



A geometric realisation of tempered representations restricted to maximal compact subgroups

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Abstract

Let G be a connected, linear, real reductive Lie group with compact centre. Let $K < G$ be maximal compact. For a tempered representation π of G , we realise the restriction $\pi|_K$ as the K -equivariant index of a Dirac operator on a homogeneous space of the form G/H , for a Cartan subgroup $H < G$. (The result in fact applies to every standard representation.) Such a space can be identified with a coadjoint orbit of G , so that we obtain an explicit version of Kirillov's orbit method for $\pi|_K$. In a companion paper, we use this realisation of $\pi|_K$ to give a geometric expression for the multiplicities of the K -types of π , in the spirit of the quantisation commutes with reduction principle. This generalises work by Paradan for the discrete series to arbitrary tempered representations.

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1 Introduction

Let G be a connected, linear, real reductive Lie group with compact centre, and let \mathfrak{g} be its Lie algebra. Let $K < G$ be a maximal compact subgroup. Harish-Chandra showed that a unitary irreducible representation π of G is determined by the corresponding actions by K and \mathfrak{g} on the K -finite vectors in the representation space of π . This means that, in a sense, half the information about π is contained in the restriction $\pi|_K$. The explicit form of this information consists of the multiplicities of the irreducible representations of K in $\pi|_K$; i.e. the multiplicities of the K -types of π .

In this paper, we consider tempered representations π , and realise $\pi|_K$ as the equivariant index of a Dirac operator on a homogeneous space of G . This is motivated by the desire to realise $\pi|_K$ using an index, or a notion of geometric quantisation, that satisfies Guillemin and Sternberg’s *quantisation commutes with reduction* principle. That principle relates multiplicities of representations to the geometry of *moment maps*: in this case, maps with values in \mathfrak{k}^* that represent conserved quantities in classical mechanics. If a representation is realised in this way, properties of the relevant moment map have consequences to the general behaviour of multiplicities; for example yielding criteria for them to equal 0 or 1. On a deeper level, the quantisation commutes with

reduction principle is a beautiful relation between geometry and representation theory, inspired by physics. It turns out that the relevant version of this principle here is the Spin^c -version, which applies more generally than the symplectic version.

In this paper we focus on the construction of representations using index theory that is natural in the context of geometric quantisation and the quantisation commutes with reduction principle. We then apply that principle to obtain a geometric expression for multiplicities of K -types in [18].

Paradan [37] did all of this for the discrete series. This paper is inspired by his work, and extends it to general tempered representations.

1.1 Background and motivation

Kirillov's orbit method is the idea that there should be a correspondence between (some) unitary irreducible representations π of a Lie group G , and (some) orbits \mathcal{O} of the coadjoint action by G on the dual \mathfrak{g}^* of its Lie algebra. See [45] for an account of the orbit method for reductive groups. On an intuitive level, the correspondence between orbits \mathcal{O} and representations π is that $\pi = Q(\mathcal{O})$, the *geometric quantisation* of \mathcal{O} . It was Bott's idea to define geometric quantisation as the equivariant index of a Dirac operator. This idea was shown to be very successful in the literature on the quantisation commutes with reduction problem. But defining such an index rigorously is a challenge if G and \mathcal{O} are noncompact.

A rigorous construction is essential for many applications, however. The scope for applications of any form of the orbit method depends on the properties of the explicit construction $\pi = Q(\mathcal{O})$ that is used. It is very useful if $Q(\mathcal{O})$ is the index of an elliptic differential operator, because this opens up the possibility to apply powerful techniques from index theory to study π . For example, if the index used satisfies a fixed point formula, then this can be used to compute the global character of π . See [19] for an application of this to discrete series representations. If the index satisfies the quantisation commutes with reduction principle, then this can be used to express the decomposition into irreducibles of the restriction of π to closed subgroups of G in terms of the geometry of \mathcal{O} .

Atiyah–Schmid [3], Parthasarathy [41] and Schmid [42] realised discrete series representations as L^2 -kernels of Dirac operators on homogeneous spaces of G . Schmid's result fits directly into the orbit method framework. However, since kernels of Dirac operators are used, rather than indices, one cannot apply index theory results, such as fixed point formulas and the quantisation commutes with reduction principle, to these realisations to obtain information about the discrete series. It is a subtle but important point here that although the realisations in [3, 41, 42] can be written as kernels minus cokernels (with one of these being zero, so the formal difference is well-defined), it is not obvious if the operators involved are Fredholm in the appropriate sense. That precludes the application of index-theoretic methods to analyse these representations.

Paradan [37] realised restrictions of discrete series representations π to K as K -equivariant indices of Dirac operators on coadjoint orbits \mathcal{O} . While his realisation only applies to the restriction of π to K , his approach has the important advantage that the index he used satisfies the quantisation commutes with reduction principle. This

principle, proved by Paradan in this setting, implies a geometric expression for the multiplicities of the K -types of π , as indices of Dirac operators on compact orbifolds. These orbifolds are *reduced spaces* for the action by K on \mathcal{O} . If $p: \mathfrak{g}^* \rightarrow \mathfrak{k}^*$ is the restriction map, then the reduced space at $\xi \in \mathfrak{k}^*$ is

$$\mathcal{O}_\xi = (p^{-1}(\text{Ad}^*(K)\xi) \cap \mathcal{O})/K.$$

For discrete series representations π , Blattner's formula, proved by Hecht–Schmid [9], is an explicit combinatorial expression for multiplicities of K -types. This was in fact used by Paradan to obtain his realisation of $\pi|_K$, and the resulting geometric multiplicity formula. But this geometric multiplicity formula has the advantage that it allows one to draw conclusions about the K -types of π from the geometry of the corresponding coadjoint orbit. For example, about the question when their multiplicities equal 0 or 1.

In this paper, we generalise Paradan's construction to arbitrary tempered representations. Tempered representations are those unitary irreducible representations whose K -finite matrix coefficients are in $L^{2+\varepsilon}(G)$, for all $\varepsilon > 0$. The set \hat{G}_{temp} of these representations occurs in the Plancherel decomposition

$$L^2(G) = \int_{\hat{G}_{\text{temp}}}^{\oplus} \pi \otimes \pi^* d\mu(\pi)$$

of $L^2(G)$ as a representation of $G \times G$. Here μ is the Plancherel measure. For this reason, tempered representations are central to harmonic analysis. Furthermore, they are used in the Langlands classification of admissible irreducible representations.

For general tempered representations, a multiplicity formula for K -types is especially valuable, because no explicit multiplicity formula exists yet in this generality. There are algorithms to compute these multiplicities, see the ATLAS software package¹ developed by du Cloux, van Leeuwen, Vogan and many others; see also [1]. But it is a challenge to draw conclusions about the general behaviour of multiplicities of K -types from these algorithms. The main difficulty is that they involve representations of disconnected subgroups, which cannot be classified via Lie algebra techniques. Also, already for the discrete series, cancellation of terms in Blattner's formula can make it nontrivial to evaluate it, and for example to see when multiplicities equal zero.

The geometric multiplicity formula we deduce from the main result of this paper in [18] allows us to use the geometry of coadjoint orbits to study multiplicities of K -types. In that paper, we find relations to multiplicity-free restrictions, and for example show that admissible representations of $\text{SU}(p, 1)$, $\text{SO}_0(p, 1)$ and $\text{SO}_0(2, 2)$ have multiplicity-free restrictions to maximal compact subgroups. This can also be deduced from Harish-Chandra's subrepresentation theorem and was proved in work by Koornwinder [26], but it illustrates our result and the kind of geometry involved. The method also imposes geometrically defined bounds on the region where nonzero multiplicities occur, besides giving a precise geometric expression for those multiplicities.

¹ See <http://www.liegroups.org/software/>.

1.2 The main result

Let π be a tempered representation of G . The main result in this paper is Theorem 3.11, a realisation of $\pi|_K$ as the K -equivariant index of a Dirac operator on G/H , for a Cartan subgroup $H < G$. Since G/H is noncompact in general, the kernel of such an operator is infinite-dimensional. This is desirable, since π is infinite-dimensional, but it makes the definition of the index more involved than on a compact manifold. We use index theory developed by Maxim Braverman [5].

Our construction involves a map

$$\Phi: G/H \rightarrow \mathfrak{k}^*$$

of the form $\Phi(gH) = (\text{Ad}^*(g)\xi)|_K$, for a fixed $\xi \in \mathfrak{g}^*$, and where $g \in G$ (see Sect. 3.5). This is a moment map in the sense of symplectic geometry, although our construction will take us into the more general almost complex or Spin^c -setting. After we identify $\mathfrak{k}^* \cong \mathfrak{k}$ via an $\text{Ad}(K)$ -invariant inner product, the map Φ induces a vector field v^Φ via the infinitesimal action by \mathfrak{k} on G/H .

We define a K -invariant almost complex structure J on G/H (see Sect. 3.4). This induces a Clifford action c by $T(G/H)$ on $\bigwedge_J T(G/H)$ (see Example 3.1). Here \bigwedge_J stands for the exterior algebra of complex vector spaces. Let D be a Dirac operator on $\bigwedge_J T(G/H)$ (see Sect. 3.1). We write $\bigwedge_J^\pm T(G/H)$ for the even and odd degree parts of this bundle, respectively. As a special case of Theorem 2.9 in [5], we find that the multiplicities m_δ^\pm of irreducible representations δ of K in

$$\ker(D - ifc(v^\Phi)) \cap L^2(\bigwedge_J^\pm T(G/H) \otimes L_\pi)$$

are finite, for a function f with suitable growth behaviour (see Sect. 3.1). Here $L_\pi \rightarrow G/H$ is a certain line bundle associated to π (see Sect. 3.5). This deformation of the Dirac operator goes back to Tian–Zhang [43], who used it to give a proof of Guillemin–Sternberg’s quantisation commutes with reduction conjecture. The difference $m_\delta^+ - m_\delta^-$ is independent of f and of the specific Dirac operator D used. This allows us to consider the K -equivariant index

$$\text{index}_K(\bigwedge_J T(G/H) \otimes L_\pi, \Phi) := \bigoplus_{\delta \in \hat{K}} (m_\delta^+ - m_\delta^-) \delta.$$

This index defines an element of the completion $\text{Hom}_{\mathbb{Z}}(R(K), \mathbb{Z})$ of the representation ring $R(K)$ of K . It equals indices defined and used by Paradan–Vergne [37, 38, 44] and Ma–Zhang [30].

The main result in this paper, Theorem 3.11, states that this index equals $\pi|_K$, up to a sign.

Theorem 1.1 *We have*

$$\pi|_K = \pm \text{index}_K(\bigwedge_J T(G/H) \otimes L_\pi, \Phi).$$

See Sect. 3.5 for the precise definition of the sign \pm . In Sect. 3.7, we illustrate this result by working out what it means for $G = \text{SL}(2, \mathbb{R})$.

1.3 Relation with geometric quantisation of coadjoint orbits

If the infinitesimal character χ of π is a regular element of $i\mathfrak{h}^*$, then Theorem 1.1 is a direct realisation of $\pi|_K$ as the K -equivariant geometric quantisation of a coadjoint orbit of G , as in the orbit method. Indeed, we may then take the map Φ to be the composition

$$\Phi: G/H \xrightarrow{\cong} \mathrm{Ad}^*(G)\chi \hookrightarrow \mathfrak{g}^* \rightarrow \mathfrak{k}^*. \quad (1.1)$$

Then Φ is the natural moment map in the symplectic sense for the action by K on the coadjoint orbit $\mathrm{Ad}^*(G)\chi$. The line bundle L_π is now such that $\bigwedge_j T(G/H) \otimes L_\pi$ is the spinor bundle of a Spin^c -structure with determinant line bundle

$$G \times_H \mathbb{C}_{2\chi} \rightarrow G/H = \mathrm{Ad}^*(G)\chi,$$

twisted by a one-dimensional representation of a finite Cartesian factor of H . In fact, Φ is a moment map in the Spin^c -sense [39] for this Spin^c -structure. Therefore, Theorem 1.1 now states that $\pi|_K$ is the K -equivariant Spin^c -quantisation [15, 37, 39] of $\mathrm{Ad}^*(G)\chi$.

Index theory of Dirac operators deformed by the vector field v^Φ , for a moment map Φ , appears frequently and naturally in geometric quantisation. (There are at least two other, but equivalent, definitions to the one we use here, used in [36, 38, 44] and [30], respectively.) In the compact case, such deformations were used to prove quantisation commutes with reduction results [36, 39, 43]. In the noncompact case, they are also used to define geometric quantisation [11, 12, 15, 30, 38, 44]. So the use of deformed Dirac operators in Theorem 1.1 is natural from the point of view of geometric quantisation. A concrete consequence of this is that it allows us to apply the quantisation commutes with reduction principle to compute the multiplicities of the K -types of π , as we do in [18].

Paradan [37] has pointed out that Spin^c -quantisation is the relevant notion of geometric quantisation to use in a construction of tempered representations (discrete series representations in his case) satisfying the quantisation commutes with reduction principle. This means that one should view $\mathrm{Ad}^*(G)\chi$ as a Spin^c -manifold rather than as a symplectic manifold. More specifically, the Spin^c -version of the quantisation commutes with reduction principle applies here, which we use in [18] to deduce a geometric expression for the multiplicities of the K -types of π from Theorem 1.1. Paradan and Vergne [39] showed that principle has a natural generalisation from the symplectic setting to the Spin^c -setting. The version we use in [18] is the result for noncompact Spin^c -manifolds proved in [15].

If χ is singular, then in the orbit method, π is associated to a nilpotent orbit (which need not be the orbit through χ). In this case, the first map in (1.1) is a fibre bundle. By using G/H rather than this nilpotent orbit, we do not directly deal with the problem of quantising nilpotent orbits, but we are able to use Theorem 1.1 to obtain a multiplicity formula for K -types for all tempered representations in [18].

1.4 Ingredients of the proof

There are several challenges in generalising Theorem 1.1 from discrete series representations to arbitrary tempered representations.

1. The space G/H does not have a naturally defined G -invariant almost complex structure.
2. We do not have an explicit result like Blattner's formula to base the construction on.
3. If $T < K$ is a maximal torus and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ is a Cartan decomposition, then in the discrete series case, we have a K -equivariant diffeomorphism $G/T = K \times_T \mathfrak{s}$. Such a "partial linearisation" is more complicated in the general case.

The last of these points takes most work to solve.

To deduce a multiplicity formula for K -types from Theorem 1.1, there are two main challenges.

1. Paradan showed in [37] that one needs a version of the quantisation commutes with reduction principle for noncompact Spin^c -manifolds. He proved such a result in the setting relevant to discrete series representations, but one needs a more general version for arbitrary tempered representations.
2. It is unclear what coadjoint orbits, or what maps Φ , one should use in general, for example for limits of the discrete series.

The first of these points was solved in [15]. There a general version of the quantisation commutes with reduction principle was proved for noncompact Spin^c -manifolds. This was based on Paradan–Vergne's result for compact Spin^c -manifolds in [39,40]. The result in [15] is an analogue of the result by Ma–Zhang [30] for noncompact symplectic manifolds in the more general Spin^c -setting. The second point will be solved in [18].

The proof of Theorem 1.1 in this paper consists of 3 steps.

1. Prove that the right hand side of the equality in Theorem 1.1 equals an index on a "partially linearised" space E that is K -equivariantly diffeomorphic to G/H . This is done in Sect. 4; see Proposition 4.1.
2. Compute this index on E explicitly. This is done in Sects. 5 and 6; see Proposition 6.1.
3. Use that explicit expression to prove that the index on E equals $\pi|_K$. This is done in Sect. 7; see Proposition 7.6.

The second step is the most elaborate. One reason for this is that the arguments we use involve deformations of Dirac operators that do not fit into the index theory developed by Braverman. That means that we have to use homotopy arguments specifically tailored to our situation, rather than the general cobordism invariance property of Braverman's index.

The representation theoretic input to our proof of Theorem 1.1 is:

- the part of Knapp and Zuckerman's classification of tempered representation that states that every tempered representation is basic (Corollary 8.8 in [23]);
- Blattner's formula for multiplicities of K -types of (limits of) discrete series representations [9].

In fact, to be precise, Theorem 1.1 applies to all basic (or standard) representations π (see Remark 3.12). The first of the above two ingredients is not necessary for the proof of the result in that formulation.

Notation

The Lie algebra of a Lie group is denoted by the corresponding lower case Gothic letter. We denote complexifications by superscripts \mathbb{C} . The dual of a finite-dimensional vector space V will be denoted by V^* . (Here V will usually be a Lie algebra, and the notation stands for its dual as a real vector space.) The unitary dual of a group H will be denoted by \hat{H} . If H is an abelian Lie group and $\xi \in \mathfrak{h}^*$ satisfies the appropriate integrality condition, then we write \mathbb{C}_ξ for the one-dimensional representation of H with weight ξ .

In Sects. 3.1 and 3.2 and in Sect. 5, the letter M will denote a manifold. In the rest of this paper, M is a subgroup of the group G . In Sect. 5, N is another manifold, whereas N denotes a subgroup of G in the rest of this paper. (There is little risk of confusion, because the group G does not play a role at all in the sections where M and N are manifolds.) The Levi–Civita connection on a Riemannian manifold M will be denoted by ∇^{TM} .

2 Tempered representations

We start by reviewing the basic properties of tempered representations that we will need, including a part of their classification by Knapp and Zuckerman [23–25].

Throughout this paper, except in Sect. 2.1, G will denote a connected, linear, real reductive Lie group with compact centre Z_G . (This is the class of groups for which tempered representations were classified in [23–25].) Let $K < G$ be a maximal compact subgroup. Let θ be the corresponding Cartan involution, with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Let $(-, -)$ be the K -invariant inner product on \mathfrak{g} defined by the Killing form and θ . We transfer this inner product to the dual spaces \mathfrak{g}^* and $i\mathfrak{g}^*$ where necessary.

A unitary irreducible representation $\pi \in \hat{G}$ is *tempered* if all of its K -finite matrix coefficients belong to $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$. If \hat{G}_{temp} is the set of tempered representations of G , then we have the Plancherel decomposition

$$L^2(G) = \int_{\hat{G}_{\text{temp}}}^{\oplus} \pi \otimes \pi^* d\mu(\pi), \quad (2.1)$$

as representations of $G \times G$, where μ is the Plancherel measure. Tempered representations are important

1. to harmonic analysis, because of (2.1);
2. because they are used in the Langlands classification of all admissible representations [28]; see also e.g. Section VIII.15 in [21].

2.1 Limits of discrete series

In this subsection and the next, we make different assumptions on G than in the rest of this paper. (The contents of these subsections will later be applied to the subgroup $M < G$ in the Langlands decomposition $P = MAN$ of a cuspidal parabolic subgroup $P < G$.) Suppose G is a real linear Lie group, not necessarily connected. Let $G_0 < G$ be the connected component of the identity element. We now assume that

1. \mathfrak{g} is reductive;
2. G_0 has compact centre;
3. G has finitely many connected components;
4. if $G^{\mathbb{C}}$ is the analytic linear Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$, and if $Z(G)$ is the centraliser of G in the full general linear matrix group containing G , then $G \subset G^{\mathbb{C}}Z(G)$.

These assumptions imply those used by Harish-Chandra in [8], see Section 1 of [20].

In addition to the above assumptions, we suppose that G has a compact Cartan subgroup $T < K$. Then it has discrete series and limits of discrete series representations. We recall the classification of those representations, taking into account the fact that G may be disconnected. We refer to Section 1 of [23] for details. See also Sections IX.7 and XII.7 in [21] for the connected case.

