

A GEOMETRIC FORMULA FOR MULTIPLICITIES OF K -TYPES OF TEMPERED REPRESENTATIONS

PETER HOCHS, YANLI SONG, AND SHILIN YU

ABSTRACT. Let G be a connected, linear, real reductive Lie group with compact centre. Let $K < G$ be compact. Under a condition on K , which holds in particular if K is maximal compact, we give a geometric expression for the multiplicities of the K -types of any tempered representation (in fact, any standard representation) π of G . This expression is in the spirit of Kirillov’s orbit method and the quantisation commutes with reduction principle. It is based on the geometric realisation of $\pi|_K$ obtained in an earlier paper. This expression was obtained for the discrete series by Paradan, and for tempered representations with regular parameters by Duflo and Vergne. We obtain consequences for the support of the multiplicity function, and a criterion for multiplicity-free restrictions that applies to general admissible representations. As examples, we show that admissible representations of $SU(p, 1)$, $SO_0(p, 1)$, and $SO_0(2, 2)$ restrict multiplicity freely to maximal compact subgroups.

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1. INTRODUCTION

1.1. Background and motivation. Let G be a connected, linear, real reductive Lie group with compact centre. Let $K < G$ be a maximal compact subgroup. A tempered representation of G is a unitary irreducible representation whose

Received by the editors May 24, 2018, and, in revised form, January 17, 2019, and March 24, 2019.

2010 *Mathematics Subject Classification.* Primary 58J20; Secondary 22E46, 53D50, 53D20, 53C27.

Key words and phrases. Tempered representation, equivariant index, multiplicity, geometric quantisation, reduction.

The first author was partially supported by the European Union, through Marie Curie fellowship PIOF-GA-2011-299300. He thanks Dartmouth College for funding a visit there in 2016.

The second author was supported by NSF grant 1800667.

The third author was partially supported by NSF grant 1564398.

K -finite matrix coefficients are in $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$. The set $\hat{G}_{\text{temp}}$ of these representations features in the Plancherel decomposition

$$L^2(G) = \int_{\hat{G}_{\text{temp}}}^{\oplus} \pi \otimes \pi^* d\mu(\pi)$$

as a representation of $G \times G$, where μ is the Plancherel measure. Tempered representations are also important because they are used in the Langlands classification [28] of admissible irreducible representations.

The restriction $\pi|_K$ of a tempered representation π to K is determined by the multiplicities of all irreducible representations of K in $\pi|_K$, i.e., the multiplicities of the K -types of π . This restriction contains a good deal of information about π . For example, if π has real infinitesimal character, then Vogan showed that it is determined by its lowest K -type (see [47, Theorem 8.1]).

If π belongs to the discrete series, then Blattner's formula (proved by Hecht and Schmid [12] and later also in [7]) is an explicit combinatorial expression for the multiplicities of the K -types of π . For general tempered representations, there exist algorithms to compute these multiplicities. See, for example, the ATLAS software package¹ and its documentation [1]. This involves representations of disconnected subgroups of G , which cannot be classified via Lie algebra methods. That is one of the reasons why it is a challenge to deduce general properties of multiplicities of K -types of tempered representations from such algorithms. Another reason is the cancellation of terms, which already occurs in Blattner's formula. That can make it hard, for example, to determine which multiplicities are zero.

Paradan [36] gave a geometric expression for the multiplicities of the K -types of discrete series representations π . This was based on a version of the *quantisation commutes with reduction* principle for a certain class of noncompact Spin^c -manifolds, and a geometric realisation of $\pi|_K$ based in turn on Blattner's formula and index theory of transversally elliptic operators. The main result in this paper, Theorem 2.7 is a generalisation of Paradan's result to arbitrary tempered representations. (In fact, it applies more generally to standard representations.) This generalisation is now possible because a quantisation commutes with reduction result for general noncompact Spin^c -manifolds proved recently by the first two authors of this paper [15]. Theorem 2.7 can in fact be generalised to more general compact subgroups $K < G$; see Corollary 2.8. For tempered representations with regular parameters, the multiplicity formula was proved by Duflo and Vergne [9] via very different methods. Our result has applications to multiplicity-free restrictions of admissible representations.

1.2. The main result. In Theorem 2.7, we use a homogeneous space of the form G/H , for a Cartan subgroup $H < G$ (depending on π). This can be identified with a coadjoint orbit $\text{Ad}^*(G)\nu \subset \mathfrak{g}^*$ through a regular element ν (depending on π) of the dual of the Lie algebra $\mathfrak{h}$ of H . (The Lie algebra of a Lie group is denoted by the corresponding lowercase Gothic letter.) First, assume that π is induced from a discrete series representation of a factor M in a cuspidal parabolic subgroup $MAN < G$. Then $G/H \cong \text{Ad}^*(G)\nu$. Consider the map

$$\Phi: G/H \xrightarrow{\cong} \text{Ad}^*(G)\nu \rightarrow \mathfrak{k}^*.$$

¹See <http://www.liegroups.org/software/>.

This is a *moment map* in the sense of symplectic geometry, although we will need to work with the more general Spin^c -geometry. This is also why we use the notation G/H rather than $\text{Ad}^*(G)\nu$.

Let δ be an irreducible representation of K , and let η be its highest weight for a maximal torus $T < K$ and a fixed positive root system for $(\mathfrak{k}, \mathfrak{t})$. Let ρ^K be half the sum of these positive roots. Via Spin^c -quantisation, the representation δ corresponds to the coadjoint orbit $\mathcal{O}_\delta := \text{Ad}^*(K)(\eta + \rho^K)/i \subset \mathfrak{k}^*$; see, for example, [39] (note that $\eta + \rho^K$ is the infinitesimal character of δ). The *reduced space* $(G/H)_{\mathcal{O}_\delta}$ is

$$(G/H)_{\mathcal{O}_\delta} := \Phi^{-1}(\mathcal{O}_\delta)/T.$$

This is a compact space, and if $(\eta + \rho^K)/i$ is a regular value of Φ , then it is an orbifold. In that case, it has a Spin^c -structure, induced by a given K -equivariant Spin^c -structure on G/H (depending on π). The index of the corresponding Spin^c -Dirac operator is denoted by

$$\text{index}((G/H)_{\mathcal{O}_\delta}) \in \mathbb{Z}.$$

This can be computed via Kawasaki's index theorem [17, formula (7)]. If $(\eta + \rho^K)/i$ is a singular value of Φ , then Paradan and Vergne [41] showed how to still define this index in a meaningful way, essentially by replacing $(\eta + \rho^K)/i$ with a nearby regular value; see Subsection 2.1. Our main result, Theorem 2.7 is the following.

Theorem 1.1. *We have*

$$[\pi|_K : \delta] = \pm \text{index}((G/H)_{\mathcal{O}_\delta}).$$

See Section 2 for precise definitions of the sign $\pm$, for the dependence on π of H , ν and the Spin^c -structure on G/H , and for the definition of the index on the right-hand side. In fact, Theorem 1.1 applies more generally to standard representations π ; see Remark 2.9.

If π is not induced from a discrete series representation of M , then its infinitesimal character is singular. In this case, the natural map $G/H \rightarrow \text{Ad}^*(G)\nu$ is a fibre bundle. We then use a different map Φ to define reduced spaces (see Subsection 2.2 for details.). This map depends on choices made, but the end result does not: Theorem 1.1 still holds in this case.

Theorem 1.1 and the results that follow, is in fact true for more general compact subgroups $K < G$: it is sufficient if the map Φ is proper. (This is true if K is maximal compact; see [34, (1.3)].) See Corollary 2.8. Duflo and Vargas showed that in the case of a discrete series representation π , properness of Φ with K replaced by a possibly noncompact, closed, reductive subgroup $H < G$ is equivalent to the restriction of π to H being admissible (i.e., decomposing into irreducibles with finite multiplicities); see [8, Proposition 4].

In the case where π is induced from the discrete series, Duflo and Vergne [9] proved a multiplicity formula for its K -types analogous to Theorem 1.1. The parametrisation part of the orbit method used by Duflo and Vergne to prove their result is the one described in [6, Section III]. The geometric/representation theoretic input is Kirillov's character formula, proved by Rossmann [42]; see also [45]. Our approach to proving Theorem 1.1 is based on the geometric realisation of $\pi|_K$ in [16] and allows us to prove it in general, i.e., even for tempered representations induced from limits of the discrete series. Furthermore, our result has applications to multiplicity-free restrictions of general admissible representations.

Theorem 1.1 allows us to use the geometry of G/H , or of the coadjoint orbit $\text{Ad}^*(G)\nu$, to draw conclusions about the general behaviour of the multiplicities of the K -types of π . One such conclusion is about the support of the multiplicity function of the K -types of π .

Corollary 1.2. *All K -types of π have highest weights in the set*

$$i\Phi(G/H) \cap i\mathfrak{t}^* - \rho^K.$$

In fact, these highest weights lie even in the relative interior of this set; see Corollary 2.11.

Applications of Theorem 1.1 to multiplicity-free restrictions are described in Subsection 1.4.

1.3. The orbit method and quantisation commutes with reduction. Theorem 1.1 is directly related to Kirillov's orbit method and Guillemin and Sternberg's quantisation commutes with reduction principle [11]. Indeed, if a representation π of G is associated to a coadjoint orbit $\mathcal{O}_\pi^G \subset \mathfrak{g}^*$, and an irreducible representation δ of a closed subgroup $G' < G$ is associated to a coadjoint orbit $\mathcal{O}_\delta^{G'} \subset \mathfrak{h}^*$, then according to this principle, one expects that

$$(1.1) \quad [\pi|_{G'} : \delta] = Q((\mathcal{O}_\pi^G \cap p^{-1}(\mathcal{O}_\delta^{G'}))/G'),$$

where $p: \mathfrak{g}^* \rightarrow \mathfrak{h}^*$ is the restriction map and Q denotes some notion of geometric quantisation. In fact, a result of this form by Heckman [13] for compact Lie groups was inspiration for Guillemin and Sternberg to develop the idea that quantisation commutes with reduction. The equality (1.1) is also related to the role that the Corwin–Greenleaf multiplicity function plays in the study of multiplicity-free restrictions (see below).

In the setting of Theorem 1.1, suppose that the infinitesimal character χ of π is a regular element of $i\mathfrak{h}^*$. Then it was shown in [16] that

$$\pi|_K = Q_K(\mathcal{O}_\pi^G),$$

where $\mathcal{O}_\pi^G = \text{Ad}^*(G)\chi$, where Q_K stands for a natural notion of K -equivariant geometric quantisation of noncompact Spin^c -manifolds [15, 30, 36, 37, 46]. If $H = K$ and $\delta \in \hat{K}$ has highest weight η (hence infinitesimal character $\eta + \rho^K$), then, as we mentioned above, $\mathcal{O}_\delta^K = \text{Ad}^*(K)(\eta + \rho^K)$ for a Spin^c -version of geometric quantisation [39]. Then Theorem 1.1 is precisely the equality (1.1), where Q is given by the index of Spin^c -Dirac operators.

We have mentioned Spin^c -quantisation several times so far. Paradan showed in [36] that it is natural to use a Spin^c -version of geometric quantisation to obtain multiplicities of K -types of representations of G , rather than the symplectic version. Paradan and Vergne showed in [41] that the quantisation commutes with reduction principle has a natural extension to the Spin^c -setting. This was generalised to a result for noncompact Spin^c -manifolds in [15] (see Theorem 3.4), which we will use to prove Theorem 1.1.

If the infinitesimal character χ is singular, then the link between Theorem 1.1 and the orbit method is less direct. Rather than using nilpotent coadjoint orbits in that case, we use G/H as a desingularisation, which allows us to still obtain an expression for multiplicities of K -types.

1.4. Multiplicity-free restrictions. The problem of determining when the restriction of an irreducible representation π of G to a closed subgroup G' is multiplicity-free is the subject of active research by a large community of mathematicians. This restriction $\pi|_{G'}$ is called multiplicity-free if the only G' -equivariant endomorphisms of the representation space of π are the scalar multiples of the identity operator. If G' is compact, as it is in our setting, then this precisely means that every irreducible representation has multiplicity 1 in $\pi|_{G'}$. We just mention a few results on multiplicity-free restrictions here that are particularly relevant to our approach. See, for example, [25] and the references given there for more information.

Many results about multiplicity-freeness apply to noncompact simple groups G of Hermitian type. This means that G/K is a Hermitian symmetric space, or equivalently, $\mathfrak{k}$ has nonzero centre. For such groups, π is said to be of scalar type if the $+i$ eigenspace of the action by a fixed central element of $\mathfrak{k}$ on the space of K -finite vectors is one-dimensional. In this setting, Kobayashi proved that π has multiplicity-free restriction to any subgroup G' such that (G, G') is a symmetric pair. See [23], and also [25, Theorem A]. There are many other results on multiplicity-free restrictions; two of many possible references are [24, 43].

Theorem 1.1 implies a geometric sufficient condition for the restriction of π to K to be multiplicity-free: this is the case when $(G/H)_{\mathcal{O}_\delta}$ is a point. In fact, one can then determine explicitly which multiplicities equal 1 and which equal 0.

Corollary 1.3. *If $(\eta + \rho^K)/i$ is a regular value of Φ and $(G/H)_{\mathcal{O}_\delta}$ is a point, then $[\pi|_K : \delta] \in \{0, 1\}$. The condition in Corollary 5.1 determines precisely when the value 0 or 1 is taken.*

If $(\eta + \rho^K)/i$ is not necessarily a regular value of Φ , but $(G/H)_{(\text{Ad}^(K)(\eta + \rho^K + \varepsilon))/i}$ is a point for all ε close enough to 0, then we still have $[\pi|_K : \delta] \in \{0, 1\}$.*

There is in fact a version of Corollary 1.3 for general admissible representations; see Corollary 5.4. By applying this version, we find that the restriction to K of every admissible representation is multiplicity-free in the examples where G is one of the groups

- $\text{SU}(p, 1)$,
- $\text{SO}_0(p, 1)$ or $\text{SO}_0(2, 2)$.

This is worked out in Section 5; see Corollary 5.16. For $\text{SU}(p, 1)$ and $\text{SO}_0(p, 1)$, this was shown by Koornwinder [27]. (In a related result for $\text{SU}(p, 1)$, van Dijk and Hille showed that the tensor product of a holomorphic discrete series representation and the corresponding antiholomorphic discrete series representation decomposes multiplicity freely; see [5, Section 12].) For $G = \text{SL}(2, \mathbb{C})$ and $\text{SL}(2, \mathbb{R})$, all reduced spaces are points so that all tempered representations have multiplicity-free restrictions to K , as is well known. We work out the case $G = \text{SL}(2, \mathbb{R})$ in detail in Subsection 5.2. Then we recover the well-known multiplicities of K -types of the tempered representations of $\text{SL}(2, \mathbb{R})$. For $\text{SL}(2, \mathbb{R})$, we show how Corollary 1.3 does not just imply multiplicity-freeness but allows us to compute precisely which representations occur.

