Pacific Journal of Mathematics

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Volume 320 No. 1 September 2022

PACIFIC JOURNAL OF MATHEMATICS
Vol. 320, No. 1, 2022
https://doi.org/10.2140/pjm.2022.320.103

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We define the *Barbasch–Evens–Magyar varieties*. We show they are isomorphic to the smooth varieties defined in [D. Barbasch and S. Evens 1994] that map generically finitely to symmetric orbit closures, thereby giving resolutions of singularities in certain cases. Our definition parallels P. Magyar's [1998] construction of the *Bott–Samelson varieties* [H. C. Hansen 1973; M. Demazure 1974]. From this alternative viewpoint, one deduces a graphical description in type A, stratification into closed subvarieties of the same kind, and determination of the torus-fixed points. Moreover, we explain how these manifolds inherit a natural symplectic structure with Hamiltonian torus action. We then express the moment polytope in terms of the moment polytope of a Bott–Samelson variety.

1. Introduction

Let X be a generalized flag variety of the form G/B, where G is a connected reductive complex algebraic group and B is a Borel subgroup of G. The left action of B on X has finitely many orbits $\mathsf{B}w\mathsf{B}/\mathsf{B}$, where w is a Weyl group element. The *Schubert variety* X_w is the closure $\overline{\mathsf{B}w\mathsf{B}/\mathsf{B}}$ of the B-orbit. The study of Schubert variety singularities is of interest; see, e.g., [1; 4; 8].

In the 1970s, H. C. Hansen [19] and M. Demazure [11] constructed a *Bott–Samelson variety* BS^Q for each reduced word Q of w, building on ideas of R. Bott and H. Samelson [6]. These manifolds are resolutions of singularities

MSC2020: 14M15.
Keywords: flag variety, K-orbit, Barbasch–Evens–Magyar variety, moment polytope, clans.

of X_w. In recent years, Bott–Samelson varieties have been used, e.g., in studies of Schubert calculus (M. Willems [44]), Kazhdan–Lusztig polynomials (B. Jones and A. Woo [22]), standard monomial theory (V. Lakshmibai, P. Littelmann and P. Magyar [28]), Newton–Okounkov bodies (M. Harada and J. Yang [20]), and matroids over valuation rings (A. Fink and L. Moci [15]).

In 1983, A. Zelevinsky [52] gave a different resolution for Grassmannian Schubert varieties, presented as configuration spaces of vector spaces prescribed by dimension and containment conditions. In 1998, P. Magyar [29] gave a new description of BS^Q in the same spirit, replacing the quotient by group action definition with an iterated fiber product.

Similar constructions have been used subsequently in, e.g.:

(I) P. Polo's proof that every polynomial $f \in 1 + q\mathbb{Z}_{\geq 0}[q]$ is a Kazhdan–Lusztig polynomial (in type A) [37].

(II) A. Cortez's proof of the singular locus theorem for Schubert varieties in type A [10] (compare [5; 23; 30]).

(III) N. Perrin's extension of Zelevinsky's resolution to minuscule Schubert varieties [33] (one application is [34]).

(IV) A. Woo's classification of "short" Kazhdan–Lusztig polynomials [45].

(V) The definition of the *brick variety*, which provides resolutions of singularities of Richardson varieties [13].

(VI) The connection [14] of Magyar's definition to S. Elnitsky's rhombic tilings [12].

We are interested in the parallel story where orbit closures for symmetric subgroups replace Schubert varieties. A *symmetric subgroup* K of G is a group comprised of the fixed points G^θ of an involution θ of G. Like B, K is *spherical*, meaning that it has finitely many orbits $\mathcal{O}$ under the left action on X. The study of the singularities of a K-orbit closure $Y = \bar{\mathcal{O}}$ is relevant to the theory of Kazhdan–Lusztig–Vogan polynomials and Harish-Chandra modules for a certain real Lie group $\mathsf{G}_\mathbb{R}$. This may be compared with the connection of Schubert varieties to Kazhdan–Lusztig polynomials and the representation theory of complex semisimple Lie algebras.

In 1994, D. Barbasch and S. Evens [3] constructed a smooth variety, using a quotient description that extends the one for Bott–Samelsons from [11; 19]. This variety comes equipped with a natural map to a particular K^0-orbit closure, where K^0 is the connected component of 1. In certain situations, this map provides a resolution of singularities of the orbit closure in question.

This paper introduces and initiates our study of the *Barbasch–Evens–Magyar variety* (BEM variety). Just as P. Magyar [29] describes, *via* a fiber product, a

variety that is equivariantly isomorphic to a Bott–Samelson variety, the BEM variety reconstructs the manifold of [3] (Theorem 3.3(I)).

Our definition gives *general-type* results about the varieties of [3]:

- A stratification (in the sense of [26, Section 1.1.2]) into smaller BEM varieties (Corollary 3.8).
- Description of its torus fixed points (Proposition 4.3).
- A symplectic structure with Hamiltonian torus action as well as analysis of the moment map image, e.g., we show that it is the convex hull of certain Weyl group reflections of the moment polytope of a Bott–Samelson variety (Theorem 4.1).
- An analogue of the brick variety (Theorem 3.3(II)).

In type A we give a diagrammatic description of the BEM varieties (Section 5) in linear algebraic terms, avoiding the algebraic group generalities. For example, we obtain more specific results (Section 6) for $\mathsf{K} = \mathsf{GL}_p \times \mathsf{GL}_q$ acting on $\mathsf{GL}_{p+q}/\mathsf{B}$. We show (Theorem 6.2) that the study of BEM polytopes can be reduced to a special case. We determine the torus weights for this special case (Theorem 6.4) which permits us to partially understand the vertices (Corollary 6.6). We also give a combinatorial characterization of the dimension of the BEM polytope (Theorem 6.8).

We anticipate that many uses of the Zelevinsky/Magyar-type construction of the Bott–Samelson variety, such as (I)–(VI) above, have BEM versions. In particular, an analogue of (II), even in the case of $\mathsf{K} = \mathsf{GL}_p \times \mathsf{GL}_q$, would bring important new information about the singularities of the symmetric orbit closures. More generally, analogues of (I)–(IV) would illuminate the combinatorics of the celebrated Kazhdan–Lusztig–Vogan polynomials.

2. Background on K-orbits

In this section we describe the background in general. See Section 5 for background on K-orbits of type A.

Let G be a connected complex reductive algebraic group and B a Borel subgroup of G containing a maximal torus T. Furthermore we assume θ is an involution of G and that B and T are θ-stable. We denote by K the symmetric subgroup G^θ. Throughout this paper we assume that K is the connected component of the fixed point set of θ.

Let $\mathsf{W} = N_{\mathsf{G}}(\mathsf{T})/\mathsf{T}$ be the Weyl group. Let r be the rank of the root system of G and $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be the system of simple roots corresponding to B, with $\{\omega_1, \dots, \omega_r\}$ the corresponding fundamental weights. Denote the simple reflection corresponding to the simple root α_i by s_i. Thus, W is generated by the simple reflections $\{s_i \mid 1 \le i \le r\}$.

Given $I \subseteq \Delta$, P_I is the standard parabolic subgroup of G corresponding to I; namely,

$$\mathsf{P}_I = \mathsf{B} \cup \Big(\bigcup_{w \in W_I} \mathsf{B} w \mathsf{B} \Big), \tag{1}$$

where W_I is the set of $w \in W$ such that $w = s_{i_1} \dots s_{i_k}$ where all $i_j \in I$. P_I is a *minimal parabolic* if $I = \{\alpha_i\}$; it is a *maximal parabolic* if $I = \{\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_r\}$. These are denoted P_i and $\mathsf{P}_{\hat{i}}$, respectively.

As described in [38, Section 3.10], the *Richardson–Springer monoid* $\mathcal{M}(\mathsf{W})$ is generated by the simple reflections s_i of W, with relations $s_i^2 = s_i$, together with the braid relations of W. As a set, this monoid may be canonically identified with W, with the ordinary product on W being replaced by the *Demazure product* $\star$, a product having the property that

$$s_i \star w = \begin{cases} s_i w & \text{if } \ell(s_i w) > \ell(w), \\ w & \text{otherwise,} \end{cases}$$

where $\ell(\cdot)$ denotes ordinary Coxeter length, and where the juxtaposition $s_i w$ denotes the ordinary product in W. A *word* $Q = (j_1, j_2, \dots, j_N)$ is an ordered tuple of numbers from $\{1, 2, \dots, r\}$. Let $\mathrm{Dem}(Q) := s_{j_1} \star s_{j_2} \star \cdots \star s_{j_N}$. If $\mathrm{Dem}(Q) = w$, then Q is a *Demazure word* for w.

Consider the natural projection $\pi_i : \mathsf{G}/\mathsf{B} \to \mathsf{G}/\mathsf{P}_i$. Given a K-orbit closure Y on G/B and a simple reflection $s_i \in \mathsf{W}$, $s_i \star Y := \pi_i^{-1}(\pi_i(Y))$ is also a K-orbit closure. This extends to an $\mathcal{M}(\mathsf{W})$-action on the set of K-orbit closures: given a Demazure word $Q = (s_{j_1}, \dots, s_{j_N})$ for w, define

$$w \star Y := s_{j_1} \star (s_{j_2} \star \cdots \star (s_{j_N} \star Y) \dots).$$

The K-orbit closure $w \star Y$ is independent of the choice of Demazure word Q for w [38, Section 4.7].

The *weak order* on the set of K-orbit closures is defined by

$$Y \leq Y' \iff Y' = w \star Y$$

for some $w \in \mathcal{M}(\mathsf{W})$. The minimal elements of this order are the *closed orbits*, i.e., those $Y_0 = \mathcal{O} = \overline{\mathcal{O}}$. The following is well known; see, e.g., [7, Proposition 2.2(i)].

Lemma 2.1. *Each closed orbit is isomorphic to* K/B' *where* B' *is a Borel subgroup of* K. *In particular, every closed orbit is smooth.*

Remark 2.2. If K is disconnected, then [7, Proposition 2.2(i)] says that Y_0 is isomorphic to a finite union of flag manifolds K^0/B', where K^0 is the connected component of 1 and B' is a Borel subgroup of K^0.

3. Barbasch–Evens–Magyar varieties

As in the previous section, G is a connected reductive complex algebraic group, $\mathsf{K} = \mathsf{G}^\theta$ is connected, and B is a θ-stable Borel subgroup of G. We begin with the definition of the manifold of Barbasch and Evens [3, Section 6].

If B^{k-1} acts on $X_k \times \cdots \times X_1$ by

$$(b_k, \dots, b_1) \cdot (x_k, \dots, x_1) = (x_k b_k, b_k^{-1} x_{k-1} b_{k-1}, \dots, b_2^{-1} p_1), \tag{2}$$

then $X_k \times^{\mathsf{B}} \cdots \times^{\mathsf{B}} X_1$ denotes the quotient of $X_k \times \cdots \times X_1$ by this action, when it exists. In [16, Section 3.3] this quotient is shown to exist in cases which include when the X_i for $i > 1$ are subgroups of G such that $\mathsf{B} \subset X_i \cap X_{i+1}$ and $X_1 = \mathsf{P}/\mathsf{B}$ for some parabolic subgroup P of G.

Let Y_0 be a closed K-orbit and $Q = (j_1, j_2, \dots, j_N)$ a (not necessarily reduced) word. The *Barbasch–Evens variety* [3, (6.3.5)] for Y_0 and Q is

$$\mathcal{BE}^{Y_0,Q} := \widetilde{Y}_0 \times^{\mathsf{B}} \mathsf{P}_{j_N} \times^{\mathsf{B}} \mathsf{P}_{j_{N-1}} \times^{\mathsf{B}} \cdots \times^{\mathsf{B}} \mathsf{P}_{j_1}/\mathsf{B}, \tag{3}$$

where $\widetilde{Y}_0$ denotes the preimage of Y_0 in G under $\mathsf{G} \to \mathsf{G}/\mathsf{B}$. By [16, Section 3.3] this quotient exists. Recall that by Lemma 2.1 Y_0 is smooth. Since $\mathcal{BE}^{Y_0,Q}$ is an iterated $\mathbb{P}^1$-bundle over Y_0, then $\mathcal{BE}^{Y_0,Q}$ is a manifold.

