Large monochromatic components in almost complete graphs and bipartite graphs

Zoltán Füredi*
Alfréd Rényi Institute of Mathematics
Budapest, Hungary
z-furedi@illinois.edu

Ruth Luo†
Department of Mathematics
University of California, San Diego
La Jolla, California, U.S.A.
ruluo@ucsd.edu

Submitted: Aug 30, 2020; Accepted: Feb 5, 2021; Published: Jun 18, 2021

Abstract

Gyárfas proved that every coloring of the edges of K_n with $t+1$ colors contains a monochromatic connected component of size at least n/t. Later, Gyárfás and Sárközy asked for which values of $\gamma = \gamma(t)$ does the following strengthening for *almost* complete graphs hold: if G is an n-vertex graph with minimum degree at least $(1-\gamma)n$, then every $(t+1)$-edge coloring of G contains a monochromatic component of size at least n/t. We show $\gamma = 1/(6t^3)$ suffices, improving a result of DeBiasio, Krueger, and Sárközy.

Mathematics Subject Classifications: 05C55, 05D10

1 Introduction, a stability of edge colorings

Erdős and Rado observed that every 2-edge-coloring of the complete graph K_n has a monochromatic spanning tree. Generalizing this result, Gyárfás [5] proved that every $(t+1)$-edge-coloring of the edge set $E(K_n)$ contains a monochromatic connected component of size at least n/t. This bound is the best possible when n is divisible by t^2 and an affine plane of order t exists.

Gyárfás and Sárközy [7] proved that Gyárfás' theorem has a remarkable stability property, the complete graph K_n can be replaced with graphs of high minimum degree.

Question 1 (Gyárfás and Sárközy [7])**.** Let $t \geqslant 2$. Which values of $\gamma = \gamma(t)$ guarantee that every $(t+1)$-edge-coloring of any n-vertex graph with minimum degree at least $(1-\gamma)n$ contains a monochromatic component of size at least n/t?

*Research is supported in part by the Hungarian National Research, Development and Innovation Office grant NKFI-133819.

†Research of this author is supported in part by NSF grant DMS-1902808.

Let $\gamma(t)$ denote the best value we can have. The case for $t = 1$ is trivial, $\gamma(1) = 0$. It is observed in [6] that *any* non-complete graph has a 2-edge-coloring without a monochromatic spanning tree: if xy is a non-edge, consider any edge-coloring where every edge incident to x is red and every edge incident to y is blue. Then there does not exist a monochromatic component containing both x and y.

The case for at least three colors (i.e., $t \geqslant 2$) is more interesting. Gyárfás and Sárközy [7] showed that $\gamma \leqslant 1/(1000t^9)$ suffices. This was improved to $1/(3072t^5)$ by DeBiasio, Krueger, and Sárközy [2].

It was also conjectured in [7] that $\gamma(t)$ could be as big as $t/(t+1)^2$. This was disproved for $t = 2$ by Guggiari and Scott [4] and by Rahimi [8], and more recently for general t by DeBiasio and Krueger [1]. The constructions of graphs in [1, 4, 8] are based on modified affine planes. They have minimum degree at least $(1 - \frac{t-1}{t(t+1)})n - 2$ and a $(t + 1)$-edge coloring in which each monochromatic component is of order less than n/t.

DeBiasio, Krueger, and Sárközy [2] proposed a version for bipartite graphs.

Question 2 (DeBiasio, Krueger, and Sárközy [2])**.** Let $t \geqslant 2$ and $n_1 \leqslant n_2$. Determine for which values of $\gamma = \gamma(t, n_1, n_2)$ the following is true: let G be an X_1, X_2-bipartite graph such that $|X_i| = n_i$ for $i \in \{1, 2\}$, for every $x \in X_1$, $d(x) \geqslant (1 - \gamma)n_2$, and for every $y \in X_2$, $d(y) \geqslant (1 - \gamma)n_1$. Then every t-edge-coloring of G contains a monochromatic component of order at least n/t.