Let $R_G = R(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ be the root system of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. For a regular element $\lambda \in i\mathfrak{t}^*$, let ρ_λ be half the sum of the elements of R_G with positive inner products with λ . Then the discrete series of G_0 is parametrised by the set of regular elements of $\lambda \in i\mathfrak{t}^*$ for which $\lambda - \rho_\lambda$ is integral (i.e. lifts to a homomorphism $e^{\lambda - \rho_\lambda} : T \rightarrow U(1)$). For such an element λ , let $\pi_\lambda^{G_0}$ be the corresponding discrete series representation. For another such element λ' , we have $\pi_\lambda^{G_0} \cong \pi_{\lambda'}^{G_0}$ if and only if there is an element w of the Weyl group $N_{G_0}(T)/Z_{G_0}(T)$ such that $\lambda' = w\lambda$.

Let $\lambda \in i\mathfrak{t}^*$ be as above. Let $\chi \in \hat{Z}_G$ be such that

$$\chi|_{T \cap Z_G} = e^{\lambda - \rho_\lambda}|_{T \cap Z_G}. \quad (2.2)$$

Then we have the well-defined representation $\pi_\lambda^{G_0} \boxtimes \chi$ of $G_0 Z_G$, given by

$$(\pi_\lambda^{G_0} \boxtimes \chi)(gz) = \pi_\lambda^{G_0}(g)\chi(z),$$

for $g \in G_0$ and $z \in Z_G$. Write

$$\pi_{\lambda, \chi}^G := \text{Ind}_{G_0 Z_G}^G (\pi_\lambda^{G_0} \boxtimes \chi)$$

(Here and in the rest of this paper, Ind denotes normalised induction.) This is a discrete series representation of G , and all discrete series representations of G are of this form. Two such representations $\pi_{\lambda, \chi}^G$ and $\pi_{\lambda', \chi'}^G$ are equivalent if and only if $\chi' = \chi$ and there is an element $w \in N_G(T)/Z_G(T)$ such that $\lambda' = w\lambda$.

Now let $\lambda \in i\mathfrak{t}^*$ be possibly singular, and choose a (non-unique) positive root system $R_G^+ \subset R_G$ of roots having nonnegative inner products with λ . Let ρ be half the sum of the elements of R_G^+ , and suppose that $\lambda - \rho$ is integral.

Let $\nu \in i\mathfrak{t}^*$ be regular and dominant with respect to R_G^+ . Let $F_{-\nu}$ be the finite-dimensional irreducible representation of $\mathfrak{g}^{\mathbb{C}}$ with lowest weight $-\nu$. Set

$$\pi_{\lambda, R_G^+}^{G_0} := p_{\lambda}(\pi_{\lambda+\nu}^{G_0} \otimes F_{-\nu}), \quad (2.3)$$

where p_{λ} denotes projection onto the subspace with infinitesimal character λ . If λ is regular, then $\pi_{\lambda, R_G^+}^{G_0} = \pi_{\lambda}^{G_0}$. And even if λ is singular, the right hand side of (2.3) is independent of ν . For singular λ , the representation $\pi_{\lambda, R_G^+}^{G_0}$ is a limit of discrete series representation of G_0 .

Let λ and R_G^+ be as above. Let χ be a unitary irreducible representation of Z_G , such that (2.2) holds. Write

$$\pi_{\lambda, R_G^+, \chi}^G := \text{Ind}_{G_0 Z_G}^G (\pi_{\lambda, R_G^+}^{G_0} \boxtimes \chi). \quad (2.4)$$

The following result is Theorem 1.1 in [23].

Theorem 2.1 *For λ , R_G^+ and χ as above, the representation $\pi_{\lambda, R_G^+, \chi}^G$ is*

- *nonzero if and only if $(\lambda, \alpha) \neq 0$ for all simple (with respect to R_G^+) compact roots α ;*
- *irreducible and tempered in that case.*

If two such representations $\pi_{\lambda, R_G^+, \chi}^G$ and $\pi_{\lambda', (R_G^+)', \chi'}^G$ are nonzero, they are equivalent if and only if $\chi' = \chi$, and there is an element $w \in N_G(T)/Z_G(T)$ such that $\lambda' = w\lambda$ and $(R_G^+)' = wR_G^+$.

The limits of discrete series representations are the nonzero representations occurring in the above theorem.

2.2 The Knapp–Zuckerman classification

We now return to the setting described at the start of this section. In particular, G is connected, linear, real reductive, with compact centre. We state the part that we need of Knapp and Zuckerman's classification of tempered representations of G , in terms of limits of discrete series representations of subgroups of G . At the same time, we fix notation that will be used in the rest of this paper.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a θ -stable Cartan subalgebra. Set

- $H := Z_G(\mathfrak{h})$;
- $\mathfrak{a} := \mathfrak{h} \cap \mathfrak{s}$;
- $A :=$ the analytic subgroup of G with Lie algebra \mathfrak{a} ;
- $\mathfrak{m} :=$ the orthogonal complement to \mathfrak{a} in $Z_{\mathfrak{g}}(\mathfrak{a})$;
- $M_0 :=$ the analytic subgroup of G with Lie algebra \mathfrak{m} ;
- $M := Z_K(\mathfrak{a})M_0$.

The subgroup M may be disconnected. But importantly, it satisfies the assumptions made on the group G in Sect. 2.1.

For $\beta \in \mathfrak{a}^*$, set

$$\mathfrak{g}_\beta := \{X \in \mathfrak{g}; \text{ for all } Y \in \mathfrak{a}, [Y, X] = \langle \beta, Y \rangle X\}.$$

Consider the restricted root system

$$\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a}) := \{\beta \in \mathfrak{a}^* \setminus \{0\}; \mathfrak{g}_\beta \neq \{0\}\}.$$

Fix a positive system $\Sigma^+ \subset \Sigma$. Consider the nilpotent subalgebras

$$\mathfrak{n}^\pm := \bigoplus_{\beta \in \Sigma^+} \mathfrak{g}_{\pm\beta}.$$

of \mathfrak{g} . We will write $\mathfrak{n} := \mathfrak{n}^+$. Let N be the analytic subgroup of G with Lie algebra \mathfrak{n} . Then $P := MAN$ is a parabolic subgroup of G .

Let $T < K$ be a maximal torus. Set

- $K_M := K \cap M$;
- $\mathfrak{k}_M := \mathfrak{k} \cap \mathfrak{m}$;
- $\mathfrak{t}_M := \mathfrak{k}_M \cap \mathfrak{t}$;
- $T_M := \exp(\mathfrak{t}_M)$.

Then $\mathfrak{t}_M \subset \mathfrak{m}$ is a Cartan subalgebra, so M has discrete series and limits of discrete series representations. That is to say, P is a cuspidal parabolic subgroup. In fact, all cuspidal parabolic subgroups occur in this way.

If we write $\mathfrak{s}_M := \mathfrak{m} \cap \mathfrak{s}$, then we obtain the Cartan decomposition $\mathfrak{m} = \mathfrak{k}_M \oplus \mathfrak{s}_M$. Set $H_M := H \cap M$. We have $T_M < H_M$. The converse inclusion does not hold in general, since T_M is connected, whereas H_M may be disconnected. More explicitly, Corollary 7.111 in [22] implies that

$$H_M = T_M Z_M. \quad (2.5)$$

In fact, $H_M = T_M Z'_M$ for a finite subgroup $Z'_M < Z_M$. So the Lie algebra of H_M is \mathfrak{t}_M .

Let $\lambda \in i\mathfrak{t}_M^*$, $R_M^+ \subset R(\mathfrak{m}^\mathbb{C}, \mathfrak{t}_M^\mathbb{C})$, and $\chi_M \in \hat{Z}_M$ be as in Sect. 2.1, with G replaced by M and T by H_M . Then we have the limit of discrete series representation $\pi_{\lambda, R_M^+, \chi_M}^M$ of M . Let $v \in i\mathfrak{a}^*$. A *basic representation* of G is a representation of the form

$$\text{Ind}_P^G(\pi_{\lambda, R_M^+, \chi_M}^M \otimes e^v \otimes 1_N),$$

where 1_N is the trivial representation of N .

Theorem 2.2 (Knapp–Zuckerman) *Every tempered representation of G is basic.*

This is Corollary 8.8 in [23]. In Theorem 14.2 in [24], Knapp and Zuckerman complete the classification of tempered representations by showing which basic representations are irreducible and tempered. (These are the ones with nondegenerate data and trivial R -groups; see Sections 8 and 12 in [24] for details on these conditions).

The main result of this paper, Theorem 3.11, is formulated for tempered representations, but in fact applies more generally to all basic representations (see Remark 3.12). The result is formulated for tempered representations, because of the special relevance of those representations.

3 Indices of deformed Dirac operators

The main result in this paper is Theorem 3.11, which states that the restriction to K of a tempered representation of G can be realised as the equivariant index of a deformed Dirac operator on G/H , for a Cartan subgroup $H < G$. In this section, we review the index theory we will use, and state the main result.

3.1 Deformed Dirac operators

Braverman [5] developed equivariant index theory for the deformations of Dirac operators on noncompact manifolds that we briefly discuss in this subsection. His index is the same as the indices defined by Paradan and Vergne [38,44] (see Theorem 5.5 in [5]) and Ma–Zhang [29] (see Theorem 1.5 in [29]). Its main applications have so far been to geometric quantisation and representation theory, see e.g. [37]. The deformation of Dirac operators (3.1) used by Braverman was introduced by Tian and Zhang [43] in their analytic proof of Guillemin and Sternberg’s *quantisation commutes with reduction* problem (which was first proved by Meinrenken [32] and Meinrenken–Sjamaar [33]; another proof was given by Paradan [36]).

In this subsection, we consider a complete Riemannian manifold M , on which a compact Lie group K acts isometrically. (In this subsection and the next, M does not denote a subgroup of G ; in fact G does not play a role at all here.) Let $S \rightarrow M$ be a \mathbb{Z}_2 -graded, Hermitian, K -equivariant complex vector bundle. Let $c: TM \rightarrow \text{End}(S)$ be a K -equivariant vector bundle homomorphism, called the *Clifford action*, such that for all $v \in TM$,

$$c(v)^2 = -\|v\|^2.$$

(Here K acts on $\text{End}(S)$ by conjugation.) Then S is called a K -equivariant *Clifford module*.

Example 3.1 In the setting we consider in the rest of this paper, M will have a K -equivariant almost complex structure J , and we will use $S = \bigwedge_J TM \otimes L$, where $L \rightarrow M$ is a line bundle and $\bigwedge_J TM$ is the complex exterior algebra bundle of TM with respect to J . This has a natural Clifford action, given by

$$c(v)x = v \wedge x - v^* \lrcorner x,$$

where $v \in T_m M$ for some $m \in M$, $x \in \bigwedge_J T_m M \otimes L_m$, $v^* \in T_m^* M$ is dual to v with respect to the Hermitian metric defined by J and the Riemannian metric, and \lrcorner denotes contraction (see e.g. page 395 in [4]). When dealing with vector bundles of the form $\bigwedge_J T M \otimes L$, we will always use this Clifford action.

Let ∇ be a K -invariant, Hermitian connection on \mathcal{S} such that for all vector fields v and w on M ,

$$[\nabla_v, c(w)] = c(\nabla_v^{TM} w),$$

where ∇^{TM} is the Levi-Civita connection on TM . If we identify $T^*M \cong TM$ via the Riemannian metric, we can view c as a vector bundle homomorphism

$$c: T^*M \otimes \mathcal{S} \rightarrow \mathcal{S}.$$

The Dirac operator D associated to ∇ is the composition

$$D: \Gamma^\infty(\mathcal{S}) \xrightarrow{\nabla} \Gamma^\infty(T^*M \otimes \mathcal{S}) \xrightarrow{c} \Gamma^\infty(\mathcal{S}).$$

It is odd with respect to the grading on \mathcal{S} ; we denote its restrictions to even and odd sections by D^+ and D^- , respectively.

If M is compact, then the elliptic operator D has finite-dimensional kernel. So it has a well-defined equivariant index

$$\text{index}_K(D) := [\ker(D^+)] - [\ker(D^-)] \in R(K),$$

where $R(K)$ is the representation ring of K , and square brackets denote equivalence classes of representations of K . If M is noncompact, one can still define an equivariant index, using a *taming map*.

Let $\psi: M \rightarrow \mathfrak{k}$ be an equivariant smooth map (with respect to the adjoint action by K on \mathfrak{k}). Let v^ψ be the vector field on M defined by

$$v^\psi(m) = \left. \frac{d}{dt} \right|_{t=0} \exp(-t\psi(m)) \cdot m,$$

for $m \in M$. The map ψ is called a taming map if the set of zeroes of the vector field v^ψ is compact. The Dirac operator deformed by ψ is the operator

$$D_\psi := D - ic(v^\psi) \tag{3.1}$$

on $\Gamma^\infty(\mathcal{S})$. As for the undeformed operator, we denote the restrictions of D_ψ to even and odd sections by D_ψ^+ and D_ψ^- , respectively.

To obtain a well-defined index, one needs to rescale the map ψ by a function with suitable growth behaviour. Let $\Phi^S \in \text{End}(\mathcal{S}) \otimes \mathfrak{k}^*$ be given by

$$\langle \Phi^S, Z \rangle = \nabla_{Z^M} - \mathcal{L}_Z, \tag{3.2}$$

for $Z \in \mathfrak{k}$. Here Z^M is the vector field on M induced by Z via the infinitesimal action. Let ∇^{TM} be the Levi-Civita connection on TM . Consider the positive, K -invariant function

$$h := \|v^\psi\| + \|\nabla^{TM} v^\psi\| + \|\langle \Phi^S, \psi \rangle\| + \|\psi\| + 1 \quad (3.3)$$

on M . A nonnegative function $f \in C^\infty(M)^K$ is said to be *admissible* (for ψ and ∇) if

$$\frac{f^2 \|v^\psi\|^2}{\|df\| \|v^\psi\| + fh + 1}(m) \rightarrow \infty$$

as $m \rightarrow \infty$ in M . It is shown in Lemma 2.7 in [5] that admissible functions always exist.

Theorem 3.2 (Braverman) *Suppose ψ is taming. Then for all admissible functions $f \in C^\infty(M)^K$, and all $\delta \in \hat{K}$, the multiplicity m_δ^\pm of δ in*

$$\ker(D_{f\psi}^\pm) \cap L^2(S)$$

is finite. The difference $m_\delta^+ - m_\delta^-$ is independent of f and ∇ .

This is Theorem 2.9 in [5]. The fact that $m_\delta^+ - m_\delta^-$ is independent of f and ∇ is a consequence of Braverman's cobordism invariance result, Theorem 3.7 in [5]. This cobordism invariance property also implies that the index is independent of the K -invariant, complete Riemannian metric on M .

Let $\hat{R}(K)$ be the abelian group

$$\hat{R}(K) := \left\{ \sum_{\delta \in \hat{K}} m_\delta \delta; m_\delta \in \mathbb{Z} \right\}.$$

It contains the representation ring $R(K)$ as the subgroup for which only finitely many of the coefficients m_δ are nonzero.

Definition 3.3 In the setting of Theorem 3.2, the *equivariant index* of the pair (S, ψ) is

$$\text{index}_K(S, \psi) := \sum_{\delta \in \hat{K}} (m_\delta^+ - m_\delta^-) \delta \in \hat{R}(K).$$

This index was generalised to proper actions by noncompact groups in Theorem 3.12 in [14]. Earlier, this was done for sections invariant under the group action in [6, 11].

3.2 Properties of the index

As mentioned above, independence of the coefficients $m_\delta^+ - m_\delta^-$ of the choices of f and ∇ follows from a cobordism invariance property. We will use the special case of this result that we describe now. For $j = 1, 2$, let $\mathcal{S}_j \rightarrow M$ be a Clifford module, and let $\psi_j: M \rightarrow \mathfrak{k}$ be a taming map.

Definition 3.4 A homotopy between (\mathcal{S}_1, ψ_1) and (\mathcal{S}_2, ψ_2) is a pair (\mathcal{S}, ψ) , where

- $\mathcal{S} \rightarrow M \times [0, 1]$ is a Clifford module, such that
 - $\mathcal{S}|_{M \times [0, 1/3[} = \mathcal{S}_1 \times [0, 1/3[$, including the Clifford actions by TM ;
 - $\mathcal{S}|_{M \times]2/3, 1]} = \mathcal{S}_2 \times]2/3, 1]$, including the Clifford actions by TM .

Let ∂_t be the unit vector field of the component \mathbb{R} in $T(M \times]0, 1]) = TM \times \mathbb{R} \times]0, 1[$. Let $c: T(M \times]0, 1]) \rightarrow \text{End}(\mathcal{S}|_{M \times]0, 1[})$ be the Clifford action. Then

- $c(\partial_t)|_{M \times [0, 1/3[} = \sqrt{-1}$;
- $c(\partial_t)|_{M \times]2/3, 1]} = -\sqrt{-1}$.

- $\psi: M \times [0, 1] \rightarrow \mathfrak{k}$ is a taming map, such that for all $m \in M$ and $t \in [0, 1]$,

$$\psi(m, t) = \begin{cases} \psi_1(m) & \text{if } t < 1/3; \\ \psi_2(m) & \text{if } t > 2/3. \end{cases}$$

Theorem 3.5 (Homotopy invariance) *If (\mathcal{S}_1, ψ_1) and (\mathcal{S}_2, ψ_2) are homotopic, then*

$$\text{index}_K(\mathcal{S}_1, \psi_1) = \text{index}_K(\mathcal{S}_2, \psi_2).$$

See Theorem 3.7 in [5]. We will only apply this result in cases where either $\mathcal{S}_1 = \mathcal{S}_2$ or $\psi_1 = \psi_2$. If (\mathcal{S}_1, ψ_1) and (\mathcal{S}_2, ψ_2) are homotopic, then if $\mathcal{S}_1 = \mathcal{S}_2$, we say that ψ_1 and ψ_2 are homotopic. Similarly, if $\psi_1 = \psi_2$, then we say that \mathcal{S}_1 and \mathcal{S}_2 are homotopic.

Corollary 3.6 *Suppose that $\mathcal{S}_1 = \mathcal{S}_2$, and that*

$$(v^{\psi_1}, v^{\psi_2}) \geq 0.$$

Then ψ_1 and ψ_2 are homotopic, so that

$$\text{index}_K(\mathcal{S}_1, \psi_1) = \text{index}_K(\mathcal{S}_2, \psi_2).$$

Proof Let $\chi: \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that $\chi(t) = 0$ if $t < 1/3$ and $\chi(t) = 1$ if $t > 2/3$. For $t \in [0, 1]$ and $m \in M$, set

$$\psi(m, t) := (1 - \chi(t))\psi_1(m) + \chi(t)\psi_2(m).$$

Then

$$\begin{aligned}\|v^{\psi_t}(m)\|^2 &= (1 - \chi(t))^2 \|v^{\psi_1}(m)\|^2 + \chi(t)^2 \|v^{\psi_2}(m)\|^2 + 2(1 - \chi(t))\chi(t)(v^{\psi_1}, v^{\psi_2}) \\ &\geq (1 - \chi(t))^2 \|v^{\psi_1}(m)\|^2 + \chi(t)^2 \|v^{\psi_2}(m)\|^2.\end{aligned}$$

This can only vanish if $v^{\psi_1}(m) = 0$ or $v^{\psi_2}(m) = 0$, so that ψ is a taming map. Hence the claim follows from Theorem 3.5. \square

Another property of the index that we will use is excision.

Proposition 3.7 (Excision) *Suppose that ψ is taming, and let $U \subset M$ be a relatively compact, K -invariant open subset, with a smooth boundary, outside which v^ψ does not vanish. Consider a K -invariant Riemannian metric on TU for which U is complete, and which coincides with the Riemannian metric on TM in a neighbourhood of the zeroes of v^ψ . Also consider a compatible Clifford action by TU on $S|_U$. Then*

$$\text{index}_K(S|_U, \psi|_U) = \text{index}_K(S, \psi).$$

For a proof, see Lemma 3.12 and Corollary 4.7 in [5].

