VARIATION OF CANONICAL HEIGHT FOR FATOU POINTS ON $\mathbb{P}^1$

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ABSTRACT. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a map of degree > 1 defined over a function field $k = K(X)$, where K is a number field and X is a projective curve over K. For each point $a \in \mathbb{P}^1(k)$ satisfying a dynamical stability condition, we prove that the Call-Silverman canonical height for specialization f_t at point a_t, for $t \in X(\overline{\mathbb{Q}})$ outside a finite set, induces a Weil height on the curve X; i.e., we prove the existence of a $\mathbb{Q}$-divisor $D = D_{f,a}$ on X so that the function $t \mapsto \hat{h}_{f_t}(a_t) - h_D(t)$ is bounded on $X(\overline{\mathbb{Q}})$ for any choice of Weil height associated to D. We also prove a local version, that the local canonical heights $t \mapsto \hat{\lambda}_{f_t,v}(a_t)$ differ from a Weil function for D by a continuous function on $X(\mathbb{C}_v)$, at each place v of the number field K. These results were known for polynomial maps f and all points $a \in \mathbb{P}^1(k)$ without the stability hypothesis [21, 14], and for maps f that are quotients of endomorphisms of elliptic curves E over k and all points $a \in \mathbb{P}^1(k)$ [32, 29]. Finally, we characterize our stability condition in terms of the geometry of the induced map $\tilde{f} : X \times \mathbb{P}^1 \dashrightarrow X \times \mathbb{P}^1$ over K; and we prove the existence of relative Néron models for the pair (f, a), when a is a Fatou point at a place γ of k, where the local canonical height $\hat{\lambda}_{f,\gamma}(a)$ can be computed as an intersection number.

1. INTRODUCTION

In this article, we study the variation of canonical height in families of maps $f : \mathbb{P}^1 \to \mathbb{P}^1$. More precisely, we fix a number field K and a smooth projective curve X defined over K. Let $k = K(X)$ be the associated function field, and let $\overline{K}$ denote an algebraic closure of K. Any map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d defined over k will specialize to a morphism $f_t : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d, defined over $\overline{K}$, for all but finitely many $t \in X(\overline{K})$. For points $a \in \mathbb{P}^1(k)$, we are interested in properties of the function $t \mapsto \hat{h}_{f_t}(a_t)$, where $\hat{h}_{f_t}$ is the Call-Silverman canonical height for f_t as defined in [5], as t varies in $X(\overline{K})$.

An important case was studied in the early 1980s. Given any elliptic surface $E \to X$ with a zero section, defined over a number field K, and given a section $P : X \to E$ also defined

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over K, the fiber-wise canonical height $t \mapsto \hat{h}_{E_t}(P_t)$ is known to define a Weil height on the base curve $X(\overline{K})$ [32]. That is, there exists a $\mathbb{Q}$-divisor $D_{E,P}$ on X, of degree equal to the geometric canonical height $\hat{h}_E(P)$ (viewing E as an elliptic curve over the function field k) so that

$$\hat{h}_{E_t}(P_t) - h_{D_{E,P}}(t) = O(1) \tag{1.1}$$

for any choice of Weil height associated to $D_{E,P}$. The notation $O(1)$ represents a bounded function, defined on the complement of finitely many points in $X(\overline{K})$; the bound depends on the pair (E, P) and the choice of Weil height $h_{D_{E,P}}$. This can be viewed as a dynamical example on $\mathbb{P}^1$ as follows. Projecting each smooth fiber E_t to $\mathbb{P}^1$ by the natural degree-two quotient that identifies a point $x \in E_t$ with its inverse $-x$, and taking, for example, the multiplication-by-2 endomorphism on E_t, we obtain a family of maps $f_t : \mathbb{P}^1 \to \mathbb{P}^1$, well-defined for all but finitely many $t \in X(\overline{K})$. See, for example, [31, §6.4]. The section P projects to an element $p \in \mathbb{P}^1(k)$, and we have $\hat{h}_{f_t}(p_t) = 2\,\hat{h}_{E_t}(P_t)$, so that

$$\hat{h}_{f_t}(p_t) - h_{D_{f,p}}(t) = O(1) \tag{1.2}$$

on the complement of finitely many points in $X(\overline{K})$, for a $\mathbb{Q}$-divisor $D_{f,p} = 2\,D_{E,P}$ on X.

For any given map $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over k of degree > 1, and each point $a \in \mathbb{P}^1(k)$, Call and Silverman proved that the specializations satisfy

$$\hat{h}_{f_t}(a_t) - h_D(t) = o(h_D(t)) \tag{1.3}$$

as $h_D(t) \to \infty$, for any choice of Weil height h_D on $X(\overline{K})$ associated to a divisor D of degree equal to the geometric (i.e., over k) canonical height $\hat{h}_f(a)$ [5, Theorem 4.1]. Recently, Ingram improved the error term $o(h_D(t))$ in (1.3) to $O(h_D^{2/3}(t))$ [22]. Inspired by (1.2) and (1.3), Call and Silverman asked if there can exist a divisor $D = D_{f,a}$ on X so that the stronger result of the form (1.2) will hold for every f and a; see the Remark after Theorem 4.1 in [5]. We give a partial answer to this question.

Definition 1.1. A point $a \in \mathbb{P}^1(k)$ is said to be **totally Fatou** for f if it is an element of the non-archimedean Fatou set at every place $\gamma \in X(\overline{K})$ of k.

We refer the reader to Section 4 for more information. We note here that throughout this article we identify the places of k with those of $k \otimes \overline{K}$ and with the points $\gamma \in X(\overline{K})$. The notion of a totally Fatou point has also appeared in [26] in the setting of number fields.

Theorem 1.2. *Let K be a number field and X a smooth projective curve over K. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a map of degree > 1 defined over $k = K(X)$, and suppose that $a \in \mathbb{P}^1(k)$ is totally Fatou for f. Then there exists a $\mathbb{Q}$-divisor $D = D_{f,a}$ on X, of degree equal to the geometric height $\hat{h}_f(a)$, so that $t \mapsto \hat{h}_{f_t}(a_t)$ defines a Weil height for D on $X(\overline{K})$. More precisely, for any choice of Weil height h_D associated to D, we have*

$$\hat{h}_{f_t}(a_t) - h_D(t) = O(1)$$

as a function of $t \in X(\overline{K}) \setminus Y$ for a finite set Y, where $\hat{h}_{f_t}(a_t)$ is well defined. The bounds on $\hat{h}_{f_t}(a_t) - h_D(t)$ depend on f, a, and the choice of Weil height h_D.

We shall see that the divisor D is given by

$$D_{f,a} = \sum_{\gamma \in X(\overline{K})} \hat{\lambda}_{f,\gamma}(a) \cdot \gamma \tag{1.4}$$

where $\hat{h}_f = \sum_\gamma \hat{\lambda}_{f,\gamma}$ is a local decomposition of the geometric canonical height for f over k. The fact that D is a $\mathbb{Q}$-divisor for totally Fatou points a, so that $\hat{\lambda}_{f,\gamma}(a) \in \mathbb{Q}$ and therefore also $\hat{h}_f(a) \in \mathbb{Q}$, is new; see Proposition 6.1, addressing a question in [10]. As a special case of Theorem 1.2 we recover (1.1) and (1.2), because all points in $\mathbb{P}^1(k)$ are totally Fatou for the maps f coming from elliptic curves.

The statement of Theorem 1.2 was proved by Ingram for polynomial maps $f(z) \in k[z]$ and for all points $a \in \mathbb{P}^1(k)$ without the totally Fatou assumption [21]. Polynomial maps have a totally invariant super-attracting fixed point at ∞, simplifying computations of the canonical height. In fact, much more is known for polynomials f and for maps f coming from elliptic curves, and we address some of this below in the context of Theorem 1.7; see the works of Favre and Gauthier [14, 15] and of Silverman [28, 29, 30]. However, even with the totally Fatou assumption, new complications arise for rational maps that do not exist for polynomials or maps coming from elliptic curves, as we discuss after Theorem 1.7 and illustrate by example in Section 7.

The totally Fatou condition. In contrast with the setting of number fields, it may be true that *every* point $a \in \mathbb{P}^1(k)$ is either preperiodic or totally Fatou for maps f defined over k. (Note that the statement of Theorem 1.2 holds trivially when a is preperiodic for f, as $\hat{h}_{f_t}(a_t) = 0$ at all points t where f_t is defined, and we can take $D = 0$.) We know of no examples, nor any mechanisms to prove existence, of maps f defined over k and points

$a \in \mathbb{P}^1(k)$ with infinite orbit for which a lies in the non-archimedean Julia set of f at a place γ of k.

Conjecture 1.3. *Let K be a number field and X a smooth projective curve over K. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree > 1 be defined over $k = K(X)$. Then every point $a \in \mathbb{P}^1(k)$ is either preperiodic or totally Fatou for f.*

Note that the conjecture remains open for polynomial maps f, though the conclusion of Theorem 1.2 is known to hold for all points $a \in \mathbb{P}^1(k)$ in that case [21]. In Section 7, we observe that for all of the previously known cases of Theorem 1.2 in the literature where the maps f are *not* polynomials (nor conjugate to polynomials), the points $a \in \mathbb{P}^1(k)$ are totally Fatou for f.

Here we prove that "most" points in $\mathbb{P}^1(k)$, from a density point of view, are totally Fatou. Let k_γ denote the completion of k at the place $\gamma \in X(\overline{K})$.

Theorem 1.4. *For any $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree > 1 defined over $k = K(X)$, the set of totally Fatou points for f in $\mathbb{P}^1(k)$ is open and dense in the product topology on $\mathbb{P}^1(k)$, coming from the embedding of k into $\prod_{\gamma \in X(\overline{K})} k_\gamma$.*

Theorem 1.4 exploits the non-local-compactness of k_γ; it is false for maps f defined over number fields K, where the Fatou set in a completion K_v can fail to be dense at archimedean or non-archimedean places v.

To understand the totally Fatou condition better, we relate it to the geometry of the induced rational map

$$\tilde{f} : X \times \mathbb{P}^1 \dashrightarrow X \times \mathbb{P}^1$$

on the complex surface $X \times \mathbb{P}^1$, defined by $(t, z) \mapsto (t, f_t(z))$. Let $I(\tilde{f})$ denote the (finite) indeterminacy set of $\tilde{f}$ in $(X \times \mathbb{P}^1)(\overline{K})$. For a point $a \in \mathbb{P}^1(k)$, let C_a denote the graph in $X \times \mathbb{P}^1$ of the associated holomorphic map $t \mapsto a(t)$ from X to $\mathbb{P}^1$.

Theorem 1.5. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be of degree > 1, defined over a function field $k = K(X)$, with the number field K chosen so that all indeterminacy points of $\tilde{f}$ lie in $(X \times \mathbb{P}^1)(K)$. A point $a \in \mathbb{P}^1(k)$ is totally Fatou for f if and only if there exists a birational morphism*

$Y \to X \times \mathbb{P}^1$, defined over K, so that the induced map

$$\begin{array}{ccc} Y & \overset{\tilde{f}_Y}{\dashrightarrow} & Y \\ \downarrow & & \downarrow \\ X \times \mathbb{P}^1 & \overset{\tilde{f}}{\dashrightarrow} & X \times \mathbb{P}^1 \end{array}$$

satisfies $C^Y_{f^n(a)} \cap I(\tilde{f}_Y) = \emptyset$ for all $n \geq 0$, where $C^Y_{f^n(a)}$ is the proper transform of the curve $C_{f^n(a)}$ in Y. Moreover, the modification Y can be chosen so that $\tilde{f}_Y$ is algebraically stable, meaning that no curve is mapped by an iterate $(\tilde{f}_Y)^n$ into the indeterminacy set $I(\tilde{f}_Y)$, and such that $C^Y_{f^n(a)}$ intersects the singular fibers of the projection $Y \to X$ only at smooth points, for all $n \geq 0$.

Remark 1.6. It was proved in [9, Theorem E] that, for every f of degree > 1 over k, there exists a modification $Y \to X \times \mathbb{P}^1$ so that the induced map $\tilde{f}_Y : Y \dashrightarrow Y$ is algebraically stable. Theorem 1.5 implies we can further modify Y so that the orbit of C_a is disjoint from the indeterminacy locus of $\tilde{f}_Y$, when a is totally Fatou. The choice of Y will depend on a.

We use Theorem 1.5 to prove that the geometric local canonical height $\hat{\lambda}_{f,\gamma}(a)$ can be computed as an intersection number in Y, assuming the point a is Fatou at γ; see Theorem 4.11 and compare with [5, Theorem 6.1]. In analogy with the study of elliptic curves and abelian varieties, the concept of a "weak Néron model" at a place γ of k was introduced in [5] for dynamical systems; but, it is known that these models often fail to exist for maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over k_γ in the absence of good reduction, for example when there is a repelling periodic point in k_γ [20]. In fact, as Ingram noted in the Introduction to [21], if $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over k is neither Lattès nor isotrivial, then it cannot have a weak Néron model at every place γ. The proof of Theorem 1.5 provides the existence of a *relative* type of weak Néron model, for a pair (f, a) with a being Fatou at γ, in which the orbit of the Fatou point can be arranged to be integral.

Theorem 1.5 follows from the proof of [9, Theorem D] and the classification of γ-adic Fatou components in the Berkovich projective line $\mathbb{P}^{1,an}_\gamma$ (over a complete and algebraically closed field $\mathbb{C}_\gamma$ containing the completion k_γ) [27] [2] [9, Appendix]; many of the ideas were already present in [20], and what remained was to show that the full orbit $\{f^n(a)\}_{n\geq 0}$ can be disjoint from the indeterminacy set after only finitely many blowups of $X \times \mathbb{P}^1$.

Local version of Theorem 1.2. In the setting of elliptic surfaces $E \to X$, Silverman strengthened Tate's result (1.1) by showing that the function $B_{E,P}(t) := \hat{h}_{E_t}(P_t) - h_{D_{E,P}}(t)$,

defined for all but finitely many $t \in X(\overline{\mathbb{Q}})$, can be expressed as a sum over all places v of the number field K of functions with good behavior [28, 29, 30]. More precisely, he proved that the local height functions for $\hat{h}_{E_t}$ on $E_t(\overline{\mathbb{Q}})$ and for $h_{D_{E,P}}$ on $X(\overline{\mathbb{Q}})$ can be chosen so that all v-adic contributions to $B_{E,P}$ extend to define *continuous* functions on $X(\mathbb{C}_v)$, even across the singular fibers, and that *all but finitely many* of the v-adic contributions are $\equiv 0$.

We also prove a local continuity result, strengthening the conclusion of Theorem 1.2:

Theorem 1.7. *Under the hypotheses of Theorem 1.2, we assume that the number field K is extended so that* $\operatorname{supp} D_{f,a} \subset X(K)$. *There are local decompositions*

$$\hat{h}_{f_t}(a_t) = \frac{1}{[K:\mathbb{Q}]} \frac{1}{|\operatorname{Gal}(\overline{K}/K)\cdot t|} \sum_{x \in \operatorname{Gal}(\overline{K}/K)\cdot t} \sum_{v \in M_K} N_v \, \hat{\lambda}_{f_x,v}(a_x)$$

and

$$h_D(t) = \frac{1}{[K:\mathbb{Q}]} \frac{1}{|\operatorname{Gal}(\overline{K}/K)\cdot t|} \sum_{x \in \operatorname{Gal}(\overline{K}/K)\cdot t} \sum_{v \in M_K} N_v \, \lambda_{D,v}(x)$$

for $t \in X(\overline{K}) \setminus \operatorname{supp} D_{f,a}$ so that the function

$$V_v(t) := \hat{\lambda}_{f_t,v}(a_t) - \lambda_{D,v}(t)$$

extends to a continuous function on the Berkovich analytification $X^{an}_{\mathbb{C}_v}$ for each place v of K.

Here, M_K denotes the set of places of the number field K, and the weights $N_v = [K_v : \mathbb{Q}_v]$ are the same as those appearing in the product formula $1 = \prod_{v \in M_K} |\alpha|_v^{N_v}$ for $\alpha \in K^*$. The conclusion of Theorem 1.7 is known for polynomial maps $f(z) \in k[z]$ and for all $a \in \mathbb{P}^1(k)$ without the totally Fatou hypothesis [19, 14]; their proofs take clever advantage of the compactness of the orbit-closures of all points in the γ-adic Julia sets, as subsets of $\mathbb{P}^1(k_\gamma)$ (see [14, Theorem 3], [20, Theorem 4.8], [33, Proposition 6.7]), which does *not* hold for general rational maps f. See, for example, the f of §7.4. Moreover, even for totally Fatou points $a \in \mathbb{P}^1(k)$, the proof of Theorem 1.7 requires a new approach. The local canonical height functions $\hat{\lambda}_{f_t,v}$ for polynomials f can be normalized so they are always non-negative. The challenge here is the absence of a uniform lower bound on the functions V_v of Theorem 1.7, independent of a. (This unboundedness was exploited in [11] to show V_v can fail to extend continuously for maps $f(z) \in k(z)$ when a point $a \in \mathbb{P}^1(k')$ is defined over a larger field such as $k' = K_v(X)$; see Remark 1.8.)

Finally, we remark that Theorem 1.7 as stated does not imply Theorem 1.2. For polynomial maps f and each $a \in \mathbb{P}^1(k)$, the functions V_v of Theorem 1.7 will satisfy $V_v \equiv 0$ at all but

finitely many places v of K [21], as is the case for sections of elliptic surfaces [30]. However, by contrast, it is *not* the case that the functions V_v will be $\equiv 0$ for all but finitely many places v for general rational maps f; there can be nontrivial contributions at infinitely many places, even for totally Fatou points $a \in \mathbb{P}^1(k)$. See the example of §7.1; such examples were studied in depth in [25]. Nevertheless, we extract the summability of the magnitudes of V_v, over all places v of K, from the proof of Theorem 1.7.

Julia points. We use the totally Fatou hypothesis on $a \in \mathbb{P}^1(k)$ in a crucial way in our proofs of Theorems 1.2 and 1.7. The study of Julia points with infinite orbit is more subtle. As we show in Section 7, there exist examples of the following:

(1) a map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 2 defined over $k = \mathbb{Q}(t)$ with bad reduction at $t = 0$, for which the non-archimedean Julia set at $t = 0$ is a Cantor set in the completion $\mathbb{P}^1(k_0)$ at $t = 0$, and the local geometric height $\hat{\lambda}_{f,0}(a)$ is in $\mathbb{R} \setminus \mathbb{Q}$ for *all* Julia points a with infinite orbit. See §7.3; compare the main results of [10].