Remark 3.1. Even though (3) looks different from [3, (6.3.5)] (3) is actually a special case. This is because by [3, (6.3.2)] any closed K-orbit Y_0 is isomorphic to a $\mathsf{K} \times_{\mathsf{K}\cap\mathsf{P}} \mathsf{P}/\mathsf{B}$ so the preimage of Y_0 in G can be taken to be $\mathsf{K} \times_{\mathsf{K}\cap\mathsf{P}} \mathsf{P}$. In general, $\mathsf{K} \times_{\mathsf{K}\cap\mathsf{P}} \mathsf{P}/\mathsf{B}$ need not be a closed K-orbit. Description (3) appears in [27, Section 5].

K acts on $\mathcal{BE}^{Y_0,Q}$ by

$$\mathsf{k} \cdot [\mathsf{g}, p_N, \dots, p_1\mathsf{B}] = [\mathsf{kg}, p_N, \dots, p_1\mathsf{B}]. \tag{4}$$

There is a K-equivariant map $\beta : \mathcal{BE}^{Y_0,Q} \to G/B$ given by

$$[\mathsf{g}, p_N, \dots, p_1\mathsf{B}] \xmapsto{\beta} \mathsf{g}p_N \dots p_1\mathsf{B}. \tag{5}$$

Indeed, both the action (4) and the map (5) are well defined, i.e., independent of choice of representative of the equivalence class $[\mathsf{g}, p_N, \dots, p_1]$.

R. W. Richardson and T. A. Springer [38] proved that for any Y, there is a closed orbit Y_0 (possibly nonunique) below it in weak order. That is, there is some $w \in \mathsf{W}$ such that $Y = w \star Y_0$ and $\dim(Y) = \ell(w) + \dim(Y_0)$. Let Y and w be as above and $Q = (j_1, j_2, \dots, j_{\ell(w)})$ be a reduced word for w. Then $\beta : \mathcal{BE}^{Y_0,Q} \to Y$ is generically finite since β is surjective onto Y (by [3, Proposition 6.4]) and since

$$\dim(\mathcal{BE}^{Y_0,Q}) = \dim(Y_0) + \ell(w) = \dim(Y).$$

When $(\mathsf{G}, \mathsf{K}) = (\mathsf{GL}_{p+q}, \mathsf{GL}_p \times \mathsf{GL}_q)$, β is a resolution of singularities for Y, again by [3, Proposition 6.4] (see also [27, Lemma 5.1]). Even for this case, not all

reduced words Q give a resolution of singularities β, although this is true if β is generically one-to-one, that is, under the condition that $\dim(Y) = \ell(w) + \dim(Y_0)$. In general, the image of β is Y and $\beta : \mathcal{BE}^{Y_0,Q} \to Y$ is a resolution of singularities for Y under certain hypotheses [3, Section 6].[1] These hypotheses are sufficiently technical that we do not wish to recall them here since we will not use them.

We now define the Barbasch–Evens–Magyar varieties. See Section 5 for a diagrammatic description in type A.

Definition 3.2 (Barbasch–Evens–Magyar variety). Let $Q = (j_1, j_2, \dots, j_N)$ is a (not necessarily reduced) word. The Barbasch–Evens–Magyar variety is

$$\mathcal{BEM}^{Y_0,Q} := Y_0 \times_{\mathsf{G}/\mathsf{P}_{j_N}} \mathsf{G}/\mathsf{B} \times_{\mathsf{G}/\mathsf{P}_{j_{N-1}}} \cdots \times_{\mathsf{G}/\mathsf{P}_{j_1}} \mathsf{G}/\mathsf{B}. \tag{6}$$

Recall that if $X_1 \xrightarrow{f} Y$ and $X_2 \xrightarrow{g} Y$ are two varieties mapping to the same variety Y, then

$$X_1 \times_Y X_2 = \{(x_1, x_2) \in X_1 \times X_2 \mid f(x_1) = g(x_2)\} \tag{7}$$

denotes the *fiber product*. In (6), each map of (7) is the natural projection $\mathsf{G}/\mathsf{B} \to \mathsf{G}/\mathsf{P}_{j_i}$ defined by $g\mathsf{B} \mapsto g\mathsf{P}_{j_i}$ (or, in the case of Y_0, the restriction of said projection).

Evidently, K acts diagonally on $\mathcal{BEM}^{Y_0,Q}$. Our next theorem asserts that the projection

$$\varphi : \mathcal{BEM}^{Y_0,Q} \to \mathsf{G}/\mathsf{B}, \quad (x_{N+1}, x_N, \dots, x_1) \mapsto x_1 \tag{8}$$

maps into Y. We remark that since we are not assuming that Q is a reduced word, this map may not be generically one-to-one.

Theorem 3.3. *Suppose that $Y = w \star Y_0$ for a closed orbit Y_0 and Q is a (not necessarily reduced) Demazure word for w.*

(I) *$\mathcal{BEM}^{Y_0,Q} \cong \mathcal{BE}^{Y_0,Q}$ as K-varieties.*

(II) *Suppose Y is the closure of the K-orbit $\mathsf{K}g\mathsf{B}$. The fiber of φ over a point of $\mathsf{K}g\mathsf{B}$ of Y is smooth of dimension $\dim(\mathcal{BEM}^{Y_0,Q}) - \dim(Y)$.*

Proof. We prove (I) by a modification of the argument of Magyar in the Schubert setting. The map

$$\phi : \mathcal{BE}^{Y_0,Q} \to Y_0 \times (\mathsf{G}/\mathsf{B})^N, \tag{9}$$

$$[g, p_N, p_{N-1}, \dots, p_1\mathsf{B}] \mapsto (g\mathsf{B}, gp_N\mathsf{B}, gp_N p_{N-1}\mathsf{B}, \dots, gp_N p_{N-1} \dots p_1\mathsf{B}), \tag{10}$$

[1] To construct a resolution of singularities, it is not necessary to take Y_0 to be a closed orbit. We need only take Y_0 to be a smooth orbit closure underneath Y in weak order [27], or take Y_0 to be the closure of a "distinguished" orbit [3]. However, closed orbits are both smooth and distinguished. Taking them as a starting point seems closest in spirit to the construction of the Bott–Samelson resolution.

is well defined (independent of choice of representative), K-equivariant, and since $p_i \in \mathsf{P}_{j_i}$, we have $\phi(\mathcal{BE}^{Y_0,Q}) \subseteq \mathcal{BEM}^{Y_0,Q}$.

ϕ is injective: If $\phi([g, p_N, p_{N-1}, \dots, p_1\mathsf{B}]) = \phi([g', p'_N, p'_{N-1}, \dots, p'_1\mathsf{B}])$, then there exist $b_0, b_1, \dots, b_N \in \mathsf{B}$ such that

$$g = g'b_0, \quad gp_N = g'p'_N b_N, \quad \dots, \quad gp_N \cdots p_1 = g'p'_N \cdots p'_1 b_1.$$

Combining these equations with the definition of $\mathcal{BE}^{Y_0,Q}$ (specifically (2)),

$$\begin{aligned}[g, p_N, p_{N-1}, \dots, p_1\mathsf{B}] &= [g'b_0, b_0^{-1}p'_N b_N, b_N^{-1}p'_{N-1}b_{N-1}, \dots, b_2^{-1}p'_1 b_1\mathsf{B}] \\ &= [g', p'_N, p'_{N-1}, \dots, p'_1\mathsf{B}],\end{aligned}$$

establishing injectivity.

ϕ is surjective: Let $(g\mathsf{B}, g_N\mathsf{B}, g_{N-1}\mathsf{B}, \dots, g_1\mathsf{B}) \in \mathcal{BEM}^{Y_0,Q}$.

Claim 3.4. $[g, g^{-1}g_N, g_N^{-1}g_{N-1}, \dots, g_2^{-1}g_1\mathsf{B}] \in \mathcal{BE}^{Y_0,Q}$.

Proof of Claim. First, by definition $g \in \widetilde{Y}_0$, as desired. Second, by (6) and (7) combined we have

$$g\mathsf{P}_{j_N} = g_N\mathsf{P}_{j_N} \Longrightarrow g^{-1}g_N \in \mathsf{P}_{j_N}.$$

Similarly, in general

$$g_i\mathsf{P}_{j_{i+1}} = g_{i+1}\mathsf{P}_{j_{i+1}} \Longrightarrow g_i^{-1}g_{i+1} \in \mathsf{P}_{j_{i+1}},$$

as required. □

Combining the claim with

$$\phi(g, g^{-1}g_N, g_N^{-1}g_{N-1}, \dots, g_2^{-1}g_1\mathsf{B}) = (g\mathsf{B}, g_N\mathsf{B}, g_{N-1}\mathsf{B}, \dots, g_1\mathsf{B}),$$

we obtain $\phi(\mathcal{BE}^{Y_0,Q}) = \mathcal{BEM}^{Y_0,Q}$.

φ maps into Y: Since β maps into Y, we have

$$\begin{aligned}\beta \circ \phi^{-1}(g\mathsf{B}, g_N\mathsf{B}, g_{N-1}\mathsf{B}, \dots, g_1\mathsf{B}) &= \beta(g, g^{-1}g_N, g_N^{-1}g_{N-1}, \dots, g_2^{-1}g_1\mathsf{B}) \\ &= g_1\mathsf{B} \in Y.\end{aligned}$$

However, by definition $\varphi(g\mathsf{B}, p_N\mathsf{B}, p_{N-1}\mathsf{B}, \dots, p_1\mathsf{B}) = p_1\mathsf{B}$ and so φ maps into Y as well.

Since $\mathcal{BEM}^{Y_0,Q}$ is smooth (and thus normal) and $\mathcal{BE}^{Y_0,Q}$ is irreducible, the bijective morphism (of $\mathbb{C}$-varieties) above is an isomorphism of varieties by Zariski's main theorem (see, e.g., [41, Theorem 5.2.8]).

For (II), we apply:

Theorem 3.5 [21, Corollary 10.7 of Chapter III]. *Let $f : X \to Y$ be a morphism of varieties over an algebraically closed field k of characteristic* 0, *and assume that X is nonsingular. There is a nonempty open subset $V \subset Y$ such that $f : f^{-1}(V) \to V$*

is smooth. In the case in which $f^{-1}(V) \neq \varnothing$, *the fiber* $f^{-1}(v)$ *is nonsingular and* $\dim(f^{-1}(v)) = \dim(X) - \dim(Y)$ *for all* $v \in V$.

Let f be the projection map $\varphi : \mathcal{BEM}^{Y_0,Q} \to Y$. Since $\mathcal{BEM}^{Y_0,Q}$ is nonsingular, by Theorem 3.5 applied to this f, there exists a nonempty $V \subset Y$ such that f restricted to $f^{-1}(V)$ is smooth. If $v \in V$ then said theorem says $\dim(f^{-1}(v))$ is of the desired dimension.

However, we want the above to be true for $p \in \mathsf{K}g\mathsf{B}$. To see this, note that everything said above holds for $f^{-1}(\mathsf{k}V)$ for all $\mathsf{k} \in \mathsf{K}$ since f is K-equivariant and multiplication by k is a smooth morphism. Let $p \in \mathsf{K}g\mathsf{B}$ be a general point. Since $\mathsf{K}p\mathsf{B}$ is dense in Y, $Y \cap \mathsf{k}V \neq \varnothing$ for all $\mathsf{k} \in \mathsf{K}$. Now we can pick k so that $p \in \mathsf{k}V$, completing the argument. □

The generic fibers of part (II) of the theorem may be considered an analogue of the *brick variety* of [13], which is the generic fiber of the Bott–Samelson map (this generic fiber being positive dimensional only when Q is not a reduced word). See *loc. cit.* for a connection to the *brick polytope* of [35; 36] and the associahedron.

Remark 3.6. $\mathcal{BEM}^{Y_0,Q}$ is an iterated $\mathbb{P}^1$-bundle over Y_0.

