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Stokes phenomena, Poisson–Lie groups and quantum groups ☆

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ABSTRACT

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{b}_\pm \subset \mathfrak{g}$ a pair of opposite Borel subalgebras, and $\mathsf{r} \in \mathfrak{b}_- \otimes \mathfrak{b}_+$ the corresponding solution of the classical Yang–Baxter equations. Let G be the simply–connected Poisson–Lie group corresponding to $(\mathfrak{g}, \mathsf{r})$, $H \subset B_\pm \subset G$ the subgroups with Lie algebras $\mathfrak{h} = \mathfrak{b}_- \cap \mathfrak{b}_+$ and $\mathfrak{b}_\pm$, and $G^* = B_+ \times_H B_-$ the Poisson–Lie group dual of G. G–valued Stokes phenomena were used by Boalch [3,4] to give a canonical, analytic linearisation of the Poisson–Lie group structure on G^*. $U\mathfrak{g}$–valued Stokes phenomena were used by the first author to construct a twist killing the KZ associator, and therefore give a transcendental construction of the Drinfeld–Jimbo quantum group $U_\hbar\mathfrak{g}$ [23]. In the present paper, we show that the former construction can be obtained as semiclassical limit of the latter. Along the way, we also show that the R–matrix of $U_\hbar\mathfrak{g}$ is a Stokes matrix for the dynamical KZ equations.

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Contents

1. Introduction

1.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $U_\hbar\mathfrak{g}$ its quantised enveloping algebra. The starting point of the present paper is the construction of $U_\hbar\mathfrak{g}$ from the *dynamical* Knizhnik–Zamolodchikov (DKZ) equations obtained by the first author [23].

Let $(\cdot,\cdot)$ be an invariant inner product on $\mathfrak{g}$, $\Omega \in \mathfrak{g}\otimes\mathfrak{g}$ the corresponding Casimir element, and $\mathfrak{h}\subset\mathfrak{g}$ a Cartan subalgebra. Consider the DKZ on $n=2$ points, that is the $\mathrm{End}(U\mathfrak{g}^{\otimes 2})$–valued connection on $\mathbb{C}\ni z = z_1 - z_2$ given by

$$d - \left(\mathsf{h}\frac{\Omega}{z} + \operatorname{ad}\mu^{(1)}\right) dz \tag{1.1}$$

where $\mu\in\mathfrak{h}$, $\mu^{(1)} = \mu\otimes 1$, and h is a formal deformation parameter. Just as its non–dynamical counterpart which is obtained for $\mu = 0$, the connection (1.1) has a regular singularity at $z=0$, and admits a canonical fundamental solution Υ_0 which is asymptotic to $z^{\mathsf{h}\Omega}$ as $z\to 0$.

1.2. The dynamical term $\operatorname{ad}\mu^{(1)}$ gives rise to an *irregular singularity* at $z=\infty$. Assuming that μ is real, so that all Stokes rays lie in $\mathbb{R}$, and regular, it is proved in [23] that (1.1) admits two canonical fundamental solutions $\Upsilon_\pm$ which are asymptotic to $e^{z\operatorname{ad}\mu^{(1)}}\cdot z^{\mathsf{h}\Omega_0}$ as $z\to\infty$ with $\operatorname{Im} z \gtrless 0$, where $\Omega_0\in\mathfrak{h}\otimes\mathfrak{h}$ is the projection of Ω.

Consider now the regularised holonomy of (1.1) from $\pm\iota\infty$ to 0 *i.e.,* the element $J_\pm\in U\mathfrak{g}^{\otimes 2}[\![\mathsf{h}]\!]$ given by $J_\pm = \Upsilon_0(z)^{-1}\cdot\Upsilon_\pm(z)$, where $\operatorname{Im} z\gtrless 0$. One of the main results of [23] is that $J_\pm$, regarded as a twist, kills the KZ associator Φ_{KZ} which arises from the (non–dynamical, reduced) KZ equations on $n=3$ points

$$d - \mathsf{h}\left(\frac{\Omega_{12}}{z} + \frac{\Omega_{23}}{z-1}\right) dz$$

Let $\Delta_\pm = J_\pm^{-1}\Delta(\cdot)J_\pm$ and $R_\pm = (J_\pm^{21})^{-1}e^{\hbar\Omega/2}J_\pm$ be the corresponding twisted coproduct and R–matrix, where $\hbar = 2\pi\iota\mathsf{h}$. It follows that $(U\mathfrak{g}[\![\hbar]\!],\Delta_\pm,R_\pm)$ is a quasitriangular Hopf algebra, which can be shown to be isomorphic to the quantum group $U_\hbar\mathfrak{g}$.

1.3. In contrast to earlier constructions of $U_\hbar\mathfrak{g}$ from the (non–dynamical) KZ equations [8,17,11–13], the above construction is entirely transcendental *i.e.,* does not rely on cohomological arguments or the representation theory of $\mathfrak{g}$, and perhaps more naturally explains how $U_\hbar\mathfrak{g}$ arises from such equations.

One additional feature is its compatibility with the Casimir equations of $\mathfrak{g}$ introduced in [6,18,20,14]. Specifically, the twist $J_\pm$ is a smooth function of $\mu \in \mathfrak{h}^{\mathbb{R}}_{\mathrm{reg}}$, and satisfies the PDE

$$dJ_\pm = \frac{\mathsf{h}}{2} \sum_{\alpha\in\Phi_+} \frac{d\alpha}{\alpha} \left(\Delta(\mathcal{K}_\alpha)J_\pm - J_\pm({\mathcal{K}_\alpha}^{(1)} + {\mathcal{K}_\alpha}^{(2)})\right)$$

where Φ_+ is a chosen system of positive roots, and $\mathcal{K}_\alpha$ the Casimir of the $\mathfrak{sl}_2$–subalgebra of $\mathfrak{g}$ corresponding to α. This compatibility is a key ingredient in proving that the monodromy of the Casimir connection of $\mathfrak{g}$ is given by Lustzig's quantum Weyl group operators [21–23,1].

1.4. Let now G be the connected and simply connected complex Lie group corresponding to $\mathfrak{g}$. Irregular singularities were exploited earlier by Boalch to linearise the Poisson structure on the Poisson–Lie group G^* dual to G [3,4].

Boalch considered connections on the holomorphically trivial G–bundle over $\mathbb{P}^1$ of the form

$$d - \left(\frac{A}{z^2} + \frac{B}{z}\right) dz \tag{1.2}$$

where $A \in \mathfrak{h}$ is regular, and $B \in \mathfrak{g}$.

Assume that A is real, so that the Stokes rays of (1.2) lie in $\mathbb{R}$,[1] and set $\mathbb{H}_\pm = \{z \in \mathbb{C}|\ \mathrm{Im}\, z \gtrless 0\}$. Then, there are unique holomorphic fundamental solutions $\gamma_\pm : \mathbb{H}_\pm \to G$ of (1.2), which are asymptotic to $e^{-A/z} \cdot z^{[B]}$ as $z \to 0$ in $\mathbb{H}_\pm$, where $[B]$ is the projection of B onto $\mathfrak{h}$.

Define the *Stokes matrices* $S_\pm \in G$ by the analytic continuation identities

$$\widetilde{\gamma_-} = \gamma_+ \cdot S_+ \qquad \text{and} \qquad \widetilde{\gamma_+} = \gamma_- \cdot S_- \cdot e^{2\pi\iota[B]}$$

where $\widetilde{\cdot}$ denotes counterclockwise analytic continuation, and the identities hold in $\mathbb{H}_+$ and $\mathbb{H}_-$ respectively. The elements $S_\pm \in G$ are unipotent. Specifically, A determines a partition $\Phi = \Phi_+ \sqcup \Phi_-$ of the root system by $\Phi_\pm = \{\alpha \in \Phi|\ \alpha(A) \gtrless 0\}$, and $S_\pm$ lies in the unipotent subgroup $N_\pm \subset G$ with Lie algebra $\mathfrak{n}_\pm = \bigoplus_{\alpha\in\Phi_\pm} \mathfrak{g}_\alpha$.

[1] Contrary to the reality assumption made in 1.2, the assumption that $A \in \mathfrak{h}^{\mathbb{R}}$ is inessential, and is only made in the Introduction to simplify the exposition.

1.5. Let $B_\pm \subset G$ be the Borel subgroups corresponding to $\Phi_\pm$, $H = B_+ \cap B_-$ the maximal torus with Lie algebra $\mathfrak{h}$, and consider the fibred product

$$B_+ \times_H B_- = \{(b_+, b_-) \in B_+ \times B_- \,|\, \pi_+(b_+)\pi_-(b_-) = 1\}$$

where $\pi_\pm : B_\pm \to H$ are the quotient maps. Following [4], we define the *Stokes map* to be the analytic map $\mathcal{S} : \mathfrak{g} \longrightarrow B_+ \times_H B_-$ given by

$$B \longrightarrow \left(S_+^{-1} \cdot e^{-\iota\pi[B]}, S_- \cdot e^{\iota\pi[B]}\right) \tag{1.3}$$

1.6. The pair (B_+, B_-) gives rise to a solution $\mathsf{r} \in \mathfrak{b}_- \otimes \mathfrak{b}_+$ of the classical Yang–Baxter equations given by

$$\mathsf{r} = x_i \otimes x^i + \frac{1}{2} t_a \otimes t^a \tag{1.4}$$

where $\{x_i\}, \{x^i\}$ are bases of $\mathfrak{n}_-, \mathfrak{n}_+$ which are dual with respect to $(\cdot,\cdot)$, and $\{t_a\}, \{t^a\}$ are dual bases of $\mathfrak{h}$. The element r gives $\mathfrak{g}$ the structure of a quasitriangular Lie bialgebra, with cobracket $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ given by $\delta(x) = [x \otimes 1 + 1 \otimes x, \mathsf{r}]$.

The dual Lie bialgebra $(\mathfrak{g}^*, \delta^t, [\cdot,\cdot]^t)$ may be identified, as a Lie algebra, with

$$\mathfrak{b}_+ \times_{\mathfrak{h}} \mathfrak{b}_- = \{(X_+, X_-) \in \mathfrak{b}_+ \oplus \mathfrak{b}_- | \pi_+(X_+) + \pi_-(X_-) = 0\}$$

where $\pi_\pm : \mathfrak{b}_\pm \to \mathfrak{h}$ is the quotient map. This endows $G^* = B_+ \times_H B_-$ with the structure of a Poisson–Lie group, which is dual to G.

1.7. Endow $\mathfrak{g}^*$ with its standard Kirillov–Kostant–Souriau Poisson structure

$$\{f, g\}(x) = \langle [d_x f, d_x g], x\rangle$$

where $d_x h \in T_x^*\mathfrak{g}^* = \mathfrak{g}$ is the differential of h at x, and $[\cdot,\cdot]$ is the Lie bracket on $\mathfrak{g}$.

Let $\nu : \mathfrak{g}^* \to \mathfrak{g}$ be the isomorphism induced by the bilinear form $(\cdot,\cdot)$, and identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by using $\nu^\vee = -1/(2\pi\iota)\nu$. The following remarkable result is due to Boalch [3,4].

Theorem. *The map $\mathcal{S} : \mathfrak{g}^* \to G^*$ is a Poisson map, and generically a local complex analytic diffeomorphism. In particular, $\mathcal{S}$ gives a linearisation of the Poisson structure on G^*.*

1.8. One of goals of the present paper is to prove that Boalch's linearisation result, specifically the fact that $\mathcal{S}$ is a Poisson map, can be obtained as a semiclassical limit of the transcendental construction of $U_\hbar\mathfrak{g}$.

Our overall strategy is the following. Since $\mathcal{S}$ is holomorphic, it suffices to show that its Taylor series $\widehat{\mathcal{S}}$ at $0 \in \mathfrak{g}^*$ is a formal Poisson map. This in turn follows if $\widehat{\mathcal{S}}$ can be quantised. We therefore seek quantisations $\mathbb{C}_\hbar[\![\mathfrak{g}^*]\!]$ and $\mathbb{C}_\hbar[\![G^*]\!]$ of the algebras of functions on the formal Poisson–Lie groups corresponding to $\mathfrak{g}^*$ and G^*, together with an algebra isomorphism $\widehat{\mathcal{S}}^*_\hbar : \mathbb{C}_\hbar[\![G^*]\!] \to \mathbb{C}_\hbar[\![\mathfrak{g}^*]\!]$ such that the following diagram is commutative

$$\begin{tikzcd} \mathbb{C}_\hbar[\![\mathfrak{g}^*]\!] \arrow[d] & \mathbb{C}_\hbar[\![G^*]\!] \arrow[l, "\widehat{\mathcal{S}}^*_\hbar"'] \arrow[d] \\ \mathbb{C}[\![\mathfrak{g}^*]\!] & \mathbb{C}[\![G^*]\!] \arrow[l, "\widehat{\mathcal{S}}^*"] \end{tikzcd} \tag{1.5}$$

where the vertical arrows are the specialisations at $\hbar = 0$, and the bottom one is the pullback of $\widehat{\mathcal{S}}$.

1.9. A formal quantisation of the dual P^* of a Poisson–Lie group P can be obtained from Drinfeld's *quantum duality principle* as follows [7,15]. Let $\mathfrak{U}$ be a quantised enveloping algebra which deforms the Lie bialgebra $\mathfrak{p}$ of P. Thus, $\mathfrak{U}$ is a topologically free Hopf algebra over $\mathbb{C}[\![\hbar]\!]$ such that $\mathfrak{U}/\hbar\mathfrak{U}$ is isomorphic to $U\mathfrak{p}$ and, for any $x \in \mathfrak{p}$ with cobracket $\delta(x) \in \mathfrak{p} \wedge \mathfrak{p}$

$$\delta(x) = \left.\frac{\Delta(\widetilde{x}) - \Delta^{21}(\widetilde{x})}{\hbar}\right|_{\hbar=0}$$

where $\widetilde{x} \in \mathfrak{U}$ is an arbitrary lift of x. Then, $\mathfrak{U}$ admits a canonical topological Hopf subalgebra $\mathfrak{U}'$ which is commutative mod $\hbar$, and endowed with a canonical Poisson isomorphism $\imath_\mathfrak{U} : \mathfrak{U}'/\hbar\mathfrak{U}' \to \mathbb{C}[\![P^*]\!]$.

The simplest example of Drinfeld duality arises when P is the Lie group G endowed with the trivial Poisson structure. The corresponding Lie bialgebra is $\mathfrak{g}$ with the trivial cobracket, and P^* is the additive abelian group $\mathfrak{g}^*$ with cobracket given by the transpose of the bracket on $\mathfrak{g}$. In this case, $\mathfrak{U}$ can be taken to be $U\mathfrak{g}[\![\hbar]\!]$ with undeformed product and coproduct. The corresponding subalgebra $\mathfrak{U}'$ is the (completed) Rees algebra of formal power series $\sum_{n\geq 0} x_n \hbar^n$ where the filtration order of x_n is at most n, and $\imath_\mathfrak{U}$ is the symbol map $\mathfrak{U}'/\hbar\mathfrak{U}' \to \prod_{n\geq 0} S^n\mathfrak{g} = \mathbb{C}[\![\mathfrak{g}^*]\!]$.

