



# Stability of Logarithmic Sobolev Inequalities Under a Noncommutative Change of Measure

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## Abstract

We generalize Holley–Stroock’s perturbation argument from commutative to finite dimensional quantum Markov semigroups. As a consequence, results on (complete) modified logarithmic Sobolev inequalities and logarithmic Sobolev inequalities for self-adjoint quantum Markov processes can be used to prove estimates on the exponential convergence in relative entropy of quantum Markov systems which preserve a fixed state. This leads to estimates for the decay to equilibrium for coupled systems and to estimates for mixed state preparation times using Lindblad operators. Our techniques also apply to discrete time settings, where we show that the strong data processing inequality constant of a quantum channel can be controlled by that of a corresponding unital channel.

**Keywords** Quantum Markov semigroup · Relative entropy · Decay estimate · Modified logarithmic Sobolev inequality

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# 1 Introduction

Quantum information theory concerns the study of information theoretic tasks that can be achieved using quantum systems (e.g. photons, electrons and atoms) as information carriers, with the long-term promise that it will revolutionize our way of computing, communicating and designing new materials. However, in realistic settings, quantum systems undergo unavoidable interactions with their environment. This gives rise to the phenomenon of decoherence, which leads to a loss of the information initially contained in the system [19]. Within the context of emerging quantum information-processing devices, gaining quantitative knowledge about the effect of decoherence is one of the main near-term challenges for the design of methods to achieve scalable quantum fault-tolerance. Quantifying decoherence is known to be a difficult problem in general, already for classical systems. Two facts make the situation even more challenging in the quantum regime: (i) the non-commutativity of quantum observables, and (ii) the potential for entanglement between systems.

Quantum Markov semigroups (QMS) constitute a particularly interesting class of noise that decomposes into successive applications of a quantum channel representing an arbitrarily small amount of time. Most recent approaches aim at quantifying decoherence from Markov semigroups using *functional inequalities* (FIs). The latter are differential versions of strong contraction properties of various distance measures under the action of the semigroup. For instance, the Poincaré inequality provides an estimate on the spectral gap of the semigroup. Exponentially faster convergence can be achieved via the existence of a logarithmic Sobolev inequality (LSI), which implies a strong contraction of weighted  $L^p$ -norms under the action of the semigroup known as hypercontractivity. Similarly, the modified logarithmic Sobolev inequality (MLSI) governs the exponential convergence in relative entropy of any initial state evolving according to the semigroup towards equilibrium.

In the commutative or classical setting, one of the key features of logarithmic Sobolev inequalities is their stability under the action of coupling with an auxiliary system. This fact implies that many such FIs can be ultimately deduced from an inequality over a two-point space. For quantum systems, entanglement with an auxiliary system may preclude the existence of LSI, but MLSI extends to the stronger notion of complete (modified) logarithmic Sobolev inequality (CLSI). CLSI plays an analogous role to LSI for studying multiplicativity properties of QMSs.

**Classical relative entropy and the Holley–Stroock perturbation argument** The focus of this paper is on inequalities in terms of relative entropy. Though we aim to show quantum inequalities in the finite-dimensional setting, we here recall the classical Holley–Stroock argument in its original, infinite-dimensional setting. In particular, our primary method is a ‘quantized version’ of an argument by Holley and Stroock [28] which allows one to transfer estimates between relative entropies with respect to different measures. For any two probability measures  $\nu \ll \mu$  on  $\mathbb{R}^n$ , their classical relative entropy is given by

$$D(\nu \parallel \mu) \equiv \text{Ent}_\mu(f) := \int f \ln f d\mu - \int f d\mu \ln \int f d\mu ,$$

where  $f$  is defined as the Radon-Nikodym derivative  $\frac{d\nu}{d\mu}$ . When  $\nu, \mu$  are probability measures on finite spaces, the relative entropy reduces to the familiar form of the Kullback-Leibler divergence given by

$$D(\nu \parallel \mu) = \sum_i v_i \ln v_i - v_i \ln \mu_i .$$

Thanks to the positivity of  $G(a, b) = a \log a - b \log b + b - a$  for  $a, b > 0$  [28], this relative entropy admits a variational characterization

$$\text{Ent}_\mu(f) = \inf_{c>0} \int (f \ln f - c \ln c + c - f) d\mu.$$

Therefore, given any other probability measure  $\mu' \ll \mu$ , the positivity of  $G$  implies the following stability property of the relative entropy:

$$\text{Ent}_{\mu'}(f) \leq \left\| \frac{d\mu'}{d\mu} \right\|_\infty \text{Ent}_\mu(f), \quad (1.1)$$

whenever the Radon-Nikodym derivative  $\frac{d\mu'}{d\mu}$  is uniformly bounded, where  $\|\cdot\|_\infty$  refers to the  $L^\infty$  norm here. A similar argument holds for the functional derivative of the relative entropy, or *Fisher Information*

$$I_\mu(f) := \int L(f) \ln f d\mu = \int \frac{|\nabla(f)|^2}{f} d\mu, \quad (1.2)$$

for any “regular enough”  $f$ , whenever the generator of a diffusion semigroup  $(T_t = e^{-tL})_{t \geq 0}$  is given as  $L(f) = -\Delta f + \nabla V \cdot \nabla f$  with respect to the derivation  $\nabla(f) = (\frac{df}{dx_1}, \dots, \frac{df}{dx_n})$  on  $\mathbb{R}^n$ , and for  $d\mu = e^{-V} dx$  and  $V \in C^2(\mathbb{R}^n)$ . Again, thanks to the positivity of  $\frac{|\nabla f|^2}{f}$ , we deduce that, if  $\mu \ll \mu'$

$$I_\mu(f) \leq \left\| \frac{d\mu}{d\mu'} \right\|_\infty I_{\mu'}(f). \quad (1.3)$$

The (*modified*) *logarithmic Sobolev inequality* (MLSI) is defined as follows: for any regular enough function  $f$ ,

$$\alpha \text{Ent}_\mu(f) \leq I_\mu(f).$$

The largest constant  $\alpha > 0$  satisfying this inequality is denoted by  $\alpha_\mu$  and called the *modified logarithmic Sobolev constant*. Note that, by the equivalent formulation of the Fisher information in terms of differential operators (1.2), this inequality can be merely interpreted as a property of the measure  $\mu$ . Hence, using the perturbation bounds previously mentioned, the Holley–Stroock perturbation bound is formulated as follows:

**Theorem 1.1** (Holley–Stroock [28]) *Let  $\mu \sim \mu'$  be equivalent measures. Then*

$$\alpha_\mu \leq \left\| \frac{d\mu}{d\mu'} \right\|_\infty \left\| \frac{d\mu'}{d\mu} \right\|_\infty \alpha_{\mu'}.$$

We refer to [32] for a wealth of interesting examples, in particular a derivation of logarithmic Sobolev inequalities at finite temperature using known estimates at infinite temperature. From a more applied angle the most impressive application of MLSI is the entropic exponential convergence of the corresponding semigroup  $(\mathcal{P}_t)_{t \geq 0}$ :

$$\text{Ent}_\mu(\mathcal{P}_t(f)) \leq e^{-\alpha t} \text{Ent}_\mu(f).$$

The best constant working for all  $f$  and  $t \geq 0$  is exactly the MLSI constant  $\alpha_\mu$ .

**Quantum (modified) logarithmic Sobolev inequalities** A standard procedure historically used to obtain estimates for the above entropy decay is to use an equivalent differential formulation of the notion of hypercontractivity, also known as *logarithmic Sobolev inequalities* (or LSI) [1, 32]. Despite the existence of logarithmic Sobolev inequalities [40, 47] for *primitive*

quantum Markov semigroups, that is for semigroups possessing a unique invariant state, it was shown in [3] that these inequalities cannot be derived for non-primitive semigroups. In particular, the natural notion of a logarithmic Sobolev inequality for semigroups of the form  $(\mathcal{P}_t \otimes \text{id}_R)_{t \geq 0}$  given some reference system  $R$ , as previously introduced in [6], is known to fail at providing entropic convergence. Introduced by Bobkov and Tetali [10] for the study of Markov chains over discrete configuration spaces, MLSI turns out to be more stable. The quantum MLSI was introduced by Kastoryano and Temme in [30] for primitive evolutions. In [2], Bardet showed that the MLSI can also be extended to non-primitive semigroups. MLSI is equivalent to LSI for classical diffusions but provides estimates on the entropy decay of non-hypercontractive jump processes [42]. In the quantum setting, however, MLSI is not necessarily stable under tensor products. Quantum entanglement between subsystems may allow composite systems to escape from the strong decay implied by MLSI. This instability led the authors of [24] to define the notion of CLSI for the study of the convergence of the tensor product evolution of a given quantum Markov semigroup with the identity map on an arbitrarily large system. CLSI is tensor-stable, allowing one to combine local estimates into global bounds on composite systems.

Quantum functional inequalities are still notoriously harder to derive than their classical analogues. For a long time, only the Poincaré inequality had been shown to hold for lattice spin systems subject to the so-called *heat-bath* and *Davies* semigroups, under some conditions on the equilibrium Gibbs state of these evolutions [29, 46]. CLSI in these settings remained unknown until several results following the initial version of this paper. After the first version of this preprint had been written it was shown by [23] that every finite dimensional  $\sigma$ -detailed balance generator satisfies  $\alpha_{\text{CLSI}}(\mathcal{L}) > 0$ . An alternative proof based on geometric arguments can be found in [25]. The second proof first shows that result for trace preserving generators and then uses the change of measure argument in this paper. The precise form of the noncommutative Holley–Stroock argument also shows that the CLSI constant is stable under small perturbations of a preserved state provided the same derivations are used. Nonetheless, the calculation of good constants remains open in many cases and is essential for many applications. Hence, it is desirable to have a quantum version of Holley–Stroock’s argument, because it would allow to transfer results from one reference state (say the completely mixed state) to another (say a Gibbs state at finite temperature). As we have seen, the main ingredients for the classical proof are (i) *variational principle*, (ii) a good understanding of the notion of *gradient*, and (iii) the pointwise positivity of the *Fisher information function*  $(\nabla f, \nabla \ln f)$ . Generalizing them to the quantum setting requires additional deep insight from the theory of quantum Markov semigroups and operator algebras. Such an approach is facilitated by recent developments of trace inequalities in quantum information theory.

For a semigroup of completely positive unital maps  $\mathcal{P}_t : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  and generator  $\mathcal{L} := \frac{d}{dt} \big|_{t=0} \mathcal{P}_t$ , we denote by  $\mathcal{P}_{t*}$  the adjoint with respect to the trace  $\text{Tr}(\mathcal{P}_{t*}(\rho)X) = \text{Tr}(\rho \mathcal{P}_t(X))$ , and  $\mathcal{E}_* = \lim_{t \rightarrow \infty} \mathcal{P}_{t*}$  [12]. Recall that a faithful quantum Markov semigroup  $(\mathcal{P}_t := e^{-t\mathcal{L}})_{t \geq 0}$ , of corresponding conditional expectation  $\mathcal{E}$  towards its fixed-point algebra and full-rank invariant state  $\sigma$ , satisfies a *weak logarithmic Sobolev inequality* (LSI) with constants  $c > 0$  and  $d \geq 0$  if the following holds: for any positive definite state  $\rho$ ,

$$D(\rho \| \mathcal{E}_*(\rho)) \leq c \mathcal{E}_{\mathcal{L}}(\sigma^{-\frac{1}{4}} \rho^{\frac{1}{2}} \sigma^{-\frac{1}{4}}) + d \left\| \sigma^{-\frac{1}{4}} \rho^{\frac{1}{2}} \sigma^{-\frac{1}{4}} \right\|_{\mathbb{L}_2(\sigma)}^2. \quad (\text{LSI}(c, d))$$

Here, the *Dirichlet form*  $\mathcal{E}_{\mathcal{L}}$  is defined as  $\mathcal{E}_{\mathcal{L}}(X) := \langle \mathcal{L}(X), X \rangle_{\sigma} = \text{Tr}(\sigma^{\frac{1}{2}} \mathcal{L}(X)^* \sigma^{\frac{1}{2}} X)$ , and the non-commutative  $\mathbb{L}_p$  norms are defined as

$$\|X\|_{\mathbb{L}_p(\Sigma)} := \left( \text{Tr} |\sigma^{1/2p} X \sigma^{1/2p}|^p \right)^{\frac{1}{p}}.$$

In analogy with the classical setting,  $\mathcal{L}$  is said to satisfy a *modified logarithmic Sobolev inequality* if there exists a constant  $\alpha > 0$  such that for all density matrices  $\rho$ :

$$D(\mathcal{P}_{t*}(\rho) \| \mathcal{E}_*(\rho)) \leq e^{-\alpha t} D(\rho \| \mathcal{E}_*(\rho)).$$