3.3 The discrete series case

We return to the setting described at the start of Sect. 2. One application of the index theory of deformed Dirac operators was Paradan's realisation in [37] of restrictions to K of discrete series representations of G . The main result in this paper, Theorem 3.11, is a generalisation of Paradan's result to arbitrary tempered representations. The main advantage of Paradan's result is that it implies a geometric formula for multiplicities of K -types of discrete series representations, Theorem 1.5 in [37]. We will use Theorem 3.11 to obtain a generalisation of this multiplicity formula to arbitrary tempered representations in a forthcoming paper [18].

Let us first state Paradan's result. Suppose G is semisimple, and has a compact Cartan subgroup $T < K$. Let $\lambda \in i\mathfrak{t}^*$ be a regular element for which $\lambda - \rho_\lambda$ is integral, and let π_λ^G be the corresponding discrete series representation. The coadjoint orbit

$$\text{Ad}^*(G)\lambda \cong G/T$$

has a G -invariant complex structure J such that

$$T_{eT}(G/T) = \mathfrak{g}/\mathfrak{t} = \bigoplus_{\alpha \in R_G^+} \mathfrak{g}_\alpha^{\mathbb{C}} \quad (3.4)$$

as complex vector spaces. Let

$$\bigwedge_J T(G/T) \rightarrow G/T$$

be the corresponding Clifford module, as in Example 3.1.

Let $\mathbb{C}_{\lambda-\rho_\lambda}$ be the one-dimensional representation of T with weight $\lambda - \rho_\lambda$, and consider the line bundle

$$L_{\lambda-\rho_\lambda} := G \times_T \mathbb{C}_{\lambda-\rho_\lambda} \rightarrow G/T.$$

Consider the map

$$\Phi: G/T = \mathrm{Ad}^*(G)\lambda \hookrightarrow \mathfrak{g}^* \rightarrow \mathfrak{k}^* \cong \mathfrak{k},$$

where $\mathfrak{g}^* \rightarrow \mathfrak{k}^*$ is the restriction map, and the identification $\mathfrak{k}^* \cong \mathfrak{k}$ is made via the K -invariant inner product chosen earlier. This map is taming by Proposition 2.1 in [35].

Theorem 3.8 (Paradan) *We have*

$$\pi_\lambda^G|_K = (-1)^{\dim(G/K)/2} \mathrm{index}(\bigwedge_J T(G/T) \otimes L_{\lambda-\rho_\lambda}, \Phi). \quad (3.5)$$

This is Theorem 5.1 in [37]. In fact, Paradan proves that the right hand side of (3.5) equals $(-1)^{\dim(G/K)/2}$ times the right hand side of Blattner's formula, as in Theorem 1.3 in [9]. The latter theorem therefore implies Theorem 3.8.

3.4 An almost complex structure

Now we drop the assumption that G is semisimple and has a compact Cartan subgroup. We consider a general tempered representation π of G , and write

$$\pi = \mathrm{Ind}_P^G \left(\pi_{\lambda, R_M^+, \chi_M}^M \otimes e^\nu \otimes 1_N \right)$$

as in Theorem 2.2. Let $H < G$ be the Cartan subgroup as in Sect. 2.2; we also use the other notation from that subsection. We will realise the restriction $\pi|_K$ as the K -equivariant index of a deformed Dirac operator on G/H (up to a sign). To do this, we will use a K -equivariant almost complex structure on G/H . (If π belongs to the discrete series, this is the G -invariant complex structure defined by (3.4), but in general it is only K -invariant, and need not be integrable.) We have

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}_M \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Recalling that $H_M = H \cap M$ (see also (2.5)), we find an H_M -invariant decomposition

$$\mathfrak{g}/\mathfrak{h} = \mathfrak{k}/\mathfrak{t}_M \oplus \mathfrak{s}_M \oplus \mathfrak{n} = \mathfrak{k}/\mathfrak{k}_M \oplus \mathfrak{k}_M/\mathfrak{t}_M \oplus \mathfrak{s}_M \oplus \mathfrak{n}. \quad (3.6)$$

The map from \mathfrak{n}^- to $\mathfrak{k}/\mathfrak{k}_M$, given by $X \mapsto \frac{1}{2}(X + \theta X)$ is an H_M -equivariant linear isomorphism. Using this, we find that, as a representation space of H_M ,

$$\mathfrak{g}/\mathfrak{h} = \mathfrak{m}/\mathfrak{t}_M \oplus \mathfrak{n}^- \oplus \mathfrak{n}^+. \quad (3.7)$$

As in the discrete series case, the positive root system R_M^+ for $(\mathfrak{m}^{\mathbb{C}}, \mathfrak{t}_M^{\mathbb{C}})$ determines an $\text{Ad}(H_M)$ -invariant complex structure $J_{\mathfrak{m}/\mathfrak{t}_M}$ on $\mathfrak{m}/\mathfrak{t}_M$ such that, as complex vector spaces,

$$\mathfrak{m}/\mathfrak{t}_M = \bigoplus_{\alpha \in R_M^+} \mathfrak{m}_{\alpha}^{\mathbb{C}}.$$

Once and for all, we fix an element $\zeta \in \mathfrak{a}$ such that $\langle \beta, \zeta \rangle > 0$ for all $\beta \in \Sigma^+$. Then the map

$$\text{ad}(\zeta): \mathfrak{n}^- \oplus \mathfrak{n}^+ \rightarrow \mathfrak{n}^- \oplus \mathfrak{n}^+$$

is invertible, with real eigenvalues. Set

$$J_{\zeta} := \theta |\text{ad}(\zeta)|^{-1} \text{ad}(\zeta): \mathfrak{n}^- \oplus \mathfrak{n}^+ \rightarrow \mathfrak{n}^- \oplus \mathfrak{n}^+.$$

Lemma 3.9 *The map J_{ζ} is an H_M -invariant complex structure.*

Proof The adjoint action by H_M commutes with θ because $H_M < K$. It commutes with $\text{ad}(\zeta)$, because $H_M < M < Z_G(\mathfrak{a})$. So the map J_{ζ} is H_M -equivariant. Let $\beta \in \Sigma^+$. The map $\text{ad}(\zeta)$ preserves the spaces $\mathfrak{g}_{\pm\beta}$, while the Cartan involution θ interchanges them. Hence, if $X_{\beta} \in \mathfrak{g}_{\beta}$,

$$J_{\zeta}^2(X_{\beta}) = J_{\zeta}(\theta X_{\beta}) = -X_{\beta}.$$

Similarly, if $X_{-\beta} \in \mathfrak{g}_{-\beta}$, then

$$J_{\zeta}^2(X_{-\beta}) = -J_{\zeta}(\theta X_{-\beta}) = -X_{-\beta}.$$

□

Let

$$J_{\mathfrak{g}/\mathfrak{h}} := J_{\mathfrak{m}/\mathfrak{t}_M} \oplus J_{\zeta} \tag{3.8}$$

be the H_M -invariant complex structure on $\mathfrak{g}/\mathfrak{h}$ defined by $J_{\mathfrak{m}/\mathfrak{t}_M}$ and J_{ζ} via the isomorphism (3.7).

Since $J_{\mathfrak{g}/\mathfrak{h}}$ is not necessarily H -invariant, it does not extend to a G -invariant almost complex structure on G/H in general. However, it does extend to a K -invariant almost complex structure in a natural way.

Lemma 3.10 *There is a unique K -invariant almost complex structure J on G/H , such that for all $k \in K$, $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$, the following diagram commutes:*

$$\begin{array}{ccc} T_{k \exp(X) \exp(Y) H} G/H & \xrightarrow{J} & T_{k \exp(X) \exp(Y) H} G/H \\ \uparrow T_{eH} k \exp(X) \exp(Y) & & \uparrow T_{eH} k \exp(X) \exp(Y) \\ T_{eH} G/H = \mathfrak{g}/\mathfrak{h} & \xrightarrow{J_{\mathfrak{g}/\mathfrak{h}}} & \mathfrak{g}/\mathfrak{h} = T_{eH} G/H. \end{array} \quad (3.9)$$

Proof If $k, k' \in K$, $X, X' \in \mathfrak{s}_M$ and $Y, Y' \in \mathfrak{n}$ such that

$$k \exp(X) \exp(Y) H = k' \exp(X') \exp(Y') H,$$

then there are $h \in H_M$ and $a \in A$ such that

$$k' \exp(X') \exp(Y') = k \exp(X) \exp(Y) h a = k h \exp(\text{Ad}(h^{-1})X) \exp(\text{Ad}(h^{-1})Y) a.$$

Since the multiplication map

$$K \times \exp(\mathfrak{s}_M) \times N \times A \rightarrow G \quad (3.10)$$

is injective (indeed, a diffeomorphism), we must have $a = e$. So

$$T_{eH} k' \exp(X') \exp(Y') = T_{eH} k \exp(X) \exp(Y) \circ T_{eH} h.$$

Since $T_{eH} h: \mathfrak{g}/\mathfrak{h} \rightarrow \mathfrak{g}/\mathfrak{h}$ is induced by $\text{Ad}(h)$, it commutes with $J_{\mathfrak{g}/\mathfrak{h}}$. Since (3.10) is a diffeomorphism, every element of G/H is of the form $k \exp(X) \exp(Y) H$ for some $k \in K$, $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$. Hence the map J is well-defined by commutativity of (3.9). This defining property directly implies that J is K -invariant. \square

3.5 Tempered representations; the main result

In the notation of the previous subsection, we have the vector bundle

$$\bigwedge_J T(G/H) \rightarrow G/H,$$

with J as in Lemma 3.10. As in the discrete series case, this vector bundle has the natural K -equivariant Clifford action of Example 3.1. Let ρ^M be half the sum of the roots in R_M^+ . Consider the one-dimensional representation

$$\mathbb{C}_{\lambda - \rho^M} \boxtimes \chi_M$$

of H_M . We extend it to a representation of H by letting A act trivially. This induces the line bundle

$$L_{\lambda - \rho^M, \chi_M} := G \times_H (\mathbb{C}_{\lambda - \rho^M} \boxtimes \chi_M) \rightarrow G/H.$$

Let $\xi \in \mathfrak{t}_M^*$ be any regular element that is dominant with respect to R_M^+ . We may, and will, assume that the elements $\zeta \in \mathfrak{a}$ and $\xi \in \mathfrak{t}_M^* \cong \mathfrak{t}_M$ are chosen so that the element $\xi + \zeta \in \mathfrak{h}$ is regular for the roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$. Define the map

$$\Phi: G/H \rightarrow \mathfrak{k}^* = \mathfrak{k}$$

by

$$\Phi(gH) = \text{Ad}^*(g)(\xi + \zeta)|_{\mathfrak{k}}.$$

(Here we identify $\zeta \in \mathfrak{a}$ with the dual element in \mathfrak{a}^* via the chosen inner product.) The map Φ is the moment map in the sense of symplectic geometry for the action by K on the coadjoint orbit $\text{Ad}^*(G)(\xi + \zeta)$. This map is taming for that action, by Proposition 2.1 in [35]. (This is true because $\xi + \zeta$ is a regular element, so $\text{Ad}^*(G)(\xi + \zeta)$ is a closed coadjoint orbit.)

Our main result is the following realisation of $\pi|_K$. Recall that

$$\pi = \text{Ind}_P^G \left(\pi_{\lambda, R_M^+, \chi_M}^M \otimes e^\nu \otimes 1_N \right). \quad (3.11)$$

Theorem 3.11 *We have*

$$\pi|_K = (-1)^{\dim(M/K_M)/2} \text{index}_K \left(\bigwedge_J T(G/H) \otimes L_{\lambda - \rho^M, \chi_M}, \Phi \right).$$

Note that if π belongs to the discrete series or limits of discrete series, then $M = G$, $H = T$ and $\mathfrak{n}^- = \mathfrak{n}^+ = \{0\}$. So if we take $\xi = \lambda$, then Theorem 3.11 reduces to Theorem 3.8. The case of limits of discrete series was not treated in [37], because the focus there was to obtain a multiplicity formula for K -types of the discrete series. But the techniques used there apply directly to this case.

Remark 3.12 The only property of the representation π that we will use to prove Theorem 3.11 is that it is of the form (3.11). This is true if π is tempered by Theorem 2.2. Since the restriction to K of the representation on the right hand side of (3.11) is independent of the parameter $\nu \in (\mathfrak{a}^{\mathbb{C}})^*$, even for non-imaginary ν , Theorem 3.11 in fact applies to every representation π equal to the right hand side of (3.11), for $\nu \in (\mathfrak{a}^{\mathbb{C}})^*$; i.e. to every standard representation. By the Langlands classification of admissible representations (see for example Theorem 14.92 in [21]), this implies that the restriction to K of every admissible representation is a quotient of an index as in Theorem 3.11.

Remark 3.13 The sign $(-1)^{\dim(G/K)/2}$ in Theorem 3.11 can be absorbed into the Clifford module used. For example, under regularity conditions on λ and ν , one can choose the Spin^c -structure on G/H whose determinant line bundle is a prequantum line bundle for a relevant coadjoint orbit. However, we prefer the explicit Clifford module in Theorem 3.11. This also avoids references to symplectic geometry, since Spin^c -geometry is more relevant here. We discuss this in more detail in Sect. 3.6.

3.6 Relation with the orbit method and geometric quantisation

In a companion paper [18], we use Theorem 3.11 together with a Spin^c -quantisation commutes with reduction result, Theorem 3.10 in [15], to obtain a geometric formula for the multiplicities of the K -types of π . Note that in Theorem 3.11, there is a large amount of freedom in choosing the elements $\xi \in \mathfrak{t}_M^*$ and $\zeta \in \mathfrak{a}$. To obtain the multiplicity formula in [18], we will need to make more specific choices.

Consider the element

$$\eta := (\lambda + \nu)/i \in \mathfrak{h}^*.$$

Then, modulo a factor i , η is the infinitesimal character of π . Suppose that η is a regular element of \mathfrak{h}^* . We will see in Section 2.2 of [18] that a natural choice is to take $\xi = \lambda$ and ζ dual to ν/i . Then Φ is the map

$$G/H \xrightarrow{\cong} \text{Ad}^*(G)\eta \hookrightarrow \mathfrak{g}^* \rightarrow \mathfrak{k}^*. \quad (3.12)$$

This is a moment map in the symplectic sense for the action by K on the coadjoint orbit $\text{Ad}^*(G)\eta$.

However, to obtain a multiplicity formula for the K -types of π from Theorem 3.11 via the quantisation commutes with reduction principle, it is better to use a Spin^c -approach to geometric quantisation. Indeed, already in the case of the discrete series, the natural complex structure on $\text{Ad}^*(G)\eta = \text{Ad}^*(G)\lambda = G/T$ is not compatible with the natural symplectic form, and the line bundle $L_{\lambda-\rho^M, \chi_M} = G \times_T \mathbb{C}_{\lambda-\rho}$ is not a prequantum line bundle for that symplectic structure. See Section 1.5 of [37]. For this reason, it is useful that, for these choices of ξ and ζ , the map Φ is a moment map in the Spin^c -sense, for the Spin^c -structure with the spinor bundle $\bigwedge_J T(G/H) \otimes L_{\lambda-\rho^M, \chi_M}$ in Theorem 3.11. This is shown in Proposition 2.4 and Lemma 4.5 in [18]. The quantisation commutes with reduction principle was shown to have a natural place in Spin^c -geometry by Paradan and Vergne [40].

The Clifford module $\bigwedge_J T(G/H) \otimes L_{\lambda-\rho^M, \chi_M}$ used in Theorem 3.11 corresponds to a K -equivariant Spin^c -structure. The manifold G/H does not admit a G -equivariant Spin^c -structure in general, because the action by G on G/H is not proper if H is noncompact. Indeed, the manifold then does not even admit a G -invariant Riemannian metric. In the context of the orbit method, the Spin^c -structure with spinor bundle

$$\bigwedge_J T(G/H) \otimes L_{\lambda-\rho^M, \chi_M} \otimes (G \times_H \mathbb{C}_\nu)$$

on $G/H \cong \text{Ad}^*(G)\eta$ is a natural one to use. This is K -equivariantly isomorphic to the Spin^c -structure with spinor bundle $\bigwedge_J T(G/H) \otimes L_{\lambda-\rho^M, \chi_M}$ in Theorem 3.11, and we leave out the factor $(G \times_H \mathbb{C}_\nu)$ to simplify our arguments. See Lemma 4.5 in [18]. (Including that factor would lead to a factor \mathbb{C}_ν in Proposition 7.1 below, which is trivial as a representation of H_M so would not change that result.)

The use of index theory of Dirac operators deformed by vector fields induced by moment maps (known as Kirwan vector fields), as in Theorem 3.11, is quite common

and natural in geometric quantisation. Already in the compact case, such deformations were used to prove that quantisation commutes with reduction in [36,43]. In the noncompact case, such deformed Dirac operators are not just used to prove this result, but also to define geometric quantisation. This was done in Vergne's conjecture [44] for actions by compact groups on noncompact symplectic manifolds, and its generalisation and proof by Ma and Zhang [30] and later by Paradan [38] (and even later in [16]). In Vergne's conjecture and Paradan's proof, and in Ma and Zhang's proof, two different but equivalent definitions of the equivariant index of deformed Dirac operators were used to the one we use here. The same deformed Dirac operators were used to prove that quantisation commutes with reduction for actions by compact groups on noncompact Spin^c -manifolds in [15], and for proper actions by noncompact groups in [11,12,17,31].

If η is not regular, then the map (3.12) is still a moment map in the Spin^c -sense, but the first arrow is a fibre bundle rather than a diffeomorphism. In this case, the map (3.12) does not have the properties that the map Φ needs to have for Theorem 3.11 to hold. In [18], we will deform the map Φ in Theorem 3.11 to a taming, proper Spin^c -moment map to obtain a geometric expression for multiplicities of K -types of π .

In representation theory, such representations with singular parameters are viewed as being associated to nilpotent coadjoint orbits. Via the deformation we use in the singular case, we lose the link with this aspect of the orbit method, but we gain a geometric formula for multiplicities of K -types.

3.7 Example: $G = \text{SL}(2, \mathbb{R})$

To illustrate Theorem 3.11, we work out the case where $G = \text{SL}(2, \mathbb{R})$ and $K = \text{SO}(2)$. This group G has three kinds of tempered representations: the (holomorphic and antiholomorphic) discrete series, the limits of the discrete series, and the (spherical and nonspherical) unitary principal series.

3.7.1 The discrete series

To realise the restrictions of the discrete series to K , we take $H = T = \text{SO}(2)$, so $M = G$. Let α be the root that maps $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to $2i$. Let $n \in \{1, 2, 3, \dots\}$, and let D_n^\pm be the discrete series representation with Harish-Chandra parameter $\lambda = \pm n\alpha/2$. Then $\rho^M = \rho_\lambda = \pm\alpha/2$. We take $\xi = \lambda$. In this case $\mathfrak{a} = \{0\}$, so $\zeta = 0$. Then the map Φ is defined by $\Phi(gT) = \text{Ad}^*(g)\lambda|_{\mathfrak{k}}$, for $g \in G$. This is the identification of G/T with the elliptic orbit $\text{Ad}^*(G)\lambda$, followed by restriction to \mathfrak{k} . We may use the orbit through any positive multiple of λ here, but this choice will turn out to be the relevant one for the computation of multiplicities of K -types.