1.10. To obtain a formal quantisation of G^*, we seek a QUE deforming the quasi-triangular Lie bialgebra $(\mathfrak{g}, \mathsf{r})$, where $\mathsf{r} \in \mathfrak{b}_- \otimes \mathfrak{b}_+$ is the canonical element (1.4). One such quantisation is the Drinfeld–Jimbo quantum group $U_\hbar\mathfrak{g}$ corresponding to $\mathfrak{g}$. That, however, shifts the problem of filling in the diagram (1.5) to one of finding an algebra isomorphism $(U_\hbar\mathfrak{g})' \to \mathfrak{U}'$, where $\mathfrak{U} = U\mathfrak{g}[\![\hbar]\!]$, and showing that the latter quantises $\widehat{\mathcal{S}}^*$.

Alternatively, we may resort to a *preferred* quantisation of $\mathfrak{g}$, that is a QUE which is *equal* to $\mathfrak{U}$ as algebras. A class of such quantisations may be obtained as a *twist quantisation*, that is by using an element $J \in 1 + \frac{\hbar}{2}\mathsf{j} + \hbar^2\mathfrak{U}^{\otimes 2}$ satisfying $\mathsf{j} - \mathsf{j}^{21} = \mathsf{r} - \mathsf{r}^{21}$, together with the twist equation

$$\Phi \cdot J_{12,3} \cdot J_{1,2} = J_{1,23} \cdot J_{2,3}$$

where Φ is a given associator. Then, $\mathfrak{U}_J = \left(\mathfrak{U}, J^{-1}\Delta_0(\cdot)J, J_{21}^{-1}e^{\hbar\Omega/2}J\right)$ is a QUE which quantises $(\mathfrak{g}, \mathsf{r})$, and $(\mathfrak{U}_J)'$ is a formal quantisation of G^*.

1.11. A general result of Enriquez–Halbout asserts that if the twist J is *admissible*, that is such that $\hbar\log(J) \in (\mathfrak{U}')^{\otimes 2}$, the Drinfeld algebras $(\mathfrak{U}_J)'$ and $\mathfrak{U}'$ *coincide* [10]. In this case, the equality $e : (\mathfrak{U}_J)' \to \mathfrak{U}'$ clearly is an algebra isomorphism, and descends to a Poisson isomorphism $e_{\mathrm{cl},J} : \mathbb{C}[\![G^*]\!] \to \mathbb{C}[\![\mathfrak{g}^*]\!]$ given by the composition

$$e_{\mathrm{cl},J} = \imath_{\mathfrak{U}} \circ e_0 \circ \imath_{\mathfrak{U}_J}^{-1}$$

where $\imath_{\mathfrak{U}} : \mathfrak{U}'/\hbar\mathfrak{U}' \to \mathbb{C}[\![\mathfrak{g}^*]\!]$ is the symbol map, $\imath_{\mathfrak{U}_J} : (\mathfrak{U}_J)'/\hbar(\mathfrak{U}_J)' \to \mathbb{C}[\![G^*]\!]$ the canonical identification mentioned in 1.9, and $e_0 = \mathrm{id}$ the reduction of e mod $\hbar$.

One of the main results of this paper is that if $J = J_+$ is (one of) the twist(s) arising from the dynamical KZ equations described in 1.2, with Φ is the KZ associator, then J is admissible, and the corresponding map $e_{\mathrm{cl},J}$ is equal to the Stokes map $\widehat{\mathcal{S}}^*$. In particular, the latter is a Poisson map.

1.12. A key ingredient in proving the identity $e_{\mathrm{cl},J} = \widehat{\mathcal{S}}^*$ is a result of Enriquez–Etingof–Marshall [9] which gives an explicit formula for $e_{\mathrm{cl},J}$, under the additional assumptions that Φ is a Lie associator and that the admissible twist J lies in $\mathfrak{U}'\otimes\mathfrak{U}\cap\mathfrak{U}\otimes\mathfrak{U}'$.

Consider to that end the quotient $\mathfrak{U} \otimes \mathfrak{U}'/\hbar\,\mathfrak{U} \otimes \mathfrak{U}' \cong U\mathfrak{g}[\![\mathfrak{g}^*]\!]$, where the latter is the algebra of $U\mathfrak{g}$–valued formal power series on $\mathfrak{g}^*$. Let $G[\![\mathfrak{g}^*]\!]_+ \subset U\mathfrak{g}[\![\mathfrak{g}^*]\!]$ be the prounipotent group of $\mathbb{C}[\![\mathfrak{g}^*]\!]$–points of G such that their value at $0 \in \mathfrak{g}^*$ is equal to 1. Then, the following holds [9]

(1) The *semiclassical limit* $\jmath = \mathrm{scl}\,(J)$, that is the image of J in $U\mathfrak{g}[\![\mathfrak{g}^*]\!]$, lies in $G[\![\mathfrak{g}^*]\!]_+$ and is therefore a formal map $\mathfrak{g}^* \to G$.
(2) Let

$$\beta : G^* \to G, \qquad (b_+, b_-) \to b_+ \cdot b_-^{-1}$$

be the *big cell map*. Then, the composition of $e_{\mathrm{cl},J}$ with β is the formal map $\mathfrak{g}^* \to G$ given by the *twisted exponential map*

$$e_\jmath(\lambda) = \jmath(\lambda)^{-1} \cdot e^{\nu(\lambda)} \cdot \jmath(\lambda) \tag{1.6}$$

where $\nu : \mathfrak{g}^* \to \mathfrak{g}$ is the isomorphism given by the inner product.

1.13. Since β is an isomorphism when regarded as a formal map, it suffices to prove that $e_{\mathrm{cl},J} \circ \widehat{\beta}^* = \widehat{\mathcal{S}}^* \circ \widehat{\beta}^* = \widehat{\beta \circ \mathcal{S}}^*$ that is, by (1.6) that

$$\widehat{\beta \circ \mathcal{S}} = \jmath(\lambda)^{-1} \cdot e^{\nu(\lambda)} \cdot \jmath(\lambda) \tag{1.7}$$

By definition of $\mathcal{S}$ (1.3), the composition $\beta \circ \mathcal{S}$ is the map $B \to S_+^{-1} \cdot e^{-2\pi\iota[B]} \cdot S_-^{-1}$, which is the *clockwise* monodromy around $z = 0$ of (1.2) expressed in the solution γ_-. By parallel transport to $z = \infty$, where (1.2) has a regular singularity with residue $-B$, $\beta \circ \mathcal{S}$ is also equal to

$$B \to C_-^{-1} \cdot e^{-2\pi\iota B} \cdot C_- \tag{1.8}$$

where $C_- = C_-(B) \in G$ is the *connection matrix*, that is the element relating γ_- to the canonical fundamental solution γ_∞ which is asymptotic to $z^{2\pi\iota B}$ near $z = \infty$.

Comparing the right–hand sides of (1.7) and (1.8), and recalling that $\mathfrak{g}^*$ and $\mathfrak{g}$ are identified by $-1/2\pi\iota \cdot \nu$, it therefore suffices to show that $B \to \widehat{C_-}$ is the semiclassical limit of the DKZ twist J.[2]

1.14. The fact that J is a quantisation of the connection matrix C_- follows from the uniqueness of canonical fundamental solutions of (1.2), when the structure group is an arbitrary affine algebraic group, specifically the prounipotent group $G[\![\mathfrak{g}^*]\!]_+$ [5]. It stems from the basic, but seemingly novel observation that the *semiclassical limit of the DKZ equation* (1.1) *is equal to the ODE* (1.2).[3]

More precisely, if Υ is a solution of

$$\frac{d\Upsilon}{dz} = \left(\operatorname{ad}\mu^{(1)} + \mathsf{h}\frac{\Omega}{z}\right)\Upsilon$$

with values in $\mathfrak{U} \otimes \mathfrak{U}'$, the semiclassical limit γ of Υ, as a formal function of $\lambda \in \mathfrak{g}^*$ with values in $U\mathfrak{g}$, is readily seen to satisfy

$$\frac{d\gamma}{dz} = \left(\operatorname{ad}\mu + \frac{\nu(\lambda)}{2\pi\iota z}\right)\gamma$$

where $\nu(\lambda) = \mathrm{id} \otimes \lambda(\Omega)$ which, after the change of variable $z \to 1/z$, and the replacement $\operatorname{ad}\mu \to -A, \nu(\lambda) \to -2\pi\iota B$ is precisely the equation (1.2).[4]

1.15. Outline of paper

In Sections 2 and 3, we review the definition of the Stokes data and map for the connection (1.2), and the transcendental construction of $U_\hbar\mathfrak{g}$ given in [23]. In Section 4,

[2] The problem of obtaining a quantisation of the connection matrix C_- formulated in [25], together with our intuition that such a quantisation should be given by the DKZ twist J, were in fact the original impetus of this project.

[3] This is related to, but different from, the fact that a different semiclassical limit of the KZ equations are the (non–linear) Schlesinger equations [19].

[4] The appearance of the factor $2\pi\iota$ is due to the fact that the identification $\mathfrak{U}'/\hbar\mathfrak{U}' \cong \widehat{S\mathfrak{g}}$ is given by mapping $x \in \mathfrak{g}$ to $\hbar x = 2\pi\iota\mathsf{h}x \in \mathfrak{U}'$.

we show that quantum R–matrix of $U_\hbar\mathfrak{g}$ is a Stokes matrix of the dynamical KZ equation. Section 5 reviews Drinfeld's duality principle. Section 6 contains the first part of our main results, namely the fact that the semiclassical limits of the DKZ equations and its canonical solutions at 0 and ∞ are equal to the connection (1.2) and its canonical solutions, after a change of variables. Section 7 describes the linearisation formula of Enriquez–Etingof–Marshall. Finally, in Section 8, we prove that the Stokes map is Poisson and, in Section 9 relate quantum and classical isomonodromic equations.

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2. Stokes phenomena and Poisson–Lie groups

2.1. G–valued irregular connections on $\mathbb{P}^1$

Let G be an affine algebraic group defined over $\mathbb{C}$, $H \subset G$ a maximal torus, and $\mathfrak{h} \subset \mathfrak{g}$ the Lie algebras of H and G respectively. Let $\Phi \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$, and $\mathfrak{h}_{\text{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha\in\Phi} \operatorname{Ker}\alpha$ the set of regular elements in $\mathfrak{h}$.

Let $\mathcal{P}$ be the holomorphically trivial, principal G–bundle on $\mathbb{P}^1$, and consider the meromorphic connection ∇ on $\mathcal{P}$ given by

$$\nabla = d - \left(\frac{A}{z^2} + \frac{B}{z}\right) dz, \tag{2.1}$$

where $A, B \in \mathfrak{g}$. We assume henceforth that $A \in \mathfrak{h}_{\text{reg}}$. By definition, the *Stokes rays* of ∇ are the rays $\mathbb{R}_{>0} \cdot \alpha(A) \subset \mathbb{C}^*$, $\alpha \in \Phi$, that is the rays through the non–zero eigenvalues of $\operatorname{ad}(A)$. A ray r is called *admissible* if it is not a Stokes ray.

2.2. Canonical fundamental solutions

To each admissible ray r, and determination of $\log z$, there is a canonical fundamental solution γ_r of ∇ with prescribed asymptotics in the open half–plane

$$\mathbb{H}_r = \left\{ ue^{\iota\phi} | \, u \in r, \phi \in (-\pi/2, \pi/2) \right\}$$

Specifically, the following result is proved in [2] for $G = GL_n(\mathbb{C})$, in [4] for G reductive, and in [5] for an arbitrary affine algebraic group.[5] Denote by $[B]$ the projection of B onto $\mathfrak{h}$ corresponding to the root space decomposition $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$.

[5] We use the formulation of [5], which does not rely on formal power series solutions.

Theorem. *Let $r = \mathbb{R}_{>0} \cdot e^{\iota\theta}$ be an admissible ray. Then, there is a unique holomorphic function $h_r : \mathbb{H}_r \to G$ such that*

(1) *h_r tends to 1 as $z \to 0$ in any closed sector of $\mathbb{H}_r$ of the form*

$$|\arg(z \cdot e^{-\iota\theta})| \leq \frac{\pi}{2} - \delta, \qquad \delta > 0$$

(2) *For any determination of $\log z$ with a cut along the ray c, the function*

$$\gamma_r = h_r \cdot e^{-A/z} \cdot z^{[B]}$$

where $z^{[B]} = \exp([B] \log z)$, satisfies $\nabla \gamma_r = 0$ on $\mathbb{H}_r \setminus c$.

2.3. Stokes phenomena

For a given determination of $\log z$, with a cut along a ray c, the canonical solution γ_r is locally constant with respect to the choice of r, so long as r does not cross a Stokes ray. More precisely, the following holds. For any subset $\Sigma \subset \mathbb{C}$, let $\mathfrak{g}_\Sigma \subseteq \mathfrak{g}$ be the direct sum of the eigenspaces of $\mathrm{ad}(A)$ corresponding to the eigenvalues contained in Σ,

$$\mathfrak{g}_\Sigma = \bigoplus_{\substack{\alpha \in \Phi \sqcup \{0\}: \\ \alpha(A) \in \Sigma}} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_0 = \mathfrak{h}$. Note that $[\mathfrak{g}_{\Sigma_1}, \mathfrak{g}_{\Sigma_2}] \subseteq \mathfrak{g}_{\Sigma_1 + \Sigma_2}$. In particular, if Σ is an open convex cone, $\mathfrak{g}_\Sigma$ is a nilpotent subalgebra of $\mathfrak{g}$.

Proposition. *Let r, r' be admissible rays such that $r \neq -r'$, so that $\mathbb{H}_r \cap \mathbb{H}_{r'} \neq \emptyset$, and denote by $\overline{\Sigma}(r, r') \subset \mathbb{C}^\times$ the closed convex cone bounded by r and r'. Let*

$$S : \mathbb{H}_r \cap \mathbb{H}_{r'} \setminus c \longrightarrow G$$

be the locally constant function defined by $\gamma_r = \gamma_{r'} \cdot S$. Then, the following holds.

(1) *S takes values in the unipotent elements of G, and $\log S$ in the nilpotent subalgebra $\mathfrak{g}_{\overline{\Sigma}(r,r')}$.*
(2) *In particular, if $\overline{\Sigma}(r, r')$ does not contain any Stokes rays, the solutions γ_r and $\gamma_{r'}$ coincide on $\mathbb{H}_r \cap \mathbb{H}_{r'} \setminus c$.*

Proof. The asymptotic behaviour of γ_r and $\gamma_{r'}$ implies that

$$z^{[B]} \cdot e^{-A/z} \cdot S \cdot e^{A/z} \cdot z^{-[B]} = \left(\gamma_{r'} \cdot e^{A/z} \cdot z^{-[B]}\right)^{-1} \cdot \gamma_r \cdot e^{A/z} \cdot z^{-[B]} \to 1 \tag{2.2}$$

as $z \to 0$ along any ray ρ in $\mathbb{H}_r \cap \mathbb{H}_{r'} \setminus c$. By [4, Lemma 6] and [5, Prop. 6.3], the restriction of S to ρ is unipotent, and $\log S$ lies in $\mathfrak{g}_{\mathbb{H}_\rho}$.

Up to a permutation, we may assume that the counterclockwise angle from r to r' is less than π, so that $\mathbb{H}_r \cap \mathbb{H}_{r'}$ is the open convex cone bound by the rays $r'e^{-i\pi/2}$ and $re^{i\pi/2}$.

If the cut c is not contained in $\mathbb{H}_r \cap \mathbb{H}_{r'}$, S takes a single value. Since the intersection of the half–planes $\mathbb{H}_\rho$ as ρ varies in $\mathbb{H}_r \cap \mathbb{H}_{r'}$ is the closed convex cone bounded by r and r', it follows that $\log S \in \mathfrak{g}_{\overline{\Sigma}(r,r')}$.