As mentioned above, for many results on multiplicity-free restrictions, the group G is assumed to be of Hermitian type. The groups $\text{SO}_0(p, 1)$ and $\text{SO}_0(2, 2)$ are not of Hermitian type and can therefore not be treated via such results. Furthermore, we do not assume the representation π to be of scalar type.

Links between multiplicity-free restrictions and the orbit method were investigated in [2, 4, 8, 26, 33]. A key role here is played by the Corwin–Greenleaf multiplicity function n . For a closed subgroup $G' < G$ and coadjoint orbits $\mathcal{O}^{G'} \in (\mathfrak{g}')^*/G'$ and $\mathcal{O}^G \in \mathfrak{g}^*/G$, this function takes the value

$$n(\mathcal{O}^G, \mathcal{O}^{G'}) = \#(\mathcal{O}^G \cap p^{-1}(\mathcal{O}^{G'})/G'),$$

where $p: \mathfrak{g}^* \rightarrow \mathfrak{h}^*$ is the restriction map. Corwin and Greenleaf showed that this function gives multiplicities of restrictions of unitary irreducible representations if G is nilpotent (see [4, Theorem 4.8]). Then Kirillov's orbit method classifies unitary irreducible representations as geometric quantisations of coadjoint orbits. In general, if π is associated to $\mathcal{O}^G$, then (1.1) suggests that the restriction $\pi|_{G'}$ should be multiplicity-free if $n(\mathcal{O}^G, \mathcal{O}^{G'}) \leq 1$ for all coadjoint orbits $\mathcal{O}^{G'}$ of G' . A precise conjecture was formulated by Kobayashi and Nasrin [26], and proved for $G' = K$ by Nasrin [33].

We conjecture the condition for multiplicity-free restrictions in Corollary 1.3 to be necessary, as well as sufficient.

Conjecture 1.4. *Let $H < G$ be a θ -stable Cartan subgroup. Suppose that every tempered representation induced from the cuspidal parabolic subgroup corresponding to H restricts multiplicity freely to K . Then all reduced spaces for all maps $\Phi: G/H \rightarrow \mathfrak{k}^*$ corresponding to those representations are points.*

Evidence for this conjecture is given under Conjecture 5.5.

1.5. Ingredients of the proof. The proof of Theorem 1.1 is based on three ingredients.

- (1) A realisation of $\pi|_K$ as a K -equivariant index of a deformed Dirac operator on G/H . This was done in [16, Theorem 3.11]. That result involves index theory of deformed Dirac operators developed by Braverman [3].
- (2) A general quantisation commutes with reduction result for noncompact Spin^c -manifolds. This is [15, Theorem 3.10]. For compact Spin^c -manifolds, this was proved by Paradan and Vergne [39–41]. For noncompact symplectic manifolds, the analogous result was proved by Ma and Zhang [30], after a conjecture by Vergne [46]. See also [37].
- (3) One needs to show that the second ingredient can be applied to the first, by using the freedom one has in the deformation of the Dirac operator on G/H to choose the particular deformation that yields the desired result. This requires some work and occupies a large part of this paper.

Notation 1.5. The Lie algebra of a Lie group is denoted by the corresponding lowercase Gothic letter. We denote complexifications by superscripts $\mathbb{C}$. The unitary dual of a group H will be denoted by $\hat{H}$. If H is an abelian Lie group and $\xi \in \mathfrak{h}^*$ satisfies the appropriate integrality condition, then we write $\mathbb{C}_\xi$ for the one-dimensional representation of H with weight ξ .

In Subsections 2.1, 3.1, and 4.4, the letter M denotes a manifold. In the rest of this paper, it denotes a subgroup of G .

2. THE MULTIPLICITY FORMULA

The main result of this paper is a multiplicity formula for K -types of tempered representations, Theorem 2.7, and its extension, Corollary 2.8. This is a geometric

formula in terms of indices on reduced spaces for the action by a maximal compact subgroup on a homogeneous space of the group in question.

2.1. Indices on reduced spaces. Let M be a complete Riemannian manifold, on which a compact Lie group K acts isometrically. Let J be a K -invariant almost complex structure on M . We write $\bigwedge_J TM$ for the complex exterior algebra bundle of TM , viewed as a complex bundle via J . Let $L \rightarrow M$ be a Hermitian, K -equivariant line bundle. The vector bundle

$$(2.1) \quad \bigwedge_J TM \otimes L \rightarrow M$$

is the spinor bundle of the Spin^c -structure on M defined by J and L ; see, e.g., [10, Proposition D.50] or [29, page 395]. In this paper, we will work only with Spin^c -structures induced by almost complex structures and line bundles, as in this case.

The determinant line bundle associated to the Spin^c -structure with spinor bundle (2.1) is

$$L_{\det} = \bigwedge_J^{\dim(M)/2} TM \otimes L^{\otimes 2} \rightarrow M.$$

Let ∇ be a K -invariant, Hermitian connection on $L_{\det}$. The corresponding moment map is the map $\Phi: M \rightarrow \mathfrak{k}^*$ such that, for all $X \in \mathfrak{k}$,

$$(2.2) \quad 2i\langle \Phi, X \rangle = \mathcal{L}_X - \nabla_{X^M}.$$

Here $\langle \Phi, X \rangle \in C^\infty(M)$ is the pairing of Φ and X , $\mathcal{L}_X$ is the Lie derivative with respect to X of smooth sections of $L_{\det}$, and X^M is the vector field on M induced by X ; our sign convention is that for $m \in M$,

$$X^M(m) = \left. \frac{d}{dt} \right|_{t=0} \exp(-tX)m.$$

The origin of the term “moment map” is that, by Kostant’s formula, Φ is a moment map in the symplectic sense if the curvature of ∇ is $-i$ times a symplectic form on M .

If $\xi \in \mathfrak{k}^*$, then the reduced space at ξ is the space

$$(2.3) \quad M_\xi := \Phi^{-1}(\xi)/K_\xi,$$

where K_ξ is the stabiliser of ξ with respect to the coadjoint action. If ξ is a regular value of Φ , then K_ξ acts on the smooth submanifold $\Phi^{-1}(\xi) \subset M$ with finite stabilisers. Then M_ξ is an orbifold. In our setting, the map Φ will be proper so that M_ξ is compact. We will express multiplicities of K -types of tempered representations as indices of Dirac operators on reduced spaces. For reduced spaces at regular values of the moment map, these are indices in the orbifold sense. For reduced spaces at singular values, one applies a small shift to a nearby regular value; see Definition 2.3 below.

The indices on reduced spaces that we will use were constructed in [41, Subsections 5.1 and 5.2] for general Spin^c -structures. We review this construction here, for Spin^c -structures induced by almost complex structures and line bundles as above. The construction is done in three steps. First, one realises a given reduced space as a reduced space for an action by a torus. For actions by tori, indices on reduced spaces at regular values of the moment map can be defined directly. For singular values, one applies a shift to a nearby regular value.

We suppose from now on that the action by K on M has abelian stabilisers. (This is true in our application of what follows.)

Let $T < K$ be a maximal torus. Fix an open Weyl chamber $C \subset \mathfrak{t}$, and let ρ^K be half the sum of the corresponding positive roots. Let $\xi \in \mathfrak{t}^*$ be dominant with respect to C . Then $\xi + \rho^K/i \in C$. We will always identify $\mathfrak{k} \cong \mathfrak{k}^*$ via the inner product equal to minus the Killing form. Let $Y \subset M$ be a connected component of $\Phi^{-1}(C)$. Consider the map

$$\Phi_Y := \Phi|_Y - \rho^K/i: Y \rightarrow \mathfrak{t}^*.$$

Set

$$Y_\xi := \Phi_Y^{-1}(\xi)/T.$$

Let $q: \Phi_Y^{-1}(\xi) \rightarrow Y_\xi$ be the quotient map. Let $\mathfrak{t}_Y \subset \mathfrak{t}$ be the generic (i.e., minimal) stabiliser of the infinitesimal action by $\mathfrak{t}$ on Y . The image of Φ_Y is contained in an affine subspace $I(Y) \subset \mathfrak{t}^*$ parallel to the annihilator of $\mathfrak{t}_Y$.

Lemma 2.1 (Paradan and Vergne). *If ξ is a regular value of $\Phi_Y: Y \rightarrow I(Y)$, then Y_ξ is an orbifold, and for every integral element $\eta \in I(Y)$, there is an orbifold Spin^c -structure on Y_ξ , with spinor bundle $\mathcal{S}_{Y,\xi}^\eta \rightarrow Y_\xi$ determined by*

$$(\bigwedge_J TM \otimes L)|_{\Phi_Y^{-1}(\xi)} = q^* \mathcal{S}_{Y,\xi}^\eta \otimes_{\mathbb{C}} \bigwedge_{\mathbb{C}} \mathfrak{k}/\mathfrak{t} \otimes_{\mathbb{C}} (\bigwedge_{\mathbb{C}} (\mathfrak{t}/\mathfrak{t}_Y \otimes_{\mathbb{R}} \mathbb{C})) \otimes_{\mathbb{C}} \mathbb{C}_\eta.$$

Here $\mathfrak{k}/\mathfrak{t}$ is viewed as a complex vector space isomorphic to the sum of the positive root spaces corresponding to C , and $\bigwedge_{\mathbb{C}}$ denotes the exterior algebra of complex vector spaces.

This is [41, Lemma 5.2].

Suppose that Φ is a proper map. Then Y_ξ is compact. In the setting of Lemma 2.1, we write

$$\text{index}(\mathcal{S}_{Y,\xi}^\eta) \in \mathbb{Z}$$

for the orbifold index [17] of a Spin^c -Dirac operator on the bundle $\mathcal{S}_{Y,\xi}^\eta$. This can be evaluated in terms of characteristic classes on Y_ξ via Kawasaki's index theorem; see [17, formula (7)].

Theorem 2.2 (Paradan and Vergne). *The integer $\text{index}(\mathcal{S}_{Y,\xi+\varepsilon}^\xi)$ is independent of $\varepsilon \in I(Y)$ such that $\xi + \varepsilon$ is a regular value of $\Phi_Y: Y \rightarrow I(Y)$ for ε small enough.*

This result is [41, Theorem 5.4]. It allows us to define

$$\text{index}(\mathcal{S}_{Y,\xi}^\xi) := \text{index}(\mathcal{S}_{Y,\xi+\varepsilon}^\xi),$$

for ε as in Theorem 2.2.

Finally, we have

$$M_{\xi+\rho^K/i} = \coprod_Y Y_\xi,$$

where Y runs over the connected components of $\Phi^{-1}(C)$.

Definition 2.3. The index of the Spin^c -Dirac operator on the reduced space $M_{\xi+\rho^K/i}$ is the integer

$$\text{index}(M_{\xi+\rho^K/i}) = \sum_Y \text{index}(\mathcal{S}_{Y,\xi}^\xi),$$

where Y runs over the connected components of $\Phi^{-1}(C)$. If $M_{\xi+\rho^K/i} = \emptyset$, then we set $\text{index}(M_{\xi+\rho^K/i}) = 0$.

Such an index on a reduced space may be viewed as the Spin^c -quantisation of that space; see [41, Definition 5.5].

2.2. Tempered representations, almost complex structures, and moment maps. Let G be a connected, linear, real reductive Lie group with compact centre. Let $K < G$ be maximal compact, and let θ be a compatible Cartan involution. A tempered representation of G is an irreducible unitary representation whose K -finite matrix coefficients are in $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$. These are the representations that occur in the Plancherel decomposition of $L^2(G)$. Let π be a tempered representation of G .

Tempered representations were classified by Knapp and Zuckerman. See [20–22] or [18, Chapter XIV] for details, or [16, Subsection 2.3] for a brief overview of the parts relevant to us here. In this classification, one parametrises π as follows. Let $P = MAN$ be the Langlands decomposition of a cuspidal parabolic subgroup $P < G$. Let $H < G$ be the θ -stable Cartan subgroup with noncompact part A . Write $\mathfrak{h} = \mathfrak{t}_M \oplus \mathfrak{a}$, with $\mathfrak{t}_M \subset \mathfrak{m}$. Then $\mathfrak{t}_M$ is a Cartan subalgebra of $\mathfrak{m}$. Set $T_M := \exp(\mathfrak{t}_M)$. Recall that we use minus the Killing form, which we denote by $(-, -)$, to identify $\mathfrak{k}^* \cong \mathfrak{k}$. Let $\lambda \in i\mathfrak{t}_M^*$, and let R_M^+ be a system of positive roots for $(\mathfrak{m}^{\mathbb{C}}, \mathfrak{t}_M^{\mathbb{C}})$ such that, for all $\alpha \in R_M^+$, we have $(\alpha, \lambda) \geq 0$. Let ρ^M be half the sum of the elements of R_M^+ . Suppose that $\lambda - \rho^M$ is integral. Let Z_M be the centre of M . Let χ_M be a one-dimensional representation of Z_M such that

$$\chi_M|_{T_M \cap Z_M} = \mathbb{C}_{\lambda - \rho^M}|_{T_M \cap Z_M}.$$

Then we have the well-defined representation $\mathbb{C}_{\lambda - \rho^M} \boxtimes \chi_M$ of $H_M = T_M Z_M$. One has the discrete series or limit of discrete series representation $\pi_{\lambda, R_M^+, \chi_M}^M$ associated to these data; see [20, page 397]. (For singular λ , $\pi_{\lambda, R_M^+, \chi_M}^M$ is a limit of discrete series representation if it is nonzero.) Let $\nu \in i\mathfrak{a}^*$. For suitable λ , R_M^+ , and χ_M as above, we have

$$(2.4) \quad \pi = \text{Ind}_{MAN}^G(\pi_{\lambda, R_M^+, \chi_M}^M \otimes e^\nu \otimes 1_N).$$

This is [20, Corollary 8.8].

We will use the K -invariant almost complex structure J on G/H defined in [16, Subsection 3.4]. This was defined via the decomposition

$$(2.5) \quad \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m}/\mathfrak{t}_M \oplus \mathfrak{n}^- \oplus \mathfrak{n}^+,$$

where $\mathfrak{n}^+ = \mathfrak{n}$ and $\mathfrak{n}^- = \theta\mathfrak{n}^+$. On $\mathfrak{m}/\mathfrak{t}_M$, we have the complex structure $J_{\mathfrak{m}/\mathfrak{t}_M}$ such that, as complex vector spaces,

$$\mathfrak{m}/\mathfrak{t}_M = \bigoplus_{\alpha \in R_M^+} \mathfrak{m}_\alpha^{\mathbb{C}}.$$

Let Σ be the set of nonzero weights of the adjoint action by $\mathfrak{a}$ on $\mathfrak{g}$. For $\beta \in \Sigma$, let $\mathfrak{g}_\beta \subset \mathfrak{g}$ be the corresponding weight space. Let $\Sigma^+ \subset \Sigma$ be the set of positive weights such that

$$\mathfrak{n} = \bigoplus_{\beta \in \Sigma^+} \mathfrak{g}_\beta.$$

Let $\zeta \in \mathfrak{a}$ be an element for which $\langle \beta, \zeta \rangle > 0$ for all $\beta \in \Sigma^+$. Then the map

$$J_\zeta := \theta| \text{ad}(\zeta)|^{-1} \text{ad}(\zeta) : \mathfrak{n}^- \oplus \mathfrak{n}^+ \rightarrow \mathfrak{n}^- \oplus \mathfrak{n}^+$$

is an H_M -invariant complex structure (see [16, Lemma 3.9]). Let $J_{\mathfrak{g}/\mathfrak{h}}$ be the complex structure on $\mathfrak{g}/\mathfrak{h}$ defined by $J_{\mathfrak{m}/\mathfrak{t}_M}$ and J_ζ via (2.5). Then J is the K -invariant almost complex structure on G/H such that, for all $k \in K$, $X \in \mathfrak{s}_M$, and

$Y \in \mathfrak{n}$, the following diagram commutes:

$$\begin{array}{ccc} T_{k \exp(X) \exp(Y)H} G/H & \xrightarrow{J} & T_{k \exp(X) \exp(Y)H} G/H \\ \uparrow T_{eH} k \exp(X) \exp(Y) & & \uparrow T_{eH} k \exp(X) \exp(Y) \\ T_{eH} G/H = \mathfrak{g}/\mathfrak{h} & \xrightarrow{J_{\mathfrak{g}/\mathfrak{h}}} & \mathfrak{g}/\mathfrak{h} = T_{eH} G/H. \end{array}$$

(See [16, Lemma 3.10].)