Then the complex structure $J_{\mathfrak{g}/\mathfrak{t}}$ is determined by the isomorphism $\mathfrak{g}/\mathfrak{t} \cong \mathfrak{g}_{\pm\alpha}^{\mathbb{C}} = \mathbb{C}E_{\pm\alpha}$, where

$$E_\alpha = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}; \quad E_{-\alpha} = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

The torus T acts on this space with weight $\pm\alpha$. We will write \mathbb{C}_l for the unitary irreducible representation of the circle with weight $l \in \mathbb{Z}$. Then $\mathfrak{g}/\mathfrak{h} = \mathbb{C}_{\pm 2}$, and $\bigwedge_{J_{\mathfrak{g}/\mathfrak{h}}} \mathfrak{g}/\mathfrak{h} = \mathbb{C}_0 \oplus \mathbb{C}_{\pm 2}$.

We have $Z_M = Z_G = \{\pm I\} \subset T$, so we have to take $\chi_M = e^{\lambda - \rho^M}|_{Z_M}$. Then

$$\mathbb{C}_{\lambda - \rho^M} \boxtimes \chi_M \cong \mathbb{C}_{\lambda - \rho^M} = \mathbb{C}_{\pm(n-1)},$$

via the map $z_1 \otimes z_2 \mapsto z_1 z_2$. So the line bundle $L_{\lambda - \rho^M, \chi_M}$ equals

$$L_{\lambda - \rho^M, \chi_M} = G \times_T \mathbb{C}_{\pm(n-1)}.$$

Theorem 3.11 now states that

$$D_n^{\pm}|_K = -\text{index}_K (G \times_T (\mathbb{C}_{\pm(n-1)} \oplus \mathbb{C}_{\pm(n+1)}), \Phi).$$

3.7.2 Limits of discrete series

For the limits of the discrete series D_0^{\pm} , we also have $H = T = \text{SO}(2)$ and $M = G$. Now $\lambda = 0$ and $R_M^+ = \{\pm\alpha\}$. We cannot take $\xi = \lambda = 0$, so we take $\xi = \pm\alpha/2$ (we can replace this element by any positive multiple). As before, we have $\zeta = 0$, so for all $g \in G$, we have $\Phi(gT) = \text{Ad}^*(g)\xi$. This is the identification of G/T with the elliptic orbit $\pm \text{Ad}^*(G)\alpha/2$, followed by restriction to \mathfrak{k} . Note that this is the same orbit as the one used for D_1^{\pm} ; this shift will be important in [18]. Now we have $\rho^M = \pm\alpha/2$, so, analogously to the discrete series case,

$$L_{\lambda - \rho^M, \chi_M} = G \times_T \mathbb{C}_{\mp 1}.$$

Theorem 3.11 therefore yields

$$D_0^{\pm}|_K = -\text{index}_K (G \times_T (\mathbb{C}_{\mp 1} \oplus \mathbb{C}_{\pm 1}), \Phi).$$

3.7.3 The principal series

For the unitary principal series P_{iv}^{\pm} , where $v \geq 0$ for the spherical principal series P_{iv}^+ and $v > 0$ for the nonspherical principal series P_{iv}^- , we have

$$H = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}; x \neq 0 \right\}.$$

Then $M = H_M = \{\pm I\}$. Now

$$\mathfrak{g}/\mathfrak{h} \cong \mathfrak{n}^- \oplus \mathfrak{n}^+,$$

where $\mathfrak{n}^\pm = \mathbb{R}E^\pm$, with

$$E^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad E^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Now $\mathfrak{t}_M = 0$, so $\xi = 0$. We take $\zeta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then for all $g \in G$, $\Phi(gH) = \text{Ad}^*(g)\zeta|_{\mathfrak{k}}$. This is the identification of G/H with the hyperbolic orbit $\text{Ad}^*(G)\zeta$, followed by restriction to \mathfrak{k} . Note that we use the same orbit for all principal series representations.

Furthermore,

$$\text{ad}(\zeta)E^\pm = \pm 2E^\pm$$

and $\theta E^\pm = -E^\mp$. So with respect to the basis $\{E^+, E^-\}$, the complex structure J_ζ has the matrix

$$J_\zeta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The almost complex structure J on G/H is given by

$$J(T_{eH}l_{k \exp(Y)}X) = T_{eH}l_{k \exp(Y)}J_\zeta X,$$

for $k \in \text{SO}(2)$, $Y \in \mathfrak{n}^+$ and $X \in \mathfrak{n}^- \oplus \mathfrak{n}^+$.

Let χ_\pm be the representations of H_M on \mathbb{C} defined by $\chi_\pm(-I) = \pm 1$. We now have $\lambda = \rho^M = 0$, so

$$L_{\lambda - \rho^M, \chi_\pm} = G \times_H \chi_\pm,$$

where H acts on χ_\pm via

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mapsto \chi_\pm(\text{sgn}(x)I).$$

This number equals 1 for χ_+ , and $\text{sgn}(x)$ for χ_- . By Theorem 3.11, we have

$$P_{iv}^\pm|_K = \text{index}_K(\bigwedge_J T(G/H) \otimes (G \times_H \chi_\pm), \Phi).$$

3.7.4 Explicit computation of the indices for $\text{SL}(2, \mathbb{R})$

The proof of Theorem 3.11 simplifies considerably in the example $G = \text{SL}(2, \mathbb{R})$. Some important points are still present, however: a partial linearisation of the space G/H , and the role of differential operators on vector spaces with index 1. We give a brief outline of the proof for $\text{SL}(2, \mathbb{R})$ to illustrate these points. In particular, we give explicit computations of the relevant indices in this case.

In the case of (limits of) discrete series representations, the proof of Theorem 3.11 is a variation on the proof of Theorem 5.1 in [37]. For the representation D_n^+ , with $n \in \{0, 1, 2, \dots\}$, the first step in this proof is to show that

$$\text{index}_K(\bigwedge_J G/T \otimes L_{\lambda-\rho^M, \chi_M}, \Phi) = \text{index}_K(\bigwedge_{\mathbb{C}} T\mathbb{C} \otimes \mathbb{C}_{\pm(n-1)}, \Phi^E), \quad (3.13)$$

where K acts on \mathbb{C} by rotation with weight 2, and $\Phi^E: \mathbb{C} \rightarrow \mathfrak{so}(2) \cong \mathbb{R}$ is the constant map with value ± 1 . Here we use that G/T is isomorphic to \mathbb{C} as a complex K -manifold, where K acts on \mathbb{C} by rotation with weight 2. The index on the right hand side of (3.13) was computed in Lemma 6.4 in [2] (see also Lemma 5.7 in [36] and the bottom of page 841 in [37]), and equals

$$-\bigoplus_{j=0}^{\infty} \mathbb{C}_{\pm(n+2j+1)}.$$

Apart from the minus sign, this is the well-known decomposition of $D_n^+|_K$. The minus sign is the factor $(-1)^{\dim(M/K_M)/2}$ in this case.

For the principal series representation P_{iv}^{\pm} , the space G/H is isomorphic to the cylinder \mathbb{C}/\mathbb{Z} as a complex K -manifold, where K acts on \mathbb{C}/\mathbb{Z} by double rotations of the cylinder in the direction \mathbb{R}/\mathbb{Z} . We will see that

$$\text{index}_K(\bigwedge_J T(G/H) \otimes L_{\lambda-\rho^M, \chi_M}, \Phi) = \text{index}_K(\bigwedge_{\mathbb{C}} T(\mathbb{C}/\mathbb{Z}) \otimes \chi_{\pm}, \tilde{\Phi}),$$

where for $x, y \in \mathbb{R}$, we have $\tilde{\Phi}(x + \mathbb{Z} + iy) = y \in \mathfrak{so}(2)$. As a special case of Proposition 5.1, we will see that the index on the right hand side equals the L^2 -kernel of the operator

$$\begin{pmatrix} 0 & -\frac{\partial}{\partial y} + y \\ \frac{\partial}{\partial y} + y & 0 \end{pmatrix}$$

on $(L^2(\mathbb{C}/\mathbb{Z}) \otimes \mathbb{C}^2 \otimes \chi_{\pm})^{\{\pm I\}}$. This kernel equals

$$\begin{aligned} \left(L^2(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C} \cdot (y \mapsto e^{-y^2/2}) \otimes \chi_{\pm} \right)^{\{\pm I\}} &\cong (L^2(K) \otimes \chi_{\pm})^{K_M} \\ &= \text{Ind}_{MAN}^G(\chi_{\pm} \otimes e^{iv} \otimes 1_N)|_K \\ &= P_{iv}^{\pm}|_K. \end{aligned}$$

Since K acts on \mathbb{R}/\mathbb{Z} by rotations with weight 2, one also finds directly that

$$\left(L^2(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C} \cdot (y \mapsto e^{-y^2/2}) \otimes \chi_{\pm} \right)^{\{\pm I\}} = \begin{cases} \bigoplus_{j \in \mathbb{Z}} \mathbb{C}_{2j} & \text{for } \chi_{+}; \\ \bigoplus_{j \in \mathbb{Z}} \mathbb{C}_{2j+1} & \text{for } \chi_{-}. \end{cases}$$

This is the usual decomposition of $P_{iv}^{\pm}|_K$. (Note that now $(-1)^{\dim(M/K_M)/2} = 1$.)

4 Linearising the index

In the rest of this paper, we prove Theorem 3.11. The first main step in this proof is to show that the index that appears there equals the index of an operator on a partially linearised version of the space G/H , see Proposition 4.1 below. This partially linearised space E is the total space of a vector bundle over the compact space K/H_M . It will then be shown that the relevant index on E equals $\pi|_K$, up to a sign, in Sects. 5–7, see Proposition 7.6. That will finish the proof of Theorem 3.11.

We continue using the notation and assumptions of Sects. 3.4 and 3.5.

4.1 The linearised index

Consider the action by H_M on $K \times (\mathfrak{s}_M \oplus \mathfrak{n})$ given by

$$h \cdot (k, X + Y) = (kh^{-1}, \text{Ad}(h)(X + Y)),$$

for $h \in H_M$, $k \in K$, $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$. Let

$$E := K \times_{H_M} (\mathfrak{s}_M \oplus \mathfrak{n})$$

be the quotient space of this action. This is the partial linearisation of G/H that we will use. (We will see in Lemma 4.2 that E is K -equivariantly diffeomorphic to G/H .)

For $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$, we have

$$T_{[e, X+Y]}E = \mathfrak{k}/\mathfrak{t}_M \oplus \mathfrak{s}_M \oplus \mathfrak{n},$$

via the map

$$(U + \mathfrak{t}_M, V + W) \mapsto \left. \frac{d}{dt} \right|_{t=0} [\exp(tU), X + Y + t(V + W)], \quad (4.1)$$

for $U \in \mathfrak{k}$, $V \in \mathfrak{s}_M$ and $W \in \mathfrak{n}$. (We spell this out explicitly here, because another identification will also be used in Sect. 4.3.) Using this identification and the first equality in (3.6), we obtain

$$T_{[e, X+Y]}E = \mathfrak{g}/\mathfrak{h}. \quad (4.2)$$

On this space, we defined the H_M -invariant complex structure $J_{\mathfrak{g}/\mathfrak{h}}$ in (3.8). We will write J^E for the K -invariant almost complex structure on E that corresponds to $J_{\mathfrak{g}/\mathfrak{h}}$ via the identification (4.2).

Consider the map $\Phi^E: E \rightarrow \mathfrak{k}$ given by

$$\Phi^E[k, X + Y] = \text{Ad}(k) \left(\xi - |\text{ad}(\zeta)|^{-1} \text{ad}(\zeta)Y_{\mathfrak{s}} \right),$$

where $k \in K$, $X \in \mathfrak{s}_M$, $Y \in \mathfrak{n}$, ζ is as in Sect. 3.4, ξ is as in Sect. 3.5, and $Y_{\mathfrak{s}} := \frac{1}{2}(Y - \theta Y)$ is the component of Y in \mathfrak{s} . We will see in Lemma 4.11 that this map is taming,

Let V be a finite-dimensional representation space of H_M . (Later, we will take $V = \mathbb{C}_{\lambda - \rho_M} \boxtimes \chi_M$.) We extend it to a representation of H by letting A act trivially. Then we obtain the associated vector bundles

$$\begin{aligned} L_V &:= G \times_H V \rightarrow G/H; \\ L_V^E &:= K \times_{H_M} (\mathfrak{s}_M \oplus \mathfrak{n} \times V) \rightarrow E. \end{aligned}$$

The linearisation result for the index in Theorem 3.11 is the following.

Proposition 4.1 *We have*

$$\text{index}_K(\bigwedge_J T(G/H) \otimes L_V, \Phi) = \text{index}_K(\bigwedge_{J^E} T E \otimes L_V^E, \Phi^E).$$

4.2 Linearising G/H

Consider the map

$$\tilde{\Psi}: K \times (\mathfrak{s}_M \oplus \mathfrak{n}) \rightarrow G/H$$

defined by

$$\tilde{\Psi}(k, X + Y) = k \exp(X) \exp(Y) H,$$

for $k \in K$, $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$.

Lemma 4.2 *The map $\tilde{\Psi}$ descends to a well-defined, K -equivariant diffeomorphism*

$$\Psi: E \xrightarrow{\cong} G/H.$$

Proof One checks directly that for $k \in K$, $X \in \mathfrak{s}_M$, $Y \in \mathfrak{n}$ and $h \in H_M$,

$$\tilde{\Psi}(kh^{-1}, \text{Ad}(h)(X + Y)) = \tilde{\Psi}(k, X + Y).$$

So $\tilde{\Psi}$ indeed descends to a map $\Psi: E \rightarrow G/H$. That map is K -equivariant by definition. To see that Ψ is surjective, we use the fact that the multiplication map (3.10) is a diffeomorphism. If $k \in K$, $X \in \mathfrak{s}_M$, $Y \in \mathfrak{n}$ and $a \in A$, then

$$k \exp(X) \exp(Y) a H = \Psi[k, X + Y],$$

so Ψ is indeed surjective. If $k, k' \in K$, $X, X' \in \mathfrak{s}_M$ and $Y, Y' \in \mathfrak{n}$ satisfy

$$\Psi[k, X + Y] = \Psi[k', X' + Y'],$$

then there are $h \in H_M$ and $a \in A$ such that

$$k' \exp(X') \exp(Y') = k \exp(X) \exp(Y) h a = k h \exp(\operatorname{Ad}(h^{-1})X) \exp(\operatorname{Ad}(h^{-1})Y) a,$$

so we must have $a = e$, and $[k', X' + Y'] = [k, X + Y]$. Hence Ψ is injective. \square

Remark 4.3 Lemma 4.2 is an explicit form of a special case of Theorem 4.1 in [34].

4.3 Linearising almost complex structures

The K -invariant almost complex structures J^E and Ψ^*J on E are different, but we will show that they are homotopic in a suitable sense.

Let $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$. The complex structures on $T_{[e, X+Y]}E$ defined by Ψ^*J and J^E correspond to $J_{\mathfrak{g}/\mathfrak{h}}$ via two different identifications

$$\varphi_0^{X+Y}, \varphi_1^{X+Y} : \mathfrak{g}/\mathfrak{h} = T_{[e, 0]}E \xrightarrow{\cong} T_{[e, X+Y]}E. \quad (4.3)$$

The map φ_0^{X+Y} is the inverse of the map (4.1), whereas φ_1^{X+Y} is defined by commutativity of the diagram

$$\begin{array}{ccc} T_{[e, 0]}E & \xrightarrow{\varphi_1^{X+Y}} & T_{[e, X+Y]}E \\ T_{[e, 0]}\Psi \downarrow & & \downarrow T_{[e, 0]}\Psi \\ T_{eH}G/H & \xrightarrow{T_{eH} \exp(X) \exp(Y)} & T_{\exp(X) \exp(Y)H}G/H. \end{array}$$

With respect to these two maps, we have

$$\begin{aligned} J_{[e, X+Y]}^E &= \varphi_0^{X+Y} J_{\mathfrak{g}/\mathfrak{h}} (\varphi_0^{X+Y})^{-1}; \\ (\Psi^*J)_{[e, X+Y]} &= \varphi_1^{X+Y} J_{\mathfrak{g}/\mathfrak{h}} (\varphi_1^{X+Y})^{-1}. \end{aligned} \quad (4.4)$$

Lemma 4.4 *We have*

$$\operatorname{index}_K (\Psi^*(\bigwedge_J T(G/H) \otimes L_V), \Psi^*\Phi) = \operatorname{index}_K (\bigwedge_{J^E} T E \otimes L_V^E, \Psi^*\Phi).$$

Proof For $X \in \mathfrak{s}_M$, $Y \in \mathfrak{n}$ and $t \in [0, 1]$, we define the linear isomorphism

$$\varphi_t^{X+Y} : T_{[e, 0]}E \xrightarrow{\cong} T_{[e, X+Y]}E$$

by commutativity of the diagram

$$\begin{array}{ccc} T_{[e,0]}E & \xrightarrow{\varphi_t^{X+Y}} & T_{[e,X+Y]}E \\ \varphi_1^{t(X+Y)} \downarrow & & \uparrow \varphi_0^{X+Y} \\ T_{[e,t(X+Y)]}E & \xleftarrow{\varphi_0^{t(X+Y)}} & T_{[e,0]}E. \end{array}$$

(Note that for $t = 0, 1$, this definition agrees with the definitions of φ_0^{X+Y} and φ_1^{X+Y} above.) Define the K -invariant almost complex structure J_t on E by the property that

$$(J_t)_{[e,X+Y]} = \varphi_t^{X+Y} J_{\mathfrak{g}/\mathfrak{h}} (\varphi_t^{X+Y})^{-1}.$$

Then by (4.4), we have $J_0 = J^E$ and $J_1 = \Psi^* J$. Using this family of almost complex structures, we obtain a homotopy in the sense of Definition 3.4 between

$$(\bigwedge_{\Psi^* J} T E \otimes L_V^E, \Psi^* \Phi) \quad \text{and} \quad (\bigwedge_{J^E} T E \otimes L_V^E, \Psi^* \Phi).$$

Since $\Psi^* L_V = L_V^E$ and

$$\Psi^* \bigwedge_J T(G/H) = \bigwedge_{\Psi^* J} T E,$$

the claim follows from Theorem 3.5. \square

4.4 Linearising taming maps

To prove Proposition 4.1, it remains to construct a homotopy between the taming maps $\Psi^* \Phi$ and Φ^E . We will do this in a number of stages.

First, for $t \in [0, 1]$, we consider the map $\Phi_1^t : E \rightarrow \mathfrak{k}$, defined by

$$\Phi_1^t[k, X + Y] = \text{Ad}(k)(\xi + t[Y_{\mathfrak{k}}, \xi] + [Y_{\mathfrak{s}}, \zeta]), \quad (4.5)$$

where $k \in K$, $X \in \mathfrak{s}_M$, $Y \in \mathfrak{n}$, $Y_{\mathfrak{k}} = \frac{1}{2}(Y + \theta Y) \in \mathfrak{k}$, and $Y_{\mathfrak{s}} = \frac{1}{2}(Y - \theta Y) \in \mathfrak{s}$. For $Z \in \mathfrak{k}$, let Z^E be the vector field on E given by the infinitesimal action by Z . In several places, we will use the fact that for all such X and Y ,

$$\begin{aligned} Z^E([e, X + Y]) &= (Z_{\mathfrak{t}_M^\perp} + \mathfrak{t}_M, -[Z_{\mathfrak{t}_M}, X + Y]) \in \mathfrak{k}/\mathfrak{t}_M \oplus \mathfrak{s}_M \oplus \mathfrak{n} \\ &= T_{[e,0]}E \cong T_{[e,X+Y]}E, \end{aligned} \quad (4.6)$$

where the last identification is made via the map φ_0^{X+Y} in (4.3). Furthermore, we write $Z = Z_{\mathfrak{t}_M} + Z_{\mathfrak{t}_M^\perp}$, where $Z_{\mathfrak{t}_M} \in \mathfrak{t}_M$ and $Z_{\mathfrak{t}_M^\perp} \in \mathfrak{t}_M^\perp$.