They proved that $\gamma(t, n_1, n_2) \leqslant (n_1/n_2)^3/(128t^5)$ suffices. For both Questions 1 and 2 the $t = 2$ case is solved completely in [4, 8] and [2], respectively. They obtained $\gamma(2) = 1/6$, $\gamma(1, n_1, n_2) < 1/2$, and $\gamma(2, n_1, n_2) < 1/3$ (independently of n), and these constants are the best possible. So from now on, we only consider $t \geqslant 3$.

It was conjectured in [2] that for general t, $\gamma(t, n_1, n_2) < \frac{1}{t+1}$. This would be best possible when n_1 and n_2 are divisible by $t + 1$ by the following construction. Consider $t + 1$ perfect matchings of $K_{t+1,t+1}$ with partite sets $X \cup Y$. Delete all the edges of the $(t + 1)$th matching. Now let G be a graph obtained by blowing up each vertex in X into $n_1/(t + 1)$ new vertices and each vertex in Y into $n_2/(t + 1)$ vertices. Color an edge with color i if its endpoints were obtained by blowing up two vertices which were matched in the ith matching. It is easy to see the degrees of vertices are either $(1 - \frac{1}{t+1})n_2$ or $(1 - \frac{1}{t+1})n_1$, and a largest monochromatic component has size $(n_1 + n_2)/(t + 1)$.

Our main result is an improvement for the bound on $\gamma(t, n_1, n_2)$ in Question 2 which in turn implies a better bound for $\gamma(t)$ in Question 1.

Theorem 1.1. Fix integers $t \geqslant 3$, n_1, n_2 such that $n_2 \geqslant n_1 \geqslant 1$ and let $\gamma \leqslant \dfrac{(n_1/n_2)}{t^3}$. Let G be an X_1, X_2-bipartite graph such that $|X_i| = n_i$ for $i \in \{1, 2\}$,

$$\text{for every } x \in X_1,\ d(x) \geqslant (1 - \gamma)n_2, \text{ and for every } y \in X_2,\ d(y) \geqslant (1 - \gamma)n_1.$$

Then every t-edge-coloring of G contains a monochromatic component of order at least n/t.

Corollary 1.2. Fix integers $n, t \geqslant 3$, and let $\gamma \leqslant 1/(6t^3)$. Suppose G is an n-vertex graph with minimum degree at least $(1 - \gamma)n$. Then any coloring of $E(G)$ with $t + 1$ colors contains a monochromatic connected component with at least n/t vertices.

Our method is very similar to that in [7] or in [2]. The major difference is that we will first collect a series of general inequalities in the next section. While these tight inequalities are seemingly unrelated to graphs, we use them to lower bound the size of a "typical" monochromatic component. Our results will imply that in every color class there exists t components that are close in size to $(n_1 + n_2)/t$, and the remaining components are very small. We prove Theorem 1.1 in Section 3 and Corollary 1.2 in Section 4.

We use standard graph theory notation. The *degree* of a vertex v in G is denoted by $d_G(v)$ or simply $d(v)$ when there is no room for ambiguity. We denote the set of integers $\{1, 2, \ldots, s\}$ by $[s]$.

2 Inequalities

In this section, we prove some inequalities for sequences of integers. While our results hold in general, the reader should think of the sequences of integers as the sizes of each part (determined by a bipartition) of a monochromatic component for a fixed color.

It was pointed out by the anonymous referee that the following lemma is an easy consequence of a result called Milne's Inequality (see [9]). We include its short proof for completeness.

Lemma 2.1. Let $a_1, \ldots, a_s, b_1, \ldots, b_s, E, M, A, B$ be non-negative real numbers such that

- $\sum_{i=1}^{s} a_i b_i \geqslant E$,
- for all $i \in [s]$, $a_i + b_i \leqslant M$,
- $\sum_{i=1}^{s} a_i \leqslant A$, and $\sum_{i=1}^{s} b_i \leqslant B$.

Then $E(A + B) \leqslant MAB$.

