Commun. Math. Phys.
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-023-04802-w

Communications in
Mathematical Physics

Lipschitz Continuous Solutions of the Vlasov–Maxwell Systems with a Conductor Boundary Condition

Yunbai Cao[1], **Chanwoo Kim**[2]

[1] Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA.
E-mail: yc1157@math.rutgers.edu
[2] Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA.
E-mail: ckim.pde@gmail.com

Received: 11 June 2022 / Accepted: 27 June 2023

Abstract: We consider relativistic plasma particles subjected to an external gravitation force in a 3D half space whose boundary is a perfect conductor. When the mean free path is much bigger than the variation of electromagnetic fields, the collision effect is negligible. As an effective PDE, we study the relativistic Vlasov–Maxwell system and its local-in-time unique solvability in the space-time locally Lipschitz space, for several basic mesoscopic (kinetic) boundary conditions: the inflow, diffuse, and specular reflection boundary conditions. We construct weak solutions to these initial-boundary value problems and study their locally Lipschitz continuity with the aid of a weight function depending on the solutions themselves. Finally, we prove the uniqueness of a solution, by using regularity estimate and realizing the Gauss's law at the boundary within Lipschitz continuous space.

Contents

Introduction

Plasma is the most abundant form of ordinary matter in universe, being mostly associated with stars. The Sun, our nearest star, is composed of 92.1% hydrogen and 7.8% helium by number, and 0.1% of heavier elements. At the central core, hydrogen burns into

Published online: 17 August 2023

helium (so-called the p-p chain of reactions starting from the fusion of two protons into a nucleus of deuterium), which is the major reaction that drives the sun's radiance (see the famous B^2FH paper [2] for details).

At the upper atmosphere of the Sun (solar corona), electrons and protons escape from the solar corona (upper atmosphere), while traces of heavier elements have been identified [41,43,57]. This emission of plasma particles is called solar wind. The solar corona can be decomposed according to the Knudsen number of plasma. At low altitude the collision is dominant (Knudsen number $\ll$ density scale height), and hence the particles are assumed to be in hydrostatic/hydrodynamic equilibrium of MHD-type systems. Above this regime (exosphere), the collision rate between particles is assumed to be negligibly small: Knudsen number is about the density scale height of the Sun, which is an order of 100 km. These two extreme Knudsen number regimes are separated by a narrow transition regime which is called the exobase (see Fig. 1 adopted from [58][1]). Above the exobase, there have been extensive research activities on the solar wind using collisionless Boltzmann equation (e.g. linear steady Vlasov model), which has been called the exospheric solar wind models. In the early 60s, Chamberlain suggested the "solar breeze model" that the radial expansion of the solar corona results from the thermal evaporation of the hot coronal protons out of the gravitational field of the Sun [11]. In this model the ambient polarization electric field is implemented as a well-known Pannekoek-Rosseland (PR) electric field [56,60], which will be discussed at (0.11). *In this paper, we are interested in a kinetic description of the exospheric solar wind using the initial-boundary value problem of the full relativistic Vlasov–Maxwell system subjected to ambient polarization electric field, geomagnetic field, and gravitation.*

When the collision effect is negligible, the master equation describing dynamics of two species plasma (an average of 95% of the solar wind ions are protons [57]) is the relativistic Vlasov–Maxwell system (RVM)

$$\begin{aligned} \partial_t f_\pm + \hat{v}_\pm \cdot \nabla_x f_\pm + \mathfrak{F}_\pm \cdot \nabla_v f_\pm = 0, \quad &\text{in } \mathbb{R}_+ \times \Omega \times \mathbb{R}^3, \\ f_\pm(0,x,v) = f_{0,\pm}(x,v), \quad &\text{in } \Omega \times \mathbb{R}^3. \end{aligned} \tag{0.1}$$

Here $f_\pm = f_\pm(t,x,v) \geq 0$ represents the density distribution functions for the proton (+) and electron (−) respectively. The relativistic velocity is

$$\hat{v}_\pm = \frac{v}{\sqrt{m_\pm^2 + |v|^2/c^2}}, \tag{0.2}$$

where $m_\pm$ is the magnitude of the masses of protons and electrons, and c is the speed of light.

