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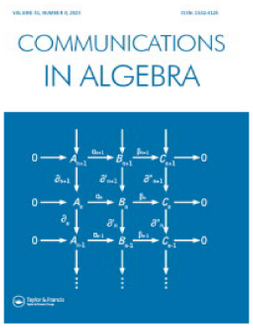

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Halperin's conjecture in formal dimensions up to 20

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ABSTRACT
A 1976 conjecture of Halperin on positively elliptic spaces has been confirmed in formal dimensions up to 16. In this article, we shorten the proof and extend the result up to formal dimension 20. We work with Meier's algebraic characterization of the conjecture, so the proof is elementary in that it involves only polynomial algebras, ideals, and derivations.

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1. Introduction

We consider Artinian complete intersection algebras

$$H^* = \mathbb{Q}[x_1, \dots, x_k]/(u_1, \dots, u_k)$$

over the rationals with a grading concentrated in even degrees. Examples include the rational cohomology of positively elliptic topological spaces, so for simplicity we refer to these algebras as *positively elliptic algebras* (see Section 2 for definitions).

Positively elliptic spaces play an important role in rational homotopy theory. In fact, they are the subject of a 1976 conjecture of Halperin that is listed as the first of seventeen open problems in [11, Chapter 39]. In 1982 Meier [25] proved that this conjecture can be reformulated algebraically as follows:

Conjecture (Halperin Conjecture). *If H^* is a positively elliptic algebra, then H^* does not admit a nontrivial derivation of negative degree.*

The conjectured nonexistence of derivations of negative degree arises in other contexts, including singularity theory where one has the conjectures of Wahl (see [5, 14, 30]) and Yau (see [8, 31]). For additional context, we refer to the survey [17], the papers [18–20], and references therein.

Evidence for Halperin's conjecture includes proofs under *geometric* assumptions such as when H^* is the rational cohomology algebra of a Kähler manifold (see [4, 24]), a homogeneous space (see [28]), or a non-negatively curved Riemannian manifold with large symmetry (see [12, 13, 15, 27]). It has also been verified under *algebraic* assumptions such as when H^* at most three generators (see [6, 21]), relations of sufficiently large degree (see [9]), or formal dimension at most 16 (see [2]). In this article, we expand on the latter result by shortening the proof and extending it as follows:

Theorem. *Halperin's conjecture holds in formal dimensions at most* 20.

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The proof simplifies and extends [2], which covers dimensions up to 16. In fact, by adopting the algebraic setup of [6, 9, 26] (see Sections 2 and 3) and proving two new lemmas, we can efficiently prove all cases in dimensions up to 16 and all but six exceptional cases in dimensions 18 and 20 (see Sections 4 and 5). The proof of those six cases, and therefore of our main theorem, is completed in Section 6.

2. Preliminaries

Let $A = \mathbb{Q}[x_1, \ldots, x_k]$ denote the polynomial ring over the rationals on k variables. Assume moreover that each x_i has a positive, even degree assigned to it that is denoted by $|x_i|$. This induces a *graded algebra* structure on $A = \bigoplus_{n\geq 0} A^n$ where the subspace A^n is spanned by monomials $x_1^{a_1} \cdots x_k^{a_k}$ satisfying $a_1|x_1| + \cdots + a_k|x_k| = n$.

Next let $I = (u_1, \ldots, u_k)$ denote the ideal generated by homogeneous polynomials $u_i \in A^{|u_i|}$, where $|u_i|$ denotes the degree of u_i. Recall that the u_i form a *regular sequence* if $u_1 \in A$ is nonzero and if the image of u_i in $A/(u_1, \ldots, u_{i-1})$ is not a zero divisor for all $2 \leq i \leq k$.

Definition 2.1. A *positively elliptic algebra* is the quotient $\mathbb{Q}[x_1, \ldots, x_k]/(u_1, \ldots, u_k)$ of a graded polynomial ring over the rationals on generators with positive, even degrees by an ideal generated by a regular sequence $u_1, \ldots, u_k$ of homogeneous polynomials.

Example 2.2. Singly generated algebras $\mathbb{Q}[x_1]/(x_1^\alpha)$ are positively elliptic. Doubly generated algebras can be positively elliptic or not, as can be seen from the examples $\mathbb{Q}[x_1, x_2]/(x_1^2 - x_2^2, x_1x_2)$ or $\mathbb{Q}[x_1, x_2]/(x_1^2, x_1x_2)$. In the latter case, the image of x_1x_2 in $\mathbb{Q}[x_1, x_2]/(x_1^2)$ is a zero divisor, so the ideal is not generated by a regular sequence.

Example 2.3. Positively elliptic algebras arise as the rational cohomology algebras of simply connected, rationally elliptic topological spaces F with positive Euler characteristic (see [11, Proposition 32.10]). Such spaces are called F_0 spaces or *positively elliptic spaces* (see [1, 3, 22]), and they were conjectured by Halperin in 1976 to satisfy the following: For any orientable fibration with fiber F, the Serre spectral sequence degenerates at the E_2-page (see [11, Chapter 39]).

In 1982, Meier [25, Theorem A] proved that Halperin's conjecture can be reformulated entirely algebraically in terms of negative degree derivations.

Definition 2.4. Given a positively elliptic algebra H^*, a *derivation* is a linear map $\delta : H^* \to H^*$ that increases degree by some integer $|\delta| \in \mathbb{Z}$ and satisfies the Leibniz rule, that is, behaves on products of homogeneous elements as follows:

$$\delta(xy) = \delta(x)y + (-1)^{|\delta||x|}x\delta(y).$$

Example 2.5. The graded algebra $H^* = \mathbb{Q}[x_1, x_2]/(x_1^2 - \lambda x_2^2, x_1x_2)$ with $|x_1| = |x_2| = 2$ and $\lambda \in \mathbb{Q} \setminus \{0\}$ is a positively elliptic algebra and admits a nontrivial derivation δ of degree 2. Indeed, if we define $\delta(x_1) = x_1^2$ and $\delta(x_2) = 0$ and extend the definition by linearity and the Leibniz rule, we obtain a well defined derivation on $\mathbb{Q}[x_1, x_2]$. In addition, $\delta(x_1^2 - \lambda x_2^2)$ and $\delta(x_1x_2)$ are in the ideal $(x_1^2 - \lambda x_2^2, x_1x_2)$, so δ descends to a well defined derivation on H^*.

This example demonstrates the way we work with derivations on H^*. They correspond to derivations on $\mathbb{Q}[x_1, \ldots, x_k]$ that map the ideal $(u_1, \ldots, u_k)$ into itself. Throughout this article, we use the same notation for the generators of $\mathbb{Q}[x_1, \ldots, x_k]$ and their images in H^*.

This example also shows the necessity of the condition that δ have negative degree. We recall Meier's reformulation of Halperin's conjecture from the introduction for easy reference:

Conjecture (Halperin Conjecture). *Positively elliptic algebras do not admit nontrivial derivations of negative degree.*

We close this preliminary section with two basic results (see [2, Lemmas 11.1 and 11.3]). Together they imply Thomas' result that the Halperin Conjecture holds when H^* is generated by at most two elements (see [29]).

