

Conditions for a bigraph to be super-cyclic

Alexandr Kostochka*

University of Illinois at Urbana–Champaign
Urbana, Illinois, U.S.A.
Sobolev Institute of Mathematics
Novosibirsk, Russia
kostochk@math.uiuc.edu

Mikhail Lavrov

Kennesaw State University
Marietta, Georgia, U.S.A.
mlavrov@kennesaw.edu

Ruth Luo[†]

University of California, San Diego
La Jolla, California, U.S.A.
ruluo@ucsd.edu

Dara Zirlin[‡]

University of Illinois at Urbana–Champaign
Urbana, Illinois, U.S.A.
zirlin2@illinois.edu

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Abstract

A hypergraph $\mathcal{H}$ is *super-pancyclic* if for each $A \subseteq V(\mathcal{H})$ with $|A| \geq 3$, $\mathcal{H}$ contains a Berge cycle with base vertex set A . We present two natural necessary conditions for a hypergraph to be super-pancyclic, and show that in several classes of hypergraphs these necessary conditions are also sufficient. In particular, they are sufficient for every hypergraph $\mathcal{H}$ with $\delta(\mathcal{H}) \geq \max\{|V(\mathcal{H})|, \frac{|E(\mathcal{H})|+10}{4}\}$.

We also consider *super-cyclic* bipartite graphs: (X, Y) -bigraphs G such that for each $A \subseteq X$ with $|A| \geq 3$, G has a cycle C_A such that $V(C_A) \cap X = A$. Super-cyclic graphs are incidence graphs of super-pancyclic hypergraphs, and our proofs use the language of such graphs.

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1 Introduction

1.1 Longest cycles in bipartite graphs and hypergraphs

For positive integers n, m , and δ with $\delta \leq m$, let $\mathcal{G}(n, m, \delta)$ denote the set of all bipartite graphs with a bipartition (X, Y) such that $|X| = n \geq 2$, $|Y| = m$ and for every $x \in X$, $d(x) \geq \delta$. In 1981, Jackson [4] proved that if $\delta \geq \max\{n, \frac{m+2}{2}\}$, then every graph $G \in \mathcal{G}(n, m, \delta)$ contains a cycle of length $2n$, i.e., a cycle that contains all vertices of X . This result is sharp. Jackson also conjectured that if $G \in \mathcal{G}(n, m, \delta)$ is 2-connected, then the lower bound on δ can be weakened.

Conjecture 1 (Jackson [4]). Let m, n, δ be integers. If $\delta \geq \max\{n, \frac{m+5}{3}\}$, then every 2-connected graph $G \in \mathcal{G}(n, m, \delta)$ contains a cycle of length $2n$.

Recently, the conjecture was proved in [8]. The restriction $\delta \geq \frac{m+5}{3}$ cannot be weakened any further because of the following example:

Example 2. Let $n_1 \geq n_2 \geq n_3 \geq 1$ be such that $n_1 + n_2 + n_3 = n$. Let $G_3(n_1, n_2, n_3) \in \mathcal{G}(n, 3\delta - 4, \delta)$ be the bipartite graph obtained from $K_{\delta-2, n_1} \cup K_{\delta-2, n_2} \cup K_{\delta-2, n_3}$ by adding two vertices a and b that are both adjacent to every vertex in the parts of size n_1, n_2 , and n_3 . Then a longest cycle in $G_3(n_1, n_2, n_3)$ has length $2(n_1 + n_2) \leq 2(n - 1)$.

Very recently [9], the bound was refined for 3-connected graphs in $G \in \mathcal{G}(n, m, \delta)$.

Theorem 3 ([9]). Let m, n, δ be integers. If $\delta \geq \max\{n, \frac{m+10}{4}\}$, then every 3-connected graph $G \in \mathcal{G}(n, m, \delta)$ contains a cycle of length $2n$.

A construction very similar to Construction 2 shows that the bound $\frac{m+10}{4}$ is sharp.

The results can be translated into the language of hypergraphs and hamiltonian Berge cycles.

Recall that a *hypergraph* $\mathcal{H}$ is a set of vertices $V(\mathcal{H})$ and a set of edges $E(\mathcal{H})$ such that each edge is a subset of $V(\mathcal{H})$. We consider hypergraphs with edges of any size. The degree, $d(v)$, of a vertex v is the number of edges that contain v .

A *Berge cycle* of length ℓ in a hypergraph is a set of ℓ distinct vertices $\{v_1, \dots, v_\ell\}$ and ℓ distinct edges $\{e_1, \dots, e_\ell\}$ such that for every $i \in [\ell]$, $v_i, v_{i+1} \in e_i$ (indices are taken modulo ℓ). The vertices $v_1, \dots, v_\ell$ are the *base vertices* of the cycle.

Naturally, a *hamiltonian Berge cycle* in a hypergraph $\mathcal{H}$ is a Berge cycle whose set of base vertices is $V(\mathcal{H})$.

Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a hypergraph. The *incidence graph* of $\mathcal{H}$ is the bipartite graph $I(\mathcal{H})$ with parts (X, Y) where $X = V(\mathcal{H})$, $Y = E(\mathcal{H})$ such that for $e \in Y, v \in X$, $ev \in E(I(\mathcal{H}))$ if and only if the vertex v is contained in the edge e in $\mathcal{H}$.

If $\mathcal{H}$ has n vertices, m edges and minimum degree at least δ , then $I(\mathcal{H}) \in \mathcal{G}(n, m, \delta)$. There is a simple relation between the cycle lengths in a hypergraph $\mathcal{H}$ and its incidence graph $I(\mathcal{H})$: If $\{v_1, \dots, v_\ell\}$ and $\{e_1, \dots, e_\ell\}$ form a Berge cycle of length ℓ in $\mathcal{H}$, then $v_1 e_1 \dots v_\ell e_\ell v_1$ is a cycle of length 2ℓ in $I(\mathcal{H})$, and vice versa.

1.2 Super-pancyclic hypergraphs and super-cyclic bigraphs

Recall that an n -vertex graph is *pancyclic* if it contains a cycle of length ℓ for every $3 \leq \ell \leq n$. There are a number of interesting results on pancyclic graphs, see e.g. survey [10]. A similar notion for hypergraphs and a strengthening of it were recently considered in [8].

A hypergraph $\mathcal{H}$ is *pancyclic* if it contains a Berge cycle of length ℓ for every $\ell \geq 3$. Furthermore, $\mathcal{H}$ is *super-pancyclic* if for every $A \subseteq V(\mathcal{H})$ with $|A| \geq 3$, $\mathcal{H}$ has a Berge cycle whose set of base vertices is A .

While the notion of super-pancyclic graphs is useless, since only complete graphs have this property, the notion for general hypergraphs is nontrivial. For example, Jackson's proof [4] that for $\delta \geq \max\{n, \frac{m+2}{2}\}$, each graph $G \in \mathcal{G}(n, m, \delta)$ has a cycle of length $2n$ yields a stronger statement. In the language of hypergraphs, it implies the following.

Theorem 4 ([4]). *If $\delta \geq \max\{n, \frac{m+2}{2}\}$, then every n -vertex hypergraph with m edges and minimum degree at least δ is super-pancyclic.*

It is interesting to find broader conditions guaranteeing that a hypergraph is super-pancyclic. The notion of super-pancyclicity translates into the language of bipartite graphs as follows.

By an (X, Y) -bigraph we mean a bipartite graph G with a specified ordered bipartition (X, Y) . An (X, Y) -bigraph is *super-cyclic* if for every $X' \subseteq X$ with $|X'| \geq 3$, G has a cycle C with $V(C) \cap X = X'$; we say that C is *based on* X' .

To state necessary conditions for an (X, Y) -bigraph to be super-cyclic, we need a new notion. For $A \subseteq X$, the *super-neighborhood* $\widehat{N}(A)$ is the set $\{y \in Y : |N(y) \cap A| \geq 2\}$.

If G is a super-cyclic (X, Y) -bigraph, $A \subseteq X$, and C is a cycle based on A , let $B = V(C) \cap Y$. Then $B \subseteq \widehat{N}(A)$ and $G[A \cup B]$ is 2-connected. Since adding a vertex of degree at least 2 to a 2-connected graph keeps the graph 2-connected, we conclude that every super-cyclic bigraph satisfies the following¹.

$$\text{For each } A \subseteq X \text{ with } |A| \geq 3: \begin{cases} |\widehat{N}(A)| \geq |A|, \text{ and} \\ G[A \cup \widehat{N}(A)] \text{ is 2-connected.} \end{cases} \quad (1)$$

We conjecture that these necessary conditions for a bigraph to be super-cyclic are also sufficient.

Conjecture 5. If G is an (X, Y) -bigraph satisfying (1), then G is super-cyclic.