Let

$$p_{Y_0,Q}(z) = \sum_{k \geq 0} z^k \dim_{\mathbb{Q}} H^{2k}(\mathcal{BEM}^{Y_0,Q}; \mathbb{Q})$$

and

$$r_{Y_0}(z) = \sum_{k \geq 0} z^k \dim_{\mathbb{Q}} H^{2k}(Y_0; \mathbb{Q})$$

be the Poincaré polynomials of $\mathcal{BEM}^{Y_0,Q}$ and Y_0, respectively.

Corollary 3.7. $p_{Y,Q}(z) = r_{Y_0}(z)(1+z)^N$, *where* $r_{Y_0}(z)$ *is known since each closed orbit* Y_0 *is isomorphic to the flag variety of* K.

Proof. In view of Remark 3.6, the claim follows by repeated applications of the Leray–Hirsch theorem. □

Following [27, Section 1.1.2], a *stratification by closed subvarieties* of a variety X is a decomposition $X = \bigcup_\xi \mathcal{S}_\xi$ into closed varieties $\mathcal{S}_\xi$ such that the intersection of any two closed strata is the union of strata. We have a stratification of $\mathcal{BEM}^{Y_0,Q}$ with strata given by subwords P of Q. A *subword* of $Q = (j_1, \ldots, j_N)$ is a list $P = (\beta_1, \ldots, \beta_N)$ such that $\beta_i \in \{-, j_i\}$.

Corollary 3.8 (of Theorem 3.3). $\mathcal{BEM}^{Y_0,Q}$ *is stratified with strata given by subwords* P *of* Q. *The stratum corresponding to a subword* P *is*

$$\mathcal{S}(P) = \{(x_{N+1}, \ldots, x_1) \in \mathcal{BEM}^{Y_0,Q} \mid x_i = x_{i+1} \text{ if } \beta_{N+1-i} = -\}.$$

This stratum is canonically isomorphic to $\mathcal{BEM}^{Y_0,\mathrm{flat}(P)}$ *where* $\mathrm{flat}(P)$ *is the word which deletes all* $-$ *appearing in* P.

Proof. The union of these strata covers $\mathcal{BEM}^{Y_0,Q}$ because $\mathcal{S}(Q) = \mathcal{BEM}^{Y_0,Q}$. For $P = (\beta_1, \dots, \beta_N)$ and $P' = (\beta'_1, \dots, \beta'_N)$ define the subword $P \vee P' = (\gamma_1, \dots, \gamma_N)$, where $\gamma_i = -$ if β_i or β'_i equals $-$. Then

$$\begin{aligned}
&\mathcal{S}(P) \cap \mathcal{S}(P') \\
&\quad = \{(x_{N+1}, \dots, x_1) \in \mathcal{BEM}^{Y_0,Q} \mid x_i = x_{i+1} \text{ if } \beta_{N+1-i} = - \text{ or } \beta'_{N+1-i} = -\} \\
&\quad = \mathcal{S}(P \vee P').
\end{aligned}$$

The isomorphism from $\mathcal{S}(P)$ to $\mathcal{BEM}^{Y_0,\mathrm{flat}(P)}$ is the projection that deletes all components of $\mathcal{S}(P)$ associated to a $-$. □

4. Moment polytopes

The projective space $\mathbb{P}^d$ is a symplectic manifold with *Fubini–Study* symplectic form. Following [40, Section 6.6], consider the restriction of the action of $T^d = (\mathbb{C}^*)^d$ on $\mathbb{P}^d$, given by coordinatewise multiplication, to the compact real subtorus

$$T^d_{\mathbb{R}} = \{(e^{i\theta_1}, \dots, e^{i\theta_d}) \in (\mathbb{C}^*)^d \mid \theta_i \in \mathbb{R} \text{ for all } i\}.$$

As explained in [25, Example 4] the action of $T^d_{\mathbb{R}}$ on $\mathbb{P}^d$ has a moment map, that is, $\mathbb{P}^d$ is a Hamiltonian $T^d_{\mathbb{R}}$-manifold.

Now let X be a smooth algebraic variety with an action of a torus $T \cong (\mathbb{C}^*)^n$ with $n \leq \dim(X)$ and a T-equivariant embedding into $\mathbb{P}^d$. Again, we restrict the T-action to the compact real subtorus $T_{\mathbb{R}}$. Since T is isomorphic to a subgroup of T^d then [25, p. 64, point 1] tells us that $\mathbb{P}^d$ is also a Hamiltonian $T_{\mathbb{R}}$-manifold. Smoothness says X is a T-invariant submanifold of $\mathbb{P}^d$. By [25, p. 64, point 1], it is a Hamiltonian $T_{\mathbb{R}}$-manifold. Hence X has a *moment map*

$$\Phi : X \to \mathfrak{t}^*_{\mathbb{R}},$$

where $\mathfrak{t}^*_{\mathbb{R}} \simeq \mathbb{R}^n$ is the dual of the Lie algebra of $T_{\mathbb{R}}$. There are only finitely many isolated fixed points since the fixed point locus X^T is closed for the Zariski topology. Therefore by [2; 17], the image $\Phi(X)$ is a polytope in $\mathfrak{t}^*_{\mathbb{R}}$; namely, it is the convex hull of the image under Φ of the $T_{\mathbb{R}}$-fixed points. $\Phi(X)$ is known as the *moment polytope* of X. A primer on moment maps which outlines their most important properties, including the ones we will use, can be found in [25, Section 2.2]. From now on, we will omit the subscript $\mathbb{R}$ from T and the Lie algebra.

Moment map images provide a source of polytopes. It is natural to consider $\Phi(BS^Q)$ for T a maximal torus of G, which is the moment polytope of the Bott–Samelson variety BS^Q. To our best knowledge, the first analysis of this polytope in the literature is [13] (an anonymous referee has suggested to us the relevance of the preprint [18] who studies T-equivariant cohomology of Bott–Samelson varieties).

We are interested on the action of $\mathsf{S} := \mathsf{T} \cap \mathsf{K}$, a maximal torus in K, on Y and on its BEM varieties. We will show in Theorem 4.1 that $\Phi(\mathcal{BEM}^{Y_0,Q})$ is the convex hull of certain reflections of $\Phi_{\mathsf{S}}(BS^Q)$, where Φ_{S} denotes the moment map of BS^Q for the S-action. The proof exploits the analogies between the descriptions of the manifolds.

In order to compute $\Phi(\mathcal{BEM}^{Y_0,Q})$ we embed $\mathcal{BEM}^{Y_0,Q}$ into a product of $\mathsf{G}/\mathsf{P}_{\hat{i}}$. To compute $\Phi(\mathcal{BEM}^{Y_0,Q})$ it is not necessary to explicitly embed $\mathcal{BEM}^{Y_0,Q}$ into projective space (via generalized Plücker embeddings followed by the Segre map). This is since the Grassmannian $\mathsf{G}/\mathsf{P}_{\hat{i}}$ is diffeomorphic to the coadjoint orbit $\mathcal{O}_{\omega_i} \cong \mathsf{G}_{\mathbb{R}}/(\mathsf{G}_{\mathbb{R}})_{\omega_i}$ where $\omega_i \in \mathfrak{t}^*$ is a fundamental weight, $\mathsf{G}_{\mathbb{R}}$ is a compact form of G, and $(\mathsf{G}_{\mathbb{R}})_{\omega_i}$ denotes the stabilizer of ω_i under the coadjoint action; see, e.g., [9]. This coadjoint orbit is already a Hamiltonian T-manifold with *Kostant–Kirillov–Souriau* symplectic form and moment map

$$\Phi_i : \mathsf{G}_{\mathbb{R}}/(\mathsf{G}_{\mathbb{R}})_{\omega_i} \to \mathfrak{g}^* \to \mathfrak{t}^* \simeq \mathbb{R}^n, \tag{11}$$

where the first map is $(g\mathsf{P}_{\hat{i}}) \mapsto g\omega_i$ and the second map is the projection induced from the inclusion $T \subset G$. For each g representing an element of the Weyl group of G we have that $\Phi_i(g\omega_i) = g\omega_i$.

Actually, if we embed $\mathcal{BEM}^{Y_0,Q}$ into projective space as indicated above we wouldn't get a different polytope anyway. This is because the Kostant–Kirillov–Souriau form coincides with the pullback of the Fubini–Study form to $\mathsf{G}/\mathsf{P}_{\hat{i}}$ under the T-equivariant embedding given by the line bundle $\mathcal{L}(\omega_i)$; see [9, Remark 3.5].

Thinking of the fundamental weights $\omega_i \in \mathfrak{t}^*$ as functions $\omega_i : \mathfrak{t} \to \mathbb{R}$, $\omega_i|_{\mathfrak{s}}$ is the restriction of ω_i to $\mathfrak{s} \subset \mathfrak{t}$.

Theorem 4.1. *$\mathcal{BEM}^{Y_0,Q}$ has an embedding into a product of G/P with P maximal as a symplectic submanifold of this product with Hamiltonian S-action; the corresponding moment polytope is*

$$\begin{aligned}
&\Phi(\mathcal{BEM}^{Y_0,Q}) \\
&\quad = \operatorname{conv}\left\{ x\left(\sum_{i=1}^{r} \omega_i|_{\mathfrak{s}} + \sum_{i=|Q|}^{1} s_{j_{|Q|}} \cdots s_{j_i} \omega_i|_{\mathfrak{s}} \right) \mid x \in Y_0^{\mathsf{S}} \text{ and } (j_1, \ldots, j_{|Q|}) \subseteq Q \right\} \\
&\quad = \operatorname{conv}\{x \cdot \Phi_{\mathsf{S}}(BS^Q) \mid x \in Y_0^{\mathsf{S}}\}.
\end{aligned}$$

Proof. $\mathcal{BEM}^{Y_0,Q}$ embeds into a product of G/P with P maximal, as follows:

Proposition 4.2. *The following map is an embedding*:

$$\delta : \mathcal{BEM}^{Y_0,Q} \hookrightarrow \prod_{i=1}^{r} \mathsf{G}/\mathsf{P}_{\hat{i}} \times \prod_{j=1}^{|Q|} \mathsf{G}/\mathsf{P}_{\hat{i}_{|Q|-j+1}},$$

$$(x\mathsf{B}, g_{|Q|}\mathsf{B}, \ldots, g_1\mathsf{B}) \mapsto (x\mathsf{P}_{\hat{1}}, \ldots, x\mathsf{P}_{\hat{r}}, g_{|Q|}\mathsf{P}_{\hat{i}_{|Q|}}, g_{|Q|-1}\mathsf{P}_{\hat{i}_{|Q|-1}}, \ldots, g_1\mathsf{P}_{\hat{i}_1}).$$

By [43, 9.2.2] the fibered product with a closed subscheme is a closed subscheme of the fibered product. From this it immediately follows that BEM is a closed subscheme of a fibered product of G/B's. It is well known that this fibered product embeds into the desired product of $\mathsf{G}/\mathsf{P}_{\hat{i}}$. Thus the proposition follows. To be more explicit, we include the proof below.

Proof. First to see that δ is injective suppose

$$\delta(x\mathsf{B}, a_{|Q|}\mathsf{B}, \dots, a_1\mathsf{B}) = \delta(y\mathsf{B}, b_{|Q|}\mathsf{B}, \dots, b_1\mathsf{B}),$$

then $x\mathsf{P}_{\hat{i}} = y\mathsf{P}_{\hat{i}}$ for $1 \le i \le r$ and therefore $y^{-1}x \in \bigcap_{i=1}^{r} \mathsf{P}_{\hat{i}} = \mathsf{B}$. Thus, $x\mathsf{B} = y\mathsf{B}$. Next, the assumption

$$a_{|Q|}\mathsf{P}_{\widehat{i_{|Q|}}} = b_{|Q|}\mathsf{P}_{\widehat{i_{|Q|}}} \Longrightarrow b_{|Q|}^{-1}a_{|Q|} \in \mathsf{P}_{\widehat{i_{|Q|}}}. \tag{12}$$

Also, using the definition of $\mathcal{BEM}^{Y_0,Q}$ (7),

$$a_{|Q|}\mathsf{P}_{i_{|Q|}} = x\mathsf{P}_{i_{|Q|}} = y\mathsf{P}_{i_{|Q|}} = b_{|Q|}\mathsf{P}_{i_{|Q|}} \Longrightarrow b_{|Q|}^{-1}a_{|Q|} \in \mathsf{P}_{i_{|Q|}}. \tag{13}$$

Combining (12) and (13) gives

$$b_{|Q|}^{-1}a_{|Q|} \in \mathsf{P}_{\widehat{i_{|Q|}}} \cap \mathsf{P}_{i_{|Q|}} = \mathsf{B} \Longrightarrow a_{|Q|}\mathsf{B} = b_{|Q|}\mathsf{B}.$$

Reasoning similarly, we see that $a_k\mathsf{B} = b_k\mathsf{B}$ for all $k = |Q|-1, |Q|-2, \dots, 1$, as required. Thus δ is injective.