Lemma 2.6 (Land in Zero Lemma). *For $i > 0$, a derivation of degree $-i$ vanishes on H^i.*

Lemma 2.7 ($k-1$ Lemma). *If δ is a derivation of negative degree on H^* such that $\delta(x_i) = 0$ for $k-1$ of the k generators x_i, then $\delta = 0$.*

3. Degree type, formal dimension, and splittings

Given a positively elliptic algebra $H^* = \mathbb{Q}[x_1, \dots, x_k]/(u_1, \dots, u_k)$, the *degree type* of H^* is the sequence of even, positive integers denoted by

$$(|x_1|, \dots, |x_k|; |u_1|, \dots, |u_k|).$$

As Example 2.5 shows, the degree type $(2, 2; 4, 4)$ can be realized in infinitely many ways, even up to isomorphism. This is a general feature. Nevertheless, it is helpful to sort positively elliptic algebras according to their degree types. In this section, we summarize previous work on degree types as they relate to Halperin's conjecture. In addition, we define pure models, formal dimension, and splittings. The first basic result is the following. It is motivated by, but not explicitly stated in, [11, Section 32]:

Theorem 3.1 (Pure model). *Given a nonzero positively elliptic algebra H^*, there exist variables x_i of positive, even degrees $|x_i|$ and homogeneous polynomials*

$$u_i \in \mathbb{Q}^{\geq 2}[x_1, \dots, x_k] = \operatorname{span}\{x_1^{a_1} \cdots x_k^{a_k} \mid a_1 + \cdots + a_k \geq 2\}$$

such that $H^* \cong \mathbb{Q}[x_1, \dots, x_k]/(u_1, \dots, u_k)$. Moreover these choices can be made to satisfy all of the following:

1. $|x_1| \leq \cdots \leq |x_k|$.
2. $|u_1| \leq \cdots \leq |u_k|$.
3. $|u_i| \geq 2|x_i|$ *for all* $1 \leq i \leq k$.

In addition, the formal dimension

$$\operatorname{fd} H^* = \sum_{k=1}^{n} (|u_i| - |x_i|)$$

is independent of the choice of presentation.

Such a presentation of H^* is called a *pure model*, and we assume from now on that our presentations of positively elliptic algebras are pure models.

Remark 3.2. The presentation (i.e., choice of generators and relations) is not unique. For example, a linear change of variables in generators of the same degree does not affect the property of being a pure model or the degree type, nor does it change the fact that the relations form a regular sequence. Somewhat more generally, we may add a polynomial of generators of lower degree to another generator, so long as we preserve the homogeneity of the generators. Similar comments apply to changes in our choice of relations.

Proof. By definition, there is some presentation of $H^* = \mathbb{Q}[x_1, \dots, x_k]/(u_1, \dots, u_k)$ with $k \geq 1$. We may assume that k is minimal.

Clearly the u_j do not have constant terms, since otherwise the ideal $I = (u_1, \ldots, u_k)$ is the entire polynomial algebra. Moreover, if some u_j has a linear term equal to a multiple of x_i, then the automorphism of $\mathbb{Q}[x_1, \ldots, x_k]$ that replaces x_i by u_j is an isomorphism. Taking the quotient by I gives rise to a presentation of H^* on $k-1$ generators. This contradicts the minimality of k, so we have that each $u_j \in \mathbb{Q}^{\geq 2}[x_1, \ldots, x_k]$.

Next, we may relabel the generators and relations so that $|x_1| \leq \cdots \leq |x_k|$ and $|u_1| \leq \cdots \leq |u_k|$. The final condition that $|u_i| \geq 2|x_i|$ for all i follows by the result of Friedlander and Halperin below (Theorem 3.4). Indeed, this result implies that some relation (and hence u_k) has degree at least twice $|x_k|$, that at least two relations (and hence u_{k-1} and u_k) have degree at least twice $|x_{k-1}|$, and so on.

For the last claim, we note that H^* satisfies Poincaré duality (see [16, Section 8]). This means that there exists $n \geq 0$ such that $H^i = 0$ for $i > n$ and $H^n \cong \mathbb{Q}$ and that the product map $H^i \times H^{n-i} \to H^n \cong \mathbb{Q}$ is a non-degenerate bilinear pairing for all $0 \leq i \leq n$. This integer n is called the *formal dimension* (or socle degree) and is denoted by $\operatorname{fd} H^*$ (cf. [7]). For our purposes, we note that H^n is generated by the Jacobian $\det\left(\partial u_i/\partial x_j\right)$ (see, for example, the remarks following Theorem B in [28]). Therefore, the formula in the theorem equals n and is therefore an invariant of the positively elliptic algebra. □

Example 3.3. The positively elliptic algebra $H^* = \mathbb{Q}[x_1, x_2]/(x_1^2 - x_2, x_2^3)$ with $|x_1| = 2$ and $|x_2| = 4$ can be more efficiently presented as $H^* \cong \mathbb{Q}[x]/(x^6)$. Indeed, an isomorphism is given by mapping $x_1 \mapsto x$ and $x_2 \mapsto x^2$.

A consequence of Theorem 3.1 is that any given formal dimension only allows for finitely many degree types. Indeed,

$$\operatorname{fd} H^* = \sum_{i=1}^{k} |u_i| - |x_i| \geq \sum_{i=1}^{k} |x_i| \geq 2k,$$

so $k \leq \frac{1}{2} \operatorname{fd} H^*$, the possible degrees $|x_i|$ are similarly bounded, and therefore the possibilities for the $|u_i|$ are finite.

A further restriction on the degree types is the following result due to Friedlander and Halperin (see [10, Corollary 1.10] or [11, Proposition 32.9]):

Theorem 3.4 (Characterization of degree types). *A sequence*

$$(A_1, \ldots, A_k; B_1, \ldots B_k)$$

of positive, even integers arises as the degree type of some positively elliptic algebra if and only if the following holds: For all $1 \leq l \leq k$ *and* $1 \leq i_1 < \cdots < i_l \leq k$, *there exist* $1 \leq j_1 < \cdots < j_l \leq k$ *such that* $B_{j_1}, \ldots, B_{j_l}$ *can be expressed as linear combinations of the form* $\lambda_1 A_{i_1} + \cdots + \lambda_l A_{i_l}$ *with non-negative integer coefficients satisfying* $\lambda_1 + \cdots + \lambda_l \geq 2$.

To illustrate, the degree type $(A_1, A_2; B_1, B_2) = (2, 4; 4, 10)$ does not satisfy this condition since $A_2 = 4$ does not properly divide any of the B_j. Similarly, the degree type $(2, 2, 4, 4; 4, 6, 8, 10)$ does not satisfy the condition and therefore does not arise as the degree type of a positively elliptic algebra.

Definition 3.5. A sequence $(A_1, \ldots, A_k; B_1, \ldots, B_k)$ as in Theorem 3.4 satisfies the condition $SAC(A_{i_1}, \ldots, A_{i_l})$ if there exist $B_{j_1}, \ldots, B_{j_l}$ as in the theorem.

In [10], the condition that $SAC(A_{i_1}, \ldots, A_{i_l})$ holds for all possible subsequences $1 \leq i_1 < \cdots < i_l \leq k$ is called the Strong Arithmetic Condition (SAC). The examples after the theorem fail the SAC(4) and the SAC(4,4), respectively, and therefore fail the SAC.

Next we discuss Markl's result, which is crucial to our inductive arguments over the formal dimension. One result we need in the proof of the main theorem is Lemma 3.8.

Definition 3.6 (Split positively elliptic algebras). A positively elliptic algebra H^* *splits* if it has a presentation as a pure model

$$H^* \cong \mathbb{Q}[x_1, \dots, x_k]/(u_1, \dots, u_k)$$

such that, for some $0 < l < k$, the polynomials $u_1, \dots, u_l$ only depend on $x_1, \dots, x_l$.