(2) a map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 2 defined over $k = \mathbb{Q}(t)$ with bad reduction at $t = 0$, and a point a defined by a formal power series in $\mathbb{Q}[[t]]$ in the non-archimedean Julia set of f at $t = 0$ for which, at the place $v = \infty$ of $\mathbb{Q}$, the function V_v of Theorem 1.7 will fail to be defined at $t = 0$. See §7.4.

For either example, if such a point a can be constructed to be *algebraic* over k, then, upon replacing k with a finite extension, it would provide a counterexample to Conjecture 1.3, and the results we prove for totally Fatou points would fail to extend to all $a \in \mathbb{P}^1(k)$. More precisely, example (1) would show that the divisor D constructed in Theorem 1.2, defined by (1.4), needs to be an $\mathbb{R}$-divisor instead of a $\mathbb{Q}$-divisor; compare Proposition 6.1. Example (2) would show that the sequences of functions converging to define the V_v of Theorem 1.7 would not always converge uniformly in the neighborhood of a singularity; compare Theorem 5.1.

Remark 1.8. It is known that, working with maps f defined over the field $\ell = \mathbb{C}(t)$, there exist points $a \in \mathbb{P}^1(\ell)$ that are totally Fatou for f but for which the analog of the (archimedean) function V_∞ of Theorem 1.7 is unbounded on the base curve $\mathbb{P}^1(\mathbb{C})$ [11]. The construction in [11] is different from the construction for example (2) and uses Baire Category. The results of Favre and Gauthier show that such examples over ℓ or examples of the types (1) and (2) above cannot exist for polynomials f [14].

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2. M_K-TERMINOLOGY

In this section, we fix some basic terminology associated to the number field K and remind the reader of fundamental facts about elements of $k = K(X)$.

Let M_K denote the set of places of the number field K, each giving rise to an absolute value $|\cdot|_v$ on K which is normalized to extend one of the standard absolute values on the field $\mathbb{Q}$ of rational numbers. The set M_K satisfies the product formula,

$$\prod_{v \in M_K} |x|_v^{N_v} = 1$$

for all $x \in K^*$. We let K_v denote the completion of K at v, so that

$$N_v = [K_v : \mathbb{Q}_v].$$

For each place v of K, we let $\mathbb{C}_v$ be the completion of an algebraic closure $\overline{K_v}$. We also fix an embedding $\overline{K} \hookrightarrow \mathbb{C}_v$. We let X_v^{an} denote the Berkovich analytification of the curve X over the field $\mathbb{C}_v$.

We will use the following terminology, as in [24, Chapter 10]: An **M_K-constant** is a function $\mathfrak{C} : M_K \to \mathbb{R}$ so that $\mathfrak{C}_v = 0$ at all but finitely many places v. An **M_K-quasiconstant** is a function $\mathfrak{C} : M_K \to \mathbb{R}$ such that

$$\sum_{v \in M_K} N_v \, |\mathfrak{C}_v| < \infty.$$

A collection of functions $\mathfrak{f}_v : Y \to \mathbb{R}$, for $v \in M_K$, defined on a set Y, is **M_K-bounded** if there exists an M_K-constant $\mathfrak{C}$ so that $|\mathfrak{f}_v(y)| \leq \mathfrak{C}_v$ for all $y \in Y$ and $v \in M_K$.

Fix a point $\gamma \in X(K)$ and a choice of $\omega_\gamma \in K(X)$ defining local coordinates for X near γ. An **M_K-neighborhood** of γ is a collection of open neighborhoods U_v of γ in $X(\mathbb{C}_v)$, for $v \in M_K$, given locally by $\{|\omega_\gamma|_v < 1\}$ for all but finitely many places v. This definition is independent of the choice of ω_γ uniformizing X near γ, as a consequence of the following proposition.

Let g denote the genus of X. For each $\gamma \in X(K)$, choose $\xi^\gamma \in K(X)$ so that ξ^γ has a pole of order $2g+1$ at γ and no other poles in X. The divisor of a function $h \in K(X)$ is

$$(h) = \sum_{\gamma \in X(\overline{K})} (\operatorname{ord}_\gamma h)\, \gamma.$$

Proposition 2.1. *For any nonconstant $h \in K(X)$ with $\operatorname{supp}(h) \subset X(K)$, there exists an M_K-constant $\mathfrak{c}$ so that*

$$e^{-\mathfrak{c}_v} \prod_{\gamma \in \operatorname{supp}(h)} \max\{1, |\xi^\gamma(t)|_v\}^{(-\operatorname{ord}_\gamma h)/(2g+1)} \leq |h(t)|_v \leq e^{\mathfrak{c}_v} \prod_{\gamma \in \operatorname{supp}(h)} \max\{1, |\xi^\gamma(t)|_v\}^{(-\operatorname{ord}_\gamma h)/(2g+1)}$$

for all $t \in X(\overline{K}) \setminus \operatorname{supp}(h)$ and $v \in M_K$. Moreover, for each $\gamma \in X(K)$, the notion of M_K-neighborhood of γ is well defined.

Remark 2.2. The proposition is a theorem of Weil [34], and its proof is contained in [24, Chapter 10] or [4, Theorem 2.2.11, Remark 2.2.13], but we include an argument here for completeness.

Proof. For each $\gamma \in \operatorname{supp}(h)$, let U_γ be the complement in X of $\operatorname{supp}(h) \setminus \{\gamma\}$ and all zeroes of ξ^γ, so that U_γ is a Zariski-open neighborhood of γ. The functions

$$h^\gamma := h^{2g+1} (\xi^\gamma)^{\operatorname{ord}_\gamma h}$$

and $1/h^\gamma$ and $1/\xi^\gamma$ and $\xi^{\gamma'}$ for $\gamma' \neq \gamma$ in $\operatorname{supp}(h)$ are all regular on U_γ. Let $U_h = X \setminus \operatorname{supp}(h)$, so that h, $1/h$ and each ξ^γ, $\gamma \in \operatorname{supp}(h)$, are regular on U_h. Note that $\mathcal{U} = \{U_h\} \cup \{U_\gamma : \gamma \in \operatorname{supp}(h)\}$ is an open cover of X.

As in [24, Chapter 10, Lemma 1.1], there exists a projective embedding of X into $\mathbb{P}^N$, defined over K, so the complement of each coordinate hyperplane in $\mathbb{P}^N$ intersects X in an open subset of some $U \in \mathcal{U}$. Indeed, letting F_U be the divisor consisting of the sum of points in the complement of $U \in \mathcal{U}$, we can find effective divisors H_U so that the elements of $\{F_U + H_U : U \in \mathcal{U}\}$ are linearly equivalent, and so that there is no point in the intersection of the supports of $F_U + H_U$. (This is because $mH - F_U$ will be very ample for any choice of ample H and every $U \in \mathcal{U}$, for all sufficiently large $m \in \mathbb{N}$.) The elements $\{F_U + H_U : U \in \mathcal{U}\}$ thus induce a morphism $\phi : X \to \mathbb{P}^k$ for some k. Choosing any projective embedding

$i : X \hookrightarrow \mathbb{P}^r$ defined over K, for some $r > 0$, our desired embedding comes from postcomposing $\phi \times i : X \to \mathbb{P}^k \times \mathbb{P}^r$ with the Segre embedding $\mathbb{P}^k \times \mathbb{P}^r \hookrightarrow \mathbb{P}^{(k-1)(m-1)-1}$.

Let $\mathbb{A}^N(\overline{K})$ denote affine space of dimension N, and let

$$\|(y_1, \ldots, y_N)\|_v = \max\{|y_1|_v, \ldots, |y_N|_v\}$$

be a v-adic norm on $\mathbb{A}^N(\overline{K})$, for each $v \in M_K$. A collection of subsets $E_v \subset \mathbb{A}^N(\overline{K})$, for $v \in M_K$, is **affine M_K-bounded** if each E_v is bounded, and if E_v lies in the unit polydisk $\{y : \|y\|_v \leq 1\}$ for all but finitely many v.

As in [24, Chapter 10, Proposition 1.2], we can now cover X by a finite collection $\{\mathbf{E}_j\}_{j=0}^N$ of affine M_K-bounded sets, subordinate to the open cover $\mathcal{U}$. Indeed, suppose that $(x_0 : x_1 : \cdots : x_N)$ are the coordinates of $\mathbb{P}^N$. For each $j = 0, \ldots, N$, let $U(j) \in \mathcal{U}$ be an element containing $X \cap \{x_j \neq 0\}$. For each $v \in M_K$, we let $E_{j,v}$ be the set of all points in $\mathbb{P}^N(\overline{K})$ with projective coordinates $(x_0 : x_1 : \cdots : x_N)$ so that $|x_j|_v$ is maximal. Then $E_{j,v}$ is the unit polydisk in the affine chart where $x_j \neq 0$ with coordinates $y_i = x_i/x_j$. For each $v \in M_K$, these affine bounded sets cover all of $\mathbb{P}^N(\overline{K})$ and so also X, and the intersection of $E_{j,v}$ with X is a subset of $U(j)$. We let $\mathbf{E}_j$ be the collection $\{E_{j,v} : v \in M_K\}$, for $j = 0, \ldots, N$.

Fix j. For $U(j) = U_\gamma$, since h^γ and $1/h^\gamma$ are both regular on U_γ, we have an M_K-constant $\mathfrak{g}_\gamma$ such that

$$e^{-\mathfrak{g}_{\gamma,v}} \;\leq\; |h^\gamma|_v = |h^{2g+1}(\xi^\gamma)^{\operatorname{ord}_\gamma h}|_v \;\leq\; e^{\mathfrak{g}_{\gamma,v}} \tag{2.1}$$

on $E_{j,v}$. It is also the case that $1/\xi^\gamma$ is regular on U_γ, and so is $\xi^{\gamma'}$ for each $\gamma' \neq \gamma$ in $\operatorname{supp}(h)$, so we can enlarge $\mathfrak{g}_\gamma$ if needed so that

$$|\xi^\gamma|_v \geq e^{-\mathfrak{g}_{\gamma,v}} \quad \text{and} \quad |\xi^{\gamma'}|_v \leq e^{\mathfrak{g}_{\gamma,v}}$$

on $E_{j,v}$. Moreover, we can also arrange that

$$|h|_v \begin{cases} \leq e^{\mathfrak{g}_{\gamma,v}} & \text{if } \operatorname{ord}_\gamma h > 0 \\ \geq e^{-\mathfrak{g}_{\gamma,v}} & \text{if } \operatorname{ord}_\gamma h < 0 \end{cases}$$

on $E_{j,v}$, because either h or $1/h$ is regular on U_γ. By increasing $\mathfrak{g}_\gamma$ yet again, it follows that

$$\begin{aligned} e^{-\mathfrak{g}_{\gamma,v}} \prod_{\beta\in\mathrm{supp}(h)} \max\{1,|\xi^\beta(t)|_v\}^{(-\operatorname{ord}_\beta h)/(2g+1)} &= e^{-\mathfrak{g}_{\gamma,v}} \max\{1,|\xi^\gamma(t)|_v\}^{(-\operatorname{ord}_\gamma h)/(2g+1)} \\ &\le |h(t)|_v \\ &\le e^{\mathfrak{g}_{\gamma,v}} \max\{1,|\xi^\gamma(t)|_v\}^{(-\operatorname{ord}_\gamma h)/(2g+1)} \\ &= e^{\mathfrak{g}_{\gamma,v}} \prod_{\beta\in\mathrm{supp}(h)} \max\{1,|\xi^\beta(t)|_v\}^{(-\operatorname{ord}_\beta h)/(2g+1)} \end{aligned}$$

for all $t \in E_{j,v}$ where $U(j) = U_\gamma$.

Similarly for $U(j) = U_h$, we can find an M_K-constant $\mathfrak{s}$ so that

$$e^{-\mathfrak{s}_v} \le |h|_v \le e^{\mathfrak{s}_v}$$

and

$$|\xi^\gamma|_v \le e^{\mathfrak{s}_v}$$

on $E_{j,v}$, for all $\gamma \in \mathrm{supp}(h)$. This completes the proof of the first statement of the proposition.

To see that the notion of M_K-neighborhood is well defined, we fix $\gamma_0 \in X(K)$ and choose any $\omega_0 \in K(X)$ with a simple zero at γ_0. For the covering $\mathcal{U}$ of X associated to ω_0, note that U_{γ_0} is the unique element containing γ_0. So for each v and j, if the set $E_{j,v}$ contains γ_0, then it must lie in U_{γ_0}. The inequality (2.1) implies that $|\omega_0|_v^{2g+1} = |\xi^{\gamma_0}|_v^{-1}$ on such $E_{j,v}$, for all but finitely many v. On the other hand, we also have that if $|\xi^{\gamma_0}(t)|_v > 1$ at a point $t \in E_{j,v}$, for some j, then $E_{j,v}$ is contained in U_{γ_0} for all but finitely many v (because $|\xi^{\gamma_0}|_v \le 1$ on the $E_{j,v}$'s in the other elements of $\mathcal{U}$). In other words, any M_K-neighborhood of γ_0 defined by ω_0 coincides with $\{t \in X(\mathbb{C}_v) : |\xi^{\gamma_0}(t)|_v > 1\}$ for all but finitely many places v of K. This completes the proof of the proposition. □

3. Escape rates and Weil heights

Throughout this section, we fix $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \ge 2$, defined over $k = K(X)$, and any point $a \in \mathbb{P}^1(k)$.

3.1. The singular set $\mathcal{S}(F, A)$ in X.

Working in homogeneous coordinates on $\mathbb{P}^1$, we let

$$F = (P, Q) : \mathbb{A}^2 \to \mathbb{A}^2$$

be a **homogeneous lift** of f, with homogeneous polynomials $P(z,w) \in k[z,w]$ and $Q(z,w) \in k[z,w]$ of degree d having no common zeroes in $\mathbb{P}^1(\overline{k})$, so that $f(z) = P(z,1)/Q(z,1)$ in local coordinates. Choose a **lift** $A = (\alpha, \beta) \in k^2 \setminus \{(0,0)\}$ of $a = (\alpha : \beta) \in \mathbb{P}^1(k)$. For each $\gamma \in X(\overline{K})$, we let

$$\operatorname{ord}_\gamma F = \min\{\operatorname{ord}_\gamma c : \text{ coefficients } c \text{ of } P \text{ and } Q\}$$

and

$$\operatorname{ord}_\gamma A = \min\{\operatorname{ord}_\gamma \alpha, \operatorname{ord}_\gamma \beta\}.$$

We let $\operatorname{Res} F \in k^*$ denote the homogeneous resultant of P and Q; see, for example, [31, §2.4].

Set

$$\mathcal{S}(F) = \{\gamma \in X(\overline{K}) : \operatorname{ord}_\gamma F \neq 0 \text{ or } \operatorname{ord}_\gamma \operatorname{Res} F \neq 0\} \tag{3.1}$$

and

$$\mathcal{S}(F, A) = \mathcal{S}(F) \cup \{\gamma \in X(\overline{K}) : \operatorname{ord}_\gamma A \neq 0\} \tag{3.2}$$

Note that $\mathcal{S}(F, A)$ is a finite set.

Convention 3.1. We enlarge the number field K, if needed, so that $\mathcal{S}(F, A) \subset X(K)$.

3.2. **Geometric escape rates and a divisor on X.** Recall here that throughout we identify the places of k with the points $\gamma \in X(\overline{K})$, with a slight abuse of terminology. For each $\gamma \in X(\overline{K})$, we work with the absolute value on k defined by

$$|z|_\gamma := e^{-\operatorname{ord}_\gamma z},$$

and the norm $\|\cdot\|_\gamma$ on k^2 given by

$$\|(z,w)\|_\gamma = \max\{|z|_\gamma, |w|_\gamma\}.$$

There is a constant $C_\gamma \geq 1$ so that

$$C_\gamma^{-1}\|(z,w)\|_\gamma^d \leq \|F(z,w)\|_\gamma \leq C_\gamma \|(z,w)\|_\gamma^d \tag{3.3}$$

for all $(z,w) \in k^2$; we can take $C_\gamma = 1$ for all $\gamma \notin \mathcal{S}(F)$ [31, Proposition 5.57].

The **escape rate** of A for F at γ is the quantity

$$G_{F,\gamma}(A) = \lim_{n\to\infty} \frac{1}{d^n} \log \|F^n(A)\|_\gamma. \tag{3.4}$$

It exists in $\mathbb{R}$, by (3.3), and it is equal to 0 for all $\gamma \notin \mathcal{S}(F,A)$; see, e.g., [31, Proposition 5.58]. We define an $\mathbb{R}$-divisor by

$$D(F,A) = \sum_{\gamma \in X(\overline{K})} G_{F,\gamma}(A)\, \gamma. \tag{3.5}$$

The support of $D(F,A)$ is contained in $\mathcal{S}(F,A)$ and so in $X(K)$ by Convention 3.1. If we had chosen different lifts of f and a, say cF and bA for $c, b \in k^*$, then

$$G_{cF,\gamma}(bA) = G_{F,\gamma}(A) - \frac{1}{d-1}\operatorname{ord}_\gamma c - \operatorname{ord}_\gamma b. \tag{3.6}$$

It follows that $D(cF, bA)$ and $D(F,A)$ are linearly equivalent $\mathbb{R}$-divisors on X.

3.3. **A Weil height associated to $D(F,A)$.** Let g denote the genus of X. For each $\gamma \in X(K)$, choose a meromorphic function $\xi^\gamma \in K(X)$ so that ξ^γ has a pole of order $2g+1$ at γ and no other poles.

Let $D = D(F,A)$ be defined by (3.5), and recall that $\operatorname{supp} D(F,A) \subset X(K)$ by Convention 3.1. For each place v of K, we define a function on $X(\mathbb{C}_v) \setminus \operatorname{supp} D$ by

$$\lambda_{D,v}(t) = \sum_{\gamma \in \mathcal{S}(F,A)} G_{F,\gamma}(A)\, \frac{\log^+ |\xi^\gamma(t)|_v}{2g+1}.$$

This function extends continuously to the Berkovich analytification $X_v^{an} \setminus \operatorname{supp} D$.

