Journal of Mathematical Physics | ARTICLE | scitation.org/journal/jmp

Multivariate trace inequalities, p-fidelity, and universal recovery beyond tracial settings

Cite as: J. Math. Phys. 63, 122204 (2022); doi: 10.1063/5.0066653
Submitted: 11 August 2021 • Accepted: 24 November 2022 •
Published Online: 13 December 2022

Marius Junge[1] and Nicholas LaRacuente[2,a)]

AFFILIATIONS

[1] University of Illinois, Champaign, Illinois 61801-2975, USA
[2] University of Chicago, Chicago, Illinois 60637, USA

[a)] Author to whom correspondence should be addressed: nlaracuente@uchicago.edu

ABSTRACT

Trace inequalities are general techniques with many applications in quantum information theory, often replacing the classical functional calculus in noncommutative settings. The physics of quantum field theory and holography, however, motivates entropy inequalities in type III von Neumann algebras that lack a semifinite trace. The Haagerup and Kosaki L_p spaces enable re-expressing trace inequalities in non-tracial von Neumann algebras. In particular, we show this for the generalized Araki–Lieb–Thirring and Golden–Thompson inequalities from the work of Sutter *et al.* [Commun. Math. Phys. **352**(1), 37 (2017)]. Then, using the Haagerup approximation method, we prove a general von Neumann algebra version of universal recovery map corrections to the data processing inequality for relative entropy. We also show subharmonicity of a logarithmic p-fidelity of recovery. Furthermore, we prove that the non-decrease of relative entropy is equivalent to the existence of an L_1-isometry implementing the channel on both input states.

I. INTRODUCTION

Trace inequalities are extremely powerful in studying quantum information and probabilities. Often, a classical inequality that follows from the functional calculus will yield a quantum generalization from an inequality on traces of matrix products. A well-known example is the Golden–Thompson inequality, stating that for a pair of Hermitian matrices x, y,

$$\mathrm{tr}\,\exp(x+y) \leq \mathrm{tr}(\exp(x)\exp(y)). \tag{1}$$

For classical vectors or simultaneously diagonalizable matrices, the equality holds almost trivially. In Ref. 1, Sutter *et al.* generalized the Golden–Thompson inequality to show that for Hermitian matrices $\{H_k\}_{k=1}^n$ and $p \geq 1$,

$$\ln\left\|\exp\sum_{k=1}^{n} H_k\right\|_p \leq \int_{\mathbb{R}} dt\beta_0(t)\ln\left\|\prod_{k=1}^{n}\exp((1+it)H_k)\right\|_p, \tag{2}$$

where $\|\cdot\|_p$ is the Schatten p-norm on matrices, and

$$\beta_0(t) = \frac{\pi}{2}(\cosh(\pi t)+1)^{-1}. \tag{3}$$

This generalized Golden–Thompson inequality follows from a generalization of the Araki–Lieb–Thirring inequality.

The four-input version of Eq. (2) implies a key result in quantum information theory. The quantum channel is a general model of how the state of an open quantum system changes when interacting with an initially uncoupled environment. Due to this environmental interaction, the effect of a channel is generally not invertible—it may lose information about the system. In some special cases, it is nonetheless possible to recover the original input state. For example, quantum error correction defines a "code space" within a larger system such that perturbations of states in the code space are effectively invertible.[2,3] In the theory of quantum communication,[4,5] one asks how many bits of information one may recover from the output of a quantum channel with arbitrarily powerful encoding and decoding. Holography in high energy physics relies on a reversible map between bulk and boundary theories.[6–8]

A key quantity in quantum information is the relative entropy between quantum density matrices, denoted as $D(\rho\|\eta)$ for densities ρ and η. One of the most fundamental inequalities in quantum information theory is the data processing inequality for relative entropy, which states that for any quantum channel Φ,

$$D(\rho\|\eta) \geq D(\Phi(\rho)\|\Phi(\eta)).$$

We recall and denote by $R_{\eta,\Phi}$ the Petz recovery map, given by a normalized and re-weighted adjoint of Φ.[9,10] It is always the case that $R_{\eta,\Phi} \circ \Phi(\eta) = \eta$. The Petz map for η, Φ sometimes acts as an inverse on ρ as well. In particular,

$$D(\rho\|\eta) = D(\Phi(\rho)\|\Phi(\eta)) \iff R_{\eta,\Phi} \circ \Phi(\rho) = \Phi(\rho). \tag{4}$$

The intuition for data processing is that no stochastic or quantum process may increase the distinction between two probability distributions or densities. Equality of relative entropy faithfully indicates that Φ also does not destroy any information in ρ relative to η.

A natural question is whether a small difference in relative entropy implies approximate recovery. Holographic theories, for instance, consider approximately invertible maps between subsystems of a bulk spacetime and the corresponding quantum boundary.[7] Quantum information applications, such as error correction and communication, may work with only approximately preserved code spaces, formally outside the strict criteria for the perfect recovery via the Petz map. A number of recent works have begun to quantitatively link a relative entropy difference to the fidelity of recovered states.

A resurgence of activity on approximate recovery started with Fawzi and Renner's approximate Markov chain result.[11] A special form of relative entropy is the conditional mutual information on a tripartite system $A \otimes B \otimes C$, given by

$$I(A:B|C)_\rho = D\left(\rho^{ABC} \middle\| \frac{\mathbf{1}}{|A|} \otimes \rho^{BC}\right) - D\left(\rho^{AC} \middle\| \frac{\mathbf{1}}{|A|} \otimes \rho^{C}\right),$$

where ρ^{BC}, ρ^{AC}, and ρ^C refer to respective marginals of ρ. Fawzi and Renner showed that

$$I(A:B|C)_\rho \geq -2 \ln f_1(\rho, R^{FW}(\rho^{AC})),$$

where $f_1(\rho,\eta) = tr(|\sqrt{\rho}\sqrt{\eta}|)$ is the usual fidelity, for some channel R^{FW} (not necessarily the Petz map). If one can perfectly recover ρ^{ABC} from ρ^{AC} by acting only on C, then the system is called a quantum Markov chain.[12] In Ref. 13, the same inequality is shown for a universal recovery map, which only depends on ρ^{AC} rather than on ρ^{ABC}. Li and Winter used this form of recovery in Ref. 14 to show a monogamy of entanglement.

Wilde extends approximate recovery to general relative entropy differences in Ref. 15, showing that

$$D(\rho\|\eta) - D(\Phi(\rho)\|\Phi(\eta)) \geq -2 \ln\left(\sup_{t\in\mathbb{R}} f_1(\rho, R^t_{\eta,\Phi}(\rho^{AC}))\right) \tag{5}$$

for a twirled recovery map $R^t_{\eta,\Phi}$ parameterized by t. In Ref. 16, Junge *et al.* showed that

$$D(\rho\|\eta) - D(\Phi(\rho)\|\Phi(\eta)) \geq -2\int_{\mathbb{R}} \ln f_1(\rho, R^t_{\eta,\Phi}(\Phi(\rho)))d\beta_0(t), \tag{6}$$

where $d\beta_0(t) = (\pi/2)(\cosh(\pi t)+1)^{-1}dt$. Using convexity, one may move the integral inside the logarithm and fidelity to construct the explicit, universal recovery map, given by

$$\tilde{R}_{\eta,\Phi}(\rho) = -2\int_{\mathbb{R}} R^t_{\eta,\Phi}(\rho)d\beta_0(t). \tag{7}$$

Another result by Sutter *et al.* (Refs. 1, Corollary 4.2) strengthens the inequality as a corollary of Eq. (2). Let $D_M(\rho\|\eta) := \sup_{\mathcal{M}\in POVMS} D(\mathcal{M}(\rho)\|\mathcal{M}(\eta))$ denote the measured relative entropy. Then,

$$D(\rho\|\eta) - D(\Phi(\rho)\|\Phi(\eta)) \geq D_M(\rho\|\tilde{R}_{\eta,\Phi} \circ \Phi(\eta)) \tag{8}$$

for a recovery map $\tilde{R}_{\eta,\Phi}$ as defined in Theorem VI.3.

More recently, Carlen and Vershynina have shown (Corollary 1.7 in Ref. 17) that

$$D(\rho\|\eta) - D(\mathcal{E}(\rho)\|\mathcal{E}(\eta)) \geq \left(\frac{\pi}{8}\right)^4 \|\Delta_{\rho,\eta}\|^{-2} \|R_{\rho,\mathcal{E}}(\mathcal{E}(\eta)) - \eta\|_1^4, \tag{9}$$

where $\Delta_{\rho,\sigma}$ is the relative modular operator and $\mathcal{E}$ is a conditional expectation that restricts a density to a matrix subalgebra. A recent work by Gilyén *et al.* suggests a quantum algorithm that implements the Petz recovery map in special cases.[18]

For recovery's applications to quantum field theory,[19] it is desirable to extend finite-dimensional results to infinite-dimensional von Neumann algebras, including type III factors that lack a finite or even semifinite trace. Applications of recovery appear in finite-dimensional analogs of the Ads/CFT correspondence.[7] Recovery may underpin eventual proofs of ideas relating the Ryu–Takayanagi conjecture and analogies to error correction, but field theories are widely believed to be type III, non-tracial algebras, in which much of the finite-dimensional quantum information machinery remains conjecture. Two very recent works address the type III extension of recovery maps. One, by Gao and Wilde, extends Eqs. (5) and (9) to the von Neumann algebra setting, also addressing generalizations to optimized f-divergences.[20] Faulkner *et al.* proved an equation in the form of (6) for subalgebraic restriction/inclusion, with applications in high energy physics.[21] In a later work, Faulkner and Hollands extended these results to two-positive channels,[22] and in a follow up, Hollands[23] derived a result in the form of Eq. (8).

A. Primary contributions

In this work, we show how the multivariate trace inequalities of Ref. 1 still hold and apply in arbitrary von Neumann algebras, surprisingly including the non-tracial types. This set of results consists of two inequalities, given as Theorems I.1 and I.2. These theorems are similar in form to those of Ref. 23, but were derived independently. First, we show a generalized Araki–Lieb–Thirring inequality extending Ref. 1, Theorem 3.2 to von Neumann algebras and slightly generalizing the form of Ref. 23, Corollary 1 to a range of Kosaki norms.

Theorem I.1 (Araki, Lieb, and Thirring). *Let ρ, η be normal, faithful states on von Neumann algebra M, $p \geq 1$, $n \in \mathbb{N}$, $w \in [0,1]$, and $\{x_k\}_{k=1}^n \subseteq M$ be positive semidefinite, bounded operators,*

$$\ln \left\| \prod_{k=1}^n x_k^r \right\|_{L^w_{p/r}(M,\rho,\eta)} \leq r \int_{-\infty}^{\infty} dt \beta_r(t) \ln \left\| \prod_{k=1}^n x_k^{1+it} \right\|_{L^w_p(M,\rho,\eta)}. \tag{10}$$

The technical version of this theorem appears as Theorem I.1. Here, the norms are Kosaki L_p norms, given for an operator $x \in M$ by

$$\|x\|_{L^w_p(M,\rho)} = \|\rho^{(1-w)/p} x \rho^{w/p}\|_{L_p(M)}. \tag{11}$$

The norm $\|\cdot\|_{L_p(M)}$ is the Haagerup L_p space norm, bypassing the potentially traceless nature of the original algebra and reducing to the usual L_p norm for tracial algebras (see Sec. II B). The weight β_r, generalizing β_0 as in Eq. (3), is given by

$$\beta_\theta(t) := \frac{\sin(\pi\theta)}{2\theta(\cosh(\pi t) + \cos(\pi t))}. \tag{12}$$

We also derive an analog of the generalized Golden–Thompson inequality [Ref. 1, Corollary 3.2; Eq. (2) in this Introduction] with a slightly different dependence on p. This inequality has a similar but not identical form to that of Ref. 23, Corollary 3.

Theorem I.2 (Golden and Thompson). *Let $\{H_k\}_{k=1}^n \subseteq M$ be bounded Hermitian operators and $\rho = \exp(H_0) \in L_1(M)$ have full support. Then,*

$$\ln \left\| \exp\left(\frac{H_0}{p} + \sum_{k=1}^n H_k \right) \right\|_{L_p(M)} \leq \int_{\mathbb{R}} dt \beta_0(t) \ln \left\| \prod_{k=1}^n \exp((1+it)H_k) \right\|_{L^1_p(M,\rho)}. \tag{13}$$

Almost immediately from the same argument, we obtain a generalization of Lieb's theorem.

Remark I.3. Let ρ be Hermitian such that $\exp(\rho) \in L_1(M)$. Then, the function $f : M \to M$ given by

$$f(X) = \| \exp(\rho/p + \ln X) \|_{L_p(M)}$$

is concave on the positive definite cone.

We then rederive Eq. (8) for arbitrary von Neumann algebras. This result is identical to but derived independently from Ref. 23, Theorem 1.

Theorem I.4. *Let $D(\rho\|\eta)$ denote the quantum relative entropy between normal states ρ and η and Φ denote a quantum channel (a completely positive, normal map). Let $D_M(\rho\|\eta) := \sup_{\mathcal{M} \in POVMS} D(\mathcal{M}(\rho)\|\mathcal{M}(\eta))$ denote the measured relative entropy. Then,*

$$D(\rho\|\eta) - D(\Phi(\rho)\|\Phi(\eta)) \geq D_M(\rho\|\tilde{R}_{\eta,\Phi} \circ \Phi(\rho)), \tag{14}$$

where $\tilde{R}_{\eta,\Phi}$ is as in Eq. (7).

Furthermore, we generalize the universal recovery map in the style of (6) to channels on all von Neumann algebras and for a p-generalization of the fidelity similar to that of Liang *et al.*'s in Ref. 24, Eq. (2.14), given by

$$f_p(\rho,\eta) = \|\sqrt{\rho}\sqrt{\eta}\|_p. \tag{15}$$

We denote a *twirled recovery map* in the equivalent form to Wilde's,[15] but parameterized by complex z,

$$R^z_{\eta,\Phi}(\hat{\rho}) = \eta^{\bar{z}/2}\Phi^\dagger(\hat{\eta}^{-\bar{z}/2}\hat{\rho}\hat{\eta}^{-z/2})\eta^{z/2}, \tag{16}$$

and a *logarithmic, twirled p-fidelity of recovery* given by

$$FR^z_{\eta,\Phi}(\hat{\rho}) = -\ln f_{1/Re(z)}(\rho^{Re(z)}, R^z_{\eta,\Phi}(\hat{\rho}^{Re(z)})). \tag{17}$$

For convenience of notation, we may denote $R_z = R^z_{\eta,\Phi}$ when η and Φ are clear from context. Our notion of fidelity of recovery is closely related to that considered earlier in the field[25] although we have included the logarithm in the quantity for convenience. Then, we show that the following holds.

Theorem I.5. *Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a normal, completely positive map from von Neumann algebra $\mathcal{M}$ to algebra $\mathcal{N}$. Let ρ, σ be densities on $\mathcal{M}$. Then,*

$$D(\rho\|\eta) \geq D(\Phi(\rho)\|\Phi(\eta)) + 2p\int_{\mathbb{R}} FR^{(1+it)/p}_{\eta,\Phi}(\rho)\beta_0(t)dt$$

for $p \geq 1$.

As with Eq. (6), we can use the convexity of the p-fidelity and the negative logarithm to move the integral inside, constructing an explicit, universal recovery map (see Theorem VI.1). Equation (6) follows as the $p = 1$ case. Theorem I.5 follows a more general result for the p-fidelity of recovery.

Theorem I.6. *$FR^z_{\eta,\Phi}$ is subharmonic.*

Theorem I.6 is justified by Remark V.6 in Sec. V. Theorem I.6 converts a mathematical comparison from complex interpolation theory into a direct bound on physical quantities.

For $p = 2$ and $M \subset \mathbb{B}(L_2(M))$ represented in the so-called standard form,[26] we may always assume that $\rho(x) = (\sqrt{d_\rho}, x\sqrt{d_\rho})$ is implemented by its natural 'purification.' Then, we deduce (see Remark X.5) that

$$\|d^{1/2}_\rho - R^{1/2}_{\eta,\Phi}(d^{1/2}_{\hat{\rho}})\|_2^2 \leq D(\rho\|\eta) - D(\Phi(\rho)\|\Phi(\eta)). \tag{18}$$

This implies that

$$\|d_\rho - R^{1/2}_{\eta,\Phi}(d^{1/2}_{\hat{\rho}})^2\|_1^2 \leq 4(D(\rho\|\eta) - D(\Phi(\rho)\|\Phi(\eta))). \tag{19}$$

Thus, using non-linear recovery maps enables us to obtain a quadratic error formula, which qualitatively resembles Eq. (9) and the results in Ref. 20.

Using the same techniques, we prove a data processing inequality for p-fidelity and that for any quantum channel Φ and pair of states ρ, η,

$$f_p(\Phi(\rho), \Phi(\eta)) \geq f_p(\rho, \eta). \tag{20}$$

Finally, we derive a new condition for equality in data processing for states with shared support.

Theorem I.7 (introduction version of XII.5)**.** *Let ρ, η be states such that $\rho \leq \lambda\eta$ and $\Phi : L_1(M) \to L_1(\hat{M})$ be a quantum channel for von Neumann algebras $M, \hat{M}$. Then, the following conditions are equivalent.*

(*i*) $D(\Phi(\rho)\|\Phi(\eta)) = D(\rho\|\eta)$.
(*ii*) *There exist an η-conditioned subalgebra $M_0 \subset M$ and a completely positive L_1-isometry $u : \hat{M} \to M_0$ such that*

$$u(\eta) = \Phi(\eta), \qquad u(\rho) = \Phi(\rho).$$

Theorem XII.5 is intuitive for finite-dimensional channels with equivalent input and output spaces, for which perfect recoverability for all states implies unitarity. In the infinite-dimensional situation and with different input and output spaces, Petz's map gives a precise recovery. However, Theorem XII.5 improves on Petz's recovery map by providing a local lift from the state space of the output back to the input, motivated by Kirchberg's work. Assuming equality in an AdS/CFT correspondence, this amounts to an exact lift from boundary to bulk states.

A first, key realization in our method is that the Haagerup L_p spaces as detailed in Ref. 27 can often serve as a substitute for the usual trace. A second is that the interpolation spaces defined by Kosaki[28] coincide with these Haagerup spaces. The trace inequalities in Ref. 1 actually follow two proof strategies: one using traditional information-theoretic techniques that mirror those of Ref. 29 and another using the complex interpolation methods roots of Ref. 16. Kosaki's interpolation results let us rederive the main trace inequalities of Ref. 1 with minor adjustments based on the Kosaki analog of the basic interpolation theorem underlying them (stated as Ref. 1, Theorem 3.1 and, in our case, as Theorem II.8). These do not lead as quickly to Corollary I.4 because the analyticity and definitions of functions such as the operator logarithm are more subtle. Instead, we return to settings with finite trace and then apply the Haagerup approximation method of Ref. 27 via the continuity results we derived previously in Ref. 30. This approach suggests the Haagerup approximation as a general method for entropy inequalities beyond tracial settings.

Section II reviews the mathematical background of the rest of the text. In Sec. III, we prove the generalized Araki–Lieb–Thirring (Theorem I.1) and Golden–Thompson (Theorem I.2) inequalities. In Sec. IV, we re-introduce the rotated recovery maps and show some necessary L_p inequalities for the recovery results. In Sec. V, we introduce the form of p-fidelity that will underlie one form of recovery inequality and prove results on differentiation of quantities that will yield the desired relative entropy comparisons. In Sec. VI, we show the finite von Neumann algebra cases of the recovery theorems (Theorems I.4 and I.5). In Sec. VII, we show continuity bounds on relative entropy, and in Sec. VIII, we prove the needed results to approximate relative entropy in type III by entropy in lower-type algebras and to remove assumptions of states sharing support. In Sec. IX, we present the technical versions and proofs of the recovery theorems (Theorems I.4 and I.5). In Sec. X, we show an analogous recovery bound for Hilbert space vectors. In Sec. XI, we show a data processing inequality for p-fidelity. In Sec. XII, we prove the L_1-isometry equivalence to saturation of data processing (Theorem I.7). We conclude with Sec. XIII.

II. BACKGROUND

By $\mathbb{B}(\mathcal{H})$, we denote the bounded operators on Hilbert space $\mathcal{H}$, and we will consider general von Neumann algebras of the form $M \subseteq \mathbb{B}(\mathcal{H})$, including infinite-dimensional and non-separable Hilbert spaces. By ρ, η, we commonly denote normal, positive semidefinite states in the predual M_*, which in finite dimension would be density matrices. By $\mathbf{1}$, we denote the identity operator. By a factor, we refer to a von Neumann algebra with a trivial center as the subalgebra of operators that commute with the whole algebra. Physically, we may think of a center as a classical probability space attached to a potentially quantum system.

The von Neumann algebra factors may have type I_d, I_∞, II_1; II_∞; III_0, III_λ, or III_1. Type I_d factors are subalgebras of the bounded operators (matrices) on finite-dimensional Hilbert spaces, and type I_∞ arises from the straightforward $d \to \infty$ limit. We denote the trace in type I by tr. In I_∞, $\mathrm{tr}(\mathbf{1}) = \infty$—here, the trace is semifinite in the sense of not being infinite on all elements of the algebra, but it is not finite. Type II_1 factors are infinite dimensional with a finite, normalized trace tr such that $\mathrm{tr}(\mathbf{1}) = 1$. Algebras of type II_∞ have the form $M \otimes \mathbb{B}(\mathcal{H})$ for M of type II_1 and infinite-dimensional $\mathcal{H}$. In type II_∞, the trace tr is semifinite, and $\mathrm{tr}(\mathbf{1}) = \infty$.

Algebras of type III are non-tracial in that there is not even a semifinite trace. For a physically motivated review of how type III arises, see the hyperfinite construction of II_1, III_λ, and III_1 factors in Ref. 19. Type III is nonetheless a relevant model of quantum field theory, matching observed divergences of the trace and other features, such as divergent entanglement between spatial subregions.

A von Neumann algebra with a non-trivial center is a direct sum (or integral) of factors. While the full algebra may have mixed type, each factor will have a type as described above. Hence, to show the results of this paper for general von Neumann algebras, it suffices to show that our constructions and results hold consistently on factors of all types. For a thorough treatment of operator algebra theory, see Ref. 31.

A. Basic modular theory

Starting from a von Neumann algebra M and state ω, the GNS construction allows one to define an inner product given by

$$\langle x|y\rangle_\omega = \omega(x^* y) \tag{21}$$

and via completion construct a corresponding Hilbert space and representation of operators in M. See Ref. 32 for an introduction with the emphasis on physical relevance.

In full generality, a von Neumann algebra M may contain bounded operators from Hilbert space $\mathcal{H}$ to Hilbert space $\mathcal{H}'$. Although one may obtain a space of bounded operators between distinct Hilbert spaces, for our purposes, we will assume that $\mathcal{H}$ and $\mathcal{H}'$ are isomorphic, ensuring closure under conjugation. Let $|\eta\rangle \in \mathcal{H}$ and $|\rho\rangle \in \mathcal{H}'$ be a pair of normalized vectors for which $|\eta\rangle$ is

1. cyclic in that $\{a|\eta\rangle : a \in M\}$ is dense in $\mathcal{H}$ and
2. separating in that if $a \in M$ and $a|\eta\rangle = 0$, then $a = 0$.

The Tomita–Takesaki operator $S_{\eta,\rho}$ is given by $S_{\eta,\rho} a|\eta\rangle = a^\dagger|\rho\rangle$. $S_{\eta,\rho}$ has a polar decomposition,

$$S_{\eta,\rho} = J_{\eta,\rho} \Delta_{\eta,\rho}^{1/2},$$

where we call $J_{\eta,\rho}$ the relative modular conjugation. $\Delta_{\eta,\rho}$ is Hermitian and is called the relative modular operator. We define $\Delta_{\eta,\rho}$ for pairs of states (in the algebra sense) $\rho, \eta \in M_*$, letting $|\rho\rangle = \rho^{1/2}$ and $|\eta\rangle = \eta^{1/2}$ as canonical purifications. In finite dimension, $\Delta_{\eta,\rho}(x) = \rho^{-1} x \eta$ for any

$x \in \mathcal{M}$. $\Delta_{\eta,\rho}^{it}$ is analogous to unitary time-evolution, leading to the interpretation of $\ln \Delta_{\rho,\eta}$ as a modular Hamiltonian in quantum field theory. For more information on modular theory in physics, see Refs. 19, 33, and 34.

B. Haagerup spaces

For a von Neumann algebra M on Hilbert space $\mathcal{H}$, faithful state ρ, and group G, we denote by $M \rtimes G = M \rtimes_{\sigma^\rho} G$ the crossed product of M by G with respect to the modular automorphism group $\sigma = \sigma^\rho$. Details of this construction appear in Ref. 27, Sec. 1.2, from which we take all subsequent constructions in this subsection. $M \rtimes G$ is the von Neumann algebra on $L_2(G, \mathcal{H})$ generated by $\pi_\sigma(x)$ for $x \in M$ and $\lambda(g)$ for $g \in G$, defined by

$$(\pi_\sigma(x)\xi)(h) = \sigma_h^{-1}(x)\xi(h), (\lambda(g)\xi)(h) = \xi(h-g) \quad \text{for } \xi \in L_2(G,\mathcal{H}), h \in G. \tag{22}$$

$M \rtimes \mathbb{R}$ is of type II_∞, so there exists a semifinite trace τ on the crossed product. For the rest of this subsection, we will assume that $G = \mathbb{R}$. Let $L_0(M \rtimes \mathbb{R}, \tau)$ denote the topological involutive algebra of all operators on $L_2(\mathbb{R}, \mathcal{H})$ that are measurable with respect to $(M \rtimes \mathbb{R}, \tau)$. Let $\hat{\sigma}_t$ be the dual automorphism of σ given by

$$\hat{\sigma}_s(\lambda(t)) = e^{its}\lambda(t) \text{ for } t \in \mathbb{R}, \qquad \hat{\sigma}_s(\pi(x)) = \pi(x) \text{ for } x \in M. \tag{23}$$

We then have the Haagerup L_p spaces, given as

$$L_p(M) = \{x \in L_0(M \rtimes \mathbb{R}, \tau) : \hat{\sigma}_s(x) = e^{-s/p}x \ \forall s \in \mathbb{R}\}. \tag{24}$$

In particular, $L_\infty(M)$ coincides with M. As we will recall in Sec. II E, Haagerup L_p spaces defined for the same M but different ρ are isometric, so we will not explicitly refer to ρ in denoting them. $L_p(M)$ is a linear subspace of M and an M-bimodule.