Note that, in this definition, H^* has a positively elliptic subalgebra

$$K^* \cong \mathbb{Q}[x_1, \dots, x_l]/(u_1, \dots, u_l)$$

and a positively elliptic quotient algebra $Q^* = H^*/K^*$ defined by $Q^n = H^n/(K^+H^*)^n$, where K^+H^* denotes the vector subspace spanned by products of an element of K^* of positive degree and an element of H^*. Note that

$$Q^* \cong \mathbb{Q}[\bar{x}_{l+1}, \dots, \bar{x}_k]/(\bar{u}_{l+1}, \dots, \bar{u}_k),$$

where the bars denote images under the projection map $H^* \to Q^*$. Also note that

$$\operatorname{fd} H^* = \operatorname{fd} K^* + \operatorname{fd} Q^*$$

and that both K^* and Q^* have formal dimension strictly less than $\operatorname{fd} H^*$.

In the proof of the Halperin conjecture up to dimension 20, we will proceed by induction over the formal dimension. In particular, the following is an important tool for dealing with the split case (see [23, Theorem 1]):

Theorem 3.7 (Markl's theorem). *Let H^* be a positively elliptic algebra with a nonzero derivation of negative degree. If H^* splits as above, then K^* or Q^* also has a nonzero derivation of negative degree.*

Markl's theorem holds in greater generality, but this statement is all we need. As the proof also simplifies somewhat in this case, we include it here.

Proof. Assume that $H^* = \mathbb{Q}[x_1, \dots, x_k]/(u_1, \dots, u_k)$ is a pure model for a positively elliptic algebra H^* with the property that $u_1, \dots, u_l \in \mathbb{Q}[x_1, \dots, x_l]$ for some $0 < l < k$. Let $K^* = \mathbb{Q}[x_1, \dots, x_l]/(u_1, \dots, u_l)$, and suppose that neither K^* nor $Q^* = H^*/K^*$ admit a nonzero derivation of negative degree. Finally, let δ be a derivation of negative degree on H^*, and note that our task is to show that $\delta = 0$.

First, since the degrees of the x_i are increasing, the derivation δ restricts to a derivation on K^*. By the assumption on K^*, we have

$$\delta(x_1) = \cdots = \delta(x_l) = 0.$$

Next, fix any vector space basis $\{\xi_\alpha\}$ for K^* consisting of monic polynomials ξ_α in the variables $x_1, \dots, x_l$. For $y \in H^*$, there exist polynomials $\delta_\alpha(y)$ in $x_{l+1}, \dots, x_k$ such that

$$\delta(y) = \sum_\alpha \xi_\alpha \delta_\alpha(y).$$

We claim that each of the maps

$$\begin{aligned} \bar{\delta}_\alpha : H^*/K^* &\to H^*/K^* \\ \bar{y} &\mapsto \overline{\delta_\alpha(y)} \end{aligned}$$

is a well defined linear map.

If this claim holds, then it is straightforward to see that $\bar{\delta}_\alpha$ is a derivation of negative degree on H^*/K^* and hence vanishes by assumption. In particular, $\overline{\delta_\alpha(y)} = \bar{\delta}_\alpha(\bar{y}) = 0$ for all α, which implies that $\delta_\alpha(y)$ is both a polynomial in $x_{l+1}, \dots, x_k$ and in the ideal $(x_1, \dots, x_l)$ for all α. It follows that $\delta_\alpha = 0$ for all α and hence that $\delta = 0$, as required.

It suffices to prove the claim, and for this it suffices to show that δ_α maps the ideal $(x_1, \ldots, x_l)$ to zero. Fix

$$z = \sum_{i=1}^{l} x_i z_i$$

in this ideal. Applying δ to both sides of this equation and noting that δ is a derivation on H^* that vanishes on $x_1, \ldots, x_l$, we obtain

$$\sum_{\alpha} \xi_\alpha \delta_\alpha(z) = \sum_{i=1}^{l} x_i \sum_{\beta} \xi_\beta \delta_\beta(z_i).$$

Extracting the coefficients of ξ_α on both sides, we obtain

$$\delta_\alpha(z) = \sum_{x_i | \xi_\alpha} \delta_{\beta(\alpha,i)}(z_i),$$

where the sum runs over $1 \leq i \leq l$ such that x_i divides ξ_α, and where $\beta(\alpha, i)$ is the index for which $x_i \xi_{\beta(\alpha,i)} = \xi_\alpha$.

If α is the index corresponding to the constant monomial $\xi_\alpha = 1$, then the sum on the right-hand side is trivial and we find that $\delta_\alpha(z) = 0$. As z was arbitrary, this proves that δ_α maps the ideal $(x_1, \ldots, x_l)$ to zero for this particular value of α. Proceeding by induction over the degree of ξ_α, we note that the right-hand side once again vanishes by the induction hypothesis since $\xi_{\beta(\alpha,i)}$ has smaller degree than ξ_α. Hence $\delta_\alpha(z) = 0$, and so by induction we conclude that δ_α vanishes on the ideal $(x_1, \ldots, x_l)$, as required. □

In the proof of our main theorem, we induct over the formal dimension. By Markl's theorem, the result holds when H^* splits since the result holds inductively for smaller formal dimensions. Therefore it is useful to have conditions that imply the existence of splittings.

Lemma 3.8 (Degree Inequality). *Let $H^* = \mathbb{Q}[x_1, \ldots, x_k]/(u_1, \ldots, u_k)$ be a positively elliptic algebra that does not split. The following hold:*

1. *If $i < k$, then $|u_i| \geq |x_1| + |x_{i+1}|$.*
2. *If $\delta(x_2) = \lambda x_1^\alpha \neq 0$ for some $\lambda \in \mathbb{Q}$, where δ is a derivation on H^* with negative degree, then $|u_1| \geq |x_1| + |x_3|$.*

Proof. The first claim is a restatement of [2, Lemma 11.4]. It follows since $|u_i| < |x_1| + |x_{i+1}|$ for some i implies that $u_1, \ldots, u_i \in \mathbb{Q}^{\geq 2}[x_1, \ldots, x_k]$ are polynomials in $x_1, \ldots, x_i$ for degree reasons. Hence $x_1, \ldots, x_i$ generate a nontrivial subalgebra, a contradiction.

The second claim is implicit in the proof of [2, Lemma 11.5]. Suppose that $\delta(x_2) = \lambda x_1^\alpha \neq 0$ for some $\lambda \in \mathbb{Q}$ and $\alpha \geq 1$. Suppose for the purpose of contradiction that $|u_1| < |x_1| + |x_3|$. As in the previous paragraph, we conclude that u_1 is a polynomial in x_1 and x_2. Write $u_1 = \sum_{i=0}^{r} p_i(x_1) x_2^i$. Since $\delta(u_1)$ is in the ideal $(u_1, \ldots, u_k)$ and has degree less than any of the u_i, we have $\delta(u_1) = 0$. On the other hand,

$$\delta(u_1) = \sum_{i=0}^{r} p_i(x_1) \delta(x_2^i) = \sum_{i=1}^{r} i p_i(x_1) \left(\lambda x_1^\alpha\right) x_2^{i-1},$$

so $p_i(x_1) = 0$ for all $i \geq 1$. Hence, $u_1 = p_0(x_1)$, x_1 generates a nontrivial subalgebra of H^*, and we have a contradiction. □

We close this section with a proposition showing how the Degree Inequality and the other preliminary facts we have established work together with the upper bound on the formal dimension to rule out a large number of cases.