A Weil height for D can be defined by

$$h_D(t) = \frac{1}{[K:\mathbb{Q}]} \frac{1}{|\operatorname{Gal}(\overline{K}/K)\cdot t|} \sum_{x \in \operatorname{Gal}(\overline{K}/K)\cdot t} \sum_{v \in M_K} N_v\, \lambda_{D,v}(x) \tag{3.7}$$

for all $t \in X(\overline{K}) \setminus \operatorname{supp} D$, and we may set $h_D(t) = 0$ for $t \in \operatorname{supp} D$. This h_D is indeed a Weil height associated to the $\mathbb{R}$-divisor D, as it is an $\mathbb{R}$-linear combination of Weil heights built from the local functions

$$\frac{1}{2g+1}\log^+ |\xi^\gamma(t)|_v$$

at each place v of K, associated to the divisor γ.

3.4. **Arithmetic escape rates.** For each place v of the number field K, we define a norm $\|\cdot\|_v$ on $\overline{K}^2$ by

$$\|(z,w)\|_v = \max\{|z|_v, |w|_v\}.$$

For each $t \in X(\overline{K}) \setminus \mathcal{S}(F)$, we let F_t denote the specializations of F. We continue to use the collection of functions $\{\xi^\gamma : \gamma \in \mathcal{S}(F)\}$ from §3.3. The following proposition appears in

various forms in the literature, e.g., in [31, Proposition 5.57] [1, Lemma 10.1] [8, Lemma 3.3], but we require an adelic version for our theorems. Recall that $\mathcal{S}(F) \subset X(K)$ by Convention 3.1.

Proposition 3.2. *For each $\gamma \in \mathcal{S}(F)$, choose $\beta^\gamma \in k$ so that $\operatorname{ord}_\gamma(\beta^\gamma F) = 0$. There is an M_K-constant $\mathfrak{b}$ and an M_K-neighborhood $\mathfrak{U}^\gamma$ of γ in X so that*

$$e^{-\mathfrak{b}_v}|\xi^\gamma(t)|_v^{-(\operatorname{ord}_\gamma \operatorname{Res}(\beta^\gamma F))/(2g+1)} \;\le\; \frac{|\beta_t^\gamma|_v \|F_t(z,w)\|_v}{\|(z,w)\|_v^d} \;\le\; e^{\mathfrak{b}_v}$$

for all $v \in M_K$, $t \in \mathfrak{U}_v^\gamma \setminus \{\gamma\}$, and for all $(z,w) \in (\mathbb{C}_v)^2 \setminus \{(0,0)\}$. We can choose the M_K-neighborhoods $\mathfrak{U}^\gamma$ for $\gamma \in \mathcal{S}(F)$ to be pairwise disjoint. Moreover, given any M_K-neighborhoods $\mathfrak{U}^\gamma$, for $\gamma \in \mathcal{S}(F)$, there exists an M_K-constant $\mathfrak{c}$ so that

$$e^{-\mathfrak{c}_v} \;\le\; \frac{\|F_t(z,w)\|_v}{\|(z,w)\|_v^d} \;\le\; e^{\mathfrak{c}_v}$$

for all $v \in M_K$, $t \in X(\mathbb{C}_v) \setminus \left(\bigcup_{\gamma\in\mathcal{S}(F)} \mathfrak{U}_v^\gamma\right)$, and for all $(z,w) \in (\mathbb{C}_v)^2 \setminus \{(0,0)\}$.

Proof. Recall that $F = (P,Q)$ for homogeneous polynomials P and Q of degree d with coefficients in k. By our choice of β^γ, there is an M_K-neighborhood $\mathfrak{U}^\gamma$ and an M_K-constant $\mathfrak{b}$ so that

$$e^{-\mathfrak{b}_v} \;\le\; \max\{|c_t|_v : \text{ coefficients } c \text{ of } \beta^\gamma P \text{ and } \beta^\gamma Q\} \;\le\; e^{\mathfrak{b}_v} \tag{3.8}$$

for each $t \in \mathfrak{U}_v^\gamma$ and each $v \in M_K$, by Proposition 2.1. By increasing the constant $\mathfrak{b}$, the upper bound on $\|\beta_t^\gamma F_t(z,w)\|_v / \|(z,w)\|_v^d$ follows from the triangle inequality.

We can enlarge $\mathfrak{b}$ at the archimedean places, if needed, so that

$$\frac{\|\beta_t^\gamma F_t(z,w)\|_v}{\|(z,w)\|_v^d} \;\ge\; e^{-\mathfrak{b}_v}|\operatorname{Res}(\beta_t^\gamma F_t)|_v$$

for each $t \in \mathfrak{U}_v^\gamma$, for all $v \in M_K$, applying [31, Proposition 5.57] and [8, Lemma 3.3]. Applying Proposition 2.1 again, this time to $\operatorname{Res}(\beta^\gamma F)$, shrinking the M_K-neighborhood and increasing the M_K-constant $\mathfrak{b}$ again if necessary, we have

$$\frac{\|\beta_t^\gamma F_t(z,w)\|_v}{\|(z,w)\|_v^d} \;\ge\; e^{-\mathfrak{b}_v}|\xi^\gamma(t)|_v^{-(\operatorname{ord}_\gamma \operatorname{Res}(\beta^\gamma F))/(2g+1)}$$

for all $t \in \mathfrak{U}_v^\gamma$ and each $v \in M_K$.

The final statements of the proposition follow from the same combination of Proposition 2.1 with [31, Proposition 5.57], because the coefficients of F will have no poles and $\operatorname{Res} F$ will have no poles or zeroes outside of $\mathcal{S}(F)$. □

Similar to the geometric escape rates of (3.4), we can define arithmetic escape rates, working at each place v of the number field K. For each $v \in M_K$, the **escape rate function for the pair** (F, A) is defined on $X(\overline{K}) \setminus \mathcal{S}(F, A)$ by

$$G_{F_t,v}(A_t) = \lim_{n\to\infty} \frac{1}{d^n} \log \|F_t^n(A_t)\|_v. \tag{3.9}$$

It exists in $\mathbb{R}$ for all $t \in X(\overline{K}) \setminus \mathcal{S}(F, A)$ by Proposition 3.2; see, e.g., [31, Proposition 5.58]. The proof of convergence for (3.9) shows it is locally uniform in t, so that $G_{F_t,v}(A_t)$ extends to a continuous function $t \in X(\mathbb{C}_v) \setminus \mathcal{S}(F, A)$ at each place $v \in M_K$. In fact, it extends to be continuous on the Berkovich analytification $X_v^{an} \setminus \mathcal{S}(F, A)$; see, e.g., [1, pp. 295–296] where the escape rate is "Berkovich-ized". If we had chosen different lifts of f and a, say cF and bA for $c, b \in k^*$, then

$$G_{c_tF_t,v}(b_tA_t) = G_{F_t,v}(A_t) + \frac{1}{d-1} \log |c_t|_v + \log |b_t|_v \tag{3.10}$$

for $t \in X(\overline{K}) \setminus (\mathcal{S}(F, A) \cup \mathcal{S}(cF, bA))$.

These escape rate functions provide local height expressions for the canonical height $\hat{h}_{f_t}$ evaluated at a_t. In particular, we have

$$\hat{h}_{f_t}(a_t) = \frac{1}{[K : \mathbb{Q}]} \frac{1}{|\operatorname{Gal}(\overline{K}/K) \cdot t|} \sum_{x \in \operatorname{Gal}(\overline{K}/K)\cdot t} \sum_{v \in M_K} N_v \, G_{F_x,v}(A_x)$$

for all $t \in X(\overline{K}) \setminus \mathcal{S}(F, A)$. See, for example, [31, Theorem 5.59]. Note that the sum over all places of K is independent of the choice of lifts F and A, by the product formula.

3.5. **Variation of canonical height.** Recall that we are trying to understand if

$$\hat{h}_{f_t}(a_t) - h_D(t)$$

is bounded, as claimed in Theorem 1.2, where h_D is a choice of Weil height for $D = D(F, A)$ defined by (3.5). Recalling that any two choices of Weil height for the same divisor are bounded from one another (and in fact, M_K-bounded) it suffices to work with the Weil height constructed in (3.7). Assuming that the point $a \in \mathbb{P}^1(k)$ is totally Fatou for f, a hypothesis which will be defined and examined in the next Section, we aim to prove three things:

(1) that the local geometric height $G_{F,\gamma}(A)$ is in $\mathbb{Q}$ at all points $\gamma \in X(\overline{K})$, so that the divisor $D = D(F, A)$ of (3.5) will be a $\mathbb{Q}$-divisor;
(2) that for all places v of K, the v-adic functions

$$V_v(t) := G_{F_t,v}(A_t) - \sum_{\gamma \in \mathcal{S}(F,A)} G_{F,\gamma}(A)\, \frac{\log^+ |\xi^\gamma(t)|_v}{2g+1}$$

on $X(\mathbb{C}_v) \setminus \mathcal{S}(F, A)$ extend to bounded – and in fact continuous – functions on the Berkovich analytification $X^{an}_{\mathbb{C}_v}$; and
(3) that the sum

$$\hat{h}_{f_t}(a_t) - h_D(t) \;=\; \frac{1}{|\operatorname{Gal}(\overline{K}/K) \cdot t|} \sum_{x \in \operatorname{Gal}(\overline{K}/K)\cdot t} \sum_v N_v\, V_v(x)$$

is uniformly bounded over all points $t \in X(\overline{K}) \setminus \mathcal{S}(F, A)$.

4. The non-archimedean Fatou set

Throughout this section, we fix $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$, defined over $k = K(X)$. For each fixed $\gamma \in X(\overline{K})$, we let k_γ be the completion of k with respect to the valuation $\operatorname{ord}_\gamma$, and let $\mathbb{L}_\gamma$ be the completion of an algebraic closure of k_γ. In this section, we introduce and study the totally Fatou condition that is assumed for Theorem 1.2, and we prove Theorems 1.4 and 1.5.

4.1. **The Fatou set.** Fix $\gamma \in X(\overline{K})$. Let $d_\gamma(x, y)$ denote the chordal distance between x and y in $\mathbb{P}^1(\mathbb{L}_\gamma)$. Explicity, if $x = (x_1 : x_2)$ and $y = (y_1 : y_2)$, then

$$d_\gamma(x, y) = \frac{|x_1 y_2 - x_2 y_1|_\gamma}{\max\{|x_1|_\gamma, |x_2|_\gamma\} \max\{|y_1|_\gamma, |y_2|_\gamma\}}.$$

The **non-archimedean Fatou set of f at γ** is the set $\Omega_\gamma(f)$ of all points $x \in \mathbb{P}^1(\mathbb{L}_\gamma)$ for which we can find an open disk D_x containing x so that the family of functions $\{f^n | D_x\}$ is equicontinuous in the distance d_γ. See, for example, [3, Chapter 5]. Its complement $\mathbb{P}^1(\mathbb{L}_\gamma) \setminus \Omega_\gamma(f)$ is the **non-archimedean Julia set of f at γ**.

This Fatou set $\Omega_\gamma(f)$ will be all of $\mathbb{P}^1(\mathbb{L}_\gamma)$ at γ where f has good reduction. In our case, this implies that $\Omega_\gamma(f) = \mathbb{P}^1(\mathbb{L}_\gamma)$ for all $\gamma \notin \mathcal{S}(F)$, the singular set defined in (3.1), for any choice of homogeneous polynomial lift F of f.

A point $a \in \mathbb{P}^1(k)$ is **totally Fatou for f** if $a \in \Omega_\gamma(f)$ at all $\gamma \in X(\overline{K})$.

4.2. **Hole-avoiding pairs.** Now fix a point $a \in \mathbb{P}^1(k)$. Fix $\gamma \in X(\overline{K})$, and choose homogeneous polynomial lift

$$F = (P, Q) : \mathbb{A}^2 \to \mathbb{A}^2$$

of f over k and lift

$$A = (\alpha, \beta) \in k^2$$

of a so that

$$\operatorname{ord}_\gamma F = \operatorname{ord}_\gamma A = 0, \tag{4.1}$$

where $\operatorname{ord}_\gamma F$ and $\operatorname{ord}_\gamma A$ are defined in §3.1, so that the specializations F_γ and A_γ are well defined. The **holes** of f at γ are the points $x = (x_1 : x_2) \in \mathbb{P}^1(\overline{K})$ for which $F_\gamma(x_1, x_2) = (0, 0)$. Holes exist if and only if $\operatorname{Res} F_\gamma = 0$. We say that the pair (f, a) is **hole-avoiding at** γ if the specializations satisfy

$$F_\gamma^n(A_\gamma) \neq (0, 0)$$

for all $n \geq 0$. In particular, the pair (f, a) is hole-avoiding at γ for all points $a \in \mathbb{P}^1(k)$ if $\operatorname{Res} F_\gamma \neq 0$. It is easy to check that this definition is independent of the choice of lifts F and A, as long as they satisfy (4.1).

Example 4.1. Consider

$$f(z) = \frac{z(z-1)}{z-t}$$

over $k = \mathbb{Q}(t)$, at $t = 0$. The polynomial map $F(z, w) = (z(z-w), (z-tw)w)$ specializes to $F_0(z, w) = (z(z-w), zw)$. The point $0 = (0 : 1) \in \mathbb{P}^1(\overline{\mathbb{Q}})$ is the unique hole for f. A point $a \in \mathbb{P}^1(k)$ will therefore *fail to be* hole-avoiding at $t = 0$ if and only if it specializes to $a_0 \in \mathbb{Z}_{\geq 0}$. Indeed, for $a_0 = n_0 \in \mathbb{Z}_{\geq 0}$, the iterates of the lift $A = (a, 1)$ will satisfy $F_0^{n_0+1}(A_0) = (0, 0)$.

Example 4.2. Consider

$$f(z) = z^2 + 1/t$$

over $k = \mathbb{Q}(t)$, at $t = 0$. The polynomial map $F(z, w) = (tz^2 + w^2, tw^2)$ specializes to $F_0(z, w) = (w^2, 0)$. The point $\infty = (1 : 0) \in \mathbb{P}^1(\overline{\mathbb{Q}})$ is the unique hole for f. There are no hole-avoiding points in $\mathbb{P}^1(k)$, because $F_0^2(z, w) = F_0(w^2, 0) = (0, 0)$ for all $(z, w) \in \overline{K}^2$.

We may view $f : \mathbb{P}^1 \to \mathbb{P}^1$ over k as a rational map

$$\tilde{f} : X \times \mathbb{P}^1 \dashrightarrow X \times \mathbb{P}^1$$

of the surface $X \times \mathbb{P}^1$ to itself, defined over the number field K by $(t, z) \mapsto (t, f_t(z))$. We may view the point $a \in \mathbb{P}^1(k)$ as a section of the projection $X \times \mathbb{P}^1 \to X$, also defined over K. The following is immediate from the definitions:

Lemma 4.3. *A pair (f, a) is hole-avoiding at $\gamma \in X(\overline{K})$ if and only if the iterates $f^n(a)$ (as sections of the fibered surface $X \times \mathbb{P}^1 \to X$) are disjoint from the indeterminacy locus of the induced map $\tilde{f} : X \times \mathbb{P}^1 \dashrightarrow X \times \mathbb{P}^1$ within the fiber $\{\gamma\} \times \mathbb{P}^1$, for all $n \geq 0$.*

Note that all of the indeterminacy points of $\tilde{f}$ in $X \times \mathbb{P}^1$ are contained in the fibers over $\mathcal{S}(F) \subset X$ for every choice of homogeneous polynomial lift F. The term "hole" for an indeterminacy point was first used in [6]; it was meant to capture the idea that the mass of the measures of maximal entropy in the family f_t, for $t \in X(\mathbb{C}) \setminus \mathcal{S}(F)$, was "falling into the holes" of f and its iterates f^n at $t = \gamma$. The same condition appears in [25].

4.3. **The action of f on the Berkovich projective line.** Fix $\gamma \in X(\overline{K})$. We now reinterpret the notion of hole-avoiding in the language of the Berkovich projective line defined over the field $\mathbb{L}_\gamma$, which we denote by $\mathbb{P}^{1,an}_\gamma$, and the extension of f to a dynamical system on $\mathbb{P}^{1,an}_\gamma$. A good basic reference for the dynamics of f on $\mathbb{P}^{1,an}_\gamma$ is [3]. Note that the definition of hole-avoiding extends naturally to elements $a \in \mathbb{P}^1(k_\gamma)$, for k_γ the completion of k at γ, with a lift A chosen in $(k_\gamma)^2 \setminus \{(0, 0)\}$.

By definition, the **hole-directions** for f from a Type II point $\zeta \in \mathbb{P}^{1,an}_\gamma$ are the connected components of $\mathbb{P}^{1,an}_\gamma \setminus \{\zeta\}$ that intersect the set of preimages $f^{-1}(\zeta)$. When $\zeta = \zeta_G$ is the Gauss point in $\mathbb{P}^{1,an}_\gamma$, the hole-directions correspond to the holes of f at γ, via the natural identification of the connected components of $\mathbb{P}^{1,an}_\gamma \setminus \{\zeta\}$ with $\mathbb{P}^1(\overline{K})$. See, for example, [3, §7.5]. (In particular, if $f(\zeta) = \zeta$, the hole-directions coincide with the "bad" directions of [3, Theorem 7.34].) This implies:

Lemma 4.4. *Let a be an element of $\mathbb{P}^1(k_\gamma)$. The pair (f, a) is hole-avoiding at γ if and only if neither a nor any iterate $f^n(a)$ lies in a hole-direction for f from the Gauss point ζ_G in $\mathbb{P}^{1,an}_\gamma$.*

Let $\Omega^{an}_\gamma(f)$ be the Berkovich Fatou set of f in $\mathbb{P}^{1,an}_\gamma$; see, e.g., [3, Chapter 8]. Any open set $U \subset \mathbb{P}^{1,an}_\gamma$ that intersects the Berkovich Julia set $J^{an}_\gamma(f) := \mathbb{P}^{1,an}_\gamma \setminus \Omega^{an}_\gamma(f)$ has the property that the union $\mathcal{U} = \bigcup_{n \geq 0} f^n(U)$ is dense in $\mathbb{P}^{1,an}_\gamma$; in fact, the set $\mathcal{U}$ omits at most 2 points, both in $\mathbb{P}^1(\mathbb{L}_\gamma)$ [3, Theorem 8.15]. Recall that $\Omega^{an}_\gamma(f) \cap \mathbb{P}^1(\mathbb{L}_\gamma) = \Omega_\gamma(f)$, the non-archimedean Fatou set as we have defined it in §4.1.