The Lorentz force $\mathfrak{F}_\pm$ consists of self-consistent field electromagnetic fields plus given polarization electric field E_{ext}, geomagnetic field B_{ext}, and gravitation:

$$\mathfrak{F}_\pm = e_\pm \left(E + E_{\text{ext}} + \frac{\hat{v}_\pm}{c} \times (B + B_{\text{ext}}) \right) - m_\pm g \mathbf{e}_3, \tag{0.3}$$

The self-consistent fields $E(t,x)$, $B(t,x)$ are coupled with $f_\pm$ through the inhomogeneous Maxwell equations

$$\begin{aligned} &\partial_t E = c\nabla_x \times B - 4\pi J, \quad \nabla_x \cdot E = 4\pi\rho, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ &\partial_t B = -c\nabla_x \times E, \quad \nabla_x \cdot B = 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \end{aligned} \tag{0.4}$$

[1] Permission to use the figure granted by https://www.agu.org/Publish-with-AGU/Publish/Author-Resources/Policies/Permission-policy#repository.

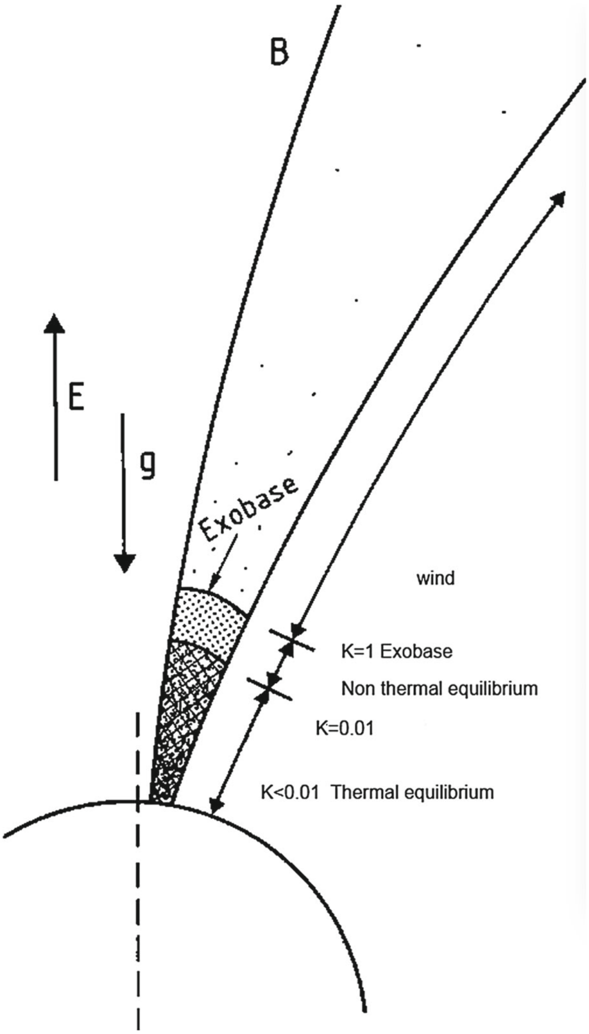

Fig. 1. The different transition regions in a stellar ionized atmosphere [58][1]

with initial conditions

$$E(0,x) = E_0(x), \quad B(0,x) = B_0(x), \quad \text{in } \Omega. \tag{0.5}$$

Here, the electric density and current are defined as

$$\rho = \int_{\mathbb{R}^3} (e_+ f_+ + e_- f_-) \mathrm{d}v, \quad J = \int_{\mathbb{R}^3} (\hat{v}_+ e_+ f_+ + \hat{v}_- e_- f_-) \mathrm{d}v. \tag{0.6}$$

Due to its importance, there have been extensive studies on the global regularity of the Cauchy problem of RVM. Here, we only overview papers relevant to our approach, and we refer to [16,44,45] for a more complete list of references. In a classical solution context, Glassey and Strauss first studied a continuation criterion of the relativistic Vlasov–Maxwell system in the whole space $\mathbb{R}^3$ in [27], using so-called the

Glassey-Strauss representation, which is a crucial tool in our analysis of this paper. It was shown that classical solution exists for all time as long as the velocity support of the particle density function f is compact. Later Klainerman and Staffilani prove the result using a different method in [36]. The work of [27] leads to substantial developments in [24–26,29–31]. Notably in [24–26], Glassey and Schaeffer proved that in the two-dimensional and two-and-a-half dimensional case, for regular initial data with compact velocity support, the system has unique global in time solution. More recently, in [44] Luk and Strain proved a new continuation criterion for the system by showing that the classical solution exists for all time if the velocity support of f is bounded after projecting to any two-dimensional plane. Then in [45], they improve the result of [24–26] the two-dimensional and two-and-a-half dimensional case by only requiring the initial data to have polynomial decay in velocity space. In addition, they showed that in the three-dimensional case, a regular solution can be extended by assuming a bound on a certain moment of f. In a weak solution context, the global weak solutions of the RVM system were obtained in [16] using a velocity averaging lemma, and the questions of its uniqueness and global regularity are still open.