To give partial support for Conjecture 5, let us somewhat refine the notion of super-cyclic bigraphs.

¹Jaehoon Kim [6] observed that to check condition (1), it is sufficient to verify that $G[A \cup \widehat{N}(A)]$ is 2-connected only when $|A| = 3$, though $|\widehat{N}(A)| \geq |A|$ still needs to be checked for all A . When $|A| > 3$, if $G[A \cup \widehat{N}(A)]$ is not 2-connected, there is a subset $A' \subseteq A$ with $|A'| = 3$ for which $G[A' \cup \widehat{N}(A')]$ is also not 2-connected.

For an integer $k \geq 3$, a bipartite graph G with partition (X, Y) is k -cyclic if for every $X' \subseteq X$ with $|X'| = k$, G has a cycle C that is based on X' . If G is k -cyclic for all $3 \leq k \leq |X|$, then it is super-cyclic.

In a series of claims, we prove the following.

Theorem 6. *If G is an (X, Y) -bigraph satisfying (1), then G is k -cyclic for $k = 3, 4, 5, 6$.*

Another result supporting Conjecture 5 was proved in [8] (in slightly different terms).

Theorem 7 ([8]). *Let $\delta \geq \max\{n, \frac{m+5}{3}\}$. If $G \in \mathcal{G}(n, m, \delta)$ satisfies (1), then G is super-cyclic.*

We use Theorem 6 and the ideas of the proof of Theorem 3 to strengthen Theorem 7 as follows.

Theorem 8. *Let $\delta \geq \max\{n, \frac{m+10}{4}\}$. If $G \in \mathcal{G}(n, m, \delta)$ satisfies (1), then G is super-cyclic.*

In terms of hypergraphs, our result is as follows.

Corollary 9 (Hypergraph version of Theorem 8). *Let $\delta \geq \max\{n, \frac{m+10}{4}\}$. If the incidence graph of an n -vertex hypergraph $\mathcal{H}$ with m edges and minimum degree $\delta(\mathcal{H})$ satisfies (1), then $\mathcal{H}$ is super-pancyclic.*

We present the main proofs in the language of bipartite graphs. We will say that an (X, Y) -bigraph G is *critical* if the following conditions hold:

- (a) G satisfies (1) but is not super-cyclic,
- (b) $\widehat{N}(X) = Y$, and
- (c) for every $X' \subset X$ with $X' \neq X$, $G[X' \cup Y]$ is super-cyclic.

Note that every graph satisfying (1) is either super-cyclic or has a critical subgraph.

Furthermore, we say that a critical (X, Y) -bigraph G is *saturated* if, after adding any X, Y -edge to G , the resulting graph is super-cyclic.

In Section 2 we prove basic properties of critical bigraphs. Based on this, in Section 3 we prove Theorem 6 for $k = 3, 4$, and 5. In Section 4 we discuss saturated critical graphs, which will be useful in the last two sections. In Section 5 we prove Theorem 6 for $k = 6$. In Section 6 we prove Theorem 8.

2 Properties of critical bigraphs

For all (X, Y) -bigraphs G below we assume $|X| \geq 3$, since G is trivially super-cyclic when $|X| \leq 2$.

Lemma 10. *Suppose that an (X, Y) -bigraph G satisfies (1). Then $|N(x) \cap N(x')| \geq 1$ for all distinct $x, x' \in X$.*

Proof. Let x'' be any vertex in $X - \{x, x'\}$ and $A = \{x, x', x''\}$. If $N(x) \cap N(x') = \emptyset$, then $G[A \cup \widehat{N}(A)] - x''$ has no x, x' -path, contradicting (1). $\square$

Claim 11. *Let G be a critical (X, Y) -bigraph. Then G is 2-connected.*

Proof. This is by the fact that $Y = \widehat{N}(X)$ and by (1). $\square$

Recall that for a vertex $v \in V(G)$ and a set $U \subseteq V(G)$, a v, U -fan of size t is a set of t paths from v to U such that the only common vertex of any two distinct paths is v . In view of Claim 11, the classical Dirac's Fan Lemma [3, 11] implies the following fact.

Lemma 12. *Let G be a critical (X, Y) -bigraph, $v \in V(G)$, and $U \subseteq V(G)$ with $|U| \geq 2$. Then G has 2 paths from v to U having only the vertex v in common.*

Let G be a critical (X, Y) -bigraph with $|X| = k + 1$ and $x_0 \in X$, where $k \geq 2$. Note that if $k < 2$, then G cannot be critical, since it is trivially super-cyclic. By definition, $G - \{x_0\}$ is super-cyclic. In particular, it has a cycle $C = x_1 y_1 x_2 y_2 \dots x_k y_k x_1$ based on $X - \{x_0\}$. We index the vertices of C modulo k ; for example, $x_{k+1} = x_1$. We derive some properties of such triples (G, x_0, C) .

Claim 13. *For all $y_i, y_j \in N(x_0)$, x_i and x_j have no common neighbor outside C . Similarly, x_{i+1} and x_{j+1} have no common neighbor outside C .*

Proof. If x_i and x_j have a common neighbor $y \notin V(C)$, then the cycle

$$x_1 y_1 \dots x_i y x_j y_{j-1} \dots y_i x_0 y_j x_{j+1} \dots x_1$$

is based on X , contrary to assumption. If x_{i+1} and x_{j+1} have such a common neighbor, consider the cycle C in reverse and apply the same argument. $\square$

Claim 14. *For every $y_i \in N(x_0)$, x_i and x_0 have no common neighbor outside C ; similarly, x_{i+1} and x_0 have no common neighbor outside C .*

Proof. If x_i and x_0 have a common neighbor $y \notin V(C)$, then we may extend C to a cycle based on X by replacing the edge $x_i y_i$ with the path $x_i y x_0 y_i$. The proof for x_{i+1} is similar. $\square$

Claim 15. *For every i , if x_i has a common neighbor y with x_0 outside C , then x_{i+1} has no common neighbor with x_0 outside C , except possibly for y .*

Proof. If x_{i+1} and x_0 have a common neighbor $y' \notin V(C)$, with $y' \neq y$, then we may extend C to a cycle based on X by replacing the path $x_i y_i x_{i+1}$ with the path $x_i y x_0 y' x_{i+1}$. $\square$

Lemma 16. *The vertex x_0 has at least two neighbors in C .*

Proof. Let A be the subset of X consisting of x_0 , together with all x_i that do not have a common neighbor with x_0 outside C .

If $|A| \geq 3$, then $G[A \cup \widehat{N}(A)]$ is 2-connected by (1), so x_0 has at least two neighbors in $\widehat{N}(A)$. Each of these neighbors must also be adjacent to at least one vertex in $A - \{x_0\}$. By our choice of A , these neighbors must be in C , and we are done.

If $|A| \leq 2$, then x_0 has a common neighbor outside C with all but at most one of $x_1, x_2, \dots, x_k$. By Claim 15, two consecutive vertices x_i, x_{i+1} cannot have different common neighbors with x_0 outside C . Therefore there is a vertex y_0 outside C adjacent to x_0 and to all but at most one of $x_1, x_2, \dots, x_k$.

By Claim 11, $d(x_0) \geq 2$. So there are two possibilities:

- If x_0 has a neighbor y_i in C , then at least one of x_i or x_{i+1} is adjacent to y_0 ; then it has a common neighbor with x_0 outside C , contradicting Claim 14.
- If x_0 has a neighbor y'_0 outside C , then y'_0 has a neighbor x_i in C because $\delta(G) \geq 2$. By Claim 15, x_{i-1} and x_{i+1} cannot have common neighbors with x_0 outside C except possibly for y'_0 . If $x_{i-1} \neq x_{i+1}$ then at least one of them is adjacent to y_0 , which is a contradiction. Otherwise, if $x_{i-1} = x_{i+1}$ then $k = 2$ and $|X| = 3$. If x_0 has a neighbor y''_0 that is adjacent to x_{i-1} , then we have a contradiction. If there is no such neighbor, then let $A = \{x_0, x_i, x_{i-1}\}$. We see that $G[A \cup \widehat{N}(A)]$ is not 2-connected, a contradiction.

Therefore the case $|A| \leq 2$ is impossible, completing the proof. $\square$

3 3-, 4-, and 5-cyclic graphs

Theorem 6 makes four claims: for $k = 3, 4, 5, 6$. In this section, we prove three of them.

Claim 17. *All (X, Y) -bigraphs G satisfying (1) are 3-cyclic.*

Proof. Suppose the claim is false and take a vertex-minimal counter-example, so that $|X| = 3$ and $Y = \widehat{N}(X)$. Then G is critical. By Claim 11, G is 2-connected, so it contains a cycle.

