Sigma^p_2$-complete, we describe a reduction from the Graph Reconstruction (GR) problem [22]. Given two sets $\mathcal{G}^+$ and $\mathcal{G}^-$ of graphs, GR determines whether there exists a graph $G_o$ such that each $G^+ \in \mathcal{G}^+$ is isomorphic to a subgraph of $G_o$, and each $G^- \in \mathcal{G}^-$ is not isomorphic to any subgraph of $G_o$. Our reduction constructs $G$ as the union of augmented $\mathcal{G}^+$ and $\mathcal{G}^-$, where each single graph $G^+_i$ in $\mathcal{G}^+$ (resp. $G^-_j \in \mathcal{G}^-$) is added an augmented edge connecting to a distinct node $v^+_i$ (resp. $v^-_j$) with unique label 'positive' (resp. 'negative'). We set $\mathcal{V}$ = $\{\mathcal{V}^+, \mathcal{V}^-\}$, where group $\mathcal{V}^+$ (resp. $\mathcal{V}^-$) contains $|\mathcal{G}^+|$ 'positive' nodes (resp. $|\mathcal{G}^-|$ 'negative' nodes), associated with constraints $[|\mathcal{G}^+|, |\mathcal{G}^+|]$ (resp. $[0, 0]$). Setting a configuration $C$ = $(r_m + 1, |\mathcal{G}^+|, |\mathcal{G}^+|)$, with $r_m$ the largest diameter of graphs in $\mathcal{G}^+$, we show there exists a solution for GR if and only if there is an $r$-summary for the FGS instance. □

## IV. Computing Summaries with Group Fairness

We next introduce practical algorithms to compute $r$-summaries with coverage and utility guarantees.

### A. Approximating Summaries

Given a configuration $\{r, k, n\}$, one wants to compute an optimal $r$-summary $\mathcal{S}$ with maximized $F(\mathcal{P}_{\mathcal{V}})$ and smallest edge coverage loss $\mathcal{C}_l$, where $\mathcal{P}_{\mathcal{V}}$ is the set of group nodes covered by $\mathcal{P}$. A naive approach enumerates and verifies all size-$k$ pattern sets, and invokes the verification process to check if each set contributes to an $r$-summary, and if so, chooses the one with $\mathcal{P}_{\mathcal{V}}$ that lead to the highest utility. This is, nevertheless, not practical for large $G$. We thus consider faster algorithms with performance guarantees.

We first represent the min-max problem FGS as the following bi-level optimization problem, in a "weaker" form:

$$\min_{V_p^* \subseteq \mathcal{P}_{\mathcal{V}}} \mathcal{C}_l(\mathcal{P}, V_p^*), \text{where} \quad (1)$$

$$V_p^* = \underset{|V_p| \leq n; |V_p \cap V_i| \in [l_i, h_i]}{\arg\max} F(V_p) \quad (2)$$

where the "lower-level" goal aims to select $n$ group nodes $V_p^*$ that maximizes utility $F(V_p^*)$, and meanwhile satisfies the coverage constraints on $\mathcal{V}$; and an "upper-level" optimization is to discover a pattern set $\mathcal{P}^*$ that minimizes $|\mathcal{C}_l(\mathcal{P}^*, V_p^*)|$, subject to cover a *fixed*, desirable set of group nodes $V_p^*$.

We then resort to compute an $r$-summary structure $\mathcal{S}$ = $(\mathcal{P}, \mathcal{C})$ of $\mathcal{V}$, that ensures the following: (1) $\mathcal{P}$ covers a set of $n$ nodes $V_p$ ($V_p \subseteq \mathcal{P}_{\mathcal{V}}$), where $V_p$ satisfies the coverage constraints of $\mathcal{V}$, (2) $F(V_p) \geq \alpha * F(V_p^*)$, and (3) $\mathcal{C}_l(\mathcal{P}, V_p) \leq \beta \mathcal{C}_l(\mathcal{P}^*, V_p)$, for a fixed selected set $V_p$. We advocate such a solution $\mathcal{P}$ as an $(\alpha, \beta)$-*approximation* for FGS. This is a weaker approximation guarantee, as a sub-optimal solution that approximates $\mathcal{P}^*$ subject to $V_p$. Nevertheless, $\mathcal{S}$ remains to be a desirable solution, treating $V_p$ as a "yardstick" solution that already has a constant approximation ratio to an optimal solution $V_p^*$ of the lower-level optimization, which ensures high utility, guaranteed group coverage constraints, and a relative bound for $\mathcal{C}_l$ (hence a bounded $|\mathcal{C}|$, as $|\mathcal{C}| \leq \mathcal{C}_l$).

Below we present our main result.

**Theorem 3:** *Given a configuration $C = \{r, n\}$ without cardinality constraint $k$, there is a $(\frac{1}{2}, ln(n))$-approximation for* FGS. *The algorithm takes $O(n \cdot N \cdot T_I \cdot |\bigcup \mathcal{V}| + n \cdot N^2 + |E|)$ time, where $N$ is the total number of verified patterns.* □

We present a constructive proof for Theorem 3. Our idea is to take a "select-and-summarize" strategy. (1) The selection phase solves the lower-level problem and computes a set of nodes $V_p$ with coverage and quality guarantee. (2) The summarization phase then explores patterns induced from the $r$-hop neighbors of $V_p$ to ensure the coverage of $V_p$ and its $r$-hop neighbors with a small reconstruction cost, by minimizing accumulated pattern-wise correction $\mathcal{C}_l$.

We next present an algorithm that implements the idea.

**Algorithm**. The algorithm, denoted as APXFGS (Fig. 3) performs the following.

(1) *Selection phase* (lines 1-4). APXFGS invokes a procedure FairSelect to compute a set of group nodes $V_p$ with high utility $F(V_p)$ and satisfy the coverage constraint (line 2; see Procedure FairSelect). It then initializes an edge set $E_r(V_p)$ to be covered by the patterns.

(2) *Summarization phase* (lines 5-13). It invokes procedure SumGen to perform a constrained graph pattern mining over $V_p$ and their $r$-hop edges $E_r(V_p)$. The process exploits established graph pattern mining, yet early terminates at patterns with a radius up to $r$ from $u_o$ (*i.e.,* those with a distance up to $r$ between $u_o$ and any other pattern nodes) (line 5). APXFGS then follows a greedy strategy to dynamically choose a pattern $P*$ that maximizes a gain determined by covered nodes $P*_{V_p}$ in $V_p$ (computed as $P^*(u_o, G) \cap V_p$) and uncovered edge counterpart $\mathcal{C}_P$ (lines 6-12). This process is guarded by an "extendable" condition that verifies the coverage constraints (line 9). The desired $r$-summary $\mathcal{S}$ is then constructed as $(\mathcal{P}, E_r \setminus \mathcal{P}_E)$ and returned (line 14).

---

**Algorithm** APXFGS

*Input:* graph $G$, groups $\mathcal{V}$ with associated coverage constraints, utility function $F$, *configuration* $\mathcal{C} = \{r, k, n\}$.

*Output:* a feasible $r$-summary $\mathcal{S}$ of $\mathcal{V}$.