It is well known that the map

$$\mathsf{G}/\mathsf{B} \to \prod_{i=1}^{r} \mathsf{G}/\mathsf{P}_{\hat{i}} : x\mathsf{B} \mapsto (x\mathsf{P}_{\hat{1}}, \dots, x\mathsf{P}_{\hat{r}})$$

is an embedding of algebraic varieties. Consequently, the map

$$\kappa : (\mathsf{G}/\mathsf{B})^{|Q|+1} \hookrightarrow \prod_{m=1}^{|Q|+1} \prod_{i=1}^{r} \mathsf{G}/\mathsf{P}_{\hat{i}},$$

$$(x_{|Q|+1}\mathsf{B}, x_{|Q|}\mathsf{B}, \dots, x_1\mathsf{B}) \mapsto ((x_{|Q|+1}\mathsf{P}_{\hat{1}}, \dots, x_{|Q|+1}\mathsf{P}_{\hat{r}}), \dots, (x_1\mathsf{P}_{\hat{1}}, \dots, x_1\mathsf{P}_{\hat{r}}))$$

is also an embedding. Let $Q = (q_1, q_2, \dots, q_N)$. The image of $\mathcal{BEM}^{Y_0,Q} \subset (\mathsf{G}/\mathsf{B})^{|Q|+1}$ under κ satisfies

$$x_m\mathsf{P}_{\hat{i}} = x_{m+1}\mathsf{P}_{\hat{i}}, \tag{14}$$

whenever $i \neq q_m$ for $m = 1, 2, \dots, |Q|$. Thus δ factors:

$$\begin{tikzcd}
\mathcal{BEM}^{Y_0,Q} \arrow[r, "\kappa"] \arrow[dr, "\delta"'] & \prod_{m=1}^{|Q+1|} \prod_{i=1}^{r} \mathsf{G}/\mathsf{P}_{\hat{i}} \arrow[d, "\psi"] \\
& \prod_{i=1}^{r} \mathsf{G}/\mathsf{P}_{\hat{i}} \times \prod_{j=1}^{|Q|} \mathsf{G}/\mathsf{P}_{\widehat{i_{|Q|-j+1}}}
\end{tikzcd}$$

where ψ is the projection that forgets the repetitions of (14). Thus, δ is an embedding. $\square$

$Gr := \prod_{i=1}^{r} \mathsf{G}/\mathsf{P}_{\hat{i}} \times \prod_{j=1}^{|Q|} \mathsf{G}/\mathsf{P}_{\widehat{i_{|Q|-j+1}}}$ is naturally a symplectic manifold, and is Hamiltonian with respect to the (diagonal) action of T. By [25, p. 64, point 1] the same is true for this action restricted to the subtorus S. As a submanifold of Gr, $\mathcal{BEM}^{Y_0,Q}$ is also symplectic, and is clearly stable under the S-action. From this it follows (compare [25, p. 65, point 4]) that the S-action is Hamiltonian, whence $\mathcal{BEM}^{Y_0,Q}$ has a moment map Φ. Then one sees from [25, p. 64, point 1 and p. 65, point 3] that Φ is given by

$$\mathcal{BEM}^{Y_0,Q} \hookrightarrow Gr \xrightarrow{\sum \Phi_i} \mathfrak{t}^* \to \mathfrak{s}^*, \tag{15}$$

where $\Phi_i : \mathsf{G}/\mathsf{P}_{\hat{i}} \to \mathfrak{t}^*$ is the moment map for $\mathsf{G}/\mathsf{P}_{\hat{i}}$ and $\mathfrak{t}^* \to \mathfrak{s}^*$ is induced from the inclusion $\mathsf{S} \subset \mathsf{T}$. The second map restricts functions $\mathfrak{t} \to \mathbb{R}$ to $\mathfrak{s}$. Therefore, by (11) and (15) combined, the moment map $\Phi : \mathcal{BEM}^{Y_0,Q} \to \mathfrak{s}^*$ is given by

$$(x\mathsf{B}, g_{|Q|}\mathsf{B}, \dots, g_1\mathsf{B}) \longmapsto \sum_{i=1}^{r} x\omega_i|_{\mathfrak{s}} + \sum_{i=1}^{|Q|} g_i\omega_i|_{\mathfrak{s}}. \tag{16}$$

Proposition 4.3 (S-fixed points of $\mathcal{BEM}^{Y_0,Q}$). *The S-fixed points of $\mathcal{BEM}^{Y_0,Q}$ are indexed by pairs $(x\mathsf{B}, J)$, where $x\mathsf{B}$ is a T-fixed point of G/B contained in Y_0, and $J = (\beta_1, \dots, \beta_{|Q|})$ is a subword of Q. Indeed, the fixed points are precisely the points*

$$p_{(x\mathsf{B},J)} := (x\mathsf{B}, xs_{\beta_{|Q|}}\mathsf{B}, xs_{\beta_{|Q|}}s_{\beta_{|Q|-1}}\mathsf{B}, \dots, xs_{\beta_{|Q|}} \cdots s_{\beta_1}\mathsf{B}) \in \mathcal{BEM}^{Y_0,Q}, \tag{17}$$

where s_{β_i} is the identity if $\beta_i = -$.

Proof. We first verify that $p_{(x\mathsf{B},J)} \in \mathcal{BEM}^{Y_0,Q}$. Note that for $i = 1, \dots, |Q|$, since by (1), $\mathsf{B}s_{\beta_i}\mathsf{B} \in \mathsf{P}_{j_i}$, in particular $s_{\beta_i} \in \mathsf{P}_{j_i}$ and hence

$$xs_{\beta_{|Q|}} \cdots s_{\beta_i}\mathsf{P}_{\beta_i} = xs_{\beta_{|Q|}} \cdots s_{\beta_{i-1}}\mathsf{P}_{\beta_i}.$$

Therefore $(xs_{\beta_{|Q|}} \cdots s_{\beta_{i-1}}, xs_{\beta_{|Q|}} \cdots s_{\beta_i})$ satisfies (7) for $i = 1, \dots, |Q|$, as needed.

Since

$$(\mathsf{G}/\mathsf{B})^{\mathsf{S}} = (\mathsf{G}/\mathsf{B})^{\mathsf{T}} \tag{18}$$

(see [7, p. 128]), the S-fixed points of Y_0 are of the form $x\mathsf{B}$ where $x \in N_{\mathsf{G}}(\mathsf{T})$. Therefore,

$$xs_{\beta_{|Q|}} \cdots s_{\beta_i} \in N_{\mathsf{G}}(\mathsf{T}). \tag{19}$$

Moreover, since $\mathsf{S} \subset \mathsf{T}$, for $\mathsf{t} \in \mathsf{S}$ we have by (19) that

$$\mathsf{t} \cdot xs_{\beta_{|Q|}} \cdots s_{\beta_i}\mathsf{B} = xs_{\beta_{|Q|}} \cdots s_{\beta_i}\mathsf{B},$$

so $p_{(x\mathsf{B},J)}$ is an S-fixed point.

Conversely, suppose $(x_{|Q|+1}\mathsf{B}, x_{|Q|}\mathsf{B}, \ldots, x_1\mathsf{B})$ is an S-fixed point of $\mathcal{BEM}^{Y_0,Q}$. Clearly, $x_{|Q|+1}\mathsf{B} \in (Y_0)^{\mathsf{S}}$. By (18), each $x_i\mathsf{B}$ is a T-fixed point so we may assume

$$x_i \in N_{\mathsf{G}}(\mathsf{T}). \tag{20}$$

By the definition (7) of $\mathcal{BEM}^{Y_0,Q}$, $x_i\mathsf{P}_{j_i} = x_{i-1}\mathsf{P}_{j_i}$. Thus, $x_{i-1}^{-1}x_i \in \mathsf{P}_{j_i}$. Hence, in view of (20) we may further assume that $x_{i-1}^{-1}x_i \in \{id, s_{\alpha_{j_i}}\}$. Therefore

$$(x_{|Q|+1}\mathsf{B}, x_{|Q|}\mathsf{B}, \ldots, x_1\mathsf{B})$$

is of the form $p_{(x\mathsf{B},J)}$, as asserted. □

Since $\Phi(\mathcal{BEM}^{Y_0,Q})$ is the convex hull of Φ applied to this set of points, the first equality of the theorem holds by Proposition 4.3 combined with (16).

Similar arguments [13] show the moment polytope of a Bott–Samelson variety is

$$\begin{aligned} &\Phi(BS^Q) \\ &\quad = \mathsf{conv}\left\{\sum_{i=1}^{r}\omega_i + \sum_{i=|Q|}^{1} s_{\beta_{|Q|}} \cdots s_{\beta_i}\omega_i \mid (\beta_1, \ldots, \beta_{|Q|}) \text{ is a subword of } Q\right\}. \end{aligned} \tag{21}$$

The second equality follows by restricting the weights to $\mathfrak{s}$. □

Define the *BEM polytope* $\mathcal{P}_{Y_0,Q}$ as $\Phi(\mathcal{BEM}^{Y_0,Q})$. In Section 6 we study $\mathcal{P}_{Y_0,Q}$ for the symmetric pair $\mathsf{G} = \mathsf{GL}_{p+q}$ and $\mathsf{K} = \mathsf{GL}_p \times \mathsf{GL}_q$.

We remark it would be interesting to study the polytopes coming from the K-action on $\mathcal{BEM}^{Y_0,Q}$. $\mathcal{BEM}^{Y_0,Q}$ is a Hamiltonian K-manifold and therefore has a moment map Φ_{K}. Two polytopes associated with the image of Φ_{K} are described in [25, Section 2.5]. One of these is the intersection of the image of Φ_{K} with the positive Weyl chamber. Kirwan's noncommutative convexity theorem [24] states that this intersection is a polytope.

5. The Barbasch–Evens–Magyar varieties in type A

In this section we describe the K-orbits and Barbasch–Evens–Magyar varieties for the case of symmetric pairs (G, K) where G is a general linear group. The three possible pairs are $(\mathsf{GL}_{p+q}, \mathsf{GL}_p \times \mathsf{GL}_q)$, $(\mathsf{GL}_{2n}, \mathsf{Sp}_{2n})$, and $(\mathsf{GL}_n, \mathsf{O}_n)$. All of these symmetric subgroups are connected; see, e.g., [48].

When $\mathsf{G} = \mathsf{GL}_n$ we take the simple roots to be $\Delta = \{\alpha_i = \vec{e}_i - \vec{e}_{i+1} \mid 1 \le i \le n-1\}$, where $\vec{e}_i \in \mathbb{R}^n$ is the standard basis vector. With this choice of root system embedding, we may identify the fundamental weight ω_i with the vector $\sum_{k=1}^{i} \vec{e}_i$. $\mathsf{W} = \mathfrak{S}_n$ is identified with the symmetric group of permutations on $\{1, 2, \dots, n\}$. Thus, s_i is the simple transposition interchanging i and $i+1$.

K-*orbits of type A.* The running example here is $(\mathsf{G}, \mathsf{K}) = (\mathsf{GL}_{p+q}, \mathsf{GL}_p \times \mathsf{GL}_q)$. Let $n = p+q$, and consider the involution θ of $\mathsf{G} = \mathsf{GL}_n$ defined by conjugation using the diagonal matrix having p-many 1's followed by q-many -1's. Then $\mathsf{K} = \mathsf{G}^\theta \cong \mathsf{GL}_p \times \mathsf{GL}_q$, embedded as block diagonal matrices with an upper-left invertible $p \times p$ block, a lower-right invertible $q \times q$ block, and zeros outside of these blocks.