If, on the other hand, c disconnects $\mathbb{H}_r \cap \mathbb{H}_{r'}$ into two open cones $\Sigma_<, \Sigma_>$, listed in counterclockwise order, then $\gamma_r = \gamma_{r'} \cdot S_{\lessgtr}$ on $\Sigma_{\lessgtr}$, for some $S_{\lessgtr} \in G$. The previous argument then shows that $S_{\lessgtr}$ are unipotent, and that

$$\log S_< \in \mathfrak{g}_{\overline{\Sigma}(e^{-\iota\pi/2}c,r')} \qquad \text{and} \qquad \log S_> \in \mathfrak{g}_{\overline{\Sigma}(r,e^{\iota\pi/2}c)}$$

Analytic continuation across c implies that $S_> = e^{2\pi\iota[B]} \cdot S_< \cdot e^{-2\pi\iota[B]}$. Since any $\mathfrak{g}_\Sigma$ is stable under $\mathrm{Ad}(e^{2\pi\iota[B]})$, this implies that

$$\log S_{\lessgtr} \in \mathfrak{g}_{\overline{\Sigma}(e^{-\iota\pi/2}c,r')} \cap \mathfrak{g}_{\overline{\Sigma}(r,e^{\iota\pi/2}c)} = \mathfrak{g}_{\overline{\Sigma}(r,r')}$$

2.4. Stokes data

For any two rays r, r', let $\sphericalangle(r, r') \subset \mathbb{C}^\times$ be the (not necessarily convex) closed sector swept by $e^{\iota\theta} \cdot r$, as θ ranges from 0 to the positive angle between r and r'. If r, r' are admissible, and different from the log cut c, define an element $S_{r'r} \in G$ by the identity

$$\left.\widetilde{\gamma_r|_r}\right|_{r'} = \gamma_{r'}|_{r'} \cdot S_{r',r} \cdot e^{2\pi\iota[B]\epsilon^c_{r'r}}$$

where the left–hand side is the counterclockwise analytic continuation to r' of the restriction of γ_r to r, and $\epsilon^c_{r'r}$ is 1 if c lies in $\sphericalangle(r, r')$, and 0 otherwise.

Proposition. *The following holds*

(1) *If the positive angle formed by r and r' is at most π, $S_{r'r}$ is unipotent, and its logarithm lies in the nilpotent subalgebra $\mathfrak{g}_{\sphericalangle(r,r')}$.*
(2) *If the admissible ray $r' \neq c$ lies in $\sphericalangle(r, r'')$, the following factorisation holds*

$$S_{r''r} = S_{r''r'} \cdot e^{2\pi\iota[B]\epsilon^c_{r''r'}} \cdot S_{r'r} \cdot e^{-2\pi\iota[B]\epsilon^c_{r''r'}}$$

Proof. (1) Let ℓ be a ray in $\mathbb{H}_r \cap \mathbb{H}_{r'} \setminus c$. Then

$$\left.\widetilde{\gamma_r|_r}\right|_\ell = \gamma_r|_\ell \cdot e^{2\pi\iota[B]\cdot\epsilon^c_{\ell r}} \qquad \text{and} \qquad \left.\widetilde{\gamma_{r'}|_\ell}\right|_{r'} = \gamma_{r'}|_{r'} \cdot e^{2\pi\iota[B]\cdot\epsilon^c_{r'\ell}}$$

By Proposition 2.3, $\gamma_r|_\ell = \gamma_{r'}|_\ell \cdot S$, where $S \in G$ is a unipotent element whose log lies in $\mathfrak{g}_{\overline{\Sigma}(r,r')}$. Computing analytic continuation in stages yields

$$\begin{aligned}\widetilde{\gamma_r|_r}\Big|_{r'} &= \widetilde{\widetilde{\gamma_r|_r}\Big|_\ell}\Bigg|_{r'} = \widetilde{\gamma_r|_\ell}\Big|_{r'} \cdot e^{2\pi\iota[B]\cdot\epsilon^c_{\ell r}} \\ &= \widetilde{\gamma_{r'}|_\ell}\Big|_{r'} \cdot S \cdot e^{2\pi\iota[B]\cdot\epsilon^c_{\ell r}} = \gamma_{r'}|_{r'} \cdot e^{2\pi\iota[B]\cdot\epsilon^c_{r'\ell}} \cdot S \cdot e^{2\pi\iota[B]\cdot\epsilon^c_{\ell r}} \\ &= \gamma_{r'}|_{r'} \cdot \mathrm{Ad}(e^{2\pi\iota[B]\cdot\epsilon^c_{r'\ell}})(S) \cdot e^{2\pi\iota[B]\cdot\epsilon^c_{r'r}}\end{aligned}$$

so that $S_{r'r} = \mathrm{Ad}(e^{2\pi\iota[B]\cdot\epsilon^c_{r'\ell}})(S)$.

(2) Computing analytic continuation from r to r'' in stages yields

$$\begin{aligned}\widetilde{\gamma_r|_r}\Big|_{r''} &= \widetilde{\widetilde{\gamma_r|_r}\Big|_{r'}}\Bigg|_{r''} = \widetilde{\gamma_{r'}|_{r'}}\Big|_{r''} \cdot S_{r'r} \cdot e^{2\pi\iota[B]\epsilon^c_{r'r}} \\ &= \gamma_{r''}|_{r''} \cdot S_{r''r'} \cdot e^{2\pi\iota[B]\epsilon^c_{r''r'}} \cdot S_{r'r} \cdot e^{2\pi\iota[B]\epsilon^c_{r'r}}\end{aligned}$$

Since the result is also equal to $\gamma_{r''}|_{r''} \cdot S_{r''r} \cdot e^{2\pi\iota[B]\epsilon^c_{r''r}}$, the result follows.

2.5. Stokes factors

Given a Stokes ray ℓ, the *Stokes factor* S_ℓ is the unipotent element of G defined by $S_\ell = S_{r'r}$, where $r, r' \neq c$ are admissible rays such that $\sphericalangle(r,r')$ contains no other Stokes rays than ℓ, and does not contain the cut c if the latter is different from ℓ. By Proposition 2.4, the definition of S_ℓ is independent of the choice of r, r'. The following is a direct consequence of Proposition 2.4.

Proposition. *The following holds*

(1) *If c does not lie in $\sphericalangle(r,r')$, then*

$$S_{r'r} = \overset{\curvearrowright}{\prod_\ell} S_\ell$$

where ℓ ranges over the Stokes rays contained in $\sphericalangle(r,r')$, and S_ℓ is placed to the left of $S_{\ell'}$ if ℓ is contained in $\sphericalangle(\ell',r')$.

(2) *If c lies in $\sphericalangle(r,r')$, then*

$$S_{r'r} = \overset{\curvearrowright}{\prod_\ell} S_\ell \cdot e^{2\pi\iota[B]} \cdot \overset{\curvearrowright}{\prod_\ell} S_\ell \cdot e^{-2\pi\iota[B]}$$

where the leftmost product ranges over the Stokes rays contained in $\sphericalangle(c,r')$, and the rightmost one over those contained in $\sphericalangle(r,c)$ except for c is the latter is a Stokes ray.

2.6. Stokes matrices

Let r be a ray such that both $\pm r$ are admissible, and distinct from the log cut c. Assume further that c lies in the cone $\sphericalangle(-r, r)$.[6] By definition, the *Stokes matrices* $S_{\pm}^{r}$ are the unipotent elements of G defined by

$$S_{+}^{r} = S_{-r\,r} \qquad \text{and} \qquad S_{-}^{r} = S_{r\,-r}$$

The pair (A, r) determines a partition $\Phi = \Phi_{+} \sqcup \Phi_{-}$ of the root system given by $\Phi_{\pm} = \{\alpha \in \Phi |\, \alpha(A) \in \sphericalangle(\pm r, \mp r)\}$. By Proposition 2.4, the Stokes matrices S_{+}^{r}, S_{-}^{r} lie in N_{+}, N_{-} respectively, where $N_{\pm} = N_{\pm}(A, r) \subset G$ is the unipotent subgroup with Lie algebra $\mathfrak{n}_{\pm} = \bigoplus_{\alpha \in \Phi_{\pm}} \mathfrak{g}_{\alpha}$. Moreover, if A is fixed, the Stokes matrices $S_{\pm}^{r}$ (and the subgroups $N_{\pm}$) are locally constant in r, so long as $\pm r$ do not cross a Stokes ray or c.

2.7. Connection matrix

Recall that the connection ∇ is said to be *non–resonant* at $z = \infty$ if none of the eigenvalues of $\mathrm{ad}(B)$ are positive integers. The following is well–know (see, *e.g.*, [24] for $G = GL_n(\mathbb{C})$).

Lemma. *If ∇ is non–resonant, there is a unique holomorphic function $g_{\infty} : \mathbb{P}^1 \setminus \{0\} \to G$ such that $g_{\infty}(\infty) = 1$ and, for any determination of $\log z$, the function $\gamma_{\infty} = g_{\infty} \cdot z^{B}$ is a solution of $\nabla \gamma_{\infty} = 0$.*

Fix a log cut c and, for any admissible ray r distinct from c, define the *connection matrix* $C_r \in G$ by

$$\gamma_{\infty} = \gamma_r \cdot C_r$$

where the identity is understood to hold on r. By Proposition 2.3, C_r is locally constant with respect to r, so long as r does not cross a Stokes ray or c.

2.8. Monodromy relation

The connection matrix C_r is related to the Stokes matrices $S_{\pm}^{r}$ by the following *monodromy relation.*

Proposition. *The following holds*

$$C_r \cdot e^{2\pi\iota B} \cdot C_r^{-1} = S_{-}^{r} \cdot e^{2\pi\iota[B]} \cdot S_{+}^{r}$$

[6] This condition is only necessary so that the monodromy relation of Proposition 2.8 is neater.

Proof. By definition of $S^r_\pm$, the monodromy of γ_r around a positive loop p_0 around 0 based at a point $z_0 \in r$ is the right–hand side of the stated identity. On the other hand, the monodromy of γ_∞ around p_0 is $e^{2\pi\iota B}$. Since $\gamma_r = \gamma_\infty \cdot C_r^{-1}$, the former monodromy is conjugate to the latter by C_r.

2.9. The Stokes map

Let $N_\pm \subset G$ be the unipotent subgroups corresponding to (A, r), and $B_\pm = H \ltimes N_\pm \subset G$ the solvable subgroups with Lie algebras $\mathfrak{b}_\pm = \mathfrak{h} \ltimes \mathfrak{n}_\pm$. Consider the fibred product

$$B_+ \times_H B_- = \{(b_+, b_-) \in B_+ \times B_- \,|\, \pi_+(b_+)\pi_-(b_-) = 1\}$$

where $\pi_\pm : B_\pm \to H$ are the quotient maps. Following [4], we define the *Stokes map* to be the analytic map $\mathcal{S}_r : \mathfrak{g} \longrightarrow B_+ \times_H B_-$ given by

$$B \longrightarrow \left((S^r_+)^{-1} \cdot e^{-\iota\pi[B]}, S^r_- \cdot e^{\iota\pi[B]}\right)$$

Note that $B_- \times_H B_+$ maps to G via the map $\beta : (b_+, b_-) \to b_+ \cdot b_-^{-1}$.[7] Moreover, by Proposition 2.8, the composition $\beta \circ \mathcal{S}_r$ is the map $\mathfrak{g} \to G$ given by

$$B \longrightarrow (S^r_+)^{-1} \cdot e^{-2\pi\iota[B]} \cdot (S^r_-)^{-1} = C_r \cdot e^{-2\pi\iota B} \cdot C_r^{-1}$$

2.10. Linearisation of G^*

Assume now that G is reductive, and fix a symmetric, non–degenerate, invariant bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$. The pair of opposite Borel subalgebras $\mathfrak{b}_\pm$ of $\mathfrak{g}$ then gives rise to a solution $\mathsf{r} \in \mathfrak{b}_- \otimes \mathfrak{b}_+$ of the classical Yang–Baxter equations given by

$$\mathsf{r} = x_i \otimes x^i + \frac{1}{2} t_a \otimes t^a \tag{2.3}$$

where $\{x_i\}, \{x^i\}$ are bases of $\mathfrak{n}_-, \mathfrak{n}_+$ respectively which are dual with respect to $(\cdot,\cdot)$, and $\{t_a\}, \{t^a\}$ are dual bases of $\mathfrak{h}$.

The element r gives $\mathfrak{g}$ the structure of a quasitriangular Lie bialgebra, with cobracket $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ given by $\delta(x) = [x \otimes 1 + 1 \otimes x, \mathsf{r}]$. The dual Lie bialgebra $(\mathfrak{g}^*, \delta^t, [\cdot,\cdot]^t)$ may be identified, as a Lie algebra, with

$$\mathfrak{b}_+ \times_{\mathfrak{h}} \mathfrak{b}_- = \{(X_+, X_-) \in \mathfrak{b}_+ \oplus \mathfrak{b}_- | \pi_+(X_+) + \pi_-(X_-) = 0\}$$

where $\pi_\pm : \mathfrak{b}_\pm \to \mathfrak{h}$ is the quotient map. This endows $G^* = B_+ \times_H B_-$ with the structure of a Poisson–Lie group, which is dual to G.

[7] β is a principal bundle over its image with structure group the order two elements in H.

Endow now $\mathfrak{g}^*$ with its standard Kirillov–Kostant–Souriau Poisson structure given by

$$\{f,g\}(x) = \langle [d_x f, d_x g], x\rangle$$

where $d_x h \in T_x^*\mathfrak{g}^* = \mathfrak{g}$ is the differential of h at x, and $[\cdot,\cdot]$ is the Lie bracket on $\mathfrak{g}$.

Let $\nu : \mathfrak{g}^* \to \mathfrak{g}$ be the identification induced by the bilinear form $(\cdot,\cdot)$, and set $\nu^\vee = -1/(2\pi\iota)\nu$. The following remarkable result is due to Boalch [3,4].

Theorem. *The map $\mathcal{S}\circ\nu^\vee : \mathfrak{g}^* \to G^*$ is a Poisson map, and generically a local analytic diffeomorphism.*

In particular, $\mathcal{S}\circ\nu^\vee$ gives a linearisation of the Poisson–Lie structure on G^*. We shall give an alternative proof of the fact that $\mathcal{S}\circ\nu^\vee : \mathfrak{g}^* \to G^*$ is a Poisson map in Section 8.

3. Stokes phenomena and quantum groups

This section is an exposition of [23]. We explain in particular how the dynamical KZ equations give rise to a twist which kills the KZ associator. Sections 3.1–3.3 contain background material required to do calculus with values in infinite–dimensional filtered vector spaces and their endomorphisms. Throughout the paper, $\mathsf{h}, \hbar$ are two formal parameters related by $\hbar = 2\pi\iota\mathsf{h}$.

3.1. Filtered vector spaces

Let $\mathcal{V}$ be a vector space over a field k endowed with a decreasing filtration

$$\mathcal{V} = \mathcal{V}_0 \supseteq \mathcal{V}_1 \supseteq \mathcal{V}_2 \supseteq \cdots$$

and $\imath$ the map $\mathcal{V} \to \varprojlim \mathcal{V}/\mathcal{V}_n$. Recall that $\mathcal{V}$ is said to be *separated* if $\imath$ is injective, and *complete* if $\imath$ is surjective.