Lemma 4.5. *For any Type II point $\zeta \in \mathbb{P}^{1,an}_\gamma$, the points of the non-archimedean Julia set of f in $\mathbb{P}^1(\mathbb{L}_\gamma)$ are contained in the union of the hole-directions from ζ for f^n, over all $n \geq 1$.*

Proof. Let U be a connected component of $\mathbb{P}^{1,an}_\gamma \setminus \{\zeta\}$. If U has non-empty intersection with the non-archimedean Julia set $J^{an}_\gamma(f) \cap \mathbb{P}^1(\mathbb{L}_\gamma)$, then the iterates of U must contain all Type II points, including ζ itself [3, Theorem 8.15]. So U must be a hole-direction from ζ for some iterate of f. □

We are now ready to prove the following result, needed to analyze the dynamics of totally Fatou points for the proof of Theorem 1.2 and Theorem 1.7:

Theorem 4.6. *Fix any $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$, defined over $k = K(X)$, a point $\gamma \in X(K)$, and any point $a \in \mathbb{P}^1(k_\gamma)$. The point a lies in the non-archimedean Fatou set $\Omega_\gamma(f)$ if and only if there exist a change of coordinates $B \in \mathrm{PGL}_2(k)$ and iterates f^n and f^m so that the pair $(Bf^nB^{-1}, B(f^m(a)))$ is hole-avoiding at γ.*

We shall see that one implication is straightforward from the definitions, that the existence of the hole-avoiding pair implies that $a \in \Omega_\gamma(f)$. To prove the converse implication, assuming $a \in \Omega_\gamma(f)$, we follow the proof of [9, Theorem D], which itself uses the Rivera-Letelier classification of Berkovich Fatou components in the Berkovich space $\mathbb{P}^{1,an}_\gamma$ [27] [9, Appendix] and the Benedetto wandering domains theorem [2], while also keeping track of the orbit of the point a.

In the language of [9], given a finite set Γ of Type II points in $\mathbb{P}^{1,an}_\gamma$, a connected component U of $\mathbb{P}^{1,an}_\gamma \setminus \Gamma$ is called a **J-component** for Γ if $f^n(U) \cap \Gamma \neq \emptyset$ for some $n > 0$. The Julia set $J^{an}_\gamma(f)$ is contained in the union of the J-components and Γ [9, Proposition 2.5]. The other connected components of $\mathbb{P}^{1,an}_\gamma \setminus \Gamma$ are called **F-components**. An F-component U for Γ is called an **F-disk** if it is a Berkovich disk; it is **wandering** if the iterates $f^n(U)$ lie in pairwise disjoint F-components for all n.

A pair (f, Γ) is **analytically stable** if, for each $\zeta \in \Gamma$, we have either $f(\zeta) \in \Gamma$ or $f(\zeta)$ is contained in an F-component for Γ.

The **k-split** Type II points ζ are those in the $\mathrm{PGL}_2(k)$-orbit of the Gauss point ζ_G. These are the Type II points that have k-rational points in infinitely many connected components of $\mathbb{P}^{1,an} \setminus \{\zeta\}$. Our proof strategy for Thoerem 4.6 also gives the following statement, which will be used in our proof of Theorem 1.5.

Theorem 4.7. *Fix any $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$ defined over $k = K(X)$, a point $\gamma \in X(K)$, and any Fatou point $a \in \Omega_\gamma(f) \cap \mathbb{P}^1(k_\gamma)$. For any finite set of Type II points Γ,*

there exists a finite set $\Gamma' \supset \Gamma$ so that the pair (f, Γ') is analytically stable and each point $f^n(a)$ of the orbit of a lies in an F-disk for Γ'. Moreover, if the elements of Γ are k-split, then we can choose Γ' so that its elements are also k-split.

Remark 4.8. In [9, Theorem D], the existence of an analytically stable $\Gamma' \supset \Gamma$ for the map f is proved, but without the additional conclusion about the orbit of the Fatou point a.

Proof of Theorems 4.6 and 4.7. Fix $a \in \mathbb{P}^1(k_\gamma)$, and assume first that there exist $B \in \mathrm{PGL}_2(k)$ and integers $n \geq 1$ and $m \geq 0$ so that the pair $(Bf^nB^{-1}, B(f^m(a)))$ is hole-avoiding at γ. Let $\zeta_B = B^{-1}(\zeta_G) \in \mathbb{P}^{1,an}_\gamma$, where ζ_G is the Gauss point. Then by Lemma 4.4, the point $f^m(a)$ and all iterates $f^{jn+m}(a)$, for $j \geq 0$, do not lie in the hole-directions of f^n from ζ_B. But the existence of such an orbit implies that either $f^n(\zeta_B) = \zeta_B$ or that $f^n(\zeta_B)$ lies in a direction from ζ_B which is not a hole-direction (for otherwise all points would either be in a hole-direction or mapped into a hole-direction under one iterate). If $f^n(\zeta_B) = \zeta_B$, then the hole-directions from ζ_B for an iterate f^{jn}, with $j \geq 1$, coincide with directions that are mapped to the hole-directions for f^n by some $f^{\ell n}$ with $\ell < j$; if $f^n(\zeta_B) \neq \zeta_B$, then the hole-directions for the iterates f^{jn} must coincide with the holes for f^n, for all $j \geq 1$. In either case, it then follows from Lemma 4.5 that $f^m(a)$ is not in $J^{an}_\gamma(f) \cap \mathbb{P}^1(\mathbb{L}_\gamma)$. In other words, a must be an element of the Fatou set $\Omega_\gamma(f)$. This proves one implication of Theorem 4.6.

To prove the converse implication in Theorem 4.6, we need to find a coordinate change B with good properties. We do this by constructing a Type II point ζ_B with the desired properties, and then we will choose any $B \in \mathrm{PGL}_2(k)$ sending ζ_B to the Gauss point ζ_G. Along the way, we will prove Theorem 4.7.

Let Γ be any finite set of Type II points. From [9, Theorem D], we know that there is a finite set of Type II points $\Gamma' \supset \Gamma$ (which can be chosen to be k-split if Γ is k-split) so that the pair (f, Γ') is analytically stable. More precisely, the main theorem of [9, §3.2] states that, for every $\zeta \in \Gamma'$, one of the following three cases must hold:

(1) the orbit of ζ lies in Γ', and $f^k(\zeta) = f^\ell(\zeta)$ for some $\ell > k \geq 0$;
(2) some iterate of ζ lies in a wandering F-disk for Γ', with a periodic boundary point $\zeta' \in \Gamma'$; or
(3) some iterate of ζ lies in an F-component for Γ' that contains an attracting periodic point.

Now fix a point $a \in \Omega_\gamma(f)$. Choose a Berkovich disk D_a containing a and contained in $\Omega^{an}_\gamma(f)$, with k-split Type II boundary point ζ_a. Choose D_a small enough so that the elements

of Γ' are disjoint from the forward iterates $f^j(D_a)$ for all $j \geq 0$; this is possible because $D_a \subset \Omega_\gamma^{an}(f)$. Following the proof in [9, §3.2], we use the classification of Fatou components (see [9, Theorem A.1]) to analyze the orbit of the disk D_a, to choose a distinguished k-split Type II point to be ζ_B, and to increase the set Γ' further so it contains ζ_a and ζ_B and remains analytically stable.

First assume that a (and so also D_a) lies in a wandering Fatou component U. From [2, Theorem 5.1], there exists $m \geq 0$ so that $V = f^m(U)$ is a wandering Berkovich disk with periodic and k-split boundary point; see also [9, Proposition 3.8]. Enlarging m if necessary, we can assume that V does not contain ζ_a and that the union $\bigcup_{j\geq 0} f^j(V)$ is disjoint from Γ'. Let ζ_B be the boundary point of V and $n \geq 1$ the period of ζ_B. We then increase Γ' to include the iterates $\zeta_a, f(\zeta_a), \ldots, f^{m-1}(\zeta_a)$ and the periodic orbit of ζ_B. Then the point $f^j(a)$ lies in an F-disk for Γ' for all $j \geq 0$, the point $f^m(a)$ lies in a wandering F-disk for Γ' with boundary point ζ_B, and (f, Γ') is analytically stable.

Now suppose that a lies in the basin of attraction of an attracting periodic point p of period n. Then there exists an integer $s \geq 0$, so that $f^s(D_a)$ is contained in the periodic Fatou component containing p. Note that p must be in the completion $\mathbb{P}^1(k_\gamma)$, because the iterates $f^{jn+s}(a) \in \mathbb{P}^1(k_\gamma)$ converge to p as $j \to \infty$. Thus, there exists a small disk D_p around p with k-split boundary point ζ_p that does not contain $f^s(\zeta_a)$ nor any element of Γ' and so that $f^n(D_p) \subsetneq D_p$. Now choose $m > s$ so that $f^m(a) \in D_p$ while $f^{m-1}(\zeta_a) \notin D_p$. We include the orbit $\zeta_a, f(\zeta_a), \ldots, f^{m-1}(\zeta_a)$ in Γ', and we also include $\zeta_p, f(\zeta_p), \ldots, f^{n-1}(\zeta_p)$. Then we set $\zeta_B = \zeta_p$. Again it follows that the point $f^j(a)$ lies in an F-disk for Γ' for all $j \geq 0$, the point $f^m(a)$ in an F-disk for Γ' containing the attracting periodic point, and (f, Γ') is analytically stable.

The final case is where an iterate $f^m(D_a)$ lies in a periodic Rivera domain V. If ζ_a is preperiodic, we include its forward orbit in Γ'. Let P be the subset of the closure $\overline{V}$ which is periodic; as explained in [9, Lemma 3.10], the set P is a closed and connected subset of $\overline{V}$ that includes the finite set of boundary points ∂V. If there exists m so that $f^m(a)$ lies in P, then we let $\zeta_B = f^m(\zeta_a)$ and let n be its period. Note that $f^m(a)$ lies in a periodic F-disk for the new Γ'. If ζ_a has infinite forward orbit, let $\zeta_B \in P$ be the Type II point which is the retraction of the iterate $f^m(a)$ to P; by increasing m if necessary, we can arrange so that no elements of Γ' lie in this component of $V \setminus P$ containing $f^m(a)$ nor in any component containing the forward orbit of $f^m(a)$. We then include $\zeta_a, f(\zeta_a), \ldots, f^{m-1}(\zeta_a)$ and the orbit

of ζ_B in Γ'. Note that $f^m(a)$ now lies in wandering F-disk for Γ'. The point ζ_B is k-split because the orbit of a intersects infinitely many directions from ζ_B.

In all cases, the pair (f, Γ') is analytically stable for the newly augmented Γ', and each element of the orbit $f^j(a)$, for $j \geq 0$, must lie in an F-disk. This proves Theorem 4.7. Moreover, in all three cases, if we set $\Gamma_B = \{\zeta_B\}$, then the pair (f^n, Γ_B) is also analytically stable, and the point $f^m(a)$ and its future iterates $f^{m+jn}(a)$, $j \geq 1$, are in F-components for Γ_B. In particular, they do not lie in hole-directions from ζ_B. We make any choice of $B \in \mathrm{PGL}_2(k)$ that takes ζ_B to the Gauss point ζ_G. In view of Lemma 4.4 this completes the proof of Theorem 4.6. □

4.4. **Proof of Theorem 1.5.** This is a consequence of Theorem 4.7, similar to the proof of [9, Theorem E] as an application of [9, Theorem D]. Fix $\gamma \in X(K)$, and assume that $a \in \mathbb{P}^1(k)$ is in the non-archimedean Fatou set of f at γ. Choose a Zariski open set $U_\gamma \subset X$ so that all indeterminacy points of

$$\tilde{f} : U_\gamma \times \mathbb{P}^1 \dashrightarrow U_\gamma \times \mathbb{P}^1$$

lie over γ. Apply Theorem 4.7 to $\Gamma = \{\zeta_G\}$, the Gauss point, in $\mathbb{P}^{1,an}_\gamma$. The analytically stable pair (f, Γ') guaranteed by Theorem 4.7 gives rise to a birational morphism $Y_\gamma \to U_\gamma \times \mathbb{P}^1$ defined over K, which is an isomorphism outside of $\{\gamma\} \times \mathbb{P}^1$, and an algebraically stable map $\tilde{f}_\gamma : Y_\gamma \dashrightarrow Y_\gamma$ lifting $\tilde{f}$. (See [9, §4] for details on the relationship between vertex sets Γ and modifications of the surface $X \times \mathbb{P}^1$.) Recall that C_a denotes the curve in $X \times \mathbb{P}^1$ defined the graph of $t \mapsto a_t$, and that $C^{Y_\gamma}_{f^n(a)}$ denotes the proper transform of the curve $C_{f^n(a)}$ in Y_γ, for each $n \geq 0$.

Let $\pi : Y_\gamma \to U_\gamma$ denote the projection. The indeterminacy points for the iterates $(\tilde{f}_\gamma)^n$ in $\pi^{-1}(\gamma)$ are identified with J-components of Γ', and the F-disks for Γ' are identified with smooth points in the fiber $\pi^{-1}(\gamma)$ that are not indeterminate for any iterate of $\tilde{f}_\gamma$. Therefore, the conclusion of Theorem 4.7 about the Fatou point a guarantees that the curves $C^{Y_\gamma}_{f^n(a)}$ are disjoint from $I(\tilde{f}_\gamma)$ and intersect the fiber over γ in smooth points, for all $n \geq 0$. Assuming that the point a is totally Fatou, we can repeat this argument over each $\gamma \in X(K)$ where $\tilde{f} : X \times \mathbb{P}^1 \dashrightarrow X \times \mathbb{P}^1$ has indeterminacy; we glue the surfaces Y_γ and maps f_γ to obtain our desired rational map

$$\tilde{f}_Y : Y \dashrightarrow Y.$$

For the converse implication, let $Y \to X \times \mathbb{P}^1$ be any choice of birational morphism defined over K, and let $\pi : Y \to X$ be the projection to the first factor. Assume that $a \in \mathbb{P}^1(k)$

lies in the non-archimedean Julia set at $\gamma \in X(K)$. Then we know that the curve C_a^Y will intersect an indeterminacy point of some iterate $(\tilde{f}_Y)^j$ in the fiber of Y over γ, by Lemma 4.5. Indeed, any small Berkovich disk around a will map, under large iterates of f, over each of the Type II points corresponding to the components of Y over γ. There are now two cases to consider. If C_a^Y or some iterate $C_{f^n(a)}^Y$ intersects a component of the fiber over γ which is mapped by $\tilde{f}_Y$ into an indeterminacy point, then we are done. If not, then since the point $p = C_a^Y \cap \pi^{-1}(\gamma)$ is indeterminate for $(\tilde{f}_Y)^j$, it must be that the point p is sent by $(\tilde{f}_Y)^m$, for some $m < j$, to an element of $I(\tilde{f}_Y)$. Consequently, $C_{f^m(a)}^Y$ intersects the indeterminacy set of $\tilde{f}_Y$, and the proof is complete. □

4.5. **Proof of Theorem 1.4.** Assume that f is defined over $k = K(X)$, for number field K. Enlarging K if necessary, we can assume that all places γ of bad reduction for f lie in $X(K)$. At each place $\gamma \in X(\overline{K})$ of k, we know that the non-archimedean Fatou set $\Omega_\gamma(f) \cap \mathbb{P}^1(k)$ is open in $\mathbb{P}^1(k)$, in the γ-adic topology. We also know that $\Omega_\gamma(f) \cap \mathbb{P}^1(k) = \mathbb{P}^1(k)$ for all but finitely many γ. We will show that $\Omega_\gamma(f) \cap \mathbb{P}^1(k)$ is dense in $\mathbb{P}^1(k)$ for the remaining γ.

Fix $\gamma \in X(K)$ and a point $b \in \mathbb{P}^1(k)$, and let ζ be any k-split Type II point in $\mathbb{P}^{1,an}_\gamma$ bounding a disk around b. Consider all the connected components of $\mathbb{P}^{1,an}_\gamma \setminus \{\zeta\}$. If one of these disks intersects $\mathbb{P}^1(k)$, we call it a **k-disk** at ζ. In the natural identification of the set of components of $\mathbb{P}^{1,an}_\gamma \setminus \{\zeta\}$ with $\mathbb{P}^1(\overline{K})$, the k-disks correspond to the rational points $\mathbb{P}^1(K)$. If a k-disk at ζ intersects the Berkovich Julia set, we call it a **Julia k-disk** at ζ.

If there are only finitely many Julia k-disks at ζ, then we can always find infinitely many k-disks at ζ that are fully contained in the Fatou set. This shows the existence of Fatou elements of $\mathbb{P}^1(k)$ in the closed Berkovich disk around b bounded by ζ.

If there are infinitely many Julia k-disks at ζ, then ζ is in the Julia set (because the Julia set is closed in $\mathbb{P}^{1,an}_\gamma$), and ζ is therefore preperiodic [9, Proposition 3.9]. We are still able to find infinitely many k-disks at ζ that are fully contained in the Fatou set. Suppose that $f^{m+n}(\zeta) = f^m(\zeta)$ for some $m \geq 0$ and $n \geq 1$. Let e be the local degree of f^n at $f^m(\zeta)$, as defined in [3, §7.4], so that $e \geq 1$. For $e > 1$, the iterate f^n induces a map $g : \mathbb{P}^1 \to \mathbb{P}^1$ of degree e, defined over K, by the natural identification of $\mathbb{P}^1(\overline{K})$ with the set of directions from $f^m(\zeta)$. The Julia set of f (which coincides with the Julia set for f^n) in $\mathbb{P}^{1,an}_\gamma$ is contained in the union of the hole-directions from $f^m(\zeta)$ for f^n and its iterates f^{jn}, $j \geq 1$, by Lemma 4.5. In other words, the Julia directions are identified with a subset of the union $\bigcup_{j\geq 0} g^{-j}(E)$ for a finite set $E \subset \mathbb{P}^1(\overline{K})$, corresponding to the hole-directions for f^n. But this implies that there are only finitely many Julia k-disks from $f^m(\zeta)$, because they correspond to a set in

$\mathbb{P}^1(K)$ with bounded Weil height, since $\deg g > 1$. It follows that there were only finitely many Julia k-disks from ζ, a contradiction. So we conclude that $e = 1$.