The largest constant  $\alpha$  such that this inequality holds for all  $\rho$  is denoted by  $\alpha_{\text{MLSI}}(\mathcal{L})$ . Similarly, we denote by  $\alpha_{\text{CLSI}}(\mathcal{L})$  the largest constant  $\tilde{\alpha}$  such that

$$D((\mathcal{P}_{t*} \otimes \text{id}_R)(\rho) \| (\mathcal{E}_* \otimes \text{id}_R)(\rho)) \leq e^{-\tilde{\alpha} t} D(\rho \| (\mathcal{E}_* \otimes \text{id}_R)(\rho))$$

holds for all  $t \geq 0$ , any reference system  $\mathcal{H}_R$ , and any density matrix  $\rho$  on  $\mathcal{H} \otimes \mathcal{H}_R$ . The advantage of the complete version is that for any two generators  $\mathcal{L}_1$  and  $\mathcal{L}_2$ :

$$\alpha_{\text{CLSI}}(\mathcal{L}_1 \otimes \text{id} + \text{id} \otimes \mathcal{L}_2) \geq \min\{\alpha_{\text{CLSI}}(\mathcal{L}_1), \alpha_{\text{CLSI}}(\mathcal{L}_2)\}.$$

In this article, we make another step towards proving CLSI for any finite dimensional quantum Markov semigroup by adapting the Holley–Stroock argument to the quantum setting, based on the seminal work of Carlen and Maas [13]. Following Carlen–Maas, the generator of a QMS satisfying the so-called *detailed balance condition* (see Sect. 2 for more details) is given by

$$\mathcal{L}(X) = - \sum_{j \in \mathcal{J}} \left( e^{-\omega_j/2} A_j^* [X, A_j] + e^{\omega_j/2} [A_j, X] A_j^* \right).$$

Here, the Bohr frequencies  $\omega_j \in \mathbb{R}$  are determined by the additional condition  $\sigma A_j \sigma^{-1} = e^{-\omega_j} A_j$ , for some full-rank state  $\sigma$  such that  $\mathcal{L}_*(\sigma) = 0$ . Choosing these frequencies to be equal to 0, we end up with the corresponding noncommutative *heat semigroup*:

$$\mathcal{L}_0(X) = - \sum_{j \in \mathcal{J}} \left( A_j^* [X, A_j] + [A_j, X] A_j^* \right) = \sum_j A_j^* A_j X + X A_j A_j^* - A_j X A_j^* - A_j^* X A_j.$$

In its simplest form, our noncommutative Holley–Stroock argument can be stated as follows:

**Theorem 1.2** *Assume that  $(\mathcal{P}_t = e^{-t\mathcal{L}})_{t \geq 0}$  is a primitive, finite dimensional quantum Markov semigroup with corresponding unique, full-rank invariant state  $\sigma = \sum_k \sigma_k |k\rangle\langle k|$  and satisfies the detailed balance condition. Then*

$$\alpha_{\text{CLSI}}(\mathcal{L}_0) \leq \frac{\max_k \sigma_k}{\min_k \sigma_k} \max_j e^{\omega_j/2} \alpha_{\text{CLSI}}(\mathcal{L}).$$

As an application of this result, we consider a primitive quantum Markov semigroup  $(\mathcal{P}_{t*} = e^{-t\mathcal{L}*})_{t \geq 0}$  on  $\mathcal{B}(\mathcal{H})$  for finite dimensional  $\mathcal{H}$ , which produces a certain full-rank state  $\sigma = \sum_k \sigma_k |k\rangle\langle k|$ :

$$\forall \rho : \lim_{t \rightarrow \infty} \mathcal{P}_{t*}(\rho) = \sigma \quad \text{and} \quad \alpha_{\text{CLSI}}(\mathcal{L}) > 0.$$

Our lower bound for  $\alpha_{\text{CLSI}}$  depends in an explicit way on the ratios  $\frac{\sigma_k}{\sigma_l}$ . On a suitably chosen inner product  $(\cdot, \cdot)$  the derivations stabilizing  $\sigma$  are exactly given by commutators with respect to matrix units  $|k\rangle\langle j|$ . In other words the density ‘determines’ its own derivation  $\delta$  and the corresponding gradient form  $(\delta(f), \delta(f))$ , in contrast to the above classical setting. In our construction, we have to work with invertible densities if we want to have complete

logarithmic Sobolev inequalities, and hence our results are complementary to the results in [31, 48] on quantum Markov semigroups producing pure states.

**Outline of the paper** In Sect. 2, we recall basic aspects of the theory of quantum Markov semigroups and complete modified logarithmic Sobolev inequalities. In particular, we derive a useful form for the entropy production of a semigroup by means of noncommutative differential calculus. The essence of the quantum Holley–Stroock perturbation argument is first provided in Sect. 3 where we compare a non-unital quantum Markov semigroup to a corresponding unital one. In Sect. 4, we extend the previous argument to (i) non-primitive quantum Markov semigroups and (ii) the logarithmic Sobolev inequality. A similar argument is given in Section 5 in order to derive strong data processing inequalities for non self-adjoint quantum channels. Sects. 6 and 6.1 focus on applications to the dissipative preparation of mixed state and Gibbs samplers.

## 2 Quantum Markov Semigroups and Entropy Decay

In this section, we briefly review the notions of quantum Markov semigroups and their related noncommutative derivations on the algebra  $\mathcal{B}(\mathcal{H})$  of bounded operators on a finite-dimensional Hilbert space, and explain how in this framework, the generator of a QMS should be interpreted as a noncommutative second order differential operator. We will have to recall and adapt some of the notations from the seminal papers by Carlen and Maas [13, 14] for Lindblad generators satisfying the detailed balance condition (see also [20]).

**Notations and definitions** Let  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  be a finite dimensional Hilbert space of dimension  $d_{\mathcal{H}}$ . We denote by  $\mathcal{B}(\mathcal{H})$  the space of bounded operators on  $\mathcal{H}$ , by  $\mathcal{B}_{\text{sa}}(\mathcal{H})$  the subspace of self-adjoint operators on  $\mathcal{H}$ , i.e.  $\mathcal{B}_{\text{sa}}(\mathcal{H}) = \{X \in \mathcal{B}(\mathcal{H}); X = X^*\}$ , and by  $\mathcal{B}_+(\mathcal{H})$  the cone of positive semidefinite operators on  $\mathcal{H}$ , where the adjoint of an operator  $Y$  is written as  $Y^*$ . The identity operator on  $\mathcal{H}$  is denoted by  $1_{\mathcal{H}}$ , dropping the index  $\mathcal{H}$  when it is unnecessary. In the case when  $\mathcal{H} \equiv \mathbb{C}^{\ell}$ ,  $\ell \in \mathbb{N}$ , we will also use the notation  $1$  for  $1_{\mathbb{C}^{\ell}}$ . Similarly, we will denote by  $\text{id}_{\mathcal{H}}$ , or simply  $\text{id}$ , resp.  $\text{id}_{\ell}$ , the identity superoperator on  $\mathcal{B}(\mathcal{H})$  and  $\mathcal{B}(\mathbb{C}^{\ell})$ , respectively. We denote by  $\mathcal{D}(\mathcal{H})$  the set of positive semidefinite, trace one operators on  $\mathcal{H}$ , also called *density operators*, and by  $\mathcal{D}_+(\mathcal{H})$  the subset of full-rank density operators. In the following, we will often identify a density matrix  $\rho \in \mathcal{D}(\mathcal{H})$  and the *state* it defines, that is the positive linear functional  $\mathcal{B}(\mathcal{H}) \ni X \mapsto \text{Tr}(\rho X)$ . By  $\|\cdot\|_p$  we denote the Schatten  $p$ -norm. In particular, we will often use the operator norm denoted  $\|\cdot\|_{\infty}$  and the trace norm denoted  $\|\cdot\|_1$ . By  $\text{supp}(\rho)$  we denote the support of density  $\rho$ .

Given two positive operators  $\rho, \sigma \in \mathcal{B}_+(\mathcal{H})$ , the relative entropy between  $\rho$  and  $\sigma$  is defined as follows:

$$D(\rho\|\sigma) := \begin{cases} \text{Tr}(\rho (\ln \rho - \ln \sigma)) & \text{supp}(\rho) \subseteq \text{supp}(\sigma) \\ +\infty & \text{else} \end{cases}$$

We recall that, given  $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$  a finite dimensional von-Neumann subalgebra of  $\mathcal{B}(\mathcal{H})$  and a full-rank state  $\sigma \in \mathcal{D}(\mathcal{H})$ , a linear map  $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{N}$  is called a *conditional expectation* with respect to  $\sigma$  of  $\mathcal{B}(\mathcal{H})$  onto  $\mathcal{N}$  if the following conditions are satisfied:

- (i) For all  $X \in \mathcal{B}(\mathcal{H})$ ,  $\|\mathcal{E}(X)\|_{\infty} \leq \|X\|_{\infty}$ ;
- (ii) For all  $X \in \mathcal{N}$ ,  $\mathcal{E}(X) = X$ ;
- (iii) For all  $X \in \mathcal{B}(\mathcal{H})$ ,  $\text{Tr}(\sigma \mathcal{E}(X)) = \text{Tr}(\sigma X)$ .

**Quantum Markov semigroups and noncommutative derivations** The basic model for the evolution of an open system in the Markovian regime is given by a quantum Markov

semigroup (or QMS)  $(\mathcal{P}_t)_{t \geq 0}$  acting on  $\mathcal{B}(\mathcal{H})$ . Such a semigroup is characterised by its generator, called the Lindblad  $\mathcal{L}$ , which is defined on  $\mathcal{B}(\mathcal{H})$  by  $\mathcal{L}(X) = \lim_{t \rightarrow 0} \frac{1}{t} (X - \mathcal{P}_t(X))$  for all  $X \in \mathcal{B}(\mathcal{H})$ , so that  $\mathcal{P}_t = e^{-t\mathcal{L}}$ .<sup>1</sup> The QMS is said to be *primitive* if it admits a unique invariant state  $\sigma$ , which in our setting we assume to have full rank. In this paper, we exclusively study QMS that satisfy the following *detailed balance condition* with respect to some full-rank invariant state  $\sigma$  (also referred to as GNS-symmetry): for any  $X, Y \in \mathcal{B}(\mathcal{H})$  and any  $t \geq 0$ ,

$$\mathrm{Tr}(\sigma X^* \mathcal{P}_t(Y)) = \mathrm{Tr}(\sigma \mathcal{P}_t(X)^* Y). \quad (\sigma\text{-DBC})$$

In particular, this condition is known to be equivalent to self-adjointness of the generator with respect to the so-called KMS inner product

$$\langle A, B \rangle_\sigma := \mathrm{Tr}(\sigma^{\frac{1}{2}} A^* \sigma^{\frac{1}{2}} B). \quad (2.1)$$

Via Theorem 2.9 in [13], the semigroup will also commute with the modular group of  $\sigma$ :  $\Delta_\sigma^{it} \circ \mathcal{L} = \mathcal{L} \circ \Delta_\sigma^{it}$  for all  $t \in \mathbb{R}$ , where  $\Delta_\sigma(X) := \sigma X \sigma^{-1}$ . It was also shown in [13] that the generator of such a semigroup has the following GKLS form [26, 35] for all  $X \in \mathcal{B}(\mathcal{H})$ ,

$$\mathcal{L}(X) = - \sum_{j \in \mathcal{J}} \left( e^{-\omega_j/2} A_j^* [X, A_j] + e^{\omega_j/2} [A_j, X] A_j^* \right). \quad (2.2)$$

where the sum runs over a finite number of *Lindblad operators*  $\{A_j\}_{j \in \mathcal{J}} = \{A_j^*\}_{j \in \mathcal{J}}$  and  $[\cdot, \cdot]$  denotes the commutator defined as  $[X, Y] := XY - YX$ ,  $\forall X, Y \in \mathcal{B}(\mathcal{H})$ , and  $\omega_j \in \mathbb{R}$ . Moreover, the Lindblad operators  $A_j$  satisfy the following relations:

$$\forall s \in \mathbb{R}, \Delta_\sigma^s(A_j) := \sigma^s A_j \sigma^{-s} = e^{-\omega_j s} A_j \quad \Rightarrow \quad \delta_{A_j}(\ln \sigma) = -\omega_j A_j, \quad (2.3)$$

where the second identity comes from differentiability of the first one at  $s = 0$ , and  $\delta_{A_j}(X) := [A_j, X]$  is a noncommutative derivation. Therefore, the reals  $\omega_j$  can be interpreted as differences of eigenvalues of the Hamiltonian corresponding to the Gibbs state  $\sigma$ , also called *Bohr frequencies*. It is important to note that  $\mathcal{L}$  is the generator in the Heisenberg picture. The generator  $\mathcal{P}_{t*} = e^{-t\mathcal{L}^*}$  in the Schrödinger picture is defined via

$$\mathrm{Tr}(\mathcal{L}_*(\rho)X) = \mathrm{Tr}(\rho \mathcal{L}(X)).$$

According to [13, Remark 3.3] the adjoint has the form

$$\begin{aligned} \mathcal{L}_*(\rho) &= - \sum_j \left( e^{-\omega_j/2} [A_j \rho, A_j^*] + e^{\omega_j/2} [A_j^*, \rho A_j] \right) \\ &= \sum_j e^{-\omega_j/2} (A_j^* A_j \rho - A_j \rho A_j^*) + e^{\omega_j/2} (\rho A_j A_j^* - A_j^* \rho A_j). \end{aligned} \quad (2.4)$$