1. set $V_p := \emptyset$; set $\mathcal{P} := \emptyset$; set $\mathcal{P}_E := \emptyset$; set $E_r := \emptyset$; set $\mathcal{P}_c := \emptyset$; set $\mathcal{P}_u := \emptyset$;
2. $V_p :=$ FairSelect$(\mathcal{V}, F, n)$;
3. **for each** $v \in V_p$ **do**
4.     $E_r := E_r \cup E_r(v)$;
5. $\mathcal{P}_c :=$ SumGen$(V_p, E_r, r)$;
6. **while** $V_p \neq \emptyset$ **do**
7.     $\mathcal{P}_u := \emptyset$;
8.     **for each** $P \in \mathcal{P}_c \setminus \mathcal{P}$ **do**
9.         **if** Extendable$(P, \mathcal{P}, \mathcal{V}, n)$ **then**
10.             $\mathcal{P}_u := \mathcal{P}_u \cup P$;
11.     $P^* := \arg\max_{P_u \in \mathcal{P}_u} \frac{|P_u(u_o, G) \cap V_p|}{\mathcal{C}_{P_u}}$;
12.     $\mathcal{P} := \mathcal{P} \cup \{P^*\}$; $V_p := V_p \setminus P^*(u_o, G)$;
13. $\mathcal{S} := (\mathcal{P}, E^r_{V_p} \setminus \mathcal{P}_E)$;
14. **return** $\mathcal{S}$;

**Procedure** FairSelect $(\mathcal{V}, F, n)$

1. set $V_p := \emptyset$;
2. **while** $|V_p| < n$ **do**
3.     set $V_u := \emptyset$
4.     **for each** $v \in \mathcal{V} \setminus V_p$ **do**
5.         **if** ExtendableM$(v, V_p, \mathcal{V}, n)$
6.             $V_u := V_u \cup \{v\}$;
7.     $v* := \arg\max_{v' \in V_u}(F(V_p \cup v') - F(V_p))$;
8.     $V_p := V_p \cup \{v*\}$;
9. **return** $\mathcal{V}_p$;

---

Fig. 3. Algorithm APXFGS

**Procedure** Extendable. Given an $r$-summary $\mathcal{S}$ = $(\mathcal{P}, \mathcal{C})$ of $\mathcal{V}$ and a pattern $P$, we say $\mathcal{S}$ is *extendable* with a pattern $P$ if $\mathcal{S}$ = $(\mathcal{P} \cup \{P\}, \mathcal{C})$ remains to be feasible. Procedure Extendable determines if a current "partial" $r$-summary $\mathcal{S}$ is extendable with $P$, by checking (1) if it violates the coverage requirement in terms of upper bound; (2) covers no new nodes (line 3), and (3) covers more than $n$ nodes (line 6).

**Procedure** FairSelect. Given graph $G$, groups $\mathcal{V}$, utility function $F$ and integer $n$, FairSelect selects a set of nodes $V_{\mathcal{S}} \subseteq \bigcup \mathcal{V}$ such that $V_{\mathcal{S}}$ maximize $F$ and covers each group $\mathcal{V}_i \in \mathcal{V}$ with desired number of nodes in $[l_i, h_i]$. To this end, it solves a *submodular maximization* problem with

```
Procedure Extendable (P, 𝒫, 𝒱, n)
    if P(u_o, G) ∩ ⋃𝒱 = ∅ then return false;
    set 𝒫_e := 𝒫 ∪ {P}; integer cov := 0;
    for each V_i ∈ 𝒱 do
        if |𝒫_e(u_o, G) ∩ V_i| > h_i then
            return false;
        cov := cov + max(|𝒫_e(u_o, G) ∩ V_i|, l_i);
        if cov > n return false;
    return true;
```

Fig. 4. Procedure Extendable

group cardinality constraints following [18], which performs an iterative greedy selection strategy over group nodes. (1) In each iteration, FairSelect initializes a candidate set $V_u$ with all the nodes in $\mathcal{V} \setminus V_p$ that can be used to "extend" $V_p$. This is determined by a procedure ExtendableM (line 5; details omitted) by checking if: (1) for any group $V_i$ in $\mathcal{V}$, $|(V_p \cup v) \cap V_i| < h_i$; (2) $\sum_{V_i \in \mathcal{V}} max(|(V_p \cup v) \cap V_i|, l_i) <= n$, similarly as in Extendable. (2) It then adds a node with maximal marginal gain of submodular function $F$ (lines 7-8) to the node set $V_p$, until up to $n$ nodes are selected.

**Example 7:** Continuing with the example in Fig. 2, we consider a configuration of $r = 2$, $n = 4$, and a same cardinality constraint $[2, 2]$ for both male and female groups.

The selection phase performs a greedy selection of the group nodes. APXFGS identifies a set of promising nodes $V_p$ = $\{v_0, v_5, v_8, v_{10}\}$, which satisfies the coverage requirement of the groups. In the summarization phase, APXFGS firstly select pattern $P_5$ due to that it introduces a minimal size of edge correction cost 0. As $\mathcal{P} = \{P_5\}$ remains extendable with $P_6 \in \mathcal{P}_c$, APXFGS next verifies $P_6$, and adds it to $\mathcal{P}$. As $V_p$ has been covered by $\{P_5, P_6\}$, APXFGS terminates and returns $\mathcal{S}$ with $\mathcal{P} = \{P_5, P_6\}$, and $\mathcal{C} = \{(v_{11}, v_{12})\}$. □

### B. Correctness and Approximability

To see the correctness and quality guarantees of FairSelect, we show that it has the following invariants.

(1) Procedure FairSelect computes a set of nodes $V_p$ such that $F(V_p) \geq \frac{1}{2}F(V_p^*)$, for all subsets of $\bigcup \mathcal{V}$ with size bounded by $n$ that also satisfy the group coverage. This can be verified by an approximation preserving reduction from the lower-level node selection problem to fair submodular maximization [18]. The reduction constructs a base set as $\bigcup \mathcal{V}$ with groups and their ranges. It has been verified that a greedy selection process ensures a $\frac{1}{2}$-approximation which is simulated by FairSelect.

(2) Algorithm APXFGS computes a set of summaries that ensures to cover $V_P$ with a induced edge cover loss $\mathcal{C}_l$, by solving the upper-level problem as a maximum coverage problem [45]. As each pattern $P$ uniquely determines a set of covered nodes $P(u_o, G)$ and an edge cover loss, a reduction treats each $P$ as a subset $P(u_o, G) \cap \bigcup V_p$ with a weight $\mathcal{C}_P$ (recall $\mathcal{C}_l$=$\sum_{P \in \mathcal{P}} \mathcal{C}_P$). It then follows a greedy strategy [45] to select $\mathcal{P}$ with $\mathcal{C}_l \leq ln(|V_p|)\mathcal{C}_l^* \leq ln(n)\mathcal{C}_l^*$. Note that this indicates a provable bound for the size of edge correction.

**Lemma 4:** APXFGS *returns an $r$-summary $\mathcal{S}$ with a size-bounded edge correction $|\mathcal{C}| \leq ln(n)\mathcal{C}_l^*$, where $\mathcal{C}_l^*$ is the optimal edge coverage loss.* □

(3) The two procedures ExtendableM and Extendable correctly implements the verification rverify of $r$-summaries (Section III-B) into selection and summarization phases. This guarantees the invariant that only feasible $r$-summaries of $\mathcal{V}$ are correctly returned.

**Time cost**. Procedure FairSelect takes $O(n \cdot |\bigcup \mathcal{V}|)$ time to select $V_p$. Procedure SumGen takes at most $N \cdot T_I |\bigcup \mathcal{V}|$ time to generate and verify the patterns and their covered group nodes, where $N$ is the total number of patterns with radius up to $r$ from $u_o$, and $T_I$ is the time cost of verifying if a single node is covered by $P$ at $u_o$. The total time cost of APXFGS is thus in $O(n \cdot N \cdot T_I \cdot |\bigcup \mathcal{V}| + n \cdot N^2 + |E|)$ time.

The above analysis verifies that APXFGS ensures (1) a solution $V_P$ that approximates a $\frac{1}{2}$ approximation ratio to the optimal solution $V_P^*$, and (2) an $r$-summary $\mathcal{S}$ with $\mathcal{C}_l$ that approximates a local optimal solution $\mathcal{C}_l^*$ given $V_p$, at a ratio $ln(n)$. It thus achieves a $(\frac{1}{2}, \ln(n))$-approximation for FGS. Theorem 3 hence follows.

## V. Computing Group Summaries with $k$ Patterns

The approximation scheme APXFGS considers a configuration $C$ = $(r, n)$ without constraints on the number of patterns, and may return an excessive number of patterns. We next consider a variant of FGS, which requires to compute an $r$-summary with at most $k$ patterns, and minimizes $|\mathcal{C}|$ instead of accumulated correction cost.

$$\min_{|\mathcal{P}| \leq k, V_p^* \subseteq \mathcal{P}_\mathcal{V}} |\mathcal{C}|, \text{where} \tag{3}$$

$$V_p^* = \underset{|V_p| \leq n; |V_p \cap V_i| \in [l_i, h_i]}{\arg\max} F(V_p) \tag{4}$$

We show that a slight revision of algorithm APXFGS achieves the following relative approximation ratio.