Proof. The case $EAB = 0$ is easy, so we may suppose $A, B, E > 0$. Apply Jensen's inequality for the convex function x^2

$$\left(\frac{\sum_{i=1}^{s} b_i a_i}{\sum_{i=1}^{s} b_i}\right)^2 \leqslant \frac{\sum_{i=1}^{s} b_i a_i^2}{\sum_{i=1}^{s} b_i}.$$

Therefore

$$\frac{(\sum_{i=1}^{s} a_i b_i)^2}{\sum_{i=1}^{s} b_i} \leqslant \sum_{i=1}^{s} a_i^2 b_i,$$

and similarly $\frac{(\sum_{i=1}^{s} a_i b_i)^2}{\sum_{i=1}^{s} a_i} \leqslant \sum_{i=1}^{s} a_i b_i^2$. So we have

$$\begin{aligned} E \sum_{i=1}^{s} a_i b_i \left(\frac{1}{A} + \frac{1}{B}\right) &\leqslant \left(\sum_{i=1}^{s} a_i b_i\right)^2 \left(\frac{1}{\sum_{i=1}^{s} a_i} + \frac{1}{\sum_{i=1}^{s} b_i}\right) \\ &\leqslant \sum_{i=1}^{s} a_i^2 b_i + a_i b_i^2 = \sum_{i=1}^{s} (a_i b_i)(a_i + b_i) \leqslant M \sum_{i=1}^{s} a_i b_i. \end{aligned}$$

Dividing by $(\sum_{i=1}^{s} a_i b_i)$ and simplifying, we have $E(A^{-1}+B^{-1}) = E(A+B)/(AB) \leqslant M$. □

Lemma 2.2. Fix $n_1, n_2, t, a_1, \dots, a_s, b_1, \dots, b_s \geqslant 0$, $\varepsilon \geqslant 0$. Suppose $t > 1$, $n_1, n_2 > 0$,

- $\sum_{i=1}^{s} a_i b_i \geqslant (1-\varepsilon)\frac{n_1 n_2}{t}$,
- $\sum_{i=1}^{s} a_i \leqslant n_1$, $\quad \sum_{i=1}^{s} b_i \leqslant n_2$, and
- $a_i + b_i < (n_1+n_2)/t$ for all $i \in [s]$.

Then for all $i \in [s]$,

$$a_i < \frac{n_1}{t} + \frac{\sqrt{\varepsilon(t-1)n_1 n_2}}{t} \text{ and } b_i < \frac{n_2}{t} + \frac{\sqrt{\varepsilon(t-1)n_1 n_2}}{t}. \tag{1}$$

Proof. We prove the statement only for a_1, as the proofs for other a_i's and b_i's are symmetric.

First, we handle the case $a_1 = n_1$. Then $a_2 = \cdots = a_s = 0$ so the first constraint gives $a_1 b_1 = n_1 b_1 \geqslant (1-\varepsilon)\frac{n_1 n_2}{t}$. Hence $(1-\varepsilon)n_2/t \leqslant b_1$. Combining this with the last constraint we get

$$n_1 + (n_2/t) - (\varepsilon n_2)/t \leqslant a_1 + b_1 < (n_1/t) + (n_2/t).$$

Rearranging we have $(t-1)n_1 < \varepsilon n_2$. Multiplying each side by $(t-1)n_1$ and taking square roots, we get $(t-1)n_1 < \sqrt{\varepsilon(t-1)n_1 n_2}$ and therefore

$$a_1 = n_1 < \frac{n_1}{t} + \frac{\sqrt{\varepsilon(t-1)n_1 n_2}}{t},$$

as desired.

Second, consider the case $b_1 = n_2$. Then the last constraint implies $a_1 < (n_1+n_2)/t - b_1 = (n_1+n_2)/t - n_2 < n_1/t$, so (1) holds. From now on, we may suppose that $n_1 - a_1$ and $n_2 - b_1$ are both positive.

Third, suppose that $\sum_{i=2}^{s} a_i b_i \geqslant \frac{(n_1-a_1)(n_2-b_1)}{t-1}$. Let $M := \max_{2 \leqslant i \leqslant s}\{a_i + b_i\}$, $A = n_1 - a_1$, $B = n_2 - b_1$. Then by Lemma 2.1, we obtain

$$\frac{(n_1-a_1)(n_2-b_1)}{t-1}(n_1 - a_1 + n_2 - b_1) \leqslant M(n_1-a_1)(n_2-b_1).$$

Simplify by the positive term $(n_1-a_1)(n_2-b_1)$

$$M \geqslant \frac{n_1 - a_1 + n_2 - b_1}{t-1} \geqslant \frac{n_1 + n_2 - (n_1+n_2)/t}{t-1} = \frac{n_1+n_2}{t},$$

a contradiction.