The action of f^n at $f^m(\zeta)$ therefore induces an automorphism $A \in \mathrm{PGL}_2(K)$ acting on $\mathbb{P}^1(\overline{K})$, the set of directions from $f^m(\zeta)$. The Julia k-directions from $f^m(\zeta)$ are contained in the union of the finitely many hole-directions of f^n at $f^m(\zeta)$ and the hole-directions for all iterates f^{nj}, $j \geq 1$. As before, these directions are identified with the union of a finite set E in $\mathbb{P}^1(\overline{K})$ and the orbit of E under A^{-1}. Now let $h : \mathbb{P}^1 \to \mathbb{P}^1$ be the map induced by f^m from ζ to $f^m(\zeta)$, defined over K, under any choice of identification of the k-directions from ζ and $f^m(\zeta)$ with $\mathbb{P}^1(K)$. Then we can find infinitely many k-disks at ζ that are fully contained in the Fatou set, as a consequence of the following:

Lemma 4.9. *For any number field K, any finite set E in $\mathbb{P}^1(K)$, any $A : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 1 defined over K, and any nonconstant $h : \mathbb{P}^1 \to \mathbb{P}^1$ defined over K, there exists an infinite set $Y \subset \mathbb{P}^1(K)$ for which*

$$\left(\bigcup_{j\geq 0} A^j(h(Y))\right) \cap E = \emptyset.$$

Proof. Choosing coordinates on $\mathbb{P}^1$ over K, we can assume that $A^{-1}(x) = \alpha x$ for some $\alpha \in K^*$ or that $A^{-1}(x) = x + 1$. If A has finite order, then there is nothing to show, as then $\bigcup_{j\geq 0} A^{-j}(E)$ is finite while $h(\mathbb{P}^1(K))$ is infinite. So if $A^{-1}(x) = \alpha x$, we can assume there exists a place v of K for which $|\alpha|_v > 1$. Then the set $\bigcup_{j\geq 0} A^{-j}(E) = \bigcup_{j\geq 0} (\alpha^j E)$ has no v-adic accumulation points except ∞. Choose any $y_0 \in \mathbb{P}^1(K)$ so that $h(y_0) \neq \infty$, and let $\{y_n\}$ be any infinite sequence in $\mathbb{P}^1(K)$ for which $y_n \to y_0$ v-adically. Then $h(y_n) \to h(y_0)$ v-adically. Therefore, letting Y be this sequence $\{y_n\}$, after excluding at most finitely many elements from the sequence, we may conclude that $\left(\bigcup_{j\geq 0} A^{-j}(E)\right) \cap h(Y) = \emptyset$. For $A^{-1}(x) = x + 1$, we work with any archimedean place of K. Let $y_0 \in \mathbb{P}^1(K)$ be a point for which $h(y_0) \neq \infty$, and select any sequence $y_n \in K$ for which $y_n \to y$ at this place. Then, as before, letting Y be the complement of finitely many points in $\{y_n\}$, we conclude that $\left(\bigcup_{j\geq 0} A^{-j}(E)\right) \cap h(Y) = \emptyset$. □

Repeating the above argument for all k-split points ζ, we see that $U_\gamma := \Omega_\gamma(f) \cap \mathbb{P}^1(k_\gamma)$ is open and dense in $\mathbb{P}^1(k_\gamma)$ in the γ-adic topology.

Let $\gamma_1, \ldots, \gamma_s \in X(K)$ denote the places for which $U_\gamma \neq \mathbb{P}^1(k_\gamma)$. Via the canonical embedding of k into $\prod_{\gamma\in X(\overline{K})} k_\gamma$, we can approximate any tuple $(x_1, \ldots, x_s) \in \prod_i U_{\gamma_i}$ by

elements in k. This shows that totally Fatou points are open and dense in $\mathbb{P}^1(k)$ in the topology induced from the product topology. □

4.6. **Intersection theory for a Fatou point.** The existence of the resolution $Y \to X \times \mathbb{P}^1$ constructed in Theorems 1.5 and 4.7 shows more. We now prove that the local geometric canonical height $\hat{\lambda}_{f,\gamma}(a)$, at each place γ of the function field $k = K(X)$, can be computed as an intersection number in Y when $a \in \mathbb{P}^1(k)$ is totally Fatou. In this way, for each place γ of k, we can view our surface Y as providing a relative type of Néron model, associated to the pair (f, a).

Fix a choice of local canonical height functions $\{\hat{\lambda}_{f,\gamma} : \gamma \in X(\overline{K})\}$ on $\mathbb{P}^1(k)$ as in [31, §3.5], so that $\hat{h}_f(a) = \sum_{\gamma \in X(\overline{K})} \hat{\lambda}_{f,\gamma}(a)$ for every $a \in \mathbb{P}^1(k)$. The local canonical height can be computed as

$$\hat{\lambda}_{f,\gamma}(a) = -\min\{0, \operatorname{ord}_\gamma(a)\}$$

at all but finitely many places γ; we enlarge the number field K so that this finite set of places is contained in $X(K)$.

Recall that, as in the statement of Theorem 1.5, the curve C_a is the section of $X \times \mathbb{P}^1 \to X$ defined by $t \mapsto a_t$, for any $a \in \mathbb{P}^1(k)$. The curve C_a^Y is its proper transform in Y.

Proposition 4.10. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be of degree $d > 1$, defined over a function field $k = K(X)$. Fix $\gamma \in X(K)$. Let $\pi : Y \to X \times \mathbb{P}^1$ be a birational morphism defined over the number field K, which is an isomorphism outside of the line $L_\gamma = \{\gamma\} \times \mathbb{P}^1$. Let $\{Y_{\gamma,i}\}_{i=1}^{m_\gamma}$ denote the irreducible components of $E_\gamma = \pi^{-1}(L_\gamma)$. Let $\tilde{f}_Y$ be the induced map on Y satisfying*

$$\begin{array}{ccc} Y & \overset{\tilde{f}_Y}{\dashrightarrow} & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X \times \mathbb{P}^1 & \overset{\tilde{f}}{\dashrightarrow} & X \times \mathbb{P}^1 \end{array}$$

and assume that $\tilde{f}_Y$ maps no component $Y_{\gamma,i}$ into an indeterminacy point of $\tilde{f}_Y$ in E_γ. Then, there exist rational numbers $c_{\gamma,i} \in \mathbb{Q}$, for $i = 1, \ldots, m_\gamma$, so the following holds. For each point $a \in \mathbb{P}^1(k)$ such that the curve $C^Y_{f^n(a)}$ is disjoint from the indeterminacy locus $I(\tilde{f}_Y) \cap E_\gamma$ and the singular locus of E_γ for every $n \geq 0$, the local geometric canonical height of a at γ is computed by

$$\hat{\lambda}_{f,\gamma}(a) = (C_a \cdot C_\infty)_\gamma + \sum_{i=1}^{m_\gamma} c_{\gamma,i}\, C_a^Y \cdot Y_{\gamma,i}$$

where $(C_a \cdot C_\infty)_\gamma$ is the intersection multiplicity of the curves C_a and C_∞ in $X \times \mathbb{P}^1$ at (γ, ∞).

Combined with Theorems 1.5 and 4.7, we obtain:

Theorem 4.11. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be of degree $d > 1$, defined over a function field $k = K(X)$, and let $a \in \mathbb{P}^1(k)$ be a totally Fatou point. Extending the number field K if necessary, let Y be the surface of Theorem 1.5, and let $\{c_{\gamma,i}\}$ be the rational numbers guaranteed by Proposition 4.10 over each $\gamma \in X(K)$. Then the geometric canonical height of a satisfies*

$$\hat{h}_f(a) = \sum_{\gamma \in X(\overline{K})} \hat{\lambda}_{f,\gamma}(a) = C_a \cdot C_\infty + \sum_{\gamma \in X(K)} \sum_{i=1}^{m_\gamma} c_{\gamma,i}\, C_a^Y \cdot Y_{\gamma,i}.$$

Proof. The theorem is almost immediate from Proposition 4.10 and the statement of Theorem 1.5, summing over all $\gamma \in X(\overline{K})$. We only need the additional input of Theorem 4.7 that the orbit of a will always lie in an F-disk for the vertex set Γ'. This guarantees that the curves $C^Y_{f^n(a)}$ intersect the singular fibers only in their smooth points. □

Proof of Proposition 4.10. As for Theorem 4.7, we continue to follow the arguments of [9], and we also build on the machinery developed in [10].

We identify the components $Y_{\gamma,i}$ with a finite set of Type II points in the Berkovich space $\mathbb{P}^{1,an}_\gamma$ over the field $\mathbb{L}_\gamma$. (We caution that some components Y_i may be non-reduced, so we need to keep track of their multiplicities as well.) See the discussion in, e.g., [9, §4]. Let $\Gamma \subset \mathbb{P}^{1,an}_\gamma$ be the union of this finite set of Type II points; note that Γ must include the Gauss point of $\mathbb{P}^{1,an}_\gamma$ because $\pi : Y \to X \times \mathbb{P}^1$ is regular.

Suppose that $a \in \mathbb{P}^1(k)$ is a point for which the curves $C^Y_{f^n(a)}$ are disjoint from the points of indeterminacy for $\tilde{f}_Y$ for all $n \geq 0$. This means, as in §4.3, that $f^n(a)$ lies in an F-component for Γ for every $n \geq 0$. Fixing a homogeneous lift F of f so that $\operatorname{ord}_\gamma F = 0$, we define the order function $\sigma(F, \cdot)$ on $\mathbb{P}^{1,an}_\gamma$ as in [10, §3.1]. Specifically, for each $n \geq 0$, we let A_n denote a homogeneous lift of $f^n(a) \in \mathbb{P}^1(k)$ so that $\operatorname{ord}_\gamma A_n = 0$, and then

$$\sigma_n := \sigma(F, f^n(a)) = \operatorname{ord}_\gamma F(A_n).$$

From [10, Lemma 3.1], the local canonical height at γ (associated to this choice of F) can be computed as

$$\hat{\lambda}_{f,\gamma}(a) = -\min\{0, \operatorname{ord}_\gamma(a)\} - \sum_{n=0}^{\infty} \frac{\sigma_n}{d^{n+1}}.$$

The key observation is contained in [10, Proposition 4.1, Theorem 4.2]: for a point a that lies in an F-disk component of $\mathbb{P}^{1,an}_\gamma \setminus \Gamma$, the order function depends only on the boundary point

of that F-disk. When each iterate of a lies in an F-disk, the sequence σ_n depends only on the sequence of boundary points of these F-disks containing $f^n(a)$, over all $n \geq 0$. However, by the stability of the pair (f, Γ), these sequences, in turn, depend only on the boundary point of the disk containing a itself. Indeed, every F-disk with boundary point ζ will map into an F-disk with the same boundary point. Moreover, the order function can only take finitely many possible values on the F-disks of Γ (by [10, Theorem 4.2]) and the stability of (f, Γ) implies that the sequence $\{\sigma_n\}$ will be eventually periodic. In other words, the sequence $\{\sigma_n\}$ depends only on the component $Y_{\gamma,i}$ that intersects C_a^Y in E_γ. The coefficient $c_{\gamma,i}$ is rational because the sequence $\{\sigma_n\}$ is eventually periodic. □

5. Near a singularity: uniform convergence to the escape rate

In this section, we fix $\gamma \in X(K)$. We assume that we are given $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$, defined over $k = K(X)$, and a point $a \in \mathbb{P}^1(k)$. We choose lifts F and A, as defined in §3.1,with $\operatorname{ord}_\gamma F = \operatorname{ord}_\gamma A = 0$ and we assume that

$$\operatorname{Res}(F_\gamma) = 0.$$

We also assume that the pair (f, a) is hole-avoiding at γ, as defined in §4.2, so that

$$F_\gamma^n(A_\gamma) \neq (0, 0)$$

for all $n \geq 0$. We set

$$A_n := F^n(A) \in k^2$$

and we study the convergence of the sequence of functions

$$g_{n,v}(t) := \frac{1}{d^n} \log \|(A_n)_t\|_v \tag{5.1}$$

in a neighborhood of $t = \gamma$ in $X(\mathbb{C}_v)$, for each $v \in M_K$. We prove:

Theorem 5.1. *Fix $\gamma \in X(K)$ and a hole-avoiding pair (f, a) at γ with lifts F and A satisfying $\operatorname{ord}_\gamma F = \operatorname{ord}_\gamma A = 0$ and $\operatorname{Res}(F_\gamma) = 0$. There exists an M_K-neighborhood $\mathfrak{U}$ of γ in X so that, for each $v \in M_K$, the functions $g_{n,v}$ converge uniformly on $\mathfrak{U}_v$ to a continuous function g_v.*

Note that the limit function g_v coincides with the escape-rate function $G_{F_t,v}(A_t)$ defined by (3.9) in §3.4, for $t \neq \gamma$. So we know that the convergence of $g_{n,v}$ to g_v is uniform on

neighborhoods where t remains *bounded away* from γ and the other singularities of f. The steps in the proof of Theorem 5.1 are inspired by the arguments in [12], [13], and [25].

5.1. Convergence of the constant terms $g_{n,v}(\gamma)$.

Proposition 5.2. *Fix $\gamma \in X(K)$. Under the hypotheses of Theorem 5.1, the limit*

$$\alpha_v := \lim_{n\to\infty} \frac{1}{d^n} \log \|(A_n)_\gamma\|_v$$

exists in $\mathbb{R}$, for all $v \in M_K$. Moreover, we have

$$\sum_{v\in M_K} N_v|\alpha_v| < \infty.$$

In other words, $\{\alpha_v : v \in M_K\}$ defines an M_K-quasiconstant.

Remark 5.3. For each fixed n, we have $\|(A_n)_\gamma\|_v = 1$ for all but finitely many v. But as n grows, the number of places for which $\|(A_n)_\gamma\|_v \neq 1$ can also grow, so that α_v can be nonzero for infinitely many $v \in M_K$. A simple example is given by the function $f(z) = z(z+1)/(z+t)$ defined over $k = \mathbb{Q}(t)$, which is similar to Example 4.1, at $t = 0$. Take $a = 1$. Fix homogeneous polynomial lift $F(z, w) = (z(z+w), (z+tw)w)$, so that $F_0(z, w) = (z(z+w), zw)$, and set $A = A_0 = (1, 1)$. Then for every prime p, we have $\|(A_n)_0\|_p = 1$ for all $n < p$, and $\|(A_n)_0\|_p < 1$ for all $n \geq p$. We show below in (5.18) that this will imply that the limit α_p of Proposition 5.2 will be negative for all primes p. Many more examples are given in [25]. Bear in mind that this does not happen for Lattès examples (the maps arising as quotients of endomorphisms of elliptic curves) or for polynomials; in other words, for those types of maps, the α_v of Proposition 5.2 always define an M_K-constant.

Proof of Proposition 5.2. Since $\mathrm{Res}(F_\gamma) = 0$, specializing F at γ, we can write

$$F_\gamma = H\,\hat{F}$$

where $H(z, w) \in K[z, w]$ is a nonconstant homogeneous polynomial of degree $k \leq d$, and $\hat{F}(z, w) \in (K[z, w])^2$ is a homogeneous polynomial map of degree $\ell = d - k < d$ inducing a morphism of degree ℓ on $\mathbb{P}^1$. The zeroes of H in $\mathbb{P}^1$ are called the holes of f at γ, as defined in §4.2.

Because the pair (f, a) is hole-avoiding, the lift A satisfies $F_\gamma^n(A_\gamma) \neq (0, 0)$ for all n. So it must be that either $\ell > 0$ or, if $\ell = 0$, the value of $\hat{F}$ is not a root of H. Consequently, as in [6, Lemma 2.2], the specialization of each iterate F^n can be expressed in terms of H and $\hat{F}$

by

$$F_\gamma^n(z,w) = \left(\prod_{i=0}^{n-1} H(\hat{F}^i(z,w))^{d^{n-1-i}}\right)\hat{F}^n(z,w) \tag{5.2}$$

for all $n \geq 1$. In particular, this shows that

$$(A_n)_\gamma = \left(\prod_{i=0}^{n-1} H(\hat{F}^i(A_\gamma))^{d^{n-1-i}}\right)\hat{F}^n(A_\gamma) \tag{5.3}$$

for every n.

For $\ell = 0$, the map $\hat{F}$ is constant, so $\hat{F}^n(A_\gamma) = (z_0, w_0) \in K^2 \setminus \{(0,0)\}$ for some point (z_0, w_0) and for all $n \geq 1$. The formula (5.3) gives

$$\begin{aligned}\frac{1}{d^n}\log\|(A_n)_\gamma\|_v &= \frac{1}{d}\log|H(A_\gamma)|_v + \sum_{i=1}^{n-1}\frac{1}{d^{i+1}}\log|H(z_0,w_0)|_v + \frac{1}{d^n}\log\|(z_0,w_0)\|_v \\ &\longrightarrow \frac{1}{d}\log|H(A_\gamma)|_v + \frac{1}{d(d-1)}\log|H(z_0,w_0)|_v\end{aligned}$$

as $n \to \infty$, for all places v of K. The statements of the proposition follow immediately in this case.

Now assume that $\ell \geq 1$. There exists an M_K-constant $\mathfrak{L}$ so that

$$e^{-\mathfrak{L}_v}\|(z,w)\|_v^\ell \leq \|\hat{F}(z,w)\| \leq e^{\mathfrak{L}_v}\|(z,w)\|_v^\ell$$

for all $(z,w) \in \overline{K}^2$ and for all $v \in M_K$ [31, Proposition 5.57]. This implies that

$$e^{-\mathfrak{L}_v(1+\ell+\cdots+\ell^{n-1})}\|A_\gamma\|_v^{\ell^n} \leq \|\hat{F}^n(A_\gamma)\|_v \leq e^{\mathfrak{L}_v(1+\ell+\cdots+\ell^{n-1})}\|A_\gamma\|_v^{\ell^n} \tag{5.4}$$

so that

$$\lim_{n\to\infty}\frac{1}{d^n}\log\|\hat{F}^n(A_\gamma)\|_v = 0 \tag{5.5}$$

for all $v \in M_K$, because $\ell < d$.