The map $\omega \mapsto d_\omega$, which maps a state $\omega \in M_*^+$ to its unique, implementing density in $L_1(M)$, extends to a linear homomorphism from M_* to $L_1(M)$. Although $\mathcal{M}$ has no semifinite trace, $L_1(M)$ is equipped with a distinguished, contractive linear functional Tr, the Haagerup trace, defined by

$$Tr(d_\omega) := \omega(\mathbf{1}) \text{ for } \omega \in M_*. \tag{25}$$

Here, d_ω is fixed by the relation that $\omega(x) = Tr(xd_\omega)$ for any $x \in M$. Hence, one may transfer the norm of M_* to a norm on $L_1(M)$, denoted as $\|\cdot\|_{L_1(M)}$. Consequently, $\|\eta\|_1 = Tr(|\eta|)$ for every $\eta \in L_1(M)$. It then holds, as expected, that

$$\|a\|_{L_p(M)} = Tr(|a|^p)^{1/p} \quad \text{and} \quad Tr(ab) = Tr(ba) \tag{26}$$

for $a \in L_p(M)$, $b \in L_{p'}(M)$, and $1 = 1/p + 1/p'$ as Hölder conjugates. The Hölder inequality holds for Haagerup L_p norms, and $L_p(M)^* = L_{p'}(M)$ for $1 \le p < \infty$. Finally, for any $a \in L_p$, there is a unique polar decomposition,

$$a = u|d_\psi|^{1/p}, \tag{27}$$

where $u \in M$, $\psi \in M_*^+$, and d_ψ implements ψ in $L_1(M)$.

If we start with a tracial von Neumann algebra M and construct Haagerup L_p spaces from (M, tr), then we will find that $Tr = tr$. Hence, as seen via Eq. (26), this L_p space coincides with the expected L_p space or Schatten class on a tracial algebra, with the norm given as $\|x\|_p = tr(|x|^p)^{1/p}$. With respect to the trace in $M \rtimes \mathbb{R}$, every normalized density in $L_p(M)$ has the same singular numbers and, hence, the same distribution, as shown in Ref. 35. Nonetheless, for quantities that depend on the L_p norms rather than directly on the detailed spectrum of densities, we are free to use the Haagerup construction everywhere.

Formally, one should distinguish between a state $\rho \in M_*^+$ and its implementing density $d_\rho \in L_1(M)$. We will however often denote d_ρ by ρ, such as in Eq. (11). As shorthand, we may denote $\|\cdot\|_{L_p(M)}$ by $\|\cdot\|_p$ when the relevant von Neumann algebra is clear from context.

Remark II.1. Let $\delta > 0$, let $\eta \in M_*^+$ be a normal, faithful state, and assume $\delta\eta \le \rho \le \delta^{-1}\eta$. The operator d_η^{it} is a unitary in $M \rtimes G$, not necessarily in M. However, the function

$$g_{\eta,\rho}(it) := d_\eta^{it} d_\rho^{-it}$$

satisfies $\hat{\sigma}_s(g_{\eta,\rho}(it)) = g_{\eta,\rho}(it)$ and, hence, does belong to $\pi(M) \cong M$. In fact, for $z = \theta + it$, $\theta \le 1/2$, we deduce from the fact that

$$d_\eta^{2\theta} \le \delta^{-2\theta} d_\rho^{-2\theta}$$

that

$$\|d_\eta^\theta d_\rho^{-\theta}\|^2 = \|d_\rho^{-\theta} d_\eta^{2\theta} d_\rho^{-\theta}\| \le \delta^{-2\theta}$$

is bounded. This implies that on $\{z|0 < \Re(z) < \frac{1}{2}\}$, the function

$$g_{\eta,\rho}(z) := d_\eta^z d_\rho^{-z}$$

is well-defined and analytic, thanks to

$$\hat{\sigma}_s(g_{\eta,\rho}(z)) = (e^{zs} d_\eta^z)(e^{-zs} d_\rho^z) = g_{\eta,\rho}(z)$$

having values in M. As noted in Ref. 36, $g_{\eta,\rho}(it)$ intertwines the modular automorphisms of η and ρ. Forms of $g_{\eta,\rho}(z)$ appear naturally and usefully in modular theory.

The same argument applies to the modular semigroup,

$$\sigma_t^{\eta,\rho}(\pi(x)) = d_\eta^{it} \pi(x) d_\rho^{-it},$$

which satisfies $\theta_s\big(\sigma_t^{\eta,\rho}(\pi(x))\big) = \sigma_t^{\eta,\rho}(\pi(x))$ and

$$g_{\eta,\rho}(it) = \sigma_t^{\eta,\rho}(\pi(1)) \in \pi(M).$$

Moreover, let $\sigma_z^{\eta,\rho}$ be the unique linear extension of the modular group. Then,

$$g_{\eta,\rho}(z) = \sigma_z^{\eta,\rho}(1) \in M$$

at least for $0 \le \Re(z) \le 1/2$.

C. The Haagerup reduction

Like the Haagerup L_p spaces, the reduction method starts with a crossed product. Instead of working with $\mathbb{R}$, we use the discrete group $G = \bigcup_n 2^{-n}\mathbb{Z} \subset \mathbb{R}$, constructing $\tilde{M} = M \rtimes_{\sigma^\eta} G$ for some normal, faithful state $\eta \in M_*^+$. The advantage of using a discrete group is that we have a conditional expectation $\mathcal{E} : \tilde{M} \to M$ given by

$$\mathcal{E}\left(\sum_g x_g \lambda(g)\right) = x_0. \tag{28}$$

$\mathcal{E}$ is norm-preserving, and a well-known result by Marie Choda[37,38] implies that such a conditional expectation may not go from a von Neumann algebra of lower type to one of higher type. Hence, $\tilde{M}$ remains of type III and will not allow us to construct Haagerup spaces. Instead, we rely on the following properties (see Ref. 27):

- (Hi) $\mathcal{E}$ and $\tilde{\eta} = \eta \circ \mathcal{E}$ are faithful.
- (Hii) There exist an increasing family of subalgebras $\tilde{M}_k$ and normal conditional expectations $F_k : \tilde{M} \to \tilde{M}_k$ such that $\tilde{\eta} F_k = \tilde{\eta}$.
- (Hiii) $\lim_k \|F_k(\psi) - \psi\|_{\tilde{M}_*} = 0$ for every normal state $\psi \in \tilde{M}_*$.
- (Hiv) For every k, there exists a normal faithful trace $\tau_k(x) = \tilde{\eta}(d_k(x))$ such that $d_k \in (\tilde{M}_k)_*^+$ and $a_k \le d_k \le a_k^{-1}$ for some scalars $a_k \in \mathbb{R}^+$. Hence, $\tilde{M}_k$ is of type II_1.

The Haagerup approximation then yields a method for proving results in type III: first, prove the result in type II_1 and then show convergence in the limit as $k \to \infty$.

D. Complex interpolation

Within finite-dimensional matrix algebras, many of the desired entropy[29] and trace[1] inequalities follow from identifying typical sets of eigenvalues. One can easily imagine that these techniques encounter challenges for infinite-dimensional operators. As noted in Ref. 1, however, the mathematical technique known as complex interpolation presents an alternate route to many of the same conclusions. Long-studied in operator theory, complex interpolation has strong results that hold without finite-dimensional assumptions. In this chapter, we review the basic tools of complex interpolation that power main results of this paper. For an in-depth treatment of the topic, the reader may consult Ref. 39.

Two Banach spaces A_0 and A_1 are compatible if both are subspaces of a Hausdorff topological space A. The sum space

$$A_0 + A_1 := \{x = x_0 + x_1 | x_0 \in A_0, x_1 \in A_1\}$$

is then a Banach space, equipped with norm

$$\|x\|_{A_0+A_1} = \inf_{x_0 \in A_0, x_1 \in A_1} \{\|x_0\|_{A_0} + \|x_1\|_{A_1}\}.$$

Let $S := \{z \in \mathbb{C} : 0 \le Re(z) \le 1\}$ be the vertical strip on the complex plane. By $\mathcal{F}(A_0, A_1)$, we denote the space of functions $f : S \to A_0 + A_1$ that are bounded and continuous on S and holomorphic on its interior such that

$$\{f(it)|t\in\mathbb{R}\}\subset A_0, \{f(1+it)|t\in\mathbb{R}\}\subset A_1.$$

$\mathcal{F}(A_0, A_1)$ is again a Banach space with norm

$$\|f\|_{\mathcal{F}} = \max\{\sup_{t\in\mathbb{R}}\|f(it)\|_{A_0}, \sup_{t\in\mathbb{R}}\|f(1+it)\|_{A_1}\}.$$

For $\theta\in[0,1]$, we define the complex interpolation space

$$[A_0, A_1]_\theta := \{x\in A_0+A_1|x=f(\theta), f\in\mathcal{F}(A_0,A_1)\} \tag{29}$$

with norm

$$\|x\|_{[A_0,A_1]_\theta} := \inf\{\|x\|_{\mathcal{F}}|f(\theta)=x\}. \tag{30}$$

For interpolation spaces, we recall Stein's interpolation theorem on norms of maps.

Theorem II.2 (Stein's interpolation, Ref. 39). *Let (A_0, A_1) and (B_0, B_1) be two couples of Banach spaces that are each compatible. Let $\{T_z|z\in S\}\subset\mathbb{B}(A_0+A_1, B_0+B_1)$ be a bounded analytic family of maps such that*

$$\{T_{it}|\,t\in\mathbb{R}\}\subset\mathbb{B}(A_0,B_0), \qquad \{T_{1+it}|\,t\in\mathbb{R}\}\subset\mathbb{B}(A_1,B_1).$$

Suppose $\Lambda_0 = \sup_t\|T_{it}\|_{\mathcal{B}(A_0,B_0)}$ and $\Lambda_1 = \sup_t\|T_{1+it}\|_{\mathcal{B}(A_1,B_1)}$ are both finite. Then, for $0<\theta<1$, T_θ is a bounded linear map from $(A_0,A_1)_\theta$ to $(B_0,B_1)_\theta$ and

$$\|T_\theta\|_{\mathbb{B}((A_0,A_1)_\theta,(B_0,B_1)_\theta)}\le\Lambda_0^{1-\theta}\Lambda_1^\theta.$$

To derive most of the results of this paper, we will rely on a different form of complex interpolation, known as Hirschman's strengthening of Hadamard's three line theorem.

Lemma II.3 (generalized Hirshman/Hadamard, Refs. 40 and 41). Let $g(z): S\to\mathbb{C}$ be bounded and continuous on S and holomorphic on its interior. Then, for $\theta\in[0,1]$,

$$\ln|g(\theta)|\le\int_{-\infty}^{\infty}\ln|g(it)|^{1-\theta}d\beta_{1-\theta}(t)+\int_{-\infty}^{\infty}\ln|g(1+it)|^{\theta}d\beta_\theta(t).$$

Here, $\beta_0(t)dt$ as in Eq. (3) is obtained as the pointwise limit of the measures $\beta_\theta(t)dt$, given in Eq. (12). For interpolation spaces, the same idea appears as follows.

Lemma II.4. Let A_0, A_1 be a pair of compatible Banach spaces and $w=\theta+is$.

1. *Let $F: S\to M$ be an analytic function vanishing at infinity. Then,*

$$\ln\|F(w)\|_{[A_0,A_1]_\theta}\le\int_{\partial S}\ln\|F(z)\|_{[A_0,A_1]_{Re(z)}}d\mu_w(z).$$

2. *$\mu_w(i\mathbb{R})=1-\theta$, $\mu_w(1+i\mathbb{R})=\theta$.*
3. *$\mu_{w|i\mathbb{R}}=f_w^0H^1$ and $\mu_{w|1+i\mathbb{R}}=f_w^1H^1$ are absolutely continuous with respect to the one-dimensional Hausdorff measure H_1, and moreover,*

$$f_w^0(it)=\frac{e^{\pi(s-t)}\sin\pi\theta}{\sin^2(\pi\theta)+(\cos(\pi\theta)-e^{-\pi(s-t)})^2}$$

and

$$f_w^1(1+it)=\frac{e^{\pi(s-t)}\sin\pi\theta}{\sin^2(\pi\theta)+(\cos(\pi\theta)+e^{-\pi(s-t)})^2}.$$

E. Kosaki spaces and norms

In Ref. 28, Kosaki constructed a family of spaces via complex interpolation, which coincide with the Haagerup spaces. In general, we will denote the Kosaki L_p norm of $x\in M$ for a pair of states ρ,η by implementing densities $d_\rho, d_\eta\in L_1(M)$ by

$$\|x\|_{L_p^w(M,\rho,\eta)}=\|d_\rho^{(1-w)/p}xd_\eta^{w/p}\|_{L_p(M)} \tag{31}$$

for $w \in [0,1]$. Multiplication by powers of densities in $L_1(M)$ enforces that the normed element is in $L_p(M)$ even if $x \in M$ may not be. When $\eta = \rho$, we denote $L^w(M,\rho) := L^w_p(M,\rho,\rho)$. At $p = \infty$, one can see that the Kosaki norm reduces to $\|\cdot\|_{L_\infty(M)}$. By the left and right Kosaki norms, we refer to, respectively, to the $w = 0$ and $w = 1$ cases.

Proposition II.5. Let $2 \le p \le \infty$ and d_ω be the density of a normal faithful state and ξ_ω be the GNS vector representing ω. Then, $i_p(x\xi_\omega) = xd_\omega^{1/2p}$ extends to isometric isomorphism between $[N, L_2(N)]_{2/p}$ and $L_p(N)$. Moreover, the map $\iota^{w,p}_\omega(x) = d_\varphi^{(1-w)/p} x d_\varphi^{w/p}$ extends to an isometry between $[\iota_{w,1}(N), L_1(N)]_{1/p}$ and $L_p(N)$.

Kosaki's original construction starts with a von Neumann algebra M and faithful state ρ, constructs the space $L_2(M,\rho)$ as the closure of M with respect to the inner product $(a,b)_\rho = \rho(ab^*)$ for $a, b \in M$ and resulting norm, and finally uses the first isomorphism of Proposition II.5 to define $L^w_{p,\rho,\eta}$. Kosaki proved a key isomorphism to the Haagerup spaces:

Theorem II.6 (Kosaki). *The map $\iota_p(x) = xd_\rho^{\frac{1}{p}}$ extends to a completely isometric isomorphism between $L^1_p(\mathcal{N},\rho)$ and the complemented subspace $L_p(\mathcal{N}e)$ of the Haagerup L_p spaces $L_p(\mathcal{N})$. Here, e is the support of ρ.*

It follows almost immediately that while the choice of the reference state in constructing the crossed product may change the map $\omega \mapsto d_\omega$, the Haagerup spaces defined for different such choices are isometric.

For our purpose, we need a slight extension of Kosaki's L_p spaces for non-faithful states φ with support projection e. This can easily be obtained by approximation. Let us assume that N is σ-finite and ψ is a normal faithful state. Then,

$$D = d_\varphi + (1-e)d_\psi(1-e)$$

is a faithful normal density in $L_1(N)$. Note that D commutes with e.

Corollary II.7. The norms $\|x\|_{L^1_p(N,\eta)}$ form an interpolation family on Ne for $1 \le p \le \infty$, as do $\|x\|_{L^0_p(M,\eta)}$ on eN.

Proof. Recall that

$$\|x\|_{L^1_p(N,\eta)} = \|x\eta^{1/p}\|_{L_p(N)}$$

form an interpolation family and the space $L_p(N)e$ is complemented in the Haagerup L_p space. Then, we observe that

$$\iota_{\eta,p}(x)e = x\eta^{1/p}e = xe\eta^{1/p} = xed_\eta^{1/p} = \iota_{\eta,p}(xe).$$

This shows that $R_e(x) = xe$ extends to a contraction from $L^1_p(N,\eta)$ to $L^1_p(Ne,\eta)$. ■

We then have a statement of Hirschmann's lemma for Kosaki L_p norms via the re-iteration theorem (see Ref. 39) and Lemma II.4.

Lemma II.8. Let $G : S \to M$ be analytic, $2 \le q_0, q_1$, and $1/q(\theta) = (1-\theta)/q_0 + \theta/q_1$. Then, for all θ in the complex strip,

$$\ln \|G(\theta)\|_{L^w_{q(\theta)}(M,\rho,\eta)} \le (1-\theta)\int \ln \|G(it)\|_{L^w_{q_0}(M,\rho,\eta)}\beta_{1-\theta}(t)dt + \theta \int \ln \|G(1+it)\|_{L^w_{q_1}(M,\rho,\eta)}\beta_\theta(t)dt.$$

For a finite von Neumann algebra M with identity $\mathbf{1}$, $\|x\|_{L^w_p(M,\mathbf{1},\mathbf{1})} = \|x\|_p$ as the usual p-norm for any $w \in [0,1]$ and $x \in M$. When M lacks finite trace, $\mathbf{1} \notin L_1(M)$ by definition. As shown in Sec. II G, it may at times be useful to take Kosaki spaces from finite von Neumann algebras, such as for proving monotonicity of relative entropies. More broadly, rewriting norm inequalities with Kosaki spaces both gives weighted generalizations and helps bypass the distinction between different algebraic types.

F. Quantum channels

A quantum channel is a general model of an open quantum process with an initially uncorrelated environment. In tracial settings, a channel is a completely positive, trace preserving (in general, normal) map $\Phi : L_1(M) \to L_1(N)$. Recall that the anti-linear duality bracket

$$(x,\rho) = Tr(x\rho^*)$$

allows us to identify $\bar{M}_*$ with $L_1(M)$ and, hence,

$$(\Phi^\dagger(x),\rho) = Tr(x\Phi(\rho)^*)$$

defines a normal, unital, completely positive map $\Phi^\dagger : N \to M$. As denoted, this construction may use the Haagerup trace.

The usual, finite-dimensional Stinespring dilation is one of the core techniques of quantum information theory, rewriting any quantum channel as an isometry followed by a partial trace. Even in semifinite von Neumann algebras, this Stinespring dilation may fail. We replace it by a more general form. The following fact is well-known. Since it is crucial for all our arguments, we indicate a short proof.

Lemma II.9. Let $\Psi : N \to M$ be a normal completely positive unital map (the dual of a channel). Then, there exist a Hilbert space H normal *-homomorphism $\pi : N \to \mathbb{B}(H)\bar{\otimes}M$ and a projection $e \in \mathbb{B}(H)$ such that

$$\Psi(x) = (e \otimes 1)\pi(x)(e \otimes 1).$$

Proof. We will use the standard GNS construction; see Ref. 42 and 43. Let $\mathcal{K} = N \otimes_\Phi M$ be the Hilbert C^*-module over M with the inner product

$$(a \otimes x, b \otimes y) = x^* \Psi(a^* b)y.$$

Let $\tilde{\mathcal{K}}$ be the closure of $\mathcal{K}$ in the strong operator topology of the module (see Ref. 44). Then, $\tilde{\mathcal{K}}$ admits a module basis and, hence, is of the form $\tilde{K} = f(H \bar{\otimes} M)$ for some projection $f \in \mathbb{B}(H)\bar{\otimes}M$. The subspace $1 \otimes M \subset \mathcal{K}$ is an M right module, and hence, the orthogonal projection q onto $(1 \otimes M)$ is in $(M^{op})' = \mathbb{B}(H)\bar{\otimes}M$. We may define the *-representation (see Ref. 42),

$$\pi(\alpha)(a \otimes x) = \alpha a \otimes x.$$

Then, we deduce that for $e = qf$, we have

$$\Psi(x) = e\pi(x)e.$$

It remains to show that π extends to the strong closure of $\tilde{K}$ and that π is normal. For simplicity, we assume that η is a normal faithful state and define the Hilbert space $L_2(\mathcal{K}, \eta)$ via the inner product,

$$(\xi, \varphi)_\eta = \eta((\xi, \varphi)).$$

Note that $L_2(\tilde{K}, \eta) = L_2(\mathcal{K}, \eta)$, and the inclusion $\tilde{K}^\eta \subset L_2(\tilde{K}, \eta)$ is dense, faithful because η is faithful. Then, we see that for all a, b, x, y, the function

$$\omega_{a,b,x,y}(\alpha) = \eta(x^* \Psi(a^* \alpha b)y)$$

is normal, thanks to Ψ being normal. By norm approximation, we deduce that π extends to a normal representation on $\mathcal{L}_2(\mathcal{K}, \eta) = L_2(\tilde{K}, \eta)$. Since this is true for all η, we see that π extends to a representation on the closure $\tilde{K}$. Finally, we observe that weak* closure of the adjointable maps on $\tilde{K}$ satisfies

$$\mathcal{L}_w(\tilde{K}) = e(\mathbb{B}\bar{\otimes}M)e.$$

Since our map $\pi : N \to \mathcal{L}_w(\tilde{K})$ is normal, we see that, after identification, $\pi : N \to (\mathbb{B}\bar{\otimes}M)$ is a normal, not necessarily unital *-homomorphism. ■

G. Relative entropy

In any von Neumann algebra M, we define the relative entropy

$$D(\rho\|\eta) := \langle \rho^{1/2}, \ln \Delta_{\eta,\rho} \rho^{1/2} \rangle, \tag{32}$$

where the inner product is given by the GNS construction for an (algebra, state) pair (M, ω) if needed. With a semifinite trace, there is an equivalent form,

$$D(\rho\|\eta) = tr(\rho(\log \rho - \log \eta)),$$

which is more familiar in quantum information.

We may write a wide variety of generalized Rényi entropies in terms of the Kosaki norms of $g_{\rho,\eta}^{1/p'}$, where p' is the Hölder conjugate of p. In particular, we recall the α–z Rényi entropies defined and analyzed in Refs. 45–48 for real $\alpha, z \geq 0$, given (up to a constant from taking the natural rather than the base two logarithm) by

$$D_{\alpha,z}(\rho\|\eta) = \frac{1}{\alpha - 1} \ln tr\Big(\Big(\rho^{\alpha/z} \sigma^{(1-\alpha)/z}\Big)^z\Big). \tag{33}$$

We recall that when $\alpha = z$, this form recovers the sandwiched Rényi relative α-entropy,[49–51] and when $z = 1$, this form recovers the Petz–Rényi relative α-entropy.[52] When $\alpha = z = 1$, these forms coincide as the usual relative entropy. Through the Kosaki norm, we re-express the α–z Rényi entropy for $z \geq \alpha \geq 1$ as

$$D_{\alpha,z}(\rho\|\eta) = \frac{z}{\alpha - 1} \ln \|g_{\rho,\eta}^{1/z'}\|_{L_z^{z-\alpha}(M,\rho,\eta)}, \tag{34}$$

where z' is the Hölder conjugate of z. When M is a general von Neumann algebra, the Kosaki form is nonetheless a sensible expression. The range of α and z may extend to $\alpha, z \geq 0$ in finite dimension by formally interpreting Eq. (31) for $w \notin [0,1]$ although it might not always be valid to construct the Kosaki space on arbitrary von Neumann algebras. The Petz–Rényi relative α-entropy corresponds to the left Kosaki norm, and the sandwiched relative entropy corresponds to the right Kosaki norm.

Kosaki L_p spaces provide an extremely convenient tool to prove data processing inequalities for the sandwiched relative entropy. Data processing for $p > 1$ was originally shown using other methods in Ref. 53. Here, we briefly sketch the Kosaki space argument. Let $\Phi : L_1(M) \to L_1(\hat{M})$ be a completely positive trace preserving map and η be a normal faithful sate, which we call the *reference* state. Let $\hat{\eta} = \Phi(\eta)$ be the image with support $\hat{e}$. By continuity, $\Phi(L_1(M)) \subset \mathcal{L}_1(\hat{e}\hat{M}\hat{e})\hat{e}$, and hence, we will assume $\hat{e} = 1$. We obtain an induced map $\Phi_\infty : M \to \hat{M}$ given by

$$\hat{\eta}^{1/2}\Phi_\infty(x)\hat{\eta}^{1/2} = \Phi(\eta^{1/2}x\eta^{1/2}).$$

More generally, it is easy to show by interpolation that the map

$$\Phi_p(\eta^{1/2p}x\eta^{1/2p}) = \hat{\eta}^{1/2p}\Phi_\infty(x)\hat{\eta}^{1/2p}$$

is a contraction. Of course, interpolation applies exactly because $\Lambda_p(\eta) = \eta^{1/2p}M\eta^{1/2p}$ is dense in the image of the symmetric Kosaki map $\iota_p^{1/2} : [\iota^{1/2}(M), L_1(M)]_{1/p} \to L_p(M)$.

We refer to Ref. 54 for the fact that Φ_∞ is indeed a normal completely positive unital map. Therefore, Φ_∞ admits a Stinespring dilation,

$$\Phi_\infty(x) = e\pi(x)e,$$

where $\pi : M \to \mathcal{L}(H_{\hat{M}})$ is obtained from the W^*-module $M \otimes_{\Phi_\infty} \hat{M}$.

Lemma II.10. Let $2 \leq p \leq \infty$ and $y \in M$. Then,

$$\|\pi(y)e\|_{L^1_{2p}(\mathcal{L},\hat{\eta})} \leq \|y\|_{L^1_{2p}(M,\eta)}.$$

Indeed, for $p = \infty$, this is obvious, and for $p = 2$, we have

$$\begin{aligned}\|\pi(y)e\|_2^2 &= \hat{\eta}(e\pi(y^*y)e) = \hat{\eta}(\Phi_\infty(y^*y)) = Tr(\eta^{1/2}\Phi_\infty(y^*y)\hat{\eta}^{1/2}) \\ &= Tr\Big(\Phi(\eta^{1/2}y^*y\eta^{1/2}\Big) \leq Tr(\eta^{1/2}y^*y\eta^{1/2}).\end{aligned}$$

Here, we only had to use the trace-reducing property of the original map Φ. In combination with Kosaki's embedding result, we deduce that

$$\begin{aligned}\|\hat{\eta}^{-1/2p'}\Phi(\eta^{1/2}y^*y\eta^{1/2})\hat{\eta}^{-1/2p'}\|_p &= \|\hat{\eta}^{-1/2p'}\hat{\eta}^{1/2}\Phi_\infty(y^*y)\hat{\eta}^{1/2}\hat{\eta}^{-1/2p'}\|_p \\ &= \|\hat{\eta}^{1/2p}\Phi_\infty(y^*y)\hat{\eta}^{1/2p}\|_p \\ &= \|\pi(y)e\|^2_{L^1_{2p}(\mathcal{L},\hat{\eta})} \\ &\leq \|y\|^2_{L^1_{2p}(M,\eta)} \\ &= \|\eta^{1/2p}y^*y\eta^{1/2p}\|_p = \|\eta^{-1/2p'}\eta^{1/2}y^*y\eta^{1/2}\eta^{-1/2p'}\|_p.\end{aligned}$$

Thus, by density, we deduce the sandwiched p-Renyi data processing inequality.