Consider the line bundle

$$L_{\lambda-\rho^M, \chi_M} := G \times_H \mathbb{C}_{\lambda-\rho^M} \boxtimes \chi_M \rightarrow G/H$$

(where we extend $\lambda - \rho^M \in i\mathfrak{t}_M^*$ to $\mathfrak{h}$ by setting it equal to zero on $\mathfrak{a}$). The vector bundle

$$(2.6) \quad \bigwedge_J T(G/H) \otimes L_{\lambda-\rho^M, \chi_M} \rightarrow G/H$$

is a spinor bundle of the form (2.1).

The positive systems R_M^+ and Σ^+ determine a set R_G^+ of positive roots α of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ that satisfy

$$(2.7) \quad (\alpha|_{\mathfrak{a}} = 0 \text{ and } \alpha|_{\mathfrak{t}_M} \in R_M^+) \quad \text{or} \quad (\alpha|_{\mathfrak{a}} \neq 0 \text{ and } \alpha|_{\mathfrak{a}} \in \Sigma^+).$$

Let ρ^G be half the sum of the elements of R_G^+ . Set

$$(2.8) \quad \xi := \lambda/i \in \mathfrak{t}_M^*.$$

Proposition 2.4. *There is a K -invariant, Hermitian connection on the determinant line bundle corresponding to (2.6) whose moment map $\Phi: G/H \rightarrow \mathfrak{k}^*$ is given by*

$$\Phi(gH) = (\text{Ad}^*(g)(\xi + \zeta))|_{\mathfrak{k}}.$$

If the infinitesimal character χ of π is regular, then ζ can be chosen such that $\xi + \zeta = \chi$ in Proposition 2.4, see Subsection 2.4. Then Φ is the moment map in the symplectic sense action by K on the coadjoint orbit $\text{Ad}^*(G)\chi$.

2.3. The main result. We initially formulate the multiplicity formula for the K -types of π in two cases separately: the case where π is induced from a discrete series representation of M , and the case where it is induced from a limit of discrete series representation of M ; see Theorems 2.5 and 2.6. Then we combine these statements into the main result of this paper, Theorem 2.7. We also formulate a generalisation, Corollary 2.8, where K is not required to be a maximal compact subgroup.

Note that the π is induced from a discrete series representation of M —i.e., $\pi_{\lambda, R_M^+, \chi_M}^M$ belongs to the discrete series of M —precisely if ξ is regular, whereas π is induced from a limit of discrete series representation of M precisely if ξ is singular.

Fix a set of positive roots of $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ compatible with R_M^+ . Let ρ^K be half the sum of these positive roots. Let $\delta \in \hat{K}$, and let $\eta \in i\mathfrak{t}^*$ be its highest weight. Then δ corresponds to the coadjoint orbit $\mathcal{O}_\delta := \text{Ad}^*(\eta + \rho^K)$ through its infinitesimal character $\eta + \rho^K$ via Spin^c -quantisation [39].

If $\xi \in \mathfrak{t}_M^*$ is regular, it has positive inner products with all roots in R_M^+ . Then we can and will choose ζ such that $\xi + \zeta$ is regular for the roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$. In this case, let

$$(G/H)_{\mathcal{O}_\delta} = \Phi^{-1}(\mathcal{O}_\delta)/K = \Phi^{-1}((\eta + \rho^K)/i)/T$$

be the reduced space at $(\eta + \rho^K)/i$, as in (2.3), for the moment map Φ of Proposition 2.4. Let

$$\text{index}((G/H)_{\mathcal{O}_\delta}) \in \mathbb{Z}$$

be the index of the Spin^c -Dirac operator on this space, as in Definition 2.3. Recall that

$$\pi = \text{Ind}_{MAN}^G(\pi_{\lambda, R_M^+, \chi_M}^M \otimes e^\nu \otimes 1_N).$$

We set $K_M := K \cap M$.

Theorem 2.5 (Multiplicity formula, regular case). *Suppose that π is induced from a discrete series representation of M . For all $\delta \in \hat{K}$ with highest weight η , the multiplicity of δ in $\pi|_K$ is*

$$[\pi|_K : \delta] = (-1)^{\dim(M/K_M)/2} \text{index}((G/H)_{\mathcal{O}_\delta}).$$

If $\xi \in \mathfrak{t}_M^*$ is singular, choose any $\tilde{\xi} \in \mathfrak{t}_M^*$ with positive inner products with the positive roots in R_M^+ , and choose ζ such that $\tilde{\xi} + \zeta$ is regular. Define the map $\psi: G/H \rightarrow \mathfrak{t}^*$ by

$$\psi(gH) = (\text{Ad}^*(g)(\tilde{\xi} + \zeta))|_{\mathfrak{t}}$$

for $g \in G$. Let v^ψ be the vector field on G/H defined by

$$v^\psi(gH) = \left. \frac{d}{dt} \right|_{t=0} \exp(-t\psi(gH))gH$$

for all $g \in G$. Next, we choose a nonnegative function $\tau \in C^\infty(G/H)^K$ that grows fast enough, as in Lemma 4.11. That lemma implies that the map $\Phi^\tau: G/H \rightarrow \mathfrak{t}$ given by

$$(2.9) \quad \langle \Phi^\tau, X \rangle = \langle \Phi, X \rangle + \tau \cdot (v^\psi, X^{G/H}),$$

for $X \in \mathfrak{t}$, is a proper moment map. In this case, we set

$$(G/H)_{\mathcal{O}_\delta} = (\Phi^\tau)^{-1}((\eta + \rho^K)/i)/T.$$

Again, let

$$\text{index}((G/H)_{\mathcal{O}_\delta}) \in \mathbb{Z}$$

be the index of the Spin^c -Dirac operator on this space.

Theorem 2.6 (Multiplicity formula, singular case). *Suppose that π is induced from a limit of discrete series representation of M . For all $\delta \in \hat{K}$ with highest weight η , the multiplicity of δ in $\pi|_K$ is*

$$[\pi|_K : \delta] = (-1)^{\dim(M/K_M)/2} \text{index}((G/H)_{\mathcal{O}_\delta}).$$

Theorem 2.6 in fact also applies if π is induced from a discrete series representation of M so that it generalises Theorem 2.5. Indeed, if ξ is regular, then we may take $\tilde{\xi} = \xi$ and $\tau = 0$. Then $\Phi^\tau = \Phi$ is proper by [34, (1.1)], taming by [34, Proposition 2.1], and trivially homotopic to $\psi = \Phi$. Hence in the regular case, these choices of τ and ψ satisfy the conditions in Lemma 4.11, and the equality in Theorem 2.6 is Theorem 2.5. The combination of these theorems is the main result of this paper.

Theorem 2.7 (Multiplicity formula for K -types of tempered representations). *For any tempered representation π of G , and all $\delta \in \hat{K}$, with highest weight η , the multiplicity of δ in $\pi|_K$ is*

$$[\pi|_K : \delta] = (-1)^{\dim(M/K_M)/2} \text{index}((G/H)_{\mathcal{O}_\delta}).$$

In other words,

$$\pi|_K = (-1)^{\dim(M/K_M)/2} \bigoplus_{\delta \in \hat{K}} \text{index}((G/H)_{\mathcal{O}_\delta}) \delta.$$

If π belongs to the discrete series, then this multiplicity formula is [36, Theorem 2.5]. The absence of the sign $(-1)^{\dim(M/K_M)/2} = (-1)^{\dim(G/K)/2}$ in that result is due to a different definition of reduced spaces and the relevant indices on them (see Remark 2.10). Our proof of Theorem 2.7 is based on a generalisation of the methods in [36], combined with Braverman's index theory described in Subsection 3.1. Via a completely different method, Duflo and Vergne [9] proved Theorem 2.7 in the regular case, where $\pi_{\lambda, R_M^+, \chi_M}^M$ belongs to the discrete series. Duflo and Vergne used Kirillov's character formula, proved by Rossmann [42]; see also [45]. This formula is based on deep results of Harish-Chandra and others. Our approach uses a geometric realisation of $\pi|_K$ instead, and in addition covers the singular case.

Theorem 2.7 can in fact be generalised to more general compact subgroups of G , using a functoriality result by Paradan [38]. Let $K < G$ now be any compact subgroup, not necessarily maximal. Let $K' < G$ be a maximal compact subgroup containing K .

Let Φ^τ be as above for the action by K' on G/H , where we take $\tau = 0$, so $\Phi^\tau = \Phi$, in the regular case. Suppose that the composition

$$(2.10) \quad \Phi_K: G/H \xrightarrow{\Phi^\tau} (\mathfrak{k}')^* \xrightarrow{p} \mathfrak{k}^*$$

is proper, where p is the restriction map. (This is true if $K' = K$ as in Theorem 2.7.) The multiplicity formula by Duflo and Vergne [9] for tempered representation induced from the discrete series holds for restrictions to compact subgroups K with this property. Let T be a maximal torus of K . For an irreducible representation $\delta \in \hat{K}$ with highest weight η , we write $\mathcal{O}_\delta^K := \text{Ad}^*(\eta + \rho^K)$ and

$$(G/H)_{\mathcal{O}_\delta^K}^K := \Phi_K^{-1}(\mathcal{O}_\delta^K)/K = \Phi_K^{-1}(\xi)/K_\xi.$$

This allows us to state the most general multiplicity formula in this paper.

Corollary 2.8. *The restriction of π to K is admissible, and we have*

$$\pi|_K = (-1)^{\dim(M/K'_M)/2} \bigoplus_{\delta \in \hat{K}} \text{index}((G/H)_{\mathcal{O}_\delta}^K) \delta.$$

Remark 2.9. In fact, Corollary 2.8 applies to every representation π of the form (2.4) with $\nu \in (\mathfrak{a}^\mathbb{C})^*$ possibly nonimaginary, i.e., to every *standard representation*. This includes the tempered representations by [20, Corollary 8.8].

Remark 2.10. The sign $(-1)^{\dim(M/K_M)/2}$ in Theorems 2.5–2.7 and Corollary 2.8 is a consequence of an implicit choice of orientations on reduced spaces. We work with almost complex (or Spin^c -structures), and the sign $(-1)^{\dim(M/K_M)/2}$ results from using orientations induced by these structures. The sign is not present if one uses symplectic orientations, but we found it more natural to use the orientations corresponding to the almost complex structures used throughout this paper.

2.4. Coadjoint orbits. Theorem 2.7 and Corollary 2.8 are instances of the Spin^c -version of the quantisation commutes with reduction principle. Indeed, we will see in Theorem 3.5, which is the main result in [16], that one can view $\pi|_K$ as the geometric quantisation in the Spin^c -sense of the action by K on G/H , with the given almost complex structure and line bundle. If the infinitesimal character χ of π is regular, then G/H is isomorphic to the coadjoint orbit $\text{Ad}^*(G)\chi$ as K -equivariant Spin^c -manifolds. Now Φ is the natural projection of this orbit onto $\mathfrak{k}^*$. See [16, Section 3.6] for this relation with the orbit method. This Φ is the moment map for the natural symplectic form on this orbit.

Nevertheless, one needs a Spin^c -version of the quantisation commutes with reduction principle ([15, Theorem 3.10]; see Theorem 3.4 rather than the symplectic version ([30, Theorem 0.1]; see also [37, Theorem 1.4]). This is because the almost complex structure J is not compatible with the Kostant–Kirillov symplectic form on the coadjoint orbit $\text{Ad}^*(G)\chi$; also, $L_{\lambda-\rho^M, \chi_M}$ is not a prequantum line bundle for this symplectic form. See [36, Subsection 1.5]. The bundle $L_{2(\lambda-\rho^M), \chi_M^2}$ is a K -equivariant prequantum line bundle for the coadjoint orbit $\text{Ad}^*(G)2\chi$, however; i.e., the spinor bundle (2.6) is a Spin^c -prequantisation of the orbit $\text{Ad}^*(G)\chi$. See Remark 4.8. In the compact case, the Spin^c -version of the quantisation commutes with reduction principle was proved by Paradan and Vergne [39–41]. Because we view G/H as a Spin^c -manifold (or an almost complex manifold) in this paper rather than as the symplectic manifold $\text{Ad}^*(G)\chi$ that it equals if χ is regular, we use the notation G/H rather than a notation that emphasises the link with coadjoint orbits.

In the orbit method, representations with singular parameters correspond to nilpotent orbits. If χ is singular, then we use the manifold G/H rather than such a nilpotent orbit. Through this desingularisation, the link with quantising nilpotent orbits is absent in our approach, but this approach does allow us to obtain the multiplicity formula in Theorem 2.7 and Corollary 2.8.

Theorem 2.7 and Corollary 2.8 allow us to deduce properties of the behaviour of the K -type multiplicities of π from the geometry of the coadjoint orbit $\text{Ad}^*(G)(\xi+\zeta)$ if ξ is regular. In general, such properties can be deduced from the geometry of the map $\Phi: G/H \rightarrow \mathfrak{k}^*$. An immediate consequence is the following fact about the support of the multiplicity function of the K -types of π . By the relative interior or relative boundary of a subset of the affine space $I(Y)$ parallel to the annihilator of $\mathfrak{t}_Y$ containing the image of Φ_Y , we mean the interior or boundary as a subset of $I(Y)$.

Corollary 2.11. *Let K be as in Corollary 2.8. All K -types of π have highest weights in the relative interior of $i\Phi(G/H) \cap i\mathfrak{k}^* - \rho^K$.*

Proof. Let $\delta \in \hat{K}$ have highest weight η . If $(\eta + \rho^K)/i$ is not in the image of Φ , then Corollary 2.8 implies that the multiplicity $[\pi|_K : \delta]$ is zero because $(G/H)_{(\eta + \rho^K)/i}$ is empty, and so are reduced spaces at elements close enough to $(\eta + \rho^K)/i$. If $(\eta + \rho^K)/i$ lies on the relative boundary of the image of Φ , then $[\pi|_K : \delta]$ is zero because the reduced space at some element close to $(\eta + \rho^K)/i$ is empty. See [41, comment below Definition 5.5]. $\square$

Remark 2.12. In the regular case, the map Φ is a moment map in the symplectic sense. So then the set $i\Phi(G/H) \cap i\mathfrak{k}^* - \rho^K$ containing the support of the multiplicity function is a convex polytope. This polytope is noncompact if G is; i.e., it is the intersection of a collection of half-spaces.