In many applications of plasma models, the particles are in contact with a different phase through a sharp interface, which can be considered as a (either solid or moving) boundary. In the solar wind model, under the top of exobase, the space is filled with fully ionized plasma particles with a very short mean free path, which can be considered as a perfect conductor. We set the altitude of the exobase $x_3 = 0$ and consider the upper half space

$$\Omega = \mathbb{R}^3_+ := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}. \tag{0.7}$$

At the top of the exobase, we assume a perfect conductor boundary condition for the self-consistent electromagnetic fields. Denote by n the outward unit normal of Ω (which is $n = -\mathbf{e}_3$ for our case); $[V]$ the jump of V across $\partial\Omega$: $[V](x_1, x_2) = \lim_{x_3\downarrow 0} V(x_1, x_2, x_3) - \lim_{x_3\uparrow 0} V(x_1, x_2, x_3)$. Then from $\partial_t B = -\nabla_x \times E$ and $\nabla_x \cdot B = 0$, we derive the jump conditions (see [13] for the details)

$$n \times [E] = 0, \quad n \cdot [B] = 0.$$

In other words, the tangential electric fields E_1, E_2, and the normal magnetic field B_3 are continuous across the interface $\partial\Omega$. Therefore, we obtain boundary conditions for a perfect conductor of the solutions (E, B) to (0.17)–(0.23).

$$E_1 = E_2 = 0, \ B_3 = 0, \ \text{on } \mathbb{R}_+ \times \partial\Omega. \tag{0.8}$$

The initial-boundary value problem of the Vlasov–Maxwell system with the perfect conductor boundary condition has been studied by Guo in [18] for general domains with boundary. By approximating the phase space via a sequence of domains and linear systems and using the compactness result of [16], he establishes a global existence of weak solutions for RVM with the perfect conductor boundary condition for various boundary conditions of f. The regularity question is highly nontrivial since the stability of the ballistic trajectory depends on the sign of the normal component of the field at the boundary. As a matter of fact, in [19], he constructs an example of the RVM system such that the solution immediately does not belong to C^1. Under the favorite sign condition of the field at the boundary, in [20], he constructs regular solutions for a 1D model of the Vlasov–Maxwell system on a half line. In the proof, he introduces an important weight function α and establishes a crucial velocity lemma. This technique motivates us to define the kinetic weight function in (0.15) and build a weighted regularity estimate for f along with it for the RVM system with boundary. We will discuss the role of kinetic weight in Definition 1 and its remarks. There are several interesting related research lines. Here we only list some of them for readers' convenience: stationary solutions of the RVM [59], initial-boundary value problem of Maxwell system in time-dependent domains [14], an inverse boundary value problem of Maxwell's equations [55], a dielectric boundary problem [3], and a non-perfect conductor boundary problem [15,51]. For other relevant studies, we refer to [12,17,23,49,52,54,61] and the reference therein.

Now we consider the gravitation (and the gravitation constant $g > 0$), an ambient polarization electric field E_{ext}, and geomagnetic field B_{ext} near the exobase. As we are only interested in the dynamics near the exobase, we can assume that E_{ext} and B_{ext} take forms of

$$E_{\text{ext}} = E_e \mathbf{e}_3 \text{ and } B_{\text{ext}} = B_e \mathbf{e}_3, \tag{0.9}$$

where E_e, B_e are the magnitude of the fields, and $\mathbf{e}_3$ is a unit vector $(0\ 0\ 1)^T$. In the early 1920s Pannekoek and Rosseland independently calculated an electric potential of a Sun-like gaseous star, which consists of fully ionized matter in isothermal equilibrium (temperature$=T$). Recall that, for the two-species model, we have $e_\pm$ and $m_\pm$ be the charge and mass of negative/positive ions, respectively. Pannekoek and Rosseland conclude that the gravitational constant $g > 0$ and the polarization electrical field $E_{\text{ext}} = E_e(0\ 0\ 1)^T$ satisfy the following condition at the exobase:

$$\frac{E_e}{-g} = -\frac{m_+ - m_-}{e_+ + |e_-|}. \tag{0.10}$$

For electron/proton gaseous star ($m_+ > 1800 m_-$ and $e_+ = 1 = -e_-$), this identity implies that the polarization electrical field is *upward*. Moreover, from $e_+ E_e = \frac{1}{2}(m_+ - m_-)g$, we derive the *Pannekoek-Rosseland condition*:

$$m_+ g > 2 e_+ E_e. \tag{0.11}$$

This condition crucially implies that the gravitation effect dominates ambient electromagnetic one so that the acceleration of particles would be *attractive to the boundary*. We will explain the importance of the Pannekoek-Rosseland condition qualitatively when defining the kinetic weight in Definition 1 and its remarks. It might be worth mentioning another important physical domain with boundary in plasma physics which is a fusion reactor such as tokamak. In a lab on the earth fusion can happen above 100 million Celsius (much higher than Sun's temperature) and no boundary materials can effectively withstand direct contact with such heat. To solve this problem, scientists have devised plasma held inside a doughnut-shaped magnetic field: if a confining external magnetic field is large enough, the plasma is localized away from the boundary. In other words, the acceleration of particles due to this external electromagnetic field is *repellent to the boundary*, which is the exact opposite effect of gravitation/polarization electric field satisfying the Pannekoek-Rosseland condition. In some sense, one can reduce the initial-boundary value problem to the Cauchy problem when the confining magnetic field is dominant [34,64].

Finally we consider a boundary condition of density distribution of plasma particles on the incoming phase boundary $\gamma_- := \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : v_3 > 0\}$. In addition let $\gamma_+ := \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : v_3 < 0\}$ and $\gamma_0 := \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : v_3 = 0\}$ denote the outgoing phase boundary and grazing phase boundary, respectively. In this paper we consider the following three simple physical boundary conditions, which were originally proposed by James Clerk Maxwell [50]. An inflow boundary condition (inflow BC) is given by a prescribed date $g_\pm : \mathbb{R}_+ \times \gamma_- \to \mathbb{R}$:

$$f_\pm(t, x, v) = g_\pm(t, x, v), \quad \text{on } \mathbb{R}_+ \times \gamma_-. \tag{0.12}$$

A diffuse boundary condition (diffuse BC) takes the form of

$$f_\pm(t, x, v) = \frac{1}{2\pi T_w^2} e^{-\frac{|v|^2}{2T_w}} \int_{u_3<0} -f_\pm(t, x, u)\hat{u}_{\pm,3} du, \quad \text{on } \mathbb{R}_+ \times \gamma_-, \tag{0.13}$$

where $T_w(x)$ is a positive smooth prescribed boundary temperature. As we are interested in a short-time dynamics from now on we assume the isothermal case $T_w(x) = 1$ for the sake of simplicity. We also have a generalized diffuse boundary condition [10]. Finally, a specular reflection boundary condition (specular BC) is given by

$$f_\pm(t, x, v_\parallel, v_3) = f_\pm(t, x, v_\parallel, -v_3) \quad \text{on } \mathbb{R}_+ \times \gamma_-. \tag{0.14}$$

For the diffuse BC and specular BC, the boundary conditions enjoy a null flux condition: $\int_{\mathbb{R}^3} f_\pm(t, x, v)\hat{v}_{\pm,3} dv = 0$ for $x \in \partial\Omega$, which implies a conservation of mass for a strong solution of RVM.

One of the advantages of kinetic theory is that we can devise different boundary conditions from the microscopic interaction law of particles and boundaries. For example in a recent solar wind model, a non-Maxwellian inflow boundary condition is used to explain coronal heating phenomena [58].

Stability of the RVM system has also been studied extensively. Notably, in [46–48], the authors study the spatially inhomogeneous equilibrium in domains without boundary. A sharp criterion for spectral stability was given in [48] and the nonlinear stability is studied in [47]. In the case of bounded domains, stability analysis of the system was carried out in [53] when the domain is a 2D disk with perfect conducting boundary which reflects particles specularly. And then later the authors consider the case when the domain is a 3D solid torus. More recently, the stability analysis was generalized to any axisymmetric domains in [63].