If $\mathsf{k} = \mathbb{C}$, and the quotients $\mathcal{V}/\mathcal{V}_n$ are finite–dimensional, we shall say that a map $F : X \to \mathcal{V}$, where X is a topological space (resp. a smooth or complex manifold) is continuous (resp. smooth or holomorphic) if its truncations $F_n : X \to \mathcal{V}/\mathcal{V}_n$ are. If $\mathcal{V}$ is separated and complete, giving such an F amounts to giving continuous (resp. smooth or holomorphic) maps $F_n : X \to \mathcal{V}/\mathcal{V}_n$ such that $F_n = F_m \mod \mathcal{V}_m/\mathcal{V}_n$, for any $n \geq m$.

3.2. Filtered endomorphisms

Let $\mathcal{V}$ be as in 3.1, and $\mathcal{E} \subset \mathrm{End}_{\mathsf{k}}(\mathcal{V})$ the subalgebra defined by

$$\mathcal{E} = \{T \in \mathrm{End}_{\mathsf{k}}(\mathcal{V}) |\, T(\mathcal{V}_m) \subseteq \mathcal{V}_m, m \geq 0\}$$

Consider the decreasing filtration $\mathcal{E} = \mathcal{E}_0 \supseteq \mathcal{E}_1 \supseteq \cdots$ where $\mathcal{E}_n \subset \mathcal{E}$ is the two–sided ideal given by $\mathcal{E}_n = \{T \in \mathcal{E} | \operatorname{Im}(T) \subseteq \mathcal{V}_n\}$. Note that if the quotients $\mathcal{V}/\mathcal{V}_n$ are finite–dimensional, the same holds for

$$\mathcal{E}/\mathcal{E}_n \cong \{T \in \operatorname{End}_{\mathsf{k}}(\mathcal{V}/\mathcal{V}_n) | \, T(\mathcal{V}_m/\mathcal{V}_n) \subseteq \mathcal{V}_m/\mathcal{V}_n, \, 0 \leq m \leq n\}$$

In particular, if $\mathsf{k} = \mathbb{C}$, we may speak of a continuous (resp. smooth, holomorphic) map with values in $\mathcal{E}$.

Lemma.

(1) *If $\mathcal{V}$ is separated, so is $\mathcal{E}$.*
(2) *If $\mathcal{V}$ is complete, so is $\mathcal{E}$.*

Proof. (1) holds because $\bigcap_{n\geq 0} \mathcal{E}_n = \{T \in \mathcal{E} | \operatorname{Im}(T) \subseteq \bigcap_{n\geq 0} \mathcal{V}_n\}$. (2) Let $T_n \in \mathcal{E}/\mathcal{E}_n$ be such that $T_n = T_m \mod \mathcal{E}/\mathcal{E}_m$ for any $n \geq m$. It suffices to find $T \in \operatorname{End}_{\mathsf{k}}(\mathcal{V})$ such that $T = T_n \mod \mathcal{E}_n$ for any $n \geq 0$, for it then follows that $T \in \mathcal{E}$. Let $\{v_i\}_{i \in I}$ be a basis of $\mathcal{V}$. For any $i \in I$, $\{T_n v_i\}$ is a well–defined element of $\lim_n \mathcal{V}/\mathcal{V}_n$. By completeness of $\mathcal{V}$, there exists $u_i \in \mathcal{V}$ such that $u = T_n v_i \mod \mathcal{V}_n$ for any n. Setting $Tv_i = u_i$ gives the required T.

3.3. Filtered algebras

Let A be a k–algebra endowed with an increasing algebra filtration $\mathsf{k} = A_0 \subseteq A_1 \subseteq \cdots$, and $A[\![\hbar]\!]^o$ the (completed Rees) algebra given by

$$A[\![\hbar]\!]^o = \{\sum_{k\geq 0} a_k \hbar^k \in A[\![\hbar]\!] | \, a_k \in A_k\}$$

Endow $A[\![\hbar]\!]^o$ with the decreasing filtration

$$A[\![\hbar]\!]^o_n = A[\![\hbar]\!]^o \cap \hbar^n A[\![\hbar]\!] \tag{3.1}$$

with respect to which it is easily seen to be separated and complete. Note that each $A[\![\hbar]\!]^o_n$ is a two–sided, $\mathbb{C}[\![\hbar]\!]$–ideal in $A[\![\hbar]\!]^o$, and that the quotients

$$A[\![\hbar]\!]^o / A[\![\hbar]\!]^o_n \cong A_0 \oplus \hbar A_1 \oplus \cdots \oplus \hbar^{n-1} A_{n-1}$$

are finite–dimensional if A is filtered by finite–dimensional subspaces.

3.4. Example

We shall be interested in the case when $A = U\mathfrak{g}^{\otimes m}$ is a tensor power of an enveloping algebra, with filtration given by $A_k = (U\mathfrak{g}_{\leq k})^{\otimes m}$. Then,

$$U\mathfrak{g}^{\otimes m}[\![\hbar]\!]^o = \mathfrak{U}' \otimes \mathfrak{U}^{\otimes m-1} \cap \mathfrak{U} \otimes \mathfrak{U}' \otimes \mathfrak{U}^{\otimes m-2} \cap \cdots \cap \mathfrak{U}^{\otimes m-1} \otimes \mathfrak{U}' \tag{3.2}$$

where $\mathfrak{U} = U\mathfrak{g}[\![\hbar]\!]$ and $\mathfrak{U}' = U\mathfrak{g}[\![\hbar]\!]^o$. Note that $U\mathfrak{g}^{\otimes m}[\![\hbar]\!]^o \cap U\mathfrak{g}^{\otimes m} = \mathsf{k}$. However, if $x \in U\mathfrak{g}_{\leq k}$, $i = 1, \ldots, m$, and

$$x^{(i)} = 1^{\otimes i-1} \otimes x \otimes 1^{\otimes m-i} \in U\mathfrak{g}^{\otimes m}$$

then $\hbar^{k-1}\mathrm{ad}\, x^{(i)}$ is a derivation of $U\mathfrak{g}^{\otimes m}[\![\hbar]\!]^o$, which preserves the filtration $U\mathfrak{g}^{\otimes m}[\![\hbar]\!]^o_n$.

3.5. The dynamical KZ equations

Let now $\mathfrak{g}$ be a complex reductive Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, and $(\cdot, \cdot)$ an invariant inner product on $\mathfrak{g}$. Let $\Phi = \{\alpha\} \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$, choose $x_\alpha \in \mathfrak{g}_\alpha$ for any $\alpha \in \Phi$ such that $(x_\alpha, x_{-\alpha}) = 1$, and set

$$\mathcal{K}_\alpha = x_\alpha x_{-\alpha} + x_{-\alpha} x_\alpha$$

Endow $\mathcal{A} = U\mathfrak{g}^{\otimes 2}[\![\hbar]\!]^o$ with the filtration $\mathcal{A}_n = U\mathfrak{g}^{\otimes 2}[\![\hbar]\!]^o \cap \hbar^n U\mathfrak{g}^{\otimes 2}$ as in (3.1), and filter $\mathcal{E} = \{T \in \mathrm{End}_{\mathbb{C}}(\mathcal{A}) |\, T(\mathcal{A}_n) \subseteq \mathcal{A}_n\}$ as in 3.2. Since the quotients $\mathcal{A}/\mathcal{A}_n$ and $\mathcal{E}/\mathcal{E}_n$ are finite–dimensional, we may speak of continuous, smooth or holomorphic functions with values in $\mathcal{A}$ and $\mathcal{E}$.

The dynamical KZ (DKZ) connection is the $\mathcal{E}$–valued connection on $\mathbb{C}$ given by

$$\nabla_{\mathrm{DKZ}} = d - \left(\mathsf{h}\frac{\Omega}{z} + \mathrm{ad}\,\mu^{(1)}\right) dz \tag{3.3}$$

where $\mu \in \mathfrak{h}$, $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ is the invariant tensor corresponding to $(\cdot, \cdot)$ and, given an element $a \in \mathcal{A}$, we abusively denote by a the corresponding left multiplication operator $L(a) \in \mathcal{E}$.

3.6. Fundamental solution at $z = 0$

Proposition.

(1) *For any $\mu \in \mathfrak{h}$, there is a unique holomorphic function $H_0 : \mathbb{C} \to \mathcal{A}$ such that $H_0(0, \mu) = 1$ and, for any determination of $\log z$, the $\mathcal{E}$–valued function*

$$\Upsilon_0(z, \mu) = e^{z\,\mathrm{ad}\,\mu^{(1)}} \cdot H_0(z, \mu) \cdot z^{\mathsf{h}\Omega}$$

satisfies $\nabla_{\mathrm{DKZ}}\Upsilon_0 = 0$.

(2) *H_0 and Υ_0 are invariant under the diagonal action of $\mathfrak{h}$.*

(3) *H_0 and Υ_0 are holomorphic functions of μ, and Υ_0 satisfies*

$$\left(d_{\mathfrak{h}} - \frac{\mathsf{h}}{2}\sum_{\alpha\in\Phi_+}\frac{d\alpha}{\alpha}\Delta(\mathcal{K}_\alpha) - z\,\mathrm{ad}\,d\mu^{(1)}\right)\Upsilon_0 = \Upsilon_0\left(d_{\mathfrak{h}} - \frac{\mathsf{h}}{2}\sum_{\alpha\in\Phi_+}\frac{d\alpha}{\alpha}\Delta(\mathcal{K}_\alpha)\right)$$

3.7. Fundamental solutions at $z=\infty$

Let $\mathbb{H}_\pm = \{z\in\mathbb{C}|\ \mathrm{Im}(z)\gtrless 0\}$.

Theorem.

(1) *For any $\mu\in\mathfrak{h}^{\mathbb{R}}_{\mathrm{reg}}$, there are unique holomorphic functions $H_\pm:\mathbb{H}_\pm\to\mathcal{A}$ such that $H_\pm(z,\mu)$ tends to 1 as $z\to\infty$ in any sector of the form $|\arg(z)|\in(\delta,\pi-\delta)$, $\delta>0$ and, for any determination of $\log z$, the $\mathcal{E}$–valued function*

$$\Upsilon_\pm(z,\mu) = H_\pm(z,\mu)\cdot z^{\mathsf{h}\Omega_0}\cdot e^{z\,\mathrm{ad}\,\mu^{(1)}}$$

satisfies $\nabla_{\mathrm{DKZ}}\Upsilon_\pm = 0$.

(2) *$H_\pm$ and $\Upsilon_\pm$ are invariant under the diagonal action of $\mathfrak{h}$.*

(3) *$H_\pm$ and $\Upsilon_\pm$ are smooth functions of μ, and $\Upsilon_\pm$ satisfies*

$$\left(d_{\mathfrak{h}} - \frac{\mathsf{h}}{2}\sum_{\alpha\in\Phi_+}\frac{d\alpha}{\alpha}\Delta(\mathcal{K}_\alpha) - z\,\mathrm{ad}\,d\mu^{(1)}\right)\Upsilon_\pm = \Upsilon_\pm\left(d_{\mathfrak{h}} - \frac{\mathsf{h}}{2}\sum_{\alpha\in\Phi_+}\frac{d\alpha}{\alpha}({\mathcal{K}_\alpha}^{(1)} + {\mathcal{K}_\alpha}^{(2)})\right)$$

3.8. Remark

The PDEs (3) in Proposition 3.6 and Theorem 3.7 do not take values in $\mathcal{E}$, since left multiplication by $\mathsf{h}\Delta(\mathcal{K}_\alpha), \mathsf{h}\mathcal{K}_\alpha^{(1)}$ and $\mathsf{h}\mathcal{K}_\alpha^{(2)}$ does not preserve $\mathcal{A}$. Let, however, $\mathcal{A}\subsetneq\widetilde{\mathcal{A}}\subset U\mathfrak{g}^{\otimes 2}[\![\hbar]\!]$ be the Rees algebra with respect to the laxer filtration $(U\mathfrak{g}^{\otimes 2})_k = \sum_{a+b=2k} U\mathfrak{g}_{\leq a}\otimes U\mathfrak{g}_{\leq b}$, and $\widetilde{\mathcal{E}}$ the corresponding algebra of endomorphisms. Then, $\Upsilon_0, \Upsilon_\pm$, and left multiplication by $\mathsf{h}\Delta(\mathcal{K}_\alpha), \mathsf{h}\mathcal{K}_\alpha^{(1)}$ and $\mathsf{h}\mathcal{K}_\alpha^{(2)}$ all lie in $\widetilde{\mathcal{E}}$, and these PDEs should be understood as holding in $\widetilde{\mathcal{E}}$.

3.9. $\mathbb{Z}_2$–equivariance

Let $\mathcal{U}\subset\mathbb{C}$ be an open subset. For any functions $F:\mathcal{U}\to\mathcal{A}$ and $G:\mathcal{U}\to\mathcal{E}$, define $F^\vee:-\mathcal{U}\to\mathcal{A}$ and $G^\vee:-\mathcal{U}\to\mathcal{E}$ by

$$F^\vee(z) = e^{z\,\mathrm{ad}(\mu^{(1)}+\mu^{(2)})}(F(-z)^{21}) \quad\text{and}\quad G^\vee(z) = e^{z\,\mathrm{ad}(\mu^{(1)}+\mu^{(2)})}\cdot G(-z)^{21}$$

where $G(-z)^{21} = (1\,2)\cdot G(-z)\cdot(1\,2)$. If F, G are local solutions of the dynamical KZ equations with values in $\mathcal{A}$ and $\mathcal{E}$ respectively, then so are $F^\vee, G^\vee$.

Lemma. *The following holds*

(1) *For* $z \in \mathbb{H}_{\pm}$,

$$\Upsilon_0^{\vee}(z) = \Upsilon_0(z) \cdot e^{\mp\pi\iota\mathsf{h}\Omega}$$

(2) *For* $z \in \mathbb{H}_{\mp}$,

$$\Upsilon_{\pm}^{\vee}(z) = \Upsilon_{\mp}(z) \cdot e^{\pm\pi\iota\mathsf{h}\Omega_0}$$

Proof. (1) The uniqueness of the holomorphic part H_0 of Υ_0 implies that $(e^{z\,\mathrm{ad}\,\mu^{(1)}} \cdot H_0)^{\vee} = e^{z\,\mathrm{ad}\,\mu^{(1)}} \cdot H_0$. It follows that $\Upsilon_0^{\vee}(z) = H_0(z) \cdot (-z)^{\mathsf{h}\Omega} = \Upsilon_0(z) \cdot e^{\mp\iota\pi\mathsf{h}\Omega_0}$ since $\log(-z) = \log z \mp \iota\pi$, depending on whether $\operatorname{Im} z \gtrless 0$.

(2) Similarly, for $z \in \mathbb{H}_{\mp}$,

$$\begin{aligned}\Upsilon_{\pm}^{\vee}(z) &= e^{z\,\mathrm{ad}(\mu^{(1)}+\mu^{(2)})} \cdot (1\,2) \cdot H_{\pm}(-z) \cdot e^{-z\,\mathrm{ad}\,\mu^{(1)}} \cdot (-z)^{\mathsf{h}\Omega_0} \cdot (1\,2)\\ &= H_{\pm}^{\vee}(z) \cdot e^{z\,\mathrm{ad}\,\mu^{(1)}} \cdot (-z)^{\mathsf{h}\Omega_0}\end{aligned}$$

The uniqueness of $H_{\pm}$ implies that $H_{\pm}^{\vee} = H_{\mp}$, from which the result follows.