Therefore, in the last case we consider, we may assume

$$\frac{(n_1-a_1)(n_2-b_1)}{t-1}+a_1b_1>\sum_{i=1}^{s}a_ib_i\geqslant(1-\varepsilon)\frac{n_1n_2}{t}.$$

Rearranging, we get

$$\begin{aligned}(n_1-a_1)(n_2-b_1)+(t-1)(a_1b_1) &> (1-\varepsilon)\frac{(t-1)(n_1n_2)}{t}\\ \Rightarrow\ n_1n_2-n_1b_1-n_2a_1+ta_1b_1 &> n_1n_2-\frac{n_1n_2}{t}-\varepsilon\frac{(t-1)n_1n_2}{t}\\ \Rightarrow\ \frac{n_1n_2}{t}+\varepsilon\frac{(t-1)n_1n_2}{t} &> n_2a_1-b_1(ta_1-n_1).\end{aligned}$$

If $a_1<n_1/t$, then we are done. So assume $a_1\geqslant n_1/t$ (so $ta_1-n_1\geqslant 0$). We add the non-positive term $(a_1+b_1-(n_1+n_2)/t)(ta_1-n_1)$ to the right hand side to obtain

$$\begin{aligned}\frac{n_1n_2}{t}+\varepsilon\frac{(t-1)n_1n_2}{t} &> n_2a_1-b_1(ta_1-n_1)+(a_1+b_1-\frac{n_1+n_2}{t})(ta_1-n_1)\\ &= n_2a_1+ta_1^2-a_1n_1-n_1a_1+\frac{n_1^2}{t}-n_2a_1+\frac{n_1n_2}{t}\\ \Rightarrow\ 0 &> ta_1^2-2n_1a_1+\left(\frac{n_1^2}{t}-\varepsilon\frac{(t-1)n_1n_2}{t}\right)\end{aligned}$$

Solving for a_1, we obtain

$$a_1<\frac{2n_1+\sqrt{4n_1^2-4(n_1^2-\varepsilon(t-1)n_1n_2)}}{2t}=\frac{n_1+\sqrt{\varepsilon(t-1)n_1n_2}}{t}.$$ □

Lemma 2.3. Fix $\varepsilon\geqslant 0$, integers $1\leqslant t\leqslant s$, and reals $a_1,\dots,a_s,b_1,\dots,b_s\geqslant 0$ such that

- $a_1\geqslant\dots\geqslant a_s\geqslant 0$,
- $\sum_{i=1}^{s}a_i=n_1,\quad \sum_{i=1}^{s}b_i=n_2>0$,
- for all $i\in[s]$, $a_i+b_i\leqslant(n_1+n_2)/t$,
- $\sum_{i=1}^{s}a_ib_i\geqslant(1-\varepsilon)n_1n_2/t$.

Let $a:=a_{t+1}+\dots+a_s$. Then

$$a\leqslant\varepsilon n_1\frac{n_1+n_2}{n_2}.$$

In particular, if $n_1\leqslant n_2$, then $a\leqslant 2\varepsilon n_1$.

Proof. We construct a new sequence $b'_1,\dots,b'_s$ with $b'_i\geqslant b_i$ for $i\in[t]$, $b'_j=0$ for $t<j\leqslant s$, such that $\sum_{i=1}^{t}b'_i=\sum_{i=1}^{s}b_i=n_2$, and $a_i+b'_i\leqslant(n_1+n_2)/t=:M$ for all $i\in[t]$. Note that these conditions together with the fact that the a_i's are non-increasing imply that

$\sum_{i=1}^{t} a_i b'_i \geqslant \sum_{i=1}^{s} a_i b_i$ since we are increasing the coefficients of larger a_i's by decreasing the coefficient of smaller a_j's.