Recalling that $\deg H = k = d - \ell$, there is also an M_K-constant $\mathfrak{H}$ so that

$$|H(z,w)|_v \leq e^{\mathfrak{H}_v}\|(z,w)\|_v^k$$

for all $(z,w) \in K^2$. So

$$|H(\hat{F}^i(A_\gamma))|_v \leq e^{\mathfrak{H}_v}\|\hat{F}^i(A_\gamma)\|_v^k \leq e^{\mathfrak{H}_v}e^{k\mathfrak{L}_v(1+\ell+\cdots+\ell^{i-1})}\|A_\gamma\|_v^{k\ell^i} \tag{5.6}$$

at all places v and for all $i \geq 1$. Note that the bound on the right side of (5.6) can be > 1 at only finitely many places v of K, independent of i. Let S_+ denote this finite set of places. Therefore, since $H(\hat{F}^i(A_\gamma)) \in K^*$ for all i, we can apply the product formula to observe that there is a constant $c > 0$ so that

$$\prod_{v \notin S_+} |H(\hat{F}^i(A_\gamma))|_v^{N_v} \geq c^{\max\{i,\ell^i\}} \tag{5.7}$$

for all $i \geq 1$. Using the formula (5.3), we combine (5.5) with (5.6) and (5.7) to deduce the existence of

$$\alpha_v = \lim_{n\to\infty} \frac{1}{d^n} \log \|(A_n)_\gamma\|_v = \sum_{i=0}^{\infty} \frac{1}{d^{i-1}} \log |H(\hat{F}^i(A_\gamma))|_v \tag{5.8}$$

at every place v, because $\ell < d$.

From (5.6) and the summation expression for α_v in (5.8), we see that $\alpha_v \leq 0$ for all $v \notin S_+$. To show that the sum over all places of the α_v is finite, we use (5.7) to estimate

$$\frac{1}{d^{i-1}} \sum_{v \notin S_+} N_v \log |H(\hat{F}^i(A_\gamma))|_v \geq \frac{\max\{i, \ell^i\}}{d^{i-1}} \log c$$

for each $i \geq 1$. Summing over all i, we can then use Fubini's theorem to deduce that

$$\sum_{v \notin S_+} N_v \, \alpha_v > -\infty,$$

so that

$$\sum_{v \in M_K} N_v |\alpha_v| < \infty.$$

This completes the proof of the proposition. □

5.2. **Proof of Theorem 5.1.** Throughout this proof, we work in an M_K-neighborhood $\mathfrak{U}$ of $\gamma \in X(K)$, so that the conclusion of Proposition 3.2 holds. For simplicity, we let $u \in K(X)$ denote a choice of local coordinate on X near γ so that $u = 0$ represents γ.

We now fix $v \in M_K$, and we drop the dependence on v to ease notation. Let δ denote the v-adic radius of the largest disk $\{|u|_v < \delta\}$ contained in the M_K-neighborhood $\mathfrak{U}_v$. Let $C = e^{\mathfrak{b}_v} \geq 1$ be the constant appearing in Proposition 3.2 at this place. For each n, we write

$A_n(u)$ for the specialization of $A_n = F^n(A)$ at u. For every $n \geq m$, we define

$$\begin{aligned} g_n(u) &= \frac{1}{d^n} \log \|A_n(u)\| \\ &= \frac{1}{d^m} \log \|A_m(u)\| + \frac{1}{d^m} \sum_{j=1}^{n-m} \frac{1}{d^j} \log \frac{\|A_{m+j}(u)\|}{\|A_{m+j-1}(u)\|^d} \end{aligned}$$

for $|u| < \delta$. Let $q = \operatorname{ord}_\gamma \operatorname{Res}(F)$. From Proposition 3.2, we have

$$g_m(u) + \frac{1}{d^m} (q \log |u| - \log C) \;\leq\; g_n(u) \;\leq\; g_m(u) + \frac{1}{d^m} \log C \tag{5.9}$$

for all $n \geq m \geq 0$ and for all $|u| < \delta$. Let

$$\alpha = \lim_{n \to \infty} \frac{1}{d^n} \log \|A_n(0)\|;$$

its existence is guaranteed by Proposition 5.2.

Step 1: a choice of N and δ_N for a uniform upper bound. Fix $\varepsilon > 0$. Choose N so that we have

$$e^{d^n(\alpha-\varepsilon)} \leq \|A_n(0)\| \leq e^{d^n(\alpha+\varepsilon)} \tag{5.10}$$

for all $n \geq N$, and so that

$$\frac{1}{d^N} \log C < \varepsilon \quad \text{and} \quad \left| \frac{1}{d^N} \log(1-\varepsilon) \right| < \varepsilon. \tag{5.11}$$

Now choose $\delta_N > 0$ so that, by continuity of $A_N(u)$, we have

$$(1-\varepsilon) e^{d^N(\alpha-\varepsilon)} \leq \|A_N(u)\| \leq e^{d^N(\alpha+2\varepsilon)} \tag{5.12}$$

for all $|u| \leq \delta_N$. Applying the upper bound of (5.9) and using (5.11), this implies that

$$g_n(u) \;\leq\; g_N(u) + \frac{1}{d^N} \log C \;\leq\; \alpha + 3\varepsilon \tag{5.13}$$

for all $n \geq N$ and for all $|u| \leq \delta_N$.

Note that the lower bound of (5.9) is not enough to get uniform control on g_n from below for $n \geq N$, because of the $\log |u|$ term.

Step 2: the Maximum Principle and lower bounds within δ_n. By the triangle inequality, we have

$$\|A_N(u) - A_N(0)\| \leq 2e^{d^N(\alpha+2\varepsilon)}$$

for all $|u| \le \delta_N$, from (5.10) and (5.12). Note that the coordinates of $A_N(u) - A_N(0)$ vanish at $t = 0$, and so the Maximum Principle (applied to $\frac{1}{u}(A_N(u) - A_N(0))$) gives

$$\|A_N(u) - A_N(0)\| \;\le\; \frac{|u|}{\delta_N}\, 2e^{d^N(\alpha+2\varepsilon)}$$

for all $|u| \le \delta_N$. For a non-archimedean Maximum Principle see e.g. [1, Proposition 8.14]. Using the upper bound of (5.13), the same argument implies that

$$\|A_n(u) - A_n(0)\| \;\le\; \frac{|u|}{\delta_N}\, 2e^{d^n(\alpha+3\varepsilon)} \tag{5.14}$$

for all $n \ge N$ and for all $|u| \le \delta_N$. This implies that

$$\begin{aligned} \|A_n(u)\| &\ge \|A_n(0)\| - \|A_n(u) - A_n(0)\| \\ &\ge e^{d^n(\alpha-\varepsilon)} - \frac{|u|}{\delta_N}\, 2e^{d^n(\alpha+3\varepsilon)} \\ &= e^{d^n(\alpha-\varepsilon)} \left(1 - \frac{|u|}{\delta_N}\, 2e^{d^n(4\varepsilon)}\right) \end{aligned}$$

for all $n \ge N$ and for all $|u| \le \delta_N$.

Now define

$$\delta_n := \frac{\delta_N \varepsilon}{2e^{4d^n\varepsilon}} \tag{5.15}$$

for all $n > N$. So we have

$$\|A_n(u)\| \;\ge\; e^{d^n(\alpha-\varepsilon)}\,(1-\varepsilon) \tag{5.16}$$

for all $|u| \le \delta_n$ and for all $n > N$. Combined with the lower bound of (5.12) and the condition on N in (5.11), this shows that

$$g_n(u) \ge \alpha - 2\varepsilon \tag{5.17}$$

for all $|u| \le \delta_n$ and for all $n \ge N$.

Step 3: Choosing larger N_0 and completing the proof. From the definition of δ_n, we see that

$$\frac{1}{d^n} \log \delta_n = \frac{1}{d^n} \log(\delta_N \varepsilon/2) - 4\varepsilon$$

for all $n > N$. Now choose $n_0 > N$ so that

$$\left| \frac{1}{d^{n_0}} \log(\delta_N \varepsilon/2) \right| < \varepsilon.$$

Recall that the sequence $\{g_n\}$ converges uniformly on neighborhoods in u that are bounded away from $u = 0$ (and any other singularities for f in X), so, by our choice of M_K-neighborhood, there exists $N_0 \geq n_0$ so that

$$|g_n - g_m| < \varepsilon$$

for all $n, m \geq N_0$, uniformly on $\{\delta_{n_0} \leq |u| < \delta\}$.

For $|u| \leq \delta_{n_0}$, we know that

$$g_n(u) \leq \alpha + 3\varepsilon$$

for all $n \geq n_0$ by (5.13). And we know that

$$g_n(u) \geq \alpha - 2\varepsilon$$

for all $|u| \leq \delta_n$ and for all $n \geq n_0$, by (5.17). On the other hand, for $\delta_n < |u| \leq \delta_{n_0}$ we can choose $n > m \geq n_0$ so that $\delta_{m+1} \leq |u| \leq \delta_m$ and then (5.9) gives

$$g_n(u) \geq g_m(u) + \frac{q}{d^m} \log \delta_{m+1} - \frac{1}{d^m} \log C \geq \alpha - 2\varepsilon - 5qd\varepsilon - \varepsilon$$

So, in particular, we have

$$\alpha - (3 + 5qd)\varepsilon \leq g_n \leq \alpha + 3\varepsilon$$

for all $n \geq N_0$ and for all $|u| \leq \delta_{n_0}$. This completes the proof of uniform convergence. □

5.3. **A summable lower bound on a disk.** We conclude this section with a consequence of Proposition 5.2 and its proof that will be used to prove Theorem 1.2.

Proposition 5.4. *Fix $\gamma \in X(K)$. Under the hypotheses of Theorem 5.1 and in the notation of Proposition 5.2, there exists a finite set $S_\gamma \subset M_K$ so that*

$$(d(\operatorname{ord}_\gamma \operatorname{Res} F) + 1)\, \alpha_v \leq g_v(t) \leq 0$$

for every $v \notin S_\gamma$ and all t in an M_K-neighborhood of γ.

Proof. We first let S_γ be the finite set of places $v \in M_K$, including all archimedean places, at which the quantities $\mathfrak{L}_v$ and $\mathfrak{H}_v$ in the proof of Proposition 5.2 differ from 0 and where $\|A_\gamma\|_v \neq 1$. It follows from the computations in Proposition 5.2 (specifically, equation (5.6) and (5.8)) that $\alpha_v \leq 0$ for all $v \notin S_\gamma$.

Recall the formula for $(A_n)_\gamma$ given in (5.3). For all $v \notin S_\gamma$, we have

$$\|(A_{n+1})_\gamma\|_v = \|F_\gamma((A_n)_\gamma)\|_v \leq \|(A_n)_\gamma\|^d$$

so that

$$\frac{1}{d^{n+1}} \log \|(A_{n+1})_\gamma\|_v \le \frac{1}{d^n} \log \|(A_n)_\gamma\|_v \tag{5.18}$$

for all n. For all $v \in M_K \setminus S_\gamma$, we also have $\|\hat{F}^n(A_\gamma)\|_v = 1$ for all n. So $\|(A_n)_\gamma\|_v < 1$ for some $n > 0$ if and only if there exists $i < n$ so that $|H(\hat{F}^i(A_\gamma))|_v < 1$. Furthermore, from (5.18), such an n exists if and only if $\alpha_v < 0$, for each $v \notin S_\gamma$.

Now let u denote a local coordinate on X with $u = 0$ representing γ. From Proposition 2.1, the coefficients of F and the coordinates of A are M_K-bounded on an M_K-neighborhood of γ. We enlarge S_γ if needed to assume that these coefficients are ≤ 1 in absolute value and so that the neighborhood is given by $\{|u|_v < 1\}$ for all $v \notin S_\gamma$. We further enlarge S_γ to include all places at which the M_K-constant $\mathfrak{b}_v$ from Proposition 3.2 differs from 1, and also so that $|u|_v^{2g+1} = |\xi^\gamma|_v$ for $v \notin S_\gamma$ on the M_K-neighborhood of γ (applying Proposition 2.1 to u).

Then, for all $v \notin S_\gamma$, the upper bound on the coefficients of F and the coordinates of A gives

$$\|A_n(u)\|_v \le 1 \tag{5.19}$$

for all $n \ge 1$ and for all $|u|_v < 1$. This implies immediately that $g_v(u) \le 0$ for all $|u|_v < 1$ with $v \notin S_\gamma$, proving the desired upper bound of the proposition.

Moreover, for $v \notin S_\gamma$ where $\alpha_v = g_v(0) = 0$, we conclude from the Maximum Principle (applied to the subharmonic g_v) that $g_v(u) = 0$ for all $|u|_v < 1$, and the estimate of the proposition holds for these v. For the rest of the proof, we fix $v \notin S_\gamma$ with $\alpha_v < 0$, and choose minimal $m \ge 0$ so that $\|(A_{m+1})_\gamma\|_v < 1$. Since $\|(A_n)_\gamma\|_v^{1/d^n}$ is a non-increasing sequence from (5.18), we see that $\|(A_n)_\gamma\|_v$ decreases to 0 as $n \to \infty$, and

$$\frac{1}{d^n} \log \|(A_n)_\gamma\|_v \ge \alpha_v \tag{5.20}$$

for all n. As $\|(A_m)_\gamma\|_v = 1$, the inequality (5.19) implies (with the Maximum Principle) that $\|A_m(u)\|_v = 1$ for all $|u|_v < 1$.

Let $q = \operatorname{ord}_\gamma(\operatorname{Res} F)$. Proposition 3.2 then gives

$$\frac{1}{d^n} \log \|A_n(u)\|_v \ge \frac{1}{d^m} \log \|A_m(u)\|_v + \frac{q}{d^m} \log |u|_v = \frac{q}{d^m} \log |u|_v \tag{5.21}$$

for all $n \geq m$ and for all $|u|_v < 1$. Therefore, for all u satisfying $\|A_{m+1}(0)\|_v \leq |u|_v < 1$, we have

$$\frac{1}{d^n} \log \|A_n(u)\|_v \;\geq\; \frac{q}{d^m} \log \|A_{m+1}(0)\|_v \;\geq\; qd\,\alpha_v$$

for all $n \geq m$, from (5.21) and (5.20). This shows that $g_v(u) \geq qd\,\alpha_v$ where $|u|_v \geq \|A_{m+1}(0)\|_v$.

On the other hand, for $|u|_v < \|A_{m+1}(0)\|_v$, we can choose $j \geq m+1$ so that

$$\|A_{j+1}(0)\|_v \leq |u|_v < \|A_j(0)\|.$$

Then, writing $A_j(u) = A_j(0) + uR_j(u)$ for u near 0, we know that $\|R_j(u)\|_v \leq 1$ for all $|u|_v < 1$, by (5.19) and the Maximum Principle, and therefore

$$\|A_j(u)\|_v = \|A_j(0) + uR_j(u)\|_v = \|A_j(0)\|_v$$

where $|u|_v < \|A_j(0)\|_v$. Therefore,

$$\begin{aligned} \frac{1}{d^n} \log \|A_n(u)\|_v &\geq \frac{1}{d^j} \log \|A_j(u)\|_v + \frac{q}{d^j} \log |u|_v \\ &\geq \frac{1}{d^j} \log \|A_j(0)\|_v + \frac{q}{d^j} \log \|A_{j+1}(0)\|_v \\ &\geq (1+dq)\alpha_v \end{aligned}$$

for all $\|A_{j+1}(0)\|_v \leq |u|_v < \|A_j(0)\|$ and for all $n \geq j$. This implies that $g_v(u) \geq (1+dq)\alpha_v$ for u values in this region. Since $\|A_n(0)\|_v \to 0$ as $n \to \infty$, the proof of the lower bound on g_v is complete, thus completing the proof of the proposition. □

6. Proofs of Theorems 1.2 and 1.7

In this section, we complete the proofs of Theorems 1.2 and 1.7. We fix $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over the field $k = K(X)$, of degree $d > 1$, and we assume that $a \in \mathbb{P}^1(k)$ is totally Fatou for f. Fix homogeneous lifts F of f and A of a as in §3.1. Recall the definitions of the finite sets $\mathcal{S}(F) \subset \mathcal{S}(F, A)$ in $X(\overline{K})$ from §3.1, and that K was enlarged (if necessary) so that $\mathcal{S}(F, A) \subset X(K)$, as stated in Convention 3.1. Recall also the definitions of the escape rates $G_{F,\gamma}(A)$ and $G_{F_t,v}(A_t)$ given in (3.4) and (3.9), respectively. The divisor

$$D = \sum_{\gamma \in X(K)} G_{F,\gamma}(A)\,\gamma$$

on X was defined in (3.5); its support lies in $\mathcal{S}(F, A)$.

For the choice of Weil height h_D defined by (3.7), we now consider the difference

$$\hat{h}_{f_t}(a_t) - h_D(t) = \frac{1}{[K:\mathbb{Q}]} \frac{1}{|\operatorname{Gal}(\overline{K}/K)\cdot t|} \sum_{x \in \operatorname{Gal}(\overline{K}/K)\cdot t} \sum_{v \in M_K} N_v \left(G_{F_x,v}(A_x) - \frac{1}{2g+1} \sum_{\gamma \in X(K)} G_{F,\gamma}(A) \log^+ |\xi^\gamma(x)|_v \right)$$

for $t \in X(\overline{K}) \setminus \mathcal{S}(F,A)$. For each place v of K, we examine the function

$$V_v(t) = G_{F_t,v}(A_t) - \frac{1}{2g+1} \sum_{\gamma \in X(K)} G_{F,\gamma}(A) \log^+ |\xi^\gamma(t)|_v \tag{6.1}$$

on $X(\overline{K})$ and its extension to $X(\mathbb{C}_v)$ and the Berkovich analytification X_v^{an}. Recall that the steps needed to complete the proofs were outlined in §3.5.