3.10. Another $\mathbb{Z}_2$–equivariance

Let $\mathcal{U} \subset \mathbb{C}$ be an open subset. For any functions $F : \mathcal{U} \to \mathcal{A}$ and $G : \mathcal{U} \to \mathcal{E}$, define $\widetilde{F} : \mathcal{U} \to \mathcal{A}$ and $\widetilde{G} : \mathcal{U} \to \mathcal{E}$ by

$$\widetilde{F}(z) = e^{-z\,\mathrm{ad}(\mu^{(1)}+\mu^{(2)})}(F(z)^{21}) \quad \text{and} \quad \widetilde{G}(z) = e^{-z\,\mathrm{ad}(\mu^{(1)}+\mu^{(2)})} \cdot (1\,2) \cdot G(z) \cdot (1\,2)$$

If F, G are local solutions of the dynamical KZ equations with parameter $\mu \in \mathfrak{h}$, then $\widetilde{F}, \widetilde{G}$ are solutions of the DKZ equations with parameter $-\mu$.

Lemma. *The following holds*

$$\widetilde{\Upsilon}_0(z;\mu) = \Upsilon_0(z;-\mu) \qquad \textit{and} \qquad \widetilde{\Upsilon}_{\pm}(z;\mu) = \Upsilon_{\pm}(z;-\mu)$$

Proof. By definition,

$$\widetilde{\Upsilon}_0(z;\mu) = e^{-z\,\mathrm{ad}(\mu^{(1)}+\mu^{(2)})} \cdot e^{z\,\mathrm{ad}\,\mu^{(2)}} \cdot H_0^{21}(z;\mu) \cdot z^{\hbar\Omega} = e^{-z\,\mathrm{ad}\,\mu^{(1)}} \cdot H_0^{21}(z;\mu) \cdot z^{\hbar\Omega}$$

which coincides with $\Upsilon_0(z;-\mu)$ by uniqueness. Similarly,

$$\begin{aligned}\widetilde{\Upsilon}_{\pm}(z;\mu) &= e^{-z\,\mathrm{ad}(\mu^{(1)}+\mu^{(2)})} \cdot H_{\pm}^{21}(z;\mu) \cdot e^{z\,\mathrm{ad}\,\mu^{(2)}} \cdot z^{\hbar\Omega_0}\\ &= H_{\pm}^{21}(z;\mu) \cdot e^{-z\,\mathrm{ad}\,\mu^{(1)}} \cdot z^{\hbar\Omega_0} = \Upsilon_{\pm}(z;-\mu)\end{aligned}$$

where the second equality uses the fact that $H_\pm$ is of weight zero, and the third follows by uniqueness.

3.11. Differential twist

Fix henceforth the standard determination of $\log z$ with a cut along the negative real axis, and let $\Upsilon_0, \Upsilon_\pm$ be the corresponding fundamental solutions of the dynamical KZ equations given in 3.6 and 3.7 respectively. We shall consider Υ_0 and $\Upsilon_\pm$ as (single–valued) holomorphic functions on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

Definition. The differential twist is the smooth function $J_\pm : \mathfrak{h}^{\mathbb{R}}_{\mathrm{reg}} \to U\mathfrak{g}^{\otimes 2}[\![\hbar]\!]^o$ defined by

$$J_\pm = \Upsilon_0(z)^{-1} \cdot \Upsilon_\pm(z)$$

where $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

Remark. $J_\pm$ takes in fact values in $\mathcal{E}$. However, the form of Υ_0 and $\Upsilon_\pm$ shows that

$$J_\pm = z^{-\mathsf{h}\Omega} \cdot H_0(z)^{-1} \cdot \exp(-z \,\mathrm{ad}\, \mu^{(1)})\,(H_\pm) \cdot z^{\mathsf{h}\Omega_0}$$

so that it is a left multiplication operator. We therefore abusively identify $J_\pm$ and $J_\pm(1^{\otimes 2})$.

Proposition. *The following holds*

$$J_- = e^{\pi\iota\mathsf{h}\Omega} \cdot J_+^{21} \cdot e^{-\pi\iota\mathsf{h}\Omega_0}$$

Proof. Let $G^\vee(z) = e^{z\,\mathrm{ad}(\mu^{(1)}+\mu^{(2)})} \cdot G(-z)^{(1\,2)}$ be the involution defined in 3.9. By definition, $J_+^{21} = (\Upsilon_0^\vee)^{-1} \cdot \Upsilon_+^\vee$, where the right–hand side is evaluated for $\mathrm{Im}\, z < 0$. By Lemma 3.9, this is equal to $e^{-\pi\iota\mathsf{h}\Omega} \cdot \Upsilon_0^{-1} \cdot \Upsilon_- \cdot e^{\pi\iota\mathsf{h}\Omega_0}$.

3.12. For any $\mu \in \mathfrak{h}^{\mathbb{R}}_{\mathrm{reg}}$, set $\Phi_+(\mu) = \{\alpha \in \Phi | \alpha(\mu) > 0\}$.

Theorem.

(1) *$J_\pm$ kills the KZ associator $\Phi_{\mathrm{KZ}} \in U\mathfrak{g}^{\otimes 3}[\![\hbar]\!]^o$, that is*

$$\Phi_{\mathrm{KZ}} \cdot \Delta \otimes \mathrm{id}(J_\pm) \cdot J_\pm \otimes 1 = \mathrm{id} \otimes \Delta(J_\pm) \cdot 1 \otimes J_\pm$$

(2) *$J_\pm = 1^{\otimes 2} + \frac{\hbar}{2} \mathsf{j}_\pm$ mod $\hbar^2$, where*

$$\mathsf{j}_\pm = \mp\Omega_- + \frac{1}{\pi\iota} \sum_{\alpha \in \Phi_+(\mu)} (\log \alpha + \gamma)\,(\Omega_\alpha + \Omega_{-\alpha})$$

with $\Omega_\alpha = x_\alpha \otimes x_{-\alpha}$, $\Omega_\pm = \sum_{\alpha\in\Phi_+(\mu)} \Omega_{\pm\alpha}$, *and* $\gamma = \lim_n(\sum_{k=1}^n \frac{1}{k} - \log(n))$ *the Euler–Mascheroni constant. In particular,*

$$\mathsf{j}_\pm - \mathsf{j}_\pm^{21} = \Omega_\pm - \Omega_\mp \tag{3.4}$$

(3) *As a function of* $\mu \in \mathfrak{h}_{\mathrm{reg}}^{\mathbb{R}}$, $J_\pm$ *satisfies*

$$d_{\mathfrak{h}} J_\pm = \frac{\mathsf{h}}{2} \sum_{\alpha\in\Phi_+(\mu)} \frac{d\alpha}{\alpha} \left(\Delta(\mathcal{K}_\alpha)J_\pm - J_\pm({\mathcal{K}_\alpha}^{(1)} + {\mathcal{K}_\alpha}^{(2)})\right)$$

Remark. Note that the PDE satisfied by $J_\pm$ is independent of the chamber which μ lies in since $d\log\alpha = d\log(-\alpha)$ and $\mathcal{K}_\alpha = \mathcal{K}_{-\alpha}$. Note also that this PDE takes values in $\mathcal{A}$. Indeed, although neither the left multiplication operator $L(\mathsf{h}\Delta(\mathcal{K}_\alpha))$ nor the right multiplication $R(\mathsf{h}\mathcal{K}_\alpha^{(1)} + \mathsf{h}\mathcal{K}_\alpha^{(2)})$ leaves $\mathcal{A}$ inavariant, the fact that $\Delta(\mathcal{K}_\alpha) = \mathcal{K}_\alpha^{(1)} + \mathcal{K}_\alpha^{(2)} + 2(\Omega_\alpha + \Omega_{-\alpha})$ implies that

$$L(\mathsf{h}\Delta(\mathcal{K}_\alpha)) - R(\mathsf{h}\mathcal{K}_\alpha^{(1)} + \mathsf{h}\mathcal{K}_\alpha^{(2)}) = 2L(\mathsf{h}\Omega_\alpha + \mathsf{h}\Omega_{-\alpha}) + \mathrm{ad}(\mathsf{h}\mathcal{K}_\alpha^{(1)} + \mathsf{h}\mathcal{K}_\alpha^{(2)})$$

which preserves $\mathcal{A}$ since $\mathsf{h}\Omega_\alpha \in \mathcal{A}$, and $\mathrm{ad}(\mathsf{h}\mathcal{K}_\alpha^{(i)})$ leave $\mathcal{A}$ invariant by 3.3.

3.13. Quantisation of $(\mathfrak{g}, \mathsf{r})$

Fix a chamber $\mathcal{C} \subset \mathfrak{h}_{\mathrm{reg}}^{\mathbb{R}}$, and set $\Phi_+ = \Phi_+(\mu)$, $\mu \in \mathcal{C}$. Let

$$\mathsf{r} = \Omega_+ + \frac{1}{2}\Omega_0 = \sum_{\alpha\in\Phi_+} x_\alpha \otimes x_{-\alpha} + \frac{1}{2}\Omega_0$$

be the Drinfeld–Sklyanin r–matrix corresponding to $\mathcal{C}$, and $(\mathfrak{g}, \mathsf{r})$ the corresponding quasitriangular Lie bialgebra.[8]

Set $R_{\mathrm{KZ}} = e^{\hbar\Omega/2}$, and let

$$(U\mathfrak{g}[\![\hbar]\!], \Delta_0, R_{\mathrm{KZ}}, \Phi_{\mathrm{KZ}})$$

be the quasitriangular quasi–Hopf algebra structure on $U\mathfrak{g}[\![\hbar]\!]$ underlying the monodromy of the KZ equations [8], where Δ_0 is the standard cocommutative coproduct on $U\mathfrak{g}$. If $\mu \in \mathcal{C}$, the differential twist $J_\pm = J_\pm(\mu)$ allows to twist this structure, and yields a quasitriangular Hopf algebra $(U\mathfrak{g}[\![\hbar]\!], \Delta_\pm, R_\pm)$, where[9]

$$\Delta_\pm(x) = J_\pm^{-1} \cdot \Delta_0(x) \cdot J_\pm \qquad \text{and} \qquad R_\pm = (J_\pm^{-1})^{21} \cdot R_{\mathrm{KZ}} \cdot J_\pm$$

[8] Note that the r considered in 2.10 corresponds to the *anti*fundamental chamber.

[9] Note that $\Delta_\pm$ and $R_\pm$ depend on the additional choice of $\mu \in \mathcal{C}$. Specifically, if $\mu_0, \mu_1 \in \mathcal{C}$, $p : [0,1] \to \mathcal{C}$ is a path with $p(0) = \mu_0, p(1) = \mu_1$, and $a_p \in U\mathfrak{g}[\![\hbar]\!]_0$ is the holonomy of the Casimir connection along p, then

Theorem.

(1) $(U\mathfrak{g}[\![\hbar]\!], \Delta_+, R_+)$ *is a quantisation of* $(\mathfrak{g}, \mathsf{r})$.
(2) $(U\mathfrak{g}[\![\hbar]\!], \Delta_-, R_-)$ *is a quantisation of* $(\mathfrak{g}, \mathsf{r}^{21})$.
(3) $(U\mathfrak{g}[\![\hbar]\!], \Delta_\pm, R_\pm)$ *is isomorphic, as a quasitriangular Hopf algebra, to the Drinfeld–Jimbo quantum group corresponding to* $\mathfrak{g}$.

Proof. (1)–(2) By (3.4), the coefficient of $\hbar$ in $R_\pm$ is $\frac{1}{2}(\Omega \pm \Omega_+ \mp \Omega_-)$, which is equal to r for R_+ and r^{21} for R_-.

(3) This follows, for example, from Drinfeld's uniqueness of the quantisation of $(\mathfrak{g}, \mathsf{r})$ [7] given that the Chevalley involution of $\mathfrak{g}$ clearly lifts to $(U\mathfrak{g}[\![\hbar]\!], \Delta_\pm, R_\pm)$.

Remark. It follows from (4) of Theorem 3.12 that

$$R_- = R^0_{\text{KZ}} \cdot R_+^{21} \cdot (R^0_{\text{KZ}})^{-1} \tag{3.5}$$

4. The R–matrix as a quantum Stokes matrix

4.1. Quantum Stokes matrices

Recall that $\mathbb{H}_\pm = \{z \in \mathbb{C} |\ \mathrm{Im}(z) \gtrless 0\}$. Define the quantum Stokes matrices $S_\pm \in U\mathfrak{g}^{\otimes 2}[\![\hbar]\!]^o$ by

$$\Upsilon_+ = \Upsilon_- \cdot S_+ \qquad \text{and} \qquad \Upsilon_- \cdot e^{\hbar\Omega_0} = \Upsilon_+ \cdot S_-$$

where the first identity is understood to hold in $\mathbb{H}_-$ after Υ_+ has been continued across the ray $\mathbb{R}_{\geq 0}$, and the second in $\mathbb{H}_+$ after Υ_- has been continued across $\mathbb{R}_{\leq 0}$.

Proposition. *The following holds*

(1) $S_- = e^{-\iota\pi\mathsf{h}\Omega_0} \cdot S_+^{21} \cdot e^{\iota\pi\mathsf{h}\Omega_0}$.
(2) $J_+^{-1} \cdot e^{2\pi\iota\mathsf{h}\Omega} \cdot J_+ = S_+^{-1} \cdot e^{2\pi\iota\mathsf{h}\Omega_0} \cdot S_-^{-1}$.
(3) *As functions of* $\mu \in \mathcal{C}$, *the quantum Stokes matrices* $S_\pm$ *satisfy*

$$d_{\mathfrak{h}} S_\pm = \frac{\mathsf{h}}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \left[\mathcal{K}_\alpha{}^{(1)} + \mathcal{K}_\alpha{}^{(2)}, S_\pm \right]$$

$$\Delta_\pm(x)(\mu_1) = a_p^{\otimes 2} \cdot \Delta_\pm(a_p^{-1} x a_p)(\mu_0) \cdot (a_p^{\otimes 2})^{-1} \qquad \text{and} \qquad R_\pm(\mu_1) = a_p^{\otimes 2} \cdot R_\pm(\mu_0) \cdot (a_p^{\otimes 2})^{-1}$$

In particular, the quasitriangular Hopf algebras corresponding to different values of $\mu \in \mathcal{C}$ are all isomorphic.

Proof. (1) Let f be a holomorphic function on $\mathbb{H}_\pm$, and denote by $\mathcal{P}_\pm(f)$ the analytic continuation of f to $\mathbb{H}_\mp$ across the half–axis $\mathbb{R}_{\gtrless 0}$. By Lemma 3.9, and the definition of S_-,

$$\mathcal{P}_-(\Upsilon_+^\vee) = \mathcal{P}_-(\Upsilon_-) \cdot e^{\iota\pi\mathsf{h}\Omega_0} = \Upsilon_+ \cdot S_- \cdot e^{-\iota\pi\mathsf{h}\Omega_0}$$

On the other hand, if $\imath : \mathbb{C} \to \mathbb{C}$ is the inversion $z \to -z$,

$$\begin{aligned}
\mathcal{P}_-(\Upsilon_+^\vee) &= e^{z\,\mathrm{ad}(\mu^{(1)}+\mu^{(2)})} \cdot (1\,2) \cdot \mathcal{P}_-(\Upsilon_+ \circ \imath) \cdot (1\,2) \\
&= e^{z\,\mathrm{ad}(\mu^{(1)}+\mu^{(2)})} \cdot (1\,2) \cdot \mathcal{P}_+(\Upsilon_+) \circ \imath \cdot (1\,2) \\
&= e^{z\,\mathrm{ad}(\mu^{(1)}+\mu^{(2)})} \cdot (1\,2) \cdot \Upsilon_- \circ \imath \cdot S_+ \cdot (1\,2) \\
&= \Upsilon_-^\vee \cdot S_+^{21} \\
&= \Upsilon_+ \cdot e^{-\iota\pi\mathsf{h}\Omega_0} \cdot S_+^{21}
\end{aligned}$$

where the last identity uses Lemma 3.9.