**Theorem 5:** *Given a configuration $C$=$(r, k, n)$, there exists an $(\frac{1}{2}, 1 + \frac{1}{e \cdot \gamma})$ approximation, where $\gamma$ = $\frac{|E_{V_p}^r|}{|\mathcal{P}_E^* \cap E_{V_p}^r|} - 1$.* □

Here $V_p$ is the approximate solution of $V_p^*$, which ensures $F(V_p) \geq \frac{1}{2}F(V_p^*)$. The constant ratio $\frac{1}{2}$ is achieved by constructing a reduction from node selection problem to fair submodular maximization [18]. The latter is $\frac{1}{2}$-approximable via an oracle-based greedy selection process. In particular, one can iteratively choose high-utility nodes subject to upper bounds, and refine the selection by adding additional nodes to fulfill the lower bounds (see [2] for detailed analysis).

Intuitively, a larger $\gamma$ inherently a better approximation ratio, yet meanwhile indicates a larger correction cost. In other words, it verifies that an optimal solution $P^*$ can be better approximated when it inherently covers a smaller fraction of $E_{V_p}^r$. For example, when $\gamma$ = 1, there is an $(\frac{1}{2}, 1 + \frac{1}{e})$ approximation, yet under the assumption that even the optimal solution can cover half of the $r$-hop edges.

**Algorithm Outline**. We next outline the variant of APXFGS. The algorithm follows the "select-and-summarize" strategy. It first computes $V_p$ with procedure FairSelect (line 2 of APXFGS), and generate patterns with procedure SumGen to

**Algorithm** Online-APXFGS

*Input:* graph $G$, groups $\mathcal{V}$ with associated coverage constraints, utility function $F$, *configuration* $\mathcal{C} = \{r, k, n\}$.
*Output:* an $r$-summary $\mathcal{S}$ of $\mathcal{V}$.

1. set $V_p := \emptyset$; set $\mathcal{P} := \emptyset$; set $\mathcal{P}_E := \emptyset$; set $E_r := \emptyset$; set $\mathcal{P}_u := \emptyset$; $\mathcal{S} := \emptyset$; $B_c := \emptyset$;
2. **for each** $v \in \bigcup \mathcal{V}$ **do** /* *streaming selection phase* */
3. $w(v) = F(V_p \cup \{v\}) - F(V_p)$;
4. **if** $v \in V_c$ **then** $B_c := B_c \cup v$;
5. **if** ExtendableM$(v, V_p, \mathcal{V}, n)$ **then**
6. $V_p := V_p \cup \{v\}$;
7. **else** /* *consult an oracle procedure* */
8. $V_p :=$ UpdateVp $(v, V_p, \mathcal{V}, F, n)$;
9. **if** $v \in V_p$ **then** /* *trigger local summarization phase* */
10. $\mathcal{P} :=$ UpdateP $(v, V_p, \mathcal{V}, F, n)$;
/* *post processing with bucket* $B_c$ */
11. **while** there is a group $V_c \in \mathcal{V}$ where $|\mathcal{P}_{V_c}| < l_i$ **do**
12. PostSelect$(G, \mathcal{V}, F, \mathcal{C}, B_c, \mathcal{P})$;
13. $\mathcal{S} := (\mathcal{P}, E_r \setminus \mathcal{P}_E)$;
14. **return** $\mathcal{S}$;

**Procedure** UpdateVp $(v, V_p, \mathcal{V}, F, n)$

1. set $U := \emptyset$;
2. **for each** $v' \in V_p$ **do**
3. $V_p^{'} := V_p \setminus \{v'\}$;
4. **if** ExtendableM$(v, V_p^{'}, \mathcal{V}, n)$ **then** $U := U \cup \{v'\}$;
5. $v^- := \arg\min_{v' \in U}(F(V_u \cup v') - F(V_u))$;
6. $V_u := V_p \setminus \{v^-\}$;
7. **if** $w(v) \geq 2(F(V_u \cup v) - F(V_u))$ **then**
8. $V_p := V_p \setminus \{v'\} \cup \{v\}$;
9. **return** $\mathcal{V}_p$;

Fig. 5. Algorithm Online-APXFGS

be verified (line 5 of APXFGS). The only differences are as follows. (1) It initializes a universal set $E^r_{V_p}$, and for each pattern $P \in \mathcal{P}_c$, a matching edge set $P_E \cap E^r_{V_p}$. (2) It revises the summarization phase (lines 6-13), and selects $k$ patterns by solving a *maximum coverage problem*, which aims to compute $k$ patterns $\mathcal{P}$ with $|\mathcal{P}| \leq k$, such that $\bigcup_{P \in \mathcal{P}} P_E \cap E^r_{V_p}$ is maximized. This equivalently leads to minimizing $|\mathcal{C}|$ for selected $\mathcal{P}$. To this end, it greedily selects the pattern $P$ that maximizes a marginal gain as the currently uncovered $r$-hop edges in $E^r_{V_p}$, *i.e.,* $|E^r_{V_p} \cap (\mathcal{P} \cup \{P\})_E|$, until $|P| = k$. (3) It verifies if the current $\mathcal{P}$ covers $V_P$ and satisfies the coverage constraint, and if so, terminates and returns $\mathcal{S}$ with $\mathcal{P}$ and $\mathcal{C}$. Otherwise, it continues (2) with greedy swapping strategy, until either fails to identify a $k$ pattern set, or early terminates with desirable $\mathcal{P}$ and an $r$-summary.

**Analysis**. The correctness and approximation analysis follows the analysis of APXFGS and a reduction from pattern selection to maximum coverage problem with a known approximation ratio $1 - \frac{1}{e}$. Specifically, let $|\mathcal{C}^*|$ be the smallest correction size achieved by optimal solution $\mathcal{P}^*$. As $|\mathcal{C}^*| = |E^r_{V_p}|$ - $|\mathcal{P}^*_E \cap E^r_{V_p}|$, $|\mathcal{C}| = |E^r_{V_p}|$ - $|\mathcal{P}_E \cap E^r_{V_p}|$, and $|\mathcal{P}_E \cap E^r_{V_p}| \geq 1 - \frac{1}{e}|\mathcal{P}^*_E \cap E^r_{V_p}|$, we have $|\mathcal{C}| \leq (1 + \frac{|\mathcal{P}^*_E \cap E^r_{V_p}|}{e \cdot (|E^r_{V_p}| - |\mathcal{P}^*_E \cap E^r_{V_p}|)})|\mathcal{C}^*|$. The algorithm takes the same time cost as APXFGS.

We present the detailed analysis in [2].

## VI. Online Group Summarization

The algorithm APXFGS requires to compute $V_p$ first, and then generates and verifies patterns from $E^r_{V_p}$. This may be expensive for large groups $\mathcal{V}$. We next introduce an online algorithm that can process $\mathcal{V}$ as a "stream" of nodes, without pre-computing $V_p$. Our idea is to (1) streamline the node selection procedure FairSelect with a streaming submodular maximization process [18], and (2) upon a group node $v$ is accepted to $V_p$, triggers ad-hoc, *localized* pattern discovery (which only involves $E^r_v$) to only perform small-scale and necessary maintenance of $\mathcal{P}$. A post processing is then performed to ensure coverage properties.