Remark 2.13. Even in the case of the discrete series, it is nontrivial to determine the support of the multiplicity function from Blattner's formula. This is because of cancellations occurring in that formula.

Applications of Theorem 2.7 and Corollary 2.8 to multiplicity-free restrictions are discussed in Section 5.

3. INGREDIENTS OF THE PROOF

3.1. Quantisation commutes with reduction. Consider the setting of Subsection 2.1. Let $\psi: M \rightarrow \mathfrak{k}$ be a smooth, K -equivariant map. It induces a vector field v^ψ , given by

$$v^\psi(m) = \left. \frac{d}{dt} \right|_{t=0} \exp(-t\psi(m))m,$$

for all $m \in M$. The map ψ is called *taming* if the set of zeros of v^ψ is compact.

The Clifford action c by TM on $\bigwedge_J TM$ is given by

$$c(v)x = v \wedge x - v^* \lrcorner x$$

for $m \in M$, $v \in T_m M$, and $x \in \bigwedge_J T_m M$. Here $v^* \in T^*M$ is dual to v with respect to the Hermitian metric defined by the Riemannian metric and J , and $\lrcorner$ denotes contraction. Let $\tilde{\nabla}$ be a K -invariant, Hermitian connection on $\bigwedge_J TM \otimes L$ such that, for all vector fields v, w on M ,

$$[\tilde{\nabla}_v, c(w)] = c(\nabla_v^{TM} w),$$

where ∇^{TM} is the Levi-Civita connection. Such a connection always exists; one is induced by the connections ∇ on $L_{\det}$ and ∇^{TM} on TM ; see, e.g., [29, Proposition D.11]. After we identify $T^*M \cong TM$ via the Riemannian metric, the Clifford action c induces a map

$$c: T^*M \otimes \bigwedge_J TM \otimes L \rightarrow \bigwedge_J TM \otimes L.$$

This allows us to define the *Dirac operator* D as the composition

$$D: \Gamma^\infty(\bigwedge_J TM \otimes L) \xrightarrow{\tilde{\nabla}} \Gamma^\infty(T^*M \otimes \bigwedge_J TM \otimes L) \xrightarrow{c} \Gamma^\infty(\bigwedge_J TM \otimes L).$$

Let $f \in C^\infty(M)^K$ be nonnegative. The *Dirac operator deformed by $f\psi$* is the operator

$$D - ifc(v^\psi)$$

on the space $\Gamma_{L^2}^\infty(\bigwedge_J TM \otimes L)$ of square-integrable smooth sections of $\bigwedge_J TM \otimes L$. For a nonnegative function $\chi \in C^\infty(M)^K$, we say that the function f is χ -admissible if, outside a compact set,

$$\frac{f^2}{\|df\| + f + 1} \geq \chi.$$

For any such function χ , there exist χ -admissible functions; see [14, Lemma 3.10]. Braverman's index theory [3] for deformed Dirac operators is based on the following result.

Theorem 3.1 (Braverman). *If ψ is taming, then there is a nonnegative function $\chi \in C^\infty(M)^K$ such that, for all χ -admissible functions f and all irreducible representations δ of K , the multiplicities m_δ^+ and m_δ^- of δ in the kernel of $D - ifc(v^\psi)$ restricted to even and odd degree forms, respectively, is finite. The difference $m_\delta^+ - m_\delta^-$ is independent of f and ∇ .*

See [3, Theorem 2.9] for a more general result.

We write $\hat{R}(K)$ for the abelian group

$$\hat{R}(K) = \left\{ \bigoplus_{\delta \in \hat{K}} m_{\delta} \delta; m_{\delta} \in \mathbb{Z} \right\}.$$

In other words, $\hat{R}(K)$ contains formal differences of possibly infinite-dimensional representations of K , in which all irreducible representations have finite multiplicities.

Definition 3.2. In the setting of Theorem 3.1, the *equivariant index* of the pair $(\bigwedge_J TM \otimes L, \psi)$ is

$$\text{index}_K(\bigwedge_J TM \otimes L, \psi) = \bigoplus_{\delta \in \hat{K}} (m_{\delta}^{+} - m_{\delta}^{-}) \delta \in \hat{R}(K).$$

A property of this index is invariance under homotopies of taming maps. Two taming maps $\psi_0, \psi_1: M \rightarrow \mathfrak{k}$ are *homotopic* if there is a taming map $\psi: M \times [0, 1] \rightarrow \mathfrak{k}$ such that, for all $m \in M$, we have $\psi(m, t) = \psi_0(m)$ if $t \in [0, 1/3]$, and $\psi(m, t) = \psi_1(m)$ if $t \in]2/3, 1]$.

Theorem 3.3 (Braverman). *If ψ_0 and ψ_1 are homotopic taming maps, then*

$$\text{index}_K(\bigwedge_J TM \otimes L, \psi_0) = \text{index}_K(\bigwedge_J TM \otimes L, \psi_1).$$

This is a special case of cobordism invariance of the index [3, Theorem 3.7].

In [15], it was proved that the index of Definition 3.2 satisfies a Spin^c -version of the quantisation commutes with reduction principle of Guillemin and Sternberg [11]. This followed results for compact symplectic manifolds [31, 32] (see also [35, 44]), for noncompact symplectic manifolds [30] (see also [37]), and for compact Spin^c -manifolds [41] (see also [39, 40]). The interpretation of the K -equivariant index of a Dirac operator deformed by a vector field such as v^{Φ} as a geometric quantisation goes back to [15, 30, 36, 37, 46].

Theorem 3.4. *In the setting of Theorem 3.1, take $\psi = \Phi$, the moment map of a K -invariant, Hermitian connection on $L_{\det}$. Suppose that Φ is taming and proper, and that the generic stabiliser of the action by K on M is abelian. For $\delta \in \hat{K}$, let $\mathcal{O}_{\delta} = \text{Ad}^*(K)(\eta + \rho^K)/i$ be the corresponding regular admissible orbit (where η is the highest weight of δ). Then*

$$m_{\delta}^{+} - m_{\delta}^{-} = \text{index}(M_{\mathcal{O}_{\delta}}).$$

In other words,

$$\text{index}_K(\bigwedge_J TM \otimes L, \Phi) = \bigoplus_{\delta \in \hat{K}} \text{index}(M_{\mathcal{O}_{\delta}}) \delta.$$

This is a special case of [15, Theorem 3.10]. In that theorem, it was not assumed that Φ is taming, that the generic stabiliser is abelian, or that the Spin^c -structure is induced by an almost complex structure and a line bundle.

3.2. A realisation of tempered representations restricted to K . As in Subsection 2.2, let π be a tempered representation of G , and write

$$\pi = \text{Ind}_{MAN}^G(\pi_{\lambda, R_M^+, \chi_M}^M \otimes e^{\nu} \otimes 1_N)$$

as in (2.4). Let H be the corresponding Cartan subgroup. In [16], we realised the restriction of π to K as an equivariant index in the sense of Definition 3.2 of a deformed Dirac operator on G/H . We briefly review the construction here.

Consider the spinor bundle (2.6) and the map Φ of Proposition 2.4 but now for any elements $\xi \in \mathfrak{t}_M^*$ and $\zeta \in \mathfrak{a}^*$ such that $(\alpha, i\xi) > 0$ for all $\alpha \in R_M^+$, and $\xi + \zeta \in \mathfrak{h}^*$ is regular for the roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$. Then the map Φ is taming by [34, Proposition 2.1].

Theorem 3.5. *We have*

$$\pi|_K = (-1)^{\dim(M/K_M)/2} \text{index}_K(\bigwedge_J T(G/H) \otimes L_{\lambda-\rho^M, \chi_M}, \Phi).$$

This is [16, Theorem 3.10]. It is the last ingredient of the proof of Theorem 2.7. Theorem 3.5 in fact applies more generally to every standard representation π ; see [16, Remark 3.12].

Proof of Theorem 2.7. Let $\delta \in \hat{K}$. Let ξ be as in (2.8). First, suppose that ξ is regular, and choose ζ such that $\xi + \zeta \in \mathfrak{h}^*$ is regular for the roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$. Then Theorem 3.5 states that

$$[\pi|_K : \delta] = (-1)^{\dim(M/K_M)/2} [\text{index}_K(\bigwedge_J T(G/H) \otimes L_{\lambda-\rho^M, \chi_M}, \Phi) : \delta].$$

By Proposition 2.4, the map Φ is a moment map for this specific choice of ξ . It is proper by [34, (1.1)] and taming, as we saw above. So Theorem 3.4 implies the claim.

If ξ is singular, let $\psi: G/H \rightarrow \mathfrak{k}^*$ be given by

$$\psi(gH) = \text{Ad}^*(g)(\tilde{\xi} + \zeta)|_{\mathfrak{k}},$$

for $\tilde{\xi} \in \mathfrak{t}_M^*$ such that $(\alpha, i\tilde{\xi}) > 0$ for all $\alpha \in R_M^+$, and $\zeta \in \mathfrak{a}^*$ such that $\xi + \zeta \in \mathfrak{h}^*$ is regular for the roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$. Let $\Phi^\tau: G/H \rightarrow \mathfrak{k}$ be as in Lemma 4.11. Then Φ^τ is a taming, proper moment map, and, by that lemma and Theorem 3.5,

$$\begin{aligned} \text{index}_K(\bigwedge_J T(G/H) \otimes L_{\lambda-\rho^M, \chi_M}, \Phi^\tau) &= \text{index}_K(\bigwedge_J T(G/H) \otimes L_{\lambda-\rho^M, \chi_M}, \psi) \\ &= (-1)^{\dim(M/K_M)/2} \pi|_K. \end{aligned}$$

In the first equality, we used Theorem 3.3. The claim again follows from Theorem 3.4. $\square$

Proof of Corollary 2.8. Applying Theorem 2.7 and [38, Theorem 1.1], we obtain

$$\begin{aligned} \pi|_K &= (-1)^{\dim(M/K'_M)/2} \left(\bigoplus_{\delta' \in \hat{K}'} \text{index}((G/H)_{\mathcal{O}_{\delta'}^{K'}}^{K'} \delta') \right) \Big|_K \\ &= (-1)^{\dim(M/K'_M)/2} \bigoplus_{\delta \in \hat{K}} \text{index}((G/H)_{\mathcal{O}_{\delta}^K}^K \delta). \end{aligned} \quad \square$$

It remains to prove Proposition 2.4 and to show how to handle the singular case; see Lemma 4.11. This is done in the next section.

4. A Spin^c -MOMENT MAP ON G/H

4.1. Spin^c -structures on linearised homogeneous spaces. As an intermediate step in the proof of Proposition 2.4, we will use a K -equivariant partial linearisation of the space G/H that was introduced in [16, Subsection 4.2]. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be the Cartan decomposition defined by θ . Write $\mathfrak{m} = \mathfrak{k}_M \oplus \mathfrak{s}_M$, with $\mathfrak{k}_M \subset \mathfrak{k}$ and $\mathfrak{s}_M \subset \mathfrak{s}$. Let $H_M := H \cap M$. Then H_M may be disconnected, but its Lie algebra is $\mathfrak{t}_M$. Let

$$E := K \times_{H_M} (\mathfrak{s}_M \oplus \mathfrak{n})$$

be the quotient of $K \times (\mathfrak{s}_M \oplus \mathfrak{n})$ be the action by H_M defined by

$$h \cdot (k, X + Y) = (kh^{-1}, \text{Ad}(h)(X + Y))$$

for $h \in H_M$, $k \in K$, $X \in \mathfrak{s}_M$, and $Y \in \mathfrak{n}$. Lemma 4.2 in [16] states that the map $\Psi: E \rightarrow G/H$ defined by

$$\Psi([k, X + Y]) = k \exp(X) \exp(Y) H,$$

for $k \in K$, $X \in \mathfrak{s}_M$, and $Y \in \mathfrak{n}$, is a well-defined, K -equivariant diffeomorphism. In this sense, E is a partial linearisation of G/H .

For every $X \in \mathfrak{s}_M$ and $Y \in \mathfrak{n}$, we have the linear isomorphism

$$(4.1) \quad T_{[e, X+Y]} E \xrightarrow{\cong} \mathfrak{k}/\mathfrak{t}_M \oplus \mathfrak{s}_M \oplus \mathfrak{n} = \mathfrak{g}/\mathfrak{h}$$

defined by

$$(U + \mathfrak{t}_M, V + W) \mapsto \left. \frac{d}{dt} \right|_{t=0} [\exp(tU), X + Y + t(V + W)]$$

for $U \in \mathfrak{k}$, $V \in \mathfrak{s}_M$, and $W \in \mathfrak{n}$. Let J^E be the K -invariant almost complex structure on E corresponding to the complex structure $J_{\mathfrak{g}/\mathfrak{h}}$ (defined in Subsection 2.2) via the isomorphism (4.1). The almost complex structure Ψ^*J on E corresponding to J via Ψ differs from J^E because it corresponds to $J_{\mathfrak{g}/\mathfrak{h}}$ via a different isomorphism (4.1). This is worked out in [16, Subsection 4.3], where it is also shown that Ψ^*J and J^E are K -equivariantly homotopic.

Lemma 4.1. *We have a complex, $\mathfrak{t}_M$ -equivariant isomorphism*

$$\mathfrak{n}^+ \oplus \mathfrak{n}^- = \bigoplus_{\alpha \in R_G, \alpha|_{\mathfrak{a}} \in \Sigma^+} \mathbb{C}_{\alpha|_{\mathfrak{t}_M}}.$$

Proof. Fix $\lambda \in \Sigma^+$. Then

$$\mathfrak{g}_\lambda = \left(\bigoplus_{\alpha \in R_G, \alpha|_{\mathfrak{a}} = \lambda} \mathfrak{g}_\alpha^{\mathbb{C}} \right) \cap \mathfrak{g}.$$

For every $\alpha \in R_G$, we have $\theta \mathfrak{g}_\alpha^{\mathbb{C}} = \mathfrak{g}_{-\bar{\alpha}}^{\mathbb{C}}$. Since $\text{ad}(\zeta)$ preserves $\mathfrak{g}_\alpha^{\mathbb{C}}$, this implies that

$$J_\zeta \mathfrak{g}_\alpha^{\mathbb{C}} = \mathfrak{g}_{-\bar{\alpha}}^{\mathbb{C}}.$$

Note that

$$\mathfrak{g}_{-\lambda} = \left(\bigoplus_{\alpha \in R_G, \alpha|_{\mathfrak{a}} = \lambda} \mathfrak{g}_{-\bar{\alpha}}^{\mathbb{C}} \right) \cap \mathfrak{g}.$$

Here we used the fact that $\alpha|_{\mathfrak{a}}$ takes values in $\mathbb{R}$, so $\alpha|_{\mathfrak{a}} = \bar{\alpha}|_{\mathfrak{a}}$. So

$$\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda} = \left(\bigoplus_{\alpha \in R_G, \alpha|_{\mathfrak{a}} = \lambda} \mathfrak{g}_\alpha^{\mathbb{C}} \oplus \mathfrak{g}_{-\bar{\alpha}}^{\mathbb{C}} \right) \cap \mathfrak{g},$$

where every term on the right-hand side is preserved by J_ζ .