Suppose the longest cycle $C = x_1y_1x_2y_2x_1$ of G has 4 vertices and does not include the vertex x_3 . By Lemma 12, there are 2 paths from x_3 to $V(C)$ having only x_3 in common. Then G would contain a cycle of length 6 unless the paths are just x_3y_1 and x_3y_2 . Suppose that $y_3 \in Y$ (note that $|Y| \geq 3$ by (1)). Again, by Lemma 12, there are 2 paths from y_3 to $V(C)$ having only y_3 in common. Then G would contain a cycle of length 6 unless the paths are just y_3x_1 and y_3x_2 . Then we get a 6-cycle $x_1y_1x_3y_2x_2y_3x_1$. $\square$

Claim 18. *All (X, Y) -bigraphs G satisfying (1) are 4-cyclic.*

Proof. Suppose the claim is false and take a vertex-minimal counter-example, so that $|X| = 4$ and $Y = \widehat{N}(X)$. Then G is critical. Let $x_0 \in X = \{x_0, x_1, x_2, x_3\}$ have maximum degree. By Claim 17, $G - \{x_0\}$ has a 6-cycle $C = x_1y_1x_2y_2x_3y_3x_1$.

Case 1: x_0 has a neighbor y_0 outside of C . By Lemma 16, x_0 is adjacent to at least two of $\{y_1, y_2, y_3\}$; since $\delta(G) \geq 2$, y_0 is adjacent to at least one of $\{x_1, x_2, x_3\}$. Then there is an edge of C both of whose endpoints are adjacent to x_0 or y_0 ; without loss of generality, it's x_1y_1 . We can replace x_1y_1 by $x_1y_0x_0y_1$, extending C .

Case 2: All of x_0 's neighbors are in C . Then the neighbors of x in C have degree at least 3, and all other vertices in Y at least 2. Since $|Y| \geq |X|$, there are also vertices of X with degree 3, and since x_0 was chosen to have maximum degree in X , its degree is at least 3. Therefore x_0 is adjacent to all of $\{y_1, y_2, y_3\}$.

Since $|Y| \geq 4$, there is $y_0 \in Y$ outside C . By Claim 11, y_0 has at least two neighbors in X , and neither of them is x_0 . Without loss of generality, y_0 is adjacent to x_1 and x_2 , and so G has a cycle $x_0y_1x_2y_0x_1y_3x_3y_2x_0$.

In both cases, we get an 8-cycle, a contradiction. $\square$

Claim 19. All (X, Y) -bigraphs G satisfying (1) are 5-cyclic.

Proof. Suppose the claim is false and take a vertex-minimal counter-example, so that $|X| = 5$ and $Y = \widehat{N}(X)$. Then G is critical. Let $x_0 \in X = \{x_0, x_1, x_2, x_3, x_4\}$ have maximum degree. By Claim 18, $G - \{x_0\}$ has an 8-cycle $C = x_1y_1x_2y_2x_3y_3x_4y_4$.

Case 1: x_0 has a neighbor y_0 outside of C . By Lemma 16, x_0 is adjacent to at least two of $\{y_1, y_2, y_3, y_4\}$; since $\delta(G) \geq 2$, y_0 is adjacent to at least one of $\{x_1, x_2, x_3, x_4\}$. In almost all cases, there is an edge of C both of whose endpoints are adjacent to x_0 or y_0 , in which case we are done as before. The remaining case is unique up to relabeling C ; without loss of generality, x_0 is adjacent to y_1 and y_2 and y_0 is adjacent to x_4 .

If x_3y_4 is an edge, then there is a 10-cycle $x_1y_1x_2y_2x_0y_0x_4y_3x_3y_4x_1$, and similarly there is a 10-cycle if x_1y_3 is an edge. If neither is an edge, then $\widehat{N}(\{x_0, x_1, x_3\})$ contains y_1 and y_2 , but not y_3 or y_4 , so it needs a third vertex (call it y_5) which is outside C , adjacent to x_1 and either to x_0 or to x_3 . In either case, we get a 10-cycle: one of

$$x_1y_5x_3y_2x_2y_1x_0y_0x_4y_4x_1 \text{ or } x_1y_5x_0y_1x_2y_2x_3y_3x_4y_4x_1.$$

Note that in Case 1, we did not use that x_0 has maximum degree.

Case 2: All of x_0 's neighbors are in C . In this case, as before, we argue that x_0 must have degree at least 3. Say x_0 is adjacent to $\{y_1, y_2, y_3\}$; we make no assumption about whether x_0 is adjacent to y_4 .

We can replace x_2 or x_3 by x_0 to get new cycles using the same vertices y_1, y_2, y_3, y_4 of Y . If x_2 or x_3 has a neighbor other than y_1, y_2, y_3, y_4 , then we can apply Case 1.

So all the other vertices of Y (and there must be at least one) must be adjacent only to x_1 and x_4 . Since they can be swapped in for y_4 to get a new cycle, if y_4 is adjacent to any of x_0, x_2, x_3 , we can also reduce to a cycle C where Case 1 applies. Therefore y_4 is also adjacent only to x_1 and x_4 .

But now $\widehat{N}(\{x_0, x_1, x_2, x_3\}) = \{y_1, y_2, y_3\}$ which violates (1). In all cases, we get a contradiction. $\square$

4 Saturated critical bigraphs

Recall that a critical (X, Y) -bigraph G is *saturated* if adding to G any X, Y -edge results in a super-cyclic bigraph.

Lemma 20. *If G is a saturated critical (X, Y) -bigraph, then for every $y \in Y$, $|N(y)| \neq |X| - 1$.*

Proof. Suppose G is a saturated critical (X, Y) -bigraph, and for $y_0 \in Y$ and $x_0 \in X$ we have $N(y_0) = X - \{x_0\}$. Since G is critical, $G - \{x_0\}$ is super-cyclic, but G has no cycles based on X . Let $|X| = k$. Since G is saturated, $G + y_0x_0$ has a $2k$ -cycle $y_0x_1y_1x_2y_2 \dots x_ky_0$ where $x_k = x_0$. Then G contains path $P = y_0x_1y_1x_2 \dots x_k$.

By the choice of y_0 , $\{x_1, \dots, x_{k-1}\} \subseteq N(y_0)$. Thus if x_k is adjacent to any y_j for $1 \leq j \leq k - 2$, then G has cycle $x_ky_jx_jy_{j-1} \dots y_0x_{j+1}y_{j+1} \dots x_k$, a contradiction. Hence x_k has only one neighbor on P . Let $N_G(x_k) = \{y_{k-1}, z_1, z_2, \dots, z_s\}$. Since G is 2-connected, $s \geq 1$. Again, if any z_i is adjacent to any x_j for $j \leq k - 2$, then G has cycle $x_kz_ix_jy_{j-1} \dots y_0x_{j+1}y_{j+1} \dots x_k$, a contradiction. Hence $N(z_i) = \{x_{k-1}, x_k\}$ for all $1 \leq i \leq s$. Switching z_1 with y_{k-1} we conclude that $N(y_{k-1}) = \{x_{k-1}, x_k\}$. So, the only vertex of $X - \{x_k\}$ at distance 2 from x_k is x_{k-1} , a contradiction to Lemma 10. $\square$

Lemma 21. *If G is a saturated critical (X, Y) -bigraph and some $x_0 \in X$ has degree 2, then*

- (a) *each of its neighbors is adjacent to all vertices in X , and*
- (b) *$d(x) \geq 4$ for every $x \in X - \{x_0\}$.*

In particular, at most one vertex in X has degree 2.

Proof. Suppose G is a saturated critical (X, Y) -bigraph, and $d(x_0) = 2$ for some $x_0 \in X$. Let $N(x_0) = \{y_1, y_2\}$. We first prove part (a):

$$N(y_1) = N(y_2) = X. \quad (2)$$

Indeed, suppose $N(y_j) \neq X$ for some $j \in \{1, 2\}$. Then by Lemma 20, $|X - N(y_j)| \geq 2$, say, $\{x, x'\} \subseteq X - N(y_j)$. Consider $A = \{x_0, x, x'\}$ and $B = \widehat{N}_G(A)$. Then $y_j \notin B$ and so $d_{G[A \cup B]}(x_0) \leq 1$, a contradiction to (1). This proves (a).

Suppose (b) does not hold and consider an $x \in X - \{x_0\}$ such that $d(x) \leq 3$. Note that by (a), x is adjacent to y_1 and y_2 . For any $x' \in X - \{x, x_0\}$, Claim 17 for $A = \{x, x', x_0\}$ yields that there is a common neighbor $y(x')$ of x and x' distinct from y_1 and y_2 . Since $d(x) \leq 3$, all $y(x')$ coincide, and hence there is a vertex y adjacent to all vertices in X apart from x_0 , a contradiction to Lemma 20. This proves (b). $\square$

Lemma 22. *If G is a saturated critical (X, Y) -bigraph, then for every $y \in Y$, $|N(y)| \neq |X| - 2$.*

Proof. Suppose G is a saturated critical (X, Y) -bigraph, and $N(y_0) = X - \{x', x''\}$ for some $y_0 \in Y$. Let $|X| = k$.