(2) By construction, the monodromy of the fundamental solution Υ_0 around a positively oriented loop γ_0 around 0 is $e^{2\pi\iota\mathsf{h}\Omega}$. Let now γ_∞ be a clockwise loop around ∞ based at $x_0 \in \mathbb{H}_+$. Since such a loop crosses the negative real axis before the positive one, the monodromy of Υ_+ around γ_+ is $S_+^{-1} \cdot e^{2\pi\iota\mathsf{h}\Omega_0} \cdot S_-^{-1}$. The result now follows from the fact that γ_∞ is homotopic to γ_0, and $\Upsilon_+ = \Upsilon_0 \cdot J_+$.

(3) follows from the PDE satisfies by Υ_0 and $\Upsilon_\pm$.

4.2. The R–matrix as a quantum Stokes matrix

Theorem. *The following holds*

$$R_+ = e^{\pi\iota\mathsf{h}\Omega_0} \cdot S_-^{-1} \qquad \textit{and} \qquad R_- = e^{\pi\iota\mathsf{h}\Omega_0} \cdot S_+^{-1}$$

Proof. By definition of S_+, $\Upsilon_+ = \Upsilon_- \cdot S_+$, when both $\Upsilon_\pm$ are considered as single–valued functions on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. On the other hand, by definition of $J_\pm$,

$$\Upsilon_+ = \Upsilon_0 \cdot J_+ = \Upsilon_- \cdot J_-^{-1} \cdot J_+$$

Using Proposition 3.11 therefore yields

$$S_+ = e^{\iota\pi\mathsf{h}\Omega_0} \cdot (J_+^{-1})^{21} \cdot e^{-\iota\pi\mathsf{h}\Omega} \cdot J_+ = e^{\iota\pi\mathsf{h}\Omega_0} \cdot (R_+^{-1})^{21}$$

where the last equality uses the fact that $R_{\mathrm{KZ}} = \exp(\pi\iota\mathsf{h}\Omega) = R_{\mathrm{KZ}}^{21}$. The first stated identity now follows from (1) of Proposition 4.1. The second one follows from the first and (3.5).

5. Quantum duality principle and semiclassical limits

5.1. Quantised universal enveloping algebras

Let k be a field of characteristic zero, and $\mathfrak{U}$ a quantised universal enveloping algebra (QUE) over k, that is a topologically free Hopf algebra over $\mathsf{k}[[\hbar]]$ such that $\mathfrak{U}/\hbar\mathfrak{U}$ is isomorphic to the enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ over k. Then, $\mathfrak{U}$ induces a Lie bialgebra structure on $\mathfrak{g}$, with cobracket $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ given by

$$\delta(x) = \left.\frac{\Delta(\widetilde{x}) - \Delta^{21}(\widetilde{x})}{\hbar}\right|_{\hbar=0}$$

where $\widetilde{x} \in \mathfrak{U}$ is an arbitrary lift of x.

5.2. The algebra $\mathfrak{U}'$

Let $\eta : \mathbb{C}[\![\hbar]\!] \to \mathfrak{U}$ and $\epsilon : \mathfrak{U} \to \mathbb{C}[\![\hbar]\!]$ be the unit and counit, respectively. $\mathfrak{U}$ splits as $\mathrm{Ker}(\epsilon) \oplus \mathbb{C}[\![\hbar]\!] \cdot 1$, with projection onto the first summand given by $\pi = \mathrm{id} - \eta \circ \epsilon$. Let $\Delta^{(n)} : \mathfrak{U} \to \mathfrak{U}^{\otimes n}$ be the iterated coproduct recursively defined by $\Delta^{(0)} = \epsilon$, $\Delta^{(1)} = \mathrm{id}$, and $\Delta^{(n)} = \Delta \otimes \mathrm{id}^{\otimes(n-2)} \circ \Delta^{(n-1)}$ for $n \geq 2$.

Following Drinfeld, define the subspace $\mathfrak{U}' \subset \mathfrak{U}$ by [7,15]

$$\mathfrak{U}' = \left\{ x \in \mathfrak{U} \,\middle|\, \pi^{\otimes n} \circ \Delta^{(n)}(x) \in \hbar^n \mathfrak{U}^{\otimes n},\ n \geq 1 \right\}$$

The definition of $\mathfrak{U}'$ extends that of the completed Rees algebra of $U\mathfrak{g}$ to an arbitrary QUE. Specifically, the following holds.

Lemma. *If $\mathfrak{U} = U\mathfrak{g}[[\hbar]]$ with undeformed coproduct, then $x = \sum_{n\geq 0} x_n \hbar^n$ lies in $\mathfrak{U}'$ if, and only if the filtration order of x_n in $U\mathfrak{g}$ is less than or equal to n.*

Proof. It is easy to see that, for any $x_1, \ldots, x_k \in \mathfrak{g}$

$$\pi^{\otimes n} \circ \Delta^{(n)}(x_1 \cdots x_k) = \sum_{\substack{I_1 \sqcup \cdots \sqcup I_n = \{1,\ldots,k\} \\ |I_i| \neq 0}} x_{I_1} \otimes \cdots \otimes x_{I_n}$$

where, for any $I = \{i_1, \ldots, i_m\}$, with $i_1 < \cdots < i_m$, we set $x_I = x_{i_1} \cdots x_{i_m}$. In particular, $\pi^{\otimes n} \circ \Delta^{(n)}(x_1 \cdots x_k) = 0$ if $n \geq k+1$. This implies that $\hbar^\ell x_1 \cdots x_k \in \mathfrak{U}'$ if, and only if $k \leq \ell$.

5.3. Quantum duality principle

Assume now that $\mathfrak{g}$ is finite–dimensional, let $(\mathfrak{g}^*, \delta^t, [\cdot,\cdot]^t)$ be the dual bialgebra, and G^* the formal Poisson–Lie group with Lie algebra $\mathfrak{g}$. By definition, the algebra of functions on G^* is the topological Poisson Hopf algebra given by $\mathsf{k}[[G^*]] = (U\mathfrak{g}^*)^*$. The following result is due to Drinfeld.

Theorem *([7,15]). $\mathfrak{U}'$ is a topologically free $\mathsf{k}[[\hbar]]$–module, and a topological sub Hopf algebra of $\mathfrak{U}$. Its multiplication is commutative mod $\hbar$, and $\mathfrak{U}'/\hbar\mathfrak{U}'$ is isomorphic, as a local, complete Poisson Hopf algebra to $\mathsf{k}[[G^*]]$.*

5.4. The isomorphism $\mathfrak{U}'/\hbar\mathfrak{U}' \cong \mathsf{k}[[G^*]]$

If $\mathfrak{U} = U\mathfrak{g}[[\hbar]]$ with undeformed coproduct, then $\delta = 0$ and $\mathfrak{g}^*$ has trivial bracket. In this case G^* is the (germ at 0 of the) abelian group $\mathfrak{g}^*$ and, by Lemma 5.2, $\mathfrak{U}'/\hbar\mathfrak{U}' = \widehat{\mathrm{gr}\, U\mathfrak{g}} = \mathsf{k}[[\mathfrak{g}^*]]$, where $\widehat{\cdot}$ is the graded completion.

More generally, the isomorphism $\mathfrak{U}/\hbar\mathfrak{U} \cong U\mathfrak{g}$ induces a canonical isomorphism

$$i_\Delta : \mathfrak{U}'/\hbar\mathfrak{U}' \longrightarrow \mathsf{k}[[G^*]]$$

as follows [10, Rem. 3.7]. Identify $U\mathfrak{g}^*$ as the quotient of the tensor algebra $T\mathfrak{g}^*$ endowed with the standard concatenation product and (cocommutative) shuffle coproduct, and $(U\mathfrak{g}^*)^*$ with a sub Hopf algebra of its dual $(T\mathfrak{g}^*)^* = \widehat{T\mathfrak{g}} = \prod_{n\geq 0} \mathfrak{g}^{\otimes n}$, where the latter is endowed with the (commutative) shuffle product and deconcatenation coproduct. Then, the isomorphism $i_\Delta : \mathfrak{U}'/\hbar\mathfrak{U}' \to \mathsf{k}[[G^*]] = (U\mathfrak{g}^*)^* \subset \widehat{T\mathfrak{g}}$ is given by noticing that if $x \in \mathfrak{U}'$, $\left(\frac{1}{\hbar^n}\pi^{\otimes n} \circ \Delta^{(n)}(x)\right)\big|_{\hbar=0}$ lies in $\mathfrak{g}^{\otimes n} \subset (U\mathfrak{g})^{\otimes n}$ for any n, and setting

$$i_\Delta(x) = \left\{ \left.\frac{\pi^{\otimes n} \circ \Delta^{(n)}(x)}{\hbar^n}\right|_{\hbar=0} \right\}_{n\geq 0} \in \prod_{n\geq 0} \mathfrak{g}^{\otimes n} \tag{5.1}$$

5.5. Semiclassical limit

If $\mathfrak{U}$ is a QUE which deforms $U\mathfrak{g}$, and $A \in \mathfrak{U} \otimes \mathfrak{U}'$, we denote by $\mathrm{scl}\,(A)$ the *semi–classical limit* of A, that is its class in $\mathfrak{U} \otimes \mathfrak{U}'/(\hbar\mathfrak{U} \otimes \mathfrak{U}')$. By Theorem 5.3, $\mathrm{scl}\,(A)$ lies in $U\mathfrak{g}\widehat{\otimes}\mathsf{k}[[G^*]]$, and is therefore a (formal) function on G^* with values in $U\mathfrak{g}$.

6. Semiclassical limit of the dynamical KZ equation

The goal of this section is to prove that the Stokes data of the ODE (2.1) are the semiclassical limits of the Stokes data of the dynamical KZ equations (3.3). Technicalities aside, this stems from the observation that if Υ is a solution of

$$\frac{d\Upsilon}{dz} = \left(\mathrm{ad}\,\mu^{(1)} + \mathsf{h}\frac{\Omega}{z}\right)\Upsilon$$

with values in $\mathfrak{U} \otimes \mathfrak{U}'$, the semiclassical limit γ of Υ, as a formal function of $\lambda \in \mathfrak{g}^*$ with values in $U\mathfrak{g}$, satisfies

$$\frac{d\gamma}{dz} = \left(\operatorname{ad}\mu + \frac{\nu(\lambda)}{2\pi\iota z}\right)\gamma$$

where $\nu(\lambda) = \mathrm{id}\otimes\lambda(\Omega)$ which, after the change of variable $z \to 1/z$, and the replacement $\operatorname{ad}\mu \to -A, \nu(\lambda) \to -2\pi\iota B$ is precisely the equation (2.1).[10]

6.1. Formal Taylor series groups

Let G be an affine algebraic group over $\mathbb{C}$. The ring of regular functions $\mathbb{C}[G]$ is a Hopf algebra, with coproduct $\Delta f(g_1, g_2) = f(g_1 g_2)$, counit $\epsilon(f) = f(1)$, and antipode $Sf(g) = f(g^{-1})$.

If $(R, m_R, 1_R)$ is a commutative, unital $\mathbb{C}$–algebra, the R–points of G are, by definition, the set of $\mathbb{C}$–algebra morphisms $G(R) = \mathrm{Alg}_{\mathbb{C}}(\mathbb{C}[G], R)$. $G(R)$ is a group, with multiplication $\phi \cdot \psi = m_R \circ \phi \otimes \psi \circ \Delta$, unit $1_R \circ \epsilon$, and inverse $\phi^{-1} = \phi \circ S$. Let $\mathfrak{m} \subset R$ be a maximal ideal, and denote by $G(R)_{\mathfrak{m}} \subset G(R)$ the normal subgroup consisting of maps $\gamma : \operatorname{Spec} R \to G(\mathbb{C})$ such that $\gamma(\mathfrak{m}) = 1$, that is

$$G(R)_{\mathfrak{m}} = \{\varphi \in \mathrm{Alg}_{\mathbb{C}}(\mathbb{C}[G], R) |\, \varphi(I) \subset \mathfrak{m}\}$$

where $I = \operatorname{Ker}\epsilon$ is the augmentation ideal. We shall need the following elementary

Lemma. *If R is a complete local ring with unique maximal ideal $\mathfrak{m}$, then $G(R)_{\mathfrak{m}}$ may be identified with the set of grouplike elements of the topological Hopf algebra*

$$U\mathfrak{g}\widehat{\otimes} R = \lim_{p} U\mathfrak{g} \otimes R/\mathfrak{m}^p$$

Proof. Let $\mathbb{C}[[G]] = \lim \mathbb{C}[G]/I^n$ be the completion of $\mathbb{C}[G]$ at the identity, and identify $U\mathfrak{g}$, as a Hopf algebra, with the continuous dual

$$\mathbb{C}[[G]]^* = \{\varphi \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[G], \mathbb{C}) |\, \varphi(I^n) = 0, n \gg 0\}$$

If $\mathfrak{m}^p = 0$ for some p, and $\phi \in G(R)_{\mathfrak{m}}$, ϕ vanishes on I^p and therefore lies in $(\mathbb{C}[G]/I^p)^* \otimes R \subset \mathbb{C}[[G]]^* \otimes R$. In general, $\mathfrak{m}$ is of finite order in $R/\mathfrak{m}^p$ for any $p \geq 1$, so that $G(R)_{\mathfrak{m}} = \lim_p G(R/\mathfrak{m}^p)_{\mathfrak{m}}$ embeds into $\lim_p U\mathfrak{g} \otimes R/\mathfrak{m}^p$.

We shall be interested below in the case when $R = \mathbb{C}[[V]]$ is the completion of the algebra of regular functions on the vector space $V = \mathfrak{g}$ or $V = \mathfrak{g}^*$ at 0. We denote in this case $G(R), G(R)_{\mathfrak{m}}$ and $U\mathfrak{g}\widehat{\otimes}R$ by $G[[V]], G[[V]]_+$ and $U\mathfrak{g}[[V]]$ respectively. As algebraic groups over $\mathbb{C}$, $G[[V]]$ and $G[[V]]_+$ are the inverse limits

$$G[[V]] = \varprojlim G[[V]]^{(m)} \qquad \text{and} \qquad G[[V]]_+ = \varprojlim G[[V]]_+^{(m)}$$

[10] The appearance of the factor $2\pi\iota$ is due to the fact that the identification $\mathfrak{U}'/\hbar\mathfrak{U}' \cong \widehat{S\mathfrak{g}}$ is given by mapping $x \in \mathfrak{g}$ to $\hbar x = 2\pi\iota\mathsf{h}x \in \mathfrak{U}'$.

where $G[[V]]^{(m)} = G(\mathbb{C}[[V]]/I^m)$, respectively, and $G[[V]]_+$ is prounipotent.

6.2. Semiclassical limit of canonical solutions of the DKZ equations

Consider the ODE

$$\frac{d\gamma}{dz} = \left(\frac{A}{z^2} + \frac{B}{z}\right)\gamma \tag{6.1}$$

and the dynamical KZ equation

$$\frac{d\Upsilon}{dz} = \left(\operatorname{ad}\mu^{(1)} + \mathsf{h}\frac{\Omega}{z}\right)\Upsilon \tag{6.2}$$

where $A, \mu \in \mathfrak{h}_{\mathrm{reg}}$, and $B \in \mathfrak{g}$.