We build our sequence greedily starting with $b_1, \ldots, b_s$. Define a set $I \subseteq [s]$ as follows

$$I(b_1, \ldots, b_s) := \{i \in [t], a_i + b_i < M\} \cup \{j : j > t, \, b_j > 0\}.$$

If for all $j \geqslant t+1$, $b_j = 0$, then we let $b'_1, \ldots, b'_s = b_1, \ldots, b_s$ and we are done. So suppose some $j \geqslant t+1$ satisfies $b_j \neq 0$, and hence $j \in I(b_1, \ldots, b_s)$. Then there exists $i \in [t]$ with $b_i + a_i < M$ (i.e., $i \in I(b_1, \ldots, b_s)$) because $\sum_{i=1}^{t}(a_i + b_i) \leqslant n_1 + n_2 - b_j = tM - b_j$. If $a_i + b_i + b_j \leqslant M$ then we update $b'_i = b_i + b_j$, $b'_j = 0$ and $b'_k = b_k$ for all $k \in [s] \setminus \{i, j\}$. Note that $j \notin I(b'_1, \ldots, b'_s)$.

If $a_i + b_i + b_j > M$ then we update $b'_i = b_i + M - (a_i + b_i) = M - a_i$, $b'_j = b_j - (M - (a_i + b_i))$ and $b'_k = b_k$ for $k \in [s] \setminus \{i, j\}$. In this case, we get $i \notin I(b'_1, \ldots, b'_s)$. Therefore in both cases we get $|I(b'_1, \ldots, b'_s)| \leqslant |I(b_1, \ldots, b_s)| - 1$, so one can continue this process at most s steps until we get $I(b'_1, \ldots, b'_s) \subset [t]$.

So suppose we have found a sequence $b'_1, \ldots, b'_t$ as desired. Apply Lemma 2.1 on the sequences $a_1, \ldots, a_t$ and $b'_1, \ldots, b'_t$. We have $\sum_{i=1}^{t} a_i = n_1 - a =: A$, $\sum_{i=1}^{t} b'_i = n_2 =: B$, $\sum_{i=1}^{t} a_i b'_i \geqslant \sum_{i=1}^{s} a_i b_i \geqslant (1-\varepsilon) n_1 n_2 / t =: E$, and $a_i + b'_i \leqslant M$ for all $i \in [t]$. Therefore,

$$\frac{(1-\varepsilon) n_1 n_2}{t}(n_1 + n_2 - a) \leqslant \frac{n_1 + n_2}{t}(n_1 - a) n_2$$

Rearranging and solving for a, we get

$$\begin{aligned} a(n_2 + \varepsilon n_1) \leqslant\ & \varepsilon n_1^2 + \varepsilon n_1 n_2 \\ \Rightarrow \quad a \leqslant\ & \varepsilon n_1 \frac{n_1 + n_2}{n_2 + \varepsilon n_1} \leqslant \varepsilon n_1 \frac{n_1 + n_2}{n_2}. \end{aligned}$$

□

3 Proof of Theorem 1.1 for almost complete bipartite graphs

Proof. Let G be an X_1, X_2-bipartite graph with $|X_1| = n_1$, $|X_2| = n_2$, and $n_2 \geqslant n_1 \geqslant 1$. Consider any coloring of the edges of G with colors $1, \ldots, t$. For a color $i \in [t]$, we denote by G^i the spanning subgraph of edges colored with i. Suppose that every monochromatic component has less than $(n_1 + n_2)/t$ vertices. We claim that $|E(G^i)| < n_1 n_2 / t$. Indeed, let $D_1, \ldots, D_s$ be the connected components of G^i. For $j \in [s]$, let $a_j = |D_j \cap X_1|$, $b_j = |D_j \cap X_2|$. Then $E := |E(G^i)| \leqslant \sum_{j=1}^{s} a_j b_j$. Apply Lemma 2.1 with $A = n_1$, $B = n_2$, $M = (n_1 + n_2 - 1)/t$. We get

$$E \leqslant (n_1 + n_2 - 1)/t \cdot (n_1 + n_2)^{-1} \cdot (n_1 n_2) < n_1 n_2 / t,$$

as desired.