The generator  $\mathcal{L}_0 := \sum_{j \in \mathcal{J}} \mathcal{L}_{A_j}$ , corresponding to taking all the Bohr frequencies to 0, satisfies the detailed balance condition with respect to the completely mixed state  $1/d_{\mathcal{H}}$ . Because of its analogy with the classical diffusive case, its corresponding QMS is usually called the *heat semigroup*. In fact, given a Lindblad operator  $A$ , the generators  $\mathcal{L}_A := [A^*, [A, \cdot]]$  satisfies the following non-commutative integration by parts:

$$\mathrm{Tr}(X^* \mathcal{L}_A(Y)) = \mathrm{Tr}(\delta_A(X)^* \delta_A(Y)) = \mathrm{Tr}(\mathcal{L}_A(X)^* Y).$$

<sup>1</sup> Let us note that our sign convention is opposite to the one usually used in the community of open quantum systems, but more common in abstract semigroup theory.

We may also consider  $B = \frac{A+A^*}{\sqrt{2}}$  and  $C = \frac{A-A^*}{\sqrt{2}i}$  and observe that

$$\mathcal{L}_A(X) := (B^2 + C^2)X + X(B^2 + C^2) - 2(BXB + CXC). \quad (2.5)$$

has the standard form of a self-adjoint Lindbladian, with corresponding self-adjoint Lindblad operators  $B$  and  $C$ , and in particular is  $*$ -preserving. In the GNS-symmetric case, the integration by parts formula reads as follows:

$$\langle \mathcal{L}(X), Y \rangle_\sigma = \sum_{j \in \mathcal{J}} \langle \delta_{A_j}(X), \delta_{A_j}(Y) \rangle_\sigma, \quad (2.6)$$

where the KMS inner product was defined in (2.1). Because of their particular symmetry property, self-adjoint semigroups (that is w.r.t. the Hilbert-Schmidt inner product) are currently better understood than their GNS-symmetric generalizations [5, 39]. The purpose of this paper is to derive a technique to transfer estimates on the entropic rate of convergence towards equilibrium of  $(e^{-t\mathcal{L}})_{t \geq 0}$  in terms of that of  $(e^{-t\mathcal{L}_0})_{t \geq 0}$ . A useful tool will be the following commuting diagram

$$\begin{array}{ccc} \mathcal{B}(\mathcal{H}) & \xrightarrow{e^{-t\mathcal{L}}} & \mathcal{B}(\mathcal{H}) \\ \downarrow \Gamma_\sigma & & \downarrow \Gamma_\sigma \\ \mathcal{T}_1(\mathcal{H}) & \xrightarrow{e^{-t\mathcal{L}_*}} & \mathcal{T}_1(\mathcal{H}) \end{array} \quad (2.7)$$

where  $\Gamma_\sigma(x) = \sigma^{1/2}x\sigma^{1/2}$  is the canonical completely positive map from the algebra  $\mathcal{B}(\mathcal{H})$  to the space  $\mathcal{T}_1(\mathcal{H})$  which can be interpreted as the predual  $\mathcal{B}(\mathcal{H})_*$  of  $\mathcal{B}(\mathcal{H})$  [30, 37, 40, 47]. Indeed, we recall from [13] that  $\mathcal{L}$  is also self-adjoint with the KMS inner product  $\langle X, Y \rangle_\sigma = \text{Tr}(\Gamma_\sigma(X^*)Y)$ , and hence

$$\text{Tr}(\mathcal{L}_*(\Gamma_\sigma(X^*))Y) = \text{Tr}(\Gamma_\sigma(X^*)\mathcal{L}(Y)) = \langle X, \mathcal{L}(Y) \rangle_\sigma = \langle \mathcal{L}(X), Y \rangle_\sigma = \text{Tr}(\Gamma_\sigma(\mathcal{L}(X^*))Y)$$

shows that indeed

$$\mathcal{L}_*(\Gamma_\sigma(X)) = \Gamma_\sigma(\mathcal{L}(X)). \quad (2.8)$$

**Entropic convergence of QMS** Under the condition of GNS-symmetry, the semigroup  $(e^{-t\mathcal{L}})_{t \geq 0}$  is known to be *ergodic* [22]: there exists a *conditional expectation*  $\mathcal{E}$  onto the fixed-point algebra  $\mathcal{F}(\mathcal{L}) := \{X \in \mathcal{B}(\mathcal{H}) : \mathcal{L}(X) = 0\}$  such that

$$e^{-t\mathcal{L}} \xrightarrow[t \rightarrow \infty]{} \mathcal{E}.$$

In this paper, we are interested in the exponential convergence in relative entropy of the semigroup towards its corresponding conditional expectation. The *entropy production* (also known as *Fisher information*) of  $(\mathcal{P}_t = e^{-t\mathcal{L}})_{t \geq 0}$  is defined as the opposite of the derivative of the relative entropy with respect to the invariant state: for any  $\rho \in \mathcal{D}(\mathcal{H})$ ,

$$\text{EP}_{\mathcal{L}}(\rho) := - \left. \frac{d}{dt} \right|_{t=0} D(\mathcal{P}_{t*}(\rho) \| \mathcal{E}_*(\rho)) = \text{Tr}(\mathcal{L}_*(\rho)(\ln \rho - \ln \sigma)),$$

where the expression on the right hand side of the above equation was first proved in [45] in the primitive setting. We will also need to extend the definition of the entropy production to non-normalized states  $\rho$  using the same expression as on the right-hand side of the above equation. In this paper, we are interested in the uniform exponential convergence in relative entropy of systems evolving according to a QMS towards equilibrium: more precisely, we

ask the question of the existence of a positive constant  $\alpha > 0$  such that the following holds, for any  $\rho \in \mathcal{D}(\mathcal{H})$ ,

$$D(\mathcal{P}_{t*}(\rho) \| \mathcal{E}_*(\rho)) \leq e^{-\alpha t} D(\rho \| \mathcal{E}_*(\rho)).$$

After differentiation at  $t = 0$  and using the semigroup property, this inequality is equivalent to the following *modified logarithmic Sobolev constant* (MLSI) [4, 7, 11, 30, 38]: for any  $\rho \in \mathcal{D}(\mathcal{H})$ ,

$$\alpha D(\rho \| \mathcal{E}_*(\rho)) \leq \text{EP}_{\mathcal{L}}(\rho). \quad (\text{MLSI})$$

The best constant  $\alpha$  achieving this bound is called the modified logarithmic Sobolev constant of the semigroup, and is denoted by  $\alpha_{\text{MLSI}}(\mathcal{L})$ . We may also consider the complete version which requires

$$\alpha_{\text{CLSI}}(\mathcal{L}) D(\rho \| (\mathcal{E}_* \otimes \text{id})(\rho)) \leq \text{EP}_{(\mathcal{L} \otimes \text{id})}(\rho). \quad (\text{CLSI})$$

to hold for all  $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$  for any system  $B$  (or even for  $\mathcal{B}(\mathcal{H}_B)$  replaced by a finite-dimensional von Neumann algebra).

**Primitive semigroups** Our main goal is to establish MLSI and CLSI for primitive semigroups, given similar knowledge for self-adjoint semigroups. Recall that  $(\mathcal{P}_t = e^{-t\mathcal{L}})_{t \geq 0}$  is called primitive if it admits a full-rank fixed point state  $\sigma$  such that  $\mathcal{P}_{t*}(\rho) = \rho$  for all  $t$  implies  $\rho = \sigma$ . This is equivalent to

$$\mathcal{L}_*(\rho) = 0 \Rightarrow \rho = \sigma.$$

We recall that  $\mathcal{L}$  in Equation (2.2) is self-adjoint with respect to the inner product  $\langle A, B \rangle_{\sigma} = \text{Tr}(A^* \sigma^{1/2} B \sigma^{1/2})$ . Therefore, we deduce that

$$\begin{aligned} 0 &= \langle \mathcal{L}(X), X \rangle_{\sigma} = \text{Tr}(\sigma^{1/2} \mathcal{L}(X)^* \sigma^{1/2} X) = \text{Tr}(\mathcal{L}(X^*) \sigma^{1/2} X \sigma^{1/2}) \\ &= \text{Tr}(X^* \mathcal{L}_*(\sigma^{1/2} X \sigma^{1/2})) \end{aligned}$$

if and only if  $[A_j, X] = 0$  for all  $j$ , by Equation (2.6). This implies that  $\mathcal{L}_*(\sigma^{1/2} X \sigma^{1/2}) = 0$  if and only if  $X \in \{A_j : j \in J\}'$ . Let us state this for later references.

**Lemma 2.1** *Let  $\mathcal{L}_*$  be given by Equation (2.4). The following are equivalent.*

- (i)  $\mathcal{L}$  is primitive with respect to  $\sigma$ ;
- (ii)  $\{A_j : j \in J\}' = \mathbb{C}1$ ;
- (iii)  $\mathcal{L}_0 = \sum_j \mathcal{L}_{A_j}$  is ergodic, i.e.  $\mathcal{L}_0(X) = 0$  implies  $X = \lambda 1$ .

The equivalence (ii)  $\leftrightarrow$  (iii) follows [13, Theorem 5.3]. Equivalence to (i) follows from the text above.

**Noncommutative differential calculus via double operator integrals** The entropy production can be written in a different form that will be more convenient for our purpose. In order to derive it, we first need to recall some notions of non-commutative differential calculus (see [8, 9, 15–18, 41, 43]). Given an operator  $L \in \mathcal{B}(H)$ , as well as any two self-adjoint operators  $X, Y \in \mathcal{B}_{sa}(\mathcal{H})$ , define the operator

$$C_A^{X,Y} := AY - XA.$$

In particular  $C_A^{X,X} := \delta_A(X)$ . Next, given a Borel function  $h : \text{sp}(X) \times \text{sp}(Y) \rightarrow \mathbb{R}$ , and writing by  $P_X$  and  $P_Y$  the spectral measures of  $X$  and  $Y$ , define the so-called *double operator*

integral

$$\mathcal{T}_h := \int h L_{P_X} R_{P_Y}.$$

where  $L_Z$ , resp.  $R_Z$ , is the operator of left, resp. right multiplication by  $Z$ . Given a differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we are exclusively interested in the restriction of the difference quotient  $\tilde{f}$  associated with  $f$  given by

$$\tilde{f}(x, y) := \begin{cases} \frac{f(x) - f(y)}{x - y} & \text{if } (x, y) \in \text{sp}(X) \times \text{sp}(Y) \\ \frac{\partial f(x)}{\partial x} & \text{else} \end{cases}. \quad (2.9)$$

**Theorem 2.2** (Noncommutative chain rule for differentiation, see [9]) *Given an operator  $A \in \mathcal{B}(\mathcal{H})$ , any two self-adjoint operators  $X, Y \in \mathcal{B}_{sa}(\mathcal{H})$  and a Borel function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , the following holds:*

$$C_A^{f(X), f(Y)} = \mathcal{T}_{\tilde{f}}^{X, Y}(C_A^{X, Y}).$$

We use the notation and fact that

$$[\Gamma_\sigma(X)]_{\omega_j}^{-1} := \mathcal{T}_{\ln}^{Z_j, Y_j}(X) = \int_0^\infty (r + e^{-\omega_j/2} L_{\Gamma_\sigma(X)})^{-1} (r + e^{\omega_j/2} R_{\Gamma_\sigma(X)})^{-1} dr$$

as in [13]. With the previous theorem at hand, the following result can be proved:

**Lemma 2.3** *Assume that the QMS  $(\mathcal{P}_t = e^{-t\mathcal{L}})_{t \geq 0}$  satisfies  $\sigma$ -DBC. Then, for any positive operator  $\rho = \Gamma_\sigma(X)$ ,*

$$\text{EP}_{\mathcal{L}}(\rho) = \sum_{j \in \mathcal{J}} (\Gamma_\sigma(\delta_{A_j}(X)), [\Gamma_\sigma(X)]_{\omega_j}^{-1} (\Gamma_\sigma(\delta_{A_j}(X))))_{\text{HS}}. \quad (2.10)$$

Moreover, the same formula holds for positive  $\rho \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$  and  $\mathcal{L}$  replaced by  $\mathcal{L} \otimes \text{id}_{\mathcal{K}}$ .