Proposition 4.5 *The vector field $v^{\Phi_1^0}$ on E vanishes at the same points as the vector field $\Psi^* v^\Phi$. In a neighbourhood of the set of these points, the taming maps Φ_1^0 and $\Psi^* \Phi$ are homotopic in the sense of Definition 3.4.*

The proof of this proposition is based on Lemmas 4.6–4.10. In the arguments below, we will use the K -invariant Riemannian metric on E induced by the inner product on \mathfrak{g} and the identification φ_0^{X+Y} in (4.3). As noted in Sect. 3.1, Braverman's index is independent of the Riemannian metric used, as long as it is complete and K -invariant. So we are free to use the Riemannian metric most suited to our purposes.

Lemma 4.6 *For all $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$,*

$$v^{\Phi_1}([e, X + Y]) = (t[Y_{\mathfrak{k}}, \xi] + [Y_{\mathfrak{s}}, \zeta] + t_M, [X + Y, \xi]).$$

Proof By ad-invariance of the Killing form B , we have for all $X \in \mathfrak{t}_M$,

$$-B([Y_{\mathfrak{s}}, \zeta], X) = B(Y_{\mathfrak{s}}, [X, \zeta]) = 0,$$

so $[Y_{\mathfrak{s}}, \zeta] \in \mathfrak{t}_M^\perp$. We also have $[Y_{\mathfrak{k}}, \xi] \in \mathfrak{t}_M^\perp$. So the claim follows from (4.5) and (4.6). \square

Lemma 4.7 *There is a constant $C > 0$ such that for all $t \in [0, 1]$, $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$,*

$$\|v^{\Phi_1}([e, X + Y])\| \geq C\|X\|.$$

Proof Note that $[X, \xi] \in \mathfrak{s}_M$. Furthermore, since the adjoint action by H_M commutes with A , it preserves the spaces \mathfrak{g}_β , and hence \mathfrak{n} . So $[Y, \xi] \in \mathfrak{n}$. Since the elements $[X, \xi]$ and $[Y, \xi]$ lie in different subspaces of \mathfrak{g} , Lemma 4.6 implies that there is a constant $C_1 > 0$ such that for all X and Y as above,

$$\|v^{\Phi_1}([e, X + Y])\| \geq \|[X + Y, \xi]\| \geq C_1\|[X, \xi]\|. \quad (4.7)$$

Now $X \in \mathfrak{s}_M \perp \mathfrak{a}$ and also $X \in \mathfrak{s} \perp \mathfrak{t}_M$. So $X \perp \mathfrak{h}$, which means that it lies in the sum of the root spaces of $(\mathfrak{g}^\mathbb{C}, \mathfrak{h}^\mathbb{C})$. Write

$$X = \sum_{\alpha \in R(\mathfrak{g}^\mathbb{C}, \mathfrak{h}^\mathbb{C})} X_\alpha,$$

where $X_\alpha \in \mathfrak{g}_\alpha^\mathbb{C}$. Since $[X, \zeta] = 0$, we have

$$\begin{aligned} \|[X, \xi]\|^2 &= \|[X, \xi + \zeta]\|^2 \\ &= \sum_{\alpha \in R(\mathfrak{g}^\mathbb{C}, \mathfrak{h}^\mathbb{C})} |\langle \alpha, \xi + \zeta \rangle|^2 \|X_\alpha\|^2 \\ &\geq \min_{\alpha \in R(\mathfrak{g}^\mathbb{C}, \mathfrak{h}^\mathbb{C})} |\langle \alpha, \xi + \zeta \rangle|^2 \sum_{\alpha \in R(\mathfrak{g}^\mathbb{C}, \mathfrak{h}^\mathbb{C})} \|X_\alpha\|^2 \\ &= \min_{\alpha \in R(\mathfrak{g}^\mathbb{C}, \mathfrak{h}^\mathbb{C})} |\langle \alpha, \xi + \zeta \rangle|^2 \|X\|^2. \end{aligned}$$

Since the element $\xi + \zeta \in \mathfrak{h}$ is regular, the factor $\min_{\alpha \in R(\mathfrak{g}^\mathbb{C}, \mathfrak{h}^\mathbb{C})} |\langle \alpha, \xi + \zeta \rangle|^2$ is positive. Together with (4.7), this implies the claim. \square

Lemma 4.8 *There is a constant $C > 0$ such that for all $t \in [0, 1]$, $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$,*

$$\|v^{\Phi'_1}([e, X + Y])\| \geq C\|Y\|.$$

Proof By Lemma 4.6, we have

$$\|v^{\Phi'_1}([e, X + Y])\| \geq \|t[Y_{\mathfrak{t}}, \xi] + [Y_{\mathfrak{s}}, \zeta]\|. \quad (4.8)$$

Since \mathfrak{t}_M and \mathfrak{a} commute, there is a simultaneous weight space decomposition of $\mathfrak{g}^{\mathbb{C}}$ for the adjoint action by these algebras. (This is just a different way of writing the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h} = \mathfrak{t}_M \oplus \mathfrak{a}$.) The set of nonzero weights of the action by \mathfrak{a} is Σ . Let $\Xi \subset i\mathfrak{t}_M^*$ be the set of weights of \mathfrak{t}_M . For $\beta \in \Sigma$ and $\delta \in \Xi$, let $\mathfrak{g}_{\beta, \delta}^{\mathbb{C}}$ be the corresponding weight space. Write

$$Y = \sum_{\beta \in \Sigma^+, \delta \in \Xi} Y_{\beta, \delta},$$

with $Y_{\beta, \delta} \in \mathfrak{g}_{\beta, \delta}^{\mathbb{C}}$. Then, since $\theta \mathfrak{g}_{\beta, \delta}^{\mathbb{C}} \subset \mathfrak{g}_{-\beta, \delta}^{\mathbb{C}}$, we have

$$[Y_{\mathfrak{t}}, \xi] = \frac{1}{2} \sum_{\beta \in \Sigma^+, \delta \in \Xi} [Y_{\beta, \delta} + \theta Y_{\beta, \delta}, \xi] = -\frac{1}{2} \sum_{\beta \in \Sigma^+, \delta \in \Xi} \langle \delta, \xi \rangle (Y_{\beta, \delta} + \theta Y_{\beta, \delta}).$$

Similarly,

$$[Y_{\mathfrak{s}}, \zeta] = \frac{1}{2} \sum_{\beta \in \Sigma^+, \delta \in \Xi} [Y_{\beta, \delta} - \theta Y_{\beta, \delta}, \zeta] = -\frac{1}{2} \sum_{\beta \in \Sigma^+, \delta \in \Xi} \langle \beta, \zeta \rangle (Y_{\beta, \delta} + \theta Y_{\beta, \delta}).$$

So for all $t \in [0, 1]$,

$$\begin{aligned} \|t[Y_{\mathfrak{t}}, \xi] + [Y_{\mathfrak{s}}, \zeta]\|^2 &= \sum_{\beta \in \Sigma^+, \delta \in \Xi} |t\langle \delta, \xi \rangle + \langle \beta, \zeta \rangle|^2 \left\| \frac{1}{2} (Y_{\beta, \delta} + \theta Y_{\beta, \delta}) \right\|^2 \\ &\geq \min_{\beta \in \Sigma^+, \delta \in \Xi} |t\langle \delta, \xi \rangle + \langle \beta, \zeta \rangle|^2 \sum_{\beta \in \Sigma^+, \delta \in \Xi} \left\| \frac{1}{2} (Y_{\beta, \delta} + \theta Y_{\beta, \delta}) \right\|^2 \\ &= \min_{\beta \in \Sigma^+, \delta \in \Xi} |t\langle \delta, \xi \rangle + \langle \beta, \zeta \rangle|^2 \|Y_{\mathfrak{t}}\|^2. \end{aligned}$$

Now for all β and δ as above, we have $\langle \delta, \xi \rangle \in i\mathbb{R}$, while $\langle \beta, \zeta \rangle \in \mathbb{R}$. So

$$|t\langle \delta, \xi \rangle + \langle \beta, \zeta \rangle| \geq |\langle \beta, \zeta \rangle| > 0,$$

by assumption on ζ . Therefore, (4.8) implies that there is a constant $C_1 > 0$ such that for all $t \in [0, 1]$, $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$,

$$\|v^{\Phi'_1}([e, X + Y])\| \geq C_1\|Y_{\mathfrak{t}}\|.$$

The projection map $\frac{1}{2}(1 + \theta): \mathfrak{n} \rightarrow \mathfrak{k}$ is injective. So there is a constant $C_2 > 0$ such that for all $Y \in \mathfrak{n}$, $\|Y_{\mathfrak{k}}\| \geq C_2\|Y\|$. This completes the proof. \square

Lemma 4.9 *There is a constant $C > 0$ such that for all $Z \in \mathfrak{k}$, $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$,*

$$\|Z^E([e, X + Y])\| \leq C(1 + \|X + Y\|)\|Z\|.$$

Proof By (4.6), we have

$$\|Z^E([e, X + Y])\|^2 = \|Z_{\mathfrak{t}_M^\perp}\|^2 + \|[Z_{\mathfrak{t}_M}, X + Y]\|^2.$$

For a constant $C_1 > 0$, this is at most equal to

$$\|Z_{\mathfrak{t}_M^\perp}\|^2 + C_1\|Z_{\mathfrak{t}_M}\|^2\|X + Y\|^2 \leq (1 + C_1)(1 + \|X + Y\|^2)\|Z\|^2.$$

\square

Lemma 4.10 *We have*

$$\Psi^*\Phi([e, X + Y]) = \Phi_1^1([e, X + Y]) + \mathcal{O}(\|X + Y\|^2).$$

as $X + Y \in \mathfrak{s}_M \oplus \mathfrak{n}$ goes to 0.

Proof We have

$$\begin{aligned} \Psi^*\Phi([e, X + Y]) &= (\exp(X)\exp(Y)(\xi + \zeta))|_{\mathfrak{k}} \\ &= \sum_{l,m=0}^{\infty} \frac{1}{l!m!} (\text{ad}(X)^l \text{ad}(Y)^m (\xi + \zeta))|_{\mathfrak{k}} \\ &= (\xi + \zeta)|_{\mathfrak{k}} + [X, \xi + \zeta]|_{\mathfrak{k}} + [Y, \xi + \zeta]|_{\mathfrak{k}} + \mathcal{O}(\|X + Y\|^2). \end{aligned}$$

Now $(\xi + \zeta)|_{\mathfrak{k}} = \xi$, $[X, \xi + \zeta]|_{\mathfrak{k}} = [X, \zeta] = 0$, and

$$[Y, \xi + \zeta]|_{\mathfrak{k}} = [Y_{\mathfrak{k}}, \xi] + [Y_{\mathfrak{s}}, \zeta].$$

\square

Proof of Proposition 4.5 By Lemmas 4.7 and 4.8, the vector field $v^{\Phi_1^i}$ vanishes precisely at the set $K \times_{H_M} \{0\} \subset E$. The map Φ is a moment map for the action by K on $G/H = \text{Ad}^*(G)(\xi + \zeta)$ with respect to the Kirillov–Kostant symplectic form. Therefore, the vector field v^{Φ} is the Hamiltonian vector field of the function $\frac{1}{2}\|\Phi\|^2$. By Proposition 2.1 in [37], this vector field therefore vanishes precisely at the set $\{kH; k \in K\} \subset G/H$. So $v^{\Psi^*\Phi} = \Psi^*v^{\Phi}$ vanishes at the set

$$\Psi^{-1}(\{kH; k \in K\}) = K \times_{H_M} \{0\}.$$

Let $C_1 > 0$ be as the constant C in Lemma 4.9. Then by Lemma 4.10, we have for all $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$,

$$\begin{aligned} & \|v^{\Phi_1^1}([e, X + Y]) - v^{\Psi^*\Phi}[e, X + Y]\| \\ & \leq C_1(1 + \|X + Y\|)\|\Phi_1^1([e, X + Y]) - \Psi^*\Phi([e, X + Y])\| \\ & = \mathcal{O}(\|X + Y\|^2), \end{aligned} \quad (4.9)$$

as $X + Y \rightarrow 0$. By Lemmas 4.7 and 4.8, there is a constant $C_2 > 0$ such that for all $t \in [0, 1]$, $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$,

$$\|v^{\Phi_1^1}([e, X + Y])\| \geq C_2\|X + Y\|.$$

Together with (4.9), this implies that in a small enough neighbourhood of $K \times_{H_M} \{0\}$ in E , we have

$$(v^{\Phi_1^1}, v^{\Psi^*\Phi}) \geq 0.$$

Corollary 3.6 implies that Φ_1^1 and $\Psi^*\Phi$ are homotopic in this neighbourhood. Lemmas 4.7 and 4.8 imply that Φ_1^0 and Φ_1^1 are homotopic, so the claim follows. \square

4.5 Proof of Proposition 4.1

We prove Proposition 4.1 by combining the earlier results in this section with a last homotopy of taming maps.

Lemma 4.11 *The map Φ^E is taming, and homotopic to the map Φ_1^0 defined in (4.5).*

Proof For $t \in [0, 1]$, consider the map $\Phi_t^E: E \rightarrow \mathfrak{k}$ defined by

$$\Phi_t^E([k, X + Y]) = \text{Ad}(k)(\xi - |\text{ad}(\zeta)|^{-t} \text{ad}(\zeta)Y_{\mathfrak{s}}).$$

Then $\Phi_0^E = \Phi_1^0$ and $\Phi_1^E = \Phi^E$.

Since $|\text{ad}(\zeta)|^{-t} \text{ad}(\zeta)Y_{\mathfrak{s}} \perp \mathfrak{t}_M$ for all t , we have by (4.6),

$$\begin{aligned} \|v^{\Phi_t^E}([e, X + Y])\|^2 &= \|(|\text{ad}(\zeta)|^{-t} \text{ad}(\zeta)Y_{\mathfrak{s}} + \mathfrak{t}_M, [X + Y, \xi])\|^2 \\ &\geq \| |\text{ad}(\zeta)|^{-t} \text{ad}(\zeta)Y_{\mathfrak{s}} \|^2. \end{aligned}$$

If we write $Y = \sum_{\beta \in \Sigma^+} Y_{\beta}$, with $Y_{\beta} \in \mathfrak{g}_{\beta}$, then

$$\| |\text{ad}(\zeta)|^{-t} \text{ad}(\zeta)Y_{\mathfrak{s}} \|^2 = \sum_{\beta \in \Sigma^+} \frac{|\langle \beta, \zeta \rangle|^2}{|\langle \beta, \zeta \rangle|^{2t}} \left\| \frac{1}{2}(Y_{\beta} + \theta Y_{\beta}) \right\|^2 \geq \min_{\beta \in \Sigma^+} \frac{|\langle \beta, \zeta \rangle|^2}{|\langle \beta, \zeta \rangle|^{2t}} \|Y_{\mathfrak{k}}\|^2.$$

As in the proof of Lemma 4.8, we conclude that there is a constant $C_1 > 0$ such that for all X and Y as above,

$$\|v^{\Phi_t^E}([e, X + Y])\| \geq C_1 \|Y\|.$$

Exactly as in the proof of Lemma 4.7, we also find a constant $C_2 > 0$ such that for all such X and Y ,

$$\|v^{\Phi_t^E}([e, X + Y])\| \geq C_2 \|X\|.$$

Hence the vector field $v^{\Phi_t^E}$ vanishes precisely at the set $K \times_{H_M} \{0\}$. So Φ_t^E is taming for all t , and the claim follows. \square

Proof of Proposition 4.1 Successively applying Lemmas 4.2, 4.4, Propositions 3.7 and 4.5, and finally Lemma 4.11, we find that

$$\begin{aligned} \text{index}_K(\bigwedge_J T(G/H) \otimes L_V, \Phi) &= \text{index}_K(\Psi^*(\bigwedge_J T(G/H) \otimes L_V), \Psi^*\Phi) \\ &= \text{index}_K(\bigwedge_{J^E} TE \otimes L_V^E, \Psi^*\Phi) \\ &= \text{index}_K(\bigwedge_{J^E} TE \otimes L_V^E, \Phi_1^0) \\ &= \text{index}_K(\bigwedge_{J^E} TE \otimes L_V^E, \Phi^E). \end{aligned}$$

\square

5 Indices on fibred products

In Sect. 6, we will explicitly compute the right hand side of the equality in Proposition 4.1, see Proposition 6.1. This involves a general result about indices of deformed Dirac operators on certain fibred product spaces, Proposition 5.1. Our goal in this section is to prove this result. In this section and the next, we will need to consider more general deformations of Dirac operators than in (3.1) in some places.

5.1 The result

Let $K' < K$ be a closed subgroup. Let N be a complete Riemannian manifold with an isometric action by K' . Let $M := K \times_{K'} N$. Let $S \rightarrow M$ be a K -equivariant Clifford module. Let $\psi: M \rightarrow \mathfrak{k}$ be a taming map for the action by K on M . Then we have the index

$$\text{index}_K(S, \psi) \in \hat{R}(K).$$

We assume that the Riemannian metric on M is induced by a K' -invariant Riemannian metric on N and a K' -invariant inner product on \mathfrak{k} . As noted below Theorem 3.2, the choice of the Riemannian metric does not influence the index.

A Clifford connection ∇ on a vector bundle $S \rightarrow M$ with a Clifford action c by definition satisfies

$$[\nabla_v, c(w)] = c(\nabla_v^{TM} w),$$

for all vector fields v and w , where ∇^{TM} is the Levi-Civita connection. Let ∇^N be a K' -invariant Hermitian Clifford connection on $S|_N \rightarrow N$. Let D^N be the Dirac operator

$$D^N : \Gamma^\infty(S|_N) \xrightarrow{\nabla^N} \Gamma^\infty(T^*N \otimes S|_N) \xrightarrow{c} \Gamma^\infty(S|_N). \quad (5.1)$$

The vector field v^ψ restricts to a section of $TM|_N$, so we have the endomorphism $c(v^\psi)|_N$ of $S|_N$. For any K' -invariant, nonnegative function $f \in C^\infty(N)^{K'}$, consider the operator

$$D_{f\psi}^N := D^N - ifc(v^\psi)|_N \quad (5.2)$$

on $\Gamma^\infty(S|_N)$.

Proposition 5.1 *If f is admissible, then the operator $D_{f\psi}^N$ is Fredholm on every K' -isotypical component of $L^2(S|_N)$. So it has a well-defined index*

$$\text{index}_{K'}(D_{f\psi}^N) \in \hat{R}(K').$$

Furthermore, for any $\delta \in \hat{K}$, the multiplicities m_δ^\pm of δ in the spaces $(L^2K \otimes \ker_{L^2}(D_{f\psi}^N)^\pm)^{K'}$ are finite, and we have

$$\text{index}_K(S, \psi) = (L^2(K) \otimes \text{index}_{K'}(D_{f\psi}^N))^{K'}.$$

Remark 5.2 Roughly speaking, Proposition 5.1 plays the role in this paper that Theorem 4.1 in [2] plays in [37].

5.2 A connection

To prove Proposition 5.1, we consider the connection ∇^M on S defined by the properties that it is K -invariant, and for all $X \in \mathfrak{h}^\perp \subset \mathfrak{k}$, $n \in N$, $v \in T_n N$ and $s \in \Gamma^\infty(S)$,

$$(\nabla_{X_n^M + v}^M s)[e, n] := (\mathcal{L}_X s)([e, n]) + (\nabla_v^N s|_N)(n). \quad (5.3)$$

Consider the projections

$$\begin{aligned} p_{K/H} : K \times N &\rightarrow K/H; \\ p_N : K \times N &\rightarrow N; \\ p_M : K \times N &\rightarrow M. \end{aligned}$$

Let $\nabla^{T(K/H)}$ and ∇^{TN} be the Levi–Civita connections on $T(K/H)$ and TN , respectively. Then the Levi–Civita connection ∇^{TM} on TM satisfies

$$p_M^* \nabla^{TM} = p_{K/H}^* \nabla^{T(K/H)} + p_N^* \nabla^{TN}. \quad (5.4)$$

(See Lemma 3.4 in [13].)