The orbits in this case are parametrized by (p, q)-clans [31; 51], which we now describe. A (p, q)-*indication* is a string of characters $\gamma = c_1 \dots c_n$ with each $c_i \in \{+, -\} \cup \mathbb{Z}_{>0}$ and such that

- if $c_i \in \mathbb{Z}_{>0}$, then there is a unique $j \neq i$ such that $c_i = c_j$; and
- $\#\{i \mid c_i = +\} - \#\{i \mid c_i = -\} = p - q$.

Consider the equivalence relation between indications given by $\gamma \sim \gamma'$ if and only if

- $c_i = c_i'$ whenever $c_i \in \{+, -\}$; and
- there exists a bijection $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $c_i = f(c_i')$ for all i with $c_i \in \mathbb{Z}_{>0}$.

The (p, q)-*clans* are the equivalence classes of this equivalence relation. By slight abuse of notation, we use the same notation for indications to denote clans; for example $1 + -1$ is the same clan as $2 + -2$. Let $\mathtt{Clans}_{p,q}$ be the set of (p, q)-clans. The closed orbits are indexed by *matchless clans*, i.e., clans using only $+, -$. Lemma 2.1 implies these closed orbits are isomorphic to $(\mathsf{GL}_p/\mathsf{B}) \times (\mathsf{GL}_q/\mathsf{B})$, the product of two flag varieties.

We briefly remark that each (p, q)-clan corresponds to an involution in $\mathfrak{S}_{p+q}$. Indeed, given a clan γ we obtain an involution w by letting $w_i = i$ whenever $c_i \in \{+, -\}$ and $w_i = j$ whenever $c_i = c_j \in \mathbb{Z}_{>0}$.

Next, we explicitly describe the orbit closures Y_γ. Fix $\gamma = c_1 \dots c_n \in \mathtt{Clans}_{p,q}$. For $i = 1, \dots, n$, define:

- $\gamma(i; +) = \#\{(j, k) \mid c_j = c_k \in \mathbb{Z}_{>0},\ 1 \le j < k \le i\} + \#\{j \mid c_j = +,\ 1 \le j \le i\}$; and
- $\gamma(i; -) = \#\{(j, k) \mid c_j = c_k \in \mathbb{Z}_{>0},\ 1 \le j < k \le i\} + \#\{j \mid c_j = -,\ 1 \le j \le i\}$.

For $1 \le i < j \le n$, define

- $\gamma(i; j) = \#\{k \in [1, i] \mid c_k = c_\ell \in \mathbb{N} \text{ with } \ell > j\}$.

Let $E_p = \mathrm{span}\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_p\}$ be the span of the first p standard basis vectors, and let $E^q = \mathrm{span}\{\vec{e}_{p+1}, \vec{e}_{p+2}, \dots, \vec{e}_n\}$ be the span of the last q standard basis vectors. Let $\rho : \mathbb{C}^n \to E_p$ be the projection map onto the subspace E_p.

Suppose $\gamma \in \mathtt{Clans}_{p,q}$ and $\epsilon \in \mathtt{Clans}_{r,s}$. Then $\epsilon = \epsilon_1 \dots \epsilon_{r+s}$ (*pattern*) *avoids* $\gamma = \gamma_1 \dots \gamma_{p+q}$ if there are no indices $i_1 < i_2 < \dots < i_{p+q}$ such that:

(1) if $\gamma_j = \pm$ then $\epsilon_{i_j} = \gamma_j$; and

(2) if $\gamma_k = \gamma_\ell$ then $\epsilon_{i_k} = \epsilon_{i_\ell}$.

A clan γ is *noncrossing* if γ avoids 1212.

Theorem 5.1 [50, Corollary 1.3; 49, Remark 3.9]. *The* K*-orbit closure Y_γ is the set of flags $F_\bullet = (V_1, \dots, V_n)$ such that*:

(1) $\dim(V_i \cap E_p) \geq \gamma(i; +)$ *for all* i.

(2) $\dim(V_i \cap E^q) \geq \gamma(i; -)$ *for all* i.

(3) $\dim(\rho(V_i) + V_j) \leq j + \gamma(i; j)$ *for all* $i < j$.

If γ is noncrossing, the third condition is redundant.

Example 5.2. Let $p = q = 2$ and $\gamma = 1 + -1$ (a noncrossing clan). In fact, $Y_\gamma = s_1 \star s_3 \star s_2 \star Y_{++--}$ and

$$Y_\gamma = \{(V_1, V_2, V_3) \in \mathsf{Gr}(1,4) \times \mathsf{Gr}(2,4) \times \mathsf{Gr}(3,4) \mid \dim(V_2 \cap E_2) \geq 1, \dim(V_3 \cap E^2) \geq 1\}.$$

A projectivized depiction of a general point in this orbit closure is given in Figure 1 (left). The blue and red lines represent E_2 and E^2 respectively. The moving flag (V_1, V_2, V_3) is the (black point, black line, front face). □

W. McGovern characterized the singular orbit closures:

Theorem 5.3 [32]. *Y_γ is smooth if and only if γ avoids the patterns* $1+-1$, $1-+1$, 1212, $1+221$, $1-221$, $122+1$, $122-1$, 122331.

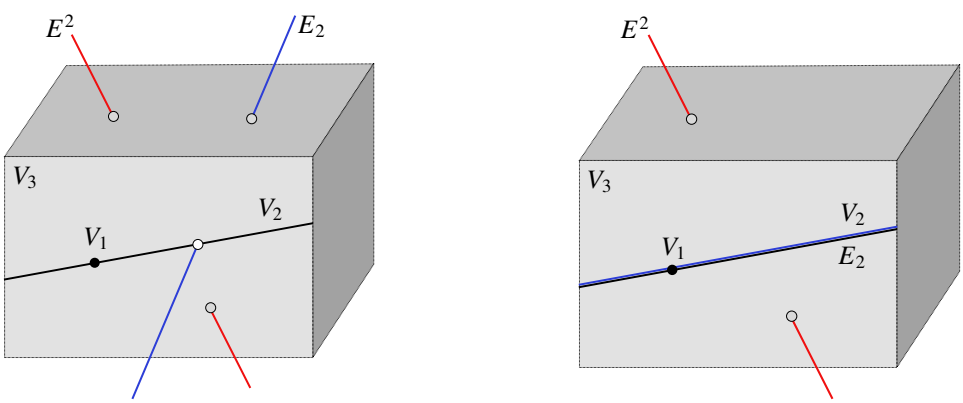

Figure 1. Y_{1+-1} and Y_{++--}.

Example 5.4. By Theorem 5.3, Y_{1+-1} is singular. One computes (e.g, using the methods of [46]) that the singular locus is the closed orbit Y_{++--} where $V_2 = E_2$ (the black and blue lines agree). In Figure 1 (left), the general picture of Y_{1+-1}, the black line V_2 has three degrees of freedom to move. Now consider the picture of Y_{++--} (Figure 1, right). Pick any point of the blue line E_2. Then the black line V_2 has two degrees of freedom to pivot and remain inside Y_{1+-1}. This is true of any other point as well. Informally, this additional degree of freedom is singular behavior. □

Barbasch–Evens–Magyar varieties of type A. In Section 3 we gave our general definition of the Barbasch–Evens–Magyar varieties. We now describe, using diagrams, these configuration spaces for the special case of symmetric pairs (G, K) where G is a general linear group.

Let $Q = (j_1, j_2, \ldots, j_N)$ be a word and $\mathsf{F} \subseteq \mathsf{G}/\mathsf{B}$. We first define the configuration space $\mathcal{C}(Q, \mathsf{F})$. Informally, a point of $\mathcal{C}(Q, \mathsf{F})$ is a collection of vector spaces forming a diagram. Figure 2 is an example for $\mathsf{G} = \mathsf{GL}_3$, $\mathsf{F} = \mathsf{G}/\mathsf{B}$, and $Q = (2, 1, 2, 1)$. The edges indicate containments among the vector spaces. For instance, we have $\mathbb{C}^0 \subset F_1 \subset F_2 \subset \mathbb{C}^3$, as well as $\mathbb{C}^0 \subset V_2 \subset V_1 \subset \mathbb{C}^3$, etc. We require that the flag $\mathbb{C}^0 \subset F_1 \subset F_2 \subset \mathbb{C}^3$ be in F.

To be precise, to define $\mathcal{C}(Q, \mathsf{F})$ we start with a vertical chain whose $n+1$ vertices are labeled by the vector spaces $\mathbb{C}^0, F_1, F_2, \ldots F_{n-1}, \mathbb{C}^n$, from south to north, such that the corresponding flag is an element of F. The *dimension* of a vertex is the dimension of the labeling vector space. At the start, this chain is declared to be the *right border of the diagram*.

We now grow the diagram as follows. Consider the last letter j_N of Q. Introduce a new vertex on the right of the diagram, labeled by V_N of dimension j_N with edges between the vertices of dimension $j_N - 1$ and $j_N + 1$ (thus indicating the containment relation $F_{j_N - 1} \subset V_N \subset F_{j_N + 1}$). We modify the current right border by replacing the vertex of the current right border of dimension j_N with the new vertex labeled by V_N. Now repeat successively with $j_{N-1}, j_{N-2}, \ldots j_2, j_1$. At step k, a

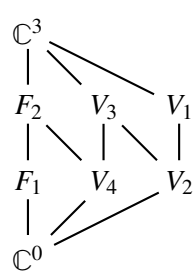

Figure 2. $\mathsf{G} = \mathsf{GL}_3$, $\mathsf{F} = \mathsf{G}/\mathsf{B}$, and $Q = (2, 1, 2, 1)$.

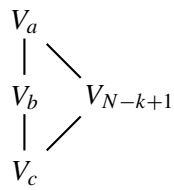

Figure 3. Quadrangle corresponding to the letter j_{N-k+1} of Q.

new vertex labeled by V_{N-k+1} is added to the right of the right border, of dimension j_{N-k+1}, and becomes the new member of the right border, replacing the unique vertex of dimension j_{N-k+1} of the current right border. Note that the letter j_{N-k+1} of Q corresponds to the *quadrangle* in Figure 3 where V_a, V_b, and V_c lie on the right border before V_{N-k+1} is added, and $\dim(V_b) = j_{N-k+1}$.

Finally, a point in $\mathcal{C}(Q, \mathsf{F})$ is a collection of vector spaces arranged in the diagram described, where the dimension of a vector space equals the length of a shortest path to $\mathbb{C}^0$ and $V_i \subset V_j$ whenever there is an upward edge $V_i - V_j$. For the example above,

$$\begin{aligned}\mathcal{C}(Q, \mathsf{F}) = \{(F_1, F_2, V_1, \dots, V_4) \mid F_1, V_4 \subset F_2,\ V_4, V_2 \subset V_3,\ V_2 \subset V_1\}\\ \subset \mathsf{Gr}(1,3) \times \mathsf{Gr}(2,3) \times \mathsf{Gr}(2,3) \times \mathsf{Gr}(1,3) \times \mathsf{Gr}(2,3) \times \mathsf{Gr}(1,3).\end{aligned}$$

The above diagram extends the configuration space used in [29] to construct the Bott–Samelson variety. The difference is that [29] takes the initial chain to be a B or GL_n-orbit, while here we take any subset F. For Bott–Samelson varieties, the initial chain corresponds to a point (usually the standard basis flag).

The following result interprets the $\mathsf{G} = \mathsf{GL}_n$ case of Theorem 3.3(I):

Theorem 5.5. *The configuration space $\mathcal{C}(Q, Y_0)$ is the image of $\mathcal{BEM}^{Y_0,Q}$ under the map δ of Proposition 4.2. Therefore, $\mathcal{C}(Q, Y_0)$ is isomorphic as a K-variety to $\mathcal{BE}^{Y_0,Q}$.*

Proof. In type A, the map δ may be interpreted as listing the vector spaces on the flags of successive right borders of the diagram for $\mathcal{BEM}^{Y_0,Q}$, but avoiding redundancy by listing only the additional new vector space introduced at each step. Thus the first part of the theorem follows. The isomorphism between $\mathcal{BE}^{Y_0,Q}$ and $\mathcal{C}(Q, Y_0)$ is the composition of the map in Theorem 3.3 and with δ. □

Definition 5.6 (Barbasch–Evens–Magyar variety for the symmetric pairs $(\mathsf{GL}_n, \mathsf{K})$). Suppose that $Q = (j_1, j_2, \dots, j_N)$ is a (not necessarily reduced) word and Y_0 is a closed K-orbit. Then we abuse notation and denote $\mathcal{C}(Q, Y_0)$ by $\mathcal{BEM}^{Y_0,Q}$.