**Procedure** UpdateP $(v, V_P, \mathcal{P}, \mathcal{V}, n)$

1. $\mathcal{P}_u :=$ SumGen$(v, E^r_v, r)$;
2. **while** $|\mathcal{P}| < k$ **do**
3. $P^* := \arg\max_{P_u \in \mathcal{P}_u} \frac{|P_u(u_o, G) \cap V_p|}{\mathcal{C}_{P_u}}$;
4. $\mathcal{P} := \mathcal{P} \cup \{P^*\}$;
5. $\mathcal{P}_u := \mathcal{P}_u \setminus \{P^*\}$;
6. $\Delta P := \emptyset$
7. **for each** $P \in \mathcal{P}_u$ **do**
8. **for each** $P' \in \mathcal{P}$ **do**
9. $\mathcal{P}' := \mathcal{P} \setminus \{P'\} \cup \{P\}$;
10. **if** $V_p \subseteq \mathcal{P}'_{\mathcal{V}}$ **then**
11. /* *ensuring covering all the nodes in* $V_p$ */
12. $\Delta\mathcal{P} := \Delta\mathcal{P} \cup \{P\}$;
13. $P^+ := \arg\max_{P' \in \Delta\mathcal{P}} \frac{|P'(u_o, G) \cap V_p|}{\mathcal{C}_P}$;
14. $P^- := \arg\min_{P \in \mathcal{P}} \frac{|P(u_o, G) \cap V_p|}{\mathcal{C}_P}$;
15. $\mathcal{P} := \mathcal{P} \setminus \{P^-\} \cup \{P^+\}$;
16. **return** $\mathcal{P}$;

Fig. 6. Procedure UpdateP

**Theorem 6:** *Given a configuration* $C = \{r, n, k\}$*, there is an online algorithm that ensures a* $(\frac{1}{4}, ln(n) + \theta)$*-approximation for* FGS*, with* $\theta \in [1, \frac{|E^r_v|}{k}]$*. The online algorithm process each group node* $v$ *in* $O(\log k + N_v \cdot T_I)$ *time, where* $N_v$ *is the number of patterns induced from* $E^r_v$*.* □

**Online Summarization**. The online algorithm, denoted as Online-APXFGS, is illustrated in Fig. 5. It maintains, for each group $V_c \in \mathcal{V}$, a bucket $B_c$, to store the processed nodes in $V_c$. Upon receiving a group node $v \in \mathcal{V}$, Online-APXFGS performs the following two major steps.

(1) Streaming selection (lines 3-8). Online-APXFGS performs streaming submodular maximization selection following [18]. In particular, it first verifies if $V_p$ is extendable (by invoking Procedure ExtendableM; line 5), and if so, either directly accept $v$ to $V_p$ (line 5); otherwise, consults a greedy streaming selection procedure (an "oracle" algorithm; lines 8-16) to decide whether to replace a node $v' \in V_P$ with $v$, following a greedy strategy, or to reject $v$, and put it in $B_c$.

(2) Local pattern update (lines 9-10). For each new node $v$ that enters $V_p$ directly or via replacement, it performs a pattern generation and verification. Unlike APXFGS which needs to verify all patterns with a radius up to $r$ from $u_o$ induced by $E^r_{V_p}$, the process only needs to verify the patterns induced by $E^r_v$. In particular, for each batch of new patterns $P_u$ derived from $E^r_v$, and current $\mathcal{P}$, it determines two processes: (a) it iteratively selects $k$-$|\mathcal{P}|$ nodes from $P_u$ that dynamically maximize the gain determined by current node coverage and correction cost $C_P$ (line 22), to add to $\mathcal{P}$, or (b) dynamically

decide a pattern set $P^+ \subseteq \mathcal{P}_u$ to be added to $\mathcal{P}$, and a pattern set $P^- \subseteq \mathcal{P}$ to be replaced out of $\mathcal{P}$, and perform the "swapping" process.

(3) *Post-processing* (lines 11-12). The above process repeats until all the nodes in $\mathcal{V}$ are processed. While ExtendableM ensures no group is "overly covered", it is possible that for some group $V_c$ with coverage constraint $[l_c, h_c]$, $|V_p \cap V_c| < l_c$ due to the rejection of the nodes. Unlike [18] that simply take random nodes to fill in the gap, for each such group, Online-APXFGS invokes a procedure PostSelect to enrich both $V_p$ and $\mathcal{P}$ with the nodes in $B_c$ to ensure coverage constraints and guarantees on correction error.

**Procedure** PostSelect (not shown). For each group $V_c \in \mathcal{V}$ where $|V_p \cap V_c| < l_c$, procedure PostSelect performs another round of "select-and-summarize" process to make $\mathcal{P}$ satisfy the coverage constraint. It (a) dynamically selects the top $(l_i - |V_p \cap V_i|$ nodes from $B_c$, where each node $v$ maximizes $F(V_p \cup \{v\})$-$F(V_p)$; and (b) follows the swapping strategy (lines 13-14, UpdateP) to update $\mathcal{P}$ with new patterns from the $r$-hop neighbors of the new nodes added to $V_p$. This repeats until $V_p$ covers all groups with desired lower bounds.

**Analysis**. The algorithm Online-APXFGS iteratively processes each group node $v \in \bigcup \mathcal{V}$ and ensures the following. (1) The dynamic decisions made by procedure updateVp on accepting or rejecting a node $v$ to $V_p$ follows the greedy streaming submodular maximization [18]. (2) Procedure updateP ensures that $V_p \subseteq \mathcal{P}_{\mathcal{V}}$ during the swapping strategy; and the procedure ExtendableM ensures that no group is overly covered by $\mathcal{P}$. (3) The procedure PostSelect ensures that no group is insufficiently covered, by enriching both $V_p$ and $\mathcal{P}$. These ensure that Online-APXFGS correctly maintains an $r$-summary of "revealed" $\mathcal{V}$ in each iteration and when terminates.

*Approximability*. The approximation guarantees follow from the $\frac{1}{4}$ approximation ensured by streaming submodular maximization, and online optimization of maximum coverage. In particular, assume $\mathcal{P}^*$ incurs $\mathcal{C}_l^*$ at each round, we show that the swapping strategy in procedure UpdateP, which exchanges $P^-$ with smallest marginal gain with a new counterpart $P^+$ having the maximized marginal gain, incurs a gap between $\mathcal{C}_l^*$ and $\mathcal{C}_l$ that is bounded by $\theta \in [1, \frac{|E_v^r|}{k}]$, given that $|\mathcal{P}| \leq k$.

*Time cost*. There are at most $|\bigcup \mathcal{V}|$ iterations, and in each iteration, (1) it takes procedure UpdateVp $O(\log k)$ time to process each group node $v \in \bigcup \mathcal{V}$; (2) procedure UpdateP takes $O(k \cdot T_I)$ time to verify if $v$ is a match, $O(|E_v^r|)$ time to update the marginal gain, and $O(N_v \cdot T_I)$ time to generate and verify patterns from $E_v^r$, with $N_v$ bounded by $N$, the total number of patterns verified. The post processing takes $\sum l_c \cdot N \cdot T_I$ time to enrich $\mathcal{P}$, where $\sum l_c$ is the sum of all the lower bounds. Hence the total cost is in $O(|\bigcup \mathcal{V}| \cdot (\log n + N \cdot T_I) + \sum l_c \cdot N \cdot T_I)$ time.

Theorem 6 follows from the above analysis (see [2]).

**Algorithm** Inc-FGS
*Input:* graph $G$, groups $\mathcal{V}$, utility function $F$, batch update $\Delta E$; configuration $C$=$\{r, n\}$, set $V_p$; $r$-summary $\mathcal{S}$=$(\mathcal{P}, \mathcal{C})$.
*Output:* an updated feasible $r$-summary $\mathcal{S}'$ for $G \oplus \Delta E$.