Theorem II.11. *Let η be faithful and $1 \leq p \leq \infty$. Then,*

$$\|\Phi(\eta)^{-1/2p'}\Phi(\rho)\Phi(\eta)^{-1/2p'}\|_p \leq \|\eta^{-1/2p'}\rho\eta^{-1/2p'}\|_p$$

for all $\rho \in L_1(M)$. Here, $\|\cdot\|_p$ may refer to Haagerup L_p norms and $^{-1/2p'}$ may refer to the pseudoinverse on the support. In terms of sandwiched Rényi entropy, the inequality is equivalent to

$$D_p(\Phi(\rho)\|\Phi(\eta)) \leq D_p(\rho\|\eta).$$

III. TRACE INEQUALITIES

From the Kosaki L_p version of Hirschmann's lemma (Lemma II.8) follows the Kosaki L_p version of the two main results of Ref. 1 and the extended Araki–Lieb–Thirring (ALT) and Golden–Thompson (GT) inequalities. First is a generalizing reproof of the former from Ref. 1, Theorem 3.2.

Proof of Theorem I.1. Assume for now that x_k are positive definite for all k and that ρ, η are faithful. When $r = 1$, $\beta_r(t)$ acts like a delta distribution at 0, and the inequality follows trivially, so suppose $r \in (0,1)$. Let $G(z) := \prod_{k=1}^n x_k^z$. Positive definiteness and boundedness of x_k for all $k \in 1 \ldots n$ ensure analyticity of G. We apply Lemma II.8 with $\theta = r, q_0 = \infty, q_1 = p$. Then, $q_\theta = p/r$,

$$\theta \ln \|G(1+it)\|_{L^w_{q_1}(M,\rho,\eta)} = r \ln \left\| \prod_{k=1}^n x_k^{1+it} \right\|_{L^w_p(M,\rho,\eta)},$$

$$(1-\theta) \ln \|G(it)\|_{L^w_{q_0}(M,\rho,\eta)} = (1-r) \ln \left\| \prod_{k=1}^n x_k^{it} \right\|_{L^w_\infty(M,\rho,\eta)},$$

and

$$\ln \|G(\theta)\|_{L^w_{p/r}(M,\rho,\eta)} = \ln \left\| \prod_{k=1}^n x_k^r \right\|_{L^w_{p/r}(M,\rho,\eta)}.$$

As $\prod_k x_k^{it}$ is unitary and because the $L^w_\infty(M,\rho,\eta)$ norm is essentially just the operator norm on M,

$$\ln \left\| \prod_{k=1}^n x_k^{it} \right\|_{L^w_\infty(M,\rho,\eta)} = 0,$$

completing the theorem.

If x_k is merely positive semidefinite, we interpret

$$\left\| \prod_{k=1}^n x_k^{1+it} \right\|_{L^w_p(M,\rho,\eta)} = \lim_{\epsilon \to 0} \left\| \prod_{k=1}^n (x_k + \epsilon \mathbf{1})^{1+it} \right\|_{L^w_p(M,\rho,\eta)}$$

for some positive definite $\mathbf{1}$. Then, the inequality holds.

If ρ, η are not faithful, we interpret $\rho = \rho + \epsilon(1-e_\rho)\omega, \eta = \eta + \epsilon(1-e_\eta)\omega$ for a faithful state $\omega \in M_*^+$ and take the limit as $\epsilon \to 0$, where e_ρ and e_η are the respective support projections of ρ and ω. ■

The generalized Golden–Thompson inequality from Ref. 1, Corollary 3.2 requires a generalized Kato–Lie–Suzuki–Trotter formula. Unfortunately, this result is not so simple when we combine elements of a type III von Neumann algebra M with an unbounded element of $L_1(M)$. Instead, we use the Trotter formula in finite algebras with the Haagerup approximation method to extend to the desired result.

Lemma III.1. Let $\{H_k\}_{k=1}^n \subseteq M$ be a collection of bounded operators in M, $\rho = \exp(H_0)$ be such that $\rho \in L_p(M)$ [equivalently, $\rho^p \in L_1(M)$], and $x_k = \exp(H_k)$ for each $k \in 1 \ldots n$. Then, we have the following.

1. $\alpha_r = (\rho^{r/2} x_1^{r/2} \ldots x_{n-1}^{r/2} x_n^r x_{n-1}^{r/2} \ldots x_1^{r/2} \rho^{r/2})^{1/r} \in L_p(M)$ *and is bounded in the L_p norm for $r = 1/\tilde{r} : \tilde{r} \in \mathbb{N}$.*
2. *Let M be a finite von Neumann algebra and x_k be bounded. Then,*

$$\lim_{r\to 0} \alpha_r = \exp\left(H_0 + \sum_k H_k\right).$$

Proof. By Hölder's inequality, we deduce that

$$\|\alpha_r\|_{L_p(M)} \leq \prod \|x_k\|_{L_\infty(M)} \|\rho\|_{L_p(M)}$$

and is thereby uniformly bounded. We use the embedding of $L_p(M)$ into $L_{p,\infty}(M \rtimes \mathbb{R}, tr)$ so that all α_r are indeed affiliated to $M \rtimes \mathbb{R}$. Let e be a spectral projection of ρ so that ρe is bounded. Using that $a \leq b$ implies $a^r \leq b^r$, we deduce that $\alpha_r e$ is also bounded. By the Trotter formula,[55,56] we deduce for the ∞ norm that

$$\lim_{r\to 0} \sigma(\alpha_r) e = \exp\left(H_0 + \sum_{k=1}^n H_k\right) e.$$

This may not hold in the general L_p spaces, where $\rho \in L_1(M)$ is unbounded. By extracting the exponential of a positive multiple of the identity, we can make all $H_0 \ldots H_k$ effectively negative operators, thereby satisfying the conditions of the Trotter formula. Hence, α_r converges in the measure topology to $\exp(H_0 + \sum_{k=1}^n H_k)$. On the image of $L_p(M)$, the norm and the measure topology coincide, so α_r converges in L_p and definitely weakly to $\exp(H_0 + \sum_k H_k)$. Note that

$$\sigma_s\left(\exp\left(H_0 + \sum_k H_k\right)\right) = \exp\left(\sigma_s(H_0) + \sum_k H_k\right).$$

Since

$$\sigma_s(\exp(H_0)) = e^{-s/p} \exp(H_0),$$

we deduce that

$$\sigma_s(H_0) = -\frac{s}{p} + H_0.$$

This implies that

$$\exp\left(\sigma_s(H_0) + \sum_k H_k\right) = e^{-s/p} \exp\left(H_0 + \sum_k H_k\right).$$

In other words, the limit is in L_p. Then, weak converge already implies

$$\| \exp\left(H_0 + \sum_k H_k\right)\|_{L_p(M)} \le \limsup_r \|\alpha_r\|_{L_p(M)}.$$

This concludes the proof for the Haagerup L_p space. ∎

Remark III.2. $\exp(H_0 + H)$ *has to be interpreted very carefully. This can be done using the embedding of* $L_1(N)$ *into* $L_{1,\infty}(N \rtimes \mathbb{R})$. *Using this formalism, the density for* $\exp(H_0 + H)$ *is the unique positive functional* ψ *such that*

$$g_{\psi,\varphi}(it) = (D\psi : D\varphi)_t = \exp(it(H_0 + H))\exp(-itH_0)$$

in the sense of Connes' cocycle. (The actual densities are then obtained by analytic continuation or by a power series.) In Ref. 9, this object is defined as ω^h, *provided that* $\omega(x) = \mathrm{tr}(\exp(H_0)x)$. *Since the density* $\exp(H_0)$ *is* L_0 *measurable, the logarithm* H_0 *is actually well-defined by the functional calculus. This construction is used in the description of relative entropy.*

Due to the subtleties therein, the generalized Golden–Thompson inequality is stated as a theorem rather than a corollary.

Proof of Theorem I.2. First, we handle the finite case, in which the proof follows simply from that of the original, Ref. 1, Corollary 3.2. Let $x_k = \exp(2H_k)$ for $k = 1 \ldots n$. Theorem I.1 implies that

$$\ln\left\|\prod_{k=1}^n x_k^r\right\|_{L^1_{2p/r}(M,\rho)} \le \int dt\beta_r(t) \ln\, tr\left(\rho^{1/2p} A_1^{\frac{1+it}{2}} \ldots A_n^{\frac{1+it}{2}}\right)^p. \tag{35}$$

For an operator $y \in L_q(M)$, it will hold generally that $\|y\|_q = \|y^* y\|_{q/2}^{1/2}$. For the Kosaki norms,

$$\|y\|_{L^1_q(\tilde{M},\rho)} = \|y\rho^{1/q}\|_{L_q(M)} = \|\rho^{1/q} y^* y\rho^{1/q}\|^{1/2}_{L_{q/2}(M)}. \tag{36}$$

Hence,

$$\begin{aligned}\ln\left\|\prod_{k=1}^n x_k^r\right\|_{L^1_{2p/r}(\tilde{M},\rho)} &= \ln\, tr\left(\left|x_1^{r/2} \ldots x_{n-1}^{r/2} x_n^{r/2} \rho^{r/2p}\right|^{2p/r}\right) \\ &= \ln\, tr\left(\left(\rho^{r/2p} x_1^{r/2} \ldots x_{n-1}^{r/2} x_n^r x_{n-1}^{r/2} \ldots x_1^{r/2} \rho^{r/2p}\right)^{p/r}\right).\end{aligned} \tag{37}$$

Compared with Ref. 1, Corollary 3.2, we must be more careful to show that the limit as $r \to 0$ exists and converges to something that is still in the correct Haagerup L_p space. Now, we consider the family of operators as follows:

$$\alpha_r = (\rho^{r/2p} x_1^{r/2} \ldots x_{n-1}^{r/2} x_n^r x_{n-1}^{r/2} \ldots x_1^{r/2} \rho^{r/2p})^{1/r}.$$

We apply Lemma III.1 to complete the finite case, substituting $\rho^{1/p}$ for ρ.

Now, we consider the general theorem in arbitrary von Neumann algebras. Let us first indicate the proof for $p = 2$. We apply the Haagerup construction for $\varphi(x) = \mathrm{tr}(dx)$ and assume $\mathrm{tr}(d) = 1$, i.e., φ is a normal faithful state. Then, $N \rtimes G = \bigcup_k M_k$ and there exists conditional expectation $E_k : N \rtimes G$ such that $E_k(x)$ converges strongly to x and $E_k(\psi)$ converges in the L_1 norm. The good news is that M_k is a finite von Neumann algebra with trace τ_k and the new extended state $\hat{\varphi}$ satisfies the following.

(i) $E_k(\hat{\varphi}) = \hat{\varphi}$.
(ii) The density $d_k = \exp(H_0(k))$ of $\hat{\varphi}$ with respect to τ_k is bounded from above and below.

This allows us to define the new bounded elements $H_j(k) = E_k(H_j)$. In this context, Lemma 3.2 2 applies and we can use the Lie–Trotter–Kato formula and deduce

$$\left\| \exp\left(\frac{H_0(k)}{2} + \sum_{j=1}^n E_k(H_j)\right)\right\| \leq \int d\beta_0(t) \left\| \prod \exp(1 + itE_k(H_j))\right\|_{L_2^1}.$$

Since $E_k(H_j)$ converges to H_j strongly and, hence, $\exp(itE_k(H_j)$ converges strongly (this series is uniformly absolutely convergence because the elements are uniformly bounded), the dominated convergence implies convergence to the correct right-hand side in $L_p(N \rtimes G)$. Applying the conditional expectation yields the correct upper bound.

Taking the limit for $k \to \infty$ on the left-hand side is more problematic, but well known thanks to the work of Araki.[36]

Let us denote $b_k = \sum_{j=1}^n E_k(H_j)$. Then,

$$\tilde{d}_k \ \exp(H_0(k) + b_k)^{1/2} \ \exp(H_0(k)/2) d_k^{1/2} = \exp\left(H_0(k) + b_k\right)^{1/2}$$

is exactly the GNS vector implementing the functional $\varphi(k)\ (x) = \tau_k(\exp(H_0(k) + b_k)x)$, and the relative modular group is given by

$$(D\varphi(k) : D\hat{\varphi})_t = \tilde{d}_k^{\ it} d_k^{-it}.$$

This particularly simple formula here is due to the trace. However, the corresponding cocycle also makes sense in the not necessarily finite von Neumann algebra $N \rtimes G$. Moreover, thanks to the work of Araki, there is a clear interpretation of the density obtained from a bounded perturbation ω^h by a bounded element $h \in N \rtimes G$. More precisely, the implementing vector is given by (see, in particular, Ref. 36, Proposition 4.12)

$$\xi_{\exp(\log(\hat{d}+h))} = \exp((\log\Delta + h)/2)(\xi_{\hat{\varphi}}).$$

Araki wrote down the explicit Feynman–Katz formula for this power series and the new density $\Psi(h)$. In the semifinite case, there is no need to use the modular operator $\Delta = L_d^{1/2} R_d^{-1/2}$ because the exponential function is additive for commuting operators. Now, we may apply Proposition 4.1 of Ref. 36, which implies the strong convergence of $\Psi(h_k)$ to $\Psi(h)$. This shows that

$$\xi_{\exp(\hat{H}_0 + \sum_j H_j)} = \Psi\left(\sum_j H_j/2\right) = \lim_k \Psi\left(\sum_j E_k(H_j)\right) = \lim_k \xi_{\varphi(k)}.$$

Here, we use the conditional expectation of $N \rtimes G \to M_k$ to define the unique embedding on the L_2 space level. Thus, passing to the limit for $k \to \infty$, the norm estimate remains true thanks to the dominated convergence theorem.

Finally, for other values of p, we may use Ricard's estimate of the Mazur map[57] to show strong convergence on the L_p level from rescaling the bounded Hamiltonian and the density. This means that the estimate is only true for $p \geq 1$. ■

Proof of Remark I.3. This inequality is immediate in the finite case, following the arguments of Ref. 58. We then apply the continuity argument from the Proof of I.2 for $\exp(\rho/p + Y)$, where in this case $Y = \ln X \in \mathcal{M}$. ■

Remark III.3. The generalization of the ALT and GT inequalities to unitarily invariant norms in Ref. 41 holds automatically in type I and with small modifications in type II, where there is a semifinite trace. In non-tracial algebras, there may not exist unitarily invariant norms in this sense.

Remark III.4. Taking a Kosaki norm on a finite von Neumann algebra M, such as of finite dimension or type II_1, we have that $\mathbf{1} \in L_p(M)$. In this case, the Haagerup trace Tr *coincides with the finite trace tr, and we may take the Kosaki norm $\|\cdot\|_{L_p^w(p,\mathbf{1},\mathbf{1})}$. Doing so recovers the original ALT and GT inequalities from Ref. 1.*

IV. L_p ESTIMATES AND RECOVERY MAPS FOR QUANTUM CHANNELS

In this section, we present *a priori* estimates on L_p spaces, which are required to formulate the recovery theorem in the von Neumann algebra setting. The arguments are very closely related to the first author's lecture notes for proving the data processing inequality for the sandwiched entropy.

In the following, we will fix $\Phi : L_1(M) \to L_1(\hat{M})$, $\Psi = \Phi^\dagger : \hat{\mathcal{M}} \to M$, $e \in \mathbb{B}(H)\bar{\otimes}M = \tilde{M}$, and the normal *-homomorphism $\pi : \hat{M} \to \tilde{M}$.

Lemma IV.1. Let $\Phi(\eta) = \hat{\eta}$ with support $s(\eta)$, $s(\hat{\eta})$, respectively. Then, for all $1 \le p \le \infty$,

$$\|\pi(y)es(\eta)\|_{L^1_{2p}(\tilde{M},\eta)} \le \|ys(\hat{\eta})\|_{L_{2p}(\hat{M},\hat{\eta})}.$$

Proof. Since Φ is trace preserving, we note that

$$\begin{aligned}\|\pi(y)e\|^2_{L_2(\eta))} &= Tr(d_\eta e\pi(y^*y)e) = Tr(d_\eta\Psi(y^*y))\\ &= Tr(\Phi(d_\eta)y^*y) = \|y\|^2_{L_2(\hat{\eta})}.\end{aligned}$$

Thus, interpolation according to Lemma II.7 implies the assertion. ∎

Proposition IV.2. Let $d \in L_1(N)$ be the density of a state η and $\hat{d} = \Phi(d)$, with support $s = s(d)$ and $\hat{s} = s(\hat{d})$. Let $1 \le p \le \infty$. Then,

$$R_p(x) = d^{1/2p}\Phi^\dagger(\hat{d}^{-1/2p}x\hat{d}^{-1/2p})d^{1/2p}$$

extends to a contraction from $L_p(\hat{M})$ to $L_p(M)$.

Proof. Let us recall the abstract Marcinkiewicz interpolation theorem: Let $(A_0, A_1) \subset V$, $(\hat{A}_0, \hat{A}_1) \subset \hat{V}$ be interpolation couples, and $T : A_0 + A_1 \to \hat{A}_0 + \hat{A}_1$ be a linear map such that $T(A_0) \subset \hat{A}_0$ and $T(A_1) \subset \hat{A}_1$. Then,

$$\|T : A_\theta \to \hat{A}_\theta\| \le \|T : A_0 \to \hat{A}_0\|^{1-\theta}\|T : A_1 \to \hat{A}_1\|^\theta.$$

For the proof, one considers the analytic function $G(z) = T(F(z))$ and then takes the infimum over F such that $F(\theta) = x$. In our situation, $A_0 = \hat{s}\hat{M}\hat{s}$ and $A_1 = \hat{s}L_1(\hat{M})\hat{s}$ and $\hat{A}_0 = sMs$, $\hat{A}_1 = sL_1(M)s$. The map is given by $T(\hat{d}^{1/2}x\hat{d}^{1/2}) = d^{1/2}\Phi^\dagger(x)d^{1/2}$. We also use the map $T_\infty(x) = s\Phi^\dagger(x)s$ and observe the following commuting diagram:

$$\begin{array}{ccc} \hat{s}\hat{M}\hat{s} & \xrightarrow{T_\infty} & M \\ \downarrow\iota_{p,\hat{d}} & & \downarrow\iota_{p,d} \\ \hat{s}L_p(\hat{M})\hat{s} & \xrightarrow{R_p} & L_p(M) \\ \downarrow\gamma_{p',\hat{d}} & & \downarrow\gamma_{p',d} \\ \hat{s}L_1(\hat{M})\hat{s} & \xrightarrow{T} & L_1(M) \end{array}.$$

Here, $\gamma_{p,d}(x) = d^{1/2p'}xd^{1/2p'}$ is chosen such that $\gamma_{p,d}\iota_{p,d} = \iota_{1,d}$ is the symmetric Kosaki embedding. We may think of T_∞ as a densely defined map on $\iota_1(\hat{s}\hat{M}\hat{s})$. Thus, it remains to show that ι_1 is indeed a contraction. By Hölder's inequality, the map $q : L_2(\hat{M}) \otimes L_2(\hat{M}) \to L_1(M)$, $q(x \otimes y) = xy$ is a contraction and, indeed, a metric surjection because the adjoint $q^* : \hat{M} \to \mathbb{B}(L_2(\hat{M}))$ is isometric. The same is true for $\hat{q}(x \otimes y) = \hat{s}xy\hat{s}$ as a map $\hat{q} : \hat{s}L_2(\hat{M}) \otimes L_2(\hat{M})\hat{s} \to \hat{s}L_1(\hat{M})\hat{s}$. Note that $\hat{M}\hat{d}^{1/2}$ is dense in $L_2(M)$. This shows that the set D_1 of elements

$$\hat{x}\hat{d}^{1/2}xy\hat{d}^{1/2}, \qquad \|\hat{d}^{1/2}x\|_2 < 1, \|y\hat{d}^{1/2}\| < 1$$

is dense in the unit ball of $\hat{s}L_1(\hat{M})\hat{s}$. Then, we recall that

$$\|\pi(y)ed^{1/2}\|_2^2 = Tr(d\Phi^\dagger(y^*y)) = Tr(\hat{d}y^*y) = \|y\hat{d}^{1/2}\|_2^2.$$

Taking *'s, we see that similarly $\|d^{1/2}e\pi(x)\|_2 = \|\hat{d}^{1/2}x\|_2$. Let $u \in M$ be a contraction. Then, we deduce (where Tr is the Haagerup trace) that

$$\begin{aligned}Tr(uT(\hat{d}^{1/2}xy\hat{d}^{1/2})) &= Tr(ud^{1/2}\Phi^\dagger(xy)d^{1/2})\\ &= Tr(ud^{1/2}e\pi(xy)ed^{1/2})\\ &= (\pi(x)ed^{1/2}, \pi(y)ed^{1/2}u).\end{aligned}$$

Thanks to the right module property of $L_2(\mathcal{M})$, we deduce that

$$\begin{aligned}|Tr\Big(uT(\hat{d}^{1/2}xy\hat{d}^{1/2})| &\le \|\pi(x)ed^{1/2}\|_{L_2(\tilde{M})}\|\pi(y)ed^{1/2}u\|_{L_2(\tilde{M})}\\ &\le \|\pi(x^*)ed^{1/2}\|_{L_2(\tilde{M})}\|\pi(y^*)ed^{1/2}\|_{L_2(\tilde{M})}\|u\|\\ &= \|u\|\,\|\hat{d}^{1/2}x\|_2\,\|y\hat{d}^{1/2}\|_2.\end{aligned}$$

Taking the supremum over $\|u\| \le 1$, we deduce that $T(D_1)$ belongs to the unit ball of $L_1(M)$, and hence, T extends to a contraction on $\hat{s}L_1(M)\hat{s}$. By the abstract Marcinkiewicz interpolation theorem, we deduce that R_p is also a contraction and the continuous extension of the map $R_p(\hat{d}^{1/2p}x\hat{d}^{1/2p}) = d^{1/2p}\Phi^\dagger(x)d^{1/2p}$. ∎

As an application, we deduce the contraction property of the (twirled) Petz recovery maps on L_p.

Lemma IV.3. Let η be a state and $\hat{\eta} = \Phi(\eta)$ be the image under η with support $\hat{e}$. Then,

$$R_z(\hat{x}) = \eta^{\bar{z}/2}\Phi^\dagger(\hat{\eta}^{-\bar{z}/2}\hat{x}\hat{\eta}^{-z/2})\eta^{z/2}$$

extends to a (completely) bounded operator on $L_{p(z)}(\hat{M})$ with values in $L_{p(z)}(M)$ for

$$\frac{1}{p(z)} = Re(z).$$

Proof. First, we handle the semifinite case. Let $\Lambda_{\hat{\eta},p(z)} = \hat{\eta}^{1/2p(z)}\hat{M}\hat{\eta}^{1/2p(z)}$ be the image of the symmetric Kosaki map in $L_{p(z)}(\hat{e}\hat{M}\hat{e})$. We consider Kosaki's right-sided interpolation space,

$$L_{2p(z)} = [\hat{M}, L_2(\hat{M},\hat{\eta})]_{1/p(z)}.$$

For an element, $\hat{x} \in L_{2p(z)}$ of norm <1. We can find an analytic function $g(z) \in \hat{M}\hat{e}$ such that

$$\|g(it)\|_\infty \le 1, \hat{\eta}(g(1+it)^*g(1+it)) \le 1$$

for all t. This allows us to consider

$$G(z) = \pi(g(z))e_N \in \mathcal{L}(H_{\mathcal{M}})$$

and deduce that

$$\|G(z)\|_{L_{2p(z)}(\mathcal{L}(H_{\mathcal{M}}),\eta)} \le 1.$$

Indeed, this is obvious for $z = it$. For $z = 1 + it$, we note that

$$\begin{aligned}\|G(1+it)\|^2_{L_2(\mathcal{L}(\mathcal{H}_{\mathcal{M}}),\eta)} &= \|\eta^{1/2}G(1+it)^*G(1+it)\eta^{1/2}\|_1\\ &= Tr(\eta^{1/2}\Phi^\dagger(g(z+it)^*g(1+it))\eta^{1/2})\\ &= Tr(\Phi(\eta)g(1+it)^*g(1+it)) = \|g(1+it)\|_{L_2(\hat{M},\hat{\eta})} \le 1.\end{aligned}$$

There, we have shown that $V_z : L_{2p(z)}(\hat{M}\hat{e}) \to L_{2p(z)}(\mathcal{L}(H_{\mathcal{M}}))$,

$$V_z(\hat{x}\hat{\eta}^{z/2}) = \pi(\hat{x})e\eta^{z/2}$$

extends to a contraction on $L_{2p(z)}(\hat{M}\hat{e})$ with values in $L_{2p(z)}(\mathcal{L}(H_{\mathcal{M}}))$. Now, we consider an element $\hat{x} \in \Lambda_{p(z),\hat{\sigma}}(\hat{M})$. Note that $L_p(\hat{M}) = L_{2p}(\hat{M})L_{2p}(\hat{M})$, i.e., we can write $\hat{x} = \hat{x}_1\hat{x}_2$ such that $\hat{e}\hat{x}_1 = \hat{x}_1$ and $\hat{x}_2\hat{e} = \hat{x}_2$. By the argument above, we know that

$$\|R_z(\hat{x}_j^*\hat{x}_j)\|_{p(z)} = \|(V_z(\hat{x}_j)^*V_z\hat{x}_j)\|_{p(z)} \le \|V_z(\hat{x}_j)\|^2_{2p(z)} \le \|x_j\|^2_{2p(z)}$$

holds for $j = 1, 2$. Therefore,

$$\|R_z(\hat{x}_2^*\hat{x}_1)\|_{p(z)} \le \|(V_z\hat{x}_2)^*\|_{2p(z)}\|V_z(\hat{x}_1)\|_{2p(z)} \le \|\hat{x}_2\|_{2p(z)}\|\hat{x}_1\|_{2p(z)}.$$

Taking the infimum over all such decompositions implies the assertion.