Consider the map from $\mathcal{BEM}^{Y_0,Q}$ to G/B that maps a point in the configuration space to the rightmost flag (corresponding to the rightmost border) in the diagram.

For example, the point of $\mathcal{BEM}^{Y_0,Q}$ depicted by the example diagram above maps to the flag $\mathbb{C}^0 \subset V_2 \subset V_1 \subset \mathbb{C}^3$. The image of this map is a K-orbit closure. Moreover, every K-orbit closure is the image of such a map for some $\mathcal{BEM}^{Y_0,Q}$. In fact, this map agrees with the map φ defined in (8).

To complete the description of $\mathcal{BEM}^{Y_0,Q}$ for the three type A cases we require a description of the flags in the closed orbit Y_0, i.e., which flags may occur on the left-hand side of the diagram.

In the case $(\mathsf{G},\mathsf{K}) = (\mathsf{GL}_{p+q}, \mathsf{GL}_p \times \mathsf{GL}_q)$ the closed orbits are indexed by matchless clans, i.e., γ consists of p $+$'s and q $-$'s. The description of these orbits is given by Theorem 5.1. Since matchless clans are clearly noncrossing, the third condition is redundant. In the case $(\mathsf{G},\mathsf{K}) = (\mathsf{GL}_{2n}, \mathsf{Sp}_{2n})$, there is a unique closed orbit Y_0 [47, Proposition 2.4.1]. This closed orbit is isomorphic to the flag variety for $\mathsf{K} = \mathsf{Sp}_{2n}$ by Lemma 2.1. In the case of $(\mathsf{G},\mathsf{K}) = (\mathsf{GL}_n, \mathsf{O}_n)$ with n there is a unique closed orbit Y_0 [47, Proposition 2.2.1 and Remark 2.3.3]. Again, this orbit is isomorphic to the flag variety for O_n. For sake of brevity, we refer the reader to [48, Section 2] for a linear algebraic description of the points of the closed orbits in these cases.

Example 5.7. The diagram for $\mathcal{BEM}^{Y_0,Q}$ where $Y_0 = Y_{++--}$ and $Q = (1,3,2)$ is

$$\begin{tikzcd}
 & \mathbb{C}^4 \arrow[d, dash] \arrow[dr, dash] & \\
 & F_3 \arrow[d, dash] \arrow[dr, dash] & V_2 \arrow[d, dash] \\
E_2 = F_2 \arrow[r, dash] & F_2 \arrow[d, dash] & V_3 \arrow[d, dash] \\
 & F_1 \arrow[d, dash] \arrow[ur, dash] & V_1 \arrow[dl, dash] \\
 & \mathbb{C}^0 &
\end{tikzcd}$$

The depiction of this variety is given in Figure 4. Here V_1, V_3, V_2 are given by the (projectivized) green point, line and plane respectively. The green spaces have the same incidence relations as the moving (black) flag in Y_{1+-1}. Thus, the projection forgetting all except the green spaces maps to Y_{1+-1}. $\square$

Example 5.8. Let us compute the Poincaré polynomial $p_{Y_0,Q}(z)$ of $\mathcal{BEM}^{Y_0,Q}$ when $\mathsf{K} = \mathsf{GL}_p \times \mathsf{GL}_q$. Recall that Corollary 3.7 says that $p_{Y_0,Q}(z) = r_{Y_0}(z)(1+z)^N$, where N is the length of Q. Let $[n]_z! = [1]_z[2]_z \cdots [n]_z$ where $[i]_z = 1+z+z^2+\cdots+z^{i-1}$. The Poincaré polynomial of GL_n/B is $[n]_z!$. We now have $r_{Y_0}(z) = [p]_z!\,[q]_z!$ for any choice of closed orbit Y_0. This is since by Lemma 2.1, we have $Y_0 \cong (\mathsf{GL}_p/\mathsf{B}) \times (\mathsf{GL}_q/\mathsf{B})$, combined with the Künneth formula.

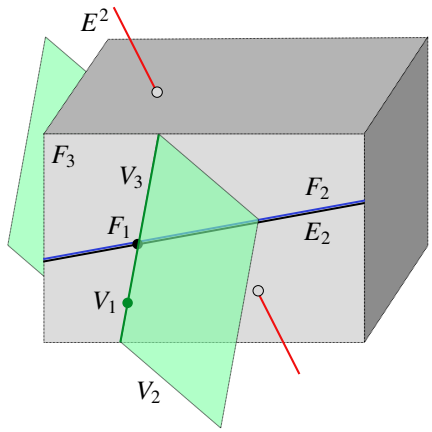

Figure 4. $\mathcal{BEM}^{Y_{++--},(1,3,2)}$: the green flag maps to Y_{1+-1}.

The standard maximal torus $\mathsf{T} \cong (\mathbb{C}^*)^n$ in GL_n consists of invertible diagonal matrices. There is a natural K-action on $\mathcal{BEM}^{Y_0,Q}$, described in Section 3, which induces an S-action, where $\mathsf{S} = \mathsf{T} \cap \mathsf{K}$. Let us describe this action in the present setting. A matrix in K acts on the Grassmannian of m-dimensional subspaces of $\mathbb{C}^n$ by change of basis. We extend this to an action of K on $\mathcal{BEM}^{Y_0,Q}$ diagonally:

$$\mathsf{k}\cdot(F_1, F_2, \ldots, F_{n-1}, V_1, V_2, \ldots V_N) = (\mathsf{k}\cdot F_1, \mathsf{k}\cdot F_2, \ldots \mathsf{k}\cdot F_{n-1}, \mathsf{k}\cdot V_1, \mathsf{k}\cdot V_2, \ldots, \mathsf{k}\cdot V_N),$$

where $\mathsf{k} \in \mathsf{K}$.

We study the moment polytope of $\mathcal{BEM}^{Y_0,Q}$ for the pair $(\mathsf{GL}_{p+q}, \mathsf{GL}_p \times \mathsf{GL}_q)$ in Section 6. To do so, we utilize the following concrete description of the S-fixed points of $\mathcal{BEM}^{Y_0,Q}$ for this symmetric pair. Notice that in this case $\mathsf{S} = \mathsf{T}$. A *subword* of $Q = (j_1, \ldots, j_N)$ is a list $P = (\beta_1, \ldots, \beta_N)$ such that $\beta_i \in \{-, j_i\}$. As we saw above, each letter of Q corresponds to a quadrangle of the associated diagram. A subword P corresponds to a subset of these quadrangles. Concretely, for each quadrangle in the set, require the two vertices associated to vector spaces of equal dimension to be the same space. For each quadrangle not in the set, insist those same vector spaces be different. Call such an assignment given a left border associated to a flag $F_\bullet$ a *P-growth of $F_\bullet$*.

Given a matchless clan γ, a permutation $\sigma \in \mathfrak{S}_{p+q}$ is *γ-shuffled* if it assigns

- $1, 2, \ldots, p$ in any order to the $+$'s;
- $p+1, p+2, \ldots, n$ in any order to the $-$'s.

Hence there are $p!q!$ such permutations (independent of γ).

Associated to any γ-shuffled permutation define $F_\bullet^{\gamma,\sigma}$ to be the σ-permuted coordinate flag, i.e., the one whose d-dimensional subspace is $\langle \vec{e}_{\sigma(1)}, \ldots \vec{e}_{\sigma(d)} \rangle$.

We will use this result, due to A. Yamamoto:

Proposition 5.9 [51]. *The* S*-fixed points of* Y_γ *are flags* $F_\bullet^{\gamma,\sigma}$ *where* $\sigma \in \mathfrak{S}_{p+q}$ *is* γ*-shuffled.*

Proposition 5.10 (S-fixed points of $\mathcal{BEM}^{Y_0,Q}$ for $(\mathsf{GL}_{p+q}, \mathsf{GL}_p \times \mathsf{GL}_q)$). *The set of* S*-fixed points of* $\mathcal{BEM}^{Y_0,Q}$ *correspond to* P*-growth diagrams whose initial vertical chain is* $F_\bullet^{\gamma,\sigma}$ *(where* $Y = Y_\gamma$*).*

Proof. The following is straightforward:

Claim 5.11. *Fix a coordinate flag* $F_\bullet$ *for the initial vertical chain, i.e.,* $F_\bullet \in (\mathsf{G}/\mathsf{B})^{\mathsf{T}}$*. There exists exactly one* P*-growth of* $F_\bullet$ *which uses only subspaces of the form* $\langle \vec{e}_{i_1}, \dots, \vec{e}_{i_d} \rangle$*.*

Clearly any such P-growth of $F_\bullet$ is an S-fixed point of $\mathcal{BEM}^{Y_0,Q}$. Conversely, consider any S-fixed point of $\mathcal{BEM}^{Y_0,Q}$ and its corresponding diagram. The left border is an S-fixed point of $Y = Y_\gamma$. The result then holds by Proposition 5.9 together with Claim 5.11. □

Proposition 4.3 gives a general form of Proposition 5.10.

Corollary 5.12. $$\#(\mathcal{BEM}^{Y_0,Q})^{\mathsf{S}} = p!\,q!\,2^{|Q|}.$$

Similar descriptions for the torus fixed points can be given for the other two symmetric pairs of the form $(\mathsf{GL}_n, \mathsf{K})$. In these cases $\mathsf{T} \neq \mathsf{S}$, however it is known that the fixed points in the respective flag varieties agree (see [7, p. 128]). In brief, in the case $(\mathsf{G}, \mathsf{K}) = (\mathsf{GL}_{2n}, \mathsf{Sp}_{2n})$, as elements of $\mathfrak{S}_{2n}$, these S-fixed points correspond to "mirrored" permutations, i.e., those permutations w having the property that $w(2n+1-i) = 2n+1-w(i)$ for each i; this is described in detail in [48]. Similarly, in the case of $(\mathsf{G}, \mathsf{K}) = (\mathsf{GL}_n, \mathsf{O}_n)$, these fixed points correspond to mirrored elements of $\mathfrak{S}_n$, as described in [48].

In [14] one considers Bott–Samelson varieties in relation to zonotopal tilings of an *Eltnitsky* polygon. This puts a poset structure on Bott–Samelson varieties (in type A) by introducing generalized Bott–Samelson varieties for which the fibers are larger flag varieties rather than $\mathbb{P}^1$'s. The diagram definition of $\mathcal{BEM}^{Y_0,Q}$ permits one to obtain similar definitions and results here *mutatis mutandis*.

6. Moment polytopes for the $\mathsf{GL}_p \times \mathsf{GL}_q$ case

Recall from the end of Section 4 that the BEM polytope $\mathcal{P}_{Y_0,Q}$ denotes the moment polytope $\Phi(\mathcal{BEM}^{Y_0,Q})$.