Let ε_i be such that $|E(G^i)| = (1 - \varepsilon_i) n_1 n_2 / t$. By Lemma 2.2, a connected component of color i contains at most $\frac{n_\alpha}{t} + \frac{\sqrt{\varepsilon_i (t-1) n_1 n_2}}{t}$ vertices from X_α, $\alpha \in \{1, 2\}$. Therefore, for any $i \in [t]$, $x \in X_1$ and $y \in X_2$,

$$d_{G^i}(x) < \frac{n_2}{t} + \frac{\sqrt{\varepsilon_i (t-1) n_1 n_2}}{t}, \qquad d_{G^i}(y) < \frac{n_1}{t} + \frac{\sqrt{\varepsilon_i (t-1) n_1 n_2}}{t}. \tag{2}$$

Since $|E(G)| \geqslant (1-\gamma)n_1n_2$, we have $\sum_{i=1}^t \varepsilon_i \leqslant t\gamma$. Without loss of generality, suppose color 1 satisfies $\varepsilon_1 \leqslant \gamma$. Let $C_1, \ldots, C_r$ be the vertex sets of the connected components of color 1, ordered so that $|X_1 \cap C_1| \geqslant \ldots \geqslant |X_1 \cap C_r|$. Define a_j, b_j as before. Note that $s \geqslant t+1$, since the C_j's cover $V(G)$ and $|C_j| < (n_1+n_2)/t$ for all j. By Lemma 2.3, $a := a_{t+1} + \ldots + a_s \leqslant 2\varepsilon_1 n_1$.

Case 1: $X_2 \cap (C_{t+1} \cup \ldots \cup C_r) \neq \emptyset$. Fix a vertex y in this set. Then $d_{G^1}(y) \leqslant 2\varepsilon_1 n_1$. We get

$$\begin{aligned}(1-\gamma)n_1 \leqslant d_G(y) &< 2\varepsilon_1 n_1 + \frac{n_1(t-1)}{t} + \sum_{i=2}^{t} \frac{\sqrt{\varepsilon_i (t-1)n_1n_2}}{t} \\ &\leqslant 2\gamma n_1 + n_1 - \frac{n_1}{t} + \frac{\sqrt{(t-1)^2(\sum_{i=2}^t \varepsilon_i)n_1n_2}}{t} \\ &\leqslant 2\gamma n_1 + n_1 - \frac{n_1}{t} + \sqrt{\gamma t n_1 n_2} \cdot \frac{t-1}{t}.\end{aligned}$$

Here we used the fact that $\sum_{i=2}^t \frac{\sqrt{\varepsilon_i}}{t-1} \leqslant \sqrt{\frac{\sum_{i=2}^t \varepsilon_i}{t-1}}$ because $\sqrt{x}$ is a concave function. Therefore

$$\frac{n_1}{t} < n_1 3\gamma + \sqrt{\gamma t n_1 n_2} \cdot \frac{t-1}{t} \leqslant n_1 3 \frac{(n_1/n_2)}{t^3} + \sqrt{t\frac{(n_1/n_2)}{t^3} n_1 n_2} \cdot \frac{t-1}{t} \leqslant \frac{n_1}{t}\left(\frac{3}{t^2} + \frac{t-1}{t}\right),$$

a contradiction when $t \geqslant 3$.

Case 2: $X_2 \cap (C_{t+1} \cup \ldots \cup C_r) = \emptyset$. Let $x \in X_1 \cap (C_{t+1} \cup \ldots \cup C_r)$. By the case, x is not incident to an edge of color 1. So we instead obtain

$$\begin{aligned}(1-\gamma)n_2 \leqslant d_G(x) &< \frac{n_2(t-1)}{t} + \sum_{i=2}^{t} \frac{\sqrt{\varepsilon_i (t-1)n_1n_2}}{t} \\ &\leqslant n_2 - \frac{n_2}{t} + \sqrt{\gamma t n_1 n_2} \cdot \frac{t-1}{t}.\end{aligned}$$

This implies that

$$\frac{n_2}{t} < n_2 \gamma + \sqrt{\gamma t n_1 n_2} \cdot \frac{t-1}{t} \leqslant n_2 \frac{(n_1/n_2)}{t^3} + \sqrt{t\frac{(n_1/n_2)}{t^3} n_1 n_2} \cdot \frac{t-1}{t} = \frac{n_1}{t}\left(\frac{1}{t^2} + \frac{t-1}{t}\right),$$

a contradiction since $n_1 \leqslant n_2$ and $t \geqslant 3$. □

4 Proof of Corollary 1.2 for almost complete graphs

Proof. Let G be an n-vertex graph with minimum degree at least $(1-\gamma)n$, and suppose the edges of G are colored with colors $0, 1, \ldots, t$ such that each monochromatic connected component has size less than n/t. Again, we use G^i to refer to the spanning subgraph of the edges of color i.