**Proof** By definition, for all positive  $\rho \in \mathcal{B}(\mathcal{H})$ , and any  $\sigma \in \mathcal{F}(\mathcal{L})$ , letting  $X := \Gamma_\sigma^{-1}(\rho)$  we have

$$\begin{aligned} \text{EP}_{\mathcal{L}}(\rho) &= \text{Tr}(\mathcal{L}_*(\rho)(\ln \rho - \ln \sigma)) \\ &= \langle \mathcal{L}(X), \ln \rho - \ln \sigma \rangle_\sigma \\ &= \sum_{j \in \mathcal{J}} \langle \delta_{A_j}(X), \delta_{A_j}(\ln \rho - \ln \sigma) \rangle_\sigma. \end{aligned}$$

Here the second line follows by Equation 2.8, whereas the third line follows by the integration by parts formula (2.6). Now, due to (2.3),  $\delta_{A_j}(\ln \sigma) = -\omega_j A_j$ , so that, denoting  $Y_j := \rho e^{-\omega_j/2}$  and  $Z_j := \rho e^{\omega_j/2}$ , we have

$$\begin{aligned} \delta_{A_j}(\ln \rho - \ln \sigma) &= A_j \ln(Y_j) - \ln(Z_j) A_j \\ &= C_{A_j}^{\ln(Z_j), \ln(Y_j)} \\ &= \mathcal{T}_{\ln}^{Z_j, Y_j}(C_{A_j}^{Z_j, Y_j}) \\ &= \mathcal{T}_{\ln}^{Z_j, Y_j}(\sigma^{1/2} \delta_{A_j}(X) \sigma^{1/2}), \end{aligned}$$

where the last equation follows once again from Equation (2.3). We end up with

$$\text{EP}_{\mathcal{L}}(\rho) = \sum_{j \in \mathcal{J}} \langle \Gamma_{\sigma}(\delta_{A_j}(X)), [\Gamma_{\sigma}(X)]_{\omega_j}^{-1}(\Gamma_{\sigma}(\delta_{A_j}(X))) \rangle_{\text{HS}}. \quad (2.11)$$

For  $\rho \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ , we observe that  $\mathcal{E}_* \otimes \text{id}_{\mathcal{K}}(\rho) = \sigma \otimes \text{Tr}_{\mathcal{H}}(\rho)$ . Since all the  $A_j$ 's act on the first register, the calculation above remains true.  $\square$

### 3 From Unital to Non-unital Quantum Markov Semigroups

We are now able to provide a quantum extension of the Holley–Stroock argument. In this section, we restrict ourselves to the primitive case and assume that  $\sigma = \sum_k \sigma_k |k\rangle\langle k|$  is a positive definite density matrix of corresponding eigenbasis  $\{|k\rangle\}$  of  $\mathcal{H}$ .

**Theorem 3.1** *Let  $\mathcal{L}$  be the generator of a primitive, GNS-symmetric QMS with respect to a full-rank state  $\sigma$ ,  $\mathcal{L}_0$  be generator of its corresponding heat semigroup, and  $\omega_j$  its Bohr frequencies. Then*

$$\alpha_{\text{MLSI}}(\mathcal{L}_0) \leq \max_{k,l} \frac{\sigma_k}{\sigma_l} \max_j e^{|\omega_j|/2} \alpha_{\text{MLSI}}(\mathcal{L}),$$

Similarly,

$$\alpha_{\text{CLSI}}(\mathcal{L}_0) \leq \max_{k,l} \frac{\sigma_k}{\sigma_l} \max_j e^{|\omega_j|/2} \alpha_{\text{CLSI}}(\mathcal{L}).$$

**Remark 3.2** Using interpolation techniques, the authors of [47] showed lower bounds on the logarithmic Sobolev constant  $\alpha_2$  of primitive QMS that are self-adjoint with respect to the KMS inner product. Moreover, since we further assume the detailed balance condition, our semigroups satisfy  $\alpha_{\text{MLSI}}(\Phi) \geq 2\alpha_2(\Phi)$ , by the so-called  $L_p$ -regularity of Dirichlet forms proved in [2]. Combining these two results, we can find that

$$\alpha_{\text{MLSI}}(\mathcal{L}) \geq \frac{2\lambda(\mathcal{L})}{\ln \|\sigma^{-1}\|_{\infty} + 2}, \quad \alpha_{\text{MLSI}}(\mathcal{L}^{(n)}) \geq \frac{2\lambda(\mathcal{L})}{\ln(d_{\mathcal{H}}^4 \|\sigma^{-1}\|_{\infty}) + 11},$$

where  $\mathcal{L}^{(n)}$  stands for the generator of the  $n$ -fold product of the semigroup  $(e^{-t\mathcal{L}})_{t \geq 0}$ , and where  $\lambda(\mathcal{L})$  denotes the spectral gap of  $\mathcal{L}$ . In the primitive setting, this means that the bounds that we derived are potentially worse than the ones provided in [47]. However, it was shown in [3] that the logarithmic Sobolev inequality does not hold for non-primitive QMS. In the next section, we provide a more general result for any finite dimensional, non-primitive GNS symmetric QMS.

As in the classical case, the proof is separated in two parts: a comparison of the relative entropies, and a comparison of the entropy productions. We review these separately in the next two paragraphs.

**Comparison of relative entropies** We are now concerned with the left-hand side of the MLSI/CLSI. First, we need to extend the definition of the relative entropy to the case where  $\rho$  and  $\sigma$  are (possibly non-normalized) positive operators [34]:

$$D_{\text{Lin}}(\rho \parallel \sigma) := \text{Tr}(\rho (\ln \rho - \ln \sigma)) + \text{Tr}(\sigma) - \text{Tr}(\rho),$$

where the right-hand side can be equal to infinity. As for its restriction to normalized density matrices, this relative entropy is positive. Moreover:

**Lemma 3.3** *Lindblad's relative entropy satisfies the following properties [34]:*

(i) *Data processing inequality: For any positive operators  $X, Y$ , and any CPTP map  $\Phi$ ,*

$$D_{\text{Lin}}(\Phi(X)\|\Phi(Y)) \leq D_{\text{Lin}}(X\|Y).$$

(ii) *Addition under direct sums: For any positive operators  $X_1, Y_1$ , resp.  $X_2, Y_2$ , on  $\mathcal{H}_1$ , resp.  $\mathcal{H}_2$ ,*

$$D_{\text{Lin}}(X_1 \oplus X_2\|Y_1 \oplus Y_2) = D_{\text{Lin}}(X_1\|Y_1) + D_{\text{Lin}}(X_2\|Y_2).$$

(iii) *Normalization: For any operators  $X, Y \geq 0$  and any constant  $\lambda > 0$*

$$D_{\text{Lin}}(\lambda X\|\lambda Y) = \lambda D_{\text{Lin}}(X\|Y).$$

We will also need the following:

**Lemma 3.4** (Chain rule for  $D_{\text{Lin}}$ ) *Let  $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{N}$  be a conditional expectation onto a  $*$ -subalgebra of  $\mathcal{B}(\mathcal{H})$ . Then, for any  $X, Y \in \mathcal{B}_+(\mathcal{H})$  such that  $Y = \mathcal{E}_*(Y)$ , we have*

$$D_{\text{Lin}}(X\|Y) = D_{\text{Lin}}(X\|\mathcal{E}_*(X)) + D_{\text{Lin}}(\mathcal{E}_*(X)\|Y).$$

**Proof** First, by the definition of  $D_{\text{Lin}}$ ,

$$D_{\text{Lin}}(X\|Y) = D(X\|Y) + \text{Tr}(Y) - \text{Tr}(X),$$

and

$$\begin{aligned} & D_{\text{Lin}}(X\|\mathcal{E}_*(X)) + D_{\text{Lin}}(\mathcal{E}_*(X)\|Y) \\ &= D(X\|\mathcal{E}_*(X)) + D(\mathcal{E}_*(X)\|Y) + \text{Tr}(\mathcal{E}_*(X)) - \text{Tr}(X) + \text{Tr}(Y) - \text{Tr}(\mathcal{E}_*(X)). \end{aligned}$$

We observe that the traces cancel, so this Lemma reduces to the well-known chain rule for relative entropy with conditional expectations. We include a short proof here for the finite-dimensional case.

$$\begin{aligned} & D(X\|\mathcal{E}_*(Y)) + D(\mathcal{E}_*(X)\|Y) \\ &= \text{Tr}(X \log X - X \log(\mathcal{E}_*(X)) + \mathcal{E}_*(X) \log(\mathcal{E}_*(X)) - \mathcal{E}_*(X) \log Y) \quad (3.1) \\ &= \text{Tr}(X \log X - X \log(\mathcal{E}_*(X)) + X \mathcal{E}(\log(\mathcal{E}_*(X))) - X \mathcal{E}(\log Y)). \end{aligned}$$

Let  $\sigma$  denote the density with respect to which  $\mathcal{E}$  is self-adjoint and  $\tilde{\sigma}$  denote the unnormalized density  $d \times \sigma$  in dimension  $d$ . By the block diagonal form of finite-dimensional conditional expectations,

$$\log \mathcal{E}_*(X) = \oplus_i (\log(X_i) \otimes 1^{E_i} + 1^{A_i} \otimes \log(\sigma_i)),$$

and

$$\mathcal{E}(\log \mathcal{E}_*(X)) = \oplus_i (\log(X_i) \otimes 1^{E_i} + 1^{A_i} \otimes 1^{E_i} \text{Tr}(\tilde{\sigma}_i^{1/2} \log(\sigma_i) \tilde{\sigma}_i^{1/2})).$$

Let

$$\eta_X := \mathcal{E}(\log \mathcal{E}_*(X)) - \log \mathcal{E}_*(X) = \oplus_i (1^{A_i} \otimes 1^{E_i} \text{Tr}(\tilde{\sigma}_i^{1/2} \log(\sigma_i) \tilde{\sigma}_i^{1/2}) - 1^{A_i} \otimes \log(\sigma_i)).$$

Since the dependence of  $\eta_X$  on  $X$  cancels in the final expression, we define  $\eta_Y$  analogously and observe that  $\eta_Y = \eta_X$ . Comparing to Equation (3.1),

$$D(X\|Y) = D(X\|\mathcal{E}_*(X)) + D(\mathcal{E}_*(X)\|Y) + \text{Tr}(X(\eta_Y - \eta_X)),$$

and  $\eta_Y - \eta_X = 0$ . □

**Proposition 3.5** (Noncommutative change of measure argument) *Let  $\mathcal{K}$  be an additional Hilbert space,  $\mathcal{E}_0 = \text{Tr}_{\mathcal{H}} \otimes \frac{1_{\mathcal{H}}}{d_{\mathcal{H}}}$  a conditional expectation onto  $\mathcal{B}(\mathcal{K}) \otimes 1_{\mathcal{H}}$  and*

$$\mathcal{E}_*(\rho) = d_{\mathcal{H}} d_{\mathcal{K}} \Gamma_{\frac{1_{\mathcal{K}}}{d_{\mathcal{K}}} \otimes \sigma} \circ \mathcal{E}_0(\rho)$$

*the conditional expectation on the space of densities. Then for all  $X \geq 0$ :*

$$D_{\text{Lin}}(\Gamma_{1_{\mathcal{K}} \otimes \sigma}(X) \| \mathcal{E}_* \circ \Gamma_{1_{\mathcal{K}} \otimes \sigma}(X)) \leq \max_k \{\sigma_k\} D_{\text{Lin}}(X \| \mathcal{E}_0(X)) .$$

**Proof** Since the following inequality holds by Lemma 3.4:

$$D_{\text{Lin}}(d_{\mathcal{H}} \Gamma_{1_{\mathcal{K}} \otimes \sigma}(X) \| d_{\mathcal{H}} \mathcal{E}_* \circ \Gamma_{1_{\mathcal{K}} \otimes \sigma}(X)) \leq D_{\text{Lin}}(d_{\mathcal{H}} \Gamma_{1_{\mathcal{K}} \otimes \sigma}(X) \| \mathcal{E}_*(X)) ,$$

it is enough to prove that

$$D_{\text{Lin}}(d_{\mathcal{H}} \Gamma_{1_{\mathcal{K}} \otimes \sigma}(X) \| \mathcal{E}_*(X)) \leq d_{\mathcal{H}} \max_k \{\sigma_k\} D_{\text{Lin}}(X \| \mathcal{E}_0(X)) .$$