Assume $d(x') \geq d(x'')$. By Lemma 21, $d(x') \geq 3$. Since G is saturated, it has a path $P = y_0x_1y_1x_2 \dots y_{k-1}x_k$ where $x_k = x'$. We may assume $x'' = x_j$ for some j .

If x_k is adjacent to any y_i for $i \in [k-2] - \{j-1\}$, then G has cycle

$$y_0x_{i+1}y_{i+1}x_{i+2} \dots x_ky_ix_i \dots y_0,$$

a contradiction. So $N(x_k) \cap V(P) \subseteq \{y_{k-1}, y_{j-1}\}$. Let $N(x_k) - V(P) = \{z_1, z_2, \dots, z_s\}$. Since $d(x_k) \geq 3$, $s \geq 1$. Let $T = X - \{x_k, x_{k-1}, x_{j-1}\}$. Again, if any z_ℓ is adjacent to any $x_i \in T$, then G has cycle $y_0x_{i+1}y_{i+1}x_{i+2} \dots x_kz_\ell x_i \dots y_0$, a contradiction. Hence

$$N(z_\ell) \cap T = \emptyset \quad \text{for all } 1 \leq \ell \leq s. \quad (3)$$

Since Claim 19 implies $k \geq 6$, $|T| \geq 3$. By Claim 17, for each $x_i, x_{i'} \in T$, G contains a 6-cycle C_1 with $V(C_1) \cap X = \{x_k, x_i, x_{i'}\}$, say $C_1 = x_ky_ix_i'y''x_k$. By (3) and the fact that $N(x_k) \cap V(P) \subseteq \{y_{j-1}, y_{k-1}\}$, $\{y, y''\} \subseteq \{y_{k-1}, y_{j-1}\}$. In particular, $x_ky_{j-1} \in E(G)$.

Similarly, if there are $x_i, x_{i'} \in T$ both not adjacent to y_{k-1} or both not adjacent to y_{j-1} , then G does not contain a 6-cycle C_1 with $V(C_1) \cap X = \{x_k, x_i, x_{i'}\}$; however, G is 3-cyclic, a contradiction. This means $|N(y_{k-1}) \cap T| \geq |T| - 1$ and $|N(y_{j-1}) \cap T| \geq |T| - 1$. Since $|T| \geq 3$, this implies that there is $x_i \in T \cap N(y_{k-1}) \cap N(y_{j-1})$. Note that $i \notin \{j-1, k-1, k\}$, since $x_{j-1} \notin T$. Since $x_{i+1} \notin \{x_j, x_k\}$, y_0 is adjacent to x_{i+1} .

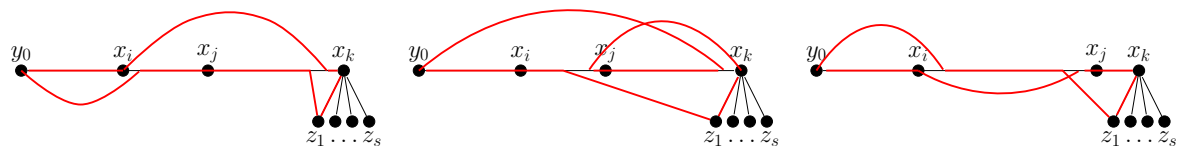


Figure 1: 3 configurations for Lemma 22

Since G is 2-connected, z_1 has a neighbor in $\{x_{k-1}, x_{j-1}\}$. If $z_1x_{k-1} \in E(G)$, then G has cycle $y_0x_1 \dots x_iy_{k-1}x_kz_1x_{k-1}y_{k-2}x_{k-2} \dots x_{i+1}y_0$. So $N(z_1) = \{x_k, x_{j-1}\}$ (see Fig. 1 (1)). If $j \neq k-1$, then G has the cycle $y_0x_1 \dots x_{j-1}z_1x_ky_{j-1}x_jy_jx_{j+1} \dots x_{k-1}y_0$ (see Fig. 1 (2)). Hence we may suppose $j = k-1$. Then by the definition of T , $i \leq k-3$. So G has the cycle $y_0x_1 \dots x_iy_{k-2}x_{k-1}y_{k-1}x_kz_1x_{k-2}y_{k-2} \dots x_{i+1}y_0$ (see Fig. 1 (3)), a contradiction. $\square$

A critical (X, Y) -bigraph G is *Y-minimal* if every proper subgraph $G' = (X', Y'; E')$ of G satisfying (1) is super-cyclic.

Lemma 23. *If a saturated critical Y-minimal (X, Y) -bigraph G has vertices $y_1, y_2 \in Y$ of degree 2, then $N(y_1) \neq N(y_2)$.*

Proof. Suppose $N(y_1) = N(y_2) = \{x_1, x_2\}$, and consider the graph $G' := G - \{y_1\}$ with partite sets X and $Y' = Y - \{y_1\}$. Note that in G , each cycle of length at least 6 contains at most one vertex in $\{y_1, y_2\}$ since the neighbors of such a vertex on the cycle must be exactly x_1 and x_2 . Hence for each cycle C of length at least 6 in G , there exists a cycle C' in G' with $C \cap X = C' \cap X$. We will show that (1) holds for G' .

Indeed, suppose there exists a set $A \subseteq X$ with $|\widehat{N}_{G'}(A)| < |A|$. Then $\{x_1, x_2\} \subseteq A$, $\widehat{N}_{G'}(A) = \widehat{N}_G(A) - \{y_1\}$, and hence $|\widehat{N}(A)| = |A|$. If $|A| \geq 4$, then $|\widehat{N}(A - \{x_1\})| \geq |A - \{x_1\}| = |A| - 1$. However, $\widehat{N}(A - \{x_1\}) \subseteq \widehat{N}(A) - \{y_1, y_2\}$, a contradiction. So $|A| = 3$, say $A = \{x_1, x_2, x_3\}$, $\widehat{N}_{G'}(A) = \{y_2, y_3\}$, and $\widehat{N}(A) = \{y_1, y_2, y_3\}$. But there is no 6-cycle in G based on A since $N(y_1) = N(y_2) = \{x_1, x_2\}$. This contradicts Claim 17.

Now suppose G' is not 2-connected. Then G' contains a cut vertex v , and $\{v, y_1\}$ is a cut set in G . This implies that x_1 and x_2 are in different components of $G - \{v, y_1\}$, and so $v = y_2$. Let $x_3 \in X - \{x_1, x_2\}$. Then there is no 6-cycle based on $\{x_1, x_2, x_3\}$ in G , a contradiction.

By the definition of critical Y -minimal bigraphs, G' is super-cyclic; but then G also is. $\square$

Lemma 24. *If G is a saturated critical Y -minimal (X, Y) -bigraph, $x \in X$ and C is a cycle based on $X - \{x\}$, then x has at least two non-neighbors in $V(C) \cap Y$.*

Proof. Let $|X| = k$ and let $C = x_1 y_1 \dots x_{k-1} y_{k-1} x_1$. Suppose for the sake of contradiction that $|N(x) \cap V(C)| \geq k - 2$. If $N(x)$ contains a vertex y that is not in C , then because G is 2-connected, y has a neighbor in $V(C)$, say x_1 . Then without loss of generality, $y_1 \in N(x)$, and we may replace the edge $x_1 y_1$ in C with the path $x_1 y x y_1$ to obtain a cycle based on X , a contradiction.

So we may assume $N(x) \subseteq V(C)$. Since $|\widehat{N}(X)| \geq |X|$, there exists a vertex $y \in \widehat{N}(X) \setminus V(C)$. Since G is 2-connected and $yx \notin E(G)$, y has some neighbors x_i and x_j in C . If $\{y_i, y_j\} \subseteq N(x)$ then we obtain the cycle $x_1 y_1 \dots x_i y x_j y_{j-1} \dots y_i x y_j x_{j+1} \dots x_1$, a contradiction. Similarly, we have that $\{y_{i-1}, y_{j-1}\} \not\subseteq N(x)$. The remaining case is $N(y) = \{x_i, x_{i+1}\}$ and $N(x) = V(C) - \{y_i\}$. By considering the cycle obtained by replacing y_i with y , we see that by symmetry, $N(y_i) = \{x_i, x_{i+1}\}$. But this contradicts Lemma 23. $\square$

5 6-cyclic graphs

In this section, we complete the proof of Theorem 6 by proving that all (X, Y) -bigraphs satisfying (1) are 6-cyclic. We will use $N_C(x)$ to denote the neighborhoods of x that are in $V(C)$.