In Haagerup spaces, let $z = \theta + it$ and $p = \theta^{-1}$. Then, we have a factorization

$$R_z = \sigma^d_{-t}R_p\sigma^{\hat{d}}_t.$$

Here, we use the L_p version of the modular group,

$$\sigma_t^d(x) = e^{-itd} x e^{itd}.$$

Note that

$$\theta_s(\sigma_t^d(x)) = e^{-itd} e^{itd} e^{-s/p} x = e^{-s/p} x.$$

Thus, by the definition of the Haagerup L_p space, σ_t^d is a contraction with inverse σ_{-t}^d. ■

V. p-FIDELITIES AND INTERPOLATION

A main tool in our analysis of recovery maps will be given by a new definition of the p-fidelity from Ref. 24,

$$F_p(x, y) = \frac{\|\sqrt{y}\sqrt{x}\|_p}{\max \|x\|_p, \|y\|_p},$$

and for $x, y \in L_p$,

$$f_p(x, y) = \|\sqrt{x}\sqrt{y}\|_p.$$

Lemma V.1. Let $1 \le p \le \infty$ *and* η *be faithful. Let* $E : \tilde{M} \to M$ *be a conditional expectation and*

$$\tilde{\rho} = \rho \circ E, \tilde{\eta} = \eta \circ E$$

such that $\tilde{\eta}$ *is also faithful. Then,*

$$f_p(\tilde{\rho}^{1/p}, \tilde{\eta}^{1/p}) = f_p(\rho^{1/p}, \eta^{1/p}).$$

Proof. We have to rewrite the fidelity by duality as follows:

$$\begin{aligned} f_p(x, y) &= \sup_{\|z\|_{p'} \le 1} Tr(z^* x^{1/2p} y^{1/2p}) \\ &= \sup_{\|ay^{1/p'}\|_{p'} \le 1} Tr(y^{1/2} a^* x^{1/2p} y^{1/2} y^{-1/2p}) \\ &= \sup_{\|ay^{1/p'}\|_{p'} \le 1} Tr(ay^{1/2}, \Delta_{x,y}^{1/2p}(y^{1/2})). \end{aligned}$$

According to our assumption, $M \subset \tilde{M}$ and also $M_2(M) \subset M_2(\tilde{M})$. According to Connes' 2×2 matrix trick (see Ref. 59), we know that $L_2(M_2(M)) \subset L_2(M_2(\tilde{M}))$. By approximation, we may assume that ρ and, hence, $\tilde{\rho}$ are also faithful. Then, $\psi(x) = \frac{\rho(x_{11})+\eta(x_{22})}{2}$ is a faithful state on $M_2(M)$ and $\tilde{\psi} = \psi \circ E$ is the corresponding extension. We also have a canonical embedding $\iota_2 : L_2(M_2(M)) \to L_2(M_2(\tilde{M}))$ given by $\iota_2(xd_\psi^{1/2}) = xd_{\tilde{\psi}}^{1/2}$ (see Ref. 59). Moreover, we have the following commutation relation:

$$\iota_2 \circ \sigma_t^\psi = \sigma_t^{\tilde{\psi}} \iota_2,$$

which implies

$$\iota_2 \Delta_\psi^z = \Delta_{\tilde{\psi}}^z \iota_2.$$

Let us also recall that for the matrix unit $e_{12} = |1\rangle\langle 2|$, we have

$$e_{12} \otimes \Delta_{\rho,\eta}(\xi) = \Delta_\psi(e_{12} \otimes \xi).$$

In particular, $\iota_2(d_\eta^{1/2}) = d_{\tilde{\eta}}^{1/2}$ and

$$\Delta_{\tilde{\rho},\tilde{\eta}}^{1/2p}(d_{\tilde{\eta}}^{1/2}) = \Delta_{\tilde{\rho},\tilde{\eta}}^{1/2p}(\iota_2(d_\eta^{1/2})) = \iota_2\Big(\Delta_{\rho,\eta}^{1/2p}(d_\eta^{1/2}).$$

Now, it is easy to conclude. The map $\iota_{p'}(ad_\eta^{1/2}) = ad_{\tilde{\eta}}^{1/2}$ extends to an isometric embedding of $L_{p'}(M) \subset L_{p'}(\tilde{M})$, and hence,

$$f_p(\rho^{1/p}, \eta^{1/p}) \le f_p(\tilde{\rho}^{1/p}, \tilde{\eta}^{1/p}).$$

On the other hand, for $a \in \tilde{M}$, we see that for $x \in M$, we have

$$(ad_{\tilde{\eta}}^{1/2}, xd_{\tilde{\eta}}^{1/2}) = (E(a)d_{\eta}^{1/2}, xd_{\eta}^{1/2}).$$

Since the conditional expectation extends to a contraction $E_{p'}(ad_{\tilde{\eta}^{1/p'}}) = E(a)d_{\eta}^{1/p'}$, we also find the reverse inequality $f_p(\tilde{\rho}^{1/p}, \tilde{\eta}^{1/p}) \le f_p(\rho^{1/p}, \eta^{1/p})$. ■

A. Interpolation formula for comparable states

In the following, we will assume that η and ρ are densities in $L_1(M)$ such that

$$\delta\eta \le \rho \le \delta^{-1}\eta.$$

Formally, we should probably write d_η for the density such that $\eta(x) = \mathrm{tr}(xd_\eta)$ holds for all x, but we decided to follow Takesaki's convention. Let $\Phi : L_1(M) \to L_1(\hat{M})$ be a completely positive and (sub-)trace preserving map, i.e., the dual map $\Phi^\dagger : \hat{M} \to M$ defined by

$$Tr(\Phi^\dagger(x^*)\eta) = Tr(x\Phi(\eta))$$

is completely positive and (sub-)unital. Let us recall the Stinespring factorization,

$$\Phi^\dagger(x) = e\pi(x)e,$$

for some normal *-homomorphism $\pi : \hat{M} \to \mathbb{B}(H)\bar{\otimes}M$ and some projection $e \in M'$. We will use the notation $\tilde{M} = e(\mathbb{B}(H)\bar{\otimes}M)e$ and f for the support of η and $\hat{f}$ for the support of $\hat{\eta} = \Phi(\eta)$ or $\hat{\rho} = \eta(\rho)$. Indeed, by positivity,

$$\delta\Phi(\eta) \le \Phi(\rho) \le \delta^{-1}\eta(\eta)$$

shows that the support projections (both in $\hat{M}$) coincide.

Lemma V.2. Let $2 \le q_0, q_1$ and $\frac{1}{q(\theta)} = \frac{1-\theta}{q_0} + \frac{\theta}{q}$. Let β_θ as given in (12) represent θ on the boundary of the strip $\{0 \le \Re(z) \le 1\}$. Then,

$$G(z) = \pi(\hat{\rho}^{z/2}\hat{\eta}^{-z/2}\hat{f})ef\eta^{z/2}\rho^{-z/2}$$

is analytic in $\tilde{M}$.

(i) *For all θ in the complex strip,*

$$\ln \|G(\theta)\|_{L^1_{q(\theta)}(\tilde{M},\rho)} \le (1-\theta)\int \ln \|G(it)\|_{L^1_{q_0}(\tilde{M},\rho)}\beta_{1-\theta}(t)dt + \theta \int \ln \|G(1+it)\|_{L^1_{q_0}(\tilde{M},\rho)}\beta_\theta(t)dt.$$

(ii) $\int -\ln \|G(1+it)\|_{q_1}\beta_\theta(t)dt \le \frac{-\ln \|G(\theta)\|_{q(\theta)}}{\theta}$.

(iii) $\int -\ln \|G(1+it)\|_{q_1}\beta_0(t)dt \le \liminf_{\theta\to 0} \frac{-\ln \|G(\theta)\|_{q(\theta)}}{\theta}$.

Proof. Let us recall that μ_θ is the unique measure such that

$$f(\theta) = (1-\theta)\int f(it)d\mu_{1-\theta}(t) + \theta\int f(1+it)d\mu_\theta(t). \tag{38}$$

Therefore, (i) is a reformulation of Lemmas II.4 and II.8 so that

$$d\mu_\theta(1+it) = \frac{1}{\theta}\beta_\theta(t)dt, d\mu_{1-\theta}(t) = \frac{1}{1-\theta}\beta_{1-\theta}(t)dt.$$

The analyticity of G follows from Remark II.1 and $\Re(z) \le 1$. For $z = it$, the element $\hat{\rho}^{it}\hat{\eta}^{-it}$ is in $\hat{M}$ and a partial isometry, and the same applies to $\eta^{it}\hat{\rho}^{-it}$, and hence,

$$\|G(it)\|_{L^1_{q_0}(\tilde{M}),\rho)} \le Tr(\rho) \le 1.$$

Thus, $\ln \|G(it)\|_{q_0} \leq 0$. Dividing by $-\theta$ yields (ii). The function $h(t) = -\ln \|G(1+it)\|_{q_1}$ is continuous, $\lim_{\theta\to 0} \frac{\sin(\pi\theta)}{\theta}$ converges to $1/\pi$, and the measures β_θ are uniformly bounded by $Ce^{-|t|}$. Thus, the dominated convergence theorem implies the assertion (see Ref. 16 for the calculation of β_0). ∎

Let us fix $0 < q_1 < q_0$ and

$$\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

We note that

$$\|G(it)\|_{L^1_{q_0}(\rho)} = \|\pi(g^{it}_{\hat\rho,\hat\eta})eg^{it}_{\eta,\rho}\rho^{1/q_0}\|_{q_0} \leq 1,$$

and recall Lemma II.8. Hence,

$$\int -\ln \|G(1+it)\|_{q_1}\beta_\theta(t)dt \leq \frac{\ln \|G(\theta)\|_{q(\theta)}}{\theta}.$$

Our abstract recovery formula is summarized in the following equation:

$$-\int \ln \|G(1+it)\|_{q_1}\beta_0(t)dt \leq \liminf_{\theta\to 0} \frac{-\ln \|G(\theta)\|_{q(\theta)}}{\theta}.$$

Before we launch into more fidelity estimates, we need a few L_p norm inequalities. These will allow us to more formally state and prove the result.

Remark V.3.

(a) For semifinite von Neumann algebras, the L_p continuity of

$$R^0_z(\hat x) = \eta^{z/2}\Phi^\dagger(\hat\eta^{-z/2}\hat x\hat\eta^{-z/2})\eta^{z/2}$$

is an immediate application of Stein's analytic family interpolation theorem. However, for non-semifinite von Neumann algebras, this map is not necessarily well-defined.

(b) We have

$$R_z(\Phi(\eta)^{Re(z)}) = \eta^{Re(z)}$$

for all z in the strip $\{z|0 \leq Re(z) \leq 1\}$.

(c) For $z = \theta + it$, we see that

$$R_z = \sigma^\eta_{t/2}R_\theta\sigma^{\hat\eta}_{-t/2}$$

is indeed a rotated, generalized Petz recovery map.

Lemma V.4. Let $z = \theta + it$. Then, the twirled Petz map (with respect to η) satisfies

$$\|G(z)\|_{L^1_{1/\theta}(\tilde M,\rho)} = f_{1/\theta}(\rho^\theta, R_z(\Phi(\rho)^\theta)).$$

Proof. Let $p = 1/\theta$. Using the calculation in the Haagerup L_p spaces, we deduce from the definition of R_z that

$$\begin{aligned}\|G(z)\|^2_{L^1_p(\tilde M,\rho)} &= \|\pi(\hat\rho^{z/2}\hat\eta^{-z/2})e\eta^{z/2}\rho^{-z/2}\rho^{1/p}\|^2_{L_p(\tilde M)} = \|\rho^{1/p}G(z)^*G(z)\rho^{1/p}\|_{p/2}\\ &= \|\rho^{1/2p-\theta}\rho^{-it/2}\eta^{-it/2}\eta^{+1/2p}\Phi^\dagger(\hat\eta^{it/2}\eta^{-\theta/2}\hat\rho^\theta\eta^{-\theta/2}\hat\eta^{-it/2})\eta^{+1/2p}\eta^{-it/2}\rho^{+it/2}\rho^{1/2p-\theta}\|_{p/2}\\ &= \|\rho^{1/2p}\eta^{1/2p}\sigma^\eta_{t/2}\Phi^\dagger(\sigma^{\hat\eta}_{-t/2}(\hat\eta^{-\theta/2}\rho^\theta\hat\eta^{-\theta/2}))\eta^{1/2p}\rho^{1/2p}\|_{p/2}\\ &= f_p(R_z(\hat\rho^{1/p}),\eta^{1/p})^2.\end{aligned}$$

∎

Corollary V.5. Let $z = \theta + it$. Then,

$$f_{1/\theta}(\eta^\theta, R_z(\Phi(\rho)^\theta)) \leq 1.$$

∎

Proof. By Hölder's inequality,

$$\|\rho^{1/2p}\eta^{1/2p}\sigma^{\eta}_{t/2}\Phi^{\dagger}\Big(\sigma^{\hat{\eta}}_{-t/2}(\hat{\eta}^{-\theta/2}\rho^{\theta}\hat{\eta}^{-\theta/2})\,\eta^{1/2p}\rho^{1/2p}\|_{p/2} \le \|\rho^{1/2p}\|^2_{2p}\|\sigma_{t/2}R_p(\sigma^{\hat{\eta}}_{-t/2}(\hat{\rho}^{1/p}))\|_p$$
$$\le \|R_p(\sigma^{\hat{\eta}}_{-t/2}(\hat{\rho}^{1/p})\Big)\|_p$$
$$\le \|\sigma^{\hat{\eta}}_{-t/2}(\hat{\rho}^{1/p})\|_p$$
$$\le \|\rho^{1/p}\|_p.$$

We use the fact that $tr(\rho) = 1$ and the modular group extends to an isometry on L_p and Proposition IV.2. ∎

The analyticity of G allows us to reformulate the interpolation formula for G as an interpolation of complex families of fidelities.

Remark V.6. Theorem I.6 then follows from Lemmas V.4 and IV.3. We use Eq. (38) as a reformulation of Lemma II.4 based on Lemma V.2, after applying the re-iteration theorem (see Ref. 39 for more information and Fig 1 for illustration), which allows us to replace the boundaries of the complex strip $i\mathbb{R}$ *and* $1 + i\mathbb{R}$ *by* $p_0 + i\mathbb{R}$ *and* $p_1 + i\mathbb{R}$.

Remark V.7. Within a finite-dimensional von Neumann algebra M, we may relate the Kosaki p-norm of $G(z)$ *to a p-norm expression in terms of modular operators. For any p,*

$$\Delta^{z/2}_{\eta,\rho}(\rho^{1/p}) = \rho^{1/p-z/2}\eta^{z/2} = \rho^{-z/2}\rho^{1/p}\eta^{z/2},$$

and for any ω *and p,*

$$\|(\hat{\rho}^{z/2}\hat{\eta}^{-z/2}\otimes \mathbf{1}^E)\omega\|_p = \|(\hat{\eta}^{-z/2}\otimes \mathbf{1}^E)\omega(\hat{\rho}^{z/2}\otimes \mathbf{1}^E)\|_p = \|(\Delta^{z/2}_{\hat{\rho},\hat{\eta}}\otimes \mathbf{1}^E)\omega\|_p.$$

Hence,

$$\|G(z)\|_{L^1_p(M,\rho)} = \|(\Delta^{z/2}_{\hat{\rho},\hat{\eta}}\otimes \mathbf{1}^E)U\Delta^{-z/2}_{\rho,\eta}\|_{L^1_p(M,\rho)},$$

where U is the finite-dimensional Stinespring isometry with environment E. This is not clear in type III, where we lack the tracial property. $G(z)$ *is a more useful form in type III due to the results we leverage from operator algebras. In particular, we have*

$$G(z) = \pi(g^{z/2}_{\hat{\rho},\hat{\eta}})eg^{z/2}_{\eta,\rho},$$

and we use in proving Lemma V.2 that $g^{it}_{\eta,\rho}$ *and* $g^{it}_{\hat{\rho},\hat{\eta}}$ *are, respectively, in M and* $\hat{M}$. *As noted in Remark II.1,* $g_{\eta,\rho}$ *has good analytic and algebraic properties that work well with the interpolation methods we require. The correspondence between* $G(z)$ *and its finite- equivalent in terms of modular operators may nonetheless merit future investigation.*

B. Differentiation

For the twirled recovery map, we have to use a suitable differentiation result, first under the additional assumption of regularity $\delta\eta \le \rho \le \delta^{-1}\eta$. More generally, we differentiate Kosaki norms for smooth functions with values in the underlying von Neumann algebra.

Lemma V.8. Let (M,τ) *be a finite von Neumann algebra with trace* τ. *Let* $h: I \to M$ *be a differentiable function such that* $h(0) = \mathbf{1}$. *Let* η *be a faithful state. Let p be a differentiable function and* $p(0) > 1$. *Then, we have the following.*

(i) $\frac{d}{d\theta}\|\eta^{1/2p(\theta)}h(\theta)\eta^{1/2p(\theta)}\|_{p(\theta)}\Big|_{\theta=0} = \lim_{\theta\to 0}\theta^{-1}(\|\eta^{1/2p(\theta)}h(\theta)\eta^{1/2p(\theta)}\|_{p(\theta)} - 1) = -\frac{\eta(h'(0))}{p(0)}.$

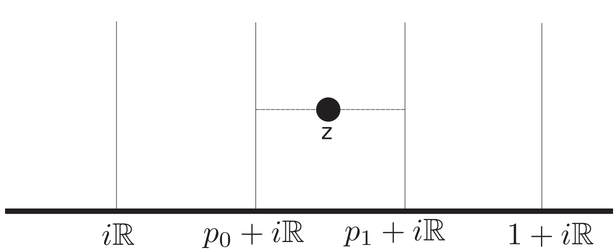

FIG. 1. Using complex interpolation and the re-iteration theorem, we estimate the value of an analytic function at point $z \in \{0 \le \mathrm{Re}(z) \le 1\}$ by the nearest points along the lines $p_0 + i\mathbb{R}$ and $p_1 + i\mathbb{R}$.

(ii) $\lim_{\theta\to 0} \frac{-\ln \|\eta^{1/2p(\theta)} h(\theta) \eta^{1/2p(\theta)}\|_{p(\theta)}}{\theta} = \eta(h'(0))$.

Proof. We consider $g(\theta) = \|\eta^{1/2p(\theta)} h(\theta)\eta^{1/2p(\theta)}\|_{p(\theta)}^{p(\theta)}$ and assume first that $p(\theta) > 1$. We may assume by continuity that $h(\theta) > 0$ in a neighborhood of $\theta = 0$. Let $H(t) = \eta^{1/2p(\theta)} h(t\theta)\eta^{1/2p(\theta)}$. Using the differentiation formula for p-norms and convexity, we get for fixed $p = p(\theta)$ that

$$
\begin{aligned}
g(\theta) - 1 &= \|H(1)\|_p^p - \|H(0)\|_p^p = p\int_0^1 \tau(H(t)^{p-1}H'(t))dt \\
&= p\theta\int_0^1 \tau(H(t)^{p-1}\eta^{1/2p}h'(t\theta)\eta^{1/2p})dt \\
&= p\theta\int_0^1 \tau((H(t)^{p-1} - H(0)^{p-1})\eta^{1/2p}h'(t\theta)\eta^{1/2p})dt + p\theta\int_0^1 \tau\Big(\eta^{\frac{p-1}{p}}\eta^{1/2p}h'(t\theta)\eta^{1/2p}\Big)dt.
\end{aligned}
$$

For the second term, we observe that

$$\tau\Big(\eta^{\frac{p-1}{p}}\eta^{1/2p}h'(t\theta)\eta^{1/2p}\Big) = \tau(\eta h'(t\theta))$$

and, hence,

$$p\theta\int_0^1 \tau\Big(\eta^{\frac{p-1}{p}}\eta^{1/2p}h'(t\theta)\eta^{1/2p}\Big)dt = p\tau(\eta(h(\theta) - h(0))).$$

As for the error (first) term, we observe that

$$|\tau((H(t)^{p-1} - H(0)^{p-1})\eta^{1/2p}h'(t\theta)\eta^{1/2p})| \le \|(H(t)^{p-1} - H(0)^{p-1})\|_{p'}\|\eta^{1/2p}h'(t\theta)\eta^{1/2p}\|_p$$

by Hölder's inequality. Now, we may use the continuity of the Mazur map (see Ref. 57, Corollary 2.3) for $\alpha = p - 1$, $p' = \frac{p}{p-1}$ and deduce that

$$
\begin{aligned}
\|(H(t)^{p-1} - H(0)^{p-1})\|_{p'} &\le 3(p-1)\|H(t) - H(0)\|_p \max\{\|H(t)\|_p, \|H(0)\|_p\}^{p-2} \\
&\le 3(p-1)\|h(t\theta) - h(0)\|_\infty \max\{\|H(t)\|_p, \|H(0)\|_p\}^{p-2} \\
&\le 3(p-1)\|h'\|_\infty t\theta \max\{\|H(t)\|_p, \|H(0)\|_p\}^{p-2}.
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
&p\int_0^1 \tau((H(t)^{p-1} - H(0)^{p-1})\eta^{1/2p}h'(t\theta)\eta^{1/2p})dt \\
&\quad\le \|h'\|_\infty 3p(p-1)\int_0^1 \max\{\|H(t)\|_p, \|H(0)\|_p\}^{p-2}\|\eta^{1/2p}h'(t\theta)\eta^{1/2p}\|_p t\theta dt \\
&\quad\le \|h'\|_\infty\|\eta^{1/2p}h'\eta^{1/2p}\|_\infty 3p(p-1)\theta\int_0^1 \max\{\|H(t)\|_p, \|H(0)\|_p\}^{p-2}tdt.
\end{aligned}
$$

The faithfulness of η and fact that $h(0) = 1$ imply that $\|H(0)\|_p > 0$ for all p, so the integral on the right-hand side remains finite. As $\theta \to 0$, this term becomes 0. Thus, for $p(0) > 1$, we can find θ_0 such that $p(\theta) - 1 > \delta$ for $\theta \le \theta_0$ and, hence,

$$\lim_{\theta\to 0}\frac{g(\theta) - 1}{\theta} = p(0)\tau(\eta h'(0)).$$

Let us now define the function $F(\theta, p) = g(\theta)^{1/p}$ in two parameters. We find that $\frac{d}{d\theta}F = -\frac{1}{p^2}g(\theta)^{1/p-1}g'(\theta)$ and $\frac{dF}{dp} = -\frac{1}{p^2}g(\theta)^{1/p}\ln g(\theta)$. As η is faithful, $g(\theta)$ is non-zero when $h(\theta)$ is always positive and not equal to zero. Hence, dF/dp is continuous and differentiable. To show that $dF(p, \theta(p))/d\theta$ is continuous and differentiable, we must also check the $dF/d\theta$ part, which involves $g'(\theta)$. We again apply separation of variables. First,

$$\frac{d}{d\theta}\|\eta^{1/2p}h(\theta)\eta^{1/2p}\|_p^p = \|\eta^{1/2p}h(\theta)\eta^{1/2p}\|_p^p\Big(\frac{d}{d\theta}\ln\|\eta^{1/2p}h(\theta)\eta^{1/2p}\|_p\Big).$$

The prefactor is continuous by the continuity of $g(\theta)$ for $p > 1$. We now use a fact of Banach spaces that for any continuous, differentiable function $H(\theta)$ and p fixed,

$$\frac{d}{d\theta}\|H(\theta)\|_p = \Big\langle\Big(\frac{H(\theta)}{\|H(\theta)\|_p}\Big)^{p/p'}, \frac{d}{d\theta}H(\theta)\Big\rangle.$$

Letting $H(\theta) = \|\eta^{1/2p(\theta)} h(\theta) \eta^{1/2p(\theta)}\|_{p(\theta)}$, the left-hand side of the bracket is again the Mazur map and, therefore, continuous. For the right-hand side,

$$\frac{d}{d\theta}(\eta^{1/2p(\theta)} h(\theta) \eta^{1/2p(\theta)}) = \eta^{1/2p} h'(\theta) \eta^{1/2p}.$$

We again see the continuity of this expression. Finally, positivity of θ and the chain rule for the natural logarithm give us the continuity of the entire expression. We still however must contend with the p derivative. Here, we apply the separation of variables yet another time, writing

$$\frac{d}{dp}\|\eta^{1/2p} h(\theta) \eta^{1/2p}\|_p^p = \frac{d}{dp}\|\eta^{1/2q} h \eta^{1/2q}\|_p^p + \frac{d}{dq}\|\eta^{1/2q} h \eta^{1/2q}\|_p^p\Big|_{p=q}.$$

First, we deal with the p-derivative, noting that the quantity inside of the norm is assumed p-independent. We obtain

$$\frac{d}{dp}\|\eta^{1/2q} h(\theta) \eta^{1/2q}\|_p^p = \frac{d}{dp}\operatorname{tr}((\eta^{1/2q} h \eta^{1/2q})^p) = \operatorname{tr}((\eta^{1/2q} h \eta^{1/2q})^p \ln(\eta^{1/2q} h \eta^{1/2q})).$$

This is finite whenever $\eta^{1/2q} h \eta^{1/2q} > 0$, so this derivative is continuous. For the q derivative,

$$\frac{d}{dq}\|\eta^{1/2q} h \eta^{1/2q}\|_p^p = \frac{d}{dq}\operatorname{tr}((\eta^{1/2q} h \eta^{1/2q})^p) = p(\eta^{1/2q} h \eta^{1/2q})^{p-1}\frac{d}{dq}(\eta^{1/2q} h \eta^{1/2q}).$$

Since we only care about the continuity and will not rely here on explicitly evaluating this derivative, we merely note that the product rule allows us to differentiate the remaining factor and that $\eta^{1/2q-1}$ is finite by the positivity of η. This term is therefore continuous.