Define the map  $\Phi(X) = \Lambda^{-1} \Gamma_{1_{\mathcal{K}} \otimes \sigma}(X)$ , where  $\Lambda := \max_k \sigma_k$ . This map is completely positive, trace non-increasing. We may also define

$$\Psi(X) := \begin{pmatrix} \Phi(X) & 0 \\ 0 & \text{Tr}((1 - \frac{1_{\mathcal{K}} \otimes \sigma}{\Lambda})X) \end{pmatrix} .$$

Then  $\Psi$  is trace-preserving and hence, see [34], we know that

$$D_{\text{Lin}}(\Psi(X) \| \Psi(\mathcal{E}_0(X))) \leq D_{\text{Lin}}(X \| \mathcal{E}_0(X)) .$$

Since  $D_{\text{Lin}}(\cdot \| \cdot)$  is positive, we deduce from the diagonal output of  $\Psi$  that indeed,

$$\begin{aligned} D_{\text{Lin}}(\Lambda^{-1} \Gamma_{1_{\mathcal{K}} \otimes \sigma}(X) \| \Lambda^{-1} \Gamma_{1_{\mathcal{K}} \otimes \sigma} \circ \mathcal{E}_0(X)) &= D_{\text{Lin}}(\Phi(X) \| \Phi(\mathcal{E}_0(X))) \\ &\leq D_{\text{Lin}}(\Psi(X) \| \Psi(\mathcal{E}_0(X))) . \end{aligned}$$

By definition,  $\mathcal{E}_* = d_{\mathcal{H}} \Gamma_{1_{\mathcal{K}} \otimes \sigma} \circ \mathcal{E}_0$ . Therefore,

$$\begin{aligned} D_{\text{Lin}}(d_{\mathcal{H}} \Gamma_{1_{\mathcal{K}} \otimes \sigma}(X) \| \mathcal{E}_*(X)) &= D_{\text{Lin}}(d_{\mathcal{H}} \Gamma_{1_{\mathcal{K}} \otimes \sigma}(X) \| d_{\mathcal{H}} \Gamma_{1_{\mathcal{K}} \otimes \sigma} \circ \mathcal{E}_0(X)) \\ &= d_{\mathcal{H}} \Lambda D_{\text{Lin}}(\Lambda^{-1} \Gamma_{1_{\mathcal{K}} \otimes \sigma}(X) \| \Lambda^{-1} \Gamma_{1_{\mathcal{K}} \otimes \sigma} \circ \mathcal{E}_0(X)) \\ &\leq d_{\mathcal{H}} \Lambda D_{\text{Lin}}(X \| \mathcal{E}_0(X)) \\ &= d_{\mathcal{H}} \max_k \sigma_k D_{\text{Lin}}(X \| \mathcal{E}_0(X)) . \end{aligned}$$

□

**Comparison of entropy productions** We are now interested in controlling the entropy production of  $\mathcal{L}_0$  in terms of that of  $\mathcal{L}$ . Extending the expression derived in Equation 2.11 for the entropy production to non-normalized states, we find for any positive operator  $X \in \mathcal{B}(\mathcal{K} \otimes \mathcal{H})$ :

$$\begin{aligned} \text{EP}_{\mathcal{L}}(\Gamma_{1_{\mathcal{K}} \otimes \sigma}(X)) &= \sum_{j \in \mathcal{J}} \langle \Gamma_{1_{\mathcal{K}} \otimes \sigma}(\delta_{A_j}(X)), [\Gamma_{1_{\mathcal{K}} \otimes \sigma}(X)]_{\omega_j}^{-1} (\Gamma_{1_{\mathcal{K}} \otimes \sigma}(\delta_{A_j}(X))) \rangle_{\text{HS}} \\ &\geq \inf_j e^{-|\omega_j|/2} \sum_{j \in \mathcal{J}} \langle \Gamma_{1_{\mathcal{K}} \otimes \sigma}(\delta_{A_j}(X)), [\Gamma_{1_{\mathcal{K}} \otimes \sigma}(X)]_0^{-1} (\Gamma_{1_{\mathcal{K}} \otimes \sigma}(\delta_{A_j}(X))) \rangle_{\text{HS}} \end{aligned}$$

where we denoted  $\delta_{A_i} = 1_{\mathcal{K}} \otimes \delta_{A_i}$  by slight abuse of notations, and where we used that, by definition  $[\rho]_{\omega_j} \leq \max_j e^{|\omega_j|/2} [\rho]_0$  as self-adjoint operators in  $\langle \cdot, \cdot \rangle_{\text{HS}}$ . Moreover, we need the following direct extension of Theorem 5 of [27] to the case of trace non-increasing maps:

**Proposition 3.6** *Let  $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  be a completely-positive, trace non-increasing map, then the following holds for any  $A > 0$  and any  $X \in \mathcal{B}(\mathcal{H})$ :*

$$\langle \Phi(X), [\Phi(A)]_0^{-1} (\Phi(X)) \rangle_{\text{HS}} \leq \langle X, [A]_0^{-1} (X) \rangle_{\text{HS}}.$$

Choose this time  $\Phi'(X) := (\Lambda')^{-1} \Gamma_{1_{\mathcal{K}} \otimes \sigma}^{-1}(X)$ , where  $\Lambda' := \max_k \{\sigma_k^{-1}\}$ . This map is completely positive and trace non-increasing. Then

$$\begin{aligned} \text{EP}_{\mathcal{L}}(\Gamma_{1_{\mathcal{K}} \otimes \sigma}(X)) &\geq \min_j e^{-|\omega_j|/2} \sum_{j \in \mathcal{J}} \langle \Gamma_{1_{\mathcal{K}} \otimes \sigma}(\delta_{A_j}(X)), [\Gamma_{1_{\mathcal{K}} \otimes \sigma}(X)]_0^{-1} (\Gamma_{1_{\mathcal{K}} \otimes \sigma}(\delta_{A_j}(X))) \rangle_{\text{HS}} \\ &\geq \min_j e^{-|\omega_j|/2} \sum_{j \in \mathcal{J}} \langle (\Lambda')^{-1} \delta_{A_j}(X), [(\Lambda')^{-1} X]_0^{-1} ((\Lambda')^{-1} \delta_{A_j}(X)) \rangle_{\text{HS}} \\ &= (\Lambda')^{-1} \min_j e^{-|\omega_j|/2} \sum_{j \in \mathcal{J}} \langle \delta_{A_j}(X), [X]_0^{-1} (\delta_{A_j}(X)) \rangle_{\text{HS}} \\ &\geq (\Lambda')^{-1} \min_j e^{-|\omega_j|/2} \text{EP}_{\mathcal{L}_0}(X). \end{aligned}$$

We have proved the following:

**Proposition 3.7** *Let  $\mathcal{L}$  be the generator of a QMS that is self-adjoint with respect to the GNS inner product associated to a full-rank state  $\sigma = \sum_k \sigma_k |k\rangle\langle k|$ . Let  $(\omega_j)$  be the Bohr frequencies as in the GKLS form, Equation (2.2). Then, the following comparison of entropy productions holds: for any positive operator  $X \in \mathcal{B}(\mathcal{K} \otimes \mathcal{H})$ ,*

$$\text{EP}_{\mathcal{L}}(\Gamma_{1_{\mathcal{K}} \otimes \sigma}(X)) \geq \min_k \{\sigma_k\} \min_j \{e^{-|\omega_j|/2}\} \text{EP}_{\mathcal{L}_0}(X).$$

Combining Propositions 3.5 and 3.7, we are now ready to prove the main result of this section, namely Theorem 3.1:

**Proof of Theorem 3.1** First notice the following: for any  $X \in \mathcal{B}_+(\mathcal{H} \otimes \mathcal{K})$ , we recall that  $\alpha_{\text{CLSI}}$  is defined as the CLSI constant. Thanks to the homogeneity of the entropy production and relative entropy, the same constant holds for  $D_{\text{Lin}}$  as for  $D$ , so

$$\alpha_{\text{CLSI}}(\mathcal{L}_0) D_{\text{Lin}}(X \| (\mathcal{E}_0 \otimes \text{id}_{\mathcal{K}})(X)) \leq \text{EP}_{\mathcal{L}_0}(X).$$

The result then comes directly from the following chain of inequalities: for any  $\rho = \Gamma_{1_{\mathcal{K}} \otimes \sigma}(X) \in \mathcal{D}(\mathcal{H} \otimes \mathcal{K})$ :

$$\begin{aligned} \alpha_{\text{CLSI}}(\mathcal{L}_0) D(\rho \| (\mathcal{E}_* \otimes \text{id}_{\mathcal{K}})(\rho)) &\leq \max_k \sigma_k \alpha_{\text{CLSI}}(\mathcal{L}_0) D_{\text{Lin}}(X \| (\mathcal{E}_0 \otimes \text{id}_{\mathcal{K}})(X)) \\ &\leq \max_k \sigma_k \text{EP}_{\mathcal{L}_0 \otimes \text{id}_{\mathcal{K}}}(X) \\ &\leq \max_k \sigma_k \max_k \sigma_k^{-1} \max e^{|\omega_j|/2} \text{EP}_{\mathcal{L} \otimes \text{id}_{\mathcal{K}}}(\rho). \end{aligned}$$

The first inequality follows from Proposition 3.5, the second one by the definition of the CLSI constant of  $\mathcal{L}_0$ , and the last one by Proposition 3.7.  $\square$

## 4 A Non-primitive Holley–Stroock Perturbation Argument

In this section, we extend Theorem 3.1 in essentially two directions: first, we do not assume that the reference semigroup  $(e^{-t\mathcal{L}_0})_{t \geq 0}$  is the heat semigroup. Second, we relax the condition of primitivity for  $(P_t)_{t \geq 0}$ . This situation will in particular extend the argument for CLSI. We will be interested in both MLSI and LSI inequalities.

#### 4.1 Perturbing the Modified Logarithmic Sobolev Inequality

More precisely, we want to compare the MLSI constants of the following two generators satisfying the detailed balance condition with respect to two different, though commuting states:

$$\mathcal{L}_\sigma(X) = - \sum_{j \in \mathcal{J}} \left( e^{-\omega_j/2} A_j^*[X, A_j] + e^{\omega_j/2} [A_j, X] A_j^* \right). \quad (4.1)$$

and

$$\mathcal{L}_{\sigma'}(X) = - \sum_{j \in \mathcal{J}} \left( e^{-\nu_j/2} A_j^*[X, A_j] + e^{\nu_j/2} [A_j, X] A_j^* \right). \quad (4.2)$$

Remark that these generators are given by the same Lindblad operators  $\{A_j\}_{j \in \mathcal{J}}$ , and only differ at the level of their Bohr frequencies. This implies in particular that they share the same fixed-point algebra  $\mathcal{F} = \{A_j : j \in \mathcal{J}\}'$ , which we decompose into matrix blocks:

$$\mathcal{F} = \bigoplus_{i \in \mathcal{I}} \mathcal{B}(\mathcal{H}_i) \otimes 1_{\mathcal{K}_i}.$$

By the detailed balance condition, the operators  $A_j$  are eigenvectors of the modular groups corresponding to two full-rank invariant states  $\sigma$ , resp.  $\sigma'$ , which can without loss of generality be taken as follows: given two families of full-rank normalized traces  $\{\tau_i\}_{i \in \mathcal{I}}$  and  $\{\tau'_i\}_{i \in \mathcal{I}}$ ,

$$\sigma := \sum_{i \in \mathcal{I}} \frac{d\mathcal{K}_i}{d\mathcal{H}} 1_{\mathcal{H}_i} \otimes \tau_i, \quad \sigma' := \sum_{i \in \mathcal{I}} \frac{d\mathcal{K}_i}{d\mathcal{H}} 1_{\mathcal{H}_i} \otimes \tau'_i,$$

In particular,  $A_j = \sum_i 1 \otimes A_i(j)$  are block diagonal with respect to the  $\mathcal{K}_i$ ,

$$\Delta_\sigma(A_j) = e^{-\omega_j} A_j, \quad \text{and} \quad \Delta_{\sigma'}(A_j) = e^{-\nu_j} A_j.$$

This implies in particular that the states  $\tau_i$  and  $\tau'_i$  commute so that:

$$\tau_i = \sum_k \lambda_k^{(i)} P_k^{(i)}, \quad \tau'_i = \sum_k \lambda_k^{(i')} P_k^{(i)}.$$