Lemma 25. *If G is a saturated critical (X, Y) -bigraph and $|X| = 6$, then X contains a vertex of degree at least 4.*

Proof. Suppose all vertices in X have degree at most 3.

Case 1: There is a vertex $y \in Y$ with $d(y) \geq 4$. By Lemma 22, $d(y) \neq 4$, so $d(y) \geq 5$. Let x_1, x_2, x_3, x_4, x_5 be five neighbors of y ; let $C = x_1y_1x_2y_2x_3y_3x_4y_4x_5y_5x_1$ be a cycle containing them.

If $y \notin V(C)$, then x_1, x_2, x_3, x_4, x_5 each have two neighbors on C and are also adjacent to y . Since the degree of each x_i is at most 3, they cannot have any other neighbors. In that case, the set $A = \{x_1, x_2, x_4\}$ contradicts (1), since $\widehat{N}(A) = \{y_1, y\}$.

Therefore $y \in V(C)$; say, $y = y_1$. Then x_3, x_4, x_5 have two neighbors on C and an edge to y_1 , so they have degree 3. By (1) applied to $A = \{x_1, x_3, x_5\}$, x_1 must have an edge to one of y_2, y_3, y_4 ; symmetrically, x_2 must have an edge to one of y_3, y_4, y_5 . This yields 3 edges incident to each of x_1, x_2, x_3, x_4, x_5 ; none of these can have any other neighbors.

By (1), $|\widehat{N}(X)| \geq 6$; however, since there is only one vertex in $X - V(C)$, $\widehat{N}(X) \subseteq N(X \cap V(C)) = Y \cap V(C)$. This only has size 5, a contradiction.

Case 2: All vertices in Y have degree at most 3. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Let $C_1 = x_1y_1x_2y_2x_3y_3x_1$ be a 6-cycle based on $\{x_1, x_2, x_3\}$ and let $C_2 = x_4y_4x_5y_5x_6y_6x_4$ be a 6-cycle based on $\{x_4, x_5, x_6\}$. We have $V(C_1) \cap V(C_2) = \emptyset$, since a vertex in $V(C_1) \cap V(C_2)$ would have degree at least 4.

In the cycle based on $\{x_1, x_2, x_4\}$, the vertex x_4 must have two common neighbors with $\{x_1, x_2\}$. Since $\Delta(G) \leq 3$, at least one of them is a neighbor of x_4 on C_2 . Without loss of generality, let y_4 be a common neighbor with x_1 , so that $x_1y_4 \in E(G)$.

Now consider the cycle based on $\{x_1, x_4, x_5\}$. By the same argument, either x_4 or x_5 must be adjacent to one of x_1 's neighbors on C_1 . Without loss of generality, let x_4y_1 be that edge; then the cycle $x_1y_4x_5y_5x_6y_6x_4y_1x_2y_2x_3y_3x_1$ is based on X , a contradiction. $\square$

Claim 26. All (X, Y) -bigraphs G satisfying (1) are 6-cyclic.

Proof. Take a vertex-minimal counterexample G with the most edges, meaning in particular that $|X| = 6$ and $Y = \widehat{N}(X)$. By Claims 17–19, G is k -cyclic for $3 \leq k \leq 5$; therefore G is critical, saturated and Y -minimal.

Let $X = \{x_1, \dots, x_6\}$ and x_6 be a vertex of maximum degree in X . By Lemma 25, $d(x_6) \geq 4$. Let $C = x_1y_1x_2y_2x_3y_3x_4y_4x_5y_5x_1$ be a cycle based on $X - \{x_6\}$. By Lemma 16 and Lemma 24, x_6 has either 2 or 3 neighbors on C , so it has at least one neighbor y_6 not on C .

By symmetry, the following two cases are exhaustive.

Case 1: $\{y_1, y_3\} \subseteq N_C(x_6)$. In this case, by Claim 14, no vertex $y \in N(x_6) - V(C)$ can be adjacent to x_1, x_2, x_3 , or x_4 , so it must be adjacent to x_5 and x_6 only. By Lemma 23, y_6 is the only such vertex. By Claim 14 again, x_6 cannot be adjacent to y_4 or y_5 . To have $d(x_6) \geq 4$, x_6 must also be adjacent to y_2 , and therefore $d(x_6) = 4$.

If $x_2y_4 \in E(G)$, then the cycle $x_2y_4x_4 \dots y_2x_6y_6x_5y_5x_1y_1x_2$ is based on X , and if $x_2y_5 \in E(G)$, the cycle $x_2y_2 \dots x_5y_6x_6y_1x_1y_5x_2$ is based on X . Thus, $x_2y_4 \notin E(G)$ and $x_2y_5 \notin E(G)$. A similar argument shows that $x_3y_5, x_3y_4 \notin E(G)$. However, applying Claim 17 to $A = \{x_2, x_3, x_5\}$, we find distinct vertices $y' \in N(x_2) \cap N(x_5)$ and $y'' \in N(x_3) \cap N(x_5)$ such that $y', y'' \notin \{y_4, y_5, y_6\}$. Therefore x_5 is adjacent to y_4, y_5, y_6, y', y'' , and $d(x_5) \geq 5 > d(x_6)$, contradicting the choice of x_6 .

Case 2: $N_C(x_6) = \{y_1, y_2\}$. In this case, in order to have $d(x_6) \geq 4$, x_6 must have neighbors y, y' outside C . By Claim 14, y and y' can only have x_4 and x_5 as neighbors. If y is adjacent to x_4 and y' is adjacent to x_5 , or vice versa, we contradict Claim 15; if both are adjacent only to x_4 or both only to x_5 , we contradict Lemma 23. $\square$

6 Bigraphs with high minimum degree

6.1 Properties of smallest counterexamples

Throughout this subsection, we assume that G is a vertex-minimal counterexample to Theorem 8 with the most edges; let $G \in \mathcal{G}(n, m, \delta)$ where $\delta \geq \max\{n, \frac{m+10}{4}\}$. Then for each $X' \subset X$ with $X' \neq X$, $G[X' \cup Y]$ also satisfies the conditions of Theorem 8 and hence is super-cyclic.

Let $G' = G[X \cup \widehat{N}(X)]$, i.e., G' is obtained by removing only the degree-1 vertices of G . Then G' is critical and saturated. In particular, for every $x \in X$, there exists a cycle C in G' (and therefore in G) such that $V(C) \cap X = X - \{x\}$. By Lemma 12, G' has an $x, V(C)$ -fan F of size 2.

Among the triples (C, x, F) where $x \in X$, C is a cycle with $V(C) \cap X = X - \{x\}$ and F is an $x, V(C)$ -fan, choose a triple such that the size of F is maximized, and subject to this, $|V(F)|$ is minimized. Let $|V(C)| = 2\ell$ (so $|X| = \ell + 1$). Let t be the size of F , and let $T = V(C) \cap V(F) = \{u_1, \dots, u_t\}$.

Fix a clockwise direction of C . For every vertex u of C , $x_C^+(u)$ (respectively, $x_C^-(u)$) denotes the closest to u clockwise (respectively, counterclockwise) vertex of X distinct from u . For a set $U \subset V(C)$, $X_C^+(U) = \{x_C^+(u) : u \in U\}$. When C is clear from the content, the subscripts could be omitted. The vertices $y^+(u), y^-(u)$ and the sets $X^-(U), Y^+(U), Y^-(U)$ are defined similarly.

Viewing F as a tree (spider) with root x , any two vertices $u, v \in V(F)$ define the unique u, v -path $F[u, v]$ in F . For $u, v \in V(C)$, let $C[u, v]$ be the clockwise u, v -path in C and let $C^-[u, v]$ be the counterclockwise u, v -path in C .

Lemma 27. $t \leq \ell - 2$.

Proof. We first show that

$$t \leq \ell - |T \cap X|. \quad (4)$$

If $w \in T \cap X$ and $y^+(w) \in T$, then the cycle $wF[w, y^+(w)]y^+(w)C[y^+(w), w]w$ is based on X , a contradiction. Similarly, $y^-(w), x^+(w), x^-(w) \notin T$. Thus, $|T \cap X| \leq \ell/2$ and $|T \cap Y| \leq \ell - 2|T \cap X|$. This proves (4).

For the remainder of the proof, note that if Claims 13–15 are applied to G' , then the conclusions hold for G as well, since they are unaffected by the addition of vertices of degree 1 in Y .