Hence, F is differentiable, and

$$\frac{d}{d\theta}F(\theta, p(\theta)) = -\frac{1}{p(\theta)^2} g(\theta)^{1/p(\theta)-1} g'(\theta) - \frac{1}{p(\theta)} g(\theta)^{1/p(\theta)} \ln g(\theta) \frac{dp(\theta)}{d\theta}.$$

For $\theta = 0$, we deduce from $g(0) = 1$ that

$$\frac{d}{d\theta}F(\theta, p(\theta))|_{\theta=0} = -\frac{1}{p(0)}\eta(h'(0)).$$

This concludes the proof of (i) in this case. For (ii), we note that

$$\frac{\ln\|\eta^{1/2p(\theta)} h(\theta) \eta^{1/2p(\theta)}\|_{p(\theta)}}{\theta} = \frac{1}{p(\theta)}\frac{\ln g(\theta)}{\theta}.$$

Using $\frac{d}{d\theta}\ln g(\theta)|_{\theta=0} = \frac{g'(0)}{g(0)}$, we deduce, indeed, (ii). ■

Theorem V.9. *Let $\delta\eta \le \rho \le \delta^{-1}\rho$ and $1 \le p < \infty$. Then,*

$$\int_{\mathbb{R}}\Big(-\ln f_p\Big(\rho^{1/p}, R_{\frac{1+it}{p}}(\Phi(\rho)^{1/p})\Big)\Big)\beta_0(t)dt \le \frac{D(\rho\|\eta) - D(\Phi(\rho)\|\Phi(\eta))}{2p}.$$

Proof. Let $q \ge 1$ and $q_0 > 2$. We define $\frac{1}{q(\theta)} = \frac{1-\theta}{q_0} + \frac{\theta}{q}$. Then, we may apply Lemma V.2 for

$$G_q(z) = G(z/q) = \pi(\hat{\rho}^{z/2q}\hat{\eta}^{-z/2q}\hat{f}) e f \eta^{z/2q}\rho^{-z/2q},$$

which remains analytic as long as $q \ge 1$. Using $\|G_q(it)\|_{q_0} \le 1$, we deduce as in Lemma V.2 that

$$\lim_{\theta\to 0}\frac{\ln\|G_q(\theta)\|_{L_1^{q(\theta)}}}{\theta} \le \int \ln\|G_q(1+it)\|_{L_q^1}\beta_0(t)dt.$$

Let us recall that, according to Lemma V.4, we have

$$\|G_q(1+it)\|_{L_q^1} = f_q\Big(\rho^{1/q}, R_{\frac{1+it}{q}}(\Phi(\rho)^{1/q})\Big).$$

However, we have used the dominated convergence theorem to interchange integral and limit, which is possible thanks to the continuity of interpolated fidelity, proved in Sec. VIII. We are left to calculate the limit. We may introduce $p(\theta) = \frac{q(\theta)}{2}$ so that $p(0) > 1$. Then, we see that

$$\begin{aligned}\|G_q(\theta)\|^2_{L_1^{q(\theta)}} &= \|\rho^{1/q(\theta)}\rho^{-1/2q(\theta)}\eta^{1/2q(\theta)}\Phi^\dagger(\hat{\eta}^{-1/2q(\theta)}\hat{\rho}^{1/q(\theta)}\hat{\eta}^{-1/2q(\theta)})\eta^{1/2q(\theta)}\rho^{-1/2q(\theta)}\rho^{1/q(\theta)}\|_{p(\theta)} \\ &= \|\rho^{1/2p(\theta)}h_q(\theta)\rho^{1/2p(\theta)}\|_{p(\theta)}\end{aligned}$$

holds for

$$h_q(\theta) = \rho^{-1/2q(\theta)}\eta^{1/2q(\theta)}\Phi^\dagger(\hat{\eta}^{-1/2q(\theta)}\hat{\rho}^{1/q(\theta)}\hat{\eta}^{-1/2q(\theta)})\eta^{1/2q(\theta)}\rho^{-1/2q(\theta)} = h\left(\frac{\theta}{q}\right).$$

For $q = 1$, our derivative of

$$h(\theta) = \rho^{-\theta/2}\eta^{\theta/2}\Phi^\dagger(\hat{\eta}^{-\theta/2}\hat{\rho}^{\theta/2}\hat{\eta}^{-\theta/2})\eta^{\theta/2}\rho^{-\theta/2}$$

satisfies

$$h'(0) = -\ln\rho + \ln\eta + \Phi^\dagger(\ln\hat{\rho}) - \Phi^\dagger(\ln\hat{\eta}).$$

This implies

$$tr(\rho h'(0)) = -tr(\rho\ln\rho) + tr(\rho\ln(\eta)) + tr(\Phi(\rho)\ln\Phi(\rho) - \ln\Phi(\eta)) = -D(\rho|\eta) + D(\Phi(\rho)|\Phi(\eta)).$$

Using the chain rule, we get

$$-q\,tr(\rho h_q'(0)) = D(\rho\|\eta) - D(\Phi(\rho)\|\Phi(\eta)).$$

■

Remark V.10. In a type III situation, it is better to write

$$h(\theta) = \Delta_{\rho,\eta}^{\theta/2}\Phi^\dagger((\Delta_{\hat{\rho},\hat{\eta}}^{\theta/2})^*\Delta_{\hat{\rho},\hat{\eta}}^{\theta/2})\Delta_{\rho,\eta}^{\theta/2}$$

and, hence,

$$h'(0) = -\ln\Delta_{\rho,\eta} + \Phi^\dagger(\ln\Delta_{\hat{\rho},\hat{\eta}}).$$

This implies again

$$\begin{aligned}tr(\rho h'(0)) &= -(\rho^{1/2},\ln\Delta_{\rho,\eta}\rho^{1/2}) + tr(\Phi(\rho)^{1/2},\Delta_{\Phi(\rho),\Phi(\eta)}\Phi(\rho)^{1/2}) \\ &= -D(\rho\|\eta) + D(\Phi(\rho)\|\Phi(\eta)).\end{aligned}$$

VI. PROOFS OF RECOVERY RESULTS IN FINITE ALGEBRAS

At this point, Theorem V.9 may appear to have nearly finished the proof of a universal recovery theorem. The remaining technical step is to remove the condition that $\delta\eta \leq \rho \leq \delta^{-1}\eta$, which absolves our analytic machinery from needing to handle infinite relative entropy. Within the finite-dimensional setting, this follows from a straightforward continuity argument. Infinite dimensions introduce additional subtleties with the continuity arguments, and it is not so simple to show that we can drop the restriction that $\delta\varphi \leq \rho \leq \delta^{-1}\eta$. Section VII resolves these issues, extending recovery to type II_1. Since the finite-dimensional case is subsumed by these continuity results, we will not include another explicit proof of continuity for the finite case. Instead, we state the following result.

Corollary VI.1. Let $1 \leq p \leq \infty$. Then,

$$\int_{\mathbb{R}}\left(-\ln f_p\left(\rho^{1/p}, R_{\frac{1+it}{p}}(\Phi(\rho)^{1/p})\right)\right)\beta_0(t)dt \leq \frac{1}{2p}(D(\rho\|\eta) - D(\Phi(\rho)\|\Phi(\eta))).$$

Moreover, the (generally non-linear) universal recovery map

$$\tilde{R}_p(x) = \left(\int R_{p,t}(x^{1/p})d\mu(t)\right)^p$$

satisfies

$$-\ln f_p(\rho,\tilde{R}_p(\Phi(\rho))) \leq \frac{1}{2p}(D(\rho\|\eta) - D(\Phi(\rho)\|\Phi(\eta))).$$

The same holds for the general von Neumann algebra version in Sec. IX.

Proof. We refer to Secs. VII and VIII for the discussion that assuming $\rho \le \lambda\eta$ is enough to justify the differentiation lemma. For the "moreover" part, we recall that ln is concave and f_p is jointly concave, and hence,

$$\begin{aligned}\int \ln f_p(\rho, R_{p,t}(\hat{\rho}^{1/p})^p)d\mu(t) \\ \le \ln \int f_p(\rho, R_{p,t}(\hat{\rho}^{1/p})^p)d\mu(t) \\ \le \ln f_p\Big(\int \rho d\mu(t), \int R_{p,t}(\hat{\rho}^{1/p})^p d\mu(t)\Big) \\ = \ln f_p(\rho, \tilde{R}_p(\hat{\rho})).\end{aligned}$$

■

A. Measured entropy recovery

Although Corollary VI.1 generalizes 6 to infinite dimensions, it does not immediately subsume the strengthened form of Eq. (8) from Ref. 1. As this entropy inequality follows from trace inequalities, we recall this original form of proof and port it to the general von Neumann algebra setting using Theorem I.2. In the infinite-dimensional setting, we define

$$D_M(\rho\|\eta) = \sup_{\Phi:L_1(M)\to\ell_1} D(\Phi(\rho)\|\Phi(\eta)),$$

replacing the POVM by an arbitrary channel from the Haagerup space $L_1(M)$ to the space ℓ_1 of probability measures. In the finite case, this definition would coincide with that using arbitrary POVMs. We use the variational forms of relative entropy (see Refs. 60 and 61),

$$D(\rho\|\eta) = \sup_{\omega>0} tr(\rho \log \omega) + 1 - tr(\exp(\log \eta + \log \omega)), \tag{39}$$

and of the measured entropy,

$$D_M(\rho\|\eta) = \sup_{\omega>0} tr(\rho \log \omega) + 1 - tr(\eta\omega). \tag{40}$$

Applying the Golden–Thompson inequality to the final term shows that the measured relative entropy is at most equal to the relative entropy, as does data processing. To justify that, this form indeed equals the measured relative entropy as defined.

Lemma VI.2. For states ρ, η on a von Neumann algebra M, where M_{sa} denotes the subspace of self-adjoint operators in M,

$$D_M(\rho\|\eta) = \sup_{\omega\in M_{sa}} \rho(\log(\omega)) + 1 - \eta(\omega).$$

Proof. Let ω be a self-adjoint element and $\pi : L_\infty(\sigma(\omega), \mu) \to M$ be the normal *-homomorphism. Let $\mathbb{E}(\cdot)$ denote the expectation of the trace of an expression over values of ω. Then, $\pi_* : L_1(M) \to L_1(\mu)$ is a quantum-classical channel. We deduce that

$$\begin{aligned}D(\Phi(\rho)\|\Phi(\eta)) &= \sup_f \mathbb{E}(\Phi(\rho)\log f) + 1 - \mathbb{E}(\exp(\pi(\eta) + f)) \\ &= \sup_f \mathbb{E}(\Phi(\rho)\log f) + 1 - \mathbb{E}(\Phi(\eta)f) \\ &= \sup_f tr(\rho(\log f(\omega))\ + 1 - tr(\eta f(\omega)).\end{aligned}$$

For $f(z) = z$, we deduce that D_M is bigger than the right-hand side, by the approximation of L_1 by finite σ-algebras. For the converse, we consider the channel $\Phi : L_1(M) \to \ell_1^m$ and $\Phi^* : \ell_\infty^m \to M$, which is unital and completely positive. Let $\Phi^*(e_j) = f_j$. Then, we find that

$$\begin{aligned}D(\Phi(\rho)\|\Phi(\eta)) &= \sup_{\beta_j}\sum_j (e_j, \Phi(\rho)) \log \beta_j) + 1 - \sum_j (e_j, \Phi(\eta))\beta_j \\ &= tr\Big(\rho\sum_j f_j \log \beta_j\Big) + 1 - tr\Big(\eta\sum_j f_j\beta_j\Big) \\ &\le tr\Big(\rho \log\Big(\sum_j f_j\beta_j\Big)\Big) + 1 - g\, tr\Big(\eta\sum_j f_i\beta_j\Big)\end{aligned}$$

thanks to the operator concavity (with respect to unital, completely positive maps) of the logarithm. ■

Through Lance's Stinespring dilation (see Lemma II.9 and Ref. 42), a quantum channel $\Phi : L_1(M) \to L_1(N)$ has the adjoint form

$$\Phi^\dagger(x) = e\pi(x)e \tag{41}$$

for some normal *-homomorphism $\pi : N \to \mathbb{B}(l_2)\bar{\otimes}M$ and projection e, where $e = e_{1,1} \otimes \mathbf{1}_M$ and $\bar{\otimes}$ is the von Neumann algebra tensor product. It also holds for states ρ, η when M is finite that

$$\begin{aligned} &tr(\Phi(\rho)\ln\,\Phi(\eta)) = tr(\rho\Phi^\dagger(\ln\,\Phi(\eta))) \\ &= Tr(\rho e\pi(\ln(\Phi(\eta)))e) = tr(\rho e\,\ln(\pi(\Phi(\eta)))e). \end{aligned} \tag{42}$$

When M is not finite, $\mathbb{B}(l_2)\bar{\otimes}M$ is not even semifinite, and the above equality may not have meaning. Here, we show an entropy bound in the style of the desired recovery inequality [Eq. (8)], but where we perturb the quantum states to ensure faithfulness and set up for use in a crossed product $M \rtimes G$.

Theorem VI.3. *Given $\rho, \eta \in M_*^+$ as states on semifinite von Neumann algebra M and a channel $\Phi : L_1(M) \to L_1(N)$,*

$$D(\rho\|\eta) - D(\Phi(\rho)\|\Phi(\eta)) \geq \int_{\mathbb{R}} \beta_0(t) D_M(\rho\|R^t_{\eta,\Phi} \circ \Phi(\eta))dt \geq D_M(\rho\|\tilde{R}_{\eta,\Phi} \circ \Phi(\eta)),$$

where

$$R^t_{\eta,\Phi}(\omega) = \eta^{(1+it)/2}\Phi^\dagger\Big(\Phi(\eta)^{(1+it)/2}\Phi^\dagger(\omega)\Phi(\eta)^{(1-it)/2}\Big)\eta^{(1-it)/2}$$

and

$$\tilde{R}_{\eta,\Phi}(\omega) = \int \beta_0(t) R^t_{\eta,\Phi}(\omega)dt.$$

Here is the same rotated Petz map as in Ref. 15, and $\tilde{R}_{\eta,\Phi}$ is the integrated, rotated Petz recovery map as in Refs. 1 and 16.

Proof. Let $\gamma \in (1-e)(\mathbb{B}(l_2)\bar{\otimes}M)(1-e)$ be a faithful state such that

$$tr(\gamma(\ln(\pi(\Phi(\eta))) - \ln(\pi(\Phi(\rho))))) < \infty,$$

$\epsilon > 0$, and

$$\rho_\epsilon = \rho + \epsilon\gamma = \begin{bmatrix} \rho & 0 \\ 0 & \epsilon\gamma \end{bmatrix}, \qquad \eta_\epsilon = \eta + \epsilon\gamma = \begin{bmatrix} \eta & 0 \\ 0 & \epsilon\gamma \end{bmatrix}.$$

Let $\hat{\rho}_\epsilon = \rho_\epsilon/tr(\rho_\epsilon), \hat{\eta}_\epsilon = \eta\epsilon/tr(\eta\epsilon)$. We define

$$c_{tr} := tr(\rho_\epsilon) = tr(\eta_\epsilon) = 1 + \epsilon tr((1-e)\gamma(1-e)).$$

We then have that $c_{tr}D(\hat{\rho}_\epsilon\|\hat{\eta}_\epsilon) = D(\rho_\epsilon\|\eta_\epsilon)$. Through the block diagonal form, $D(\rho_\epsilon\|\eta_\epsilon) = D(\rho\|\eta)$. We consider

$$\begin{aligned} I &:= \frac{1}{c_{tr}} D(\rho_\epsilon\|\exp(\ln\,\rho_\epsilon - \ln\,\eta_\epsilon - \ln\,\pi(\Phi(\rho)) + \ln\,\pi(\Phi(\eta)))) \\ &= \frac{1}{c_{tr}}\,tr(\rho_\epsilon(\ln\,\rho_\epsilon - \ln\,\eta_\epsilon - \ln\,\pi(\Phi(\rho)) + \pi(\Phi(\eta)))). \end{aligned}$$

We then use the variational form

$$c_{tr}I = \sup_{\omega\in\mathbb{B}(l_2)\bar{\otimes}M:\omega>0} tr(\rho_\epsilon\,\ln\,\omega) + 1 - tr\left(\exp(\ln\,\omega + \ln(\pi(\Phi(\eta)))) - \ln(\pi(\Phi(\rho))) - \ln\,\eta_\epsilon\right)). \tag{43}$$

To use Eq. (43), we apply the four-term version of the generalized Golden–Thompson inequality for $p = 2$, which states for real, faithful $\exp(H_0) \in \tilde{M}$ and Hermitian H_1, H_2, H_3 that

$$\begin{aligned} &tr(\exp(H_0/2 + H_1 + H_2 + H_3)) \\ &\leq \int dt\beta_0(t)\ln\,Tr(\exp(H_0/2)\exp((1+it)H_1/2)\exp((1+it)H_2/2) \\ &\quad\times\exp(H_3)\exp((1-it)H_2/2)\exp((1-it)H_1/2)\exp(H_0/2)) \end{aligned} \tag{44}$$

using Eq. (36). We identify

$$H_0/2 \leftarrow \ln\,\omega, \quad H_1 \leftarrow -\ln\,\eta_\epsilon, \quad H_2 \leftarrow \ln\,\pi(\Phi(\eta)), \quad H_3 \leftarrow -\ln\,\pi(\Phi(\rho)). \tag{45}$$

Through the supremum and positivity of ω, we can replace ω by $\sqrt{\omega}$ or ω^2 in Eq. (43) without changing the value. Hence,

$$c_{tr}I \geq \sup_{\omega>0} tr(\rho_\epsilon\,\ln\,\omega) + 1 - \int dt\beta_0(t)\,\ln\,tr\Big(\eta_\epsilon^{(1+it)/2}\pi(\Phi(\eta))^{(1+it)/2}\pi(\Phi(\rho))\pi(\Phi(\eta))^{(1-it)/2}\eta_\epsilon^{(1-it)/2}\omega\Big).$$

As π is a homomorphism,

$$\ldots = \sup_{\omega>0} tr(\rho_\epsilon\,\ln\,\omega) + 1 - \int dt\beta_0(t)\,\ln\,tr\Big(\eta_\epsilon^{(1+it)/2}\pi\Big(\Phi(\eta)^{(1+it)/2}\Phi(\rho)\Phi(\eta)^{(1-it)/2}\Big)\eta_\epsilon^{(1-it)/2}\omega\Big).$$

Through the supremum over ω, this expression only decreases if we assume that $\omega = e\tilde{\omega}e$ for some $\tilde{\omega}$, and observing that $[e,\eta] = 0$, we have

$$\ldots \geq \sup_{\tilde{\omega}>0} tr(\rho\,\ln\tilde{\omega}) + 1 - \int dt\beta_0(t)\ln\,tr\Big(\eta^{(1+it)/2}\Phi^\dagger\Big(\Phi(\eta)^{(1+it)/2}\Phi(\rho)\Phi(\eta)^{(1-it)/2}\Big)\eta^{(1-it)/2}\tilde{\omega}\Big).$$

This step conveniently takes care of both eliminating the $\epsilon\gamma$ corrections and resulting in a recovery map form. We may compare directly to $R^t_{\eta,\Phi}$ and to Eq. (40) to see that

$$\ldots = \int_{\mathbb{R}} \beta_0(t) D_M(\rho\|R^t_{\eta,\Phi}\circ\Phi(\rho))dt$$

as sought on the right-hand side of the recovery inequality. We may also use the concavity of the logarithm to move the integral inside the logarithm, obtaining the sought form in terms of $\tilde{R}_{\eta,\Phi}$.

For the left-hand side, using Eq. (42) and the fact that $\rho = e\rho e$ and that $D(\rho\|\eta) = \mathrm{tr}(\rho(\ln\rho - \ln\eta))$,

$$\begin{aligned} c_{tr}I &= tr(\rho_\epsilon(\ln\,\rho_\epsilon - \ln\,\eta_\epsilon - \ln(\pi(\Phi(\rho))) + \ln(\pi(\Phi(\eta))))) \\ &= D(\rho_\epsilon\|\eta_\epsilon) - D(\Phi(\rho)\|\Phi(\eta)) + \epsilon tr(\gamma(\ln(\pi(\Phi(\eta))) - \ln(\pi(\Phi(\rho))))) \\ &= D(\rho\|\eta) - D(\Phi(\rho)\|\Phi(\eta)) + \epsilon tr(\gamma(\ln(\pi(\Phi(\eta))) - \ln(\pi(\Phi(\rho))))). \end{aligned} \tag{46}$$

Then, we note that as $\epsilon \to 0$, the correction term that is linear in ϵ vanishes. This limit completes the theorem. ■

The obvious barrier in type III is the lack of a trace. Were this the only barrier, the Haagerup L_p spaces and corresponding trace would suffice. The deeper problem is that the differentiability of $h(\theta)$ as used in Lemma V.8 and the continuity of the trace of the operator logarithm are not clear without a finite trace. Hence, we must approximate the crossed product by finite von Neumann algebras in Sec. VIII, our main use of the techniques of Ref. 27.

VII. CONTINUITY FOR FIDELITY OF RECOVERY

In this section, we show some continuity results for the fidelity of recovery, which are not immediate in infinite dimension. We continue to use our standard assumptions on η, ρ, and Φ.

Lemma VII.1. Let A be a (possibly unbounded) positive operator on a Hilbert space H, ξ in the domain of $A^{1/2}$, and $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions such that

$$|f_n(x)| \leq C(1 + |x|^{1/2})$$

and $\lim_n f_n(x) = f(x)$ *for all x. Then,*

$$\lim_n \|(f_n(A) - f(A))(\xi)\|_H = 0,$$

where f_n extends to operators by the elementary functional calculus.

Proof. Let $d\mu_\xi(x)$ be the spectral measure of A, i.e.,

$$(\xi, f(A)\xi) = \int f(x)d\mu_\xi(x)$$

for all measurable f. Then, we observe by the triangle inequality that $|f_n(x) - f(x)|^2 \leq 16C^2(1 + |x|)$ holds for all $n \in \mathbb{N}$, and moreover,

$$\|A^{1/2}\xi\|_H^2 = (A^{1/2}\xi, A^{1/2}\xi) = \int |x|d\mu_\xi(x).$$

Since ξ has a finite norm, we deduce that $x \mapsto (1+|x|)$ is in $L_1(\mu_\xi)$. By the dominated convergence theorem, we deduce that

$$\lim_n \|(f_n(A) - f(A))\xi\|_H^2 = \lim_n \int |f_n(x) - f(x)|^2 d\mu_\xi(x) = 0.$$

■

Proposition VII.2. Let $\delta\eta \le \rho \le \delta^{-1}\rho$. Then, the function

$$F(z) = f_{Re(z)}(\rho^{Re(z)}, R_z(\Phi(\rho)^{Re(z)}))$$

is continuous in z on $\{z|0 < Re(z) \le 1\}$.

Proof. Here, we recall Lemma VII.1 as a general fact.

Let ρ and η be states and $\psi(x) = \frac{\eta(x_{11})+\rho(x_{22})}{2}$ be the corresponding positive functional on $M_2(M)$ considered by Connes.[62] Then,

$$\eta^{1/2} = \Delta_{\eta,\rho}(\rho^{1/2}) = \Delta_\psi\left(\begin{pmatrix} 0 & \rho^{1/2} \\ 0 & 0 \end{pmatrix}\right)$$

belongs to the domain of $\Delta_\psi^{1/2}$. In addition, hence,

$$\lim_{z\to w} \|\Delta_\psi^z - \Delta_\psi(|1\rangle\langle 2| \otimes \rho^{1/2})\| = 0$$

as long as $\Re(z), \Re(w) \le 1/2$. Note that thanks to the calculation in the core $M \rtimes \mathbb{R}$, we know that

$$\eta^{z/2}\rho^{-z/2}\rho^{1/2} = \eta^{z/2}\rho^{1/2}\rho^{-z/2} = \Delta_{\eta,\rho}^{z/2}(\rho^{1/2}) \cong \Delta_\psi(|1\rangle\langle 2| \otimes \rho^{1/2}).$$

This means that we have the L_2 convergence in z for $0 \le \Re(z) \le 1$. Using Kosaki's interpolation result, we deduce that

$$\|(\eta^{z/2}\rho^{-z/2} - \eta^{w/2}\rho^{-w/2})\rho^{1/2p}\|_p \le |(\eta^{z/2}\rho^{-z/2} - \eta^{w/2}\rho^{-w/2})\|_\infty^{1-1/p} \|(\eta^{z/2}\rho^{-z/2} - \eta^{w/2}\rho^{-w/2})\rho^{1/2}\|_2^{1/p}.$$

Therefore, we deduce that $0 \le \Re(z), \Re(w) \le 1$ and we have that

$$\lim_{z\to w} \|(\eta^{z/2}\rho^{-z/2} - \eta^{w/2}\rho^{-w/2})\rho^{1/2p}\|_p = 0$$

holds uniformly on compact sets.