Our general perturbation theorem follows:

**Theorem 4.1** (Non-primitive Holley–Stroock for MLSI) *With the above notations, the following holds:*

$$\alpha_{\text{MLSI}}(\mathcal{L}_{\sigma'}) \leq \max_{i \in \mathcal{I}, k} \frac{\lambda_k^{(i)}}{\lambda_k^{(i')}} \max_{i \in \mathcal{I}, k} \frac{\lambda_k^{(i')}}{\lambda_k^{(i)}} \max_{j \in \mathcal{J}} e^{|\omega_j - \nu_j|/2} \alpha_{\text{MLSI}}(\mathcal{L}_\sigma).$$

Similarly,

$$\alpha_{\text{CLSI}}(\mathcal{L}_{\sigma'}) \leq \max_{i \in \mathcal{I}, k} \frac{\lambda_k^{(i)}}{\lambda_k^{(i')}} \max_{i \in \mathcal{I}, k} \frac{\lambda_k^{(i')}}{\lambda_k^{(i)}} \max_{j \in \mathcal{J}} e^{|\omega_j - \nu_j|/2} \alpha_{\text{CLSI}}(\mathcal{L}_\sigma).$$

The proof of the theorem follows the same lines as for that of Theorem 3.1. We compare relative entropies and entropy productions separately:

## Comparison of relative entropies

**Proposition 4.2** (Change of measure) *Denote  $\mathcal{E}_\sigma := \lim_{t \rightarrow \infty} e^{-t\mathcal{L}_\sigma}$  and  $\mathcal{E}_{\sigma'} := \lim_{t \rightarrow \infty} e^{-t\mathcal{L}_{\sigma'}}$ . Then, for all  $X \geq 0$ :*

$$D_{\text{Lin}}(\Gamma_\sigma(X) \| \mathcal{E}_{\sigma*} \circ \Gamma_\sigma(X)) \leq \max_{i \in \mathcal{I}, k} \frac{\lambda_k^{(i)}}{\lambda_k^{(i)'}} D_{\text{Lin}}(\Gamma_{\sigma'}(X) \| \mathcal{E}_{\sigma'*} \circ \Gamma_{\sigma'}(X)) .$$

**Proof** The conditional expectations  $\mathcal{E}_\sigma$  and  $\mathcal{E}_{\sigma'}$  take the following form:

$$\mathcal{E}_{\sigma*} = \sum_i \text{Tr}_{\mathcal{K}_i}(P_i \cdot P_i) \otimes \tau_i , \quad \mathcal{E}_{\sigma'*} = \sum_i \text{Tr}_{\mathcal{K}_i}(P_i \cdot P_i) \otimes \tau'_i ,$$

where for each  $i$ ,  $P_i$  is the projection onto the block  $i$  in the decomposition of  $\mathcal{F}$ . This implies in particular the following relation:  $\Gamma_{\sigma'}^{-1} \circ \mathcal{E}_{\sigma'*} = \Gamma_\sigma^{-1} \circ \mathcal{E}_{\sigma*}$ . Next, consider the completely positive map  $\Phi := \frac{1}{r} \Gamma_\sigma \circ \Gamma_{\sigma'}^{-1}$ , with  $r := \max_{i \in \mathcal{I}, k} \lambda_k^{(i)} / \lambda_k^{(i)'}$ . One can readily verify that  $\Phi$  is trace non-increasing. Moreover, by Lemma 3.4:

$$\begin{aligned} D_{\text{Lin}}(\Gamma_\sigma(X) \| \mathcal{E}_{\sigma*} \circ \Gamma_\sigma(X)) &= D_{\text{Lin}}(r \Phi \circ \Gamma_{\sigma'}(X) \| r \mathcal{E}_{\sigma*}(\Phi \circ \Gamma_{\sigma'}(X))) \\ &\leq D_{\text{Lin}}(r \Phi \circ \Gamma_{\sigma'}(X) \| \mathcal{E}_{\sigma*}(\Gamma_{\sigma'}(X))) \\ &= D_{\text{Lin}}(r \Phi \circ \Gamma_{\sigma'}(X) \| r \Phi \circ \mathcal{E}_{\sigma'*} \circ \Gamma_{\sigma'}(X)) . \end{aligned}$$

Next, by homogeneity and data processing inequality after proper normalization of the channel as in the proof of Theorem 3.5, we find that

$$D_{\text{Lin}}(\Gamma_\sigma(X) \| \mathcal{E}_{\sigma*} \circ \Gamma_\sigma(X)) \leq r D_{\text{Lin}}(\Gamma_{\sigma'}(X) \| \mathcal{E}_{\sigma'*} \circ \Gamma_{\sigma'}(X)) ,$$

which is what needed to be proved.  $\square$

**Comparison of entropy productions** We are now interested in comparing the entropy productions of  $\mathcal{L}_\sigma$  and  $\mathcal{L}_{\sigma'}$ . In this section we use the double operator integral notation from Sect. 2, in particular the definition of  $[\Gamma_\sigma(X)]_{\omega_j}^{-1}$ .

**Proposition 4.3** *In the above notations, for any  $X \geq 0$ :*

$$\text{EP}_{\mathcal{L}_{\sigma'}}(\Gamma_{\sigma'}(X)) \leq \max_{i \in \mathcal{I}, k} \frac{\lambda_k^{(i)'}}{\lambda_k^{(i)}} \max_j e^{|\omega_j - \nu_j|/2} \text{EP}_{\mathcal{L}_\sigma}(\Gamma_\sigma(X)) .$$

**Proof** Using the expression derived in 2.11 for the entropy production for non-normalized states, we find for any positive operator  $X \in \mathcal{B}(\mathcal{H})$ :

$$\begin{aligned} \text{EP}_{\mathcal{L}_\sigma}(\Gamma_\sigma(X)) &= \sum_{j \in \mathcal{J}} \langle \Gamma_\sigma(\delta_{A_j}(X)), [\Gamma_\sigma(X)]_{\omega_j}^{-1} (\Gamma_\sigma(\delta_{A_j}(X))) \rangle_{\text{HS}} \\ &\geq \min_j e^{-|\omega_j - \nu_j|/2} \sum_{j \in \mathcal{J}} \langle \Gamma_\sigma(\delta_{A_j}(X)), [\Gamma_\sigma(X)]_{\nu_j}^{-1} (\Gamma_\sigma(\delta_{A_j}(X))) \rangle_{\text{HS}} \end{aligned}$$

where we used that, by definition  $[\rho]_{\omega_j} \leq \max_j e^{|\omega_j - \nu_j|/2} [\rho]_{\nu_j}$  as self-adjoint operators in  $\langle \cdot, \cdot \rangle_{\text{HS}}$ . Next, we observe that  $\Phi(X) = \frac{1}{R} \Gamma_{\sigma'} \circ \Gamma_\sigma^{-1}(X)$  is a completely positive trace preserving map for  $R := \max_{i \in \mathcal{I}, k} \lambda_k^{(i)'} / \lambda_k^{(i)}$ . Hence, given  $Y_j := \Gamma_\sigma(X) e^{-\nu_j/2}$  and  $Z_j :=$

$\Gamma_\sigma(X) e^{v_j/2}$ , and  $Y'_j := \Gamma_{\sigma'}(X) e^{-v_j/2}$  and  $Z'_j := \Gamma_{\sigma'}(X) e^{v_j/2}$ :

$$\begin{aligned} R^{-1} \text{EP}_{\mathcal{L}_{\sigma'}}(\Gamma_{\sigma'}(X)) &= R^{-1} \sum_{j \in J} \langle \Gamma_{\sigma'}(\delta_{A_j}(X)), [\Gamma_{\sigma'}(X)]_{v_j}^{-1} (\Gamma_{\sigma'}(\delta_{A_j}(X))) \rangle_{\text{HS}} \\ &= R^{-1} \sum_{j \in J} \langle \Gamma_{\sigma'}(\delta_{A_j}(X)), \mathcal{T}_{\text{in}}^{Y'_j, Z'_j} \circ \Gamma_{\sigma'}(\delta_{A_j}(X)) \rangle_{\text{HS}} \\ &= \sum_{j \in J} \langle \Phi \circ \Gamma_\sigma(\delta_{A_j}(X)), \mathcal{T}_{\text{in}}^{\Phi(Y_j), \Phi(Z_j)} \circ \Phi \circ \Gamma_\sigma(\delta_{A_j}(X)) \rangle_{\text{HS}} \\ &\leq \sum_{j \in J} \langle \Gamma_\sigma(\delta_{A_j}(X)), \mathcal{T}_{\text{in}}^{Y_j, Z_j} \circ \Gamma_\sigma(\delta_{A_j}(X)) \rangle_{\text{HS}} \\ &= \sum_{j \in J} \langle \Gamma_\sigma(\delta_{A_j}(X)), [\Gamma_\sigma(X)]_{v_j}^{-1} \Gamma_\sigma(\delta_{A_j}(X)) \rangle_{\text{HS}} . \end{aligned}$$

The inequality in the calculation above follows from the version of Lieb's concavity stated as Theorem 5 of [27]. The Proposition follows from this Equation and the first in the proof.  $\square$

**Proof of Theorem 4.1** Combine the previous two propositions as in the proof of Theorem 3.1.  $\square$

## 4.2 Perturbing the Logarithmic Sobolev Inequality

The above Holley–Stroock argument can be easily adapted to the setting of the logarithmic Sobolev inequality. Such an inequality was shown to provide similar decoherence times as the MLSI in the primitive [30, 40, 47] and non-primitive [3] settings.

**Theorem 4.4** Let  $\mathcal{L}_\sigma$  and  $\mathcal{L}_{\sigma'}$  be defined as in Sect. 4.1. Assume that  $\mathcal{L}_{\sigma'}$  satisfies LSI with constants  $(c', d')$ . Then,  $\mathcal{L}_\sigma$  satisfies LSI with constants  $(c, d)$  such that

$$c \leq \max_{i \in \mathcal{I}, k} \frac{\lambda_k^{(i)}}{\lambda_k^{(i)'}} \max_{i \in \mathcal{I}, k} \frac{\lambda_k^{(i)'}}{\lambda_k^{(i)}} \max_{j \in \mathcal{J}} e^{|\omega_j - v_j|/2} c' , \quad d \leq \max_{i \in \mathcal{I}, k} \frac{\lambda_k^{(i)}}{\lambda_k^{(i)'}} \max_{i \in \mathcal{I}, k} \frac{\lambda_k^{(i)'}}{\lambda_k^{(i)}} d' .$$

**Proof** Given  $X \geq 0$ , the entropic term on the left-hand side of **LSI**( $c, d$ ) is taken care of the exact same way as in Proposition 4.2:

$$D_{\text{Lin}}(\Gamma_\sigma(X) || \mathcal{E}_{\sigma*} \circ \Gamma_\sigma(X)) \leq \max_{i \in \mathcal{I}, k} \frac{\lambda_k^{(i)}}{\lambda_k^{(i)'}} D_{\text{Lin}}(\Gamma_{\sigma'}(X) || \mathcal{E}_{\sigma'*} \circ \Gamma_{\sigma'}(X)) .$$

Next, by assumption and homogeneity of the LSI, we have that

$$D_{\text{Lin}}(\Gamma_{\sigma'}(X) || \mathcal{E}_{\sigma'*} \circ \Gamma_{\sigma'}(X)) \leq c' \mathcal{E}_{\mathcal{L}_{\sigma'}}(\sigma'^{-\frac{1}{4}} \Gamma_{\sigma'}(X)^{\frac{1}{2}} \sigma'^{-\frac{1}{4}}) + d' \|\sigma'^{-\frac{1}{4}} \Gamma_{\sigma'}(X)^{\frac{1}{2}} \sigma'^{-\frac{1}{4}}\|_{\mathbb{L}_2(\sigma')}^2 .$$