Let $C = x_1y_1 \dots x_\ell y_\ell x_1$, and suppose $t \geq \ell - 1$. By (4), $|T \cap X| \leq 1$. If $T \cap X = \emptyset$, we may assume that $xy_i \in E(G)$ for all $1 \leq i \leq \ell - 1$. By (1), $|\widehat{N}(X)| \geq \ell + 1$, so there is $y \in Y - V(C)$ with at least two neighbors in X . This will contradict one of Claims 13–15 (possibly, in reversed orientation of C), unless all such y are adjacent to

only $x_\ell = x^-(y_\ell)$ and $x_1 = x^+(y_\ell)$. Fix such a vertex y . Let $A = X - \{x_\ell\}$. There exists a vertex $y' \in (Y - V(C)) \cup \{y_\ell\}$ such that $y' \in \widehat{N}(A)$, i.e., y' has two neighbors other than x_ℓ (so $y' \neq y$). Let C' be the cycle obtained by replacing y_ℓ with y . Then the vertex y' violates one of Claims 13–15 with respect to C' .

If $|T \cap X| = 1$, then by (4), we may assume that $xy_i \in E(G)$ for all $1 \leq i \leq \ell - 2$ and that x has a common neighbor $y \in Y - V(C)$ with x_ℓ . By (1), $|\widehat{N}(X - x_\ell)| \geq \ell$, so there is $y_0 \in (Y - V(C)) \cup \{y_{\ell-1}, y_\ell\}$ with at least two neighbors in $X - x_\ell$. If $y_0 \in (Y - V(C))$, this again will contradict one of Claims 13–15, unless $N(y_0) = \{x_{\ell-1}, x_1\}$. In this case, we obtain the longer cycle $y_1 C[y_1, y_{\ell-2}] y_{\ell-2} x y x_\ell y_{\ell-1} x_{\ell-1} y_0 x_1$. So suppose without loss of generality $y_0 = y_\ell$ has a neighbor $z \in X - \{x_\ell, x_1\}$. By the case, $z \neq x$, so suppose $z = x_j$ for some $2 \leq j \leq \ell - 1$. Then G has cycle $y_\ell C[y_\ell, y_{j-1}] y_{j-1} x y x_\ell C^-[x_\ell, x_j] x_j y_\ell$ based on X , a contradiction. $\square$

Given a cycle C and distinct $x_1, x_2, x_3 \in X \cap V(C)$, we say that x_1 and x_2 **cross** at x_3 if the cyclic order is x_1, x_3, x_2 and $x_1 y^+(x_3), x_2 y^-(x_3) \in E(G)$ or if the cyclic order is x_1, x_2, x_3 and $x_1 y^-(x_3), x_2 y^+(x_3) \in E(G)$. In this case, we also say that x_3 is **crossed** by x_1 and x_2 .

The following is Lemma 2.8 in [9]. It holds for each bipartite graph G (no restrictions).

Lemma 28 ([9]). *Let C be a cycle of an (X, Y) -bigraph G , and let $u, v \in V(C) \cap X$. If u and v have at most r crossings, then $d_C(u) + d_C(v) \leq |V(C)|/2 + 2 + r$.*

Proof. We induct on r . Suppose $r = 0$. Consider the two paths $P_1 = C[u, v]$ and $P_2 = C^-[u, v]$. In $P_1 = v_1 \dots v_k$ ($v_1 = u, v_k = v$), each $v_i \in X$ satisfies at most one of the following: $v_{i+1}u \in E(G)$ or $v_{i-1}v \in E(G)$. So $d_{P_1}(u) + d_{P_1}(v) \leq |V(P_1) \cap X|$. Similarly, $d_{P_2}(u) + d_{P_2}(v) \leq |V(P_2) \cap X|$. Since $(X \cap V(P_1)) \cap (X \cap V(P_2)) = \{u, v\}$ and $V(P_1) \cup V(P_2) = V(C)$, we get $d_C(u) + d_C(v) \leq |V(C)|/2 + 2$.

For $r \geq 1$, delete an edge incident to u that is used in a crossing, and induct. $\square$

Lemma 29. *If $u_i \in X \cap T$, then $y^+(u_i)$ has no neighbors in $(F - V(C)) \cup X^+(T) \setminus \{x^+(u_i)\}$.*

Proof. Suppose $y^+(u_i)$ has a neighbor z in $F - V(C)$. Then the cycle

$$u_i F[u_i, z] z y^+(u_i) C[y^+(u_i), u_i] u_i$$

is based on X , a contradiction.

Suppose now that $y^+(u_i)$ has a neighbor x_1 in $X^+(T) \setminus \{x^+(u_i)\}$, where $u \in T$ satisfies $x^+(u) = x_1$. Then the cycle $x_1 y^+(u_i) C[y^+(u_i), u] u F[u, u_i] u_i C^-[u_i, x_1] x_1$ is based on X , a contradiction. $\square$

Lemma 30. *If $x_1 \in X^+(T)$, then x_1 cannot have a neighbor in $F - V(C)$.*

Proof. Suppose x_1 has a neighbor y' in $F - V(C)$. Let $u_1 \in T$ be such that $x_1 = x^+(u_1)$ and z be a neighbor of u_1 in F . Let P be a z, y' -path in F and the cycle C' be defined by $C' = x_1 C[x_1, u_1] u_1 z P y' x_1$. If $y' \neq z$, then C' is based on X and we are done. Thus $z = y'$ and hence $u_1 \in X$. Let $F' = F - u_1$. Note that F' is an $x, V(C')$ -fan such that $|V(F \cap C)| = |V(F' \cap C')|$, but $|V(F')| < |V(F)|$, contradicting the choice of C and F . $\square$

Lemma 31. *Suppose that $x_1, x_2 \in X^+(T)$. Then*

- (i) x_1 and x_2 share no neighbors in $Y - V(C)$;
- (ii) neither x_1 nor x_2 share a neighbor in $Y - V(C)$ with x .

Proof. Part (i) follows from Claim 13. From Lemma 30, if x_1 and x have a common neighbor outside of C , it is not in F . Suppose they share some neighbor $y \in Y - V(C)$. Let $x_1 = x^+(u_1)$. Then we have a longer cycle $x_1 C[x_1, u_1] u_1 F[u_1, x] x y x_1$. The same holds for x_2 and x . This proves (ii). $\square$

Lemma 32. *Suppose $u_1, u_2 \in T$. If $x_1 = x^+(u_1)$ and $x_2 = x^+(u_2)$ cross at $x_3 \in X \cap V(C)$, then*

- (i) $x_3 \notin T$;
- (ii) G has a cycle C' containing $(X \cap V(C) - \{x_3\}) \cup \{x\}$ such that $|C'| \geq |C|$;
- (iii) x_3 shares no neighbors in $Y - V(C)$ with any vertex in the set $\{x\} \cup X^+(T)$;
- (iv) x_3 has at most t neighbors on C .

Proof. Suppose that the cyclic order is x_1, x_3, x_2 and $x_1 y^+(x_3), x_2 y^-(x_3) \in E(G)$ (the other case is symmetric).

For part (i), let y be a neighbor of x_3 in F . Let z be a neighbor of u_1 in F . Let P be a z, y -path in F and the cycle C' be defined by

$$C' := x_1 y^+(x_3) C[y^+(x_3), u_1] u_1 z P y x_3 C^-[x_3, x_1] x_1.$$

Then C' is based on X . This contradiction proves (i).

The cycle

$$C_1 := x_1 y^+(x_3) C[y^+(x_3), u_2] u_2 F[u_2, u_1] u_1 C^-[u_1, x_2] x_2 y^-(x_3) C^-[y^-(x_3), x_1] x_1$$

proves (ii).

To prove (iii), assume that $y \in Y - V(C)$ is a common neighbor of x_3 and a vertex in $\{x\} \cup X^+(T)$, and consider all cases. If $yx \in E(G)$, let

$$C' = x_1 y^+(x_3) C[y^+(x_3), u_1] u_1 F[u_1, x] x y x_3 C^-[x_3, x_1] x_1.$$

If y is not in $F[x, u_1]$, then C' is a cycle based on X , a contradiction. Otherwise, let F'' be $F - F[u_1, y]$. Note F'' is an $x, V(C'')$ -fan where

$$C'' = x_1 y^+(x_3) C[y^+(x_3), u_1] u_1 F[u_1, y] y x_3 C^-[x_3, x_1] x_1,$$

and $|V(F \cap C)| = |V(F'' \cap C'')|$, but $|V(F'')| < |V(F)|$, contradicting the choice of C and F .

If $u_j \in T$, $x_j = x^+(u_j)$, $yx_j \in E(G)$, and $x_j \in C[y^+(x_3), u_1]$, then the cycle

$$C' := x_1C[x_1, x_3]x_3yx_jC[x_j, u_1]u_1F[u_1, u_j]u_jC^-[u_j, y^+(x_3)]y^+(x_3)x_1$$

is based on X . Similarly, if $x_j \in C[u_1, y^-(x_3)]$, then the cycle

$$C' := x_2C[x_2, u_j]u_jF[u_j, u_2]u_2C^-[u_2, x_3]x_3yx_jC[x_j, y^-(x_3)]y^-(x_3)x_2$$

is based on X , a contradiction. This proves (iii).