Lemma 5.3 *The relation (5.3) indeed defines a well-defined, K -invariant, Hermitian Clifford connection ∇^M on \mathcal{S} .*

Proof Every tangent vector in $T_{[e,n]}M$ can be represented in a unique way as $X_n^M + v$, with X and v as above. To show that the K -invariant extension is well-defined, we note that ∇^M as defined above is H -invariant. Indeed, ∇^N is H -invariant by assumption, while for all $h \in H$, $T_n h(X_n^M) = (\text{Ad}(h)X)_n^M$, and

$$\mathcal{L}_{\text{Ad}(h)X} = h\mathcal{L}_X h^{-1}.$$

To verify the Leibniz rule, note that for all $\varphi \in C^\infty(M)$,

$$\begin{aligned} (\mathcal{L}_X \varphi s)([e, n]) + (\nabla_v^N \varphi s|_N)(n) \\ = \varphi(n) \left((\mathcal{L}_X s)([e, n]) + (\nabla_v^N s|_N)(n) \right) + (X^M(f)([e, n]) + v(f)(n))s([e, n]). \end{aligned}$$

The property that ∇^M is Hermitian follows from the facts that ∇^N is, and that the metric on \mathcal{S} is K -invariant.

It remains to show that ∇^M is a Clifford connection. Let $X, Y \in \mathfrak{h}^\perp$, and let v and w be vector fields on N . The space of vector fields on M is isomorphic to the subspace

$$\Gamma^\infty(K \times N, p_N^* TN \oplus p_{K/H}^* T(K/H))^H \subset \Gamma^\infty(K \times N, p_N^* TN \oplus p_{K/H}^* T(K/H)).$$

We work in the larger space on the right hand side, where we have the elements $p_N^* v$ and $p_N^* w$. Connections on vector bundles on M define operators on this larger space via the pullback along p_M . Let $s \in \Gamma^\infty(\mathcal{S})$. Since \mathcal{L}_X commutes with $c(p_N^* w)$ and $p_N^* \nabla_v^N$ commutes with $c(Y^M)$, we have

$$\left(\left[\nabla_{X^M + p_N^* v}^M, c(Y^M + p_N^* w) \right] s \right) |_N = ([\mathcal{L}_X, c(Y^M)]s)|_N + [\nabla_v^N, c(w)](s|_N). \quad (5.5)$$

Now ∇^N is a Clifford connection, so by (5.4), $[\nabla_v^N, c(w)] = c(\nabla_v^{TM} w)$. Furthermore, equivariance of the Clifford action implies that

$$[\mathcal{L}_X, c(Y^M)] = c([X, Y]^M).$$

Now by (5.4), $[X, Y]^M = \nabla_{X^M}^{TM} Y^M$, so we conclude that the right hand side of (5.5) equals

$$\left(c \left(\nabla_{X^M}^{TM} Y^M \right) s \right) |_N + c \left(\nabla_v^{TM} w \right) (s|_N) = \left(c \left(\nabla_{X^M + v}^{TM} (Y^M + p_N^* w) \right) s \right) |_N.$$

Here we used that $\nabla_{X^M}^{TM} w = \nabla_v^{TM} Y^M = 0$. \square

5.3 Estimates for Dirac operators

Let D^M be the Dirac operator on $\Gamma^\infty(S)$ associated to ∇^M . Write

$$D_{f\psi}^M := D^M - ifc(v^\psi),$$

for a nonnegative function $f \in C^\infty(M)^K = C^\infty(N)^H$. Note that

$$S = K \times_H (S|_N), \quad (5.6)$$

via the map $[k, x] \mapsto k \cdot x$ for all $k \in K$ and $x \in S|_N$. So

$$\Gamma^\infty(S) = (C^\infty(K) \otimes \Gamma^\infty(S|_N))^H. \quad (5.7)$$

Let $\{X_1, \dots, X_l\}$ be an orthonormal basis of \mathfrak{h}^\perp . Then, with respect to the decomposition (5.7), we have

$$D^M - 1 \otimes D^N = \sum_{j=1}^l c(X_j^M) \mathcal{L}_{X_j}. \quad (5.8)$$

For $\delta \in \hat{K}$, we denote the δ -isotypical subspace of $L^2(S)$ by $L^2(S)_\delta$.

Lemma 5.4 *For every $\delta \in \hat{K}$,*

- (a) *the operator $D^M - 1 \otimes D^N$ is bounded on $L^2(S)_\delta$ (with norm depending on δ);*
- (b) *there is a constant $C^\delta > 0$ such that for all $f \in C^\infty(M)^K$, we have on $L^2(S)_\delta$,*

$$\begin{aligned} & -C^\delta(1 + f\|v^\psi\|) \\ & \leq D_{f\psi}^M(D^M - 1 \otimes D^N) + (D^M - 1 \otimes D^N)D_{f\psi}^M \\ & \leq C^\delta(1 + f\|v^\psi\|). \end{aligned}$$

Proof Let $\delta \in \hat{K}$. We use (5.8). Note that \mathcal{L}_{X_j} is bounded on $L^2(S)_\delta$, with norm depending on δ . For each j , because $X_j \in \mathfrak{h}^\perp$, and the Riemannian metric is of the form mentioned on page 35, we have

$$\|c(X_j^M)\| = \|X_j^M\| = \|(X_j, 0)\| = \|X_j\|,$$

which is constant. So part (a) follows.

For part (b), we note that K -invariance of $D_{f\psi}^M$ implies that

$$\begin{aligned} D_{f\psi}^M (D^M - 1 \otimes D^N) + (D^M - 1 \otimes D^N) D_{f\psi}^M \\ = \sum_{j=1}^l \left(D_{f\psi}^M c(X_j^M) + c(X_j^M) D_{f\psi}^M \right) \mathcal{L}_{X_j}. \end{aligned} \quad (5.9)$$

Now for every j , \mathcal{L}_{X_j} is bounded on $L^2(\mathcal{S})_\delta$, while

$$D_{f\psi}^M c(X_j^M) + c(X_j^M) D_{f\psi}^M = D^M c(X_j^M) + c(X_j^M) D^M - 2if(v^\psi, X_j^M).$$

By a straightforward computation (see (1.26) in [43] and Lemma 9.2 in [5]), we have for any local orthonormal frame $\{e_1, \dots, e_{\dim(M)}\}$ of TM , and any j ,

$$D^M c(X_j^M) + c(X_j^M) D^M = i \sum_j c(e_j) c \left(\nabla_{e_j}^{TM} X_j^M \right) - 2i(\mathcal{L}_{X_j} + \langle \Phi^S, X_j \rangle),$$

with Φ^S as in (3.2).

Now for every j , we have $\langle \Phi^S, X_j \rangle = 0$ by definition of ∇^M in (5.3). And

$$|(v^\psi, X_j^M)| \leq \|v^\psi\|$$

by definition of the Riemannian metric used. Finally, we claim that $\|\nabla^{TM} X_j^M\|$ is bounded. Indeed, recall the form (5.4) of the Levi-Civita connection on a fibred product space like M . Let $\Phi^{T(K/H)} \in \text{End}(T(K/H)) \otimes \mathfrak{k}^*$ be such that for all $Z \in \mathfrak{k}$,

$$\langle \Phi^{T(K/H)}, Z \rangle = \nabla_{Z+\mathfrak{h}}^{T(K/H)} - \mathcal{L}_Z. \quad (5.10)$$

We have for all $k \in K$ and $n \in N$,

$$X_j^M(kn) = (\text{Ad}(k)X_j, 0) \in \mathfrak{h}^\perp \oplus T_n N \cong T_{kn} M.$$

So for all $w \in TN$, we have

$$\nabla_w^{TM} X_j^M = (p_N^* \nabla^{TN})_w X_j^M = 0.$$

And for all $Z \in \mathfrak{k}$, and $n \in N$,

$$\left(\nabla_{Z^M}^{TM} X_j^M \right) (n) = \left((\mathcal{L}_Z + \langle \Phi^S, Z \rangle) X_j^M \right) (n) = ([Z, X_j] + \langle \Phi^S, Z \rangle) X_j, 0).$$

This is constant in n , so we find that $\|\nabla^{TM} X_j^M\|$ is indeed bounded.

By combining the above arguments, we find that the claim in part (b) is true. \square

5.4 Proof of Proposition 5.1

For $t \in [0, 1]$, consider the operator

$$D_{t,f\psi} := D_{f\psi}^M + t(1 \otimes D^N - D^M).$$

on $\Gamma^\infty(\mathcal{S})$. We view it as an unbounded operator on $L^2(\mathcal{S})$.

Lemma 5.5 *If $f \in C^\infty(M)^K$ is admissible, then for all $t \in [0, 1]$, the operator $D_{t,f\psi}^2$ has discrete spectrum on every K -isotypical subspace of $L^2(\mathcal{S})$.*

Proof We have

$$\begin{aligned} D_{t,f\psi}^2 &= \left(D_{f\psi}^M\right)^2 + t^2(1 \otimes D^N - D^M)^2 \\ &\quad - t \left(D_{f\psi}^M(D^M - 1 \otimes D^N) + (D^M - 1 \otimes D^N)D_{f\psi}^M\right). \end{aligned}$$

Let $\delta \in \hat{K}$. Braverman shows in the proof of Theorem 2.9, on page 22 of [5], there is a constant $C_1^\delta > 0$ such that for all $f \in C^\infty(M)^K$, we have on $L^2(\mathcal{S})_\delta$,

$$(D_{f\psi}^M)^2 \geq (D^M)^2 + f^2\|v^\psi\|^2 - C_1^\delta(\|df\|\|v^\psi\| + fh),$$

with h as in (3.3). Using Lemma 5.4, we let C_2^δ be the norm of $D^M - 1 \otimes D^N$ on $L^2(\mathcal{S})_\delta$, and C_3^δ as the constant C^δ in part (b) of that lemma. Then we find that on $L^2(\mathcal{S})_\delta$, all $f \in C^\infty(M)^K$, and all $t \in [0, 1]$,

$$\begin{aligned} D_{t,f\psi}^2 &\geq (D^M)^2 + f^2\|v^\psi\|^2 - C_1^\delta(\|df\|\|v^\psi\| + fh) - C_2^\delta - C_3^\delta(1 + f\|v^\psi\|) \\ &\geq (D^M)^2 + f^2\|v^\psi\|^2 - C^\delta(\|df\|\|v^\psi\| + fh + 1), \end{aligned}$$

for $C^\delta := C_1^\delta + C_2^\delta + C_3^\delta$. If f is admissible, then the function

$$f^2\|v^\psi\|^2 - C^\delta(\|df\|\|v^\psi\| + fh + 1)$$

goes to infinity as its argument goes to infinity in M . This implies the claim. \square

Proof of Proposition 5.1 Fix $\delta \in \hat{K}$. Suppose $f \in C^\infty(M)^K$ is admissible. Then by Lemma 5.5, the operator

$$\frac{D_{t,f\psi}}{D_{t,f\psi} + i}$$

defines a Fredholm operator on $L^2(\mathcal{S})_\delta$ for every $t \in [0, 1]$. This path of bounded operators is continuous in the operator norm, because (5.8) is bounded on $L^2(\mathcal{S})_\delta$. So the operators $D_{f\psi}^M = D_{0,f\psi}$ and $D_{1,f\psi}$ on $L^2(\mathcal{S})_\delta$ have finite-dimensional kernels, and the same index.

Because of the form of the isomorphism (5.6) and K -equivariance of ψ and the Clifford action, the operator $c(v^\psi)$ on $\Gamma^\infty(\mathcal{S})$ corresponds to the operator $1 \otimes c(v^\psi)|_N$ on $(C^\infty(K) \otimes \Gamma^\infty(\mathcal{S}|_N))^H$. So $D_{1,f\psi} = 1 \otimes D_{f\psi}^N$. This operator is Fredholm on

$$L^2(\mathcal{S})_\delta \cong \delta \otimes (\delta^* \otimes L^2(\mathcal{S}|_N))^H$$

and equals $1_\delta \otimes 1_{\delta^*} \otimes D_{f\psi}^N$ on this space. Now

$$(\delta^* \otimes L^2(\mathcal{S}|_N))^H = \bigoplus_{\delta' \in \hat{H}} [\delta|_H : \delta'] L^2(\mathcal{S}|_N)_{\delta'}.$$

So the operator $D_{f\psi}^N$ is Fredholm on $L^2(\mathcal{S}|_N)_{\delta'}$, and therefore has a well-defined equivariant index in $\hat{R}(H)$. Furthermore,

$$\text{index}_K(D_{1,f\psi}) = \left(L^2(K) \otimes \text{index}_H(D_{f\psi}^N) \right)^H.$$

□

Remark 5.6 A variation on this proof of Proposition 5.1 is to note that Lemma 5.5 implies that the operator $D_{t,f\psi}$ defines a class in the K -homology of the group C^* -algebra C^*K , for all $t \in [0, 1]$.

If N is compact (so we may take $f = 0$), then Proposition 5.1 is a consequence of homotopy invariance of the index of transversally elliptic operators.

6 Computing the linearised index

In this section, we prove an explicit expression for the right hand side of the equality in Proposition 4.1.

Let $R_n^+ \subset R_M^+$ be the set of noncompact positive roots of $(\mathfrak{m}^\mathbb{C}, \mathfrak{t}_M^\mathbb{C})$. Then, as complex vector spaces, we have

$$\mathfrak{s}_M = \bigoplus_{\alpha \in R_n^+} \mathbb{C}_\alpha.$$

Define

$$\left(\bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M \right)^{-1} := \bigotimes_{\alpha \in R_n^+} \bigoplus_{n=0}^{\infty} \mathbb{C}_{n\alpha} \in \hat{R}(H_M).$$

This notation is motivated by the equality

$$\left(\bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M \right)^{-1} \otimes \bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M = \mathbb{C},$$

the trivial representation of H_M . Let ρ_n^M be half the sum of the roots in R_n^+ .

Proposition 6.1 *We have*

$$\begin{aligned} \text{index}_K \left(\bigwedge_{J^E} TE \otimes L_V^E, \Phi^E \right) \\ = (-1)^{\dim(M/K_M)/2} \left(L^2(K) \otimes \mathbb{C}_{2\rho_n^M} \otimes \left(\bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M \right)^{-1} \otimes \bigwedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M \otimes V \right)^{H_M}. \end{aligned}$$

In Sect. 7, we will use this proposition to complete the proof of Theorem 3.11.

6.1 A constant admissible function

We will apply Proposition 5.1 and then use a further decomposition of the resulting index to prove Proposition 6.1. This decomposition is possible because the constant function 1 turns out to be admissible in our setting. We prove this in the current subsection.

Let W_1 and W_2 be finite-dimensional representation spaces of a group H . Consider the H -equivariant vector bundle

$$W_1 \times W_2 \rightarrow W_1.$$

Let ∇^{W_2} be the connection on this bundle defined by

$$\nabla_v^{W_2} s = Ts(v)$$

for all vector fields v on W_1 and all $s \in C^\infty(W_1, W_2)$. Here Ts denotes the derivative of s . Explicitly, at a vector $w_1 \in W_1$, we have

$$(Ts(v))(w_1) = T_{w_1}s(v(w_1)) \in T_{s(w_1)}W_2 = W_2.$$

By a straightforward computation, this connection is H -invariant.

Now we take

$$\begin{aligned} W_1 &= \mathfrak{s}_M \oplus \mathfrak{n}; \\ W_2 &= \bigwedge_{J^E} T_{[e,0]}E \otimes V, \end{aligned}$$

and consider the H -invariant connection $\nabla^{\bigwedge_{J^E} T_{[e,0]}E \otimes V}$ on

$$(\mathfrak{s}_M \oplus \mathfrak{n}) \times \bigwedge_{J^E} T_{[e,0]}E \otimes V = \left(\bigwedge_{J^E} TE \otimes L_V^E \right) |_{\mathfrak{s}_M \oplus \mathfrak{n}} \rightarrow \mathfrak{s}_M \oplus \mathfrak{n},$$

defined as above. (We identify $\mathfrak{s}_M \oplus \mathfrak{n}$ with $\{[e, X + Y]; X \in \mathfrak{s}_M, Y \in \mathfrak{n}\} \subset E$.) Let ∇^E be the connection on $\bigwedge_{J^E} TE \otimes L_V^E$ induced by $\nabla^{\bigwedge_{J^E} T_{[e,0]}E \otimes V}$ as in (5.3).

Proposition 6.2 *The constant function 1 on E is admissible for the connection ∇^E and the taming map Φ^E .*

The claim is that the function

$$\frac{\|v^{\Phi^E}\|^2}{\|\Phi^E\| + \|v^{\Phi^E}\| + \|\nabla^{TE} v^{\Phi^E}\| + \|\langle \Phi \wedge_{J^E} TE \otimes L_V^E, \Phi^E \rangle\| + 1} \quad (6.1)$$

goes to infinity as its argument goes to infinity in M . Here $\Phi \wedge_{J^E} TE \otimes L_V^E$ is as in (3.2). By (4.6) and the definition of Φ^E , we have for all $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$,

$$v^{\Phi^E}([e, X + Y]) = \left(|\operatorname{ad}(\zeta)|^{-1} \operatorname{ad}(\zeta) Y_{\mathfrak{s}} + \mathfrak{t}_M, -[\xi, X + Y] \right). \quad (6.2)$$

Lemma 6.3 *We have*

$$\left\| \left(\nabla^{TE} v^{\Phi^E} \right) ([k, X + Y]) \right\| = \mathcal{O}(\|X + Y\|)$$

as $X + Y \rightarrow \infty$ in $\mathfrak{s}_M \oplus \mathfrak{n}$.

Proof Recall the expression (5.4) for ∇^{TE} , which we now apply with $H = H_M$ and $N = \mathfrak{s}_M \oplus \mathfrak{n}$. Also recall the definition (5.10) of $\Phi^{T(K/H)} \in \operatorname{End}(T(K/H_M)) \otimes \mathfrak{k}^*$. Since v^{Φ^E} is K -invariant, we have for all $Z \in \mathfrak{k}$, and all $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$,

$$\left(p_{K/H_M}^* \nabla_{Z+\mathfrak{t}_M}^{K/H_M} v^{\Phi^E} \right) ([e, X + Y]) = \langle \Phi^{T(K/H)}, Z \rangle v^{\Phi^E}([e, X + Y]).$$

Hence

$$\left\| \left(p_{K/H_M}^* \nabla_{Z+\mathfrak{t}_M}^{K/H_M} v^{\Phi^E} \right) ([e, X + Y]) \right\| = \mathcal{O} \left(\left\| v^{\Phi^E}([e, X + Y]) \right\| \right) = \mathcal{O}(\|X + Y\|)$$

as $X + Y \rightarrow \infty$ in $\mathfrak{s}_M \oplus \mathfrak{n}$.