1. initializes set $\Delta\mathcal{V}$ with $\mathcal{V}$ and $\Delta E$;
2. **if** $\Delta\mathcal{V}$:=$\emptyset$ **return** $\mathcal{S}$;
3. **else** update $\mathcal{V}$; set $\Delta E_r$:=$\emptyset$; set $\mathcal{P}_u$:=$\emptyset$;
   /* *incrementalized node selection* */
4. set $V_p'$ := IncFairSel $(V_p, \Delta\mathcal{V}, \mathcal{V}, F, n)$; set $\Delta V_p$ := $V_p' \setminus V_p$;
5. **for each** $P \in \mathcal{P}$ **do**
6.     **if** $P(u_o, G \oplus \Delta E) \cap \mathcal{V} = \emptyset$ **then** $\mathcal{P} := \mathcal{P} \setminus \{P\}$;
7. **for each** $v \in \Delta V_p$ **do**
8.     $\Delta E_r := \Delta E_r \cup E_v^r$;
9. set $\Delta P_c$ := SumGen $(V_p', \Delta E_r, r)$;
   /* *incrementalized summarization* */
10. **while** $\Delta V_p \neq \emptyset$ **do**
11.     $\mathcal{P}_u$:=$\emptyset$;
12.     **for each** $P \in \Delta P_c \setminus \mathcal{P}$ **do**
13.         **if** Extendable $(P, \mathcal{P}, \mathcal{V}, n)$ **then** $\mathcal{P}_u$:=$\mathcal{P}_u \cup \{P\}$;
14.     $P^*$:=$\arg\max_{P_u \in \mathcal{P}_u} \frac{|P_u(u_o, G \oplus \Delta E) \cap V_p'|}{\mathcal{C}_{P_u}}$;
15.     $\mathcal{P}$:=$\mathcal{P} \cup \{P^*\}$; $\Delta V_p$:=$\Delta V_p \setminus P^*(u_o, G \oplus \Delta E)$;
16. set $\mathcal{S}'$ = $(\mathcal{P}, \Delta E_r \setminus \mathcal{P}_E)$;
17. **return** $\mathcal{S}'$;

Fig. 7. Algorithm Inc-FGS

## VII. Maintenance of Group Summaries

Real graphs are constantly changing. When new nodes join the interested groups $\mathcal{V}$ or new links are formed among group nodes, it is expensive to recompute an $r$-summary from scratch. We next present an incremental algorithm to maintain a feasible $r$-summary upon the arrival of new nodes and edges, and preserves "anytime" utility guarantee and group fairness.

Our idea is to incrementalize the "selection-and-summarization" phases. (1) Upon a batch of edge insertions $\Delta E$, it updates the current $V_p$ to $V_p'$ that satisfies both coverage constraints with approximated optimal utility $F$. This is achieved by invoking a streaming process to select new nodes induced from $\Delta E$, following [18]. (2) The summarization phase then finds the new nodes $\Delta V_p$ not in $V_p$, and their $r$-hop edges and creates a small instance for the summarization task. Due to the strong data locality of subgraph isomorphism, it suffices to incrementally update $\mathcal{P}$ to ensure the coverage of $\Delta V_p$.

**Algorithm**. The algorithm, denoted as Inc-FGS and illustrated in Fig. 7, processes edge insertions in batches $\Delta E$. (1) It first verifies if the edge insertions affect group nodes and their neighbors. If not, the original $r$-summary $\mathcal{S}$ is returned as there is no need to update the summary (lines 1-2). Otherwise, it updates $\mathcal{V}$ and invokes a procedure IncFairSel that follows a streaming fair submodular maximization process [18] to update $V_p$ to $V_p'$ (lines 3-4). It also refines $\mathcal{P}$ by removing any patterns that do not contribute to group coverage in $G \oplus \Delta E$, where $\oplus$ means "applying" the edge insertions to $G$ (lines 5-6). (2) Inc-FGS then invokes SumGen only in a (small) bounded fraction of affected nodes and $r$-hop neighbors, to generate new patterns. It incrementalizes the pattern selection as in APXFGS (lines 10-16) and update $\mathcal{P}$ necessarily with patterns that incurs small $\mathcal{C}_l$. This repeats until $\Delta V_p$ is covered by $\mathcal{P}$ (line 10). The updated $\mathcal{S}'$ is then returned (line 17).

| Dataset | $\|V\|$ | $\|E\|$ | # node types | # edge types | avg. # attr |
|---|---|---|---|---|---|
| DBP | 1M | 3.18M | 115 | 398 | 10 |
| LKI | 3M | 26M | 2 | 2 | 7 |
| Cite | 4.9M | 46M | 2 | 2 | 6 |

TABLE II
OVERVIEW OF REAL-LIFE GRAPHS

**Analysis**. It has been verified that an $n$-set for fair submodularity maximization can be maintained at a competitive ratio $\frac{1}{4}$, with a greedy swapping strategy [18]. At any time, algorithm Inc-FGS maintains a feasible $r$-summary which has the matching quality guarantees on $F$ and covers the updated $\mathcal{V}$ that satisfies the coverage constraints.

The delay time on processing a batch of edge insertion takes (1) $O(\min(n, |\Delta E|) \cdot |\bigcup \mathcal{V}|)$ to update $V_p$, (2) $O(|\Delta E| \cdot T_I \cdot |\mathcal{P}|)$ to refine $\mathcal{P}$ (lines 5-6), and (3) $O(n \cdot m^2)$ to update $\mathcal{S}$, where $m$ is the number of newly generated patterns from the small fraction of $E^r_{\Delta \mathcal{V}}$. Our tests verified that Inc-FGS can process batch update efficiently and significantly outperforms APXFGS in efficiency with comparable utility (see Section VIII).

## VIII. EXPERIMENTS

Using real-world attributed graphs, we experimentally verify the effectiveness and efficiency of our algorithms[1].

**Experiment Setting.** We used the following setting.

*Datasets and Groups.*

We use three real-life data graphs (summarized in Table II). (1) DBP [27] is a movie knowledge graph induced from DBpedia. Each node has a label (*e.g.,* movie, director or actors) and attributes such as title, genre, and years, and on average 10 attributes. Each relation has a label (*e.g.,* directed, collaboration). We induce up to 5 movie groups based on their genres or countries for diversified and fair movie recommendations. We induced group constraints varied from $[20, 30]$ to $[50, 60]$. (2) For talent search, we use LKI [49]. with nodes denoting *e.g.,* users organizations and edges denoting *e.g.,* co-review, works. Each node has attributes such as “Major”. we induced 6 groups of users in LKI based on their gender and degree, *e.g.,* ({gender:male; degree:BS},{gender:female; degree: BS},{gender:female; degree:MS}). We induced group constraints varied from $[50, 60]$ to $[250, 260]$ (3) For diversified and fair academic recommendation, we induced up to 4 groups of papers with different topics (*e.g.,* “Machine Learning”,“Networking”) and from Cite and same group constraints with LKI.

*Utility Functions*. We used the following functions.

(1) For fair movie recommendation, We set the utility function $F$ for a set of movies $V_{\mathcal{S}}$ in DBP as $F(V_{\mathcal{S}}) = \sum_{v \in V_{\mathcal{S}}} Rating(v)$. (2) For diversified and fair talent search, we defined $F$ over LKI as $F(V_{\mathcal{S}}) = |\bigcup_{v \in V_{\mathcal{S}}} N(v)|$, where $N(v) = \{u : (u, v) \in E\}$. This function, adapted from social influence maximization [20], [16], favors candidates that maximize the professional impact across their peers via “co-reviewed” relation. (3) We adopted the same impact function $F$ for Cite, yet via relation “citation”, to summarize interdisciplinary papers of desired coverage of topics and their influenced research.