Now, it is time we address the fidelity. We will use the functional calculus and observe that

$$\eta^{z/2}\rho^{-z/2} - \eta^{w/2}\rho^{-w/2} = \eta^{z/2}(1 - \eta^{w-z/2}\rho^{(z-w)/z})\rho^{-z/2}.$$

Let us define the * homomorphism $\pi : C(\mathbb{R}^2) \to B(L_2(M))$ given by $\pi(F_1 \otimes F_2) = L_{F_1(\rho)} R_{F_2(\eta)}$. Using $|e^a - 1| \le ae^{|a|}$, we observe that

$$|(x/y)^w - (x/y)^z| = |(e^{\ln x - \ln y}(w-z) - 1)(x/y)^z| \le |w-z||\ln(x/y)||(x/y)^z|.$$

Let $\delta \le D \le \delta^{-1}$ be a bounded operator. Using $|e^x - 1| \le xe^{|x|}$ and the functional calculus, we deduce that

$$\|D^w - D^z\| = \|(e^{(\ln D)w-z} - 1)D^z\| \le |w-z||\ln \delta|e^{|\ln \delta\|w-z|}\delta^{-|Re(z)|} = |w-z|(\delta^{-1})^{|w-z|+|z|}.$$

This allows us to estimate

$$\begin{aligned}\|G(z) - G(w)\| &= \|\pi(\Delta_{\hat\rho,\hat\eta}^{z/2})\Delta_{\eta,\rho}^{z/2} - \pi(\Delta_{\hat\rho,\hat\eta}^{w/2})\Delta_{\eta,\rho}^{w/2}\| \\ &\le 2(\delta^{-1})^{|w-z|+|z|}|w-z|.\end{aligned}$$

Let us now consider the case $p \le p_1$ where $\frac{1}{p} = Re(w)$, $Re(z) = \frac{1}{p_1}$. Then, we find that

$$\begin{aligned}\|G(w)\|_{L_p(\mathcal{L}(\mathcal{H}_\mathcal{M}),\rho)} &\le \|G(w) - G(z)\|_{L_p(\mathcal{L}(\mathcal{H}_\mathcal{M}),\rho)} + \|G(z)\|_{L_p(\mathcal{L}(\mathcal{H}_\mathcal{M}),\rho)} \\ &\le C(\delta, w, z)|w-z| + \|G(z)\|_{L_{p_1}(\mathcal{L}(\mathcal{H}_\mathcal{M}),\rho)}.\end{aligned}$$

Since $C(\delta, w, z)$ is bounded in bounded regions of $\mathbb{C}$, we deduce the continuity for $\mathrm{Re}(w) \ge \mathrm{Re}(z)$. More precisely, we have continuity for fixed $\mathrm{Re}(z)$, and moreover,

$$F(w) \le \liminf_{z\to w, Re(w)\ge Re(z)} F(z) \le \limsup_{z\to w, Re(w)\ge Re(z)} F(z), \tag{47}$$

$$\limsup_{z\to w, Re(z)\ge Re(w)} F(z) \le F(w). \tag{48}$$

To prove the missing inequality in (47), we may assume $Im(z) = Im(w) = 0$. Let us now assume that $Re(w) = \frac{1}{p} > Re(z) = \frac{1}{p_1}$, i.e., $p_1 > p$ for fixed p. Let $p_2 \ge 1$. Then, we can find ϵ such that $\frac{1}{p_1} = \frac{1-\epsilon}{p} + \frac{\epsilon}{p_2}$. We use that standard interpolation estimate and deduce from $\|G(1/p_2)\| \le 1$ that

$$\|G(1/p_1)\|_{p_1} \le \left(\int_{\mathbb{R}} f_{p(\epsilon)}\Big(\rho^{1/p}, R_{\frac{1+it}{p}}(\Phi(\rho)^{1/p_1(\epsilon)})\Big)\beta_\epsilon(t)dt\right)^{1-\epsilon}.$$

Here, $\frac{1}{q} = \frac{1}{p} - \frac{1}{p_2}$. We may now send $\epsilon \to 0$. Thanks to the continuity in the imaginary part and the explicit form of the measure (see Ref. 39, p. 93),

$$d\mu_\epsilon(t) = h_\epsilon(t)dt, h_\epsilon(t) = \frac{e^{-\pi t}\sin \pi\epsilon}{(1-\epsilon)(\sin^2\pi\epsilon + (\cos\pi\epsilon - e^{-\pi t})^2)},$$

we deduce that

$$\begin{aligned}\limsup_{\epsilon\to 0}\|G(1/p_1(\epsilon))\|_{p_1(\epsilon)} &\le \limsup_{\epsilon\to 0}\left(\int_{\mathbb{R}} f_p\Big(\rho^{1/p}, R_{\frac{1+it}{p}}(\Phi(\rho)^{1/p})\Big)\beta_\epsilon(t)dt\right)^{1-\epsilon}\\ &= f_p\Big(\rho^{1/p}, R_{\frac{1+it}{p}}(\Phi(\rho)^{1/p})\Big).\end{aligned}$$

This shows that

$$\limsup_{z\to w, Re(z)>Re(w)} F(z) \le F(w).$$

Similarly, we prove the missing inequality

$$F(w) \le \liminf_{z\to w, Re(z)>Re(w)} F(z).$$

in (48) using the uniform continuity in the imaginary axes. All four inequalities together then yield continuity. ∎

Lemma VII.3. Let $2 \le p < \infty$. The function

$$h(z) = g_{\eta,\rho}(z)$$

is continuous in $L^1_p(M,\rho)$.

Proof. We will first prove the assertion for $p = 2$. Following Connes, we consider $M_2(N)$ and the state $\psi(x) = \frac{1}{2}(\eta(x_{11}) + \rho(x_{22}))$. Let $e_{i,j} = |i\rangle\langle j|$ be the matrix units in M_2. Then, we see that

$$\Delta_\psi(e_{12}\otimes\xi) = e_{12}\otimes\eta\xi\rho^{-1} = e_{12}\otimes\Delta_{\eta,\rho}(\xi).$$

Moreover, $\Delta^{1/2}(\rho^{1/2}) = \eta^{1/2}$ shows that $e_{12}\otimes\rho^{1/2}$ belongs to the domain. Note, however, that thanks to the calculation in the core $M \rtimes \mathbb{R}$, we have

$$g_{\eta,\rho}(z)\rho^{1/2} = \eta^{z/2}\rho^{-z/2}\rho^{1/2} = \Delta^{z/2}_{\eta,\rho}(\rho^{1/2}).$$

Let $\lim_n z_n = z$ such that $0 \le \Re(z_n) \le 1$. Then, $f_n(x) = x^{z_n/2}$ and $f(x) = x^{z/2}$ satisfy the assumption of Lemma VII.1, and hence, we have convergence. For $2 < p < \infty$, we deduce from Kosaki's interpolation theorem that we also have

$$\|a\|_{L^1_p} \le \|a\|^{1-\theta}_{L^1_2}\|a\|^\theta_\infty,$$

provided that a is bounded and $\frac{1}{p} = \frac{1-\theta}{2}$. We apply this to $a = g_{\eta,\rho}(z_n) - g_{\eta,\rho}(z)$, which is uniformly bounded; see Remark II.1. Therefore, convergence in L_2 implies convergence for all $2 \le p < \infty$. ∎

Lemma VII.4. Let $a \in M$. Then,

$$h(p) = \|a\|_{L^1_p(\rho)}$$

is continuous.

Proof. Let $p \le q \le p_0$ and $\theta(q)$ such that

$$\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{p_0}.$$

Then, we deduce from Kosaki's interpolation theorem that

$$\|a\|_p \le \|a\|_q \le \|a\|_p^{1-\theta(q)} \|a\|_{p_0}^{\theta(q)}.$$

Note that q converges to p iff $\theta(q)$ converges to 0. This implies the assertion. ■

Proof (VII.2). Let us consider $G_1(z) = \pi(g_{\hat{\rho},\hat{\eta}}(z/2))e$ and $G_2(z) = g_{\eta,\rho}(z/2)$ such that

$$G(z) = G_1(z)G_2(z).$$

Let the notation $\frac{1}{p(z)} = \Re(z)$. From the triangle inequality, we deduce that

$$\begin{aligned} |\|G(z)\|_{2p(z)} - \|G(w)\|_{2p(w)}| &\le |\|G(z)\|_{2p(z)} - \|G(w)\|_{2p(z)}| + |\|G(w)\|_{2p(z)} - \|G(w)\|_{2p(w)}| \\ &\le \|G(z) - G(w)\|_{2p(z)} + |\|G(w)\|_{2p(z)} - \|G(w)\|_{2p(w)}|. \end{aligned}$$

A glance at the Proof of Lemma VII.4 shows that because $\|G(w)\| \le M$ uniformly for $\mathrm{Re}(w) \le 1$ (see Remark II.1), we do have

$$\lim_{w\to z} |\|G(w)\|_{2p(z)} - \|G(w)\|_{2p(w)}| = 0.$$

For the first part, we use Kosaki's interpolation result and get

$$\|G(z) - G(w)\|_{2p(z)} \le \|G(z) - G(w)\|_2^{1-Re(z)}.$$

Thus, for $\mathrm{Re}(z) > 0$, it suffices to show the L_2 estimate. Then, we observe that

$$\begin{aligned} \|G(z) - G(w)\|_2 &= \|G_1(z)G_2(z) - G_1(w)G_2(w)\|_2 \\ &\le \|G_1(z)(G_2(z) - G_2(w))\|_2 + \|(G_1(z) - G_1(w))G_2(w)\|_2 \\ &\le \|G_1(z)\|_\infty \|g_{\eta,\rho}(z/2)\rho^{1/2} - g_{\eta,\rho}(w/2)\rho^{1/2}\|_2 + \|(G_1(z) - G_1(w))G_2(w)\|_2. \end{aligned}$$

Thanks to Remark II.1, we deduce the convergence for the first of the two terms from Lemma VII.3. Let us consider the remaining term and $w = 1/q + it$. Then, we deduce from Hölder's inequality and interpolation that

$$\begin{aligned} \|aG_2(w)\|_2 &= \|a\eta^{w/2}\rho^{1/2-w/2}\|_2 = \|a\eta^{1/2q}\eta^{it/2}\rho^{-it/2}\rho^{1/2-1/q}\|_2 \\ &\le \|a\eta^{1/2q}\|_{2q} \le \|a\|_\infty^{1-1/q}\|a\eta^{1/2}\|_2^{1/q}. \end{aligned}$$

Therefore, we are left with an L_2-norm estimate. In our case, $a = \pi(G_1(z) - G_1(w))e$, and hence, for $b = G_1(z) - G_1(w)$, we find that

$$\begin{aligned} \|a\eta^{1/2}\|_2^2 &= Tr(\eta^{1/2}\Phi^\dagger(b^*b)\eta^{1/2}) = Tr(\hat{\eta}(b^*b)) \\ &= \|\hat{\rho}^{z/2}\hat{\eta}^{-z/2}\hat{\eta}^{1/2} - \hat{\rho}^{w/2}\hat{\eta}^{-w/2}\hat{\eta}^{1/2}\|_2^2. \end{aligned}$$

Therefore, Lemma VII.3 concludes the proof. ■

VIII. APPROXIMATION OF RELATIVE ENTROPY

In this section, we will work with Lindblad's definition of relative entropy,

$$D_{\mathrm{Lin}}(\rho\|\eta) = (\sqrt{\rho}, \log \Delta_{\rho,\eta}(\sqrt{\rho})) + \eta(1) - \rho(1).$$

Indeed, D_{Lin} is the unique homogeneous joint extension of the relative D entropy, i.e., we have the following.

(i) $D_{\mathrm{Lin}}(t\rho\|\eta) = tD_{\mathrm{Lin}}(\rho\|\eta)$.
(ii) $D_{\mathrm{Lin}}(\rho\|\eta) = D(\rho\|\eta)$ if $\rho(1) = \eta(1) = 1$.

A. Finite von Neuman algebras

Proposition VIII.1. Let (N,τ) be a finite von Neumann algebra and $a \le d_\eta \le a^{-1}$. Let d_ψ be a density of a state ψ. Then, $d_{M,\delta} = 1_{[0,M]}(d_\psi)d_\psi + \delta d_\eta$ satisfies

- *(0)* $\delta d_\eta \le d_{M,\delta} \le (a+\delta)d_\eta$,
- *(i)* $\lim_{M\to\infty}\lim_{\delta\to 0}\|d_{M,\delta} - d_M\|_1 = 0$, *and*
- *(ii)* $\lim_{M\to\infty}\lim_{\delta\to 0} D_{\rm Lin}(d_{M,\delta}\|d_\eta) = D(d\|d_\eta)$.

Proof. In the tracial setting, we have (see Ref. 19) that

$$D(\psi|\eta) = D(d_\psi\|d_\eta) = \tau(d_\psi \ln d_\psi) - \tau(d_\psi \ln d_\eta) = D_{\rm Lin}(d_\psi\|d_\eta).$$

For fixed M, we denote by $d_M = 1_{[0,M]}(d_\psi)d_\psi$ the density obtained by the functional calculus. Then, $d_{M,\delta} = d_M + \delta d_\eta$ converges in the operator norm and L_1 norm to d_M. Therefore, the continuity of $f(x) = x\ln x$ implies that

$$\lim_{\delta\to 0}\tau(d_M + \delta d_\eta \ln d_M + \delta d_\eta) - \tau((d_M + \delta d_\eta)\ln d_\eta) + \tau(d_\eta) - \tau(d_M + \delta d_\eta) = D_{\rm Lin}(d_M\|d_\eta).$$

Here, we use that d_η is bounded below and above, and hence, $\ln d_\eta$ is in $L_\infty(N)$. Using this fact again, we deduce from Fatou's lemma that

$$\tau(d_\psi \ln d_\psi) - \tau(d_\psi \ln d_\eta) + \tau(d_\eta) - \tau(d_\psi) = \lim_{M\to\infty}\tau(d_M \ln d_M) - \tau(d_M \ln d_\eta) + \tau(d_\eta) - \tau(d_M).$$

Note here that $D(d_\psi\|d_\eta)$ is finite iff $\tau(d_\psi \ln d_\psi)$ is finite. ■

For the convenience of the reader, let us briefly review how to transition from trace free definition to the one using trace. Indeed, in $L_2(\mathcal{N},\tau)$, the vector $\sqrt{d_\eta}$, the purification of the state η, implements the GNS representation with respect to the usual left-regular representation $\pi(x)\sqrt{d_\eta} = x\sqrt{d_\eta}$ for $x \in N$. We will use π again in the Haagerup construction, Sec. VIII B. Moreover, using Connes' 2×2 matrix trick (see, e.g., Ref. 62), we know for $\xi \in L_2(\mathcal{N},\tau)$ that

$$\Delta_{\eta,\psi}(\xi) = d_\eta \xi d_\psi^{-1}$$

and, hence,

$$\Delta_{\psi,\eta}^{it}(x) = d_\psi^{it} x d_\eta^{-it}.$$

This implies that

$$\ln \Delta_{\psi,\eta}(d_\psi^{1/2}) = \ln d_\psi d_\psi^{1/2} - d_\psi^{1/2} \ln d_\eta.$$

Taking the inner product, we find

$$(d_\psi^{1/2}, \ln \Delta_{\psi,\eta}(d_\psi^{1/2})) = \tau(d_\psi \ln d_\psi) - \tau(d_\psi \ln d_\eta) = D_\tau(d_\psi\|d_\eta).$$

B. Haagerup construction

Haagerup's construction for type *III* algebras provides a convenient tool to deduce properties of type *III* algebras from finite von Neumann algebras.

Remark VIII.2. Let us recall two possible ways to represent the crossed product $M \rtimes G$ for an action α of a discrete group on Hilbert space. We may assume that $M \subset B(H)$ and consider $\ell_2(G,H)$. Then, $M \rtimes G = \langle \lambda_H(G), \pi(M)\rangle$ is generated by a copy of $\lambda(G)$, the left regular representation of G, and $\pi(M)$. Here, we may assume that

$$\pi(x) = \sum_g |g\rangle\langle g| \otimes \alpha_{g^{-1}}(x)$$

is given by a twisted diagonal representation and $\lambda_H(g) = \lambda(g)\otimes 1_H$. Alternatively, we may choose $\hat{\pi}(x) = 1\otimes x$ and $\hat{\lambda}_H(g) = \lambda(g)\otimes u_g$ such that $u_g^* x u_g = \alpha_{g^{-1}}(x)$. Both of these representations are used in the literature, and their equivalence is used in the proof of Takai's theorem. For the equivalence, we note that

$$\lambda_H(g)^{-1}\pi(x)\lambda_H(g) = \pi(\alpha_{g^{-1}}(x)).$$

Similarly, $\lambda(g)^{-1}\otimes u_g^{-1}(1\otimes x)\lambda(g)\otimes u_g = 1\otimes \alpha_g^{-1}(x)$. This shows that the algebraic relations of these two representations coincide. Using a GNS construction, this extends to the generated von Neumann algebras.

Lemma VIII.3. Let ρ, η be states on the von Neumann algebra M with corresponding $\tilde{\rho}, \tilde{\eta}$ in $\tilde{M}_*$. Then, $D(\tilde{\rho}\|\tilde{\varphi}) = D(\rho\|\varphi)$.

Proof. We consider the Hilbert space $H = \ell_2(G, L_2(M))$ and still use the symbol $\lambda(g)$ instead of $\lambda_{L_2(M)}(g)$. Our first goal is to calculate the modular operator for an analytic state η with density d in $L_1(M)$ and $\tilde{\eta} = \eta \circ E$, $E : M \rtimes G \to M$ being the canonical conditional expectation. Then, $\xi = |1\rangle \otimes d^{1/2}$ implements the state $\tilde{\eta}$ on the crossed product. In order to calculate the modular operator $\Delta = S^* S$, we recall that

$$(y\xi, \Delta(x\xi)) = (x^*\xi, y^*\xi).$$

We start with finitely supported $y = \sum_g \lambda(g)\pi(y_g), z = \sum_g \lambda(g)\pi(z_g)$ and observe that

$$(y\xi, z\xi) = \left(\sum_g |g\rangle y_g d^{1/2}, \sum_g |g\rangle z_g d^{1/2}\right) = \sum_g \eta(y_g^* x_g).$$

On the other hand, we find that

$$(x^*\xi, y^*\xi) = \left(\sum_g |g^{-1}\rangle \alpha_g(x_g^*) d^{1/2}, \sum_g |g^{-1}\rangle \alpha_g(y_g^*) d^{1/2}\right) = \sum_g \eta(\alpha_g(x_g y_g^*)).$$

Let $d_{g^{-1}} = \alpha_g^{-1}(d)$. Then, we see that

$$\eta(\alpha_g(x_g y_g^*)) = tr(d_{g^{-1}} x_g y_g^*) = tr(d^{1/2} y_g^* d_{g^{-1}} x_g d^{-1} d^{1/2}) = (y_g d^{1/2}, d_{g^{-1}} x_g d^{-1} d^{1/2}).$$

This means that the diagonal operator $\Delta_g(\xi_g) = \Delta_{d_{g^{-1}}, d}$ is a good candidate for the modular operator and is indeed well-defined for finitely supported sequences of $\sigma^t_{\alpha_g^{-1}(\eta),\eta}$-analytic elements, which are dense. Now, it is easy to identify the polar composition using the isometry $J(\sum_g |g\rangle \xi_g) = \sum_g |g^{-1}\rangle \alpha_g(\xi_g^*)$ on $\ell_2(G, L_2(M))$ because α_g extends to an isometry on $L_2(M)$. This formula $S = J\Delta^{1/2}$ follows by calculation. Finally, we use Connes' 2×2 matrix trick for two states η, ψ and the diagonal state $\hat{\eta}(x_{ab}) = \eta(x_{11}) + \psi(x_{22})$. Note that $M_2(M) \rtimes G = M_2(M \rtimes G)$, and hence, $\Delta_{\tilde{\eta},\tilde{\psi}}$ is the 1,2 entry given by the G-diagonal operator $\Delta_{\alpha_g^{-1}(\eta),\psi}$. This implies that

$$\begin{aligned} D(\tilde{\eta}\|\tilde{\psi}) &= (\xi_\psi, \log \Delta_{\tilde{\eta},\tilde{\psi}}(\xi_\psi)) = (d_\psi^{1/2}, \Delta_{\alpha_1^{-1}(\eta),\psi}(d_\psi^{1/2})) \\ &= (d_\psi^{1/2}, \log \Delta_{\eta,\psi}(d_\psi^{1/2})) = D(\eta\|\psi). \end{aligned}$$

Here, we use the fact that the relative entropy can be calculated on any representing Hilbert space. However, the representation of $M \rtimes G$ is in the standard form, which may be used as a definition of the relative entropy. ∎

A similar result holds for the fidelity.

Theorem VIII.4. *Let η be a faithful state. Then, there exists a sequence of states ρ_α such that*

(i) $\delta_\alpha \eta \le \rho_\alpha \le \delta_\alpha^{-1}\eta$ *for some* $\delta_\alpha > 0$,
(ii) $\lim_\alpha \rho_\alpha = \rho$, *and*
(iii) $D(\rho\|\eta) = \lim_\alpha D(\rho_\alpha\|\eta)$.

Proof. Let us define $\psi_k = F_k(\tilde{\rho})$. Thanks to the Haagerup construction, we know that $\lim_k \psi_k = \tilde{\rho}$. We may apply Proposition VIII.1 and find $d_{k,m,\delta} = \alpha_{k,m,\delta}(1_{[0,m]}(d_{\psi_k})d_{\psi_k} + \delta d_{\eta_k})$, where $\alpha_{k,m,\delta}$ is chosen such that $d_{k,m,\delta}$ has trace 1. Denote by $\psi^0_{k,m,\delta}$ the corresponding state on $\tilde{M}_k$ and $\psi_{k,m,\delta} = \psi^0_{k,m,\delta} \circ F_k$. Let $\rho_{k,m,\delta}$ be the restriction to M. Certainly, we find condition (i). Moreover, by the data processing inequality (see Witten's notes[19]),

$$D(\rho_{k,m,\delta}\|\eta) \le D(\psi_{k,m,\delta}\|\eta),$$

and hence,

$$\begin{aligned} \limsup_{k\to\infty, m\to\infty, \delta\to 0} D(\rho_{k,m,\delta}\|\eta) &\le \limsup_k D(\psi_k\|\tilde{\eta}) \\ &\le D(\tilde{\rho}\|\tilde{\eta}) = D(\rho\|\eta). \end{aligned}$$

However, we deduce from *(iii)* and Proposition VIII.1 that

$$\lim_k \lim_m \lim_\delta \psi_{k,m,\delta} = \tilde{\rho}.$$

Taking the conditional expectation $\mathcal{E}$ by restricting these states to M preserves this property. Thus, by the semicontinuity of D_{Lin}, we deduce that

$$D(\rho\|\eta) \le \liminf_{k,m,\delta} D(\rho_{k,m,\delta}\|\eta) \le \limsup_{k,m,\delta} D(\psi_{k,m,\delta}\|\tilde{\eta}) \le D(\rho\|\eta).$$

This allows us to find a suitable convergent subsequence. ■

IX. RECOVERY RESULTS

Finally, we are ready to show the general recovery results of this paper. In the following diagram, we illustrate the relationship of densities on the original algebra, crossed product, and approximating finite algebras used to derive the final result,

$$\begin{array}{llll} \Phi: & L_1(M) & \rightarrow & L_1(N) \\ & \uparrow \pi_M^\dagger & & \downarrow \mathcal{E}_N^\dagger \\ & L_1(M \rtimes G) & & L_1(N \rtimes G). \\ & \uparrow \mathcal{E}_j^\dagger & & \downarrow \pi_k^\dagger \\ \Phi_{j,k}: & L_1(M_j) & \rightarrow & L_1(N_k) \end{array} \tag{49}$$

Here, π_m, π_k are inclusion maps in the Haagerup approximation, and $\mathcal{E}_N, \mathcal{E}_j$ are conditional expectations. We define an approximating sequence of quantum channels $\Phi_{j,k} : L_1(M_j) \rightarrow L_1(N_k)$ in the finite von Neumann algebras and apply Theorem VI.3. Lemma IX.2 shows that the relative entropies in the crossed product converge to that of the original relative entropy in the von Neumann algebra with which we started. Theorem VIII.4 shows that we can construct an increasing sequence $L_1(M_j)$ and $L_1(N_k))$ in the finite algebras that converges to the relative entropy in the crossed product. We also may check that $\lim_{j,k} R_{\eta_j,\Phi_{j,k}} \rightarrow R_{\eta,\Phi}$. These steps follow those of Ref. 30, introducing no new concepts, so we do not repeat them in detail here.

For the p-fidelities, we have the following.

Theorem IX.1 (technical version of I.5)**.** *Let η and ρ be states such that the corresponding support projections satisfy $e_\rho \le e_\eta$. Let d_η, d_ρ denote their densities in $L_1(M)$. Let $\Phi : L_1(M) \rightarrow L_1(\hat{M})$ be a complete positive trace preserving map with adjoint $\Phi^\dagger$. Then, it holds for $1 \le p < \infty$,*

$$-2p \ln \|\rho^{1/2p}\eta^{1/2p}\Phi^\dagger(\Phi(\eta)^{-1/2p}\Phi(\rho)^{1/p}\Phi(\eta)^{-1/2p})\eta^{1/2p}\rho^{1/2p}\|_{p/2} + D(\Phi(\rho)\|\Phi(\eta)) \le D(\rho\|\eta).$$

Proof. Let ρ_α be as in Theorem VIII.4. We also need to fix k and consider $F_k(\tilde{\rho})$ together with states $d_{k,m,\delta}$ and the density $\eta_k = F_k(\tilde{\eta})$ on the $\tilde{M}_k$. Then, $d_{k,m,\delta}$ and $\tilde{\eta}_k$ satisfy the assumptions and keep the notation of the Proof of Theorem VIII.4. Moreover, the map $\Phi_k = \Phi \circ E \circ F_k : L_1(\tilde{M}_k) \rightarrow L_1(\hat{M})$ is completely positive and trace preserving. This allows us to apply Theorem V.9 and deduce

$$D(\Phi_k(d_{k,m,\delta})\|\Phi_k(\tilde{\eta}_k)) - 2p \ln f_p(d_{k,m,\delta}^{1/p}, R_{1/p}(\Phi_k(d_{k,m,\delta})^{1/p})) \le D(d_{k,m,\delta}\|\eta_k). \tag{50}$$

Using the lower semi-continuity, we deduce that

$$D(\Phi(\rho)\|\Phi(\eta)) \le \liminf_{k,m,\delta} D(\Phi_k(d_{k,m,\delta})\|\Phi_k(\tilde{\eta}_k)).$$

We also know that $\lim_{k,m,\delta} D(d_{k,m,\delta}\|\eta_k) = D(\rho\|\eta)$. Note that $\lim_{km,\delta} d_{k,m,\delta} = \tilde{\rho}$. Thus, by the norm continuity of the map $R_{1/p}$ and the Mazur map, we deduce that

$$\lim_{k,m,\delta} f_p(d_{k,m,\delta}^{1/p}, R_{1/p}(\Phi_k(d_{k,m,\delta})^{1/p})) = f_p\Big(\tilde{\rho}^{1/p}, R_{1/p}(\Phi(\tilde{\rho}^{1/p}))\Big).$$

By the definition of $R_{1/p}$ and Lemma V.1, we deduce that

$$f_p(\rho^{1/p}, \eta^{1/p}) = \lim_{k,m,\delta} f_p(d_{k,m,\delta}^{1/p}, R_{1/p}(\Phi_k(d_{k,m,\delta})^{1/p})).$$

Thus, taking the limit in (50) implies the assertion. ■

Here, we recall a shortened and slightly modified version of Lemma VIII.3, which uses the Haagerup approximation method to relate the semifinite and type III relative entropies.