Main theorems. As a major goal of this paper, we construct a weak solution of RVM in a locally Lipschitz space, in which we can guarantee a *uniqueness!* The major difficulty is that the density distribution $f_\pm$ is singular at the grazing set γ_0 in general. Notably, a solution is discontinuous at the grazing set γ_0 [37], and a derivative $\nabla_{x,v} f_\pm$ blows up at γ_0 [20,21]. If a trajectory emanating from the grazing set can propagate inside the domain (either if the domain is not convex or the field is repellent to the boundary) then such singularities propagate inside the domain and the regularity of solutions become restrictive [19,22,37,39]. In particular, following the proof of Guo-Kim-Tonon-Trescases [21], we can deduce that a global $H^1(\Omega)$ bound is not possible for solutions $f_\pm$ of (0.17)–(0.23) & (0.24), in general. Unfortunately, such low regularity hardly guarantees uniqueness due to the nonlinear term $\mathfrak{F}_\pm \cdot \nabla_v f_\pm$. To overcome such obstacle, we adopt a kinetic weight function $\alpha_\pm(t, x, v)$ in the regularity estimate of a locally Lipschitz space, inspired by [19,21].

Definition 1 (*Kinetic weight*). Recall the Lorentz force $\mathfrak{F}_\pm$ in (0.19) with $(E_{\rm ext}, B_{\rm ext})$ in (0.20). We define

$$\begin{aligned}\alpha_\pm(t, x_\parallel, x_3, v) &:= \sqrt{(x_3)^2 + (\hat{v}_{\pm,3})^2 - 2\mathfrak{F}_{\pm,3}(t, x_\parallel, 0, v)\frac{x_3}{\langle v_\pm\rangle}}\\ &= \sqrt{(x_3)^2 + (\hat{v}_{\pm,3})^2 + 2\Big(m_\pm g - e_\pm\Big(E_3 + E_e + \frac{1}{c}(\hat{v}_\pm \times B)_3\Big)_{x_3=0}\Big)\frac{x_3}{\langle v_\pm\rangle}},\end{aligned} \tag{0.15}$$

where we have used that $(\hat{v}_\pm \times B_{\rm ext})_3 = 0$ for (0.20).

Remark 1. Clearly, $\alpha_\pm$ is well-defined when $-\mathfrak{F}_{\pm,3}(t, x_\parallel, 0, v)$ is positive. In this paper, we assume this condition on the initial data at the boundary:

$$m_\pm g - e_\pm\Big(E_{0,3}(x) + E_e + \frac{1}{c}(\hat{v}_\pm \times B_0(x))_3\Big)_{x_3=0} > c_1, \quad \text{for some } \ c_1 > 0. \tag{0.16}$$

Remark 2. The condition (0.16) is not very restrictive under the Pannekoek-Rosseland condition (0.11). Note that $-\mathfrak{F}_{\pm,3}(t, x_\parallel, 0, v)$ equals

$$\underbrace{\big(m_\pm g - e_\pm E_e\big)}\frac{x_3}{\langle v_\pm\rangle} - e_\pm\Big(E_{0,3} + (\frac{\hat{v}_\pm}{c} \times B_0)_3\Big)_{x_3=0}\frac{x_3}{\langle v_\pm\rangle}.$$

If the Pannekoek-Rosseland condition (0.11) holds then the underlined coefficients of the first term, which corresponds to the net force at the equilibrium, has lower bounds:

$$\big(m_+ g - e_+ E_e\big) > \frac{m_+ g}{2}, \quad \big(m_- g - e_- E_e\big) > |e_-| E_e.$$

From (0.10), we know that both lower bounds are of the same size. If $E_{0,3}|_{x_3=0}$ and $B_{0,1}|_{x_3=0}$, $B_{0,2}|_{x_3=0}$ are smaller than such lower bounds then the condition (0.16) holds. It is the case when the initial state of plasma is either close to the neutral state or vacuum at the boundary.

Remark 3. Since being introduced in [20], such weight function α and its variants have served important roles in the regularity analysis for various kinetic equations with boundary such as [4–6,8–10,21,32]. Notably in [21], an α-weighted C^1 solution for the Boltzmann equation was constructed in convex domains. In [9], the authors used a different version of kinetic weight to construct the global strong solution to the Vlasov-Poisson-Boltzmann (VPB) system in convex domains with diffuse BC. The result was generalized to the two-species case in [5], and to the case of the presence of the external field in [4]. A generalized diffuse boundary condition (namely the Cercignani-Lampis boundary condition) for the VPB system is studied in [10]. A survey on the recent development in this direction can be found in [8].