[1] Source code: github.com/PanCakeMan/FGS

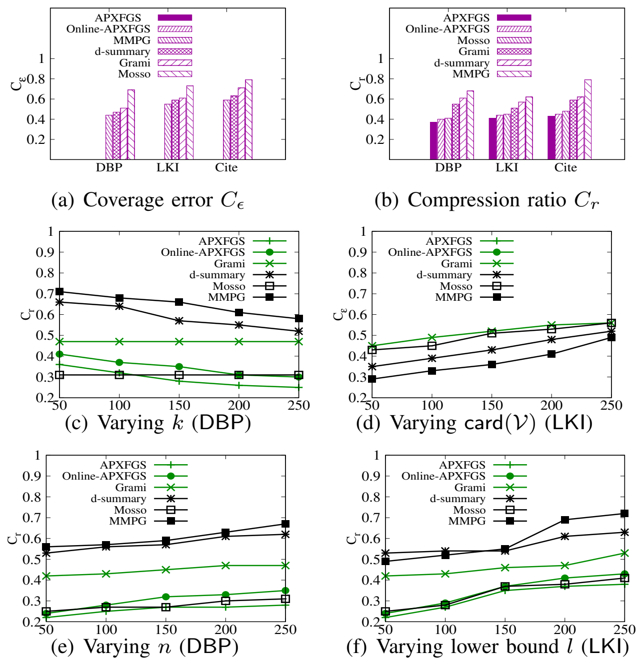


Fig. 8. Effectiveness of Fair Graph Summarization

We set $r$ in a principled manner to preserve comprehensive information. For each group, we inspected their $r$-hop neighbors as $r$ grows from 1 (one-hop neighbors) until no new information *e.g.,* node labels are included. For DBP, LKI and Cite, $r$ is set to be 3, 5 and 3, respectively. We set diameter $d = r$ consistently for $d$-summaries in d-sum. We also investigated the impact of $k$, the number of patterns in $\mathcal{S}$, and set $k$=20 by default to allow a large enough coverage of groups $\mathcal{V}$.

*Algorithms.* We implemented the following summarization approaches for FGS, all in Java. (1) Our lossless approaches are APXFGS, Online-APXFGS and Inc-FGS. (2) Grami is adapted from [12]. It mines top-$k$ frequent subgraphs as summary patterns. (3) d-sum is a lossy summary approach adapted from [41]. It generates $k$ graph patterns that approximately match their counterparts. d-sum encourages “larger” patterns to balance the informativeness and frequency of summary structures. (4) MMPG is a lossy summary approach adapting [34]. It computes $k$ reformulated patterns from a specified one to diversify the nodes they cover. (5) Mosso [21] is a lossless graph compression method. It incrementally updates super nodes and edges to summarize a dynamic graph with small edge corrections. APXFGS was compared with Grami, MMPG, and d-sum as pattern-based summarization approaches [26]. Inc-FGS is compared with Mosso (both lossless) over dynamic edge streams to evaluate online performances.

**Experimental results.** We next presented our findings.

**Exp-1: Effectiveness.** Given a summarization algorithm $\mathcal{A}$, groups $\mathcal{V}$ and a summary structure $\mathcal{S}$, we used two normalized measures below. (1) The *coverage error* of $\mathcal{A}$, adapted from set selection with fairness [18], quantifies the accumulated “gap” between the nodes covered by $\mathcal{S}$ from $\mathcal{A}$ (denoted as $V_{\mathcal{S}}$) and the required coverage of all groups. It is defined as $C_\epsilon(\mathcal{A})$ = $\frac{\sum_{V_i \in \mathcal{V}} \max\{|V_{\mathcal{S}} \cap V_i| - h_i, l_i - |V_{\mathcal{S}} \cap V_i|, 0\}}{|\mathcal{V}|}$. $C_\epsilon(\mathcal{A}) \in [0, 1]$. The smaller $C_\epsilon(\mathcal{A})$ is, the better. (2) We adapted the *compression ratio* of $\mathcal{A}$ consistently with Mosso [21]. It quantifies the representation size of $S$ (including the edge size of summary patterns $|S|$ and correction sizes $|S.C|$), normalized by the edge size of $r$-hop graphs of $\mathcal{V}$ (denoted as $|G_{\mathcal{V}}|$). It is defined as $C_r(\mathcal{S}) = \frac{|\mathcal{S}| + |\bigcup_{S \in \mathcal{S}} S.C|}{|G_{\mathcal{V}}|}$ ($C_r(\mathcal{S}) \in [0, 1]$). Smaller $C_r(\mathcal{S})$ indicates more "compact" $\mathcal{S}$ with smaller reconstruction cost.

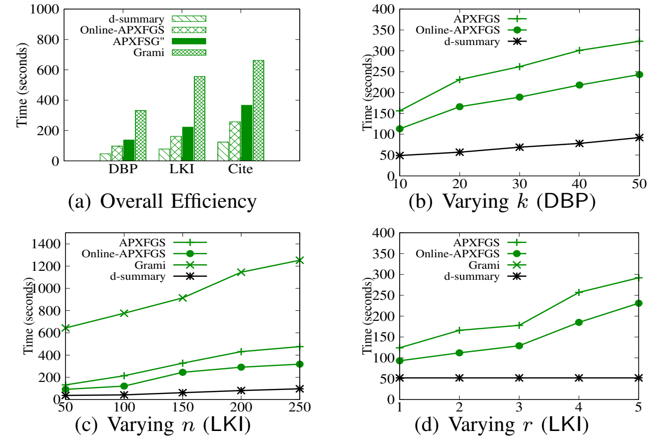
*Effectiveness*. Setting group size card($\mathcal{V}$) = 2, $r = 2$, $k = 20$, and $n = 100$, and the cardinality constraint as $[40, 60]$ for both groups, we reported the coverage error and compression ratio of all the algorithms in Figs. 8(a) and 8(b), respectively. Fig. 8(a) verifies the following. (1) Our algorithm APXFGS and Online-APXFGS achieve the optimal coverage with $C_\epsilon$ = 0, as they compute the summaries that satisfy the group coverage constraints. (2) Grami discovers summary patterns as frequent subgraphs and is more sensitive to cover major population. Mosso focuses on compressed representation of dense edge connections rather than group coverage. Both have higher coverage errors. (3) d-sum and MMPG have a comparable performance in $C_\epsilon$, and both outperform Mosso and Grami. This is because they both optimize a bi-criteria objectives that balance pattern size and diversity, and allow better node coverage compared with Mosso and Grami.

Fig. 8(b) tells us the following. (1) While achieving the optimal coverage, APXFGS achieves the smallest (on average 0.39) compression ratio. It outperforms Online-APXFGS,Grami, d-sum and MMPG by 8%, 27%, 41% and 79% in $C_r$, respectively. (2) Mosso has comparable performance and achieves 0.44 on average, due to its design for compact summaries. (3) MMPG favors larger summaries (by adding edges) to diversify the covered nodes. On the other hand, d-sum introduces more compact structures with approximate pattern matching.

We next evaluated the impact of several factors.

*Varying $k$*. Fixing card $(\mathcal{V}) = 2$, $n = 100$, and $r = 2$, we varied $k$ from 10 to 50 and evaluated its impact to compression ratio over DBP (Fig. 8(c)). (1) Larger $k$ allows APXFGS, d-sum and MMPG to summarize the neighborhood better with more summary patterns and smaller edge errors. On the other hand, using 50 patterns, APXFGS achieves a compression ratio at 0.26 for an underlying graph of $1M$ nodes and $3.2M$ edges, and outperforms Online-APXFGS, d-sum and MMPG. (2) Mosso only generates a single summary graph and is insensitive to $k$. APXFGS achieves a comparable compression ratio, and outperforms Mosso when $k \geq 20$. This shows that APXFGS can effectively exploit extendable summaries and edge corrections to avoid introducing too many new ones. Our results over other datasets are consistent, thus omitted.

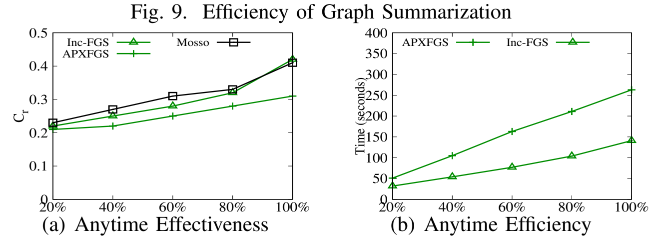
*Varying card($\mathcal{V}$)*. Fixing $n = 240$ $r = 2$, and $k = 20$, we varied card($\mathcal{V}$) from 2 to 6 and evaluated its impact to $C_\epsilon$ over LKI. For all groups, APXFGS ensures optimal group coverage




(a) Overall Efficiency (b) Varying $k$ (DBP)

(c) Varying $n$ (LKI) (d) Varying $r$ (LKI)

Fig. 9. Efficiency of Graph Summarization




(a) Anytime Effectiveness (b) Anytime Efficiency

Fig. 10. Online Graph Summarization

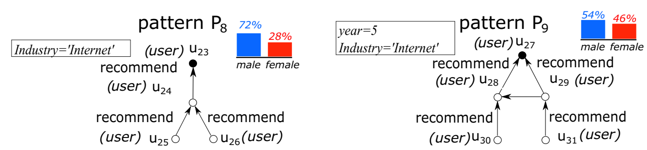



Fig. 11. Case study: Graph Summaries for Talent Search

($C_\epsilon$ = 0) as its capability to guarantee the group coverage. It outperforms d-sum, Grami and Mosso by 8%, 17% and 19% on average in $C_\epsilon$, respectively. As card($\mathcal{V}$) increases, all other pattern-based summarization methods have degraded performance in coverage, due to that it becomes more difficult for them to maintain the coverage error over more groups.