Lemma IX.2. Let G be a discrete group and N be a von Neumann algebra such that $E : N \rtimes G \rightarrow N$ is a conditional expectation. Let $\tilde{\Phi} = w \circ E$ and $\tilde{\rho} = \rho \circ E$. Then,

$$D_M(\tilde{\rho}\|\tilde{\varphi}) = D_M(\rho\|\varphi).$$

Proof. Since $N \subset N \rtimes G$, we deduce that

$$D_M(\rho\|\varphi) \le D_M(\tilde{\rho}\|\tilde{\varphi}).$$

For the converse, consider $\Phi : L_1(N \rtimes G) \to \ell_1^m$ and the ucp-map $\Phi^* : \ell_\infty^m \to N \rtimes G$. The relative entropy is calculated with the help of the coefficients

$$\alpha_j = \tilde{\rho}(\Phi^*(e_j)) = \rho(E\Phi^*(e_j))$$

and $\beta_j = \varphi\big(E(\Phi^*(e_j))\big)$. Since $E\Phi^* : \ell_\infty^m \to N$ is a normal ucp map, we deduce the assertion. ■

We also recall Theorem VIII.4.

Corollary IX.3. Let $\rho, \eta \in M_*^+$ be a pair of states on a von Neumann algebra M, and let Φ be a quantum channel. Then,

$$D(\rho\|\eta) - D(\Phi(\rho)\|\Phi(\eta)) \geq \int_{\mathbb{R}} \beta_0(t) D_M(\rho\|R^t_{\eta,\Phi} \circ \Phi(\eta))dt \geq D_M(\rho\|\tilde{R}_{\eta,\Phi} \circ \Phi(\eta)).$$

Corollary IX.3 is the technical version of Theorem VI.3.

X. RECOVERY OF POSITIVE VECTORS

In this section, we explain how to recover certain vectors in a Hilbert space from a Petz recovery map. Our starting point is the representation of a von Neumann algebra $M \subset \mathbb{B}(H)$ and a separating vector $h \in M$, i.e., the map $x \mapsto xh$ is injective. This implies that the corresponding normal state $\eta(x) = (h, xh)$ has full support in M_*. Then, we may apply the GNS construction and a partial isometry $U : \overline{Mh} \to L_2(M)$ via

$$U(xh) = x\eta^{1/2}.$$

Indeed,

$$(U(xh), U(yh)) = Tr(\eta^{1/2}x^*y\eta^{1/2}) = \eta(x^*y) = (xh, yh)$$

shows that U extends to an isometry between Mh and $L_2(M)$. Recall that the inclusion $M \subset \mathbb{B}(L_2(M))$ is in the standard position. This means that there is a real subspace $L_2(M)_+ \subset L_2(M)$ and partial isometry J such that $J|_{L_2(M)_+} = id$. In fact, all these objects can be constructed by Tomita–Takesaki theory and $J_\eta = U^*JU$ is indeed the anti-linear part of $S = J\Delta^{1/2}$ in the polar decomposition of $S(xh) = x^*h$. Of particular importance here is the real subspace,

$$H_+ = U^*(L_2(M)_+).$$

The space of positive vectors is the range of Mazur map. Let us be more precise. For every norm one vector $k \in H$, we may consider the state

$$\omega_k(x) = (k, xk),$$

which admits a density $d_k \in L_1(M)$ such that

$$\omega_k(x) = Tr(d_k x).$$

Thanks to Størmer's inequality, the map $d_k \mapsto d_k^{1/2}$ is continuous and hence

$$|k| = U^* d_k^{1/2} \in H_+.$$

This allows us to reformulate the usual polar decomposition theorem.

Proposition X.1. Let h be a separating vector and $H_h = Mh$. Then, every element $k \in Mh$ admits a polar decomposition,

$$k = v|k|,$$

where $v \in M$ is a partial isometry, uniquely determined by $v^*v = \mathrm{supp}(\omega_k)$.

Remark X.2. Since $U^* : L_2(M) \to \overline{Mh}$, we can also work with polar decomposition for the adjoint

$$U(k) = |U(k)^*|w = R_w(|U(k)^*),$$

where w belongs to M and R_w is the right multiplication, and hence,

$$k = U^* R_w UU^*(|U(k)^*) \in M'H_+$$

admits a polar decomposition with respect to the commutant. In this form, the theorem extends to all of H. Indeed, let

$$H = \sum_i \overline{Mh_i}$$

be a direct sum of irreducible subspaces with projections $e_iH = \overline{Mh_i}$ in M'. Then, $Mh_i \cong L_2(M)f_i$ for some projection f_i corresponding to the support of h_i. Using an isomorphism V between H and $\oplus_i L_2(M)f_i$, we see that $M'(Mh) = M'h$ is dense in H. Using this isomorphism, we now deduce that

$$k = wV^*(|V(k)^*|)$$

admits a polar decomposition with a partial isometry $w \in M'$ and $V^*(|V(k)^*|) \in H_+$.

For $1 \le p \le \infty$, we may now consider the Kosaki interpolation space $L_p^1(M, \omega_h)$ as embedded in H. Indeed, we have already the inclusion

$$L_\infty(M, \omega_h) \cong Mh \subset H \cong L_2^1(M, \omega_h),$$

and by interpolation, we find an injective map,

$$U_p^* : L_p^1(M, \omega_h) \to H.$$

This allows us to define the corresponding p-norm

$$\|k\|_p = \sup\{|(ah, h)|| \|a\omega_h^{1/p'}\|_p < \infty\}$$

for $1 \le p \le \infty$. For $1 \le p \le 2$, the space

$$H^p = \{k| \|k\|_p < \infty\}$$

is dense in H and isomorphic $L_p(M)$. Therefore, we find natural cones

$$H_+^p = H^p \cap H_+$$

as the range of $U^*(L_p(M)_+)$. Let us explain how these cones appear naturally in the context of Petz maps. We will assume that $\Phi : L_1(M) \to L_1(\hat{M})$ is a completely positive trace preserving map and, for simplicity, that η and $\hat{\eta} = \Phi(\eta)$ have full support. Then, the Petz map

$$R_{1/p} : L_p(\hat{M}) \to L_p(M), R_{1/p}(\hat{\eta}^{1/2p} x \hat{\eta}^{1/2p}) = \eta^{1/2p}\Phi^\dagger(x)\eta^{1/2p}$$

is a contraction and sends $L_p(\hat{M})_+$ to $L_p(M)_+$. Therefore, we also find a contraction

$$R_{1/p} : \hat{H}_+^p \to H_+^p.$$

Let us describe this map more explicitly by assuming that $\omega_k \le C\omega_h$, and hence, as above,

$$a(z) = \omega_k^{z/2}\omega_h^{-z/2}, \qquad \hat{a}(z) = \hat{\omega}_k^{z/2}\hat{\omega}_h^{-z/2}$$

are well defined. Then, we find that

$$\begin{aligned} R_{1/p}(\hat{\omega}_k^{1/p}) &= \omega_h^{1/2p}\Phi^\dagger(\hat{a}(1/2p)^*\hat{a}(1/2p))\omega_h^{1/2p} \\ &= \Delta_{\omega_h}^{1/2p}(\Phi^\dagger(\hat{a}(1/2p)^*\hat{a}(1/2p))\omega_h^{1/p}). \end{aligned}$$

If we define $b = \Phi^\dagger(\hat{a}(1/2p)^*\hat{a}(1/2p))$, we see that

$$R_{1/p}(\hat{\omega}_k^{1/p}) = \Delta_{\omega_h}^{1/2p}(bh) \in H_+^p.$$

On the other hand, we see that $k \in \hat{H}_+^p$ is represented $\hat{U}(k) = \hat{\omega}_k^{1/p}\hat{\omega}_h^{-1/p}\hat{\omega}_h^{1/p}$. This implies that

$$\hat{a}(1/2p)^*\hat{a}(1/2p) = \hat{\omega}_h^{-1/2p}\hat{\omega}_k^{1/p}\hat{\omega}_h^{-1/2p} = \Delta_{\hat{\omega}_h}^{-1/2p}(\hat{\omega}_k^{1/p}\hat{\omega}_h^{-1/p}).$$

Let us recall the map

$$\Phi_p^\dagger(b\hat{\omega}_h^{1/p}) = \Phi^\dagger(b)\omega_h^{1/p},$$

which we extend to a densely map on H_p as follows:

$$\Phi_p^\dagger(b\hat{h}) = \Phi^\dagger(b)h.$$

Then, we can combine the calculations above and find that

$$R_{1/p} = \Delta_{\omega_h}^{1/2p} \Phi_p^\dagger \Delta_{\hat{\omega}_h}^{-1/2p}. \tag{51}$$

Our fidelity result can be formulated as follows.

Corollary X.3. Let h be a separating vector for M with associated vector state ω_h, and let $\Phi^\dagger : \hat{M} \to M$ be a normal, unital completely positive map, and let $\hat{\omega}_h = \omega_h \circ \Phi^\dagger$ be the associated vector state. Then, map $R_{1/p} : \hat{H}_p \to H_p$,

$$R_{1/p} = \Delta_{\omega_h}^{1/2p} \Phi_p^\dagger \Delta_{\hat{\omega}_h}^{-1/2p}$$

extends to a contraction and satisfies

$$-\ln f_p(k, R_{1/p}(\hat{k})) \le \frac{1}{2p}(D(\omega_k \| \omega_h) - D(\hat{\omega}_k \| \hat{\omega}_h))$$

for every $k \in H_+^p$.

Our next application tells us that if we use the standard form of representing a state on von Neumann algebras, then we may recover the implementing vector.

Corollary X.4. Let $H = L_2(M)$. Then, implementing vectors ξ_ρ for ρ and $\xi_{\hat{\rho}}$ satisfies

$$\|\xi_\rho - R_{1/2}(\xi_{\hat{\rho}})\|_2^2 \le D(\rho|\eta) - D(\Phi(\rho)\|\Phi(\eta)).$$

Proof. Let us first consider $a, b \in L_2(M)_+$ of norm 1 and $h = b - a$. Then,

$$0 = \|b\|^2 - \|a\|^2 = \|a + h\|^2 - \|a\|^2 = 2(a, h) + \|h\|^2.$$

On the other hand,

$$1 - f_2(a, b)^2 = \|a\|^2 - \|a^{1/2} b^{1/2}\|_2^2 = tr(a^2) - tr(ab) = tr(a(a - b))$$
$$= -(a, h) = \frac{\|h\|^2}{2}.$$

Then, $\ln(1 + x) \le x$ implies for $a = \rho^{1/2}$ and $b = R_{1/2}(\hat{\rho}^{1/2})$ that

$$-\ln f_2(a, b)^2 = -\ln(1 - (1 - f_2(a, b)^2)) \ge (1 - f_2(a, b)^2) \ge \frac{\|a - b\|_2^2}{2}.$$

The assertion then follows from Theorem V.9. ■

Remark X.5. The proof of Eqs. (18) and (19) in the Introduction follows via the triangle and Cauchy–Schwarz inequalities.

As an illustration, we will now assume that $\hat{M} \subset M$ is a subalgebra and that there exists a normal conditional expectation $E : M \to \hat{M}$ such that

$$\omega_h = \omega_h|_{\hat{M}} \circ E.$$

In this case, $\Phi^\dagger = \iota$ is just the inclusion map $\hat{M} \subset M$, and moreover, $\Phi^\dagger$ commutes with the modular group (see Ref. 59). Then, E extends to map $E : L_2(M)_+ \to L_2(\hat{M})_+$ via

$$E(x\omega_h^{1/2}) = E(x)\hat{\omega}_h^{1/2}.$$

Under these additional assumptions, we see that $R_{1/p} : \hat{H}^p \to H^p$ is simply the inclusion map. In this particular case, the fidelity can also be expressed easily. Indeed, according to the Proof of Lemma V.1, we know that

$$f_p(k', k) = \sup_{\|ak\|_{p'} \le 1} |(ak, \Delta_{k',k}^{1/2p}(k))|.$$

The case $p = 2$ is particularly interesting and gives the self-polar form,

$$f_2(x, y)^2 = \|x^{1/4} y^{1/4}\|_2^2 = Tr(x^{1/2} y^{1/2}).$$

For elements $k, k' \in H_+$, we may assume $k = a\omega_h^{1/2}$ and $k' = b\omega_h^{1/2}$ and $x^{1/2} = U(a\omega_h^{1/2})$, $y^{1/2} = U(b\omega_h^{1/2})$. This means that

$$f_2(x,y) = Tr(\omega_h b^* a) = (h, b^* ah) = (bh, ah) = (k', k).$$

Corollary X.6. In addition to the assumption of X.3, *assume that* $\omega_h = \hat{\omega}_h \circ E$ *holds for a normal conditional expectation. For* $k \in H_+$,

$$-\ln(k, E(k)) \leq D(\omega_h \| \omega_k) - D(\hat{\omega}_h \| \hat{\omega}_k).$$

Remark X.7. Without assuming the existence of E, we can still describe the Petz map for L_2 in this special case. Indeed, let us assume that $\hat{M} \subset M$ and denote by $\hat{\imath} : \hat{M}h \to Mh$ the canonical inclusion map. We will assume that $k \in H_+(\hat{M})$ and $\omega_k \leq C\omega_h$ (which implies that $\hat{\omega}_k \leq C\hat{\omega}_h$). Then,

$$\hat{\omega}_k^{1/2} = \hat{\omega}_k^{1/2} \hat{\omega}_h^{-1/2} \hat{\omega}_h^{1/2}$$

implies

$$k = \hat{\omega}_k^{1/2} \hat{\omega}_h^{-1/2} h$$

and

$$\hat{\Delta}_{-1/4}(k) = \hat{\Delta}_{-1/4}(\hat{\omega}_k^{1/2} \hat{\omega}_h^{-1/2})h.$$

Thanks to (51), this implies

$$\xi = R_{1/2}(k) = \Delta_{1/4}\big(\hat{\imath}((\hat{\Delta}_{-1/4}(k))\big).$$

Let $P_{1/4}$ be the orthogonal projection onto the rotated space $\hat{H}_{1/4} = \Delta_{1/4}(\hat{M}h)$. Then, $\xi \in \hat{H}_{1/4}$ implies

$$(|k|, \xi) = (P_{1/4}|k|, \xi) = \|P_{1/4}|k|\| \|\xi\| \leq \|P_{1/4}|k|\|.$$

Therefore, we deduce that

$$-\ln \|P_{1/4}|k|\| \leq D(\omega_h \| \omega_k) - D(\hat{\omega}_h \| \hat{\omega}_k).$$

In particular, if the relative entropy difference is small, then $P_{1/4}|k| \approx |k|$ implies that $U(|k|)$ almost commutes with ω_h.

XI. DATA PROCESSING INEQUALITY FOR *p*-FIDELITY

Theorem XI.1. *Let* $\Phi : L_1(M) \to L_1(\hat{M})$ *be a channel. Then,*

$$f_p(\Phi(\rho), \Phi(\sigma)) \geq f_p(\rho, \sigma).$$

We need the following L_p norm inequality.

Proposition XI.2. Let $\Phi^\dagger$ *be a normal, unital, completely, positive adjoint map of a channel* Φ *and* η *be a normal state on* M *such that* $\Phi(\eta) = \hat{\eta}$. *Then,* $\Phi_p : L_p(\hat{M}) \to L_p(M)$ *given by*

$$\Phi_p(x) = \eta^{1/2p} \Phi^\dagger(\hat{\eta}^{1/2p} x \hat{\eta}^{1/2p}) \eta^{1/2p}$$

is a completely positive contraction.

Proof. We may assume that the density η of a given state has full support; let $\hat{e}$ be the support of $\hat{\eta}$ so that we may assume that Φ_p is defined on $\hat{e}L_p(\hat{M})\big)\hat{e}$. This allows us to use the Kosaki isomorphism $L_p(\hat{M}) = L_p(\hat{M}, \hat{\eta})$. With the help of this automorphism, we consider the densely defined map,

$$T(\hat{\eta} x \hat{\eta}) = \eta^{1/2} \Phi(x) \eta^{1/2}.$$

Since $\Phi^\dagger : \hat{M} \to M$ is a contraction, we see that

$$\|T(x)\|_\infty \leq \|x\|_\infty.$$

On the other hand, let us assume that $x = ab$. Then, we see deduce from the Cauchy–Schwarz inequality for completely positive maps that

$$\begin{aligned}\|T(\hat{\eta}^{1/2}ab\hat{\eta}^{1/2})\| &= \|\eta^{1/2}\Phi^\dagger(ab)\eta^{1/2}\|_1 \le \|\eta\Phi^\dagger(aa^*)\eta\|_1^{1/2}\|\eta\Phi^\dagger(b^*b)\eta\|_1^{1/2}\\ &= tr(\eta\Phi^\dagger(aa^*))^{1/2}tr(\Phi^\dagger(b^*b)\eta)^{1/2}\\ &= tr(\hat{\eta}(aa^*))^{1/2}tr(\hat{\eta}b^*b)^{1/2}\\ &= \|\hat{\eta}a\|_2\|b\hat{\eta}\|_2.\end{aligned}$$

By the density of $\hat{M}\hat{\eta}^{1/2}$ in $L_2(\hat{M})\hat{e}$, we deduce that

$$\|T(\xi\varphi)\|_1 \le \|\xi\|_2\|\varphi\|_2$$

for any ξ and φ. Thus, T extends to a completely positive contraction on $\hat{e}L_1(\hat{M})\hat{e}$. By the general Riesz–Thorin theorem (see Ref. 39), we deduce that $T : L_p(\hat{M},\hat{\eta}) \to L_p(M,\eta)$ is a contraction. By Kosaki's theorem, this completes the proof. ■

Corollary XI.3. Let η,ρ be two densities of states. Then,

$$T_p^{\eta,\rho}(x) = \eta^{1/2p}\Phi^\dagger(\hat{\eta}^{-1/2p}x\hat{\rho}^{-1/2p})\rho^{1/2p}$$

extends to a contraction from $L_p(\hat{M})$ to $L_p(M)$.

Proof. We use Connes' matrix trick and consider $\sigma = \begin{pmatrix}\rho & 0\\ \eta & 0\end{pmatrix}$ on $M_2(M)$ for $\Phi_2 = id_{M_2}\otimes\Phi$. The assertion follows from applying Proposition XI.2 to $y = \begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix}$. ■

Proof of XI.1. Let $x = \hat{\eta}^{1/2p}\hat{\rho}^{1/2p}$. Then, we deduce that

$$T_p^{\eta,\rho}(\hat{\eta}^{1/2p}\hat{\rho}^{1/2p}) = \eta^{1/2p}\rho^{1/2p}.$$

Since $T_p^{\eta,\rho}$ is a contraction, we deduce that

$$\begin{aligned}f_p(\eta,\rho) &= \|\eta^{1/2p}\rho^{1/2p}\|_p\\ &\le \|\hat{\eta}^{1/2p}\hat{\rho}^{1/2p}\|_p = f_p(\hat{\eta},\hat{\rho}).\end{aligned}$$

■

Corollary XI.4. If $D(\rho\|\eta) = D(\Phi(\rho)\|\Phi(\eta))$ for a channel $\Phi : L_1(M)\to L_1(\hat{M})$, then

$$T_p^{\eta,\eta}(\sigma_s^{\hat{\eta}}) = \sigma_s^\eta(\rho^{1/p})$$

holds for all $1\le p\le\infty$ and $s\in\mathbb{R}$. Moreover, there exists a modular group intertwining channel $\Psi : L_1(M)\to L_1(\hat{M})$ such that $\hat{T}_p(x) = \sigma^{1/2p}\Psi^\dagger(\hat{\sigma}^{-1/2p}x\hat{\sigma}^{-1/2p})\sigma^{1/2p}$ satisfies

$$\hat{T}_p(\hat{\rho}^{1/p}) = \rho^{1/p}$$

and

$$\Psi(\rho) = \Phi(\rho).$$

Proof. In this case,

$$-\ln f_p(\rho, R_{p,t}(\hat{\rho}^{1/p})^p) = 0$$

holds μ almost everywhere. By continuity, this holds for all t. In other words, thanks to the Mazur map, we get

$$\rho^{1/2p}\sigma^{1-it/2p}\Phi^\dagger\Big(\hat{\sigma}^{-(1-it/2p}\hat{\rho}^{1/p}\hat{\sigma}^{-(1+it)/2p}\Big)\sigma^{(1+it)/2p}\rho^{1/2p} = \rho^{2/p}$$

for all t. This implies

$$T_p^\eta(\sigma_{\hat{\eta}}(s)(\hat{\rho}^{1/p})) = \sigma_\eta(s)\rho^{1/p}$$

for all s. For the moreover part, we consider the family $R_p(x) = \hat{\sigma}^{-1/2p'}\Phi(\sigma^{1/2p'}x\sigma^{1/2p'})\hat{\sigma}^{1/2p}$. Thanks to data processing inequality for sandwiched relative entropy, this map is a contraction, and hence,

$$\Psi_2(x) = \lim_{T,\mathcal{U}} \int_{-T}^{T} \sigma_{\hat{\eta}}(s)\Phi_2(\sigma_\eta(-s)(x))\frac{ds}{2T}$$

exists as a bounded operator on L_2. By the density of L_2 in L_1, we deduce that

$$\Psi_1(\eta^{1/4}x\eta^{1/4}) = \hat{\eta}^{1/4}\Psi_2(x)\hat{\eta}^{1/4} = \lim_{T,\mathcal{U}} \int_{-T}^{T} \sigma_{\hat{\eta}}(s)\Phi_2(\sigma_\eta(-s)(x))\frac{ds}{2T}$$

is a completely positive map on $L_1(M)$. Its adjoint $\Psi_1^\dagger$ is a normal, unital completely positive map, defined as a point weak* limit of averages. Hence, our assumption shows that $\hat{T}_p(x) = \eta^{1/2p}\Psi^\dagger(\hat{\eta}^{-1/2p}x\hat{\eta}^{-1/2p})\eta^{1/2p}$ also satisfies

$$\hat{T}_p(\hat{\rho}^{1/p}) = \rho^{1/p}$$

for $1 \le p \le \infty$. For the final assertion, we have to establish a simple duality relation. Using Kosaki L_p spaces, we see that the family of maps

$$\Phi_p \cong \Phi|_{\iota_p(L_p(M,\eta))}$$

is really the same map through the topological embedding $\iota_p(x) = \eta^{1/2p}x\eta^{1/2p}$. Similarly,

$$\eta^{1/2p'}T_p(\hat{\eta}^{1/2p}x\eta^{1/2p})\eta^{1/2p'} = T_1(\hat{\eta}^{1/2}x\hat{\eta}^{1/2})$$

shows that $T_p = T_1|_{\iota_p(L_p)}$ is also the same map. Moreover,

$$Tr(\Phi(\eta^{1/2}x\eta^{1/2}y)) = tr(\eta^{1/2}x\eta^{1/2}\Phi^\dagger(y)) = tr(xT_1(\hat{\sigma}^{1/2}y\hat{\sigma}^{1/2}))$$

shows that $T_p = \Phi_{p'}^\dagger$ by density. The same holds for $\hat{T}_p = \Psi_{p'}^\dagger$. Now, it is easy to conclude. Our assumption implies

$$\begin{aligned} 1 &= Tr(\rho^{1/p}\rho^{1/p'}) = Tr(\rho^{1/p}\hat{T}_{p'}(\hat{\rho}^{1/p'})) \\ &= (\iota_p(\rho^{1/p}), \hat{T}_1(\iota_{p'}(\hat{\rho}^{1/p'}))) \\ &= (\Psi(\iota_p(\rho^{1/p}), \iota_{p'}(\hat{\rho}^{1/p'}))) \\ &= Tr(\Psi_p(\rho^{1/p})\rho^{1/p'}). \end{aligned}$$

By the uniform convexity of L_p, we deduce that

$$\Psi_p(\rho^{1/p}) = \hat{\rho}^{1/p} = \Phi(\rho)^{1/p}.$$

For $p \to 1$, we deduce the assertion. ∎

XII. L_1 ISOMETRIES

In the theory of von Neumann algebras, completely isometric embeddings of $L_1(N)$ into $L_1(M)$ are completely characterized (see Ref. 63 for more information on the crucial work by Kirchberg). Indeed, a map $u : L_1(N) \to L_1(M)$ is completely isometric iff there exists a normal conditional expectation $E : M \to N \subset N_0$, an *-homomorphism $\pi : M \to N_0$, and $J \in N_0'$ such that

$$u(\eta^{1/2}x\eta^{1/2}) = \hat{\eta}\pi(x)J\hat{\eta}.$$

Such a map is completely positive if J is completely positive. Moreover, the inverse u^{-1} extends to $L_1(M)$. Let us formulate a simple consequence of the data processing inequalities.

Lemma XII.1. Let u be a completely positive complete isometry $u : L_1(N) \to L_1(\tilde{N})$. Then,

$$D(u(\eta)\|u(\rho)) = D(\eta\|\rho),$$

provided that they are finite. Moreover,

$$f_p(u(\rho), u(\eta)) = f_p(\rho, \eta).$$

Lemma XII.2. Let $\hat{M}$ and $\hat{N}$ be semifinite and $\Phi : L_1(M) \to L_1(\hat{M})$, $\rho \le C\eta$ such that

$$D(\rho\|\eta) = D(\Phi(\rho)\|\Phi(\eta)).$$

Then, there exists completely positive L_1-isometry u such that $\hat{\eta} = u(\eta)$ and $\hat{\rho} = u(\rho)$.