Fix throughout the standard determination of the logarithm, with a cut along $\mathbb{R}_{<0}$. The following result shows that the semiclassical limits of the canonical fundamental solutions of (6.2) at $z = 0, \infty$ are the canonical fundamental solutions of (6.1) at $z = \infty, 0$, after the change of variable $z \to 1/z$.

Proposition. *Let $\nu : \mathfrak{g}^* \to \mathfrak{g}$ be the isomorphism given by $\lambda \to \lambda \otimes \mathrm{id}(\Omega)$, and set $\nu^\vee = -\nu/2\pi\iota$.*

(1) *Let γ_∞ be the canonical solution of (6.1) near $z = \infty$, and write*

$$\gamma_\infty = e^{-A/z} \cdot h_\infty \cdot z^B$$

where $h_\infty : \mathbb{P}^1 \setminus 0 \to G$ is such that $h_\infty(\infty) = 1$. Regard h_∞ as a holomorphic function of $B \in \mathfrak{g}_{\mathrm{nr}}$ such that $h_\infty(z)|_{B=0} \equiv 1$, and let

$$\widehat{h}_\infty : \mathbb{P}^1 \setminus 0 \longrightarrow G[[\mathfrak{g}]]_+$$

be its formal Taylor series at $B = 0$.
Let $\Upsilon_0 = e^{z \operatorname{ad}\mu^{(1)}} \cdot H_0 \cdot z^{\mathsf{h}\Omega}$ be the canonical solution of (6.2) near $z = 0$. Then, the semiclassical limit of H_0 takes values in $G[[\mathfrak{g}^*]]_+ \subset U\mathfrak{g}[[\mathfrak{g}^*]]$. Moreover, if $\mu = -A$, then

$$\operatorname{scl}(H_0(z))(\lambda) = \widehat{h}_\infty(1/z; \nu^\vee(\lambda))$$

(2) *Assume now that $A \in \mathfrak{h}_{\mathrm{reg}}^{\mathbb{R}}$. Let*

$$\gamma_\pm = h_\pm \cdot e^{-A/z} \cdot z^{[B]} : \mathbb{H}_\pm \to G$$

be the canonical solution of (6.1) *at* $z=0$ *corresponding to the half–plane* $\mathbb{H}_\pm = \{z \in \mathbb{C} |\ \mathrm{Im}(z) \gtrless 0\}$. *Regard* $h_\pm$ *as a holomorphic function of* $B \in \mathfrak{g}$ *such that* $h_\pm(z)|_{B=0} \equiv 1$, *and let*

$$\widehat{h}_\pm : \mathbb{H}_\pm \longrightarrow G[[\mathfrak{g}]]_+$$

be its formal Taylor series at $B=0$.

Let $\Upsilon_\pm = H_\pm \cdot e^{z\,\mathrm{ad}\,\mu^{(1)}} \cdot z^{\mathsf{h}\Omega_0}$, *be the canonical solution of* (6.2) *at* $z=\infty$ *corresponding to the half–planes* $\mathbb{H}_\pm$. *Then, the semiclassical limit of* $H_\pm$ *takes values in* $G[[\mathfrak{g}^*]]_+ \subset U\mathfrak{g}[[\mathfrak{g}^*]]$. *Moreover, if* $\mu = -A$, *then*

$$\mathrm{scl}\,(H_\pm(z))(\lambda) = \widehat{h}_\mp(1/z; \nu^\vee(\lambda))$$

Proof. (1) By definition, H_0 is a solution of

$$\frac{dH_0}{dz} = \frac{\mathsf{h}}{z}\left(\ell(e^{-z\,\mathrm{ad}\,\mu^{(1)}}(\Omega)) - \rho(\Omega)\right) H_0$$

where ℓ, ρ denote left and right multiplication respectively. Thus, as en element of $\mathfrak{U}\otimes\mathfrak{U}'/\hbar\mathfrak{U}\otimes\mathfrak{U}' = U\mathfrak{g}[[\mathfrak{g}^*]]$, the semiclassical limit h_0 of H_0 satisfies

$$\frac{dh_0}{dz} = \frac{1}{2\pi\iota z}\left(\ell(e^{-z\,\mathrm{ad}\,\mu}(\nu) - \rho(\nu)\right) h_0$$

together with the initial condition $h_0(0) = 1$. We claim that h_0 takes values in $G[[\mathfrak{g}^*]]_+ \subset U\mathfrak{g}[[\mathfrak{g}^*]]$. Indeed, both $\Delta\otimes\mathrm{id}(h_0)$ and $h_0^{13}h_0^{23}$ satisfy

$$\frac{dh}{dz} = \frac{1}{2\pi\iota z}\left(\ell(e^{-z\,\mathrm{ad}\,\mu}(\nu^1+\nu^2)) - \rho(\nu^1+\nu^2)\right) h$$

and the result follows by uniqueness. The claimed equality now follows from the uniqueness statement of Lemma 2.7, applied to the affine algebraic groups $G[\mathbb{C}[\mathfrak{g}^*]/I^m]$, $m \geq 1$.

(2) is proved similarly.

6.3. Semiclassical limit of the differential twist

Theorem. *Assume that* $A \in \mathfrak{h}^{\mathbb{R}}_{\mathrm{reg}}$, *and let* $C_\pm = \gamma_\pm^{-1}\cdot\gamma_\infty$ *be the connection matrix of* (6.1) *(see* 2.7*). Regard* $C_\pm$ *as a* G*–valued holomorphic function of* $B \in \mathfrak{g}_{\mathrm{nr}}$ *such that* $C_\pm|_{B=0} = 1$, *and let* $\widehat{C}_\pm \in G[[\mathfrak{g}]]_+$ *be its formal Taylor series at* $B = 0$.

Let $J_\pm = \Upsilon_0^{-1}\cdot\Upsilon_\pm$ *be the differential twist defined in* 3.11. *Then, if* $\mu = -A$, *the semiclassical limit of* $J_\pm$ *is given by*

$$\mathrm{scl}\,(J_\pm)(\lambda) = \widehat{C}_\mp(\nu^\vee(\lambda))^{-1}$$

Proof. By definition, $J_\pm = z^{-\mathsf{h}\Omega} \cdot H_0(z)^{-1} \cdot \exp(-z\,\mathrm{ad}\,\mu^{(1)})\,(H_\pm) \cdot z^{\mathsf{h}\Omega_0}$, where $z \in \mathbb{H}_\pm$. By Proposition 6.2,

$$\mathrm{scl}\,(J_\pm) = z^{-\nu/2\pi\iota} \cdot \widehat{h}_\infty(1/z; -\nu/2\pi\iota)^{-1} \cdot \exp(z\,\mathrm{ad}(A))\left(\widehat{h}_\mp(1/z; -\nu/2\pi\iota)\right) \cdot z^{[\nu]/2\pi\iota}$$

On the other hand,

$$C_\pm(B) = w^{-[B]} \cdot e^{A/w} \cdot h_\pm(w)^{-1} \cdot e^{-A/w} \cdot h_\infty(w) \cdot w^B$$

where $w \in \mathbb{H}_\pm$.

6.4. Semiclassical limit of the quantum Stokes matrices

Theorem. *Let $A \in \mathfrak{h}_{\mathrm{reg}}^{\mathbb{R}}$, and $S_\pm$ the Stokes matrices of the ODE (6.1) relative to the ray $-\iota\mathbb{R}_{>0}$ (see 2.6). Regard $S_\pm$ as a G–valued holomorphic function of $B \in \mathfrak{g}$ such that $S_\pm|_{B=0} = 1$, and let $\widehat{S}_\pm \in G[[\mathfrak{g}]]_+$ be its formal Taylor series at 0.*

Let $\mu \in \mathfrak{h}_{\mathrm{reg}}^{\mathbb{R}}$, and $S_\pm^\hbar$ the Stokes matrices of the dynamical KZ equation (6.2) (see 4.1). Then, $S_\pm^\hbar$ take values in $\mathfrak{U}\widehat{\otimes}\mathfrak{U}'$, and its semi–classical limit in $G[[\mathfrak{g}^*]]_+ \subset U\mathfrak{g}[[\mathfrak{g}^*]]$. Moreover, if $\mu = -A$, then

$$\mathrm{scl}\,(S_\pm^\hbar)(\lambda) = \widehat{S}_\pm(\nu^\vee(\lambda))$$

Proof. Let $\Upsilon_\pm = H_\pm \cdot e^{z\,\mathrm{ad}\,\mu^{(1)}} \cdot z^{\mathsf{h}\Omega_0}$ be the canonical solutions of the DKZ equations corresponding to the halfplanes $\mathbb{H}_\pm$, and $\widetilde{\Upsilon}_+ = \widetilde{H}_+ \cdot e^{z\,\mathrm{ad}\,\mu^{(1)}} \cdot z^{\mathsf{h}\Omega_0}$ the analytic continuation of Υ_+ across $\mathbb{R}_{>0}$. By definition,

$$S_+^\hbar = \Upsilon_-^{-1} \cdot \widetilde{\Upsilon}_+ = z^{-\mathsf{h}\Omega_0} \cdot \exp(-z\,\mathrm{ad}\,\mu^{(1)})\left(H_-^{-1} \cdot \widetilde{H}_+\right) \cdot z^{\mathsf{h}\Omega_0}$$

for $z \in \mathbb{H}_-$. By Proposition 6.2,

$$\mathrm{scl}\,(S_+^\hbar) = z^{-[\nu]/2\pi\iota} \cdot \exp(-z\,\mathrm{ad}(\mu))\left(\widehat{h}_+(1/z; -\nu/2\pi\iota)^{-1} \cdot \widetilde{\widehat{h}}_-(1/z; -\nu/2\pi\iota)\right) z^{[\nu]/2\pi\iota}$$

where $\widetilde{\widehat{h}}_-$ is the analytic continuation of $\widehat{h}_-$ across $\mathbb{R}_{>0}$.

On the other hand, if $\gamma_\pm(w) = h_\pm(w) \cdot e^{-A/w} \cdot w^{[B]}$ are the canonical solutions of (6.1) corresponding to $w \in \mathbb{H}_\pm$, and $\widetilde{\gamma}_-$ is the analytic continuation of γ_- across $\mathbb{R}_+$ then, by definition

$$S_+ = \gamma_+^{-1} \cdot \widetilde{\gamma}_- = w^{-[B]} \cdot e^{A/w} \cdot h_+(w)^{-1} \cdot \widetilde{h}_-(w) \cdot e^{-A/w} \cdot w^{[B]}$$

The Taylor series of S_+ at $B = 0$ therefore coincides with $\mathrm{scl}\,(S_+^\hbar)$ provided $A = -\mu$, $w = 1/z$, and B is replaced by $-\nu(\lambda)/2\pi\iota$. The proof that $\mathrm{scl}\,(S_-^\hbar) = \widehat{S}_-(-\nu/2\pi\iota)$ is identical.

7. Formal linearisation via quantisation

7.1. Let $(\mathfrak{p}, \mathsf{r})$ be a finite–dimensional quasitriangular Lie bialgebra over a field k of characteristic zero. Thus, $\mathfrak{p}$ is a Lie algebra, $\mathsf{r} \in \mathfrak{p} \otimes \mathfrak{p}$ satisfies the classical Yang–Baxter equations (CYBE)

$$[\mathsf{r}_{12}, \mathsf{r}_{23} + \mathsf{r}_{13}] + [\mathsf{r}_{13}, \mathsf{r}_{23}] = 0$$

and is such that $\Omega = \mathsf{r} + \mathsf{r}^{21}$ is invariant under $\mathfrak{p}$. In particular, $\mathfrak{p}$ is a Lie bialgebra with cobracket $\delta : \mathfrak{p} \to \mathfrak{p} \wedge \mathfrak{p}$ given by $\delta(x) = [x \otimes 1 + 1 \otimes x, \mathsf{r}]$.

Let $\mathfrak{p}^*$ be the dual Lie bialgebra to $\mathfrak{p}$, and P, P^* the formal Poisson–Lie groups with Lie algebras $\mathfrak{p}, \mathfrak{p}^*$. The CYBE imply that the maps $\ell, \rho : \mathfrak{p}^* \to \mathfrak{p}$ given by

$$\ell(\lambda) = \lambda \otimes \mathrm{id}(\mathsf{r}) \qquad \text{and} \qquad \rho(\lambda) = -\,\mathrm{id} \otimes \lambda(\mathsf{r})$$

are morphisms of Lie algebras. We denote the corresponding morphisms of formal groups $P^* \to P$ by $\mathcal{L}$ and $\mathcal{R}$ respectively, and by $\beta : P^* \to P$ the *big cell map*

$$g^* \longrightarrow \mathcal{L}(g^*) \cdot \mathcal{R}(g^*)^{-1}$$

The differential of β at 1 is $\ell - \rho : \lambda \to \lambda \otimes \mathrm{id}(\Omega) =: \nu(\lambda)$. In particular, β is an isomorphism of formal manifolds if r is *non–degenerate*, that is such that $\nu : \mathfrak{p}^* \to \mathfrak{p}$ is an isomorphism.

7.2. Set $\mathfrak{U} = U\mathfrak{p}[[\hbar]]$ and let $\Phi \in \mathfrak{U}^{\otimes 3}$ be an associator, that is an element satisfying $\Phi \in 1 + \frac{\hbar^2}{24}[\Omega_{12}, \Omega_{23}] + \hbar^3 \mathfrak{U}^{\otimes 3}$, and such that $(\mathfrak{U}, \Delta_0, e^{\hbar\Omega/2}, \Phi)$ is a quasitriangular quasi–Hopf algebra. Let $J \in 1 + \frac{\hbar}{2}\mathsf{j} + \hbar^2 \mathfrak{U}^{\otimes 2}$ be a twist such that $\mathsf{j} - \mathsf{j}^{21} = \mathsf{r} - \mathsf{r}^{21}$, and the following twist equation holds

$$\Phi \cdot J_{12,3} \cdot J_{1,2} = J_{1,23} \cdot J_{2,3} \tag{7.1}$$

Then, $\mathfrak{U}_J = \big(\mathfrak{U}, \Delta_J = J^{-1}\Delta_0(\cdot)J, R_J = J_{21}^{-1} e^{\hbar\Omega/2} J\big)$ is a quasitriangular Hopf algebra, which is a quantisation of $(\mathfrak{p}, \mathsf{r})$. By Theorem 5.3, $(\mathfrak{U}_J)'$ is therefore a quantisation of the Poisson algebra $\mathsf{k}[[P^*]]$.

7.3. Assume that the twist J is *admissible*, that is such that $\hbar \log(J) \in (\mathfrak{U}')^{\otimes 2}$. The following linearisation result is due to Enriquez–Halbout [10, Prop. 4.2].

Proposition. *The subalgebras $\mathfrak{U}'$ and $(\mathfrak{U}_J)'$ of $\mathfrak{U}$ coincide. Their equality therefore induces a formal Poisson isomorphism $\pi_J : \mathfrak{p}^* \to P^*$ given by the composition*

$$\mathsf{k}[[P^*]] \cong (\mathfrak{U}_J)'/\hbar(\mathfrak{U}_J)' = \mathfrak{U}'/\hbar\mathfrak{U}' \cong \mathsf{k}[[\mathfrak{p}^*]] \tag{7.2}$$

where the first and last isomorphisms are given by (5.1).

The explicit form of the isomorphism π_J is given by the following result of Enriquez–Etingof–Marshall [9, §3.3.2].