6.1. **Changing coordinates and lifts.** If we change the lifts F and A, multiplying each by an element of k^*, it follows from (3.6) and (3.10) that

$$\begin{aligned} G_{cF_t,v}(\alpha A_t) - \frac{1}{2g+1} G_{cF,\gamma}(\alpha A) \log^+ |\xi^\gamma(t)|_v \;&=\; G_{F_t,v}(A_t) - \frac{1}{2g+1} G_{F,\gamma}(A) \log^+ |\xi^\gamma(t)|_v \\ &\quad + \frac{1}{d-1} \log |c_t \alpha_t^{d-1}|_v + \\ &\quad \frac{1}{(2g+1)(d-1)} \operatorname{ord}_\gamma(c\alpha^{d-1}) \log^+ |\xi^\gamma(t)|_v \end{aligned} \tag{6.2}$$

for any choice of $\gamma \in X(K)$. Moreover, the sum of the last two terms is M_K-bounded on an M_K-neighborhood of γ, as a consequence of Proposition 2.1.

If we conjugate F by an element $B \in \mathrm{GL}_2(k)$, we have

$$G_{BFB^{-1},\gamma}(B(A)) = G_{F,\gamma}(A) \;\text{ and }\; G_{(BFB^{-1})_t,v}(B(A)_t) = G_{F_t,v}(A_t) \tag{6.3}$$

from the definitions of the escape rates, for each $\gamma \in X(K)$, and each place v of K and all $t \in X(\overline{K}) \setminus \big(\mathcal{S}(F,A) \cup \mathcal{S}(BFB^{-1}, B(A)) \cup \mathcal{S}(B)\big)$. Replacing F or A by an iterate gives

$$G_{F^n,\gamma}(F^m(A)) = d^m\, G_{F,\gamma}(A) \;\text{ and }\; G_{F_t^n,v}(F^m(A)_t) = d^m\, G_{F_t,v}(A_t) \tag{6.4}$$

for all $n \geq 1$ and $m \geq 0$, again immediate from the definitions.

6.2. **The divisor $D = D(F,A)$ is a $\mathbb{Q}$-divisor.** We need to show that $G_{F,\gamma}(A) \in \mathbb{Q}$ for each $\gamma \in \mathcal{S}(F,A)$. This is immediate from the following proposition. (It also follows from the

statement of Proposition 4.10.) We present an alternative short argument in the following proposition. Recall that k_γ denotes the completion of k at γ.

Proposition 6.1. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be of degree $d \geq 2$, defined over $k = K(X)$, γ a point in $X(K)$, and $a \in \mathbb{P}^1(k_\gamma)$. If the point a is an element of the non-archimedean Fatou set $\Omega_\gamma(f)$ at γ, then the geometric escape rate $G_{F,\gamma}(A)$ is a rational number, for any choice of lifts F and A.*

Remark 6.2. In [10] it was shown, for maps f defined over k, that there can exist points $a \in \mathbb{P}^1(k_\gamma)$ with irrational local canonical height. Proposition 6.1 implies that these points must always lie in the non-archimedean Julia set of f at γ. We provide examples in Section 7. It is not known if the Julia points can be algebraic over k.

Proof. If the pair (f, a) is hole-avoiding at γ, as defined in §4.2, and if βF and αA are lifts of f and a, respectively, so that $\operatorname{ord}_\gamma \beta F = \operatorname{ord}_\gamma \alpha A = 0$ for $\alpha, \beta \in k^*$, then

$$G_{F,\gamma}(A) = G_{\beta F,\gamma}(\alpha A) + \frac{1}{d-1} \operatorname{ord}_\gamma \beta + \operatorname{ord}_\gamma \alpha$$

from (3.6), so that

$$G_{F,\gamma}(A) = 0 + \frac{1}{d-1} \operatorname{ord}_\gamma \beta + \operatorname{ord}_\gamma \alpha \ \in \ \mathbb{Q}$$

because $\operatorname{ord}_\gamma(\beta F)^n(\alpha A) = 0$ for all $n \geq 0$.

If the pair (f, a) is not hole-avoiding at γ, then Theorem 4.6 implies the existence of a change of coordinates $B \in \mathrm{GL}_2(k)$ and iterates so that the pair $(Bf^nB^{-1}, B(f^m(a)))$ is hole-avoiding at γ. The conclusion then follows from (6.3) and (6.4). □

6.3. **Variation of canonical height: proofs of the main theorems.** Assume that $a \in \mathbb{P}^1(k)$ is totally Fatou for f, and let $D = D(F, A)$. Proposition 6.1 implies that D is a $\mathbb{Q}$-divisor, so it remains to study properties of the functions V_v, defined in (6.1), associated to this divsor D on the curve X at each place v of the number field K.

We begin by proving Theorem 1.7, which states that the functions V_v are continuous on the Berkovich analytification X_v^{an} at all places v. This implies, in particular, the existence of a uniform bound C_v so that $|V_v| \leq C_v$ at all points of $X(\overline{K})$. (Recall that we have fixed an embedding of $\overline{K} \hookrightarrow \mathbb{C}_v$ for each place v.) Towards proving Theorem 1.2, we then find a finite set of places $S \subset M_K$ outside of which we have strong bounds on V_v, so that we can show the sum $\sum_{v \in M_K \setminus S} N_v V_v(t)$ is uniformly bounded on $X(\overline{K})$. Combined with the

bound C_v for each place v, we obtain a uniform bound on the sum $\sum_{v\in M_K} N_v\, V_v(t)$, for all $t \in X(\overline{K})$. Averaging over Galois orbits will complete the proof of Theorem 1.2.

Proof of Theorem 1.7. Fix $\gamma \in \mathcal{S}(F, A)$. First assume that the pair (f, a) is hole-avoiding at γ, as defined in §4.2. Choose functions $\alpha, \beta \in k$ so that $\operatorname{ord}_\gamma \beta F = \operatorname{ord}_\gamma \alpha A = 0$. This places us in the setting required for the results of Section 5. For the function g_v defined there, note that

$$g_v(t) - V_v(t) = \frac{1}{d-1} \log |\beta_t \alpha_t^{d-1}|_v + \frac{1}{(2g+1)(d-1)} \operatorname{ord}_\gamma(\beta\alpha^{d-1}) \log^+ |\xi^\gamma(t)|_v \tag{6.5}$$

on an M_K-neighborhood of γ, as a consequence of (6.2), and this difference is continuous at all places $v \in M_K$ and an M_K-bounded function.

If the pair (f, a) fails to be hole-avoiding at γ, then from Theorem 4.6, we can find a change of coordinates $B \in \mathrm{GL}_2(k)$ and pass to iterates so that the pair $(Bf^nB^{-1}, B(f^m(a)))$ is hole-avoiding at γ. From properties (6.3) and (6.4) of the escape rates, we can replace the lifts (F, A) with $(BF^nB^{-1}, BF^m(A))$, and these changes do not affect the computation of V_v on an M_K-neighborhood of γ (outside of γ itself, where the specialization B_γ may fail to be invertible), except to multiply it by d^m at every place v. So we can assume that (f, a) is hole-avoiding at γ.

We can apply Theorem 5.1 to conclude that V_v is a continuous function on an M_K-neighborhood of γ, for every $v \in M_K$, and that it extends to a continuous function on the closure of this neighborhood in the Berkovich analytification of X, for each v. This completes the proof of Theorem 1.7, because the continuity of V_v – when bounded away from the elements of $\mathcal{S}(F, A)$ in X_v^{an} – is immediate from the definitions of the escape rates $G_{F_t,v}(A_t)$ and the local height functions for h_D. □

Proof of Theorem 1.2. Fix $\gamma \in \mathcal{S}(F, A)$. As in the proof of Theorem 1.7, it suffices to assume that (f, a) is hole-avoiding at γ. Choose functions $\alpha, \beta \in k$ so that $\operatorname{ord}_\gamma \beta F = \operatorname{ord}_\gamma \alpha A = 0$. Let S_γ be a finite set of places of the number field K so that the function in (6.5) vanishes on an M_K-neighborhood of γ for all $v \in M_K \setminus S_\gamma$. The function V_v for the given pair (F, A) then coincides with the function V_v for the pair $(\beta F, \alpha A)$ for all $v \in M_K \setminus S_\gamma$ on an M_K-neighborhood of γ and is equal to g_v at these places.

We can enlarge the finite set S_γ so that Propositions 5.4 and 5.2 imply the existence of an M_K-quasiconstant $\mathfrak{a}(\gamma)$ for which

$$|V_v(t)| = |g_v(t)| \le \mathfrak{a}_v(\gamma) \tag{6.6}$$

for all $v \notin S_\gamma$ and t in an M_K-neighborhood of γ.

Let $\mathfrak{U}$ be the union of these M_K-neighborhoods over all $\gamma \in \mathcal{S}(F,A)$. From Proposition 3.2, we know that there exists an M_K-constant $\mathfrak{c}$ so that

$$e^{-\mathfrak{c}_v} \ \le \ \frac{\|F_t(z,w)\|_v}{\|(z,w)\|_v^d} \ \le \ e^{\mathfrak{c}_v} \tag{6.7}$$

for all $t \in X(\mathbb{C}_v)$ outside of $\mathfrak{U}_v$ and all $v \in M_K$. From Proposition 2.1, we can increase the M_K-constant $\mathfrak{c}$ so that

$$e^{-\mathfrak{c}_v} \ \le \ \|A_t\|_v \ \le \ e^{\mathfrak{c}_v} \tag{6.8}$$

for all $t \in X(\mathbb{C}_v) \setminus \mathfrak{U}_v$.

Let $S \subset M_K$ be a finite set containing S_γ for each $\gamma \in \mathcal{S}(F,A)$ and containing all places for which $\mathfrak{c}_v \neq 0$ and for which $\mathfrak{U}_v$ is not equal to the union $\bigcup_{\gamma \in \mathcal{S}(F,A)} \{|\xi^\gamma|_v > 1\}$. Now fix $t \in X(\overline{K})$. If $t \notin \mathfrak{U}_v$ at $v \notin S$, we have

$$V_v(t) = G_{F_t,v}(A_t) = 0 \tag{6.9}$$

because $\|F_t^n(A_t)\|_v = 1$ for all n, from (6.7) and (6.8). On the other hand, if $t \in \mathfrak{U}_v$ for $v \notin S$, then we still have the bound

$$|V_v(t)| \le \mathfrak{a}_v(\gamma) \tag{6.10}$$

from (6.6). Recalling the summability of the bounds in (6.6) near each $\gamma \in \mathcal{S}(F,A)$, inequalities (6.9) and (6.10) yield

$$\sum_{v \notin S} N_v \, |V_v(t)| \ \le \ C := \sum_{\gamma \in \mathcal{S}(F,A)} \sum_{v \notin S} N_v \, \mathfrak{a}_v(\gamma) < \infty,$$

for each $t \in X(\overline{K})$.

By the continuity of V_v on X_v^{an} for every $v \in M_K$, from Theorem 1.7, there is a constant C_v for each $v \in S$ so that $|V_v(t)| \le C_v$ for all $t \in X(\overline{K})$. This gives

$$\sum_{v \in M_K} N_v \, |V_v(t)| \ \le \ C + \sum_{v \in S} N_v \, C_v < \infty$$

for all $t \in X(\overline{K})$. It follows that, taking averages over the Galois orbit of t, we have

$$\frac{1}{|\operatorname{Gal}(\overline{K}/K) \cdot t|} \sum_{x \in \operatorname{Gal}(\overline{K}/K) \cdot t} \sum_{v \in M_K} N_v \, |V_v(t)| \ \le \ C + \sum_{v \in S} N_v \, C_v$$

for all $t \in X(\overline{K})$. This completes the proof of Theorem 1.2. □

7. Examples

In this final section, we present examples to illustrate some of the subtle phenomena that can arise for non-polynomial maps $f : \mathbb{P}^1 \to \mathbb{P}^1$, even in the simplest setting of degree $d = 2$, with $K = \mathbb{Q}$ and $k = \mathbb{Q}(t)$.

7.1. **The difference of heights in Theorem 1.2 is bounded but not M_K-bounded.** We first present an example, already seen in Remark 5.3, where the conditions of Theorem 1.2 hold but the functions V_v of Theorem 1.7 are nontrivial at infinitely many places v of $K = \mathbb{Q}$. A mechanism to construct many other such examples appears in [25]. Consider

$$f(z) = \frac{z(z+1)}{z+t}$$

defined over $k = \mathbb{Q}(t)$. Put

$$F(z, w) = (z(z + w), (z + tw)w),$$

so that $\mathcal{S}(F)$ consists only of the three points $t = 0, 1, \infty$ in $X = \mathbb{P}^1$. Let $a = 1$, and take $A = (1, 1)$ so that $\mathcal{S}(F, A) = \mathcal{S}(F) = \{0, 1, \infty\}$. The point a is totally Fatou as a consequence of Theorem 4.6, because the pair (f, a) is hole-avoiding at all points of $\mathcal{S}(F)$. Indeed, at $t = 0$, we have $F_0(z, w) = (z(z + w), zw)$ with hole at $z/w = 0$ and orbit $f_0^n(a) = n + 1$ for all $n \geq 0$. At $t = 1$, we have $F_1(z, w) = (z(z + w), (z + w)w)$ with hole at $z/w = -1$, and orbit $f_1^n(a) = 1$ for all $n \geq 0$. Finally, at $t = \infty$, we can choose a new lift $F' = \frac{1}{t}F$ so that $(F')_\infty(z, w) = (0, w^2)$ with hole at $z/w = \infty$ and orbit $f_\infty^n(a) = 0$ for all $n \geq 1$.

It follows from these computations that $D = D(F, A) = (\infty)$ is the divisor of degree 1 on $X = \mathbb{P}^1$ supported at the point $t = \infty$. This implies, in particular, that $\hat{h}_f(a) = 1$.

Fix a prime p of $\mathbb{Q}$. To see that the function V_p is nontrivial on X, it suffices to show that $V_p(0) \neq 0$. Let $A_n = F^n(A)$. As explained in Remark 5.3, we have $\|(A_n)_0\|_p = 1$ for all $n < p$, and $\|(A_n)_0\|_p < 1$ for all $n \geq p$. As computed in (5.18), we know that $\|(A_n)_0\|_p^{1/2^n}$ is a decreasing sequence for all primes p, so that the α_p of Proposition 5.2 (defined as $g_p(0)$ for the function $g_p(t) = \lim_{n\to\infty} 2^{-n} \log \|(A_n)_t\|_p$ in a p-adic neighborhood of $t = 0$) is non-zero for all primes p. Moreover, as explained in the proof of Theorem 1.7, we also have that $V_p(0) = g_p(0)$ and so $V_p(0) = \alpha_p < 0$ for all primes p.

7.2. **All known non-polynomial examples are totally Fatou.** Here we survey the results in the literature where the conclusions of Theorems 1.2 and 1.7 were known for examples $f : \mathbb{P}^1 \to \mathbb{P}^1$ that are *not* polynomial maps (nor conjugate to a polynomial). In every case, the points $a \in \mathbb{P}^1(k)$ that were treated satisfy our totally Fatou hypothesis.

The first example is the one presented in the Introduction, where the variation of canonical height $t \mapsto \hat{h}_{f_t}(p_t)$ for a family of Lattès maps f_t – those arising as quotients of endomorphisms of ellptic curves – is known to differ from a Weil height for a $\mathbb{Q}$-divisor on the base curve X by a bounded amount, for any choice of $p \in \mathbb{P}^1(k)$ [32]. The continuity of the local contributions V_v, as defined in Theorem 1.7, was shown by Silverman in [29]. Also as mentioned in the Introduction, it is well known that all points are totally Fatou for these maps; see, e.g., the computation of the Berkovich Julia set in [16, §5]. Alternatively, note that the existence of a Néron model forces all points to be hole-avoiding in appropriate coordinates.

In [17], the authors prove Theorem 1.2 for rational maps f defined over $k = K(X)$ for a curve X and points $c \in \mathbb{P}^1(k)$, under the assumptions that

(1) there exists $t_0 \in X$ so that the map f has good reduction at all $t \neq t_0$;
(2) f has a super-attracting fixed point at $z = \infty$; and
(3) the point c satisfies $\operatorname{ord}_{t_0} f^n(c) \to -\infty$.

Condition (3) implies that c is in the basin of attraction of the super-attracting fixed point at ∞, so it is clearly Fatou at t_0. (The hypothesis (3) is stated in [17, Theorem 5.4] as $\{\deg f^n(c) : n \geq 0\}$ is unbounded, but for a notion of degree defined in their Section 5 on the regular functions on $X \setminus \{t_0\}$ and extended to k after equation (5.4).)

In [18], the authors studied maps of the form

$$f(z) = \frac{z^d + t}{z}$$

over $k = \overline{\mathbb{Q}}(t)$, for $d \geq 3$, and they prove Theorem 1.2 for all points $a \in \mathbb{P}^1(k)$. (The map f for $d = 2$ is isotrivial, making the theorem true but much easier.) In this example, the point $z = \infty$ is a super-attracting fixed point, and there are two places of bad reduction, at $t = 0$ and $t = \infty$. All points $a \in \mathbb{P}^1(k)$ are totally Fatou. Indeed, at $t = 0$, the reduction is $f_0(z) = z^{d-1}$ with only hole at $z = 0$. So the only points we need to consider are those which vanish at $t = 0$. But for any integer $m \geq 1$, if $\operatorname{ord}_0 a = m$, then $\operatorname{ord}_0 f(a) = 1 - m \leq 0$, so $f(a)$ which will no longer specialize to 0 at $t = 0$; this implies that the pair $(f, f(a))$ is hole-avoiding at $t = 0$ for all $a \in \mathbb{P}^1(k)$. At $t = \infty$, a computation shows that if $\operatorname{ord}_\infty a = r < 0$, then $\operatorname{ord}_\infty f(a) = (d-1)r$; iterating implies that $f^n(a) \to \infty$ in the ∞-adic topology, so the

point a will be Fatou at $t=\infty$. Moreover, if $\operatorname{ord}_\infty a = r \geq 0$, then $\operatorname{ord}_\infty f(a) = -r-1 < 0$, and again a is Fatou at ∞.