Proof. Let $\Psi^\dagger : \hat{M} \to \hat{N}$ be the averaged map. Then, we see that

$$\Psi^\dagger(\hat{\rho}^{1/2}) = \rho^{1/2}$$

and, hence,

$$\rho = \Psi^\dagger(\hat{\rho}^{1/2})\Psi^\dagger(\hat{\rho}^{1/2}) \le \Psi^\dagger(\hat{\rho}) = \rho.$$

Thus, we find equality in Kadison's inequality, and $\hat{\rho}$ belongs to the (extended) multiplicative domain $m \subset \hat{M}$. Since Ψ is normal and invariant under $\sigma_{\hat{\eta}}$, we see that the multiplicative domain m admits an η-invariant conditional expectation $E : \hat{M} \to m$ such that $\hat{\eta}E = \hat{\eta}$; see, e.g., Ref. 62 and also Ref. 59. In particular, we have completely isometric and completely positive inclusion $\iota : L_1(m) \to L_1(\hat{M})$ such that

$$\iota(\hat{\eta}^{1/2}x\hat{\eta}^{1/2}) = \hat{\eta}^{1/2}x\hat{\eta}^{1/2}.$$

Let us denote by $\hat{M}(\hat{\rho},\hat{\eta}) \subset m$ the smallest von Neumann algebra generated by $C^*(\hat{\rho})$ and $\sigma_t^{\hat{\eta}}$, which remains $\hat{\eta}$-complemented. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function. Then, we deduce that

$$\Psi^\dagger(f(\rho)) = f(\rho), \Psi^\dagger(\sigma_{\hat{\eta}}(t)(f(\rho))) = \sigma_t f(\rho).$$

This means that $\Psi^\dagger$ extends to a natural isomorphism between $\hat{M}(\hat{\rho},\hat{\eta})$ and $M(\rho,\eta)$ such that

$$tr(\eta\Psi^\dagger(x)) = tr(\Psi(\eta)x) = tr(\hat{\Phi}(x)).$$

The adjoint of $u = (\Psi^\dagger|_{\hat{M}(\hat{\rho},\hat{\eta})})^\dagger$ satisfies $u(\eta) = \hat{\psi}$ and

$$tr(u(\rho)x) = tr(\rho\Psi^\dagger(x)) = tr(\hat{\rho}(x)).$$

Since $M(\rho,\eta)$ is also η-conditioned, we deduce the assertion. ∎

Remark XII.3. It follows easily that

$$u(\eta)^{1/p} = \hat{\eta}^{1/p}$$

and

$$u(\rho)^{1/p} = \hat{\rho}^{1/p}$$

hold for all $1 \le p \le \infty$ under the assumptions above.

We want to extend this result to type *III* von Neumann algebras. For this, we need the notion of the multiplicative domain. For a completely positive unital map $\Phi : M \to N$ with Stinespring dilation $\Phi(x) = V^*\pi(x)V$, we recall that x belongs to the right domain if

$$\Phi(x)^*\Phi(x) = \Phi(x^*x) \tag{52}$$

or, equivalently, $V^*\pi(x)(1 - VV^*)\pi(x)V = 0$. If x and x^* satisfy (52), then $[V,\pi(x)] = 0$ holds for a minimal Stinespring dilation. The set

$$\mathrm{mdom}(\Phi) = \{x|[V,\pi(x)] = 0\} = \{x|\Phi(x^*)\Phi(x) = \Phi(x^*x) \quad \text{and} \quad \Phi(x)\Phi(x^*) = \Phi(xx^*)\}$$

is a sub-C^*-algebra of M, and for normal Φ and, hence, normal π (see Refs. 44 and 54), this is even a sub-von Neumann algebra.

Lemma XII.4. Let $\Phi_n : \hat{M} \to M$ be a sequence of normal completely positive maps such that we have the following.

(i) *The weak* limit*

$$\Phi_\infty(x) = \lim_n \Phi_n(x).$$

(ii) $\Phi_n^\dagger(\sigma) = \hat{\sigma}$ *for normal faithful states σ and $\hat{\sigma}$.*

(iii) $(\sigma^{1/2}\Phi_n(x), \Phi_m(y)\sigma^{1/2}) = (\sigma^{1/2}\Phi_{\min(n,m)}(x), \Phi_{\min(n,m)}(y)\sigma^{1/2})$.

Let (a_n) be a bounded sequence in the multiplicative domain of Φ_n, converging strongly to a. Then, a belongs to the multiplicative domain of Φ.

Proof. We follow Kirchberg and use the C^*-algebra $C(\hat{\mathcal{M}})$ of all bounded sequences (a_n) such that a_n converges in the strong and strong *-algebra. Similarly, we consider $C(\hat{M})$ and the corresponding quotient maps $\hat{q}$ and $q : C(M) \to M$ given by $q((a_n)) = w^* \lim_n a_n$. We claim that $\Phi^{\cdot} C(\hat{M}) \subset C(M)$. Indeed, assume that $\lim_n a_n - a$ converges to 0 strongly. Then, $a_n - a\hat{\sigma}$ converges to 0 in $L_2(\hat{M})$. Let us fix $n \leq mm$. We find that

$$\begin{aligned}\|(\Phi_n(a_n) - \Phi_m(a_m))\sigma^{1/2}\|_2 &= Tr(\sigma^{1/2}\Phi_n(a_n^* a_n)\hat{\sigma}^{1/2}) + Tr(\sigma^{1/2}\Phi_m(a_m^* a_m)\hat{\sigma}^{1/2}) \\ &\quad - Tr(\sigma^{1/2}\Phi_n(a_n^*)\Phi_m(a_m)\sigma^{1/2}) - Tr(\sigma^{1/2}\Phi_m(a_m)^*\Phi_n(a_n)\sigma^{1/2}) \\ &= Tr(\Phi_n^*(\sigma)(a_n^* a_n)) + Tr(\Phi_m^*(\sigma)(a_m^* a_m)) - Tr(\Phi_n^*(\sigma)(a_n^* a_m)) - Tr(\Phi_n^*(\sigma)(a_m^* a_n)) \\ &= Tr(\hat{\sigma}(a_n^* a_n + a_m^* a_m - a_n^* a_m - a_m^* a_n)) \\ &= \|(a_n - a_m)\hat{\sigma}\|_2^2.\end{aligned}$$

Since σ is faithful and $(\Phi_n(a_n))$ is bounded, we deduce that $\Phi_n(a_n)$ is also strongly convergent.

Let $\hat{M}_n \subset M$ be the multiplicative domain of $\mathcal{A} = \{(x_n) | x_n \in \hat{M}_n\}$, be the corresponding subalgebra of $\ell_\infty(\hat{M})$. Then, $\Phi^{\cdot} : \mathcal{A} \to \ell_\infty(M)$ is an *-homomorphism, and we may define $A = C(\hat{M}) \cap \mathcal{A}$. Then,

$$\Phi^\infty|_A : A \to C(M)$$

is a C^*-homomorphism. Let $\hat{J} \subset C(\hat{M})$ be the kernel of the quotient map $\hat{q}$. Since Φ^∞ preserves strong convergence, we deduce that $\Phi^\infty(\hat{J}) \subset J$, where J is the kernel q. We deduce that there exists an *-homomorphism $\pi : \hat{q}(A) \subset C(\hat{M})/\hat{J} = \hat{M}$ to $M = C(M)/J$ such that

$$q\Phi^\infty(a_n) = \sigma(q(a_n)).$$

Note that σ is the restriction of the completely positive map $\tilde{\Phi} : C(\hat{M})/\hat{J} \to C(M)/J$. By applying this map to the constant sequence $(b_n) = b$, we deduce that $\tilde{\Phi} = \Phi^\infty$. Thus, for every strongly convergent sequence in A, we deduce that $a = \lim_n a_n$ belongs to the multiplicative domain of Φ^∞ because $\sigma(a^* a) = \sigma(a)^*\sigma(a)$ and $\sigma(a)^*\sigma(a) = \sigma(aa^*)$. ■

Theorem XII.5 (technical version of Theorem I.7)**.** *Let $\rho \leq \lambda\eta$ for some $\lambda > 0$, and $\Phi : L_1(M) \to L_1(\hat{M})$. Then, the following conditions are equivalent:*

(i) $D(\Phi(\rho)\|\Phi(\eta)) = D(\rho\|\eta)$.

(ii) *There exist an η-conditioned subalgebra $M_0 \subset M$ and a completely positive L_1-isometry u such that*

$$u(\eta) = \Phi(\eta), \qquad u(\rho) = \Phi(\rho).$$

Proof. Thanks to Lemma XII.1, we only have to prove *(i) ⇒ (ii)*. In view of Corollary XI.4, we may assume that $\Phi = \Psi$ intertwines σ_η and $\sigma_{\hat{\eta}}$. Let $G = \bigcup_k 2^{-k}\mathbb{Z}$. Since Ψ is σ-invariant, we know that $\Psi_G = \Psi \rtimes G$ extends to the cross product. Recall that $\eta_G = \eta \circ E_G$ and $\rho_G = \rho \circ E_G$ naturally extend to the discrete crossed product. Let us recall that Ψ^G extends to a map $T_1^G : L_1(\hat{M}_G) \to L_1(M_G)$ via

$$T_G(\hat{\eta}_G^{1/2} x \hat{\eta}_G^{1/2}) = \eta_G^{1/2}\Phi_G^\dagger \eta_G^{1/2}.$$

Since $D(\rho_G\|\eta_G) = D(\rho\|\eta)$ and $D(\Psi_G(\rho)\|\Psi_G(\eta_G)) = D(\Psi(\rho)\|\Psi(\eta))$, we deduce that

$$T_1^G(\hat{\rho}_G) = \rho_G.$$

Let $\mathcal{E}_n$ be the conditional expectation given by the Haagerup construction. Note that $T_1^G E_n = E_n T_1^G$ follows from the fact that Ψ commutes with the modular group. Thus, for every $n \in \mathbb{N}$, we may apply Lemma XII.2 and find $A_n = \hat{M}_n(E_n(\rho_G)), E_n(\eta_G))$ in the multiplicative domain, which is modular group invariant.

Let us now assume that $\rho = \eta^{1/2} h \eta^{1/2}$ for a bounded h and, hence (using the map Ψ instead of Φ), that

$$\hat{\rho} = \hat{\eta}^{1/2}\hat{h}\hat{\eta}, \qquad \hat{\rho}_G = \hat{\eta}_G\hat{h}\hat{\eta}_G.$$

Let d_n and $\hat{d}_n$ be the densities of $\hat{\eta}_G|_{\hat{M}_n}$ and $\eta_G|_{M(n)}$, respectively. Recall that $\hat{d}_n$ and d_n belong to the center of $\hat{M}(n)$ and $M(n)$. Then,

$$E_n(\hat{\rho}_G) = \hat{d}_n^{1/2} E_n(\hat{h})\, \hat{d}_n^{1/2}$$

implies that $\hat{h}_n = E_n(\hat{h})$ also belongs to the multiplicative domain of $\Psi_n^\dagger = \Psi^\dagger E_n$. In order to apply the lemma, we recall that η_G and $\hat{\eta}_G$ are E_n invariant. Since $\hat{M}_n$ are increasing, we deduce that for $n \le m$,

$$\begin{aligned} Tr(\eta_G^{1/2} E_n \Psi^\dagger(a) E_m(b) \eta_G^{1/2}) &= Tr(\eta_G^{1/2} \Psi^\dagger(E_n(a) E_m(b)) \eta_G^{1/2}) \\ &= Tr(\Psi(\eta_G) E_m(E_n(a) b)) = Tr(\hat{\eta}_G(E_n(a) b)) \\ &= Tr(\hat{\eta}_G(E_n(a) E_n(b))) = Tr(\eta_G^{1/2} E_n \Psi^\dagger(a) E_n(b) \eta_G^{1/2}). \end{aligned}$$

Note that for $\Phi_n = \Psi_G^\dagger E_n$, we have $\Phi_\infty = \Psi^\dagger$ and, hence, $\hat{k} = \lim_n \hat{k}_n$ belongs to the multiplicative domain of $\Psi_G^\dagger$ and, hence, to the multiplicative domain of Ψ. Indeed, we may consider $a_n \hat{k}_n^{1/2}$. Then, $a_n^* a_n$ converges weakly to $\hat{k}$ if $a_n - \hat{k}^{1/2}$ converges strongly to 0. Using

$$E_n(\hat{\rho}_G) = \hat{\eta}_G^{1/2} E_n(\hat{k}) \hat{\eta}_G^{1/2},$$

we deduce the weak-convergence from the crucial inequality

$$\lim_n \| E_n(\hat{\rho}_G) - \hat{\rho}_G \|_1$$

in the Haagerup construction. Note also that

$$\sqrt{\hat{d}_n^{1/2} \hat{k}_n \hat{d}_n^{1/2}} = \hat{d}_n^{1/4} \hat{k}_n^{1/2} \hat{d}_n^{1/4}$$

because $\hat{d}_n$ belongs to the center of $\hat{M}(n)$, which allows us to use Størmer's inequality. Since the multiplicative domain of $\Phi^\dagger$ is invariant under the modular group of $\hat{\eta}$ and $\hat{k}$ belongs to the smallest modular group invariant von Neumann subalgebra $\hat{M}_0$, which is mapped to M_0, the smallest modular group invariant generated by h, we can now conclude as in Lemma XII.2. ■

XIII. CONCLUSIONS AND OUTLOOK

The proofs in Ref. 29 and more traditionally information-theoretic proofs in Ref. 1 use an approach called the method of types[64] (not to be confused with von Neumann algebra types). Classically, the key innovation of typicality in Shannon theory turns many copies of a complicated vector of different probabilities into a distribution that is nearly uniform on a set of typical outcomes and nearly unsupported elsewhere. The number of distinct eigenvalues of many copies of a density matrix grows only polynomially, while the dimension grows exponentially. The method of types is thereby powerful on quantities that scale linearly with tensored copies of a matrix.

A more mathematical approach to entropy bounds, used in Refs. 15, 16, and 65 and in the second proof style of Ref. 1, uses complex interpolation to compare entropies as limits and logarithms of p-norms. These techniques are further from classical intuition, can lead to breakthroughs on problems that had resisted traditional information theoretic techniques, and often yield automatic p-Rényi generalizations. Furthermore, they naturally generalize to Kosaki spaces and do not rely on finite-dimensional assumptions.

Apparent in Ref. 1 are direct correspondences between some information-theoretic methods and their interpolation analogs. Deeper work on this analogy may lead to a more intuitive understanding or mathematical duality. A renewed understanding of Shannon theory through analysis on operator algebras helps escape classical intuition and generalizes beyond finite dimension, while the Shannon-theoretic analogy of results on operators may help clarify the physical justification of obtained inequalities. The Haagerup approximation method and Kosaki interpolation spaces add to the understanding of this connection.

Holography in high energy physics proposes duality between entropy and geometry, suggesting spatial correspondences and the intuition for famous entropy inequalities[66,67] and operational techniques, such as error correction.[68,69] Many of these connections would manifest physically in field theories modeled as type III von Neumann algebras. The theory of entropy in holography will therefore benefit from an intuitive method of traceless entropy results.

ACKNOWLEDGMENTS

N.L. was supported by IBM as a Postdoctoral Scholar at the University of Chicago and The Chicago Quantum Exchange. N.L. was previously supported by the Department of Physics at the University of Illinois at Urbana-Champaign. M.J. was partially supported by the NSF under Grant Nos. DMS 1800872 and Raise-TAQS 1839177.

We acknowledge the correspondence with David Sutter as inspiring some inquiries within this project. We also acknowledge conversations with Li Gao, Nima Lashkari, and Mark Wilde.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Marius Junge: Conceptualization (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Supervision (equal); Validation (equal); Writing – original draft (equal); Writing – review & editing (equal). **Nicholas LaRacuente**: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Validation (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

REFERENCES

[1] D. Sutter, M. Berta, and M. Tomamichel, "Multivariate trace inequalities," Commun. Math. Phys. **352**, 37–58 (2017).
[2] B. M. Terhal, "Quantum error correction for quantum memories," Rev. Mod. Phys. **87**, 307–346 (2015).
[3] H. Barnum and E. Knill, "Reversing quantum dynamics with near-optimal quantum and classical fidelity," arXiv:quant-ph/0004088 (2000).
[4] M. Wilde, *Quantum Information Theory* (Cambridge University Press, 2013), google-Books-ID: T36v2Sp7DnIC.
[5] S. Beigi, N. Datta, and F. Leditzky, "Decoding quantum information via the petz recovery map," J. Math. Phys. **57**, 082203 (2016).
[6] S. Ryu and T. Takayanagi, "Holographic derivation of entanglement entropy from the anti–de sitter space/conformal field theory correspondence," Phys. Rev. Lett. **96**, 181602 (2006).
[7] J. Cotler, P. Hayden, G. Penington, G. Salton, B. Swingle, and M. Walter, "Entanglement wedge reconstruction via universal recovery channels," Phys. Rev. X **9**, 031011 (2019).
[8] G. Penington, S. H. Shenker, D. Stanford, and Z. Yang, "Replica wormholes and the black hole interior," J. High Energy Phys. **2022**(3), 1–87 (2022).
[9] D. n. Petz, "Sufficient subalgebras and the relative entropy of states of a von Neumann algebra," Commun. Math. Phys. **105**, 123–131 (1986).
[10] D. Petz, "Sufficiency of channels over von Neumann algebras," Q. J. Math. **39**, 97–108 (1988).
[11] O. Fawzi and R. Renner, "Quantum conditional mutual information and approximate Markov chains," Commun. Math. Phys. **340**, 575–611 (2015).
[12] D. Sutter, "Approximate quantum Markov chains," in *Approximate Quantum Markov Chains* (Springer, Cham, 2018) pp. 75–100.
[13] D. Sutter, O. Fawzi, and R. Renner, "Universal recovery map for approximate Markov chains," Proc. R. Soc. A **472**, 20150623 (2016).
[14] K. Li and A. Winter, "Squashed entanglement, k-extendibility, quantum Markov chains, and recovery maps," Found. Phys. **48**, 910–924 (2018).
[15] M. M. Wilde, "Recoverability in quantum information theory," Proc. R. Soc. A **471**, 20150338 (2015).
[16] M. Junge, R. Renner, D. Sutter, M. M. Wilde, and A. Winter, "Universal recovery maps and approximate sufficiency of quantum relative entropy," Ann. Henri Poincaré **19**, 2955–2978 (2018).
[17] E. A. Carlen and A. Vershynina, "Recovery map stability for the data processing inequality," J. Phys. A: Math. Theor. **53**, 035204 (2020).
[18] A. Gilyén, S. Lloyd, I. Marvian, Y. Quek, and M. M. Wilde, "Quantum algorithm for petz recovery channels and pretty good measurements," Phys. Rev. Lett. **128**(22), 220502 (2022).
[19] E. Witten, "APS medal for exceptional achievement in research: Invited article on entanglement properties of quantum field theory," Rev. Mod. Phys. **90**, 045003 (2018).
[20] L. Gao and M. M. Wilde, "Recoverability for optimized quantum f-divergences," arXiv:2008.01668 [hep-th, physics:math-ph, physics:quant-ph] (2020).
[21] T. Faulkner, S. Hollands, B. Swingle, and Y. Wang, "Approximate recovery and relative entropy I. general von Neumann subalgebras," Commun. Math. Phys. **389**(1), 349–397 (2022).
[22] T. Faulkner and S. Hollands, "Approximate recoverability and relative entropy II: 2-positive channels of general v. neumann algebras," Lett. Math. Phys. **112**(2), 1–24 (2022).
[23] S. Hollands, "Trace- and improved data processing inequalities for von Neumann algebras," arXiv:2102.07479 [hep-th, physics:math-ph, physics:quant-ph] version: 2 (2021).
[24] Y.-C. Liang, Y.-H. Yeh, P. E. M. F. Mendonça, R. Y. Teh, M. D. Reid, and P. D. Drummond, "Quantum fidelity measures for mixed states," Rep. Prog. Phys. **82**, 076001 (2019).
[25] M. Berta and M. Tomamichel, "The fidelity of recovery is multiplicative," IEEE Trans. Inf. Theory **62**, 1758–1763 (2016).
[26] U. Haagerup, "The standard form of von Neumann algebras," Math. Scand. **37**, 271–283 (1975).
[27] U. Haagerup, M. Junge, and Q. Xu, "A reduction method for noncommutative L_p-spaces and applications," Trans. Am. Math. Soc. **362**(4), 2125–2165 (2010).
[28] H. Kosaki, "Applications of the complex interpolation method to a von Neumann algebra: Non-commutative L_p-spaces," J. Funct. Anal. **56**, 29–78 (1984).
[29] D. Sutter, M. Tomamichel, and A. W. Harrow, "Strengthened monotonicity of relative entropy via pinched petz recovery map," IEEE Trans. Inf. Theory **62**, 2907–2913 (2016).
[30] M. Junge and N. LaRacuente, "Universal recovery and p-fidelity in von Neumann algebras," arXiv:2009.11866 [hep-th, physics:math-ph, physics:quant-ph] (2020).
[31] M. Takesaki, *Theory of Operator Algebras I,II,III*, Encyclopaedia of Mathematical Sciences, Theory of Operator Algebras (Springer-Verlag, 2002).
[32] F. Strocchi, *An Introduction to the Mathematical Structure of Quantum Mechanics: A Short Course for Mathematicians* (World Scientific, 2008), google-Books-ID: Bn7MaT3X8fkC.
[33] S. J. Summers, "Tomita-takesaki modular theory," arXiv:math-ph/0511034 (2005).
[34] L. Zhang and J. Wu, "Tomita-takesaki modular theory vs. quantum information theory," arXiv:1301.1836 [math-ph, physics:quant-ph] (2013).
[35] T. Fack and H. Kosaki, "Generalized s-numbers of τ-measurable operators," Pac. J. Math. **123**, 269–300 (1986).
[36] H. Araki, "Relative Hamiltonian for faithful normal states of a von Neumann algebra," Publ. Res. Inst. Math. Sci. **9**, 165–209 (1973).
[37] M. Choda, "A remark on the normal expectations," Proc. Jpn. Acad. **44**, 462–466 (1968).
[38] M. Choda, "A remark on the normal expectations, II," Proc. Jpn. Acad. **49**, 252–256 (1973).
[39] J. Bergh and J. Lofstrom, *Interpolation Spaces: An Introduction, Grundlehren der mathematischen Wissenschaften* (Springer-Verlag, 1976).

[40] I. I. Hirschman, "A convexity theorem for certain groups of transformations," J. Anal. Math. **2**, 209–218 (1952).
[41] F. Hiai, R. König, and M. Tomamichel, "Generalized log-majorization and multivariate trace inequalities," Ann. Henri Poincaré **18**, 2499–2521 (2017).
[42] E. C. Lance, *Hilbert C^*-Modules: A Toolkit for Operator Algebraists*, London Mathematical Society Lecture Note Series (Cambridge University Press, 1995).
[43] V. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge Studies in Advanced Mathematics (Cambridge University Press, 2003).
[44] M. Junge and D. Sherman, "Noncommutative L_p modules," J. Operator Theory **53**, 3–34 (2005); available at https://www.jstor.org/stable/24715642.
[45] K. M. R. Audenaert and N. Datta, "α-z-Rényi relative entropies," J. Math. Phys. **56**, 022202 (2015).
[46] H. Zhang, "Equality conditions of data processing inequality for α-z Rényi relative entropies," J. Math. Phys. **61**, 102201 (2020).
[47] H. Zhang, "From Wigner-Yanase-Dyson conjecture to Carlen-Frank-Lieb conjecture (new title)," Adv. Math. **365**, 107053 (2020).
[48] S. Chehade, "Saturating the data processing inequality for α–z renyi relative entropy," arXiv:2006.07726 [math-ph] (2020).
[49] M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr, and M. Tomamichel, "On quantum Rényi entropies: A new generalization and some properties," J. Math. Phys. **54**, 122203 (2013).
[50] M. M. Wilde, A. Winter, and D. Yang, "Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy," Commun. Math. Phys. **331**, 593–622 (2014).
[51] A. Jencova, "Rényi relative entropies and noncommutative L_p-spaces," Ann. Henri Poincaré **19**, 2513–2542 (2018).
[52] D. Petz, "Quasi-entropies for finite quantum systems," Rep. Math. Phys. **23**, 57–65 (1986).
[53] S. Beigi, "Sandwiched Rényi divergence satisfies data processing inequality," J. Math. Phys. **54**, 122202 (2013).
[54] M. Junge and Z.-J. Ruan, "Decomposable maps on non-commutative L_p-spaces," *Operator Algebras, Quantization, and Noncommutative Geometry* (American Mathematical Society, Providence, RI, 2004), pp. 355–381.
[55] M. Suzuki, "Convergence of exponential product formulas for unbounded operators," Rev. Math. Phys. **08**, 487–502 (1996).
[56] M. Suzuki, "Compact exponential product formulas and operator functional derivative," J. Math. Phys. **38**, 1183–1196 (1997).
[57] E. Ricard, "Hölder estimates for the noncommutative mazur maps," Archiv der Mathematik **104**(1), 37–45 (2015).
[58] J. A. Tropp, "From joint convexity of quantum relative entropy to a concavity theorem of lieb," Proc. Am. Math. Soc. **140**, 1757–1760 (2012).
[59] M. Junge and Q. Xu, "Noncommutative Burkholder/Rosenthal inequalities," Ann. Probab. **31**, 948–995 (2003).
[60] D. n. Petz, "A variational expression for the relative entropy," Commun. Math. Phys. **114**, 345–349 (1988).
[61] M. Berta, O. Fawzi, and M. Tomamichel, "On variational expressions for quantum relative entropies," Lett. Math. Phys. **107**, 2239–2265 (2017).
[62] G. K. Pedersen and M. Takesaki, "The Radon-Nikodym theorem for von Neumann algebras," Acta Math. **130**, 53–87 (1973).
[63] M. Junge, Z.-J. Ruan, and D. Sherman, "A classification for 2-isometries of noncommutative L_p-spaces," Isr. J. Math. **150**, 285 (2005).
[64] I. Csiszar, "The method of types [information theory]," IEEE Trans. Inf. Theory **44**, 2505–2523 (1998), conference Name: IEEE Transactions on Information Theory.
[65] L. Gao, M. Junge, and N. LaRacuente, "Capacity estimates via comparison with TRO channels," Commun. Math. Phys. **364**, 83–121 (2018).
[66] M. Headrick and T. Takayanagi, "Holographic proof of the strong subadditivity of entanglement entropy," Phys. Rev. D **76**, 106013 (2007).
[67] M. Headrick, "Lectures on entanglement entropy in field theory and holography," arXiv:1907.08126 [cond-mat, physics:gr-qc, physics:hep-th] (2019).
[68] A. Almheiri, X. Dong, and D. Harlow, "Bulk locality and quantum error correction in AdS/CFT," J. High Energy Phys. **2015**, 163.
[69] F. Pastawski, B. Yoshida, D. Harlow, and J. Preskill, "Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence," J. High Energy Phys. **2015**, 149.