Example 6.1. Let $Q = (3, 2)$ and $Y_0 = Y_{++--}$. Following the construction in Section 5, and applying Theorem 5.1, $\mathcal{BEM}^{Y_0,Q}$ is described by the following diagram.

$$\begin{tikzcd}
\mathbb{C}^4 \arrow[d, dash] \arrow[dr, dash] & \\
W_3 \arrow[d, dash] \arrow[dr, dash] & V_1 \arrow[d, dash] \\
E_2 \arrow[d, dash] & V_2 \arrow[dl, dash] \\
W_1 \arrow[d, dash] & \\
\mathbb{C}^0 &
\end{tikzcd}$$

By Corollary 5.12, $\mathcal{BEM}^{Y_0,Q}$ has $4\cdot 2^2 = 16$ S-fixed points. We apply Theorem 4.1 to construct the moment polytope. First, by (21), $\Phi(BS^Q)$ is the convex hull of the following points:

(β_1, β_2)	$\sum_{i=1}^{r} \omega_i + \sum_{i=\lvert Q\rvert}^{1} s_{\beta_{\lvert Q\rvert}} \cdots s_{\beta_i}\omega_i$
$(-,-)$	$(5,4,2,0) = (3,2,1,0) + s_-\cdot(1,1,0,0) + s_-s_-\cdot(1,1,1,0)$
$(3,-)$	$(5,4,1,1) = (3,2,1,0) + s_-\cdot(1,1,0,0) + s_-s_3\cdot(1,1,1,0)$
$(-,2)$	$(5,3,3,0) = (3,2,1,0) + s_2\cdot(1,1,0,0) + s_2s_-\cdot(1,1,1,0)$
$(3,2)$	$(5,2,3,1) = (3,2,1,0) + s_2\cdot(1,1,0,0) + s_2s_3\cdot(1,1,1,0)$

The polytope $\Phi(BS^Q)$ is the white quadrilateral in Figure 5. We consider the reflections of $\Phi(BS^Q)$ by the T-fixed points of Y_0, corresponding to the $++--$ shuffled permutations:

$$[1,2,3,4],\ [2,1,3,4],\ [1,2,4,3],\ \text{and}\ [2,1,4,3].$$

By Theorem 4.1, $\mathcal{P}_{Y_0,Q}$ is the convex hull of the reflections

$$[1,2,3,4]\cdot\Phi(BS^Q) = \mathsf{conv}\{(5,4,2,0),(5,4,1,1),(5,3,3,0),(5,2,3,1)\},$$
$$[2,1,3,4]\cdot\Phi(BS^Q) = \mathsf{conv}\{(4,5,2,0),(4,5,1,1),(3,5,3,0),(2,5,3,1)\},$$
$$[1,2,4,3]\cdot\Phi(BS^Q) = \mathsf{conv}\{(5,4,0,2),(5,4,1,1),(5,3,0,3),(5,2,1,3)\},\ \text{and}$$
$$[2,1,4,3]\cdot\Phi(BS^Q) = \mathsf{conv}\{(4,5,0,2),(4,5,1,1),(3,5,0,3),(2,5,1,3)\}.$$

By the discussion of K-orbits of type A, the number of choices of closed orbits $Y_0 = Y_\gamma$ equals the number of matchless clans in $\mathtt{Clans}_{p,q}$ and this number is $\binom{p+q}{p}$. However, as we verify in the following theorem, if we fix Q and let Y_γ vary, then all the BEM-polytopes are isometric, being reflections of one other.

Theorem 6.2 (Reduction to $+\cdots+-\cdots-$ case). *$\mathcal{P}_{Y_\gamma,Q}$ is a w-reflection of $\mathcal{P}_{Y_{+\cdots+-\cdots-},Q}$ where w is the smallest permutation such that $w\cdot(+\cdots+-\cdots-) = \gamma$.*

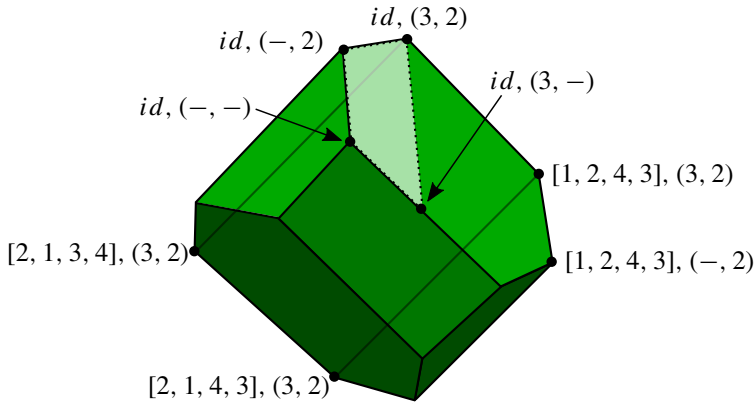

Figure 5. $\mathcal{P}_{Y_0,Q}$ for $Y_0 = Y_{++--}$ and $Q = (3, 2)$ is the convex hull of four reflections in $\mathbb{R}^3$ of the Bott–Samelson polytope (white). We have labeled some of the points $\Phi(p_{(x,J)})$ using x, J; all other points can be inferred from these.

Proof. Suppose $\gamma \in \texttt{Clans}_{p,q}$ is matchless and there exists an i such that $\gamma_i = -$ and $\gamma_{i+1} = +$. Let $\gamma' \in \texttt{Clans}_{p,q}$ be obtained by interchanging $-+ \mapsto +-$ at those positions.

By Proposition 5.9, the T-fixed points of Y_γ are the γ-shuffled permutations; call this set $\mathcal{A}$. Similarly, the T-fixed points of $Y_{\gamma'}$ are the γ'-shuffled permutations; call this set $\mathcal{B}$.

Claim 6.3. $$\mathcal{A}s_i = \mathcal{B}.$$

Proof of Claim 6.3. Let $\sigma \in \mathcal{A}$. Since $\gamma_i = -$, by definition $\sigma(i) \in \{p+1, p+2, \dots, n\}$. Also, since $\gamma_{i+1} = +$, $\sigma(i+1) \in \{1, 2, \dots, p\}$. Thus if $\sigma' = \sigma s_i$ then $\sigma'(i) \in \{1, 2, \dots, p\}$ and $\sigma'(i+1) \in \{p+1, p+2, \dots, n\}$, as is required for $\sigma' \in \mathcal{B}$. The claim follows. □

The claim, combined with Proposition 5.10 imply that the T-fixed points of $\mathcal{BEM}^{Y_{\gamma'},Q}$ are the s_i-reflection of those of $\mathcal{BEM}^{Y_\gamma,Q}$. Since the moment map images are determined by these T-fixed points, the respective polytopes must be an s_i reflection of one another. Now iterate this process down to the case $+\cdots+-\cdots-$. □

The Table 1 summarizes some information about the resulting polytopes for $p = q = 2$. In view of Theorem 6.2, we only need to consider $\gamma = ++--$. We have restricted to Q reduced and $|Q| \le 3$ for brevity. Actually, based on such calculations, it seems true that if Q and Q' are Demazure words for the same w then the BEM polytopes are combinatorially equivalent. For example, $Q = (1), (1, 1), (1, 1, 1)$ are all two-dimensional with $(V, E, F) = (4, 4, 1)$. However, we have no proof of this at present.

Q	dim	V	E	F
(1)	2	4	4	1
(2)	3	8	12	6
(3)	2	4	4	1
$(1, 2)$	3	12	18	8
$(1, 3) \equiv (3, 1)$	2	4	4	1
$(2, 1)$	3	8	12	6
$(2, 3)$	3	8	12	6
$(3, 2)$	3	12	18	8
$(1, 2, 1) \equiv (2, 1, 2)$	3	12	18	8
$(1, 2, 3)$	3	12	18	8
$(1, 3, 2) \equiv (3, 1, 2)$	3	8	12	6
$(2, 1, 3) \equiv (2, 3, 1)$	3	8	12	6
$(2, 3, 2)$	3	12	18	8
$(3, 2, 1)$	3	12	18	8
$(3, 2, 3)$	3	12	18	8

Table 1. BEM polytope data for $(\mathsf{GL}_4, \mathsf{GL}_2 \times \mathsf{GL}_2)$ where Q is reduced and $|Q| \leq 3$.

Following [42, Section 5], let X be a projective algebraic variety with a torus action T. Suppose $p \in X^{\mathsf{T}}$. Let $T_p(X)$ be the tangent space; this too carries a T action and a $\mathsf{T}_{\mathbb{R}}$ action. The $\mathsf{T}_{\mathbb{R}}$-decomposition is $T_p(X) = \bigoplus_\alpha E_\alpha$, where E_α are the eigenspaces with eigenvalues $\alpha \in \mathfrak{t}^*$. These $\{\alpha\}$ are the T-*weights*. The nonnegative cone spanned by these T-weights of $T_p(X)$ is equal to the cone spanned by the edges of the moment polytope $\Phi(X)$ incident to $\Phi(p)$.

For $w = s_{j_1} s_{j_2} \dots s_{j_\ell}$ a reduced expression of w we define

$$\mathrm{inv}(w) := \{\alpha_{j_1}, s_{j_1}(\alpha_{j_2}), \dots, s_{j_1} s_{j_2} \dots s_{j_{\ell-1}}(\alpha_{j_\ell})\}.$$

Theorem 6.4 (combinatorial description of T-weights). *Let $Q = (j_1, j_2, \dots, j_N)$ be a word and $J = (\beta_1, \dots, \beta_N)$ be a subword of Q. The T-weights of the tangent space of $\mathcal{BEM}^{Y_{+\cdots+-\cdots-}, Q}$ at $p_{(u\mathsf{B}, J)}$, where $u\mathsf{B}$ is a T-fixed point of $Y_{+\cdots+-\cdots-}$, are*

$$u \cdot (-\mathrm{inv}(w)) \cup u \cdot \{s_{\beta_N} \cdot (-\alpha_{j_N}), s_{\beta_N} s_{\beta_{N-1}} \cdot (-\alpha_{j_{N-1}}), \dots, s_{\beta_N} \dots s_{\beta_1} \cdot (-\alpha_{j_1})\},$$

where $w = [p, p-1, \dots, 1, n, n-1, \dots, p+1]$.

Proof. We apply:

Theorem 6.5 [16, Corollary 3.11]. *Let $Q_0, \dots, Q_n$ be subgroups of an algebraic group G and let T be a torus in G. Suppose that $R_0, \dots, R_n$ are subgroups of G*

with $R_i \subset Q_{i-1} \cap Q_i$ for $i > 0$ and $R_0 \subset Q_0$. Let

$$X = Q_n \times^{R_n} Q_{n-1} \times^{R_{n-1}} \cdots \times^{R_2} Q_1 \times^{R_1} Q_0/R_0$$

and $[q_n, \ldots, q_0] \in X$ a T-fixed point. Assume in addition that for every i, $q_i^{-1} \cdots q_n^{-1}$ is in the normalizer of T. Then the weights of T acting on the tangent space $T_{[q_n,\ldots,q_0]}X$ is the multiset union of

$$q_n q_{n-1} \ldots q_i \cdot \{\textit{weights of } T \textit{ acting on } Q_i/R_i\},$$

where i runs from n to 0.

More precisely, we apply this result to T and

$$\mathcal{BE}^{Y_0,Q} = \widetilde{Y}_0 \times^{\mathsf{B}} \mathsf{P}_{j_N} \times^{\mathsf{B}} \mathsf{P}_{j_{N-1}} \times^{\mathsf{B}} \cdots \times^{\mathsf{B}} \mathsf{P}_{j_1}/\mathsf{B},$$

where $\widetilde{Y}_0$ is the preimage of $Y_0 = Y_{+\cdots+-\cdots-}$ in G under $\mathsf{G} \to \mathsf{G}/\mathsf{B}$.

Let us verify that $\mathcal{BE}^{Y_0,Q}$ satisfies the required hypotheses. The orbit

$$Y_{+\cdots+-\cdots-} = \{\mathbb{C}^0 \subset F_1 \subset \cdots \subset F_{p-1} \subset \mathbb{C}^p \subset F_{p+1} \subset \cdots \subset \mathbb{C}^{p+q}\}.$$

Thus $\widetilde{Y}_0$ is the maximal parabolic subgroup $P_{\hat{p}}$. We have that $\widetilde{Y}_0, \mathsf{P}_{j_N}, \ldots, \mathsf{P}_{j_1}$ are subgroups of GL_n. Since B is a Borel subgroup then $\mathsf{B} \subset \widetilde{Y}_0 \cap \mathsf{P}_{j_N}$ and $\mathsf{B} \subset \mathsf{P}_{j_{t-1}} \cap \mathsf{P}_{j_t}$ for $1 \le t \le N$.