First, notice that

$$\begin{aligned} \|\sigma'^{-\frac{1}{4}} \Gamma_{\sigma'}(X)^{\frac{1}{2}} \sigma'^{-\frac{1}{4}}\|_{\mathbb{L}_2(\sigma')}^2 &= \text{Tr}(\sigma' X) \leq \max_{i \in \mathcal{I}, k} \frac{\lambda_k^{(i)'}}{\lambda_k^{(i)}} \text{Tr}(\sigma X) \\ &= \max_{i \in \mathcal{I}, k} \frac{\lambda_k^{(i)'}}{\lambda_k^{(i)}} \|\sigma^{-\frac{1}{4}} \Gamma_\sigma(X)^{\frac{1}{2}} \sigma^{-\frac{1}{4}}\|_{\mathbb{L}_2(\sigma)}^2 , \end{aligned}$$

which directly gives the expected upper bound on  $d$ . For  $c$ , we control the Dirichlet form in a way that is completely analogous to what we did for the entropy production in the proof of Proposition 4.3: we have

$$\begin{aligned} \mathcal{E}_{\mathcal{L}_{\sigma'}}(\sigma'^{-\frac{1}{4}} \Gamma_{\sigma'}(X)^{\frac{1}{2}} \sigma'^{-\frac{1}{4}}) &= \sum_{j \in \mathcal{J}} \langle C_{A_j}^{Y'_j, Z'_j}, T_{f_{1/2}^2}^{Y'_j, Z'_j} (C_{A_j}^{Y_j, Z_j}) \rangle_{\text{HS}} \\ &= \sum_{j \in \mathcal{J}} \langle \Gamma_{\sigma'}(\delta_{A_j}(X)), T_{f_{1/2}^2}^{Y'_j, Z'_j} (\Gamma_{\sigma'}(\delta_{A_j}(X))) \rangle_{\text{HS}}, \end{aligned}$$

where  $Y'_j := e^{\frac{v_j}{2}} \Gamma_{\sigma'}(X)$ ,  $Z'_j := e^{-\frac{v_j}{2}} \Gamma_{\sigma'}(X)$  and  $f_{1/2} : x \mapsto x^{1/2}$ . We conclude by first noticing that

$$T_{f_{1/2}^2}^{v_j, -v_j} \tilde{X} e^{-\frac{v_j}{2}} \tilde{X} \leq \max_{j \in \mathcal{J}} e^{|v_j - \omega_j|/2} T_{f_{1/2}^2}^{\omega_j, -\omega_j} \tilde{X}$$

and by invoking Theorem 5 of [27] in the same way as we did in Proposition 4.3.  $\square$

## 5 Strong Data Processing Inequality

In this section, we adapt the proof of Sect. 3 to the discrete time setting. Let  $\Phi_* : \mathcal{T}_1(\mathcal{H}) \rightarrow \mathcal{T}_1(\mathcal{H})$  be a quantum channel. Assume that  $\Phi_*$  has a full-rank invariant state  $\sigma$ , and that the following detailed balance condition holds for its dual map  $\Phi$ : for all  $X, Y \in \mathcal{B}(\mathcal{H})$ ,

$$\text{Tr}(\sigma \Phi(X)^* Y) = \text{Tr}(\sigma X^* \Phi(Y)). \quad (5.1)$$

It was shown in [7] that  $\Phi$  has the following Kraus decomposition  $\Phi(X) := \sum_{j \in \mathcal{J}} \lambda_j K_j X K_j^*$  for some normalization constants  $\lambda_j > 0$ , and where the Kraus operators  $\{K_j\} = \{K_j^*\}$  satisfy  $\sigma K_j = e^{-\omega_j} K_j \sigma$ ,  $\omega_j \in \mathbb{R}$ . The normalization constants are defined in such a way that the map  $\Phi_0$  defined by

$$\Phi_0(X) = \sum_{j \in \mathcal{J}} K_j^* X K_j \quad (5.2)$$

is unital. It is easy to see that the choice  $\lambda_j = e^{\omega_j}$  works. Therefore, we have

$$\Phi(X) = \sum_{j \in \mathcal{J}} e^{\omega_j} K_j X K_j^*. \quad (5.3)$$

Now, in complete analogy with the continuous time setting, there exists a conditional expectation, call it  $\mathcal{E}$ , onto the fixed point algebra  $\mathcal{F}(\Phi) := \{X \in \mathcal{B}(\mathcal{H}) : \Phi(X) = X\}$ , such that

$$\Phi^n \xrightarrow{n \rightarrow \infty} \mathcal{E}.$$

In this section, we are interested in estimating a particular form of the *strong data processing inequality* (SDPI) constant  $c(\Phi)$ , in our case the largest constant  $c$  such that for any  $\rho \geq 0$ ,

$$D_{\text{Lin}}(\Phi_*(\rho) \| \mathcal{E}_*(\rho)) \leq c D_{\text{Lin}}(\rho \| \mathcal{E}_*(\rho)).$$

By the data processing inequality, and since  $\Phi_* \circ \mathcal{E}_* = \mathcal{E}_*$ ,  $c \leq 1$  necessarily. Moreover, when  $\Phi_*$  is unital, the constant  $c$  can be estimated in terms of the logarithmic Sobolev constant of a related quantum Markov semigroup, see [39]. Now, a direct adaptation of the Holley–Stroock

argument of Sect. 3 allows us to reduce to this setting. For sake of simplicity, we state our result in the primitive case so that  $\sigma$  is the unique invariant state of  $\Phi$ :

**Proposition 5.1** *Let  $\Phi_*$  be a primitive quantum channel of unique invariant state  $\sigma$  and satisfying Equation (5.1), and let  $\Phi_0$  be the corresponding unital channel defined as in Equation (5.2). Then,*

$$c(\Phi) \leq \min\{1, \|\sigma\|_\infty \|\sigma^{-1}\|_\infty c(\Phi_0)\}.$$

**Proof** First of all, we notice that  $\Phi_* \circ \Gamma_\sigma = \Gamma_\sigma \circ \Phi_0$ . Indeed:

$$\Gamma_\sigma \circ \Phi_0(X) = \sigma^{\frac{1}{2}} \sum_{j \in \mathcal{J}} K_j^* X K_j \sigma^{\frac{1}{2}} = \sum_{j \in \mathcal{J}} e^{\omega_j} K_j^* \Gamma_\sigma(X) K_j = \Phi_*(\Gamma_\sigma(X)).$$

Then, given any  $\rho := \Gamma_\sigma(X) \geq 0$ ,

$$\begin{aligned} D_{\text{Lin}}(\Phi_*(\rho) \|\sigma) &\leq D_{\text{Lin}} D(\Phi_*(\rho) \|\mathcal{E}_*(X)/d_{\mathcal{H}}) \\ &= D_{\text{Lin}}(\Gamma_\sigma \circ \Phi_0(X) \|\Gamma_\sigma(\mathcal{E}_0(X))) \\ &\leq \|\sigma\|_\infty D_{\text{Lin}}(\Phi_0(X) \|\mathcal{E}_0(X)) \\ &\leq \|\sigma\|_\infty c(\Phi_0) D_{\text{Lin}}(X \|\mathcal{E}_0(X)) \\ &\leq \|\sigma\|_\infty c(\Phi_0) D_{\text{Lin}}(X \|1) \\ &= \|\sigma\|_\infty c(\Phi_0) D_{\text{Lin}}(\Gamma_\sigma^{-1}(\rho) \|\Gamma_\sigma^{-1}(\sigma)) \\ &\leq \|\sigma\|_\infty \|\sigma^{-1}\|_\infty c(\Phi_0) D_{\text{Lin}}(\rho \|\sigma). \end{aligned}$$

where the first and fourth inequalities follow from Lemma 3.4, whereas the second and last inequalities follow the same way as in Proposition 3.5.  $\square$

In [36], Miclo devised a technique to estimate the SDPI constant of a doubly stochastic, primitive Markov chain in terms of the logarithmic Sobolev inequality of a corresponding Markov semigroup. This result was later generalized in [44]. An extension to the tracial quantum setting was recently provided in [39]. Combining their result with our Proposition 5.1, we arrive at the following corollary. Given a unital, self-adjoint quantum Markov semigroup  $(e^{-t\mathcal{L}_0})_{t \geq 0}$ , its *logarithmic Sobolev constant* is defined as

$$\alpha_2(\mathcal{L}_0) := \inf_{X > 0} \frac{\frac{1}{d} \langle X, \mathcal{L}_0(X) \rangle_{\text{HS}}}{\text{Tr}\left(\frac{X^2}{d} \ln X^2\right) - \text{Tr}\left(\frac{X^2}{d}\right) \ln \text{Tr}\left(\frac{X^2}{d}\right)}.$$

**Corollary 5.2** *Let  $\Phi, \Phi_0$  be defined as in (5.1) and assume that  $\Phi_0$  is primitive. Then,*

$$c(\Phi) \leq \min\{\|\sigma\|_\infty \|\sigma^{-1}\|_\infty, 1\}.$$

**Proof** Theorem 4.2 in [39] estimates  $c_0 \leq (1 - \alpha_2(\Phi_0^2 - \text{id}))$  in this case, as their result shows a strong data processing inequality for the case in which the fixed point subspace is spanned by the completely mixed state. The bound then follows Proposition 5.1.  $\square$

## 6 Preparation of Mixed Densities

In this section our goal is to identify certain quantum Markov semigroups that can be used to prepare a given mixed state and satisfy CLSI estimates. While decoherence is an important problem for which CLSI estimates may seem to paint a pessimistic picture, fast decay

could be optimistic when one hopes to generate certain states. Though CLSI for symmetric semigroups already yields strong bounds on the preparation times of states with completely mixed or dephased components, the ability to prepare thermal and other non-maximal mixtures is far more general and useful. Furthermore, these techniques allow us to study thermal equilibration and extend decoherence estimates to not just unital noise but to processes that decay states toward biased mixtures.

CLSI estimates will allow us to estimate the waiting time for a good approximation of the state. As the current work considers mixed fixed point states, our results complement those of [31]. For the rest of this section we will assume that

$$\sigma = \sum_{k=1}^m \sigma_k P_k$$

is a full-rank state on an  $n$ -dimensional Hilbert space, the  $P_k$  are the projections onto eigenspaces.

**Eigenvalues of multiplicity one/state transition graphs** Here we assume in addition that  $\text{Tr}(P_k) = 1$ . Following [13], we know that the operators  $A_j$  are eigenvectors of the modular operator  $\Delta_\sigma$ . In this particular case, these are given by the matrix units  $E_{rs} := |r\rangle\langle s|$  corresponding to a subset of graph edges  $E \subset \{1, \dots, m\}^2$ . Hence we may think of such semigroups as corresponding to a graph representing transitions between quantum states in the given basis. One may find analogies between graph-based Lindbladians and graph Laplacians as studied in the classical literature, though such are not necessary to understand the examples we consider here. We may also choose

$$A_{rs} = \chi_{rs} E_{rs} \quad \text{and} \quad A_{sr} = \chi_{rs} E_{sr},$$

given some arbitrary constants  $\chi_{rs}$  such that  $\chi_{rs} = \chi_{sr}$ . The Bohr frequencies are given by  $\Delta_\sigma(E_{rs}) = \frac{\sigma_r}{\sigma_s} E_{rs}$  and hence  $\omega_{rs} = \ln \sigma_s - \ln \sigma_r$ . Therefore the generator of the semigroup is given by

$$\mathcal{L}_E(X) = \sum_{rs \in E} |\chi_{rs}|^2 \left( \left( \frac{\sigma_r}{\sigma_s} \right)^{1/2} (E_{ss} X - E_{sr} X E_{rs}) + \left( \frac{\sigma_s}{\sigma_r} \right)^{1/2} (X E_{rr} - E_{rs} X E_{sr}) \right). \quad (6.1)$$

Note that both terms are necessary for  $\mathcal{L}_E$  to be the generator of a semigroup and we assume  $\chi_{rs} \neq 0$ . When the Bohr frequencies are all equal to 0, the corresponding generator is denoted by  $\mathcal{L}_{E0}$ .

**Definition 6.1** Let  $E \subset \{(r, s) | 1 \leq r < s \leq m\}$  be a subset of edges and  $\tilde{E} = E \cup \{(s, r) | (r, s) \in E\}$ . Then  $E$  is said to be *irreducible* if the graph with vertices  $\{1, \dots, m\}$  and edges  $\tilde{E}$  is connected.

**Lemma 6.2**  $\mathcal{L}_E$  leaves the diagonal matrices  $\ell_\infty^m \subset \mathbb{M}_m$  with respect to the basis  $\{|r\rangle\}_r$  invariant. Moreover, if  $E$  is irreducible, then  $\mathcal{L}_E$  is primitive.

**Proof** For a diagonal matrix  $X$  we have

$$E_{ss} X - E_{sr} X E_{rs} = (X_{ss} - X_{rr}) E_{ss}, \quad X E_{rr} - E_{rs} X E_{sr} = (X_{rr} - X_{ss}) E_{rr}.$$

Thus  $\mathcal{L}_E(\ell_\infty^m) \subset \ell_\infty^m$ .