Let us determine the weight of $\mathfrak{t}_M$ on $\mathfrak{g}_\alpha^{\mathbb{C}} \oplus \mathfrak{g}_{-\bar{\alpha}}^{\mathbb{C}}$ for $\alpha \in R_G$. For $Y \in \mathfrak{t}_M$, $X_\alpha \in \mathfrak{g}_\alpha^{\mathbb{C}}$, and $X_{-\bar{\alpha}} \in \mathfrak{g}_{-\bar{\alpha}}^{\mathbb{C}}$, we have

$$\text{ad}(Y)(X_\alpha + X_{-\bar{\alpha}}) = \langle \alpha, Y \rangle X_\alpha - \overline{\langle \alpha, Y \rangle} X_{-\bar{\alpha}}.$$

Since $\langle \alpha, Y \rangle \in i\mathbb{R}$, this equals

$$\langle \alpha, Y \rangle (X_\alpha + X_{-\bar{\alpha}}).$$

So $\mathfrak{t}_M$ acts on $\mathfrak{g}_\alpha^{\mathbb{C}} \oplus \mathfrak{g}_{-\bar{\alpha}}^{\mathbb{C}}$ with weight $\alpha|_{\mathfrak{t}_M}$.

We therefore obtain a complex, $\mathfrak{t}_M$ -equivariant isomorphism

$$\begin{aligned} \mathfrak{n}^+ \oplus \mathfrak{n}^- &= \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda} \\ &\cong \bigoplus_{\lambda \in \Sigma^+} \bigoplus_{\alpha \in R_G, \alpha|_{\mathfrak{a}} = \lambda} \mathbb{C}_{\alpha|_{\mathfrak{t}_M}} \\ &= \bigoplus_{\alpha \in R_G, \alpha|_{\mathfrak{a}} \in \Sigma^+} \mathbb{C}_{\alpha|_{\mathfrak{t}_M}}. \end{aligned}$$

Here we used that the space $(\mathfrak{g}_\alpha^{\mathbb{C}} \oplus \mathfrak{g}_{-\bar{\alpha}}^{\mathbb{C}}) \cap \mathfrak{g}$ is complex one-dimensional and preserved by J_ζ and $\mathfrak{t}_M$. $\square$

Lemma 4.2. *The group T_M acts trivially on the highest exterior power of $\mathfrak{n}^- \oplus \mathfrak{n}^+$.*

Proof. By Lemma 4.1, the group T_M acts on the highest exterior power of $\mathfrak{n}^- \oplus \mathfrak{n}^+$ with infinitesimal weight

$$(4.2) \quad \sum_{\alpha \in R_G, \alpha|_{\mathfrak{a}} \in \Sigma^+} \alpha|_{\mathfrak{t}_M}.$$

If $\alpha \in R_G$ satisfies $\alpha|_{\mathfrak{a}} \in \Sigma^+$, then also $\bar{\alpha} \in R_G$ and $\bar{\alpha}|_{\mathfrak{a}} = \alpha_{\mathfrak{a}} \in \Sigma^+$. And $\bar{\alpha}|_{\mathfrak{t}_M} = -\alpha|_{\mathfrak{t}_M}$. Hence the sum (4.2) consists of pairs of terms $(\alpha|_{\mathfrak{t}_M}, \bar{\alpha}|_{\mathfrak{t}_M})$ that cancel. $\square$

Lemma 4.3. *As a complex representation of T_M , the space $\mathfrak{g}/\mathfrak{h}$ decomposes as*

$$\mathfrak{g}/\mathfrak{h} = \bigoplus_{\alpha \in R_G^+} \mathbb{C}_{\alpha|_{\mathfrak{t}_M}}.$$

Its highest exterior power equals $\mathbb{C}_{2\rho^M}$ as a representation of T_M .

Proof. We have a complex, T_M -invariant decomposition

$$\mathfrak{g}/\mathfrak{h} \cong \mathfrak{m}/\mathfrak{t}_M \oplus \mathfrak{n}^- \oplus \mathfrak{n}^+ = \left(\bigoplus_{\beta \in R_M^+} \mathfrak{m}_\beta^{\mathbb{C}} \right) \oplus \mathfrak{n}^- \oplus \mathfrak{n}^+.$$

So by definition (2.7) of R_G^+ , the first claim follows from Lemma 4.1, and the second from Lemma 4.2. $\square$

Consider the spinor bundle

$$\bigwedge_{J^E} TE \rightarrow E$$

of the Spin^c -structure defined by J^E . Let $L_{\det}^E$ be its determinant line bundle. Let $p: E \rightarrow K/H_M$ be the natural projection.

Lemma 4.4. *We have an isomorphism of complex, K -equivariant line bundles*

$$L_{\det}^E \cong p^*(K \times_{H_M} \mathbb{C}_{2\rho^M}).$$

Proof. Lemma 4.3 implies that

$$L_{\det}^E \cong p^*(K \times_{H_M} \mathbb{C}_{2\rho^M} \boxtimes \chi),$$

where χ is the representation of Z_M in the highest complex exterior power of $\mathfrak{g}/\mathfrak{h}$. And Z_M acts trivially on $\mathfrak{m}/\mathfrak{t}_M$. It also acts trivially on the highest exterior power of $\mathfrak{n}^- \oplus \mathfrak{n}^+$, analogously to Lemma 4.2. $\square$

Remark 4.5. In Lemma 4.4, the fact that Z_M acts trivially on the highest exterior power of $\mathfrak{n}^- \oplus \mathfrak{n}^+$ is not necessary in what follows.

4.2. Line bundles.

Lemma 4.6. *The determinant line bundle of the Spin^c -structure on G/H with spinor bundle (2.6) is*

$$G \times_H (\mathbb{C}_{2\lambda} \boxtimes \chi_M^2).$$

Proof. The almost complex structures J and $\Psi_* J^E$ are K -equivariantly homotopic (see [16, proof of Lemma 4.3]). Hence the induced Spin^c -structures have equivariantly isomorphic determinant line bundles. By Lemma 4.4, the determinant line bundle corresponding to $\Psi_* J^E$ is

$$\Psi_* p^*(K \times_{H_M} \mathbb{C}_{2\rho^M}) \rightarrow G/H.$$

One can check directly that this line bundle is K -equivariantly isomorphic to $G \times_H \mathbb{C}_{2\rho^M}$. Therefore, the determinant line bundle of the Spin^c -structure in the statement of the lemma is

$$L_{\det} = G \times_H (\mathbb{C}_{2\rho^M} \otimes \mathbb{C}_{\lambda-\rho^M}^2 \boxtimes \chi_M^2) = G \times_H (\mathbb{C}_{2\lambda} \boxtimes \chi_M^2). \quad \square$$

Lemma 4.7. *Let $\sigma \in \mathfrak{it}_M^*$ be integral. Let $\zeta \in \mathfrak{ia}^*$, and set*

$$L_\zeta := G \times_H \mathbb{C}_{\sigma+\zeta}.$$

Then there is a K -equivariant isomorphism of line bundles $L_\zeta \cong G \times_H \mathbb{C}_\sigma$.

Proof. The multiplication map defines a diffeomorphism

$$K \times \exp(\mathfrak{s}_M) \times N \times A \cong G.$$

Define the map

$$\Xi: G \times \mathbb{C}_{\sigma+\zeta} \rightarrow G \times \mathbb{C}_\sigma$$

by

$$\Xi(khna, z) = (khna, e^\zeta(a)z)$$

for $k \in K$, $h \in \exp(\mathfrak{s}_M)$, $n \in N$, $a \in A$, and $z \in \mathbb{C}_{\sigma+\zeta}$. We claim that this is map is H -equivariant, and that the induced map $L_\zeta \rightarrow G \times_H \mathbb{C}_\sigma$ is a K -equivariant isomorphism of line bundles.

To show that Ξ is H -equivariant, let $t \in T_M$ and $a_0 \in A$. Then for an element $(khna, z) \in G \times \mathbb{C}_{\sigma+\zeta}$ as above,

$$(ta_0) \cdot (khna, z) = (khna(ta_0)^{-1}, e^\sigma(t)e^\zeta(a_0)z) = ((khnt^{-1})aa_0^{-1}, e^\sigma(t)e^\zeta(a_0)z).$$

The adjoint action by T_M preserves the restricted root spaces of the system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ because this action commutes with $\mathrm{ad}(\mathfrak{a})$. So this action preserves $\mathfrak{n}$. Furthermore, since $T_M \subset K_M$, this action also preserves $\mathfrak{s}_M$. So if $h = \exp(X)$ and $n = \exp(Y)$, then

$$khnt^{-1} = kt^{-1} \exp(\mathrm{Ad}(t)X) \exp(\mathrm{Ad}(t)Y) \in K \exp(\mathfrak{s}_M)N,$$

Therefore,

$$\begin{aligned} \Xi((khnt^{-1})aa_0^{-1}, e^\sigma(t)e^\zeta(a_0)z) &= ((khnt^{-1})aa_0^{-1}, e^\zeta(aa_0^{-1})e^\sigma(t)e^\zeta(a_0)z) \\ &= (khna(ta_0)^{-1}, e^\sigma(t)e^\zeta(a)z) \\ &= (ta_0) \cdot \Xi(khna, z). \end{aligned}$$

Since Ξ is H -equivariant, it indeed descends to a map $L_\zeta \rightarrow G \times_H \mathbb{C}_\sigma$. This map is immediately seen to be a K -equivariant isomorphism of line bundles. $\square$

Remark 4.8. Lemmas 4.6 and 4.7 imply that the determinant line bundle of the K -equivariant Spin^c -structure with spinor bundle (2.6) is K -equivariantly isomorphic to

$$G \times_H \mathbb{C}_{2(\xi+\zeta)} \boxtimes \chi_M^2 \rightarrow G/H,$$

with ξ and ζ as in Proposition 2.4. That is to say, modulo the representation χ_M of group Z_M , the map Φ is the symplectic moment map for the action by K on the coadjoint orbit $\text{Ad}^*(G)(\xi + \zeta)$, while the Spin^c -structure with spinor bundle (2.6) is a K -equivariant Spin^c -prequantisation of this coadjoint orbit. If the infinitesimal character χ of π is regular, then we may take ζ to be the component of χ in $\mathfrak{a}^*$ so that $\xi + \zeta = \chi$.

4.3. Proof of Proposition 2.4. We start with a general, well-known comment about moment maps on homogeneous spaces. For now, let G be any Lie group, and let $H < G$ be a possibly disconnected, closed subgroup. Let $\mathbb{C}_\sigma$ be a one-dimensional unitary representation of H , with differential $\sigma \in i\mathfrak{h}^*$. Consider the line bundle

$$L_\sigma := G \times_H \mathbb{C}_\sigma \rightarrow G/H.$$

Then $\Gamma^\infty(L_\sigma) \cong (C^\infty(G) \otimes \mathbb{C}_\sigma)^H$. Let $V \subset \mathfrak{g}$ be an H -invariant subspace such that $\mathfrak{g} = \mathfrak{h} \oplus V$. Extend σ linearly to $\mathfrak{g}$ by setting it equal to zero on V .

Lemma 4.9. *For $X \in \mathfrak{g}$ and $s \in (C^\infty(G) \otimes \mathbb{C}_\sigma)^H$, set*

$$(\nabla_{X+\mathfrak{h}} s)(e) := \mathcal{L}_{-X}(s)(e) - \langle \sigma, X \rangle s(e).$$

Here $\mathcal{L}$ is the left regular representation. This extends to a well-defined G -invariant connection on L_σ . The associated moment map $\Phi^\sigma: G/H \rightarrow \mathfrak{g}^$ is given by*

$$\Phi^\sigma(gH) = \text{Ad}^*(g)\sigma/2i.$$

Proof. To see that ∇ is well-defined, note that if $s \in (C^\infty(G) \otimes \mathbb{C}_\sigma)^H$ and $X \in \mathfrak{h}$, then $\mathcal{R}_X s = \langle \sigma, X \rangle s$, with $\mathcal{R}$ being the right regular representation. So at e ,

$$\mathcal{L}_{-X}(s)(e) - \langle \sigma, X \rangle s(e) = \mathcal{R}_X(s)(e) - \langle \sigma, X \rangle s(e) = 0.$$

The moment map Φ^σ satisfies

$$2i\langle \Phi^\sigma, X \rangle s(e) = (\mathcal{L}_X - \nabla_{X^{G/H}})s(e).$$

Now note that, with respect to the identification $T_{eH}G/H = \mathfrak{g}/\mathfrak{h}$,

$$X_{eH}^{G/H} = \left. \frac{d}{dt} \right|_{t=0} \exp(-tX)H = -X + \mathfrak{h}.$$

Hence

$$(\mathcal{L}_X - \nabla_{X^{G/H}})s(e) = (\mathcal{L}_X - \nabla_{-X+\mathfrak{h}})s(e) = \langle \sigma, X \rangle s(e).$$

So $\Phi^\sigma(e) = \sigma/2i$, and the claim about Φ^σ follows by G -equivariance. $\square$

Importantly, even if H is disconnected—so the representation $\mathbb{C}_\sigma$ of H is not determined by σ —Lemma 4.9 still gives us a connection with the desired moment map. This means we can apply it to the representation $\mathbb{C}_{2\lambda+i\zeta} \boxtimes \chi_M^2$ of the Cartan subgroup H .

Proof of Proposition 2.4. By Lemmas 4.6 and 4.7, we have K -equivariant isomorphisms of line bundles

$$L_{\det} \cong G \times_H \mathbb{C}_{2\lambda} \boxtimes \chi_M^2 \cong G \times_H \mathbb{C}_{2(\lambda+i\zeta)} \boxtimes \chi_M^2.$$

Let ∇ be the connection of Lemma 4.9 on the line bundle on the right-hand side; we use the same notation for the connection on $L_{\det}$ corresponding to ∇ via the above isomorphism. The moment map for the action by K on G/H associated to ∇ is the map $\Phi^{2(\lambda+i\zeta)}$ in Lemma 4.9, composed with restriction to $\mathfrak{k}$. This is precisely the map Φ in Proposition 2.4. $\square$

Remark 4.10. In the proof of Proposition 2.4, we used Lemma 4.7 to replace 2λ with $2(\lambda+i\zeta)$. The reason for introducing the extra term in Proposition 2.4 is that the moment map Φ is taming if $\lambda+i\zeta$ is regular.

4.4. The singular case. If $\xi+\zeta$ is singular, then the moment map of Proposition 2.4 is not necessarily proper or taming. But then we can still find a proper, taming Spin^c -moment map such that the associated index equals $(-1)^{\dim(M/K_M)}\pi|_K$.