By the choice of (C, x, F) and (ii), x_3 has at most t neighbors on C_1 . The only vertices in $Y \cap V(C) - V(C_1)$ are $y^-(x_1)$ and $y^-(x_2)$. If $x_3y^-(x_1) \in E(G)$, then the cycle

$$y^-(x_1)C[y^-(x_1), y^-(x_3)]y^-(x_3)x_2C[x_2, u_1]u_1F[u_1, u_2]u_2C^-[u_2, x_3]x_3y^-(x_1)$$

is based on X . If $x_3y^-(x_2) \in E(G)$, then the cycle

$$x_1C[x_1, x_3]x_3y^-(x_2)C[y^-(x_2), u_1]u_1F[u_1, u_2]u_2C^-[u_2, y^+(x_3)]y^+(x_3)x_1$$

is based on X . This proves (iv). $\square$

Lemma 33. Suppose $u_1, u_2 \in T$, $x_1 = x^+(u_1)$, and $x_2 = x^+(u_2)$. Then no two vertices $x_3, x_4 \in V(C)$ crossed by x_1 and x_2 have a shared neighbor in $Y - V(C)$.

Proof. Suppose vertices $x_3, x_4 \in V(C) \cap X$ are crossed by x_1 and x_2 and there is some $y \in (N(x_3) \cap N(x_4)) - V(C)$. By Lemma 32, $y \notin V(F)$.

We consider two cases. If x_3 and x_4 both are on $C[x_1, x_2]$ or both are on $C[x_2, x_1]$, then we may assume that their cyclic order is x_1, x_3, x_4, x_2 . In this case, the cycle

$$x_1C[x_1, x_3]x_3yx_4C[x_4, u_2]u_2F[u_2, u_1]u_1C^-[u_1, x_2]x_2y^-(x_4)C^-[y^-(x_4), y^+(x_3)]y^+(x_3)x_1$$

is based on X .

If one of x_3 and x_4 is on $C[x_1, x_2]$ and the other is on $C[x_2, x_1]$, then we may assume that their cyclic order is x_1, x_3, x_2, x_4 . In this case, the cycle

$$x_1C[x_1, x_3]x_3yx_4C^-[x_4, x_2]x_2y^+(x_4)C[y^+(x_4), u_1]u_1F[u_1, u_2]u_2C^-[u_2, y^+(x_3)]y^+(x_3)x_1$$

is based on X . This proves the lemma. $\square$

Lemma 34. Let $A \subseteq X^+(T)$. Then $\sum_{w \in A} d_C(w) \leq |A|(\ell - 2) + 2$.

Proof. Let $x_1, x_2 \in A$ such that $x_1 = x^+(u_1)$ and $x_2 = x^+(u_2)$ for some $u_1, u_2 \in T$. We first prove that

$$\text{if } u_2 \in Y \text{ and } y^+(x_2)x_1 \in E(G), \text{ then } d_C(x_2) \leq \ell - 2. \quad (5)$$

The cycle $C' = x_1y^+(x_2)C[y^+(x_2), u_1]u_1F[u_1, u_2]u_2C^-[u_2, x_1]x_1$ contains all vertices in C except x_2 and possibly $y^+(u_1)$ (in the case that $u_1 \in X$). By Lemma 29 and Lemma 30, $N_C(x_2) = N_{C'}(x_2)$. By Lemma 27 applied to C' and x_2 , $d_C(x_2) = d_{C'}(x_2) \leq \ell - 2$.

In particular, if $d_C(x_1) = \ell$, i.e., x_1 is adjacent to every y vertex in C , then by Lemma 29, each $x_2 \in X^+(T) - \{x_1\}$ satisfies $u_2 \in Y$. Therefore by (5), $d_C(x_2) \leq \ell - 2$. It follows that $\sum_{w \in A} d_C(w) \leq |A|(\ell - 2) + 2$.

So suppose every $w \in A$ has $d_C(w) \leq \ell - 1$, and there exist two vertices $x_1, x_2 \in A$ with equality. Define u_1, u_2 as before. Then for every $x_3 \in X^+(T) - \{x_1, x_2\}$, either $u_1 \in Y$ and $x_3y^+(x_1) \notin E(G)$ by (5), or $u_1 \in X$ and $x_3y^+(x_1) \notin E(G)$ by Lemma 29. The same holds for x_3 and x_2 . Therefore $d_C(x_3) \leq \ell - 2$, and again $\sum_{w \in A} d_C(w) \leq |A|(\ell - 2) + 2$. $\square$

Lemma 35. *Suppose $t \geq 4$, $u_1, u_2 \in T$, $x_1 = x^+(u_1)$, and $x_2 = x^+(u_2)$. Then at most one vertex in C is crossed by x_1 and x_2 .*

Proof. Suppose vertices $x_3, x_4 \in V(C) \cap X$ are crossed by x_1 and x_2 .

Let $A = X^+(T) \cup \{x, x_3, x_4\}$ (possibly, $X^+(T) \cap \{x_3, x_4\} \neq \emptyset$), and $A' = A - \{x, x_3, x_4\}$. Note that $|A'| \geq t - 2$, and by Lemma 34 applied to A' , $\sum_{w \in A'} d_C(w) \leq |A'|(\ell - 2) + 2$.

Since x can have at most t neighbors on C , $|N(x) - V(C)| \geq \delta - t$. By Lemma 32(iv), $|N(x_3) - V(C)| \geq \delta - t$ and $|N(x_4) - V(C)| \geq \delta - t$. By Claims 13–15 (applied to G') and Lemmas 32(iii) and 33, no two distinct vertices in A have a common neighbor in $Y - V(C)$. Thus, remembering the ℓ vertices in $Y \cap V(C)$, we get

$$\begin{aligned} |Y| &\geq \ell + \sum_{u \in A} |N(u) - V(C)| \\ &= \ell + |N(x) - V(C)| + |N(x_3) - V(C)| + |N(x_4) - V(C)| + \sum_{u \in A'} |N(u) - V(C)| \\ &\geq \ell + 3(\delta - t) + \delta|A'| - (\ell - 2)|A'| - 2 \\ &\geq \ell + 3\delta - 3t + (\delta - \ell + 2)|A'| - 2 \\ &\geq \ell + 3\delta - 3t + (\delta - \ell + 2)(t - 2) - 2 \\ &\geq \ell + 3\delta - 3t + (\delta - \ell + 2) + (\delta - \ell + 2)(t - 3) - 2 \\ &\geq \ell + 3\delta - 3t + (\delta - \ell + 2) + 3(t - 3) - 2 \\ &= 4\delta - 3t + 2 + 3(t - 3) - 2 = 4\delta - 9, \end{aligned}$$

as claimed. $\square$

Lemma 36. *For any $x_1, x_2 \in X$, x_1 and x_2 cannot be separated by a set of two vertices.*

Proof. Recall that G is a vertex-minimum counterexample to Theorem 8, and $G' = G[X \cup \widehat{N}(X)]$ is critical and saturated.

Suppose that for some $x_1, x_2 \in X$, $u_1, u_2 \in V(G)$, x_1 and x_2 are in different components of $G - \{u_1, u_2\}$. Note that $u_1, u_2 \in V(G')$, since $V(G) - V(G')$ contains only vertices of degree 1 in Y .

If there also exists $x_3 \in X - \{x_1, x_2\}$ such that x_3 is in a different component of $G - \{u_1, u_2\}$ than both x_1 and x_2 , then G cannot contain a 6-cycle based on $\{x_1, x_2, x_3\}$, since these vertices are separated by a set of size two. Hence we may assume $G - \{u_1, u_2\}$ contains exactly two components containing vertices in X . Call these components D_1 and D_2 where $x_1 \in V(D_1)$ and $x_2 \in V(D_2)$.

Choose any two vertices $x, x' \in X - \{u_1, u_2\}$; then choose a third vertex $x'' \in X - \{u_1, u_2\}$ such that not all three of x, x', x'' are in the same component of $G - \{u_1, u_2\}$. Let C be a cycle based on $A = \{x, x', x''\}$.

Since $\{u_1, u_2\}$ separates one of the vertices of A from the others, $u_1, u_2 \in V(C)$; since $V(C) \cap X = A$ and neither u_1 nor u_2 is in A , we must have $u_1, u_2 \in Y$.

Moreover, u_1 must have an edge to either x or x' in C , and therefore in G . Since $x, x' \in X$ were arbitrary, $|N(u_1)| \geq |X| - 1$. By Lemma 20 applied to G' , $N_{G'}(u_1) = X$, and therefore $N(u_1) = X$. By symmetry, we also obtain $N(u_2) = X$.