For $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$, we have

$$\nabla_{X+Y}^{\mathfrak{s}_M \oplus \mathfrak{n}} = X + Y,$$

where on the right hand side, X and Y are viewed as differential operators on $C^\infty(\mathfrak{s}_M \oplus \mathfrak{n})$. Since (6.2) is linear in X and Y , this implies that $(p_{\mathfrak{s}_M \oplus \mathfrak{n}}^* \nabla^{\mathfrak{s}_M \oplus \mathfrak{n}} v^{\Phi^E})([e, X + Y])$ is linear in X and Y . \square

Lemma 6.4 *The function $\|\langle \Phi \wedge_{J^E} TE \otimes L_V^E, \Phi^E \rangle\|$ is constant.*

Proof If $Z \in \mathfrak{t}_M$, then for all $s \in \Gamma^\infty(\wedge_{J^E} TE \otimes L_V^E)$,

$$\left(\left(\nabla_{Z^E}^{\wedge_{J^E} TE \otimes L_V^E} - \mathcal{L}_Z \right) s \right) |_{\mathfrak{s}_M \oplus \mathfrak{n}} = \nabla_{Z^{\mathfrak{s}_M \oplus \mathfrak{n}}} (s|_{\mathfrak{s}_M \oplus \mathfrak{n}}) - \mathcal{L}_Z (s|_{\mathfrak{s}_M \oplus \mathfrak{n}}).$$

Now for $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$, we have

$$\begin{aligned} \mathcal{L}_Z s([e, X + Y]) &= \frac{d}{dt} \bigg|_{t=0} \exp(tZ) s([e, \exp(-tZ)(X + Y)]) \\ &= \operatorname{ad}(Z)(s([e, X + Y])) + (T(s|_{\mathfrak{s}_M \oplus \mathfrak{n}})(Z^{\mathfrak{s}_M \oplus \mathfrak{n}}))(X + Y). \end{aligned}$$

So, for $Z \in \mathfrak{t}_M$,

$$\langle \Phi \wedge_{JE} TE \otimes L_V^E, Z \rangle = \text{ad}(Z).$$

If $Z \in \mathfrak{t}_M^\perp$, then the left hand side of this equality equals 0. We conclude that for all $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$,

$$\langle \Phi \wedge_{JE} TE \otimes L_V^E, \Phi^E \rangle([e, X + Y]) = \text{ad}(\xi),$$

which is constant in X and Y . \square

Proof Proposition 6.2 By Lemmas 4.7 and 4.8, there is a constant $C > 0$ such that for all $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$,

$$\|v^{\Phi_1^0}([e, X + Y])\| \geq C(\|X\| + \|Y\|).$$

The proofs of these lemmas also directly show that this estimate holds with Φ_1^0 replaced by Φ^E . Furthermore, we have

$$\|\Phi^E([e, X + Y])\| = \mathcal{O}(\|X + Y\|)$$

as $X + Y \rightarrow \infty$ in $\mathfrak{s}_M \oplus \mathfrak{n}$. Together with Lemmas 6.3 and 6.4, these estimates imply that (6.1) goes to infinity as its argument goes to infinity. \square

6.2 Indices on vector spaces

The starting point of the proof of Proposition 6.1 is an application of Proposition 5.1. Note that \mathfrak{s}_M is a complex subspace of $\mathfrak{m}/\mathfrak{t}_M$ with respect to the complex structure $J_{\mathfrak{m}/\mathfrak{t}_M}$; let $J_{\mathfrak{s}_M}$ be the restriction of $J_{\mathfrak{m}/\mathfrak{t}_M}$ to \mathfrak{s}_M . Consider the trivial vector bundle

$$\mathfrak{s}_M \times \bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M \rightarrow \mathfrak{s}_M.$$

We view this bundle as a nontrivial H_M -vector bundle. The constant map $\Phi_{\mathfrak{s}_M} : \mathfrak{s}_M \rightarrow \mathfrak{t}_M$ with value ξ is a taming map for the action by H_M on \mathfrak{s}_M . Therefore, we have the equivariant index

$$\text{index}_{H_M}(\mathfrak{s}_M \times \bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M, \Phi_{\mathfrak{s}_M}) \in \hat{R}(H_M).$$

Next, consider the trivial vector bundle

$$\mathfrak{n} \times \bigwedge_{J_\zeta} (\mathfrak{n}^- \oplus \mathfrak{n}^+) \rightarrow \mathfrak{n}.$$

(Recall that $\mathfrak{n} = \mathfrak{n}^+$; we use the notation \mathfrak{n}^+ where this space appears in the direct sum with \mathfrak{n}^- .) This bundle is also a nontrivial H_M -vector bundle. We write $c \circ J_\zeta$ for the endomorphism of this bundle give by

$$c \circ J_\zeta(Y) = c(J_\zeta Y),$$

for $Y \in \mathfrak{n}$. Here c denotes the Clifford action by $\mathfrak{n}^- \oplus \mathfrak{n}^+$ on $\bigwedge_{J_\zeta}(\mathfrak{n}^- \oplus \mathfrak{n}^+)$ as in Example 3.1. Note that for $Y \in \mathfrak{n}$, the vector $J_\zeta Y \in \mathfrak{n}^-$ does not lie in the tangent space $T_Y \mathfrak{n}$. Hence $Y \mapsto J_\zeta Y$ is not a vector field on \mathfrak{n} , so the endomorphism $c \circ J_\zeta$ is not of the type of the deformation term in (3.1). Let $\{Y_1, \dots, Y_s\}$ be an orthonormal basis of \mathfrak{n} . Consider the Dirac operator

$$D^\mathfrak{n} := \sum_{j=1}^s Y_j \otimes c(Y_j)$$

on

$$\Gamma^\infty(\mathfrak{n} \times \bigwedge_{J_\zeta}(\mathfrak{n}^- \oplus \mathfrak{n}^+)) = C^\infty(\mathfrak{n}) \otimes \bigwedge_{J_\zeta}(\mathfrak{n}^- \oplus \mathfrak{n}^+).$$

The operator

$$D^\mathfrak{n} - ic \circ J_\zeta \tag{6.3}$$

is not of the form (3.1), but we will see in Lemma 6.7 that it has a well-defined H_M -equivariant index (which is actually just the trivial representation of H_M).

Proposition 6.5 *The L^2 -kernel of the operator (6.3) is finite-dimensional, and*

$$\begin{aligned} & \text{index}_K \left(\bigwedge_{J^E} T E \otimes L_V^E, \Phi^E \right) \\ &= \left(L^2(K) \otimes \text{index}_{H_M}(\mathfrak{s}_M \times \bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M, \Phi_{\mathfrak{s}_M}) \otimes \text{index}_{H_M}(D^\mathfrak{n} - ic \circ J_\zeta) \right. \\ & \quad \left. \otimes \bigwedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M \otimes V \right)^{H_M}, \end{aligned}$$

where $J_{\mathfrak{k}_M/\mathfrak{t}_M}$ is the restriction of $J_{\mathfrak{m}/\mathfrak{t}_M}$ to the complex subspace $\mathfrak{k}_M/\mathfrak{t}_M \subset \mathfrak{m}/\mathfrak{t}_M$.

We will prove this proposition in the rest of this section. But first, we show how it allows us to prove Proposition 6.1.

Lemma 6.6 *We have*

$$\text{index}_{H_M}(\mathfrak{s}_M \times \bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M, \Phi_{\mathfrak{s}_M}) = (-1)^{\dim(M/K_M)/2} \mathbb{C}_{2\rho_n^M} \otimes \left(\bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M \right)^{-1},$$

where ρ_n^M is half the sum of the elements of R_n^+ .

Proof See the bottom of page 841 in [37], and also Lemma 5.7 in [36]; these in turn are based on Proposition 6.2 in [2]. In the notation of those references, we use the positive expansion of the inverse, because $(\xi, \alpha) > 0$ for all $\alpha \in R_n^+$. \square

In Sect. 6.4, we will compute the index on \mathfrak{n} that occurs in Proposition 6.5, and reach the following conclusion.

Lemma 6.7 *We have*

$$\text{index}_{H_M}(D^n - ic \circ J_\zeta) = \mathbb{C},$$

the trivial representation of H_M .

Combining Proposition 6.5 and Lemmas 6.6 and 6.7, we conclude that Proposition 6.1 is true.

6.3 Decomposing the index on E

Let us prove Proposition 6.5. In Proposition 5.1, we take $H = H_M$, $N = \mathfrak{s}_M \oplus \mathfrak{n}$, $S = \bigwedge_{J^E} TE$, and $\psi = \Phi^E$. By Proposition 6.2, we may take $f = 1$ in Proposition 5.1. That proposition then implies that

$$\text{index}_K \left(\bigwedge_{J^E} TE \otimes L_V^E, \Phi^E \right) = \left(L^2(K) \otimes \text{index}_{H_M}(D_{\Phi^E}^{\mathfrak{s}_M \oplus \mathfrak{n}}) \right)^{H_M}, \quad (6.4)$$

with $D_{f\Phi^E}^{\mathfrak{s}_M \oplus \mathfrak{n}}$ as in (5.2), defined with respect to the connection $\nabla \bigwedge_{J^E} T_{[e,0]}E \otimes V$ as in Sect. 6.1. The operator $D_{\Phi^E}^{\mathfrak{s}_M \oplus \mathfrak{n}}$ acts on sections of the vector bundle

$$\begin{aligned} & \left(\bigwedge_{J^E} TE \otimes L_V^E \right) |_{\mathfrak{s}_M \oplus \mathfrak{n}} \\ &= (\mathfrak{s}_M \oplus \mathfrak{n}) \times \left(\bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M \otimes \bigwedge_{J_\zeta} (\mathfrak{n}^- \oplus \mathfrak{n}^+) \otimes \bigwedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M \right) \otimes V \\ &= \left(\mathfrak{s}_M \times \bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M \right) \boxtimes \left(\mathfrak{n} \times \bigwedge_{J_\zeta} (\mathfrak{n}^- \oplus \mathfrak{n}^+) \right) \otimes \bigwedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M \otimes V. \end{aligned}$$

Here \boxtimes denotes the exterior tensor product of vector bundles, we use graded tensor products everywhere, and, as before, we identify

$$\mathfrak{s}_M \oplus \mathfrak{n} \cong \{[e, X + Y]; X \in \mathfrak{s}_M, Y \in \mathfrak{n}\} \subset E.$$

In terms of this decomposition, (6.2) implies that for all $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$,

$$\begin{aligned} & c(v^{\Phi^E})(X + Y) \\ &= c(-[\xi, X]) \otimes 1_{\bigwedge_{J_\zeta} (\mathfrak{n}^- \oplus \mathfrak{n}^+)} \otimes 1_{\bigwedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M} \otimes 1_V \\ &+ 1_{\bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M} \otimes \left(c(-[\xi, Y]) + c(|\text{ad}(\zeta)|^{-1} \text{ad}(\zeta)Y_{\mathfrak{s}}) \right) \otimes 1_{\bigwedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M} \otimes 1_V. \end{aligned} \quad (6.5)$$

Here we used the fact that the element

$$|\text{ad}(\zeta)|^{-1} \text{ad}(\zeta)Y_{\mathfrak{s}} \in \mathfrak{t}_M^\perp$$

induces a term in $\mathfrak{k}/\mathfrak{t}_M$ orthogonal to \mathfrak{k}_M , hence in the space $\mathfrak{k}/\mathfrak{k}_M$ that is identified with \mathfrak{n}^- . So the Clifford action by this vector only acts nontrivially on $\bigwedge_{J_\zeta}(\mathfrak{n}^- \oplus \mathfrak{n}^+)$, and trivially on $\bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M$.

Define the map $v^n: \mathfrak{n} \rightarrow \mathfrak{n}^- \oplus \mathfrak{n}^+$ by

$$v^n(Y) = -[\xi, Y] + |\mathrm{ad}(\zeta)|^{-1} \mathrm{ad}(\zeta)Y_{\mathfrak{s}},$$

for $Y \in \mathfrak{n}$. Then (6.5) becomes

$$\begin{aligned} c(v^{\Phi^E})(X + Y) \\ = c(v^{\Phi_{\mathfrak{s}_M}}(X)) \otimes 1_{\bigwedge_{J_\zeta}(\mathfrak{n}^- \oplus \mathfrak{n}^+)} \otimes 1_{\bigwedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M} \otimes 1_V \\ + 1_{\bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M} \otimes c(v^n(Y)) \otimes 1_{\bigwedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M} \otimes 1_V. \end{aligned}$$

(Recall that $\Phi_{\mathfrak{s}_M}$ is the constant map with value ξ .)

Let $\{X_1, \dots, X_r\}$ be an orthonormal basis of \mathfrak{s}_M . Let $D^{\mathfrak{s}_M}$ be the Dirac operator

$$D^{\mathfrak{s}_M} := \sum_{j=1}^r X_j \otimes c(X_j)$$

on

$$\Gamma^\infty(\mathfrak{s}_M \times \bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M) = C^\infty(\mathfrak{s}_M) \otimes \bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M.$$

Then for the choice of the connection on $(\bigwedge_{J^E} TE \otimes L_V^E)|_{\mathfrak{s}_M \oplus \mathfrak{n}}$ that we made at the start of Sect. 6.1, the deformed Dirac operator $D_{\Phi^E}^{\mathfrak{s}_M \oplus \mathfrak{n}}$ as in (5.2) equals

$$\begin{aligned} D_{\Phi^E}^{\mathfrak{s}_M \oplus \mathfrak{n}} \\ = (D^{\mathfrak{s}_M} - ic(v^{\Phi_{\mathfrak{s}_M}})) \otimes 1_{\bigwedge_{J_\zeta}(\mathfrak{n}^- \oplus \mathfrak{n}^+)} \otimes 1_{\bigwedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M} \otimes 1_V \\ + 1_{\bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M} \otimes (D^n - ic(v^n)) \otimes 1_{\bigwedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M} \otimes 1_V. \end{aligned} \quad (6.6)$$

Lemma 6.8 *The index $\mathrm{index}_{H_M}(D^n - ic(v^n))$ is well-defined, so is its tensor product with $\mathrm{index}_{H_M}(\mathfrak{s}_M \times \bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M, \Phi_{\mathfrak{s}_M})$, and we have*

$$\begin{aligned} \mathrm{index}_{H_M}(D_{\Phi^E}^{\mathfrak{s}_M \oplus \mathfrak{n}}) \\ = \mathrm{index}_{H_M}(D^n - ic(v^n)) \otimes \mathrm{index}_{H_M}(\mathfrak{s}_M \times \bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M, \Phi_{\mathfrak{s}_M}) \otimes \bigwedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M \otimes V. \end{aligned}$$

Proof Because the tensor products in (6.6) are graded, we have

$$\begin{aligned} \left(D_{\Phi^E}^{\mathfrak{s}_M \oplus \mathfrak{n}}\right)^2 &= (D^{\mathfrak{s}_M} - ic(v^{\Phi_{\mathfrak{s}_M}}))^2 \otimes 1_{\bigwedge_{J_\zeta}(\mathfrak{n}^- \oplus \mathfrak{n}^+)} \otimes 1_{\bigwedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M} \otimes 1_V \\ &+ 1_{\bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M} \otimes (D^n - ic(v^n))^2 \otimes 1_{\bigwedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M} \otimes 1_V. \end{aligned}$$

Since all operators occurring here are nonnegative, this implies that

$$\ker(D_{\Phi^E}^{\mathfrak{s}_M \oplus \mathfrak{n}}) = \ker(D^{\mathfrak{s}_M} - ic(v^{\Phi_{\mathfrak{s}_M}})) \otimes \ker(D^{\mathfrak{n}} - ic(v^{\mathfrak{n}})) \otimes \bigwedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M \otimes V.$$

Here the kernels are L^2 -kernels, and the equality includes gradings. \square

To prove Proposition 6.5, we will show that we may replace $D^{\mathfrak{n}} - ic(v^{\mathfrak{n}})$ by $D^{\mathfrak{n}} - ic \circ J_{\zeta}$ in the above result. Consider the map $\tilde{v}^{\mathfrak{n}}: \mathfrak{n} \rightarrow \mathfrak{n}^- \oplus \mathfrak{n}^+$ given by

$$\tilde{v}^{\mathfrak{n}}(Y) = |\mathrm{ad}(\zeta)|^{-1} \mathrm{ad}(\zeta)Y_{\mathfrak{s}},$$

for $Y \in \mathfrak{n}$.

Lemma 6.9 *Let $Y \in \mathfrak{n}$. Under the identification*

$$T_{[e,Y]}E \cong T_{[e,0]}E = \mathfrak{s}_M \oplus \mathfrak{n}^- \oplus \mathfrak{n}^+ \oplus \mathfrak{k}_M/\mathfrak{t}_M$$

via the map φ_0^Y in (4.3), we have

$$\tilde{v}^{\mathfrak{n}}(Y) = J_{\zeta}Y.$$

Proof Let $Y \in \mathfrak{n}$. Since $\mathrm{ad}(\zeta)$ anticommutes with θ , we have

$$|\mathrm{ad}(\zeta)|^{-1} \mathrm{ad}(\zeta)Y_{\mathfrak{s}} = \frac{1}{2}(1 + \theta)|\mathrm{ad}(\zeta)|^{-1} \mathrm{ad}(\zeta)Y = \frac{1}{2}(1 + \theta)\theta J_{\zeta}Y = \frac{1}{2}(1 + \theta)J_{\zeta}Y.$$

The identification $\mathfrak{n}^- \cong \mathfrak{k}/\mathfrak{k}_M$ is made via the map $\frac{1}{2}(1 + \theta)$, so the claim follows. \square

Proposition 6.10 *The multiplicities of all irreducible representations of H_M in the L^2 -kernels of the operators*

$$D^{\mathfrak{n}} - ic(v^{\mathfrak{n}}) \quad \text{and} \quad D^{\mathfrak{n}} - ic(\tilde{v}^{\mathfrak{n}}) \tag{6.7}$$

are finite, and we have

$$\mathrm{index}_{H_M}(D^{\mathfrak{n}} - ic(v^{\mathfrak{n}})) = \mathrm{index}_{H_M}(D^{\mathfrak{n}} - ic(\tilde{v}^{\mathfrak{n}})) \in \hat{R}(H_M).$$

This proposition will be proved in Sect. 6.5.

Proof of Proposition 6.5 By (6.4) and Lemma 6.8, we have

$$\begin{aligned} & \mathrm{index}_K \left(\bigwedge_{J^E} T E \otimes L_V^E, \Phi^E \right) \\ &= \left(L^2(K) \otimes \mathrm{index}_{H_M}(\mathfrak{s}_M \times \bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M, \Phi_{\mathfrak{s}_M}) \otimes \mathrm{index}_{H_M}(D^{\mathfrak{n}} - ic(v^{\mathfrak{n}})) \right. \\ & \quad \left. \otimes \bigwedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M \otimes V \right)^{H_M}. \end{aligned}$$

By Lemma 6.9, we have $c(\tilde{v}^{\mathfrak{n}}) = c \circ J_{\zeta}$, so the proposition follows from Proposition 6.10. \square

6.4 The index on \mathbb{R}^n

The proof of Lemma 6.7 is a direct computation on a vector space. Let e_0 and e_1 be the generators of the complex exterior algebra $\bigwedge_{\mathbb{C}} \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$ in degrees 0 and 1, respectively. The Clifford action c by \mathbb{C} on $\bigwedge_{\mathbb{C}} \mathbb{C}$ as in Example 3.1 now has the form

$$\begin{aligned} c(z)e_0 &= ze_1; \\ c(z)e_1 &= -\bar{z}e_0, \end{aligned}$$

for $z \in \mathbb{C}$. Let $c \circ i$ be the endomorphism of the trivial bundle $\mathbb{R} \times \bigwedge_{\mathbb{C}} \mathbb{C} \rightarrow \mathbb{R}$ given by

$$(c \circ i)(x) = c(ix)$$

for $x \in \mathbb{R}$. Consider the Dirac operator

$$D^{\mathbb{R}} := c(1) \frac{d}{dx}$$

on $C^\infty(\mathbb{R}, \bigwedge_{\mathbb{C}} \mathbb{C})$.