Theorem. *Assume further that $\Phi = \Psi(\hbar\Omega_{12}, \hbar\Omega_{23})$ where Ψ is a Lie associator, and that J lies in $\mathfrak{U}\otimes\mathfrak{U}'\cap\mathfrak{U}'\otimes\mathfrak{U}$. Then,*

(1) *The semiclassical limit $\jmath = J \mod \hbar\mathfrak{U}\otimes\mathfrak{U}'$ lies in $P[[\mathfrak{p}^*]]_+ \subset U\mathfrak{g}\widehat{\otimes}\mathsf{k}[[\mathfrak{p}^*]]$, that is, is a formal map $\mathfrak{p}^* \to P$.*
(2) *The composition of the Poisson isomorphism $\pi_J : \mathfrak{p}^* \to P^*$ with the big cell map $\beta : P^* \to P$ is the map $e_\jmath : \mathfrak{p}^* \to P$ defined by*

$$e_\jmath(\lambda) = \jmath(\lambda)^{-1} \cdot e^{\nu(\lambda)} \cdot \jmath(\lambda)$$

In particular, if r is non–degenerate, the map $\beta^{-1}\circ e_\jmath : \mathfrak{p}^* \to P^*$ is an isomorphism of formal Poisson manifolds.

Proof. We outline the proof for the reader's convenience. By assumption, $R = J_{21}^{-1}\cdot e^{\hbar\Omega/2}\cdot J$ lies in $\mathfrak{U}\otimes\mathfrak{U}' = \mathfrak{U}_J\otimes(\mathfrak{U}_J)'$, and similarly $R_{21}\in\mathfrak{U}_J\otimes(\mathfrak{U}_J)'$.

Consider now the identity

$$R_{21}\cdot R = J^{-1}\cdot e^{\hbar\Omega}\cdot J \tag{7.3}$$

Let $b \in \mathfrak{U}_J\otimes(\mathfrak{U}_J)'/\hbar\mathfrak{U}_J\otimes(\mathfrak{U}_J)' \cong U\mathfrak{p}[[P^*]]$ be the semiclassical limit of the left–hand side, and $a\in\mathfrak{U}\otimes\mathfrak{U}'/\hbar\mathfrak{U}\otimes\mathfrak{U}'\cong U\mathfrak{p}[[\mathfrak{p}^*]]$ that of the right–hand side. Clearly, $b\circ\pi_J = a$. It therefore suffices to show that $b=\beta$ and $a = e_\jmath$.

The identity $\Delta_J\otimes\mathrm{id}(R) = R_{13}\cdot R_{23}$ implies that the semiclassical limit R' of R lies in $P[[P^*]]_+$, and $\mathrm{id}\otimes\Delta_J(R) = R_{13}\cdot R_{12}$ that R' is an antihomomorphism $P^*\to P$. Its differential at the identity is readily seen to be the map $\mathfrak{p}^*\to\mathfrak{p}$ given by $\lambda\to\mathrm{id}\otimes\lambda(\mathsf{r})$, so that $R'(g^*) = \mathcal{R}(g^*)^{-1}$. Similarly, the semiclassical limit of R^{21} is the homomorphism $\mathcal{L} : P^*\to P$, and it follows that $b=\beta$.

Since the semiclassical limit of $e^{\hbar\Omega}$ is $e^\nu\in P[[\mathfrak{p}^*]]_+$, we have $a = e_\jmath$ and there remains to prove that $\jmath$ lies in $P[[\mathfrak{p}^*]]_+$, that is satisfies $\Delta_0\otimes\mathrm{id}(\jmath) = \jmath_{1,3}\cdot\jmath_{2,3}$. This is a consequence of the reduction of the twist equation (7.1) mod $\hbar\mathfrak{U}\otimes\mathfrak{U}\otimes\mathfrak{U}'$, as follows. Note first that since $J\in 1+\hbar\mathfrak{U}\otimes\mathfrak{U}$, $J_{1,2}\in 1+\hbar\mathfrak{U}\otimes\mathfrak{U}\otimes\mathfrak{U}'$. Next, it is easy to see that for any $x\in\mathfrak{U}'$, $\Delta_0(x)\in 1\otimes x+\hbar\mathfrak{U}\otimes\mathfrak{U}'$, hence $J_{1,23}\in J_{1,3}+\hbar\mathfrak{U}\otimes\mathfrak{U}\otimes\mathfrak{U}'$. Finally, $\hbar\Omega_{12}\in\hbar\mathfrak{U}\otimes\mathfrak{U}\otimes\mathfrak{U}'$, hence $\Phi = \Psi(\hbar\Omega_{12},\hbar\Omega_{23}) = \Psi(0,\hbar\Omega_{23}) = 1 \mod \hbar\mathfrak{U}\otimes\mathfrak{U}\otimes\mathfrak{U}'$.

8. Analytic linearisation via Stokes data

Let G be a complex reductive group, and $B_\pm\subset G$ a pair of opposite Borel subgroups intersecting along the maximal torus H. Let $\mathfrak{g},\mathfrak{b}_\pm,\mathfrak{h}$ be the Lie algebras of $G, B_\pm$, and H respectively, $\Phi\subset\mathfrak{h}^*$ the corresponding root system, and $\Phi_\pm\subset\Phi$ the set of positive and negative roots, so that $\mathfrak{b}_\pm = \mathfrak{h} \bigoplus_{\alpha \in \Phi_\pm} \mathfrak{g}_\alpha$. Fix an invariant inner product $(\cdot, \cdot)$ on $\mathfrak{g}$, and let $\mathsf{r} \in \mathfrak{b}_+ \otimes \mathfrak{b}_-$ be the corresponding canonical element (see (2.3)). Then, $(\mathfrak{g}, \mathsf{r})$ is a quasitriangular Lie bialgebra, and G and $G^* = B_- \times_H B_+$ are dual Poisson–Lie groups. Moreover, the homomorphisms $L, R : G^* \to G$ defined in 1.12 correspond to the first and second projection, respectively.

Let $A \in \mathfrak{h}_{\mathrm{reg}}$, and consider the connection

$$\nabla = d - \left(\frac{A}{z^2} + \frac{B}{z}\right) dz$$

Set $\mathfrak{h}^{\mathbb{R}} = \{t \in \mathfrak{h} | \alpha(t) \in \mathbb{R}, \alpha \in \Phi\}$, and let $\mathcal{C} = \{t \in \mathfrak{h} | \alpha(t) > 0,\ \alpha \in \Phi_+\} \subset \mathfrak{h}^{\mathbb{R}}_{\mathrm{reg}}$ be the fundamental chamber corresponding to Φ_+. Note that the rays $\pm\iota\mathbb{R}_{>0}$ are admissible if $A \in \mathfrak{h}^{\mathbb{R}}_{\mathrm{reg}} + \iota\mathfrak{h}^{\mathbb{R}} \subset \mathfrak{h}_{\mathrm{reg}}$. Moreover, by 2.6, the Stokes matrices $S_\pm$ corresponding to $r = -\iota\mathbb{R}_{>0}$ lie in $B_\pm(A, r) = B_\mp$ if $A \in -\mathcal{C} + \iota\mathfrak{h}^{\mathbb{R}}$. Let

$$\mathcal{S} : \mathfrak{g} \to G^* \qquad\qquad B \to \left(S_+^{-1} \cdot e^{-\iota\pi[B]}, S_- \cdot e^{\iota\pi[B]}\right)$$

be the Stokes map defined in 2.9.

Let $\nu : \mathfrak{g}^* \to \mathfrak{g}$ be the identification determined by $(\cdot, \cdot)$, and set $\nu^\vee = -\nu/2\pi\iota$.

Theorem. *If $A \in -\mathcal{C}$, the map $\mathcal{S} \circ \nu^\vee : \mathfrak{g}^* \to G^*$ is a Poisson map.*

Proof. Since $\mathcal{S} \circ \nu^\vee$ is complex analytic, it is sufficient to prove that its formal Taylor series at 0 is a Poisson map.

Set $\mu = -A \in \mathcal{C}$, and let $J_+ = J_+(\mu)$ the differential twist defined in 3.11. By Theorem 3.12, $J_+ \in 1 + \frac{\hbar}{2}\mathsf{j}_+ + \hbar^2\mathfrak{U}^{\otimes 2}$, where $\mathsf{j}_+ - \mathsf{j}_+^{21} = \mathsf{r} - \mathsf{r}^{21}$, and J_+ kills the KZ associator Φ_{KZ}.

Write $\Omega = \Omega_0 + \sum_{\alpha \in \Phi} \Omega_\alpha$, where $\Omega_0 = \sum_i t_i \otimes t^i$, with $\{t_i\}, \{t^i\}$ dual bases of $\mathfrak{h}$ with respect to $(\cdot, \cdot)$, and $\Omega_\alpha = x_\alpha \otimes x_{-\alpha}$, with $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ such that $(x_\alpha, x_{-\alpha}) = 1$. Then, one can show that $\log J_+$ is a Lie series in the variables $\hbar\Omega_0, \hbar\Omega_\alpha$. Since the subspace $\mathfrak{A}_n = \{x \in \mathfrak{U}^{\otimes n} | \, \hbar x \in (\mathfrak{U}')^{\otimes n}\}$ is a Lie algebra for any $n \geq 1$, and $\hbar\Omega_0, \hbar\Omega_\alpha \in \mathfrak{A}_2$, it follows that $\log J_+ \in \mathfrak{A}_2$.

Since J_+ lies in $\mathfrak{U}' \otimes \mathfrak{U} \cap \mathfrak{U} \otimes \mathfrak{U}'$ by 3.11, we may apply Theorem 7.2 to the pair $(\Phi_{\mathrm{KZ}}, J_+)$. Let $\jmath_+ \in \mathfrak{U} \otimes \mathfrak{U}'/\mathfrak{U} \otimes \mathfrak{U}' = G[[\mathfrak{g}^*]]_+$ be the semiclassical limit of J_+, and $e_{\jmath_+} \in G[[\mathfrak{g}^*]]_+$ the map $\lambda \to \jmath_+(\lambda)^{-1} \cdot e^{\nu(\lambda)} \cdot \jmath_+(\lambda)$. By Theorem 6.3

$$\begin{aligned}
e_{\jmath_+}(\lambda) &= \widehat{C}_-(-\nu(\lambda)/2\pi\iota; -\mu) \cdot e^{\nu(\lambda)} \cdot \widehat{C}_-(-\nu(\lambda)/2\pi\iota; -\mu)^{-1} \\
&= \left(\widehat{C}_-(\nu^\vee(\lambda); A) \cdot e^{2\pi\iota\nu^\vee(\lambda)} \cdot \widehat{C}_-(\nu^\vee(\lambda); A)^{-1}\right)^{-1} \\
&= \left(\widehat{S}_-(\nu^\vee(\lambda); A) \cdot e^{2\pi\iota[\nu^\vee(\lambda)]} \cdot \widehat{S}_+(\nu^\vee(\lambda); A)\right)^{-1} \\
&= \left(\widehat{S}_+(\nu^\vee(\lambda); A)^{-1} \cdot e^{-\pi\iota[\nu^\vee(\lambda)]}\right) \cdot \left(\widehat{S}_-(\nu^\vee(\lambda); A) \cdot e^{\pi\iota[\nu^\vee(\lambda)]}\right)^{-1}
\end{aligned}$$

$$= L(\widehat{\mathcal{S}}(\nu^\vee(\lambda); A)) \cdot R(\widehat{\mathcal{S}}(\nu^\vee(\lambda); A))^{-1}$$

where the third equality follows from the monodromy relation (Proposition 2.8), and the fifth from the definition of the Stokes map, as well as the assumption that $A \in -\mathcal{C}$, so that $S_\pm(B; A) \in N_\pm(A, r) = N_\mp$.

It follows that the composition $\beta^{-1} \circ e_{\jmath_+}$ is equal to $\widehat{\mathcal{S}} \circ \nu^\vee$, and is therefore a Poisson map by Theorem 7.2.

9. Isomonodromic deformations

Let $S_\pm \in U\mathfrak{g}^{\otimes 2}[\![\hbar]\!]^o$ be the Stokes matrices of the dynamical KZ equations, and $S_\pm^{\mathrm{scl}} \in G[[\mathfrak{g}^*]]_+$ their semiclassical limit.

For any $\alpha \in \Phi$, let $Q_\alpha \in S^2\mathfrak{g} \subset \mathbb{C}[\mathfrak{g}^*]$ be given by $Q_\alpha = x_\alpha \cdot x_{-\alpha} = Q_{-\alpha}$.

Proposition.

(1) *As a function of $\mu \in \mathfrak{h}_{\mathrm{reg}}^{\mathbb{R}}$, $S_\pm^{\mathrm{scl}}$ satisfies the following PDE*

$$d_{\mathfrak{h}} S_\pm^{\mathrm{scl}} = \frac{1}{2\pi\iota} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \{Q_\alpha, S_\pm^{\mathrm{scl}}\}$$

(2) *Regard $B \in \mathfrak{g}$ as a function of $\mu \in \mathfrak{h}_{\mathrm{reg}}^{\mathbb{R}}$. Then, the Stokes matrices (of the classical ODE) are locally constant as μ varies in $\mathfrak{h}_{\mathrm{reg}}$ if, and only if B satisfies the nonlinear PDE*

$$d_{\mathfrak{h}} B = -\frac{1}{2\pi\iota} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} H_\alpha$$

where $H_\alpha = \{Q_\alpha, -\}$ is the Hamiltonian vector field corresponding to Q_α.

Proof. (1) By Proposition 4.1, $S_\pm$ satisfy

$$d_{\mathfrak{h}} S_\pm = \frac{1}{4\pi\iota} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \left[\hbar\mathcal{K}_\alpha{}^{(1)} + \hbar\mathcal{K}_\alpha{}^{(2)}, S_\pm\right]$$

Note that $\hbar^2\mathcal{K}_\alpha \in \mathfrak{U}'$, and that its image in $\mathfrak{U}'/\hbar\mathfrak{U}'$ is $2Q_\alpha$. As pointed out in 3.3, $\hbar\,\mathrm{ad}(\mathcal{K}_\alpha)$ is a derivation of $\mathfrak{U}'$. Since $[\hbar\mathcal{K}_\alpha, -] = \hbar^{-1}[\hbar^2\mathcal{K}_\alpha, -]$, $\hbar\,\mathrm{ad}(\mathcal{K}_\alpha)$ induces the derivation $\{Q_\alpha, -\}$ on $\mathbb{C}[\mathfrak{g}^*]$. The result now follows from the fact that $\hbar\mathcal{K}_\alpha^{(1)} \in \hbar\mathfrak{U} \otimes \mathfrak{U}'$, so that its image in $U\mathfrak{g}[\![\mathfrak{g}^*]\!]$ is zero.

(2)

$$
\begin{aligned}
d_{\mathfrak{h}} S_{\pm}^{\mathrm{scl}} &= \frac{1}{2\pi\iota} \sum_{\alpha\in\Phi_+} \frac{d\alpha}{\alpha} \{Q_\alpha, S_{\pm}^{\mathrm{scl}}\} + d_{\mathfrak{g}^*} S_{\pm}^{\mathrm{scl}} (d_{\mathfrak{h}} B) \\
&= d_{\mathfrak{g}^*} S_{\pm}^{\mathrm{scl}} \left(\frac{1}{2\pi\iota} \sum_{\alpha\in\Phi_+} \frac{d\alpha}{\alpha} H_\alpha + d_{\mathfrak{h}} B \right)
\end{aligned}
$$

This is the time–dependent Hamiltonian description of the isomonodromic deformation given by [16,4]. Here, we give a quantum algebra proof, which enables us to interpret the symplectic nature of the isomonodromic deformation from the perspective of the gauge action of Casimir operators on quantum Stokes matrices.

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