*Varying $n$*. Fixing card($\mathcal{V}$) = 2, $k = 20$, and $r = 2$, we varied $n$ (the size of nodes to be summarized) from 50 to 250 and evaluate its impact on compression ratio over LKI. Fig. 8(e) verifies that it is more difficult to maintain the compression ratio when more representative nodes are to be covered even when the group size is fixed. This is due to that larger amount of neighborhood information needs to be summarized, causing more edges to be either missed or added to edge correction.

*Varying lower bounds*. Fixing $|\mathcal{V}| = 2$, $k = 20$ $r = 2$ and $n = 500$, we varied the lower bound $l$ from 50 to 250, while keeping the upper bound $h$ to be 260, and evaluated impact of coverage requirement to compression ratio over LKI. As shown in Fig. 8(f), all methods perform worse in compression ratio as more nodes and neighbors are required to be summarized. With fixed number of summary patterns, APXFGS responses with larger representation size to ensure optimal coverage, yet still achieves a comparable compactness with Mosso. This verifies its effectiveness under various coverage requirements.

**Exp-2: Efficiency**. For a fair comparison, we only compared the cost of pattern-based summarization APXFGS, Online-APXFGS, Grami and d-sum. Using the same setting as in Figs. 8(a) and 8(b), we report the efficiency of APXFGS, Grami and d-sum in Fig. 9(a). APXFGS outperforms Grami by,1.13 times on average. Indeed, APXFGS discovers summary patterns over selected representative nodes and their neighbors, while Grami performs frequent pattern mining over all group nodes. On the other hand, d-sum takes the least time with lossy graph pattern matching, at the cost of high coverage error (see Fig. 8(a)). Besides, Online-APXFGS outperforms APXFGS by 1.2 times due to that Online-APXFGS only incrementally evaluates patterns that are generated locally.

*Varying $k$.* Using the same setting as in Fig. 8(c), we report the efficiency of APXFGS, and d-sum in Fig. 9(b). Grami always output all the frequent subgraphs and is not sensitive to $k$ (thus not shown). All the algorithms take more time to output the $k$ summaries as $k$ becomes larger. APXFGS outperforms Grami 1.85 times and Online-APXFGS outperforms APXFGS 1.25 times om average. d-sum remains to be the fastest due to lossy matching, yet does not guarantee group coverage.

*Varying $n$ and $r$.* Using the same setting as in Fig. 8(e), Fig. 9(c) reports the efficiency of APXFGS, Online-APXFGS, Grami and d-sum. As $n$ increases, all four methods take more time to summarize more nodes from the group, due to larger underlying neighborhood graphs to be covered. APXFGS outperforms Grami by 1.6 times on average. Fixing $n = 50$, $k = 20$ and card($\mathcal{V}$) = 2, we varied the hop constraint $r$ from 1 to 5. Fig. 9(d) shows that APXFGS takes more time, and up to 310 seconds, to generate larger patterns that can cover the $r$-hop neighbors of the group nodes as needed. Grami (resp. d-sum with $d$=5) lacks the support to such flexibility and yields same results as top-$k$ most frequent (resp. diversified and lossy) summary patterns without group coverage guarantees.

**Exp-3: Online Summarization**. We simulated a sequence of edges of LKI by (a) randomly selecting from 2 groups of in total $10K$ nodes, and (b) inducing $r$-hop neighbor graphs ($r = 2$) of the selected nodes and extract edges. We reported the anytime performance of Inc-FGS and Mosso upon the "seen" fraction of the graph. (1) As more subgraphs arrive, all the algorithms require larger summary structures that lead to higher compression ratio (Fig. 10(a)). APXFGS recomputes the summaries from scratch with smaller error corrections and summary patterns, and outperforms Mosso by 19% in $C_r$. (2) On the other hand, Inc-FGS outperforms APXFGS by 1.56 times in processing batches of edge insertions, with summaries of comparable size and optimal coverage ($C_e = 0$), and improves the efficiency of APXFGS better as larger batches arrive (Fig. 10(b)). This verifies the incremental summarization strategy in Inc-FGS is feasible for large graphs.

**Exp-4: Case Study.** We also report two case analysis to evaluate how $r$-summaries support graph analytics with group fairness requirement.

*Talent Search.* A pattern query $P_8$ aims to search for candidates in Internet industry. Over LKI, it retrieves 15200 candidates, among which 77% are males, and 23% are females, which is not very desirable for the need of equal opportunity. (Given $r = 2$, $n = 100$ and two gender groups $\mathcal{V}$ where each group $V_i \in \mathcal{V}$ is enforced with an equal constraint $[40, 60]$, APXFGS identifies $\mathcal{S}_9$ as a 2-summary with pattern $P_9$. Treating $S_9$ as a "materialized view" over LKI, $S_8$ efficiently retrieves a smaller, representative, high quality candidates with 57% male candidates and 43% female candidates over 90 candidates in "Internet" industry. $S_9$ helps reduced 82% of the time cost for processing the query $P_8$. Finally, $P_9$ suggests revisions to $P_8$ to understand the results towards new queries.

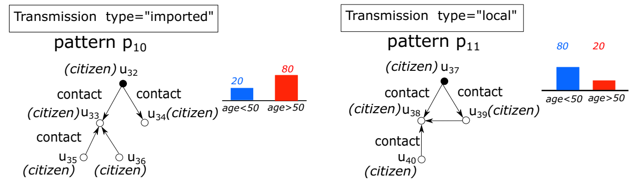


Fig. 12. Summarizing Pandemic Spreading for Immunization Strategies

*Pandemic Analysis*. Our second case study investigates how $r$-summaries help pandemic analysis and group immunization. Recall the real-world pandemic spreading network G' [1] in Example 3. There are 10000 citizens, among which 58% are young citizens ($age < 50$) and 42% are seniors ($age \geq 50$). Given 10 seed nodes, and 100 vaccine budgets, we simulated the group immunization [48] over $G'$. We tested different configurations for the group immunization, and report two vaccine distributions $[80, 20]$ (by setting the bound (80,80) for age group 1 and (20,20) for age group 2) and $[20, 80]$, respectively. By setting vaccine distribution as $[80, 20]$, 315 citizens are infected while 116 are infected for $[20, 80]$, indicating the latter a possible better vaccine strategy. The patterns $P_{10}$ and $P_{11}$ further suggests frequent social contact patterns from the selected seeds. Both well summarize the spreading patterns close to "individual popularity" ($P_{10}$) and "nominations contact" ($P_{11}$), as consistently observed in [19].

In general, 85% (resp. 90%) summaries are trees, and 15% (resp. 10%) are DAGs, for LKI (resp. pandemic network). We remark that these results only apply to the selected requirements. Our algorithms readily apply to various configurations.

## IX. Conclusions

We have introduced a class of $r$-summaries to summarize node groups and their neighbors with fairness constraints (in terms of group coverage constraints) in graphs. We have verified the hardness of the summarization problem, and provided feasible approximation algorithms and incremental algorithms to compute and maintain $r$-summaries, with guarantees on coverage and quality properties. As verified analytically and experimentally, our methods are feasible to support graph summarization among other applications. A future topic is to support more types of fairness constraints.