Next, let us define  $\delta_{rs}(x) = [E_{rs}, x]$ . Thanks to Lemma (2.1) we note that  $\mathcal{L}_E(X) = 0$  if and only if  $[E_{rs}, X] = 0$  for all  $(r, s) \in \tilde{E}$ . Since the graph is irreducible, we can find a chain  $(t, t_1), (t_1, t_2), \dots, (t_k, v)$  connecting any  $t$  and  $v$  and write

$$E_{tv} = E_{tt_1} E_{t_1 t_2} \cdots E_{t_k v}.$$

Thus  $X$  commutes with all matrix units and hence is a multiple of the identity.  $\square$

In the following we will assume that  $\chi_{rs} = 1$  for all  $r, s$ .

**Remark 6.3** Since the initial version of this paper, the results of [25] showed that for every irreducible  $E$ ,  $\mathcal{L}_{E0}$  satisfies the CLSI. For the complete graph, i.e., when all edges are included, we see that

$$\mathcal{L}_{E0}(X) = 2m \left( X - \frac{1}{m} \sum_{k=1}^m X_{kk} 1 \right).$$

This semigroup directly replaces the input state by the fixed point in convex combination. As in [2], we deduce that  $\alpha_{\text{CLSI}}(\mathcal{L}_{E0}) \geq 2m$ . For the cyclic graph  $E = \{(j, j+1)\}$  we know that  $\alpha_{\text{CLSI}}(\mathcal{L}_{E0}) \geq \frac{c}{m^2}$  for some universal constant  $c$  via the results of [33].

**Corollary 6.4** Let  $E \subset \{1, \dots, m\}^2$  be an irreducible graph with CLSI constant  $\alpha_{\text{CLSI}}(\mathcal{L}_{E0})$ . Assume further that  $\sigma = \sum_{k=1}^m \sigma_k |k\rangle\langle k|$  is nondegenerate. Then the CLSI constant of  $\mathcal{L}_E$  satisfies

$$\alpha_{\text{CLSI}}(\mathcal{L}_{E0}) \leq \max_{kl} \frac{\sigma_k}{\sigma_l} \max_{(rs) \in \bar{E}} \left( \frac{\sigma_s}{\sigma_r} \right)^{1/2} \alpha_{\text{CLSI}}(\mathcal{L}_E).$$

**Proof** This follows directly from Theorem 3.1 and Lemma 6.2.  $\square$

**Remark 6.5** For Lindblad operators with coefficients  $\chi_{rs}$ , we obtain a similar result. Similar generators were considered in [33].

**Eigenvalues of larger multiplicity and locality** We will now extend Corollary (6.4) to the general case following the same procedure. Recalling that  $\sigma = \sum_k \sigma_k P_k$ , let us define the subspaces  $\mathcal{H}_k = P_k \mathcal{H}$  and write

$$\{1, \dots, n\} = \bigcup_k I_k,$$

where each subset  $I_k$  corresponds to the eigenspace  $\mathcal{H}_k$ . We will also assume that the eigenvalues  $\sigma_k$  of  $\sigma$  are defined in decreasing order. Then we choose edges  $E \subset \{1, \dots, n\}^2$  and consider

$$\mathcal{L}_E(X) = \sum_{(r,s) \in E} \left( e^{-\omega_{rs}/2} (E_{ss} X - E_{sr} X E_{rs}) + e^{\omega_{rs}/2} (X E_{rr} - E_{rs} X E_{sr}) \right), \quad (6.2)$$

where

$$e^{-\omega_{rs}/2} = \begin{cases} 0 & \text{if there exists a } k \text{ such that } r, s \in I_k \\ \left( \frac{\sigma_k}{\sigma_j} \right)^{1/2} & \text{if there exists } k \neq j \text{ such that } r \in I_k, s \in I_j. \end{cases}$$

**Corollary 6.6** Let  $E$  be an irreducible graph and  $\sigma = \sum_k \sigma_k P_k$ . Then the semigroup  $\mathcal{L}_E$  satisfies CLSI and

$$\alpha_{\text{CLSI}}(\mathcal{L}_{E0}) \leq \max_{k,l} \frac{\sigma_k}{\sigma_l} \max_{I_k \times I_j \cap \bar{E} \neq \emptyset} \left( \frac{\sigma_k}{\sigma_l} \right)^{1/2} \alpha_{\text{CLSI}}(\mathcal{L}_E).$$

Note here that the graph structure is by no means necessary. In particular, we could use any nice set of generators to produce primitive semigroups in  $\mathcal{B}(\mathcal{H}_k)$ . Moreover, once this is achieved we just need sufficiently many links  $A_{kj} \in \mathcal{B}(\mathcal{H}_k, \mathcal{H}_j)$  to guarantee primitivity of  $\mathcal{L}_0$ .

We assume now that  $\mathcal{H}^{\otimes d}$  is a  $d$ -partite system, and denote by  $\mathcal{H}_k$  the eigensubspaces of  $\sigma \in \mathcal{D}(\mathcal{H}^{\otimes d})$ . We say that a subspace  $\mathcal{K} \subset \mathcal{H}^{\otimes d}$  is  $l$ -local if there exists a permutation  $\pi : \{1, \dots, d\} \rightarrow \{1, \dots, d\}$  and a projection  $Q \in \mathcal{B}(\mathcal{H}^{\otimes l})$  such that  $P_{\mathcal{K}} = \Sigma_{\pi}^{-1}(Q \otimes 1_{\mathcal{H}^{\otimes(d-l)}})\Sigma_{\pi}$ , where  $\Sigma_{\pi}$  is obtained by permuting registers:  $\Sigma_{\pi}(h_1 \otimes \dots \otimes h_d) = h_{\pi(1)} \otimes \dots \otimes h_{\pi(d)}$ . Similarly we say that an operator  $A$  is  $l$ -local if  $A \cong B \otimes 1_{\mathcal{H}^{\otimes(d-l)}}$  holds up to a permutation of registers. The same definition holds for superoperators.

**Lemma 6.7** Assume that for all  $k$  the subspaces  $\mathcal{H}_k$  are  $l$ -local, and that for  $I_k \times I_j \cap E$  the space  $\mathcal{H}_k + \mathcal{H}_j$  is  $l$ -local. Then  $\mathcal{L}_{E*}$  is  $l$ -local.

**Proof** We recall that

$$\mathcal{L}_{E*}(\rho) = \sum_{(r,s) \in E} e^{-\omega_{rs}/2} (A_{rs}^* A_{rs} \rho - A_{rs} \rho A_{rs}^*) + e^{+\omega_{rs}/2} (\rho A_{rs} A_{rs}^* - A_{rs}^* \rho A_{rs}).$$

Here we may replace the matrix units by  $E_{rs} \otimes 1$  for  $r \in I_l$  and  $s \in I_l$  up to permutation. Similarly, we can stabilize the space  $1 \otimes \mathcal{H}^{\otimes d-l}$  using a Lindbladian

$$\mathcal{L} = \sum_{i=1}^{d-l} 1 \otimes \dots \otimes \underbrace{\mathcal{L}_H}_{i\text{-th position}} \otimes \dots \otimes 1$$

for a primitive Lindbladian on  $\mathcal{B}(\mathcal{H})$  given by commutators. Then  $\mathcal{L}_{E0}$  and  $\mathcal{L}_{E*}$  will only use local operators  $A_{rs}$  and second order differential operators of the form  $[A_{rs}\rho, A_{rs}^*]$ .  $\square$

**Remark 6.8** 1) The semigroups  $(\mathcal{P}_t = e^{-t\mathcal{L}_0})_{t \geq 0}$  for unital  $\mathcal{L}_0$  can be obtained in the form  $\mathcal{P}_t(X) = \mathbb{E}[U_t^* X U_t]$  for some random unitaries  $U_t$ . The approximation of  $e^{-t\mathcal{L}_{E*}} \approx 1 - t\mathcal{L}_{E*}$  with unitary operations will be investigated in a forthcoming publication.

2) Nevertheless in the local situation operators  $A_{rs}$  do not really depend on the state per se, just on its eigen-projections. The Bohr-frequencies however, drive the QMS specifically in  $\sigma$ .

## 6.1 Estimating Rates of Thermal Equilibration

A system in contact with a heat bath at fixed temperature will decay toward a Gibbs state given by

$$\sigma_{\beta} = \frac{e^{-\beta H}}{Z_{\beta}} = \frac{1}{Z_{\beta}} \sum_{k=1}^m e^{-\beta E_k} |k\rangle\langle k| = \frac{1}{Z_{\beta}} \sum_{E \in \text{sp}(H)} e^{-\beta E} \sum_{k: E_k=E} |k\rangle\langle k|, \quad (6.3)$$

where  $H$  is the corresponding Hamiltonian,  $\beta$  is the unitless inverse temperature, and energies are indexed in increasing order. Here the last expression explicitly separates the sum over possibly degenerate energies. The partition function  $Z_{\beta} = \sum_k \exp(-\beta E_k)$  normalizes the probabilities.

In practice, we often expect matter qubits to decay toward a low-temperature thermal state. On the preparation side, we may wish to heat or cool a system to a desired temperature. Directly following Theorem 3.1,

**Corollary 6.9** *Let  $\omega_\beta$  be a thermal state as in Equation (6.3) and the fixed point of QMS  $(P_t)_{t \geq 0}$  generated by Lindbladian  $\mathcal{L}$  with  $m$  energy levels  $0 = E_1 < E_2 < \dots < E_m$ . Let  $\mathcal{L}_0$  be the corresponding self-adjoint Lindbladian. Then*

$$\alpha_{\text{CLSI}}(\mathcal{L}_0) \leq e^{\beta E_m} \max_j e^{\beta(E_{j+1}-E_j)/2} \alpha_{\text{CLSI}}(\mathcal{L}). \quad (6.4)$$

The completely mixed state is equivalent to the infinite temperature Gibbs state  $\sigma_0$ , so we might think of the CLSI constant comparison as perturbing the infinite-temperature limit. With a finite maximum energy and for temperatures substantially above that scale, this CLSI constant is close to that for decay toward complete mixture.

In general, our CLSI constant estimate depends exponentially on the largest energy scale and becomes trivial with an infinite spectrum. This appears to be not a flaw of the estimate, but a property of the relative entropy. When high-energy elements of the Gibbs state are close to 0, even small fluctuations into the high-energy regime can result in enormous relative entropy. In thermodynamics, a usual solution to relative entropy blowup on rare states is to work with smoothed relative entropies [21], which discount contributions from highly unlikely configurations. While it is beyond the scope of this paper to fully formulate CLSI for smoothed relative entropy, we may nonetheless consider an analogous approach for states that rarely occur.

A simple strategy is to replace the Gibbs state by

$$\tilde{\sigma} = \frac{1}{\tilde{Z}_\beta} \left( \sum_{k=1}^{l-1} e^{-\beta E_k} |k\rangle\langle k| + \sum_{k=l}^m e^{-\beta E_l} |k\rangle\langle k| \right), \quad (6.5)$$

for which the associated Lindbladian  $\tilde{\mathcal{L}}$  has

$$\alpha_{\text{CLSI}}(\mathcal{L}_0) \leq e^{\beta E_l} \max_{j \leq l} e^{\beta(E_{j+1}-E_j)/2} \alpha_{\text{CLSI}}(\tilde{\mathcal{L}}).$$

Physically, this is equivalent to artificially compressing high energies to a single, degenerate level. We do not claim that this accurately represents the high-energy parts of the thermal state or decay of states with substantial support above  $E_l$ . Rather,  $\tilde{\mathcal{L}}$  is an example of a Lindbladian with the same transitions as  $\mathcal{L}$  and similar low-energy behavior at short timescales. It hence naturally has the same locality properties.

The distance  $\|\sigma_\beta - \tilde{\sigma}\|_1$  increases with the value of  $E_{j+1}$  and higher levels. We can overestimate it by assuming  $E_{l+1} = \infty$ , as though  $\sigma_\beta$  had no support in the high-energy space. We can easily check that  $|\tilde{Z}_\beta - Z_\beta| \leq (m-l)e^{-\beta E_l}$ , so  $\|\sigma_\beta - (Z_\beta/\tilde{Z}_\beta)\sigma_\beta\|_1 \leq (m-l)e^{-\beta E_l}/\tilde{Z}_\beta$ . Similarly,  $\|(Z_\beta/\tilde{Z}_\beta)\sigma_\beta - \tilde{\sigma}\|_1 \leq (m-l)e^{-\beta E_l}/\tilde{Z}_\beta$ . Hence

$$\|\tilde{\sigma} - \sigma_\beta\|_1 \leq \frac{2}{\tilde{Z}_\beta} (m-l) e^{-\beta E_l},$$

which decreases exponentially with  $E_l$ .

Estimating rates of thermal equilibration is an active area of research. The techniques of this paper allow one to directly transfer estimates from the infinite to finite temperature setting.

## Declarations

**Conflict of interest** Data sharing not applicable to this article as no datasets were generated or analysed during the current study. The authors have no competing interests to declare.

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