Now suppose each component of $G - \{u_1, u_2\}$ has at least 2 vertices in X . For $i \in \{1, 2\}$, set $X_i = X \cap D_i$. By the minimality of G , there exists a cycle C_1 of G based on $X_1 \cup \{x_2\}$ and a cycle C_2 based on $X_2 \cup \{x_1\}$. Since D_1 and D_2 are separated by $\{u_1, u_2\}$, $N_{C_1}(x_2) = N_{C_2}(x_1) = \{u_1, u_2\}$. Therefore $(C_1 - \{x_2\}) \cup (C_2 - \{x_1\})$ is a cycle in G which is based on X , a contradiction.

Thus we may assume without loss of generality that $V(D_1) \cap X = \{x_1\}$. Note that this implies all neighbors of x_1 other than u_1 and u_2 have degree 1. Let G_1 be obtained from G by deleting x_1 and all of its neighbors except for u_1 .

We will show that G_1 is a counterexample that has fewer vertices than G . Set $X' = X - \{x_1\} = X \cap V(G_1)$. If there exists $A \subseteq X'$ with $|A| \geq 3$ such that $|\widehat{N}_{G_1}(A)| < |A|$, then in G , $\widehat{N}_G(A \cup \{x_1\}) = \widehat{N}_{G_1}(A) \cup \{u_2\} < |A \cup \{x_1\}|$, a contradiction.

Next, we will show that for all A with $|A| \geq 3$, $G_1[A \cup \widehat{N}_{G_1}(A)]$ is 2-connected. Recall that $G_1 - \{u_1\} = D_2$. The subgraph of D_2 obtained by removing all vertices in Y of degree 1 is still connected. Call this subgraph H . If $A = X'$, then $G_1[A \cup \widehat{N}_{G_1}(A)] = G_1[H \cup \{u_1\}]$. Since H is connected and u_1 is adjacent to all vertices in X' , $G_1[H \cup \{u_1\}]$ is 2-connected. Now suppose $A \neq X'$. Then by the choice of G as a minimum counterexample, there exists a cycle C in G with $V(C) \cap X = A \cup \{x_1\}$, where $N_C(x_1) = \{u_1, u_2\}$. In particular, $P := C - \{x_1, u_1, u_2\}$ is a path containing all vertices of A . In G_1 , $G_1[A \cup \widehat{N}_{G_1}(A)]$ can be obtained from P by adding u_1 , which is adjacent to all of $V(P) \cap X$, and possibly adding some additional vertices in Y with degree at least 2. Hence it is 2-connected.

Next, suppose that it is super-cyclic. By the minimality of G , G contains no cycle C based on X ; however, because G_1 is super-cyclic, we may find a cycle $C' = v_1 v_2 \dots v_{|X'|} v_1$ in G_1 (and therefore in G) based on $X - \{x_1\}$ such that $u_2 \notin V(C')$. If $u_1 \notin V(C')$, then we may replace in C' any segment $v_i v_{i+1} v_{i+2}$ (for $v_i \in X$) with the path $v_i u_1 x_1 u_2 v_{i+2}$ to obtain a contradiction. Otherwise, if $u_1 = v_i$ for some i , we replace the path $v_{i-1} u_1 v_{i+2}$ with $v_{i-1} u_1 x_1 u_2 v_{i+2}$.

Finally, we have $|Y \cap G_1| \leq |Y| - (\delta - 1) \leq (4\delta - 10) - (\delta - 1) \leq 4(\delta - 1) - 10$. The last inequality holds because we may assume that $|X| \geq 7$ and therefore $\delta \geq 7$, since Theorem 6 implies Theorem 8 for $|X| \leq 6$. This shows that G_1 is a counterexample for Theorem 8 (with $\delta' = \delta - 1$) which has fewer vertices than G , contradicting the choice of G . $\square$

6.2 Proof of Theorem 8

Proof of Theorem 8. As in the previous subsection, suppose for the sake of contradiction that G is a vertex-minimum, edge-maximal counterexample to Theorem 8. By the choice of G , for each $x \in X$, there exists some cycle C with $V(C) \cap X = X - \{x\}$. We may also assume that $|X| \geq 7$ and therefore $\delta \geq 7$, since Theorem 6 implies Theorem 8 for $|X| \leq 6$.

Letting $G' = G[X \cup \widehat{N}(X)]$, it follows from our choice of G that G' is critical and saturated.

If there exists a pair (x, C) with an $x, V(C)$ -fan F of size at least 4, then choose such a triple which maximizes $t = |V(F) \cap V(C)|$, and subject to this, minimizes $|V(F)|$. Let $T = V(F) \cap V(C)$. By Lemmas 31 and 35, no two vertices in $X^+(T) \cup \{x\}$ share a neighbor outside of $V(C)$, and no two vertices in $X^+(T)$ cross more than one time. By Lemma 28, for each pair $x_1, x_2 \in X^+(T)$, $d_C(x_1) + d_C(x_2) \leq |V(C) \cap Y| + 3 = |X| + 2$. Therefore

$$\begin{aligned} |Y| &\geq |V(C) \cap Y| + \sum_{w \in X^+(T) \cup \{x\}} d_{Y-V(C)}(w) \\ &\geq |X| - 1 + \delta(t + 1) - \sum_{w \in X^+(T) \cup \{x\}} d_C(w) \\ &\geq |X| - 1 + \delta(t + 1) - t - t(|X| + 2)/2. \end{aligned}$$

Since the coefficient at t is at least $\delta - 1 - (\delta + 2)/2 > 0$ (assuming, as we do, that $\delta > 4$), this quantity is minimized whenever t is minimized, i.e., $t = 4$. We obtain $|Y| \geq |X| - 1 + 5\delta - 4 - 2(|X| + 2)$, which is minimized when $|X| = \delta$. So $|Y| \geq 4\delta - 9$, a contradiction.

Now suppose that for all $x \in X$ and cycles C with $V(C) \cap X = X - \{x\}$, the largest $x, V(C)$ -fan has size at most 3. Choose $x \in X$ with the maximum number of neighbors of degree at least 2. If every $x \in X$ has at most 3 neighbors of degree at least 2 (and at least $\delta - 3$ neighbors of degree 1), then we have $|Y| \geq |X|(\delta - 3) + 3$; since $|X| \geq 4$, $|Y| \geq 4\delta - 9$, a contradiction.

Therefore x has at least 4 neighbors of degree at least 2. Let F be a maximum $x, V(C)$ -fan of G and set $T = F \cap V(C)$. By Lemma 36, $|T| \geq 3$, since x cannot be separated from $X - x'$ by a set of size 2. So $|T| = 3$.

By Lemma 16, $|T \cap Y| \geq 2$ (we apply this lemma to G' , but the conclusion carries over to G). If $|T \cap Y| = 3$, then since x has at least 4 neighbors of degree at least 2, there exists $y \in N(x) - V(C)$. Since all vertices in $X - \{x\}$ are contained in C , y has a neighbor $x' \in C$. But then $F \cup xyx'$ is an $x, V(C)$ -fan of size 4, a contradiction.

Finally, we may assume $T \cap V(C) = \{x_1, y_1, y_2\}$, where x and x_1 have at least 2 common neighbors outside C . In particular, $\{y_1, y_2\} \subset N(x)$, and for any $x' \neq x, x_1$, we have $N(x) \cap N(x') \subseteq \{y_1, y_2\}$, otherwise we could find a larger $x, V(C)$ -fan. We will show that $N(y_1) = N(y_2) = X$. If there exists $x', x'' \in X - \{x_1\}$ such that $x'y_i, x''y_i \notin E(G)$ for some $i \in \{1, 2\}$, then there cannot exist a 6-cycle based on $\{x, x', x''\}$, a contradiction.

Hence $|N(y)| \geq |X| - 2$ which implies $N(y) = X$ by Lemma 22 (again, we apply this lemma to G' , but the conclusion carries over to G).

Consider $y \in \widehat{N}(x) - \{y_1, y_2\}$. Since there is no x , $V(C_x)$ -fan of size 4, $N(y) \subseteq T \cup \{x\}$. That is, $N(y) = \{x, x_1\}$ and so $\widehat{N}(x) \subseteq N(x_1)$. Recall that we chose x to have a maximum number of neighbors of degree at least 2. Additionally, note that $V(C) \cap Y \subseteq \widehat{N}(X)$. Thus $N_C(x_1) = \{y_1, y_2\}$, since otherwise $|\widehat{N}(x_1)| > |\widehat{N}(x)|$. But then $\{y_1, y_2\}$ separates $\{x, x_1\}$ from the rest of the vertices in X , contradicting Lemma 36. $\square$

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