Let $V_1, \dots, V_r$ be the vertex sets of the connected components of G^0. We will split the vertex set into two almost equal parts X_1 and X_2 such that the size of each part is in the range $[n(\frac{1}{2} - \frac{1}{2t}), n(\frac{1}{2} + \frac{1}{2t})]$, and each set V_i is contained either entirely in X_1 or entirely in X_2. To see that this is possible, arbitrarily add entire sets V_i to X_1 until $|X_1| < n(\frac{1}{2} + \frac{1}{2t})$ but adding any additional set to X_1 causes the size of X_1 to be at least $n(\frac{1}{2} + \frac{1}{2t})$. Then let $X_2 = V(G) - X_1$. At this point, $|X_1| > n(\frac{1}{2} - \frac{1}{2t})$, otherwise all sets V_j not contained in X_1 have size at least n/t, a contradiction.

Now let $|X_1| = n_1$, $|X_2| = n_2$, where without loss of generality, $|X_1| \leqslant |X_2| < 2|X_1|$ (and $n = n_1 + n_2$). By construction, there are no edges of color 0 between X_1 and X_2. Hence, the edges of the bipartite subgraph $G[X_1, X_2]$ are colored with t colors. (Here $G[X, Y]$ denotes the spanning bipartite subgraph of G in which we include only edges with endpoints in both X and Y.)

For simplicity, set $G' = G[X_1, X_2]$. Let $x \in X_1$ and $y \in X_2$. Then

$$d_{G'}(x) \geqslant n_2 - \gamma n = n_2 - \gamma(n_1 + n_2) \geqslant (1 - 2\gamma)n_2,$$

and

$$d_{G'}(y) \geqslant n_1 - \gamma n = n_1 - \gamma(n_1 + n_2) \geqslant n_1 - \gamma(n_1 + 2n_1) = (1 - 3\gamma)n_1.$$

Since G' does not have a monochromatic component of size at least $n/t = (n_1 + n_2)/t$, Theorem 1.1 implies that

$$3\gamma \geqslant \frac{(n_1/n_2)}{t^3} > \frac{1/2}{t^3} = \frac{1}{2t^3}.$$

We get a contradiction when $\gamma \leqslant 1/(6t^3)$.

□

Acknowledgement

We would like to thank the anonymous referees for their helpful comments.

References

[1] L. DeBiasio, R. A. Krueger. A note about monochromatic components in graphs of large minimum degree, arXiv:2006.08775 (2020).

[2] L. DeBiasio, R. A. Krueger, G. N. Sárközy. Large monochromatic components in multicolored bipartite graphs, J. Graph Theory **94** (2020), 117–130.

[3] A. Girão, S. Letzter, J. Sahasrabudhe. Partitioning a graph into monochromatic connected subgraphs, J. Graph Theory **91** (2019), 353–364.

[4] H. Guggiari, A. Scott. Monochromatic components in edge-coloured graphs with large minimum degree, arXiv:1909.09178 (2019), 18 pp.

[5] A. Gyárfás. Partition coverings and blocking sets in hypergraphs (in Hungarian), *Commun. Comput. Autom. Inst. Hungar. Acad. Sci.* **71** (1977): 62 pp.

[6] A. Gyárfás, G. N. Sárközy. Star versus two stripes Ramsey numbers and a conjecture of Schelp, Comb. Probab. Comput. **21**, (2012), 179–186.

[7] A. Gyárfás, G. N. Sárközy. Large monochromatic components in edge colored graphs with a minimum degree condition, Electron. J. Comb., **24**, no. 3 (2017), #P3.54.

[8] Z. Rahimi. Large monochromatic components in 3-colored non-complete graphs. J. Comb. Theory Ser. A. **172** (2020), 105256.

[9] J. M. Steele. The Cauchy–Schwarz master class: an introduction to the art of mathematical inequalities. Cambridge University Press (2004).