Proposition 3.9. *Let $H^* = \mathbb{Q}[x_1, \dots, x_k]/(u_1, \dots, u_k)$ be a positively elliptic algebra with no nontrivial subalgebra and $\operatorname{fd} H^* \leq 20$. If there exists a nonzero derivation of negative degree and $|x_{k-1}| + |x_k| \geq 12$, then the degree type is*

$$(2, 4, 6, 6; 6, 8, 12, 12) \text{ or } (2, 2, 6, 6; 4, 8, 12, 12).$$

Proof. First, suppose that $k \leq 3$. By the Land in Zero Lemma, $\delta(x_1) = 0$, so we may assume $k \geq 2$. Moreover by the $k-1$ Lemma, we may assume that $k = 3$ and that $\delta(x_2)$ and $\delta(x_3)$ are linearly independent since otherwise we could choose new generators so that $\delta(x_2) = 0$. In particular, it follows for degree reasons that $|x_1| < |x_2| < |x_3|$. On one extreme, these degrees could be 2, 4, and 6, but this contradicts the assumption that $|x_{k-1}| + |x_k| \geq 12$. We may assume therefore that $|x_3| \geq 8$. We put this into the formula for the formal dimension in Theorem 3.1 and we estimate the summands using the Degree Inequality (Lemma 3.8):

$$\operatorname{fd} H^* = \sum_{i=1}^{3} (|u_i| - |x_i|) \geq |x_3| + \max(|x_1| + |x_3| - |x_2|, |x_2|) + |x_3|.$$

Since the maximum is at least the average, this implies $\operatorname{fd} H^* > 20$, a contradiction.

Next, suppose that $k \geq 4$ and $|x_k| \geq 8$. Using the Degree Inequality to estimate $|u_i|$ for $i \leq k-1$ and the estimate $|u_k| \geq 2|x_k|$, we obtain

$$\operatorname{fd} H^* \geq \sum_{i=1}^{k-1} (|x_1| + |x_{i+1}| - |x_i|) + |x_k| = (k-2)|x_1| + 2|x_k| \geq 20.$$

Hence equality holds everywhere, and we have $k = 4$, $|x_1| = 2$, $|x_4| = 8$, and $|u_3| = |x_1| + |x_4| = 10$. Now $|x_3| \leq \frac{1}{2}|u_3|$, so $|x_3| \leq 4$. Since additionally $|x_3| \geq 4$ by the Land in Zero and $k-1$ Lemmas, we have $|x_3| = 4$. Using equality in the above estimate once more, we have $|u_2| = |x_1| + |x_3| = 6$, so we have a contradiction to the $SAC(4, 8)$ condition since u_4 is the only relation properly divisible by four.

Finally, suppose that $k \geq 4$ and $|x_k| \leq 6$. By the assumption in the proposition, we have $|x_{k-1}| = |x_k| = 6$. Estimating as in the previous case, except replacing the estimate for the $i = k-1$ term by the estimate $|u_{k-1}| \geq 2|x_{k-1}|$, we see that

$$\operatorname{fd} H^* \geq (k-3)|x_1| + 2|x_{k-1}| + |x_k| \geq 2 + 3(6) = 20.$$

Hence equality holds, and the degree type is of the form

$$(2, A_2, 6, 6; 2 + A_2, 8, 12, 12)$$

where $A_2 \in \{2, 4\}$. These two possibilities correspond to the two degree types in the conclusion of the proposition, so the proof is complete. □

4. The Large Relations Lemma

In this section, we prove the Large Relations Lemma and Proposition 4.2, which verifies the Halperin conjecture in formal dimensions up to 20 in all but three exceptional cases when the degrees of the two largest generators satisfy $|x_{k-1}| + |x_k| \leq 8$.

Lemma 4.1 (Large Relations Lemma). *Let $H^* \cong \mathbb{Q}[x_1, \dots, x_k]/(u_1, \dots, u_k)$ be a positively elliptic algebra that does not split. Assume that H^* admits a derivation δ of degree -2 such that the map $\delta : H^4 \to H^2$ has rank $m \geq 1$.*

Let g_i denote the number of generators with degree i, and let r_j denote the number of relations with degree j. The following hold:

1. *If $g_6 + g_{10} + g_{14} + \cdots = 0$, then*

$$r_{12} + r_{16} + \cdots \geq \big(k - g_2 - g_4\big) + \max(1, m - r_4).$$

2. If $g_6 + g_{10} + g_{14} + \cdots \geq 1$ *and* $\delta^2(H^6) = 0$, *then*

$$r_{10} + r_{12} + \cdots \geq \left(k - g_2 - g_4\right) + \max(1, m - r_4).$$

In particular, $|u_k| \geq 12$ *in the first case and* $|u_{k-1}| \geq 10$ *in the second.*

Proof. We prove the claims simultaneously. By the Land in Zero Lemma, we may assume that $\delta(x_i) = 0$ for $1 \leq i \leq g_2$. In addition, we may change basis so that

$$\delta(x_{g_2+i}) = \begin{cases} x_i & \text{if} \quad 1 \leq i \leq m \\ 0 & \text{if} \quad m < i \leq g_4 \end{cases}$$

Finally, if x_h is a generator in degree six, then the condition $\delta^2(H^6) = 0$ implies that $\delta(x_h)$ has no x_{g_2+i} term with $1 \leq i \leq m$.

Let $\{u_j | j \in J\}$ denote the relations with degree 8. Write each of these as

$$u_j = p_j(x_{g_2+1}, \ldots, x_{g_2+m}) + q_j$$

where p_j is a quadratic polynomial and where q_j is in the ideal

$$I_0 = (x_1, \ldots, x_{g_2}) + (x_{g_2+m+1}, \ldots, x_{g_2+g_4}).$$

Fix $J' \subseteq J$ such that $\{p_j \mid j \in J'\}$ is a basis for the span of $\{p_j \mid j \in J\}$.

We claim that $|J'| \leq m-1$, and we prove this by contradiction. Note that $\delta(H^2) = 0$ and $\delta^2(H^4) = 0$ by the Land in Zero Lemma. In addition, $\delta^2(H^6) = 0$ by assumption in Case (2) of the lemma and for degree reasons in Case (1) since there are no generators in degree six. Noting next that δ vanishes on the generators of the ideal I_0, we have $\delta^2(q_j) = 0$ and hence

$$\delta^2(u_j) = 2p_j(x_1, \ldots, x_m)$$

for $j \in J'$. Now $\delta^2(u_j)$ has degree four and lies in the ideal $(u_1, \ldots, u_k)$. Hence $p_j(x_1, \ldots, x_m)$ lies in the r_4-dimensional span of $\{u_i \mid |u_i| = 4\}$. Since the polynomials p_j with $j \in J'$ are linearly independent, we may perform a change of basis on the degree-four relations u_i such that $\{u_1, \ldots, u_{|J'|}\} = \{p_j(x_1, \ldots, x_m) \mid j \in J'\}$. If $|J'| \geq m$, then $u_1, \ldots, u_m \in \mathbb{Q}[x_1, \ldots, x_m]$ and hence that $x_1, \ldots, x_m$ generate a subalgebra K^*. Since moreover $1 \leq m \leq \frac{k}{2}$, we see that H^* splits, and we have a contradiction to the assumptions of the lemma.

We may assume now that $|J'| \leq m-1$. By the argument in the previous paragraph, $|J'| \leq \min(m-1, r_4)$ by choice of J'. We can perform a change of basis on the u_j for $j \in J$ so that $p_j = 0$ for $j \in J \setminus J'$.

To finish the proof of Claim 1, consider the ideal

$$I = I_0 + \left(\{u_j \mid j \in J'\}\right) + \left(\{u_j \mid |u_j| \in \{12, 16, \ldots\}\right).$$

If a relation u_i has degree less than eight or not divisible by four, then it lies in I_0 for degree reasons since there are no generators in degrees 6, 10, etc. If $|u_i| = 8$, then it lies in $I_0 + \left(\{u_j \mid j \in J'\}\right)$ by choice of J'. Finally it is clear that $u_i \in I$ for all other relations u_i. Hence H^* projects onto $\mathbb{Q}[x_1, \ldots, x_k]/I$. Since H^* is finite-dimensional, I must have at least k generators. Therefore

$$(g_2 + g_4 - m) + \min(m-1, r_4) + (r_{12} + r_{16} + \cdots) \geq k,$$

which implies the desired bound in Claim 1.