Consider a general setting, where M is a complete Riemannian manifold with an action by a compact Lie group K , and $\Phi: M \rightarrow \mathfrak{k}$ is the moment map for a connection ∇ on a line bundle (defined as in (2.2)), and $\psi: M \rightarrow \mathfrak{k}$ is a taming map. For $\tau \in C^\infty(M)^K$, define the connection

$$\nabla^\tau := \nabla + 2i\tau(v^\psi, -).$$

Let Φ^τ be the associated moment map.

Lemma 4.11. *For τ large enough, the map Φ^τ is proper, taming, and homotopic to ψ as taming maps.*

Proof. Let $\{X_1, \dots, X_n\}$ be an orthonormal basis of $\mathfrak{k}$. Then

$$v^{\Phi^\tau} = v^\Phi + \tau w^\psi,$$

where

$$w^\psi := \sum_{j=1}^n (v^\psi, X_j^M) X_j^M.$$

Let $m \in M$, and suppose that $v^\psi(m) \neq 0$. The definition of w^ψ is independent of the basis of $\mathfrak{k}$, so we may suppose that $X_1, \dots, X_d \in \mathfrak{k}_m$ and $X_{d+1}, \dots, X_n \in \mathfrak{k}_m^\perp$. Since $\{X_{d+1}^M(m), \dots, X_n^M(m)\}$ is a basis of the subspace $T_m(K \cdot m) \subset T_m M$ containing $v^\psi(m)$, we have

$$w^\psi(m) = \left(\sum_{j=1}^n (v^\psi, X_j^M) X_j^M \right)(m) = \left(\sum_{j=d+1}^n (v^\psi, X_j^M) X_j^M \right)(m) \neq 0.$$

So w^ψ vanishes exactly at the points where v^ψ vanishes.

Now note that

$$\|v^{\Phi^\tau}\| \geq \tau \|w^\psi\| - \|v^\Phi\|.$$

Let U be a relatively compact, K -invariant neighbourhood of the vanishing set of v^ψ . Choose τ so that, outside U , $\tau \|w^\psi\| > \|v^\Phi\|$. Then v^{Φ^τ} does not vanish outside U .

To show that Φ^τ is homotopic to ψ , first note that, by the previous arguments, the vector field

$$tv^\Phi + \tau w^\psi$$

is nonzero outside U for all $t \in [0, 1]$. Hence Φ^τ is homotopic to the taming map $\Phi^\tau - \Phi$. And

$$(v^{\Phi^\tau - \Phi}, v^\psi) = (\tau w^\psi, v^\psi) = \tau \sum_{j=1}^n (v^\psi, X_j^M)^2.$$

If $\tau \geq 0$, then this is nonnegative. This implies that $\Phi^\tau - \Phi$ is homotopic to ψ (this is elementary; see, for example, [16, Corollary 3.5]).

Finally, by adding a function θ as in [15, Subsection 5.1] to τ , we can ensure that the resulting moment map is proper, as well as taming, and homotopic to ψ . $\square$

Remark 4.12. In [15, Proposition 5.1], it was shown how to replace a taming moment map by a proper, taming moment map, without changing the corresponding indices. (This was the last step in the proof of Lemma 4.11.) In Lemma 4.11, we show how to replace any taming map by a proper, taming moment map without changing the index. The additional step here is to replace any taming map by a taming moment map that is homotopic to it.

5. MULTIPLICITY-FREE RESTRICTIONS

Throughout this section, $K < G$ is a compact subgroup satisfying the condition of Corollary 2.8. That is, the map Φ_K in (2.10) is proper. In particular, what follows is true if K is a maximal compact subgroup, as we will assume from Subsection 5.2 onwards. We will omit the subscript K from Φ_K and write $\Phi := \Phi_K$ from now on. We will also write $(G/H)_\xi := (G/H)_\xi^K$ for $\xi \in \mathfrak{t}^*$.

Recall that if $\xi \in \mathfrak{t}_M^*$, defined in (2.8), is regular for the roots of $(\mathfrak{m}^\mathbb{C}, \mathfrak{t}_M^\mathbb{C})$, then Φ is simply the projection of the coadjoint orbit $\text{Ad}^*(G)(\xi + \zeta)$ onto $\mathfrak{k}^*$:

$$(5.1) \quad \Phi(gH) = (\text{Ad}^*(g)(\xi + \zeta))|_{\mathfrak{k}}$$

If ξ is singular, then Φ is as in (2.9), with τ being as in Lemma 4.11.

5.1. Reduced spaces that are points. In the setting of Corollary 2.8, we obtain multiplicities equal to 0 or 1 if the reduced space $(G/H)_{(\eta+\rho^K)/i}$ is a single point. Indeed, the orbifold index on $(G/H)_{(\eta+\rho^K)/i}$ then lies in $\{-1, 0, 1\}$. It takes only these values because, up to a sign, it is the dimension of the trivial part of a one-dimensional representation of a finite group. We can make this more explicit using [41, expression (5.36)] for indices on reduced spaces that are points.

Let $C \subset \mathfrak{t}^*$ be the open positive Weyl chamber. Set $Y := \Phi^{-1}(C)$. Set $\Phi_Y := \Phi|_Y - \rho^K$. Let $\delta \in \hat{K}$ have highest weight η . Then

$$(G/H)_{(\eta+\rho^K)/i} = Y_{\eta/i} := \Phi_Y^{-1}(\eta/i)/T.$$

Let $\mathfrak{t}_Y \subset \mathfrak{t}$ be the generic stabiliser of the infinitesimal action by $\mathfrak{t}$ on Y . Let $I(Y) \subset \mathfrak{t}^*$ be the affine space parallel to the annihilator of $\mathfrak{t}_Y$, containing the image of Φ_Y . Let $T_Y < T$ be the subtorus with Lie algebra $\mathfrak{t}_Y$ (note that this subgroup is connected). Fix $g_0H \in \Phi_Y^{-1}(\eta/i)$, and let $\Gamma < T/T_Y$ be its stabiliser. This is a finite group.

Corollary 5.1.

- (a) Suppose that η/i is a regular value of $\Phi_Y: Y \rightarrow I(Y)$, and that $\Phi^{-1}(\eta/i + \rho^K)/K$ is a point. Then

$$[\pi|_K : \delta] = \begin{cases} 1 & \text{if } \Gamma \text{ acts trivially on } \mathbb{C}_{\lambda - \eta - \rho^M} \boxtimes \chi_M, \\ 0 & \text{otherwise.} \end{cases}$$

- (b) If η/i is not necessarily a regular value of Φ_Y but $\Phi^{-1}(\eta/i + \rho^K + \varepsilon)/K$ is a point for all $\varepsilon \in I(Y)$ close enough to zero, then we still have

$$[\pi|_K : \delta] \in \{0, 1\}.$$

Proof. First, note that for any $\sigma \in \mathfrak{t}^*$, by construction $\Phi^{-1}(\sigma + \rho^K)$ is a single K -orbit if and only if $\Phi_Y^{-1}(\xi)$ is a single T -orbit.

We have

$$T(G/H)|_Y = TY \oplus (Y \times \mathfrak{k}/\mathfrak{t}).$$

By the two-out-of-three lemma, we have a spinor bundle $\mathcal{S}_{Y,\eta} \rightarrow Y$ such that

$$(5.2) \quad (\bigwedge_J T(G/H) \otimes L_{\lambda-\rho^M, \chi_M} \otimes \mathbb{C}_{-\eta})|_Y = \mathcal{S}_{Y,\eta} \otimes \bigwedge_{\mathbb{C}} \mathfrak{k}/\mathfrak{t}.$$

Here the complex structure on $\mathfrak{k}/\mathfrak{t}$ is the one defined by the positive compact roots. Let V_η be the one-dimensional representation of Γ such that, as representations of Γ ,

$$(5.3) \quad (\mathcal{S}_{Y,\eta})_{g_0H} = \bigwedge_{\mathbb{C}} T_{g_0H} Y \otimes V_\eta,$$

for some Γ -invariant complex structure on $T_{g_0H} Y$. This V_η exists since $(\mathcal{S}_{Y,\eta})_{g_0H}$ and $\bigwedge_{\mathbb{C}} T_{g_0H} Y$ are irreducible, Γ -equivariant modules over the Clifford algebra of $T_{g_0H} Y$; see also [41, end of Section 5.1]. Then [41, (5.36)] states that

$$(5.4) \quad \text{index}((G/H)_{(\eta+\rho^K)/i}) = \text{index}(Y_\eta) = \dim V_\eta^\Gamma.$$

Now by (5.2) and (5.3),

$$\bigwedge_{J_{g_0H}} (T_{g_0H} G/H) \otimes \mathbb{C}_{\lambda-\eta-\rho^M} \otimes \chi_M = \bigwedge_{\mathbb{C}} (T_{g_0H} G/H) \otimes V_\eta.$$

Here on the right-hand side, the complex structure on $T_{g_0H} G/H$ is defined by the complex structures on $T_{g_0H} Y$ and $\mathfrak{k}/\mathfrak{t}$ via the isomorphism $T_{g_0H} G/H \cong T_{g_0H} Y \oplus \mathfrak{k}/\mathfrak{t}$. This may be a different complex structure from J_{g_0H} . We conclude that V_η equals $\mathbb{C}_{\lambda-\eta-\rho^M} \otimes \chi_M$ or its dual. So the claim follows from (5.4) and Corollary 2.8. $\square$

Remark 5.2. We have implicitly used that $\Phi_Y^{-1}(\eta)$ is connected because it is a single T -orbit.

Example 5.3. If $G = \text{SL}(2, \mathbb{C})$, then one can check that all reduced spaces are points. This is compatible with the fact that the multiplicities of the K -types of the principal series of $\text{SL}(2, \mathbb{C})$ are 1.

We work out the example $G = \text{SL}(2, \mathbb{R})$ in detail in Subsection 5.2, and we discuss the groups $\text{SU}(p, 1)$, $\text{SO}_0(p, 1)$, and $\text{SO}_0(2, 2)$ in Subsection 5.5.

Corollary 5.1 implies a criterion for multiplicity-free restrictions of general *admissible* representations. Let π be an irreducible admissible representation of G . By the Langlands classification, π is a quotient of an induced representation as on the right-hand side of (2.4), where now $\nu \in (\mathfrak{a}^{\mathbb{C}})^*$ may be nonimaginary. Let $\Phi: G/H \rightarrow \mathfrak{k}^*$ be the corresponding moment map as in Proposition 2.4

Corollary 5.4. *Let $\delta \in \hat{K}$ with highest weight η . Suppose that $\Phi^{-1}(\eta + \rho^K)/K$ is a point if η/i is a regular value of Φ_Y , or $\Phi^{-1}(\eta + \rho^K + \varepsilon)/K$ is a point for all η small enough if η/i is a singular value of Φ_Y . Then $[\pi|_K, \delta] \in \{0, 1\}$.*

In particular, if all reduced spaces for Φ are points, then π restricts multiplicity freely to K .

Proof. Corollary 2.8, and hence Corollary 5.1, applies to any standard representation π ; see Remark 2.9. So under the conditions stated, $\pi|_K$ is a quotient of a multiplicity-free representation and hence is multiplicity-free itself. $\square$

We end this subsection with a conjecture that is a partial converse to Corollary 5.1

Conjecture 5.5. *Let $H < G$ be a θ -stable Cartan subgroup. Let $P = MAN < G$ be a cuspidal parabolic subgroup corresponding to H (so that A is the noncompact part of H). Then all tempered representations π induced from P restrict multiplicity freely to K if and only if all reduced spaces for all maps $\Phi: G/H \rightarrow \mathfrak{k}^*$ corresponding to such representations are points.*

The “if” part of this conjecture follows from Corollary 5.1. Evidence for the “only if” part is the following. Let $\delta \in \hat{K}$ have highest weight η . Let H , π , and Φ be as in the conjecture. If the reduced space $(G/H)_{\eta+\rho^K}$ is smooth, the Atiyah–Singer index theorem and Theorem 2.7 imply that

$$[\pi|_K : \delta] = (-1)^{\dim(M/K_M)/2} \int_{(G/H)_{\eta+\rho^K}} e^{\frac{1}{2}c_1(L_{\det}^{\eta+\rho^K})} \hat{A}((G/H)_{\eta+\rho^K}).$$

Here $L_{\det}^{\eta+\rho^K} \rightarrow (G/H)_{\eta+\rho^K}$ is induced by the determinant line bundle on G/H from Lemma 4.6. If $(G/H)_{\eta+\rho^K}$ is not a point, then the right-hand side depends on $c_1(L_{\det}^{\eta+\rho^K})$. Then one expects that number to vary with π and δ , and hence not to equal 1 for all π and δ .

Remark 5.6. In some other work about multiplicity-free restrictions, for example, [23], it is assumed that G is of Hermitian type, and that the representation π in question is of scalar type. We do not make these assumptions in Corollaries 5.1 and 5.4. In Subsection 5.5, we work our examples for the groups $\mathrm{SO}_0(p, 1)$ and $\mathrm{SO}_o(2, 2)$ that are not of Hermitian type (in addition to the group $\mathrm{SU}(p, 1)$, which is).

5.2. Example: $G = \mathrm{SL}(2, \mathbb{R})$. If $G = \mathrm{SL}(2, \mathbb{R})$ and $K = \mathrm{SO}(2)$, then Theorem 2.7 implies the usual multiplicity formulas for the K -types of tempered representations of $\mathrm{SL}(2, \mathbb{R})$. This example illustrates the essential point that indices on reduced spaces that are points may be zero (as in Corollary 5.1) because these indices are orbifold indices.

5.2.1. The discrete series. Consider the holomorphic discrete series representation D_n^+ of $G = \mathrm{SL}(2, \mathbb{R})$ for $n \in \{1, 2, 3, \dots\}$. Then $H = T = \mathrm{SO}(2)$, $M = G$, and $\lambda = n\alpha/2$, where $\alpha \in i\mathfrak{t}^*$ is the root mapping $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to $2i$. So $\rho^G = \rho^M$, and $\xi = n\alpha/2i$. This element is regular, so Φ is the projection of $G/T \cong G \cdot \xi$ onto $\mathfrak{k}^*$.

Let $\delta_l = \mathbb{C}_l$ be the irreducible representation of $K = \mathrm{SO}(2)$ with weight $l \in \mathbb{Z}$; i.e., $\mathbb{C}_l = \mathbb{C}_{l\alpha/2}$. If $l \leq n$, then by Corollary 2.11,

$$[D_n^+ : \delta_l] = 0.$$

If $l > n$, then $l\alpha/2i$ is a regular value of Φ , and $\Phi^{-1}(l\alpha/2i)$ is a circle, acted on by $T = \mathrm{SO}(2)$ by rotations with weight 2. Now $\mathfrak{t}_Y = \{0\}$, so $T_Y = \{I\}$ and $\Gamma = \{\pm I\}$ in Corollary 5.1. Since $Z_M \subset T_M = T$, we have $\mathbb{C}_{\lambda-\rho^M} \otimes \chi_M = \mathbb{C}_{\lambda-\rho^M} = \mathbb{C}_{(n-1)\alpha/2}$. Hence Γ acts trivially on

$$\mathbb{C}_{\lambda-\eta-\rho^M} \otimes \chi_M = \mathbb{C}_{n-l-1}$$

precisely if $n - l$ is odd. We conclude that

$$[D_n^+ : \delta_l] = \begin{cases} 1 & \text{if } l = n + s \text{ for a positive odd integer } s, \\ 0 & \text{otherwise.} \end{cases}$$

In the same way, we find that, for the antiholomorphic discrete series representation D_n^- ,

$$[D_n^- : \delta_l] = \begin{cases} 1 & \text{if } l = -n - s \text{ for a positive odd integer } s, \\ 0 & \text{otherwise.} \end{cases}$$

See [37, Example 2.21] for a symplectic version of the computation of indices on reduced spaces in this example.