Lemma 6.11 *The kernel of the operator $D^{\mathbb{R}} - ic \circ i$ intersected with $L^2(\mathbb{R}, \bigwedge_{\mathbb{C}} \mathbb{C})$ is one-dimensional, and spanned in degree zero by the function*

$$x \mapsto e^{-x^2/2}.$$

Proof Let $s \in C^\infty(\mathbb{R}, \bigwedge_{\mathbb{C}} \mathbb{C})$, and write $s = s_0e_0 + s_1e_1$, for (complex-valued) $s_0, s_1 \in C^\infty(\mathbb{R})$. The equation $(D^{\mathbb{R}} - ic \circ i)s = 0$ then becomes

$$\begin{aligned} s'_0 + xs_0 &= 0; \\ s'_1 - xs_1 &= 0. \end{aligned}$$

This is to say that there are complex constants a and b such that for all $x \in \mathbb{R}$,

$$\begin{aligned} s_0(x) &= ae^{-x^2/2}; \\ s_1(x) &= be^{x^2/2}. \end{aligned}$$

□

Lemma 6.11 directly generalises to higher dimensions. Let $n \in \mathbb{N}$, and consider the endomorphism $c \circ i$ of the trivial bundle

$$\mathbb{R}^n \times \bigwedge_{\mathbb{C}} \mathbb{C}^n \rightarrow \mathbb{R}, \quad (6.8)$$

given by $c \circ i(x) = c(ix)$, for $x \in \mathbb{R}^n$. Let $\{v_1, \dots, v_n\}$ be the standard basis of \mathbb{R}^n , with corresponding coordinates (x_1, \dots, x_n) . Consider the Dirac operator

$$D^{\mathbb{R}^n} := \sum_{j=1}^n c(v_j) \frac{\partial}{\partial x_j}$$

on $C^\infty(\mathbb{R}^n, \bigwedge_{\mathbb{C}} \mathbb{C}^n)$. We now also consider an isometric action by a compact Lie group H on \mathbb{R}^n (with the Euclidean metric), and suppose this action lifts to the bundle (6.8) so that c is H -invariant.

Lemma 6.12 *The kernel of the operator $D^{\mathbb{R}^n} - ic \circ i$ intersected with $L^2(\mathbb{R}^n, \bigwedge_{\mathbb{C}} \mathbb{C}^n)$ is one-dimensional in even degree, zero-dimensional in odd degree, and the action by H on this kernel is trivial.*

Proof It follows from Lemma 6.11 that $\ker(D^{\mathbb{R}^n} - ic \circ i) \cap L^2(\mathbb{R}^n, \bigwedge_{\mathbb{C}} \mathbb{C}^n)$ is one-dimensional, spanned in degree zero by the function

$$x \mapsto e^{-\|x\|^2/2}.$$

Since H acts isometrically, this function is H -invariant. \square

Proof of Lemma 6.7 Via a complex-linear isomorphism $\mathfrak{n}^- \oplus \mathfrak{n}^+ \cong \mathbb{C}^s$ mapping \mathfrak{n}^+ to \mathbb{R}^s , the operator $D^n - ic \circ J_\zeta$ corresponds to the operator $D^{\mathbb{R}^n} - ic \circ i$ in Lemma 6.12. That lemma therefore implies that the kernel of the operator $D^n - ic \circ J_\zeta$ intersected with $L^2(\mathfrak{n}, \bigwedge_{J_\zeta} \mathfrak{n}^- \oplus \mathfrak{n}^+)$ is one-dimensional in even degree, zero-dimensional in odd degree, and the action by H_M on this kernel is trivial. \square

Remark 6.13 From a higher viewpoint, Lemma 6.7 is a special case of Bott periodicity. Indeed, the operator $D^n - ic \circ J_\zeta$ is of Callias-type [7, 27]. Therefore, it is Fredholm, and by Proposition 2.18 in [7] or Lemma 3.1 in [27], its index is the Kasparov product of the classes

$$[D^n] \in KK_H(C_0(\mathfrak{n}), \mathbb{C}) \quad \text{and} \\ [ic \circ J_\zeta] \in KK_H(\mathbb{C}, C_0(\mathfrak{n})).$$

The latter class is the Bott generator, and the fact that the index of $D^n - ic \circ J_\zeta$ is the trivial representation of H_M means that $[D^n]$ is the inverse of that class.

6.5 Proof of Proposition 6.10

We cannot directly apply Theorem 3.5 to prove Proposition 6.10, because the operators involved are not of the kind considered in Sect. 3.1. (The deformation terms are not given by Clifford multiplication by vector fields on \mathfrak{n} .) So we need slightly modified arguments.

Let $\|\cdot\|_{\mathfrak{n}}$ be the norm function on \mathfrak{n} .

Lemma 6.14 *We have*

$$(D^{\mathfrak{n}} - ic \circ J_{\zeta})^2 \geq (D^{\mathfrak{n}})^2 + \|\cdot\|_{\mathfrak{n}}^2 - \dim(\mathfrak{n}).$$

Proof By a direct computation, we have

$$(D^{\mathbb{R}} - ic \circ i)^2 = -\frac{d^2}{dx^2} + \begin{pmatrix} x^2 - 1 & 0 \\ 0 & x^2 + 1 \end{pmatrix} \geq -\frac{d^2}{dx^2} + x^2 - 1,$$

with respect to the basis $\{e_0, e_1\}$ of $\bigwedge_{\mathbb{C}} \mathbb{C}$. By factorising $D^{\mathbb{R}^n} - ic \circ i$, we deduce that

$$(D^{\mathbb{R}^n} - ic \circ i)^2 \geq -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} + \|x\|^2 - n.$$

□

For $t \in [0, 1]$, set

$$v^t := tv^{\mathfrak{n}} + (1-t)\tilde{v}^{\mathfrak{n}}.$$

Define the map $\xi^{\mathfrak{n}}: \mathfrak{n} \rightarrow \mathfrak{n}^- \oplus \mathfrak{n}^+$ by

$$\xi^{\mathfrak{n}}(Y) = -[\xi, Y],$$

for $Y \in \mathfrak{n}$.

Lemma 6.15 *We have for all $t \in [0, 1]$, for any orthonormal basis $\{Y_1, \dots, Y_s\}$ of \mathfrak{n} ,*

$$(D^{\mathfrak{n}} - ic(v^t))^2 = (D^{\mathfrak{n}} - ic \circ J_{\zeta})^2 + t^2 \|\xi^{\mathfrak{n}}\|^2 - it \sum_j c(Y_j)c(Y_j(\xi^{\mathfrak{n}})) + 2it\mathcal{L}_{\xi}.$$

Proof By Lemma 6.9, we have for all $Y \in \mathfrak{n}$,

$$v^t(Y) = t\xi^{\mathfrak{n}}(Y) + J_{\zeta}Y.$$

So

$$(D^{\mathfrak{n}} - ic(v^t))^2 = (D^{\mathfrak{n}} - ic \circ J_{\zeta})^2 + t^2 \|\xi^{\mathfrak{n}}\|^2 - it \left((D^{\mathfrak{n}} - ic \circ J_{\zeta})c(\xi^{\mathfrak{n}}) + c(\xi^{\mathfrak{n}})(D^{\mathfrak{n}} - ic \circ J_{\zeta}) \right).$$

Now if $Y \in \mathfrak{g}_{\beta}$, for $\beta \in \Sigma^+$, then $J_{\zeta}Y \in \mathfrak{g}_{-\beta}$, whereas $\xi^{\mathfrak{n}}(Y) \in \mathfrak{g}_{\beta}$. Since the adjoint action by \mathfrak{a} is symmetric, it follows that $J_{\zeta}Y \perp \xi^{\mathfrak{n}}(Y)$. So

$$(c \circ J_{\zeta})c(\xi^{\mathfrak{n}}) + c(\xi^{\mathfrak{n}})(c \circ J_{\zeta}) = 0.$$

Hence

$$(D^n - ic \circ J_\zeta)c(\xi^n) + c(\xi^n)(D^n - ic \circ J_\zeta) = D^n c(\xi^n) + c(\xi^n)D^n.$$

Let $\{Y_1, \dots, Y_s\}$ be an orthonormal basis of \mathfrak{n} . Then by a direct computation,

$$D^n c(\xi^n) + c(\xi^n)D^n = \sum_j c(Y_j)c(Y_j(\xi^n)) - 2\mathcal{L}_\xi.$$

□

Lemma 6.16 *For all $\delta \in \hat{K}$, there is a constant $C_\delta > 0$ such that on $L^2(\mathfrak{n}, \bigwedge_{J_\zeta}(\mathfrak{n}^- \oplus \mathfrak{n}^+))_\delta$, we have for all $t \in [0, 1]$,*

$$(D^n - ic(v^t))^2 \geq (D^n)^2 + \|\cdot\|_{\mathfrak{n}}^2 + t^2 \|\xi^n\|^2 - C_\delta.$$

Proof By Lemmas 6.14 and 6.15, we have for all $t \in [0, 1]$,

$$\begin{aligned} (D^n - ic(v^t))^2 &\geq (D^n)^2 + \|\cdot\|_{\mathfrak{n}}^2 - \dim(\mathfrak{n}) + t^2 \|\xi^n\|^2 \\ &\quad - it \sum_j c(Y_j)c(Y_j(\xi^n)) + 2it\mathcal{L}_\xi. \end{aligned}$$

On isotypical components, the Lie derivative \mathcal{L}_ξ is bounded. And for all j ,

$$c(Y_j)c(Y_j(\xi^n)) \geq -\|Y_j(\xi^n)\|.$$

Since $\xi^n = -\text{ad}(\xi)$ is a linear map from \mathfrak{n} to itself, its derivatives are constant. □

Proof of Proposition 6.10 Fix $\delta \in \hat{K}$. Let C_δ be as in Lemma 6.16. For $t \in \mathbb{R}$, set $D_t := D^n - ic(v^t)$. This operator is essentially self-adjoint by a finite propagation speed argument, see e.g. Proposition 10.2.11 in [10]. Also, it is K -equivariant, so it preserves isotypical components. Hence the operator

$$F_t := \frac{D_t}{D_t + iC_\delta} \in \mathcal{B}(L^2(\mathfrak{n}, \bigwedge_{J_\zeta}(\mathfrak{n}^- \oplus \mathfrak{n}^+))_\delta)$$

is well-defined. This operator is Fredholm for all t by Lemma 6.16. It has the same kernel as D_t , so the claim follows if we prove that the path $t \mapsto F_t$ is continuous in the operator norm.

To prove this, we first show that $c(\xi^n)(D_t + iC_\delta)^{-1}$ is bounded. Let $C_\xi > 0$ be a constant such that

$$\|\xi^n\| \leq C_\xi \|\cdot\|_{\mathfrak{n}}.$$

(Explicitly, we can take $C_\xi = |\langle \alpha, \xi \rangle|$ for the root α of $(\mathfrak{g}^\mathbb{C}, \mathfrak{h}^\mathbb{C})$ for which this number is maximal.) Let $s \in \Gamma_c^\infty(\mathfrak{n}, \bigwedge_{J_\zeta}(\mathfrak{n}^- \oplus \mathfrak{n}^+))_\delta$. Then by Lemma 6.16 and

self-adjointness of D_t ,

$$\|c(\xi^n)s\|_{L^2}^2 \leq C_\xi^2 \| \cdot \|_n s\|_{L^2}^2 \leq C_\xi^2 (\|D_t s\|_{L^2}^2 + C_\delta \|s\|_{L^2}^2) = C_\xi^2 \|(D_t + iC_\delta)s\|_{L^2}^2.$$

In other words,

$$\|c(\xi^n)(D_t + iC_\delta)^{-1}(D_t + iC_\delta)s\|_{L^2}^2 \leq C_\xi^2 \|(D_t + iC_\delta)s\|_{L^2}^2.$$

Since $(D_t + iC_\delta)$ is invertible, every element of $L^2(\mathfrak{n}, \bigwedge_{J_\zeta}(\mathfrak{n}^- \oplus \mathfrak{n}^+))_\delta$ is of the form $(D_t + iC_\delta)s$ for some $s \in (D_t + iC_\delta)s$. So we conclude that $c(\xi^n)(D_t + iC_\delta)^{-1}$ is indeed bounded, with norm at most C_ξ^2 .

Now let $t_1, t_2 \in \mathbb{R}$. Then

$$F_{t_1} - F_{t_2} = C_\delta(t_2 - t_1)(D_{t_1} + iC_\delta)^{-1}c(\xi^n)(D_{t_2} + iC_\delta)^{-1}.$$

By the above argument, the operator

$$(D_{t_1} + iC_\delta)^{-1}c(\xi^n)(D_{t_2} + iC_\delta)^{-1}$$

is bounded uniformly in t_1 and t_2 . We conclude that the path $t \mapsto F_t$ is indeed continuous in the operator norm. \square

7 Rewriting $\pi|_K$

In this section, we rewrite the restricted representation $\pi|_K$ as follows. Let ρ_c^M and ρ_n^M be half the sums of the compact and noncompact roots in R_M^+ , respectively.

Proposition 7.1 *We have*

$$\pi|_K = \left(L^2(K) \otimes \left(\bigwedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M \right)^{-1} \otimes \bigwedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M \otimes \mathbb{C}_{\lambda - \rho_c^M + \rho_n^M} \boxtimes \chi_M \right)^{H_M}.$$

Together with Propositions 6.1 and 4.1, this will allow us to prove Theorem 3.11 in Sect. 7.2.

7.1 Rewriting Blattner's formula

In this section only, suppose that G is a Lie group satisfying the assumptions made in Sect. 2.1. In particular, we suppose that G has discrete series and limits of discrete series representations.

Paradan gave the following reformulation of Blattner's formula.

Lemma 7.2 (Paradan) *Let $\pi_{\lambda, R_G^+}^{G_0}$ be a discrete series or limit of discrete series representation of the connected group G_0 , with parameters (λ, R_G^+) as in Sect. 2.1. Then*

$$\pi_{\lambda, R_G^+}^{G_0}|_{K_0} = \left(L^2(K_0) \otimes (\bigwedge_{J_s} \mathfrak{s})^{-1} \otimes \bigwedge_{J_{\mathfrak{k}/\mathfrak{t}}} \mathfrak{k}/\mathfrak{t} \otimes \mathbb{C}_{\lambda - \rho_c + \rho_n} \right)^T.$$

For discrete series representations, this is Lemma 5.4 in [37]. The proof given there extends directly to limits of discrete series, because Blattner's formula applies to those representations as well. (See the bottom of page 131 and the top of page 132 in [9].)

We will need a generalisation of this result to disconnected groups. Let

$$\pi_{\lambda, R_G^+, \chi}^G = \text{Ind}_{G_0 Z_G}^G (\pi_{\lambda, R_G^+}^{G_0} \boxtimes \chi) \quad (7.1)$$

be a discrete series or limit of discrete series representation of G , as in (2.4).

Proposition 7.3 *We have*

$$\pi_{\lambda, R_G^+, \chi}^G|_K = \left(L^2(K) \otimes (\bigwedge_{J_s} \mathfrak{s})^{-1} \otimes \bigwedge_{J_{\mathfrak{k}/\mathfrak{t}}} \mathfrak{k}/\mathfrak{t} \otimes \mathbb{C}_{\lambda - \rho_c + \rho_n} \boxtimes \chi \right)^{TZ_G}. \quad (7.2)$$

Proof Note that

$$\begin{aligned} \pi_{\lambda, R_G^+, \chi}^G|_K &= \text{Ind}_{G_0 Z_G}^G \left(\pi_{\lambda, R_G^+}^{G_0} \boxtimes \chi \right)|_K \\ &= \text{Ind}_{K_0 Z_G}^K \left(\left(\pi_{\lambda, R_G^+}^{G_0} \boxtimes \chi \right)|_{K_0 Z_G} \right) \\ &= \text{Ind}_{K_0 Z_G}^K \left(\left(\pi_{\lambda, R_G^+}^{G_0} \right)|_{K_0} \boxtimes \chi \right). \end{aligned}$$

Consider the element

$$V := (\bigwedge_{J_s} \mathfrak{s})^{-1} \otimes \bigwedge_{J_{\mathfrak{k}/\mathfrak{t}}} \mathfrak{k}/\mathfrak{t} \otimes \mathbb{C}_{\lambda - \rho_c + \rho_n} \in \hat{R}(T).$$

then by Lemma 7.2, and by Lemma 7.5 below, we have

$$\begin{aligned} \text{Ind}_{K_0 Z_G}^K \left(\left(\pi_{\lambda, R_G^+}^{G_0} \right)|_{K_0} \boxtimes \chi \right) &= \text{Ind}_{K_0 Z_G}^K \left(\text{Ind}_T^{K_0} (V) \boxtimes \chi \right) \\ &= \text{Ind}_{K_0 Z_G}^K \left(\text{Ind}_{TZ_G}^{K_0 Z_G} (V \boxtimes \chi) \right) \\ &= \text{Ind}_{TZ_G}^K (V \boxtimes \chi), \end{aligned}$$

which is the right hand side of (7.2). \square

Remark 7.4 Note that Z_G acts on \mathfrak{s} and $\mathfrak{k}/\mathfrak{t}$ trivially, hence the character $e^{2\rho_n}$ is equal to the trivial character when restricted to $T \cap Z_G$. Therefore $e^{\lambda - \rho_c + \rho_n} = e^{\lambda - \rho}$ when

restricted to $T \cap Z_G$ and the right hand side of (7.2) makes sense given the condition (2.2) on χ .

Lemma 7.5 *For all $V \in \hat{R}(T)$, we have*

$$\mathrm{Ind}_T^{K_0}(V) \boxtimes \chi = \mathrm{Ind}_{TZ_G}^{K_0 Z_G}(V \boxtimes \chi). \quad (7.3)$$

Proof Since $K_0/T \cong K_0 Z_G/T Z_G$, it can be shown that the restrictions of the representations on both sides of (7.3) to K_0 are equal to $\mathrm{Ind}_T^{K_0}(V)$. Since Z_G commutes with all elements in K_0 , the restrictions of the representations on both sides to Z_G are equal to χ . The lemma follows. \square

7.2 Proof of Theorem 3.11

We are now prepared to prove Proposition 7.1 and Theorem 3.11. First, we consider the setting of Proposition 7.1. We have

$$\pi|_K = \mathrm{Ind}_P^G \left(\pi_{\lambda, R_M^+, \chi_M}^M \otimes e^v \otimes 1_N \right) |_K = \left(L^2(K) \otimes \pi_{\lambda, R_M^+, \chi_M}^M \right)^{K_M}.$$

By Proposition 7.3, this equals

$$\begin{aligned} & \left(L^2(K) \otimes \left(L^2(K_M) \otimes (\wedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M)^{-1} \otimes \wedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M \otimes \mathbb{C}_{\lambda - \rho_c^M + \rho_n^M} \boxtimes \chi_M \right)^{T_M Z_M} \right)^{K_M} \\ &= \left(L^2(K) \otimes (\wedge_{J_{\mathfrak{s}_M}} \mathfrak{s}_M)^{-1} \otimes \wedge_{J_{\mathfrak{k}_M/\mathfrak{t}_M}} \mathfrak{k}_M/\mathfrak{t}_M \otimes \mathbb{C}_{\lambda - \rho_c^M + \rho_n^M} \boxtimes \chi_M \right)^{T_M Z_M}. \end{aligned}$$

Proposition 7.1 now follows from (2.5).

Combining Propositions 6.1 and 7.1, we obtain a realisation of $\pi|_K$ as an index on E .

Proposition 7.6 *In Proposition 4.1, if we take $V = \mathbb{C}_{\lambda - \rho^M} \boxtimes \chi_M$, then*

$$\pi|_K = (-1)^{\dim(M/K_M)/2} \mathrm{index}_K \left(\wedge_{J_E} T E \otimes L_V^E, \Phi^E \right).$$

By combining this with Proposition 4.1, we conclude that Theorem 3.11 is true.

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