To finish the proof of Claim 2, we use a similar argument with I replaced by

$$I = I_0 + \left(\{u_j \mid j \in J'\}\right) + \left(\{u_j \mid |u_j| \in \{10, 12, \ldots\}\right).$$

It is clear that relations of degree four or degree eight or larger lie in I. Relations of degree six are also in I_0 and hence in I because they are polynomials in $\mathbb{Q}^{\geq 2}[x_1, \ldots, x_k]$ (see Theorem 3.1). Hence again all relations are in I, and the claim follows as before. □

Next, we apply Lemma 4.1 to prove our main theorem when $|x_{k-1}| + |x_k| \leq 8$ in all but three exceptional cases.

Proposition 4.2. *Let H^* be a positively elliptic algebra that does not split. If H^* admits a nonzero derivation of negative degree and $|x_{k-1}| + |x_k| \leq 8$, then either $\operatorname{fd} H^* > 20$ or the degree type is one of the following:*

$$(2,2,4,4;4,6,8,12),\ (2,2,4,4;4,8,8,12),\ \text{or}\ (2,2,2,4,4;4,4,6,8,12),$$

Proof. By the Land in Zero and $k-1$ Lemmas (2.6 and 2.7), we may assume that $|x_{k-1}| \geq 4$ since otherwise δ vanishes on the first $k-1$ generators and hence on all of them. By the assumptions of the proposition, we have $|x_{k-1}| = |x_k| = 4$. Similarly, we may assume that δ has degree -2 and that the map $\delta : H^4 \to H^2$ has rank $m \geq 2$.

By Lemma 4.1, $|u_k| \geq 12$, and this forces the formal dimension

$$\operatorname{fd} H^* = \sum_{i=1}^{k} (|u_i| - |x_i|)$$

to be large. Indeed, let $g_4 \geq m$ be the number of generators of degree four. We have $|x_i| = 2$ for $i \leq k - g_4$ and $|x_i| = 4$ otherwise. Additionally the Degree Inequality (Lemma 3.8) implies

$$|u_i| - |x_i| \geq |x_1| + |x_{i+1}| - |x_i|$$

for all $1 \leq i \leq k - g_4$, and Theorem 3.1 implies

$$|u_i| - |x_i| \geq |x_i| \geq 4$$

for $i > k - g_4$. Putting these into the above formula and summing gives the estimate

$$\operatorname{fd} H^* \geq \big(2k - 2g_4 + 2\big) + (4g_4 - 4) + (12 - 4) = 2k + 2g_4 + 6.$$

If $g_4 \geq 3$, then $k \geq m + g_4 \geq 5$ and hence $\operatorname{fd} H^* > 20$. This is what we wish to show, so we may assume that $g_4 = m = 2$. The degree type is of the form

$$(2,\ldots,2,4,4;B_1,\ldots,B_k)$$

with $B_i \geq 2|x_i| = 4$ for $1 \leq i \leq k-3$, $B_{k-2} \geq |x_1| + |x_{k-1}| = 6$, $B_{k-1} \geq 2|x_{k-1}| = 8$, and $B_k \geq 12$. Going back to the estimate on $\operatorname{fd} H^*$, we see that $k \in \{4,5\}$.

If $k = 4$, then $\operatorname{fd} H^* = \sum_{i=1}^{4} B_i - 12 \geq 18$. Since we may assume that $\operatorname{fd} H^* \leq 20$, it follows either that we have equality in all four of the lower bounds on the B_i or that we have equality in three of the four bounds and we are off by two in the fourth. This gives rise to five possibilities for the degree type. Two of these are ruled out by the SAC(4,4) condition, one is ruled out by the bound $r_{12} + r_{16} + \cdots \geq m - r_4$ from Lemma 4.1, and the remaining two appear in the conclusion of the proposition.

If instead $k = 5$, then we estimate as above: $\operatorname{fd} H^* = \sum_{i=1}^{5} B_i - 14 \geq 20$. Hence equality holds in all five of the lower bounds on the B_i, and we find that the degree type is the last one shown in statement of the proposition. □

5. The Top-to-Bottom Lemma

In this section, we prove the Top-to-Bottom Lemma and use it to prove Proposition 5.2, which verifies the Halperin conjecture for formal dimensions up to 20 in all but one exceptional case when the largest two generator degrees satisfies $|x_{k-1}| + |x_k| = 10$.

Lemma 5.1 (Top-to-Bottom Lemma). *Let $H^* \cong \mathbb{Q}[x_1,\ldots,x_k]/(u_1,\ldots,u_k)$ be a positively elliptic algebra that does not split and that satisfies $|u_k| < 3|x_k|$. If there exists a derivation δ on H^* and $l \geq 1$ such that the map*

$$\delta^l \colon H^{|x_k|} \to H^{|x_1|}$$

exists and is nonzero, then in fact this map has rank at least two.

This lemma is reminiscent of the $k-1$ Lemma, which states that a derivation with negative degree is nonzero only if it has rank at least two.

Proof. Without loss of generality, we may assume that $|\delta|$ divides $|x_k| - |x_1|$, and we may fix $l \geq 1$ such that δ^l maps $H^{|x_k|}$ into $H^{|x_1|}$. We may also assume that this map has rank exactly one and change basis, if necessary, so that $\delta^l(x_k) = x_1$ and that $\delta^l(x_i) = 0$ for $i < k$.

Consider the ideal in H^* generated by $x_1, \ldots, x_{k-1}$. Since H^* is finite-dimensional, there exists some relation u_i not in this ideal. Since $|u_i| < 3|x_k|$, we must have

$$u_i = \lambda x_k^2 + x_k f + g$$

for some nonzero $\lambda \in \mathbb{Q}$ and some $f, g \in \mathbb{Q}[x_1, \ldots, x_{k-1}]$. By scaling u_i, we may assume $\lambda = 1$, and then completing the square and replacing x_k by $x_k + \frac{1}{2}f$, we may assume $f = 0$.

We apply δ^{2l} to this equation. On the left-hand side, we see that $\delta^{2l}(u_i)$ is in the ideal $(u_1, \ldots, u_k)$ and has (minimal) degree $2|x_1|$. In particular, $\delta^{2l}(u_i)$ is a rational linear combination of the u_j with minimal degree. Hence either it is zero or it is u_1 after possibly replacing u_1 by this linear combination.

On the right-hand side, note that

$$\delta^{2l}(x_k^2) = \binom{2l}{l}\left(\delta^l x_k\right)^2 = \binom{2l}{l}x_1^2.$$

If it is the case that $\delta^{2l}(g) = 0$, then we have that $u_1 \in \mathbb{Q}[x_1]$, a contradiction to the assumption that H^* does not split. Hence, we may assume that $\delta^{2l}(g) \neq 0$.

Now g is a polynomial in $x_1, \ldots, x_{k-1}$, so there exists a monomial $x_{i_1} \cdots x_{i_p}$ appearing in g such that

$$\delta^{2l}(x_{i_1} \cdots x_{i_p}) \neq 0.$$

Furthermore, by the Leibniz rule, there exists $j_1 + \cdots + j_p = 2l$ such that

$$\delta^{j_1}(x_{i_1}) \cdots \delta^{j_p}(x_{i_p}) \neq 0.$$

Each term in this product is nonzero and hence has degree at least $|x_1|$. Summing, we have

$$p|x_1| \leq \left(|x_{i_1}| + j_1|\delta|\right) + \cdots + \left(|x_{i_p}| + j_p|\delta|\right) = 2|x_k| + 2l|\delta| = 2|x_1|.$$

Hence $p \leq 2$. At the same time, x_k has maximal degree among the generators, so $p = 2$ and equality holds in the estimate above. It follows that some $\delta^l(x_i) \neq 0$ with $x_i \neq x_k$, and this implies a contradiction to our choice of basis at the beginning of the proof. □

Using the Top-to-Bottom Lemma, we can nearly prove the theorem under the condition $|x_{k-1}| + |x_k| = 10$. The exceptional case given in Proposition 5.2 is proved in Section 6.