In [13], the authors consider

$$f(z) = \frac{\lambda z}{z^2+tz+1} \tag{7.1}$$

for a fixed $\lambda \neq 0$ in $\overline{\mathbb{Q}}$, defined over $k = \overline{\mathbb{Q}}(t)$, having a fixed point of multiplier λ. For λ not a root of unity or for $\lambda = 1$, the result of Theorem 1.2 is obtained there for the critical points $c_\pm = \pm 1$ of f. The critical points will be totally Fatou for any choice of λ. It suffices to check the dynamics of f at $t=\infty$. For $\lambda = 1$, we can conjugate f by $B(z) = 1/(tz)$ so the new map $z+1+1/(t^2 z)$ specializes to $z \mapsto z+1$ with hole at $z=0$, and the critical values in the new coordinate system $B(f(c_\pm)) = (\pm 2 + t)/t$ specialize to $z=1$, so the pairs $(f, f(c_\pm))$ are seen to be hole-avoiding in the new coordinate system. For λ not a root of unity, the map f can be conjugated to a map that specializes to $z \mapsto \lambda\, z$ with a hole at $z=1$, and so that the critical values $f(c_\pm)$ in the new coordinates will specialize to $z=\lambda$. Again the pairs $(f, f(c_\pm))$ are hole-avoiding in the new coordinate system. These facts appear in the proof of [13, Proposition 2.2] and in [7, §5]; these cases are also covered by [23, Lemma 3.4]. Finally, in [25], the authors obtain the result of Theorem 1.2 for the maps f of the form (7.1) when λ is a root of unity and for a large class of points $c \in \overline{\mathbb{Q}}(t)$ satisfying a hole-avoiding condition at the place of bad reduction $t=\infty$. This includes in particular maps of the form (7.1) with $c_\pm = \pm 1$. As explained in Theorem 4.6 above, this means that the points considered are totally Fatou.

7.3. **Julia points: irrational local heights and $\mathbb{R}$-divisors.** This next example is a map of degree 2 defined over the field $k = \mathbb{Q}(t)$, with the property that all points with infinite orbit that lie in a local non-archimedean Julia set at the place $t=0$ of k will have an irrational local canonical height. If such a point can be algebraic over k, it would show that the conclusion of Theorem 1.2 would fail for Julia points, as stated; the divisor D should be an $\mathbb{R}$-divisor on the curve X. Set

$$f(z) = \frac{(t^2+t+1)z^2 + tz + t^2 - 1}{(2t^2+t)z + t}. \tag{7.2}$$

This map has fixed points at $z=1$ (with multiplier $1/t$) and at $z=-1$ (with multiplier $1/t^2$), and at $z=\infty$. At the place $t=0$ of k, these fixed points at ± 1 are repelling, and the fixed point at ∞ is attracting (but not super-attracting). The Julia set in $\mathbb{P}^{1,an}_{t=0}$, defined over

the field $\mathbb{L}$ of formal Puiseux series in t, is a Cantor set of Type I points, and f is conjugate to the full 2-shift [23, Theorem 3(1)]. All points outside of the Julia set will tend to ∞ under iteration. This map f exhibits a polynomial-like behavior near its Julia set, and it can be computed that the Julia set is a subset the formal completion $k_0 := \mathbb{Q}[[t]]$. But this example is not *strongly* polynomial-like, in the sense of [10, Theorem 1.5], because the multipliers at the two repelling fixed points have distinct absolute values.

We can compute local canonical heights over the field k with the procedure described in [10]. In homogeneous coordinates, put

$$F(z,w) = ((t^2+t+1)z^2 + tzw + (t^2-1)w^2,\ (2t^2+t)zw + tw^2).$$

At $t=0$, we have $F_0(z,w) = (z^2-w^2,\ 0)$, and the Julia set is contained in the hole-directions ± 1 from the Gauss point ζ_G, i.e., in the union of the two disks $D_\pm = \{z \in \mathbb{L} : |z-(\pm 1)|_0 < 1\}$. The conjugacy between f on its Julia set and the shift map on 2 symbols is given by the itinerary of a point as it moves between D_+ and D_-. For a point $a \in k_0$ with lift $A \in (k_0)^2 \setminus \{(0,0)\}$, a sequence of orders is defined by

$$\tau_n := \operatorname{ord}_{t=0} F^n(A) = 2\,\tau_{n-1} + \sigma_{n-1} \tag{7.3}$$

so that

$$G_{F,0}(A) = -\lim_{n\to\infty} \frac{\tau_n}{2^n} = -\tau_0 - \sum_{n=1}^{\infty} \frac{\sigma_{n-1}}{2^n}.$$

From the formula for F, we can compute that $\sigma_n = 1$ if $f^n(a) \in D_+$ and $\sigma_n = 2$ if $f^n(a) \in D_-$, for all $n \geq 0$. Because of the conjugation to the shift map, we see that the sequence $\{\sigma_n\}$ is eventually periodic if and only if the point a is eventually periodic. Therefore, $G_{F,0}(A)$ (and so also any presentation of the geometric local canonical height $\hat{\lambda}_{f,0}(a)$ at $t=0$) is irrational for all Julia points with infinite orbit.

Remark 7.1. The function f of (7.2) is conjugate to $z \mapsto \frac{t^2z^2+z}{tz+t^2}$, in a standard normal form for quadratic rational maps, with fixed points at 0 and ∞ of specified multipliers (in this case, having multiplier $1/t^2$ at 0 and $1/t$ at ∞). We then moved the two repelling fixed points to 1 and -1 and the attracting fixed point to ∞.

7.4. **Julia points with divergent escape rates.** Our final example is

$$f(z) = \frac{z^2 + (t^2-t-1)z - t^3 - 2t^2 + t}{z - t^2 - 1}, \tag{7.4}$$

defined over the field $k = \mathbb{Q}(t)$, at the place γ corresponding to $t = 0$. As for the example (7.2), all Julia points at $t = 0$ with infinite orbit for f will have an irrational local canonical height at $t = 0$. This can be seen from the proof of [10, Theorem 1.3], because this f is conjugate to the map $z \mapsto \frac{(z+1)(z-t)}{z+t}$ studied there, combined with an identification of the Julia set with the shift on 2 symbols [23, Proposition 4.2].

We construct (formal) points $a \in \mathbb{Q}[[t]]$ in the Julia set of f at $t = 0$ so that the sequence of functions (5.1) that define V_∞ (at the archimedean place) will diverge at $t = 0$. We do not know if the points a we construct can be algebraic over k, nor even if the series will converge on a disk around $t = 0$. We use these examples to illustrate some of the features of Julia points that do not arise for the Fatou points.

Remark 7.2. As we shall see, taking any unbounded sequence of positive integers $\{m_k\}_{k\geq 0}$ in the construction below, this example also shows that the orbits of points in $\mathbb{P}^1(k_\gamma)$ can have non-locally-compact closures. This is distinct from what happens for polynomials; compare [14, Theorem 3].

More precisely, we construct examples so that the sequence

$$\alpha_n := \frac{1}{2^n} \log \|(A_n)_0\|_v,$$

as defined and studied in Proposition 5.2, will diverge to $-\infty$ at the place $v = \infty$. In particular, this would show that – if the point a defines a convergent series in $\mathbb{Q}[[t]]$ – the conclusion of Theorem 5.1 would fail. That is, the sequence of functions

$$g_n(t) := \frac{1}{d^n} \log \|(A_n)_t\|$$

would converge, locally uniformly on a punctured disk around $t = 0$, and we know that the limit function $g(t)$ must be bounded by $o(\log |t|)$ as $t \to 0$ [8, Proposition 3.1]. But the convergence to g would not be uniform in a neighborhood of $t = 0$.

For the construction, note first that f specializes to the identity transformation $f_0(z) = z$ at $t = 0$ with a hole at $z = 1$. The Berkovich Julia set is contained in the direction $z = 1$ from the Gauss point, and all Julia points have the form $1 + m\,t + O(t^2)$ for some integer $m \geq 0$. We have

$$f(1 + w\,t + O(t^2)) = 1 + (w - 1)\,t + O(t^2)$$

for all $w \neq 0$, and

$$f(1 + w\,t^2 + O(t^3)) = 1 + \frac{1 + w}{1 - w}\,t + O(t^2)$$

for all $w \neq 1$.

For any sequence of positive integers $\{m_k\}_{k\geq 0}$, there is a unique point $a \in \mathbb{Q}[[t]]$ so that

$$a = 1 + m_0\, t + O(t^2)$$

and

$$f^{m_0+\cdots+m_{k-1}+k}(a) = 1 + m_k\, t + O(t^2)$$

for all $k \geq 1$ [10, Theorem 1.3], [23, Proposition 4.2]. Let

$$F(z,w) = \left(z^2 + (t^2 - t - 1)zw + (-t^3 - 2t^2 + t)w^2,\ zw - (t^2+1)w^2\right)$$

be a lift of f. Set

$$A = (a, 1) = (1 + m_0\, t + O(t^2),\ 1)$$

for the a associated to a given sequence $\{m_k\}_{k\geq 0}$. For each $n \geq 1$, we set

$$A_n = t^{-\sigma_{n-1}} F(A_{n-1}),$$

where σ_{n-1} is chosen so that $\operatorname{ord}_{t=0} A_n = 0$, as above in (7.3). In fact, $\sigma_n = 1$ whenever $f^n(a) = 1 + wt + O(t^2)$ for $w \neq 0$, and $\sigma_n = 2$ for $f^n(a) = 1 + O(t^2)$. For each $n \geq 1$, we let $(A_n)_0$ denote the specialization at $t = 0$, and set

$$\alpha_n = \frac{1}{2^n} \log \|(A_n)_0\|$$

in the archimedean norm.

We now show that, by choosing the sequence $\{m_k\}_{k\geq 0}$ to grow to infinity sufficiently fast, we can have $\liminf_{n\to\infty} \alpha_n = -\infty$. For a lift $(y_0 + x_1 t + O(t^2),\ y_0 + y_1 t + O(t^2))$ of the point $1 + m\, t + O(t^2)$, with $m > 0$, we have $(x_1 - y_1)/y_0 = m$, and this gives

$$\begin{aligned} F\left(y_0 + x_1 t + O(t^2),\ y_0 + y_1 t + O(t^2)\right) &= t\ (y_0(x_1 - y_1),\ y_0(x_1 - y_1)) + O(t^2) \\ &= t\ \left(m\, y_0^2,\ m\, y_0^2\right)) + O(t^2) \end{aligned}$$

for $m \neq 0$. Now suppose we take any lift $(y_0 + y_1 t + x_2 t^2 + O(t^3),\ y_0 + y_1 t + y_2 t^2 + O(t^3))$ with $y_0 \neq 0$ of a point $p = 1 + c\, t^2 + O(t^3)$, with $c = \frac{m-1}{m+1}$ for $m \geq 1$ so that $f(p) = 1 + m\, t + O(t^2)$.

Then we must have $\frac{x_2-y_2}{y_0} = \frac{m-1}{m+1}$, and this gives

$$\begin{aligned} F\left(y_0 + y_1 t + x_2 t^2 + O(t^3),\, y_0 + y_1 t + y_2 t^2 + O(t^3)\right) \\ &= t^2\ (y_0(x_2 - y_0 - y_2),\, y_0(x_2 - y_0 - y_2)) + O(t^3) \\ &= t^2\ \left(\frac{-2y_0^2}{m+1},\, \frac{-2y_0^2}{m+1}\right) + O(t^3). \end{aligned}$$

Iterating the point A, we find that

$$(A_1)_0 = (m_0,\, m_0)$$

$$(A_2)_0 = \left(m_0^2(m_0-1),\, m_0^2(m_0-1)\right)$$

all the way to

$$(A_{m_0})_0 = \left(m_0^{2^{m_0-1}}(m_0-1)^{2^{m_0-2}}\cdots 1,\, m_0^{2^{m_0-1}}(m_0-1)^{2^{m_0-2}}\cdots 1\right) =: (Y_{m_0}, Y_{m_0})$$

so that

$$\alpha_{m_0} = \frac{1}{2^{m_0}}\log|Y_{m_0}| = \sum_{j=1}^{m_0}\frac{1}{2^j}\log(m_0 - j + 1).$$

The point A_{m_0} is a lift of $f^{m_0}(a) = 1 + 0\,t + \frac{m_1-1}{m_1+1}t^2 + O(t^3)$, so that

$$(A_{m_0+1})_0 = \left(\frac{-2}{m_1+1}Y_{m_0}^2,\, \frac{-2}{m_1+1}Y_{m_0}^2\right).$$

Thus,

$$\alpha_{m_0+1} = \alpha_{m_0} + \frac{1}{2^{m_0+1}}\log\left(\frac{2}{m_1+1}\right).$$

Note that this α_{m_0+1} can be made as negative as desired by choosing $m_1 \gg m_0$. Continuing to iterate, we have

$$\alpha_{m_0+1+s} = \alpha_{m_0+1} + \sum_{j=1}^{s}\frac{1}{2^{m_0+1+j}}\log(m_1 - j + 1)$$

for all $s = 1, \ldots, m_1$. Then

$$\alpha_{m_0+m_1+2} = \alpha_{m_0+m_1+1} + \frac{1}{2^{m_0+m_1+2}}\log\left(\frac{2}{m_2+1}\right).$$

Again, we can make $\alpha_{m_0+m_1+2}$ as negative as desired by choosing $m_2 \gg m_1$. We see that the pattern continues, and so, by choosing the sequence $\{m_k\}_{k\geq 0}$ to grow to infinity very fast, we conclude that the sequence $\{\alpha_n\}_{n\geq 0}$ is unbounded from below.

References

[1] Matthew Baker and Robert Rumely. *Potential theory and dynamics on the Berkovich projective line*, volume 159 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[2] Robert L. Benedetto. Wandering domains and nontrivial reduction in non-Archimedean dynamics. *Illinois J. Math.* **49**(2005), 167–193.

[3] Robert L. Benedetto. *Dynamics in one non-archimedean variable*, volume 198 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2019.

[4] Enrico Bombieri and Walter Gubler. *Heights in Diophantine geometry*, volume 4 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2006.

[5] G. S. Call and J. H. Silverman. Canonical heights on varieties with morphisms. *Compositio Math.* **89**(1993), 163–205.

[6] Laura DeMarco. Iteration at the boundary of the space of rational maps. *Duke Math. Journal.* **130**(2005), 169–197.

[7] Laura DeMarco. The moduli space of quadratic rational maps. *J. Amer. Math. Soc.* **20**(2007), 321–355.

[8] Laura DeMarco. Bifurcations, intersections, and heights. *Algebra Number Theory.* **10**(2016), 1031–1056.

[9] Laura DeMarco and Xander Faber. Degenerations of complex dynamical systems II: analytic and algebraic stability. *Math. Ann.* **365**(2016), 1669–1699. With an appendix by Jan Kiwi.

[10] Laura DeMarco and Dragos Ghioca. Rationality of dynamical canonical height. *Ergodic Theory Dynam. Systems.* **39**(2019), 2507–2540.

[11] Laura DeMarco and Yûsuke Okuyama. Discontinuity of a degenerating escape rate. *Conform. Geom. Dyn.* **22**(2018), 33–44.

[12] Laura DeMarco, Xiaoguang Wang, and Hexi Ye. Torsion points and the Lattès family. *Amer. J. Math.***138**(2016), 697–732.

[13] Laura DeMarco, Xiaoguang Wang, and Hexi Ye. Bifurcation measures and quadratic rational maps. *Proc. London Math. Soc.* **111**(2015), 149–180.

[14] Charles Favre and Thomas Gauthier. Continuity of the Green function in meromorphic families of polynomials. *Algebra Number Theory* **12**(2018), 1471–1487.

[15] Charles Favre and Thomas Gauthier. *The Arithmetic of Polynomial Dynamical Pairs*. Annals of Mathematics Studies, 214. Princeton University Press, Princeton, NJ, 2022.

[16] Charles Favre and Juan Rivera-Letelier. Théorie ergodique des fractions rationnelles sur un corps ultramétrique. *Proc. Lond. Math. Soc. (3)* **100**(2010), 116–154.

[17] D. Ghioca, L.-C. Hsia, and T. Tucker. Preperiodic points for families of rational maps. *Proc. London Math. Soc.* **110**(2015), 395–427.

[18] Dragos Ghioca and Niki Myrto Mavraki. Variation of the canonical height in a family of rational maps. *New York J. Math.* **19**(2013), 873–907.

[19] Dragos Ghioca and Hexi Ye. A dynamical variant of the André-Oort conjecture. *Int. Math. Res. Not. IMRN* **8**(2018), 2447–2480.

[20] Liang-Chung Hsia. A weak Néron model with applications to p-adic dynamical systems. *Compositio Math.* **100**(1996), 277–304.

[21] P. Ingram. Variation of the canonical height for a family of polynomials. *J. Reine Angew. Math.* **685**(2013), 73–97.

[22] P. Ingram. Variation of the canonical height in a family of polarized dynamical systems *Preprint,* arXiv:2104.12877 [math.NT].

[23] Jan Kiwi. Puiseux series dynamics of quadratic rational maps. *Israel J. Math.* **201**(2014), 631–700.

[24] Serge Lang. *Fundamentals of Diophantine geometry*. Springer-Verlag, New York, 1983.

[25] Niki Myrto Mavraki and Hexi Ye. Quasi-adelic measures and equidistriution on $\mathbb{P}^1$. *Preprint,* arXiv:1502.04660v3 [math.DS].

[26] Clayton Petsche. S-integral preperiodic points by dynamical systems over number fields. Bull. Lond. Math. Soc. **40**(2008), 749–758.

[27] Juan Rivera-Letelier. Dynamique des fonctions rationnelles sur des corps locaux. *Astérisque* (2003), xv, 147–230. Geometric methods in dynamics. II.

[28] Joseph H. Silverman. Variation of the canonical height on elliptic surfaces. I. Three examples. *J. Reine Angew. Math.* **426**(1992), 151–178.

[29] Joseph H. Silverman. Variation of the canonical height on elliptic surfaces. II. Local analyticity properties. *J. Number Theory* **48**(1994), 291–329.

[30] Joseph H. Silverman. Variation of the canonical height on elliptic surfaces. III. Global boundedness properties. *J. Number Theory* **48**(1994), 330–352.

[31] Joseph H. Silverman. *The Arithmetic of Dynamical Systems*, volume 241 of *Graduate Texts in Mathematics*. Springer, New York, 2007.

[32] J. Tate. Variation of the canonical height of a point depending on a parameter. *Amer. J. Math.* **105**(1983), 287–294.

[33] Eugenio Trucco. Wandering Fatou components and algebraic Julia sets. *Bull. Soc. Math. France* **142**(2014), 411–464.

[34] André Weil. Arithmetic on algebraic varieties. *Ann. of Math. (2)* **53**(1951), 412–444.

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