The T-fixed point of $\mathcal{BE}^{Y_0,Q}$ corresponding to $p_{(u\mathsf{B},J)}$ is $[u, s_{\beta_N}, s_{\beta_{N-1}}, \ldots, s_{\beta_1}\mathsf{B}]$, where $u \in N(\mathsf{T})$. Therefore $(us_{\beta_N}s_{\beta_{N-1}} \ldots s_{\beta_i})^{-1}$ is in the normalizer of T for all i. We have now verified that $\mathcal{BE}^{Y_0,Q}$ satisfies the required hypotheses.

Since Y_0 is the Schubert variety for w, the T-weights of $\widetilde{Y}_0/\mathsf{B} = Y_0$ at B are the negatives of the inversions of w. The T-weight of $\mathsf{P}_{\alpha_i}/\mathsf{B}$ at B is the simple root α_i. By Theorem 6.5, the T-weights of $\mathcal{BE}^{Y_0,Q}$ at the fixed point $[u, s_{\beta_N}, s_{\beta_{N-1}}, \ldots, s_{\beta_1}]$ is the multiset-union

$$u \cdot (-\operatorname{inv}(w)) \cup \{us_{\beta_N} \cdot (-\alpha_{j_N})\} \cup \{us_{\beta_N}s_{\beta_{N-1}} \cdot (-\alpha_{j_{N-1}})\} \cup \ldots \{us_{\beta_N} \ldots s_{\beta_1} \cdot (-\alpha_{j_1})\}$$

By Theorem 3.3, the T-weights for the tangent spaces of $\mathcal{BE}^{Y_0,Q}$ are the same as those for $\mathcal{BEM}^{Y_0,Q}$. □

Corollary 6.6. *The point $\Phi(p_{(u\mathsf{B},J)})$ is a vertex of $\mathcal{P}_{Y_{+\cdots+-\cdots-},Q}$ if and only if $\Phi(p_{(\mathsf{B},J)})$ is a vertex of $\mathcal{P}_{Y_{+\cdots+-\cdots-},Q}$.*

Proof. $\Phi(p_{(w\mathsf{B},J)})$ is a vertex whenever there is not a line in the cone spanned by the T-weights of the tangent space $T_{p_{(w\mathsf{B},J)}}(\mathcal{BEM}^{Y_{+\cdots+-\cdots-},Q})$. By Theorem 6.4,

$$\mathsf{T}\text{-weights of } T_{p_{(u\mathsf{B},J)}}(\mathcal{BEM}^{Y_{+\cdots+-\cdots-},Q}) = u \cdot (\mathsf{T}\text{-weights of } T_{p_{(\mathsf{B},J)}}(\mathcal{BEM}^{Y_{+\cdots+-\cdots-},Q})).$$

The claim follows since a cone contains a line if and only any reflection contains a line. □

Example 6.7. Consider the BEM polytope $\mathcal{P}_{Y_0,Q}$ of Example 6.1 and Figure 5. The vertices of $\mathcal{P}_{Y_0,Q}$ adjacent to $\Phi(p_{(\mathsf{B},(3,2))})$ are

$$\Phi(p_{(\mathsf{B},(-,2))}), \quad \Phi(p_{([1,2,4,3]\mathsf{B},(3,2))}), \quad \text{and} \quad \Phi(p_{([2,1,3,4]\mathsf{B},(3,2))}).$$

The cone spanned by the edges of $\mathcal{P}_{Y_0,Q}$ incident to $\Phi(p_{(\mathsf{B},(3,2))})$ is

$$\begin{aligned}\mathsf{pos}\{&\Phi(p_{(\mathsf{B},(-,2))}) - \Phi(p_{(\mathsf{B},(3,2))}), \Phi(p_{(s_3\mathsf{B},(3,2))}) - \Phi(p_{(\mathsf{B},(3,2))}),\\ &\qquad\qquad \Phi(p_{([s_1\mathsf{B},(3,2))}) - \Phi(p_{(\mathsf{B},(3,2))})\}\\ &= \mathsf{pos}\{(0,0,-1,1),(0,1,0,-1),(-1,1,0,0)\}.\end{aligned}$$

Let us compute the T-weights for the tangent space of $\mathcal{BEM}^{Y_0,Q}$ at $p_{(\mathsf{B},(3,2))}$. We have that $w = [2,1,4,3] = s_1 s_3$ so

$$\mathrm{inv}(w) = \{\alpha_1, s_1(\alpha_3)\} = \{\alpha_1, \alpha_3\} = \{(1,-1,0,0),(0,0,1,-1)\}.$$

Since $J = (3,2)$, then by Theorem 6.4 the T-weights are

$$\begin{aligned}&\{-\alpha_1, -\alpha_3\} \cup \{s_2(-\alpha_2), s_2 s_3(-\alpha_3)\}\\ &\quad = \{-\alpha_1, -\alpha_3, \alpha_2, \alpha_2+\alpha_3\}\\ &\quad = \{(-1,1,0,0),(0,0,-1,1),(0,1,-1,0),(0,1,0,-1)\}.\end{aligned}$$

The cone spanned by the T-weights coincides with the cone spanned by the edges incident to $\Phi(p_{(\mathsf{B},(3,2))})$. Since this cone does not contain a line it follows that $\Phi(p_{(\mathsf{B},(3,2))})$ is a vertex of $\mathcal{P}_{Y_0,Q}$.

Now consider the T-fixed point $p_{(\mathsf{B},(3,-))}$. The T-weights for the tangent space of $\mathcal{BEM}^{Y_0,Q}$ at $p_{(\mathsf{B},(3,-))}$ are

$$\begin{aligned}&\{-\alpha_1, -\alpha_3\} \cup \{s_-(-\alpha_2), s_- s_3(-\alpha_3)\}\\ &\quad = \{-\alpha_1, -\alpha_3, -\alpha_2, \alpha_3\}\\ &\quad = \{(-1,1,0,0),(0,0,-1,1,),(0,-1,1,0),(0,0,1,-1)\}.\end{aligned}$$

By Theorem 6.4 the cone spanned by these vectors is the cone spanned by the edges incident to $\phi(p_{(\mathsf{B},(3,-))})$. Since this cone contains the line spanned by α_3 then this point is not a vertex of $\mathcal{P}_{Y_0,Q}$. □

Although we have not done so here, it should be possible to give a combinatorial description of the vertices of $\mathcal{P}_{Y_0,Q}$. Doing so is equivalent to classifying the T-fixed points for which the cone spanned by the T-weights does not contain a line.

We conclude this paper with:

Theorem 6.8 (dimension of $\mathcal{P}_{Y_0,Q}$). *For* $(\mathsf{G},\mathsf{K}) = (\mathsf{GL}_{p+q}, \mathsf{GL}_p \times \mathsf{GL}_q)$,

$$\dim(\mathcal{P}_{Y_0,Q}) = \begin{cases} p+q-1 & \textit{if } p \textit{ is in } Q,\\ p+q-2 & \textit{if } p \textit{ is not in } Q.\end{cases}$$

Proof. A T-action on a space X is *effective* if each element of T, other than the identity, moves at least one point of X. In the proof of [39, Corollary 27.2] it is shown that for an effective Hamiltonian T-action the dimension of the corresponding moment polytope equals the dimension of the torus. If the T-action is not effective it is known that it can be reduced it to an effective action with the same moment polytope. The *stabilizer* of the T-action is the normal subgroup

$$S_T(X) := \{t \in T \mid t \cdot x = x \text{ for all } x \in X\}.$$

The T-action on X reduces to the effective action of $T/S_T(X)$ given by $tS_T(X)\cdot x := t \cdot x$. To prove the theorem we will consider the cases in which p is in Q and when it isn't separately. In each case we will explicitly compute the stabilizer of the T-action on $\mathcal{BEM}^{Y_0,Q}$. First, note that $\mathsf{T}_1 = \{\mathsf{t} \in \mathsf{T} \mid t_1 = \cdots = t_n\}$ acts trivially on $\mathcal{BEM}^{Y_0,Q}$ and therefore it is a subgroup of $S_{\mathsf{T}}(\mathcal{BEM}^{Y_0,Q})$.

In view of Theorem 6.2, we assume without loss of generality, that $Y_0 = Y_{+\cdots+-\cdots-}$, i.e.,

$$Y_0 = \{(F_1, F_2, \ldots, F_{n-1}) \in \mathsf{G}/\mathsf{B} \mid F_p = E_p\}.$$

Consider the projection $\varphi_0 : \mathcal{BEM}^{Y_0,Q} \to Y_0$ that maps a point in the configuration space to the leftmost flag (corresponding to the leftmost border) in the diagram. Since the projection is T-equivariant, $S_{\mathsf{T}}(\mathcal{BEM}^{Y_0,Q})$ is a subgroup of $S_{\mathsf{T}}(Y_0)$. Let us first describe $S_{\mathsf{T}}(Y_0)$. Note that

$$\mathsf{T}_2 = \{\mathsf{t} \in \mathsf{T} \mid t_1 = \cdots = t_p,\ t_{p+1} = \cdots = t_n\}$$

acts trivially on Y_0. Furthermore, since Y_0 is isomorphic to $\mathsf{GL}_p/\mathsf{B} \times \mathsf{GL}_q/\mathsf{B}$ then $S_{\mathsf{T}}(Y_0)$ is isomorphic to $S_{\mathsf{T}}(\mathsf{GL}_p/\mathsf{B}) \times S_{\mathsf{T}}(\mathsf{GL}_q/\mathsf{B})$. It follows that $S_{\mathsf{T}}(Y_0) = \mathsf{T}_2$ and therefore $S_{\mathsf{T}}(\mathcal{BEM}^{Y_0,Q})$ is either equal to T_1 or T_2.

First suppose that p is in Q and let k be such that $j_k = p$, where $Q = (j_1, \ldots, j_N)$. It is straightforward from Section 5 that if we set

$$V_k = \operatorname{span}\{\vec{e}_1, \ldots, \vec{e}_{p-1}, \vec{e}_p + \vec{e}_{p+1}\},$$

then the following holds:

$$(E_1, E_2, \ldots, E_{n-1}, E_{j_1}, E_{j_2}, \ldots, E_{j_{k-1}}, V_k, E_{j_{k+1}}, \ldots, E_{j_N}) \in \mathcal{BEM}^{Y_0,Q}.$$

Furthermore, for $\mathsf{t} \in \mathsf{T}_2$

$$\mathsf{t} \cdot V_k = \operatorname{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_{p-1}, \vec{e}_p + t_n\vec{e}_{p+1}\} \neq V_k,$$

so T_2 does not act trivially on $\mathcal{BEM}^{Y_0,Q}$ and $S_{\mathsf{T}}(\mathcal{BEM}^{Y_0,Q}) = \mathsf{T}_1$. From this and the first paragraph it follows that

$$\dim(\mathcal{P}_{Y_0,Q}) = p + q - 1.$$

Now suppose that p is not in Q. This implies that every vector space in

$$(F_1, F_2, \dots, F_{n-1}, V_1, V_2, \dots, V_N) \in \mathcal{BEM}^{Y_0, Q}$$

must either be a subspace of E_p or contain E_p. So T_2 acts trivially on $\mathcal{BEM}^{Y_0, Q}$ and $S_{\mathsf{T}}(\mathcal{BEM}^{Y_0, Q}) = \mathsf{T}_2$. We conclude that $\dim(\mathcal{P}_{Y_0, Q}) = p + q - 2$. □

Example 6.9. The data of Table 1 is consistent with Theorem 6.8. Furthermore, note that the dimension characterization only depends on p and not q. Indeed, if $p = 2$ and $q = 3$, one can check $\mathcal{P}_{Y_{++---},(3)}$ has dimension 3, also in agreement with the theorem. □

Acknowledgements

We thank D. Barbasch, B. Elek, S. Evens, W. Graham, A. Knutson, E. Lerman, L. Li, J. Pascaleff, and A. Woo for helpful conversations. We thank S. Evens for clarifying and correcting our understanding of the hypotheses of [3, Proposition 6.4]. We also thank referees for suggestions. Escobar was supported by an AMS-Simons travel grant, NSF grant DMS 1855598 and NSF CAREER grant DMS 2142656. Yong was supported by NSF grant DMS 1500691, an NSF RTG in combinatorics DMS 1937241, and Simons Collaboration Grant 582242.

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Received August 27, 2020. Revised July 4, 2022.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 320 No. 1 September 2022