Proposition 5.2. *Let H^* be a positively elliptic algebra that does not split. If there exists a nonzero derivation of negative degree and $|x_{k-1}| + |x_k| = 10$, then either* $\operatorname{fd} H^* > 20$ *or the degree type is equal to*

$$(2, 2, 2, 4, 6; 4, 4, 6, 10, 12).$$

Proof. Since $|x_{k-1}|$ and $|x_k|$ are positive, even numbers summing to 10, and since $|x_{k-1}| \neq 2$ by the Land in Zero and $k-1$ Lemmas, we may assume that $|x_{k-1}| = 4$ and $|x_k| = 6$. In addition, we may assume that

$$\delta(x_{k-1}) = x_1$$

up to a change in basis. Note also that $k \geq 3$.

First suppose that $|u_k| > 12$. By the condition $SAC(6)$, there is a relation whose degree is properly divisible by six. In particular, $|u_{k-1}| \geq 12$ or $|u_k| \geq 18$, and hence

$$\sum_{i=k-1}^{k} (|u_i| - |x_i|) = |u_{k-1}| + |u_k| - 10 \geq 16.$$

Note also that

$$|u_{k-2}| - |x_{k-2}| \geq \max\left(|x_{k-2}|, |x_1| + |x_{k-1}| - |x_{k-2}|\right).$$

Since the maximum is at least the average, and since the left-hand side is even, this is at least four. Substituting these estimates into the formula for the formal dimension and applying the Degree Inequality, we have

$$\operatorname{fd} H^* \geq \sum_{i=k-2}^{k} (|u_i| - |x_i|) \geq 20.$$

Since we may assume that $\operatorname{fd} H^* \leq 20$, we have equality everywhere. In particular, $k = 3$ and $|u_1| - |x_1| = 4$. But the $k-1$ Lemma implies that $\delta(x_2) = \lambda x_1 \neq 0$, so Part 2 of the Degree Inequality implies that $|u_1| \geq |x_1| + |x_3|$. This is a contradiction, and we may assume that $|u_k| = 12$.

The Top-to-Bottom Lemma implies that $\delta^2(x_k) = 0$. After replacing x_k by something of the form $x_k - l(x_1, \ldots, x_{k-2})x_{k-1}$, we may assume that

$$\delta(x_k) = p(x_2, \ldots, x_{k-2}) \neq 0.$$

In particular, $k \neq 3$, since otherwise this expression implies that $\delta(x_3) = 0$, a contradiction to the $k-1$ Lemma. Assume then that $k \geq 4$.

The condition $\delta^2(x_k) = 0$ also means that we can apply the second part of Large Relations Lemma (Lemma 4.1). Hence, $|u_{k-1}| \geq 10$.

Suppose first that $k \geq 5$. Since $|u_{k-1}| - |x_{k-1}| \geq 6$ and $|u_k| - |x_k| = 6$, we can estimate the formal dimension as above to obtain

$$\operatorname{fd} H^* \geq (k-3)|x_1| + 4 + 6 + 6 \geq 20.$$

Hence we may assume that equality holds in these estimates. It follows that the degree type is of the form

$$(2, A_2, A_3, 4, 6; 2 + A_2, 2 + A_3, 6, 10, 12).$$

But now the bounds $|u_i| \geq 2|x_i|$ for all i imply that $A_3 = 2$ and $A_2 = 2$, so this is the exceptional case given in the conclusion of the proposition.

We may assume therefore that $k = 4$. In particular,

$$\delta(x_3) = x_1 \text{ and } \delta(x_4) = p(x_2),$$

where p is linear if $|x_2| = 4$ and quadratic if $|x_2| = 2$.

Since H^* is finite-dimensional, not all of the u_i lie in the ideal $I = (x_1, x_2, x_4)$, since otherwise H^* projects onto the infinite-dimensional algebra $\mathbb{Q}[x_1, \ldots, x_4]/I$. Hence there exists a relation (up to scaling) of the form

$$u_i = x_3^2 + q \text{ or } u_i = x_3^3 + q$$

for some $q \in I$. For degree reasons, the structure of δ implies that $q \in \ker(\delta^2)$ in the first case or $q \in \ker(\delta^3)$ in the second. Applying δ^2 or δ^3, we see that

$$2x_1^2 = \delta^2(u_i) \text{ or } 6x_1^3 = \delta^3(u_i).$$

Since δ preserves the ideal $(u_1, \ldots, u_4)$, the right-hand side of each expression lies in this ideal. In the first case, we may perform a change of basis on the degree four u_i to obtain $u_1 = 2x_1^2$. This gives rise to a splitting by the subalgebra generated by x_1, a contradiction to the assumptions of the proposition.

Similarly, the second case gives rise to a contradiction if it is possible to change basis so that some $u_j = 6x_1^3$. Therefore we may assume that

$$6x_1^3 = \sum l_j u_j$$

where the l_j are linear polynomials in the degree two generators and the u_j are degree four relations. Now if u_1 is the only one degree four relation, then u_1 is a multiple of x_1^2, which is again a contradiction. But then we must have that $|u_1| = |u_2| = 4$, so we have that $|u_2| < 6 = |x_1| + |x_3|$. By the Degree Inequality, it follows that x_1 and x_2 generate a subalgebra of H^* that induces a splitting. This is a contradiction, so the proof is complete. □

6. Proof of the main theorem

In this section, we finish the proof of the Halperin Conjecture for formal dimensions at most 20. We are given a positively elliptic algebra

$$H^* \cong \mathbb{Q}[x_1, \ldots, x_k]/(u_1, \ldots, u_k)$$

as in Theorem 3.1, and we assume the existence of a nonzero derivation δ on H^* of negative degree. We seek a contradiction.

If the formal dimension is two, then Theorem 3.1 implies that $k = 1$ and the Land in Zero Lemma implies that $\delta = 0$, a contradiction. Hence we may inductively assume that $2 < \operatorname{fd} H^* \leq 20$ and the Halperin Conjecture holds for formal dimensions less than $\operatorname{fd} H^*$.

By Markl's theorem, we may assume that H^* does not split. In particular, Propositions 3.9, 4.2, and 5.2 apply, and together they imply that the degree sequence of H^* must fall into one of six exceptional cases. To finish the proof, therefore, it suffices to prove Halperin's conjecture in each of these six cases.

We first consider the three exceptional cases that arose in the case $|x_{k-1}| + |x_k| = 8$ (see Proposition 4.2):

Proposition 6.1. *If H^* is a positively elliptic algebra that does not split and has degree type*

$$(2, 2, 4, 4; 4, 6, 8, 12), (2, 2, 4, 4; 4, 8, 8, 12), \text{ or } (2, 2, 2, 4, 4; 4, 4, 6, 8, 12),$$

then there does not exist a nonzero derivation with negative degree.

Proof. We adopt the notation from Lemma 4.1, with a slight modification. We may assume

$$\delta(x_{k-1}) = x_1 \text{ and } \delta(x_k) = x_2$$

and that $\delta(x_i) = 0$ for $1 \leq i \leq k-2$. In addition, after possibly swapping the two degree eight relations in the second case, we may assume that

$$u_{k-1} = p_{k-1}(x_{k-1}, x_k) + q_{k-1}$$

with $p_{k-1} \neq 0$ and $q_{k-1} \in (x_1, \ldots, x_{k-2})$. Indeed, if $p_{k-1} = 0$ (and $p_{k-2} = 0$ in the second case), then H^* admits a quotient map onto $\mathbb{Q}[x_1, \ldots, x_k]/(x_1, \ldots, x_{k-2}, u_k)$, a contradiction to finite-dimensionality.