5.2.2. Limits of discrete series. Consider the limit of discrete series representation D_0^+ . Then, as in the discrete series case, $H = T = \mathrm{SO}(2)$ and $M = G$. But now $\lambda = 0$, which is singular. So we have to use the taming moment map from Lemma 4.11. Taking $\psi(gT) = (\mathrm{Ad}^*(g)\alpha/2i)|_{\mathfrak{t}}$, we have, for all $\tau \in C^\infty(G/T)^K$,

$$\Phi_X = \Phi_X^\tau = \tau \cdot (v^\psi, X^{G/T}).$$

Let $\varphi: G/T \rightarrow [1, \infty[$ be the function such that, for all $g \in G$,

$$\psi(gT) = \varphi(gT)\alpha/2i.$$

Then

$$\Phi^\tau = \tau\varphi\|(\alpha/2i)^{G/T}\|^2\alpha/2i.$$

The factor $\|(\alpha/2i)^{G/T}\|^2$ only vanishes at the point eT . So we can choose τ so that $\Phi^\tau = f\alpha/2i$ for a surjective, proper, K -invariant map $f: G/T \rightarrow [0, \infty[$ whose level sets are circles. (In fact, we may take $\tau \equiv 1$.) Then Φ^τ is K -invariant, proper, taming, homotopic to ψ , and surjective onto the closed Weyl chamber containing α . For all integers $l \geq 1$, $l\alpha/2i$ is a regular value of Φ^τ , and $(\Phi^\tau)^{-1}(l\alpha/2i)$ is a circle, acted on by $T = \mathrm{SO}(2)$ with rotations with weight 2. So in the same way as for the discrete series, we find that

$$[D_0^+ : \delta_l] = \begin{cases} 1 & \text{if } l \text{ is a positive odd integer,} \\ 0 & \text{otherwise.} \end{cases}$$

And analogously,

$$[D_0^- : \delta_l] = \begin{cases} 1 & \text{if } l \text{ is a negative odd integer,} \\ 0 & \text{otherwise.} \end{cases}$$

5.2.3. The principal series. Consider the spherical principal series representation $P_{i\nu}^+$ for $\nu \geq 0$. We now have

$$H = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}; x \neq 0 \right\},$$

$M = \{\pm I\}$, $\lambda = 0$, and $\chi_M = \chi_+$, the trivial representation of M . Now $\mathfrak{t}_M = 0$, so $\xi = 0$. For any nonzero $\zeta \in \mathfrak{a}$, the element $\xi + \zeta = \zeta$ is regular. So $\Phi: G/H \rightarrow \mathfrak{k}^*$ is the projection map of the hyperbolic coadjoint orbit $G/H \cong G \cdot \zeta$ onto $\mathfrak{k}^*$. Therefore, for all $l \in \mathbb{Z}$, $\Phi^{-1}(l\alpha/2i)$ is a circle, on which $T = \mathrm{SO}(2)$ acts by rotations with weight 2. Also, $l\alpha/2i$ is a regular value of Φ .

In Corollary 5.1, we have

$$\mathbb{C}_{\lambda - \eta - \rho^M} \otimes \chi_M|_\Gamma = \mathbb{C}_{-l} \otimes \chi_+|_\Gamma = \mathbb{C}_{-l}|_\Gamma.$$

The group $\Gamma = \{\pm I\}$ acts trivially on this space precisely if l is even. Hence

$$[P_{i\nu}^+ : \delta_l] = \begin{cases} 1 & \text{if } l \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

For a nonspherical principal series representation $P_{i\nu}^-$ (with $\nu > 0$), we have $\chi_M = \chi_-$, the nontrivial representation of $Z_M = M$. Hence

$$\mathbb{C}_{\lambda - \eta - \rho^M} \otimes \chi_M|_{\Gamma} = \mathbb{C}_{-l} \otimes \chi_-|_{\Gamma} = \mathbb{C}_{-l+1}|_{\Gamma}.$$

Now $\Gamma = \{\pm I\}$ acts trivially on this space precisely if l is odd. Hence

$$[P_{i\nu}^- : \delta_l] = \begin{cases} 1 & \text{if } l \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

5.3. Multiplicity-freeness via dimension counts. Corollary 5.1 implies a dimension-counting criterion for the restriction of any admissible representation to K to be multiplicity-free. Let π be an admissible representation. By the Langlands classification of admissible representations and the fact that any tempered representation is a subrepresentation of a representation induced from a discrete series representation, π is a subrepresentation of a quotient of a representation of the form

$$\tilde{\pi} := \text{Ind}_{MAN}^G(\pi_{\lambda, \chi_M}^M \otimes e^{\nu} \otimes 1_N),$$

for a cuspidal parabolic $MAN < G$, where π_{λ, χ_M}^M belongs to the discrete series of M , and $\nu \in (\mathfrak{a}^{\mathbb{C}})^*$ may be nonimaginary. Let $\Phi: G/H \rightarrow \mathfrak{k}^*$ be the moment map from Proposition 2.4 for this situation. We write $\dim(\text{im}(\Phi))$ for the dimension of the relative interior of $\text{im}(\Phi)$.

Corollary 5.7. *If*

$$(5.5) \quad \dim(\text{im}(\Phi)) = \dim(G) - \text{rank}(G) - \dim(T),$$

then $[\pi|_K : \delta] \in \{0, 1\}$ for all $\delta \in \hat{K}$.

In particular, if $\text{im}(\Phi)$ has nonempty interior in $\mathfrak{k}^$ and*

$$\dim(G) \leq \text{rank}(G) + \dim(T) + \dim(K),$$

then $[\pi|_K : \delta] \in \{0, 1\}$ for all $\delta \in \hat{K}$.

Proof. For a map Φ as in the corollary, the condition (5.5) implies that the reduced space $\Phi^{-1}(\sigma)/T$ is zero-dimensional for every σ in the relative interior of $\text{im}(\Phi)$. Since $\xi \in \mathfrak{k}_M^*$ is regular, Φ is a moment map in the symplectic sense, so $\Phi^{-1}(\sigma)/T$ is connected for such σ , and hence a point. So by Corollary 5.1(b), which applies to representations like $\tilde{\pi}$, that representation restricts multiplicity freely to K . Hence so does π . $\square$

In Subsection 5.5, we show that admissible representations of $\text{SU}(p, 1)$, $\text{SO}_0(p, 1)$, and $\text{SO}_0(2, 2)$ with regular infinitesimal characters have multiplicity-free restrictions to maximal compact subgroups. This is based on Corollary 5.7 and techniques for computing the dimension of the image of Φ developed in Subsection 5.4.

From now on, suppose that $K < G$ is a maximal compact subgroup. The condition

$$\dim(G) \leq \operatorname{rank}(G) + \dim(T) + \dim(K)$$

in Corollary 5.7 holds for the following classical semisimple groups:

- $\mathrm{SL}(2, \mathbb{C})$,
- $\mathrm{SO}(n, \mathbb{C})$ for $n \leq 4$,
- $\mathrm{SL}(2, \mathbb{H})$,
- $\mathrm{SL}(2, \mathbb{R})$,
- $\mathrm{SO}^*(4)$,
- $\mathrm{SU}(p, 1)$ for all p ,
- $\mathrm{SO}_0(p, 1)$ for all p , and
- $\mathrm{SO}_0(2, 2)$.

So for these groups, any admissible representation for which $\operatorname{im}(\Phi)$ has nonempty interior in $\mathfrak{k}^*$ has multiplicity-free restriction to a maximal compact subgroup. To determine the dimension of the image of Φ , we use the equality

$$\dim(\operatorname{im}(\Phi)) = \dim(K/T) + \dim(\operatorname{im}(\Phi) \cap \mathfrak{t}).$$

5.4. Computing the dimension of $\operatorname{im}(\Phi)$. The following proposition is a tool to compute $\dim(\operatorname{im}(\Phi) \cap \mathfrak{t})$.

Let $\mathfrak{h}_c \subset \mathfrak{g}$ be a maximally compact, θ -stable Cartan subalgebra. Let $R_n^+ \subset R(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}_c^{\mathbb{C}})$ be a choice of positive, imaginary, noncompact roots. For every $\alpha \in R_n^+$, let $E_\alpha \in \mathfrak{g}_\alpha^{\mathbb{C}}$ be any nonzero vector. Let $\bar{E}_\alpha$ be its complex conjugate with respect to the real form $\mathfrak{g}$, and set $H_\alpha := [E_\alpha, \bar{E}_\alpha]$.

Proposition 5.8. *Let $\mathfrak{h} \subset \mathfrak{g}$ be any θ -stable Cartan subalgebra. Suppose that $\Phi: G/H \rightarrow \mathfrak{k}$ is given by*

$$\Phi: G/H \cong \operatorname{Ad}(G)(\xi + \zeta) \hookrightarrow \mathfrak{g} \rightarrow \mathfrak{k}$$

for $\xi \in \mathfrak{t} \cap \mathfrak{h}$ and $\zeta \in \mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$ such that $\xi + \zeta \in \mathfrak{h}$ is regular. Then $\operatorname{im}(\Phi) \cap \mathfrak{t}$ contains the convex hull of the set

$$\bigcup_{\alpha \in R_n^+} \xi + I_\alpha i H_\alpha,$$

where for all $\alpha \in R_n^+$, the set I_α equals either $\mathbb{R}$, $[0, \infty)$ or $(-\infty, 0]$.

We will use Lemmas 5.9 and 5.11 below to prove Proposition 5.8.

Lemma 5.9. *Consider the map*

$$\Phi: G/H \cong \operatorname{Ad}(G)(\xi + \zeta) \hookrightarrow \mathfrak{g} \rightarrow \mathfrak{k},$$

for $\xi \in \mathfrak{t} \cap \mathfrak{h}$ and $\zeta \in \mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$, such that $\xi + \zeta \in \mathfrak{h}$ is regular. Suppose that there is a set of roots $S \subset R(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$, and for every $\alpha \in S$, that there are $X_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}^{\mathbb{C}}$ such that

- $X_\alpha + X_{-\alpha} \in \mathfrak{g}$,
- $X_\alpha - X_{-\alpha} \in \mathfrak{t}$,
- $\eta_\alpha := [X_\alpha, X_{-\alpha}] \in \mathfrak{a}$, and
- $\langle \alpha, \eta_\alpha \rangle > 0$.

Then $\operatorname{im}(\Phi) \cap \mathfrak{t}$ contains

$$\xi + \operatorname{span}_{\mathbb{R}}\{(X_\alpha - X_{-\alpha}); \alpha \in S\}.$$

Proof. Fix $\alpha \in R(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$, and fix $X_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}^{\mathbb{C}}$. Write $\eta_{\alpha} := [X_{\alpha}, X_{-\alpha}]$. One proves by induction that, for every positive integer j ,

$$\begin{aligned} \operatorname{ad}(X_{\alpha} + X_{-\alpha})^{2j}(\xi + \zeta) &= 2^j \langle \alpha, \xi + \zeta \rangle \langle \alpha, \eta_{\alpha} \rangle^{j-1} \eta_{\alpha}, \\ \operatorname{ad}(X_{\alpha} + X_{-\alpha})^{2j+1}(\xi + \zeta) &= -2^j \langle \alpha, \xi + \zeta \rangle \langle \alpha, \eta_{\alpha} \rangle^j (X_{\alpha} - X_{-\alpha}). \end{aligned}$$

Suppose that $\langle \alpha, \eta_{\alpha} \rangle > 0$. Then the above equalities imply that, for all $t \in \mathbb{R}$,

$$\begin{aligned} (5.6) \quad & \operatorname{Ad}(\exp(t(X_{\alpha} + X_{-\alpha}))) (\xi + \zeta) \\ &= \xi + \zeta + \frac{\langle \alpha, \xi + \zeta \rangle}{\langle \alpha, \eta_{\alpha} \rangle} \eta_{\alpha} \sum_{j=1}^{\infty} \frac{1}{(2j)!} t^{2j} 2^j \langle \alpha, \eta_{\alpha} \rangle^j \\ &\quad - \langle \alpha, \xi + \zeta \rangle (X_{\alpha} - X_{-\alpha}) \sum_{j=0}^{\infty} \frac{1}{(2j+1)!} t^{2j+1} 2^j \langle \alpha, \eta_{\alpha} \rangle^j \\ &= \xi + \zeta \\ &\quad + \langle \alpha, \xi + \zeta \rangle \left(\frac{\cosh(t\sqrt{2\langle \alpha, \eta_{\alpha} \rangle}) - 1}{\langle \alpha, \eta_{\alpha} \rangle} \eta_{\alpha} - \frac{\sinh(t\sqrt{2\langle \alpha, \eta_{\alpha} \rangle})}{\sqrt{2\langle \alpha, \eta_{\alpha} \rangle}} (X_{\alpha} - X_{-\alpha}) \right). \end{aligned}$$

Suppose that $\alpha \in S$, and let $X_{\pm\alpha}$ be as in the lemma. Then, using (5.6) and the fact that both sides of this equality lie in $\mathfrak{g}$ (so the component of the right-hand side in $i\mathfrak{g}$ is zero), we find that

$$\Phi(\exp(t(X_{\alpha} + X_{-\alpha}))H) = \xi - \frac{\sinh(t\sqrt{2\langle \alpha, \eta_{\alpha} \rangle})}{\sqrt{2\langle \alpha, \eta_{\alpha} \rangle}} (X_{\alpha} - X_{-\alpha}) \in \mathfrak{t}.$$

So

$$\xi + \mathbb{R}(X_{\alpha} - X_{-\alpha}) \in \operatorname{im}(\mu) \cap \mathfrak{t}.$$

And since Φ is a moment map in the symplectic sense, its image intersected with $\mathfrak{t}$ is convex. $\square$

Example 5.10. If $G = \operatorname{SL}(2, \mathbb{R})$, $\xi = 0$, $\zeta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathfrak{h} = \mathfrak{a} = \mathbb{R}\zeta$, $\langle \alpha, \zeta \rangle = 2$, $S = \{\alpha\}$, and

$$X_{\alpha} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then $X_{\alpha} - X_{-\alpha} \in \mathfrak{t}$, $\eta_{\alpha} = \zeta \in \mathfrak{a}$, and Lemma 5.9 states that $\operatorname{im}(\Phi)$ contains the line $\mathbb{R}\zeta$ and is therefore surjective.