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## REFERENCES


[1] covid-19-india-data. https://github.com/imdevskp/covid-19-india-data/, 2018.

[2] Full version. https://github.com/CWRU-DB-Group/FGS/blob/main/full.pdf, 2022.

[3] Z. Abbassi, V. Mirrokni, and M. Thakur. Diversity maximization under matroid constraints. In *KDD*, 2013.

[4] L. Akoglu, M. McGlohon, and C. Faloutsos. Oddball: Spotting anomalies in weighted graphs. In *PAKDD*, 2010.

[5] G. Bagan, A. Bonifati, R. Ciucanu, G. H. Fletcher, A. Lemay, and N. Advokaat. gmark: Schema-driven generation of graphs and queries. *TKDE*, 2016.

[6] F. Bian, D. Kempe, and R. Govindan. Utility based sensor selection. In *IPSN*, 2006.

[7] A. Bonifati, I. Holubová, A. Prat-Pérez, and S. Sakr. Graph generators: State of the art and open challenges. *CSUR*, 2020.

[8] D. Chakrabarti, S. Papadimitriou, D. S. Modha, and C. Faloutsos. Fully automatic cross-associations. KDD '04, 2004.

[9] A. Das and M. Lease. A conceptual framework for evaluating fairness in search. *ArXiv*, 2019.

[10] C. Dunne and B. Shneiderman. Motif simplification: improving network visualization readability with fan, connector, and clique glyphs. In *SIGCHI*, 2013.

[11] M. El Halabi, S. Mitrović, A. Norouzi-Fard, J. Tardos, and J. M. Tarnawski. Fairness in streaming submodular maximization: Algorithms and hardness. *NeurIPS*, 2020.

[12] M. Elseidy, E. Abdelhamid, S. Skiadopoulos, and P. Kalnis. Grami: Frequent subgraph and pattern mining in a single large graph. *VLDB*, 2014.

[13] W. Fan, X. Wang, and Y. Wu. Answering graph pattern queries using views. In *ICDE*, 2014.

[14] M. Feldman, S. A. Friedler, J. Moeller, C. Scheidegger, and S. Venkatasubramanian. Certifying and removing disparate impact. KDD '15, 2015.

[15] Y. Ge, S. Zhao, H. Zhou, C. Pei, F. Sun, W. Ou, and Y. Zhang. Understanding echo chambers in e-commerce recommender systems. In *SIGIR*, pages 2261–2270, 2020.

[16] S. Gershtein, T. Milo, and B. Youngmann. Multi-objective influence maximization. *algorithms*, 20:33, 2021.

[17] S. C. Geyik, S. Ambler, and K. Kenthapadi. Fairness-aware ranking in search & recommendation systems with application to linkedin talent search. In *KDD*, 2019.

[18] M. E. Halabi, S. Mitrović, A. Norouzi-Fard, J. Tardos, and J. Tarnawski. Fairness in streaming submodular maximization: Algorithms and hardness. *arXiv preprint arXiv:2010.07431*, 2020.

[19] M.-G. Hâncean, J. Lerner, M. Perc, M. C. Ghiţă, D.-A. Bunaciu, A. A. Stoica, and B.-E. Mihăilă. The role of age in the spreading of covid-19 across a social network in bucharest. *Journal of Complex Networks*.

[20] D. Kempe, J. Kleinberg, and E. Tardos. Maximizing the spread of influence through a social network. In *KDD*, 2003.

[21] J. Ko, Y. Kook, and K. Shin. Incremental lossless graph summarization. In *KDD*, 2020.

[22] K.-I. Ko and W.-G. Tzeng. Three $\Sigma_2^P$-complete problems in computational learning theory. *Computational Complexity*, 1(3):269–310, 1991.

[23] D. Koutra, U. Kang, J. Vreeken, and C. Faloutsos. Vog: Summarizing and understanding large graphs. In *SIAM*, 2014.

[24] J. Leskovec, A. Krause, C. Guestrin, C. Faloutsos, J. VanBriesen, and N. Glance. Cost-effective outbreak detection in networks. In *KDD*, 2007.

[25] C.-T. Li and S.-D. Lin. Egocentric information abstraction for heterogeneous social networks. In *ASONAM*, 2009.

[26] Y. Liu, T. Safavi, A. Dighe, and D. Koutra. Graph summarization methods and applications: A survey. *CSUR*, 2018.

[27] J. Lu, J. Chen, and C. Zhang. Helsinki Multi-Model Data Repository. https://www2.helsinki.fi/en/researchgroups/unified-database-management-systems-udbms/, 2018.

[28] H. Ma, S. Guan, C. Toomey, and Y. Wu. Diversified subgraph query generation with group fairness. *WSDM*, 2022.

[29] A. Maccioni and D. J. Abadi. Scalable pattern matching over compressed graphs via dedensification. KDD '16, 2016.

[30] A. S. Maiya and T. Y. Berger-Wolf. Sampling community structure. In *WWW*, 2010.

[31] N. Mehrabi, F. Morstatter, N. Saxena, K. Lerman, and A. Galstyan. A survey on bias and fairness in machine learning. *CSUR*, 2021.

[32] R. Mehrotra and E. Yilmaz. Representative & informative query selection for learning to rank using submodular functions. In *SIGIR*, 2015.

[33] B. Mirzasoleiman, A. Badanidiyuru, and A. Karbasi. Fast constrained submodular maximization: Personalized data summarization. In *ICML*, 2016.

[34] D. Mottin, F. Bonchi, and F. Gullo. Graph query reformulation with diversity. In *KDD*, 2015.

[35] Z. Moumoulidou, A. McGregor, and A. Meliou. Diverse data selection under fairness constraints. In *ICDT*, 2021.

[36] S. Navlakha, R. Rastogi, and N. Shrivastava. Graph summarization with bounded error. In *SIGMOD*, 2008.

[37] M. Pan, R.-H. Li, Q. Zhang, Y. Dai, Q. Tian, and G. Wang. Fairness-aware maximal clique enumeration. *arXiv preprint arXiv:2107.10025*, 2021.

[38] A. Prasad, S. Jegelka, and D. Batra. Submodular meets structured: Finding diverse subsets in exponentially-large structured item sets. *NeurIPS*, 2014.

[39] A. Rahmattalabi, P. Vayanos, A. Fulginiti, E. Rice, B. Wilder, A. Yadav, and M. Tambe. Exploring algorithmic fairness in robust graph covering problems. In *NeurIPS*, 2019.

[40] K. Shin, A. Ghoting, M. Kim, and H. Raghavan. Sweg: Lossless and lossy summarization of web-scale graphs. In *WWW*, 2019.

[41] Q. Song, Y. Wu, P. Lin, L. X. Dong, and H. Sun. Mining summaries for knowledge graph search. *TKDE*, 2018.

[42] J. Stoyanovich, S. Abiteboul, and G. Miklau. Data, responsibly: Fairness, neutrality and transparency in data analysis. In *EDBT*, 2016.

[43] J. Stoyanovich, K. Yang, and H. Jagadish. Online set selection with fairness and diversity constraints. In *EDBT*, 2018.

[44] A. Tsang, B. Wilder, E. Rice, M. Tambe, and Y. Zick. Group-fairness in influence maximization. *IJCAI*, 2019.

[45] V. V. Vazirani. *Approximation algorithms*. Springer, 2013.

[46] Y. Wu, Z. Zhong, W. Xiong, and N. Jing. Graph summarization for attributed graphs. In *2014 International Conference on Information Science, Electronics and Electrical Engineering*, 2014.

[47] Y. Yang, N. V. Chawla, and B. Uzzi. A network's gender composition and communication pattern predict women's leadership success. *Proceedings of the National Academy of Sciences*, 116(6):2033–2038, 2019.

[48] Y. Zhang, A. Adiga, A. Vullikanti, and B. A. Prakash. Controlling propagation at group scale on networks. In *ICDM*, 2015.

[49] Y. Zhang, J. Tang, Z. Yang, J. Pei, and P. S. Yu. Cosnet: Connecting heterogeneous social networks with local and global consistency. In *KDD*, 2015.