Applying δ^2 as in the proof of Lemma 4.1, we find that

$$p_{k-1}(x_1, x_2) = u_1$$

after possibly changing basis in the degree four relations. In addition, in the case where u_{k-2} also has degree eight, we find that p_{k-2} is a multiple of u_1, where $u_{k-2} = p_{k-2}(x_{k-1}, x_k) + q_{k-2}$ and $q_{k-2} \in (x_1, \ldots, x_{k-2})$. In this case, we can replace u_{k-2} by $u_{k-2} - \mu u_{k-1}$ for some $\mu \in \mathbb{Q}$ so that $p_{k-2} = 0$. In any case, we have shown that

$$u_1, \ldots, u_{k-2} \in (x_1, \ldots, x_{k-2}).$$

We extend the argument from Lemma 4.1 by considering the degree 12 relation u_k. Write

$$u_k = p_k(x_{k-1}, x_k) + q_k$$

for some cubic polynomial p_k and some $q_k \in (x_1, \ldots, x_{k-2})$. For degree reasons, we have that $\delta^3(q_k) = 0$ and hence that

$$6p_k(x_1, x_2) = \delta^3(u_k) \in (u_1, \ldots, u_k).$$

Note that $p_k(x_1, x_2)$ has degree six and can be expressed as

$$p_k(x_1, x_2) = \sum_{i=1}^{k} h_i u_i$$

where $h_i \in \mathbb{Q}[x_1, \dots, x_k]$ is a linear polynomial in the first $k-2$ variables if $|u_i| = 4$, where $h_i \in \mathbb{Q}$ if $|u_i| = 6$, and where $h_i = 0$ if $|u_i| \geq 8$.

We further claim that $h_i = 0$ when $|u_i| = 6$. Indeed, otherwise we can replace u_i by the expression $\sum h_i u_i$ so that $u_i = p_k(x_1, x_2)$. For the degree types under consideration, this implies that $x_1, x_2, \dots, x_{k-2}$ generate a subalgebra K^* that induces a splitting of H^*, a contradiction. We may therefore assume that $p_k(x_1, x_2) = h_1 u_1$ in the first two cases and that $p_k(x_1, x_2) = h_1 u_1 + h_2 u_2$ in the third.

To derive a contradiction in the first two cases (where $k = 4$), recall that $u_1 = p_3(x_1, x_2)$ and hence that $p_4(x_3, x_4)$ is in the ideal

$$I = (x_1, x_2, p_3(x_3, x_4)).$$

For degree reasons, it follows that I contains all four of the u_i and hence that there exists a projection of H^* onto $\mathbb{Q}[x_1, \dots, x_4]/I$. Since the latter space has infinite dimension, this is a contradiction.

To derive a contradiction in the last case (where $k = 5$), we consider the expression

$$p_5(x_1, x_2) = h_1 u_1 + h_2 u_2.$$

Write $h_i = l_i(x_1, x_2) + k_i x_3$ for some linear polynomials l_i and some $k_i \in \mathbb{Q}$, and write $u_2 = u_{2,0}(x_1, x_2) + x_3 u_{2,1}(x_1, x_2, x_3)$. We break the proof into cases.

- Suppose $u_{2,1} = 0$. This implies that u_2 is a polynomial in x_1 and x_2. Since $u_1 = p_4(x_1, x_2)$ as well, we see that x_1 and x_2 generate a subalgebra that induces a splitting of H^*, a contradiction.
- Suppose $h_2 = 0$. This implies that $u_1 = p_4$ divides p_5. Hence the ideal

  $$I = (x_1, x_2, x_3, p_4(x_4, x_5))$$

  contains all of the u_j, a contradiction to finite-dimensionality of H^*.
- Suppose instead that $u_{2,1} \neq 0$ and that $h_2 \neq 0$. Comparing coefficients of x_3^2 and x_3^3 in the above equation, we see that $h_2 = l_2 \neq 0$. Similarly, comparing coefficients of x_3, we find that $k_1 \neq 0$.

  Now l_2 divides $p_5 - h_1 p_4$, which can be written as

  $$\left(p_5 - l_1 p_4\right) - x_3 \left(k_1 p_4\right).$$

  It follows that l_2 divides both $p_5 - l_1 p_4$ and $k_1 p_4$ and hence p_4 and p_5 as well. Hence the ideal

  $$I = (x_1, x_2, x_3, l_2(x_4, x_5))$$

  contains all five of the relations u_j, and we once again have a contradiction to the finite-dimensionality of H^*.

We have derived a contradiction in all cases, so the proof is complete. □

Next we consider the exceptional case arising in the case where $|x_{k-1}| + |x_k| = 10$ (see Proposition 5.2):

Proposition 6.2. *Let $H^* \cong \mathbb{Q}[x_1, \dots, x_k]/(u_1, \dots, u_k)$ be a positively elliptic algebra that does not split. If the degree type is*

$$(2, 2, 2, 4, 6; 4, 4, 6, 10, 12),$$

then H^* does not admit a nonzero derivation of negative degree.

Proof. As in the proof of Proposition 5.2, we may assume that

$$\delta(x_4) = x_1 \text{ and } \delta(x_5) = p(x_2, x_3).$$

Consider the ideal $I = (x_1, x_2, x_3, u_5)$. For degree reasons, $u_j \in I$ for all $j \neq 4$. Since H^* is finite-dimensional, it follows that $u_4 \notin I$. After scaling u_4, if necessary, we have

$$u_4 = x_4 x_5 + q_4$$

with $q_4 \in I$.

Note that q_4 has degree ten. For degree reasons, it is a polynomial in $\mathbb{Q}^{\geq 3}[x_1, \ldots, x_5]$. Note that δ preserves this subspace. Since, in addition, $x_4\delta(x_5) = x_4 p(x_2, x_3)$ is in this subspace, we have that

$$\delta(u_4) \in x_1x_5 + \mathbb{Q}^{\geq 3}[x_1, \ldots, x_5].$$

On the other hand, $\delta(u_4)$ is a degree eight element of the ideal $(u_1, \ldots, u_5)$. For degree reasons, this implies that

$$\delta(u_4) = \sum_{i=1}^{3} h_i u_i$$

with $h_i \in \mathbb{Q}^{\geq 1}[x_1, \ldots, x_5]$. But each u_j is an element of $\mathbb{Q}^{\geq 2}[x_1, \ldots, x_5]$, so $\delta(u_j)$ is as well. Hence this equation shows that $\delta(u_4) \in \mathbb{Q}^{\geq 3}[x_1, \ldots, x_5]$, a contradiction. □

Finally, we consider the remaining two exceptional cases, which arise in the case where $|x_{k-1}|+|x_k| \geq 12$ (see Proposition 3.9). Note that, for the first time, the possibility that δ has degree -4 is nontrivial. Indeed, in all previous cases, it is immediate to see that δ having degree -4, -6,…implies that δ is zero on at least $k-1$ generators for degree reasons and hence that $\delta = 0$ by the $k-1$ Lemma.

The first of the two remaining cases is simpler and uses ideas similar to previous proofs.

Proposition 6.3. *If $H^* = \mathbb{Q}[x_1, \ldots, x_k]/(u_1, \ldots, u_k)$ is a positively elliptic algebra with no nontrivial subalgebra and degree type*

$$(2, 4, 6, 6; 6, 8, 12, 12),$$

then H^* does not admit a nonzero derivation with negative degree.