Lemma 5.11. Consider the map

$$\Phi: G/H \cong \operatorname{Ad}(G)(\xi + \zeta) \hookrightarrow \mathfrak{g} \rightarrow \mathfrak{k},$$

for $\xi \in \mathfrak{t} \cap \mathfrak{h}$ and $\zeta \in \mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$, such that $\xi + \zeta \in \mathfrak{h}$ is regular. Suppose that there is a set of roots $S \subset R(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$, and for every $\alpha \in S$, there are $X_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}^{\mathbb{C}}$ such that

- $X_{\alpha} + X_{-\alpha} \in \mathfrak{p}$,
- $\eta_{\alpha} := [X_{\alpha}, X_{-\alpha}] \in i\mathfrak{t}$, and
- $\langle \alpha, \eta_{\alpha} \rangle > 0$.

Then $\operatorname{im}(\Phi) \cap \mathfrak{t}$ contains the convex hull of

$$\bigcup_{\alpha \in S} (\xi + \mathbb{R}_{\geq 0} \langle \alpha, \xi \rangle \eta_{\alpha}).$$

Proof. As in the proof of Lemma 5.9, we find that, for all $t \in \mathbb{R}$,

$$\begin{aligned}
 (5.7) \quad & \Phi(\exp(t(X_\alpha + X_{-\alpha}))H) = \cosh(\operatorname{ad}(t(X_\alpha + X_{-\alpha})))\xi + \sinh(\operatorname{ad}(t(X_\alpha + X_{-\alpha})))\zeta \\
 & = \xi + \frac{\langle \alpha, \xi \rangle}{\langle \alpha, \eta_\alpha \rangle} (\cosh(t\sqrt{2\langle \alpha, \eta_\alpha \rangle}) - 1)\eta_\alpha \\
 & \quad - \frac{\langle \alpha, \zeta \rangle}{\sqrt{2\langle \alpha, \eta_\alpha \rangle}} \sinh(t\sqrt{2\langle \alpha, \eta_\alpha \rangle})(X_\alpha - X_{-\alpha}).
 \end{aligned}$$

The left-hand side and the first term on the right hand side lie in $\mathfrak{g}$; hence so does the second term on the right-hand side. But

$$X_\alpha - X_{-\alpha} = \frac{1}{\langle \alpha, \eta_\alpha \rangle} [\eta_\alpha, X_\alpha + X_{-\alpha}] \in i\mathfrak{g}.$$

And $\langle \alpha, \zeta \rangle \in \mathbb{R}$, so the second term on the right-hand side of (5.7) lies in $i\mathfrak{g} \cap \mathfrak{g} = \{0\}$. We conclude that

$$\Phi(\exp(t(X_\alpha + X_{-\alpha}))H) = \xi + \frac{\langle \alpha, \xi \rangle}{\langle \alpha, \eta_\alpha \rangle} (\cosh(t\sqrt{2\langle \alpha, \eta_\alpha \rangle}) - 1)\eta_\alpha \in \mathfrak{t}.$$

So

$$\xi + \mathbb{R}_{\geq 0}\langle \alpha, \xi \rangle \eta_\alpha \subset \operatorname{im}(\Phi) \cap \mathfrak{t}.$$

The claim again follows by convexity of $\operatorname{im}(\Phi) \cap \mathfrak{t}$. $\square$

Example 5.12. If $G = \operatorname{SL}(2, \mathbb{R})$, $\xi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\zeta = 0$, $\mathfrak{h} = \mathfrak{t} = \mathbb{R}\xi$, $\langle \alpha, \xi \rangle = 2i$, $S = \{\alpha\}$, and

$$X_\alpha = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad X_{-\alpha} = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix},$$

then $X_\alpha + X_{-\alpha} \in \mathfrak{p}$, $\eta_\alpha = -i\xi \in i\mathfrak{t}$, and Lemma 5.11 states that $\operatorname{im}(\Phi)$ contains the half-line $[1, \infty)\xi$. (In this case, we actually find that $\operatorname{im}(\Phi)$ equals that half-line.)

Proof of Proposition 5.8. For every $\alpha \in R_n^+$, the element $H_\alpha = [E_\alpha - \bar{E}_\alpha, E_\alpha + \bar{E}_\alpha]/2$ is imaginary, and it lies in $\mathfrak{h}_c^\mathbb{C}$ and in $[\mathfrak{p}^\mathbb{C}, \mathfrak{p}^\mathbb{C}] \subset \mathfrak{k}^\mathbb{C}$. Hence $H_\alpha \in i\mathfrak{t}$. Therefore, applying Lemma 5.11 with $S = R_n^+$, $X_\alpha = E_\alpha$, and $X_{-\alpha} = \bar{E}_\alpha$ shows that the claim holds for $\mathfrak{h} = \mathfrak{h}_c$.

Now fix $\alpha \in R_n^+$. Consider the Cayley transform

$$c_\alpha := \operatorname{Ad}\left(\exp\left(\frac{\pi}{4}(\bar{E}_\alpha - E_\alpha)\right)\right).$$

(For the properties of Cayley transforms that we use, see, for example, [19, Section VI.7].) Set $\mathfrak{h}_1 := c_\alpha(\mathfrak{h}_c) \cap \mathfrak{g}$, and set

$$\begin{aligned}
 X_\alpha &:= ic_\alpha(E_\alpha), \\
 X_{-\alpha} &:= -ic_\alpha(\bar{E}_\alpha).
 \end{aligned}$$

These elements lie in root spaces for $\mathfrak{h}_1$. They satisfy

- (1) $X_\alpha + X_{-\alpha} = i(E_\alpha - \bar{E}_\alpha) \in \mathfrak{g}$,
- (2) $X_\alpha - X_{-\alpha} = ic_\alpha(E_\alpha + \bar{E}_\alpha) = -iH_\alpha \in \mathfrak{t}$, and
- (3) $[X_\alpha, X_{-\alpha}] = c_\alpha(H_\alpha) = E_\alpha + \bar{E}_\alpha \in \mathfrak{h}_1 \cap \mathfrak{p}$.

Hence Lemma 5.9 implies that, with Φ as in the proposition for $\mathfrak{h} = \mathfrak{h}_1$,

$$\xi + i\mathbb{R}H_\alpha \in \operatorname{im}(\Phi) \cap \mathfrak{t}.$$

As in the first paragraph of this proof, by applying Lemma 5.11 with $S = R_n^+ \setminus \{\alpha\}$, we find that

$$\bigcup_{\alpha \in R_n^+ \setminus \{\alpha\}} \xi + I_\alpha i H_\alpha \subset \text{im}(\Phi) \cap \mathfrak{t},$$

with I_α equal to $[0, \infty)$ or $(-\infty, 0]$. If $\xi + \zeta$ is regular, then Φ is a moment map in the symplectic sense, so its image intersected with $\mathfrak{t}$ is convex. Hence the claim follows for $\mathfrak{h} = \mathfrak{h}_1$.

Continuing in this way, removing noncompact, imaginary roots until there are none left, one proves the claim for all θ -stable Cartan subalgebras. $\square$

5.5. Examples: $\text{SU}(p, 1)$, $\text{SO}_0(p, 1)$, and $\text{SO}_0(2, 2)$.

Lemma 5.13. *Let $G = \text{SU}(p, q)$. Let $H < G$ be a θ -stable Cartan subgroup, and let $\mu \in \mathfrak{h}$ be regular. The image of the map*

$$\Phi: G/H \cong \text{Ad}(G)\mu \hookrightarrow \mathfrak{g} \rightarrow \mathfrak{k}$$

has nonempty interior.

Proof. Let $H_c < G$ be the compact Cartan of diagonal elements. Then a choice of positive imaginary noncompact roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}_c^{\mathbb{C}})$ is

$$R_n^+ = \{\alpha_{jk}; 1 \leq j \leq p, p+1 \leq k \leq p+q\},$$

where α_{jk} maps the diagonal matrix with entries $(t_1, \dots, t_{p+q})$ to $t_j - t_k$. A root vector in $\mathfrak{g}_{\alpha_{jk}}^{\mathbb{C}}$ is the matrix E_{jk} with a 1 in position (j, k) and zeros in the other positions. The complex conjugation of E_{jk} with respect to the real form $\mathfrak{su}(p, q)$ is E_{kj} . And

$$[E_{jk}, E_{kj}] = h_{jk},$$

where h_{jk} is the diagonal matrix with entry with 1 in the j th position and -1 in the k th position, and zeros everywhere else. Together, these span $i\mathfrak{h}_c$. So Proposition 5.8 implies that $\text{im}(\Phi) \cap \mathfrak{t}$ has nonempty interior in $\mathfrak{t}$ so that $\text{im}(\Phi)$ has nonempty interior in $\mathfrak{k}$. $\square$

Lemma 5.14. *Let $G = \text{SO}_0(p, q)$, with p and q even. Let $H < G$ be a θ -stable Cartan subgroup, and let $\mu \in \mathfrak{h}$ be regular. The image of the map*

$$\Phi: G/H \cong \text{Ad}(G)\mu \hookrightarrow \mathfrak{g} \rightarrow \mathfrak{k}$$

has nonempty interior.

Proof. Write $p = 2r$, $q = 2s$, and $l = r + s$. Consider the compact Cartan subgroup $H_c = \text{SO}(2)^l < G$. For $j = 1, \dots, l$, let $h_j \in H_c$ be the matrix with a block $X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ as the j th 2×2 block on the diagonal, and zeros everywhere else. For $j, k = 1, \dots, l$ with $j < k$, define positive roots $\alpha_{jk}^\pm$ by

$$\begin{aligned} \langle \alpha_{jk}^\pm, h_j \rangle &= i, \\ \langle \alpha_{jk}^\pm, h_k \rangle &= \pm i, \end{aligned}$$

and $\langle \alpha_{jk}^\pm, h_m \rangle = 0$ for all other m . Then

$$R_n^+ = \{\alpha_{jk}^\pm; 1 \leq j \leq r, r+1 \leq k \leq l\}$$

is a choice of positive, noncompact, imaginary roots. A root vector for $\alpha_{jk}^\pm$ is the matrix $E_{\alpha_{jk}^\pm}$ with a 2×2 block

$$Y_\pm = \frac{1}{2} \begin{pmatrix} 1 & \mp i \\ -i & \mp 1 \end{pmatrix}$$

as the 2×2 block in position (j, k) and a block $-Y_\pm^T$ in position (k, j) if we divide $n \times n$ matrices into $l \times l$ blocks of size 2×2 . And

$$[E_{\alpha_{jk}^\pm}, \bar{E}_{\alpha_{jk}^\pm}] = i(h_j \pm h_k).$$

The set

$$\{h_j \pm h_k; 1 \leq j \leq r, r+1 \leq k \leq l\}$$

spans $\mathfrak{h}_c$. So Proposition 5.8 implies that $\text{im}(\Phi) \cap \mathfrak{t}$ has nonempty interior in $\mathfrak{t}$ so that $\text{im}(\Phi)$ has nonempty interior in $\mathfrak{k}$. $\square$

Lemma 5.15. *Let $G = \text{SO}_0(p, 1)$. Let $H < G$ be a θ -stable Cartan subgroup, and let $\mu \in \mathfrak{h}$ be regular. The image of the map*

$$\Phi: G/H \cong \text{Ad}(G)\mu \hookrightarrow \mathfrak{g} \rightarrow \mathfrak{k}$$

has nonempty interior.

Proof. Write $p = 2l$ or $p = 2l + 1$ depending on the parity of p . Set

$$A := \left\{ \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}; t \in \mathbb{R} \right\}.$$

Consider the maximal torus $T = \text{SO}(2)^l$ of $K = \text{SO}(p)$. A maximally compact Cartan subgroup of G is $H_c = T$ if p is even, and $H_c = T \times A$ if p is odd.

For $j = 1, \dots, l$, let h_j be the matrix whose j th 2×2 block on the diagonal is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and with all other entries zero. Consider the root α_j of $(\mathfrak{g}^\mathbb{C}, \mathfrak{h}_c^\mathbb{C})$ given by $\langle \alpha_j, h_j \rangle = i$, $\langle \alpha_j, h_k \rangle = 0$ if $k \neq j$, and, if p is odd, $\alpha_j|_{\mathfrak{a}} = 0$. A root vector for α_j is the matrix

$$E_{\alpha_j} = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ & & & -i \\ & & & 1 \\ \vdots & & & \vdots \\ 0 & \cdots & i & -1 & \cdots & 0 \end{pmatrix},$$

where the two nonzero entries in the last column are in rows $2j - 1$ and $2j$, and the two nonzero entries in the bottom row are in columns $2j - 1$ and $2j$. So α_j is an imaginary, noncompact root. The matrices $[E_{\alpha_j}, \bar{E}_{\alpha_j}] = -2ih_j$, where $j = 1, \dots, l$, span $\mathfrak{it}$. Proposition 5.8 implies that $\text{im}(\Phi) \cap \mathfrak{t}$ has nonempty interior in $\mathfrak{t}$ so that $\text{im}(\Phi)$ has nonempty interior in $\mathfrak{k}$. $\square$

Combining Corollary 5.7 and Lemmas 5.13–5.15 with the list of groups in Subsection 5.3, we obtain the following consequence of Theorem 2.7

Corollary 5.16. *If $G = \text{SU}(p, 1)$, $G = \text{SO}_0(p, 1)$, or $G = \text{SO}_0(2, 2)$, then any admissible representation of G has multiplicity-free restriction to a maximal compact subgroup.*

Koornwinder [27] proved the cases $G = \mathrm{SU}(p, 1)$ and $G = \mathrm{SO}_0(p, 1)$. We give a geometric explanation of this fact here, include the case $G = \mathrm{SO}_0(2, 2)$, and also include a geometric criterion that determines which multiplicities equal 1 (see Corollary 5.1). Using Proposition 5.8, one can investigate the groups listed at the start of this section in a similar way.

Note that $\mathrm{SU}(p, q)$ is of Hermitian type (meaning that G/K is a Hermitian symmetric space), but $\mathrm{SO}_0(p, 1)$ and $\mathrm{SO}_0(2, 2)$ are not. Therefore, Corollary 5.16 illustrates the fact that our method applies beyond the Hermitian case considered, for example, in [23]. Furthermore, $\mathrm{SO}_0(p, 1)$ has no discrete series for p odd, so we find that the method yields nontrivial results for such groups as well.

ACKNOWLEDGMENTS

The authors are grateful to Maxim Braverman, Paul-Émile Paradan, and David Vogan for their hospitality and inspiring discussions at various stages. The authors thank Michèle Vergne for the useful advice, and for input on Conjecture 5.5. The referee made several helpful suggestions, which we gratefully acknowledge.

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SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF ADELAIDE, ADELAIDE, AUSTRALIA

Email address: peter.hochs@adelaide.edu.au

DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY IN ST. LOUIS, ST. LOUIS, MISSOURI

Email address: yanlisong@wustl.edu

SCHOOL OF MATHEMATICAL SCIENCES, XIAMEN UNIVERSITY, FUJIAN, CHINA

Email address: turingfish@gmail.com