Proof. Suppose first that $\delta(x_2) = x_1$, after possibly rescaling. By the Top-to-Bottom Lemma, we see that $\delta(x_i) = \lambda_i x_1^2$ for some $\lambda_i \in \mathbb{Q}$ for $i \in \{3, 4\}$. Replacing x_i by $x_i - \lambda_i x_1 x_2$, we find that $x_1, x_3, x_4 \in \ker(\delta)$ in contradiction to the $k-1$ Lemma. Hence, we may assume that

$$\delta(x_1) = 0 \text{ and } \delta(x_2) = 0.$$

Furthermore, we may assume that $\delta(x_3)$ and $\delta(x_4)$ are linearly independent elements in degree four. In particular, δ cannot have degree -4 (or smaller), so δ has degree -2. After choosing a suitable basis, we may assume that

$$\delta(x_3) = x_1^2 \text{ and } \delta(x_4) = x_2.$$

Write

$$u_j = p_j(x_3, x_4) + q_j$$

for $j \in \{3, 4\}$, where $q_j \in (x_1, x_2)$. Note that $\delta^2(q_j) = 0$ for degree reasons, so

$$2p_j(x_1^2, x_2) = \delta^2(u_j) \in (u_1, \ldots, u_4).$$

This is an equation in degree eight, so we have

$$2p_j(x_1^2, x_2) = ax_1u_1 + bu_2$$

for some $a, b \in \mathbb{Q}$. Note that $b = 0$, since otherwise u_1 and u_2 are polynomials in x_1 and x_2, which contradicts the assumption that H^* does not have a nontrivial subalgebra.

Since $b = 0$, we find that x_1 divides $p_j(x_1^2, x_2)$ for $j \in \{3, 4\}$. This implies that x_1^2 divides $p_j(x_1^2, x_2)$, and hence both $p_3(x_3, x_4)$ and $p_4(x_3, x_4)$ are divisible by x_3. It follows that

$$u_1, \ldots, u_4 \in (x_1, x_2, x_3),$$

which is a contradiction to the finite-dimensionality of H^*. □

Finally, we prove the last exceptional case. We wish to highlight that the proof in this case differs from all of the previous arguments. Specifically, we do not choose our basis in order to simplify the action of δ, as this does not appear to help us. Rather we choose our basis in order to simplify the form of the relations.

Proposition 6.4. *If $H^* = \mathbb{Q}[x_1, \dots, x_k]/(u_1, \dots, u_k)$ is a positively elliptic algebra with no nontrivial subalgebra and degree type*

$$(2, 2, 6, 6; 4, 8, 12, 12),$$

then H^* does not admit a nonzero derivation with negative degree.

Proof. Suppose δ is a nonzero derivation of negative degree, and note that δ has degree -2 or -4 by the Land in Zero Lemma. For $j \in \{3, 4\}$, write

$$u_j = p_j(x_3, x_4) + q_j$$

where $q_j \in (x_1, x_2)$. Since q_j has degree 12 and hence at most one x_3 or x_4 in each of its monomials, $q_j \in \ker(\delta^2)$.

Note that p_3 and p_4 are coprime polynomials. Indeed, if $g(x_3, x_4)$ were a non-constant common factor, then all relations u_j are in the ideal $I = (x_1, x_2, g(x_3, x_4))$ and H^* projects onto the infinite-dimensional space $\mathbb{Q}[x_1, \dots, x_4]/I$, a contradiction.

Since $p_3(x_3, x_4)$ and $p_4(x_3, x_4)$ are coprime, quadratic polynomials, we can choose bases of $\operatorname{span}\{x_3, x_4\}$ and $\operatorname{span}\{u_3, u_4\}$ such that one of the following cases occurs:

1. $p_3 = x_3^2$ and $p_4 = x_4^2$, or
2. $p_3 = x_3^2 - \lambda x_4^2$ and $p_4 = x_3x_4$ for some $\lambda \neq 0$.

Indeed, up to relabeling and scaling, we may assume that p_3 contains an x_3^2 term. Completing the square and replacing x_3 by something of the form $x_3 + \mu x_4$, we find that $p_3 = x_3^2 - \lambda x_4^2$ for some $\lambda \in \mathbb{Q}$. Subtracting a multiple of u_3 from u_4 corresponds to subtracting the same multiple of p_3 from p_4. We can do this so that $p_4 = \mu x_3x_4 + \nu x_4^2$ for some $\mu, \nu \in \mathbb{Q}$. If $\mu = 0$, the claim follows by rescaling u_4 and subtracting a multiple of u_4 from u_3. If $\mu \neq 0$, we may replace x_3 by $\mu x_3 + \nu x_4$. This results in $p_4 = x_3x_4$. Subtracting now a multiple of u_4 from u_3 and scaling u_3 once more, we find that we are in the second case of the claim. Note here that $\lambda \neq 0$ because p_3 and p_4 are coprime.

Returning to the expressions for u_j, we apply δ^2 to get

$$2p_j(\delta(x_3), \delta(x_4)) = \delta^2(u_j) \in (u_1, \dots, u_4).$$

Suppose first that δ has degree -4, so that $\delta(x_j) \in \operatorname{span}\{x_1, x_2\}$ for $j \in \{3, 4\}$. Without loss of generality, we may assume $\delta(x_3) = x_1$ and $\delta(x_4) = x_2$. Since p_3 and p_4 are coprime polynomials, so are

$$\delta(u_3) = 2p_3(x_1, x_2) \quad \text{and} \quad \delta(u_4) = 2p_4(x_1, x_2).$$

But $\delta(u_3), \delta(u_4) \in \operatorname{span}\{u_1\}$, so we have a contradiction.

Suppose instead that δ has degree -2. Since the expressions for $p_j(\delta(x_3), \delta(x_4))$ are in degree eight, we have equations of the form

$$2p_j(\delta(x_3), \delta(x_4)) = l_j(x_1, x_2)u_1 + k_ju_2$$

for $j \in \{3, 4\}$, where the l_j are linear polynomials and the $k_j \in \mathbb{Q}$.

If some $k_j \neq 0$, we may replace u_2 by $l_j(x_1, x_2)u_1 + k_ju_2$ and conclude that u_1 and u_2 are polynomials in x_1 and x_2. This implies the existence of nontrivial subalgebra, a contradiction.

We may assume that $k_3 = k_4 = 0$, so that u_1 divides both $p_3(\delta(x_3), \delta(x_4))$ and $p_4(\delta(x_3), \delta(x_4))$. Using the simple formulas for p_3 and p_4, we see that one of the following happens:

1. u_1 divides both $\delta(x_3)^2$ and $\delta(x_4)^2$.
2. u_1 divides both $\delta(x_3)^2 - \lambda\delta(x_4)^2$ and $\delta(x_3)\delta(x_4)$ for some $\lambda \in \mathbb{Q} \setminus \{0\}$.

In either case, if u_1 is irreducible, it follows that u_1 divides both $\delta(x_3)$ and $\delta(x_4)$. Since all of these elements have degree four, we find that $\delta(x_3)$ and $\delta(x_4)$ are linearly dependent. After changing basis once more, we find a contradiction to the $k-1$ Lemma.

Next if $u_1 = l_1 l_2$ is a product of coprime irreducibles, then each irreducible factor divides both $\delta(x_3)$ and $\delta(x_4)$ by a similar argument. Moreover, since l_1 and l_2 are coprime, it follows that u_1 divides both of these elements, and we again have a contradiction.

Finally, if neither of these cases occurs, then $u_1 = \lambda l^2$ for some $\lambda \in \mathbb{Q}$ and some linear polynomial $l = l(x_1, x_2)$. But now we can replace x_1 or x_2 by $l(x_1, x_2)$ and derive the existence of a nontrivial subalgebra of H^*, so we again have a contradiction. □

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