Although the problem has been set already (RVM system (0.1)–(0.6) under the perfect conductor boundary condition of electromagnetic field (0.8)), we list them here redundantly for the sake of the reader's convenience: Let Ω the half space (0.7). We read the RVM system

$$\begin{aligned}\partial_t f_\pm + \hat{v}_\pm \cdot \nabla_x f_\pm + \mathfrak{F}_\pm \cdot \nabla_v f_\pm = 0, &\quad \text{in } \mathbb{R}_+ \times \Omega \times \mathbb{R}^3,\\ f_\pm(0, x, v) = f_{0,\pm}(x, v), &\quad \text{in } \Omega \times \mathbb{R}^3.\end{aligned} \tag{0.17}$$

with the relativistic velocity, Lorentz force, and the external fields

$$\hat{v}_\pm = v \Big/ \sqrt{m_\pm^2 + |v|^2/c^2}, \tag{0.18}$$

$$\mathfrak{F}_\pm = e_\pm \left(E + E_{\text{ext}} + \frac{\hat{v}_\pm}{c} \times (B + B_{\text{ext}}) \right) - m_\pm g \mathbf{e}_3, \tag{0.19}$$

$$E_{\text{ext}} = E_e \mathbf{e}_3 \text{ and } B_{\text{ext}} = B_e \mathbf{e}_3. \tag{0.20}$$

The Maxwell's equations solve

$$\begin{aligned} \partial_t E &= c\nabla_x \times B - 4\pi J, \quad \nabla_x \cdot E = 4\pi\rho, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ \partial_t B &= -c\nabla_x \times E, \quad \nabla_x \cdot B = 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \end{aligned} \tag{0.21}$$

$$E(0,x) = E_0(x), \quad B(0,x) = B_0(x), \quad \text{in } \Omega. \tag{0.22}$$

where the electric density and current are defined as

$$\rho = \int_{\mathbb{R}^3} (e_+ f_+ + e_- f_-) \mathrm{d}v, \quad J = \int_{\mathbb{R}^3} (\hat{v}_+ e_+ f_+ + \hat{v}_- e_- f_-) \mathrm{d}v. \tag{0.23}$$

Finally we impose the perfect conductor boundary condition

$$E_1 = E_2 = 0, \; B_3 = 0, \; \text{on } \mathbb{R}_+ \times \partial\Omega, \tag{0.24}$$

and consider the inflow BC, diffuse BC, and specular BC on the incoming boundary γ_-:

$$f_\pm(t,x,v) = g_\pm(t,x,v) \qquad \text{on } \mathbb{R}_+ \times \gamma_-, \tag{0.25}$$

$$f_\pm(t,x,v) = \frac{1}{2\pi T_w^2} e^{-\frac{|v|^2}{2T_w}} \int_{u_3<0} -f_\pm(t,x,u)\hat{u}_{\pm,3} du \qquad \text{on } \mathbb{R}_+ \times \gamma_-, \tag{0.26}$$

$$f_\pm(t,x,v_\parallel,v_3) = f_\pm(t,x,v_\parallel,-v_3) \qquad \text{on } \mathbb{R}_+ \times \gamma_-. \tag{0.27}$$

We define a notation of weak solutions to this initial-boundary value problem.

Definition 2 (*Definition 1.5 of* [18]). Let $f_\pm \in L^1_{\text{loc}}((0,T)\times\Omega\times\mathbb{R}^3) \cap L^1_{\text{loc}}((0,T)\times\gamma_+)$, $f_{0,\pm} \in L^1_{\text{loc}}(\Omega\times\mathbb{R}^3)$, $g \in L^1_{\text{loc}}((0,T)\times\gamma_-)$. Let $E, B \in L^1_{\text{loc}}((0,T)\times\Omega)$, $E_0, B_0 \in L^1_{\text{loc}}(\Omega)$. Then $(f_\pm, E, B)$ is a weak solution of (0.17)–(0.23) under the perfect conductor boundary condition of electromagnetic field (0.24) and different boundary conditions for $f_\pm$ (0.25), (0.26), or (0.27), if for any test functions

$$\begin{aligned} &\phi(t,x,v) \in C_c^\infty([0,T)\times\Omega\times\mathbb{R}^3), \text{ with} \\ &\quad \operatorname{supp}\phi \subset \{[0,T)\times\bar{\Omega}\times\mathbb{R}^3\}\backslash\{(\{0\}\times\gamma)\cup(0,T)\times\gamma_0\}, \text{ and} \\ &\Psi(t,x) \in C_c^\infty([0,T)\times\bar{\Omega};\mathbb{R}^3), \; \Phi(t,x) \in C_c^\infty([0,T)\times\Omega;\mathbb{R}^3), \end{aligned}$$

we have

$$\begin{aligned} &\iint_{\Omega\times\mathbb{R}^3} f_{0,\pm}\phi(0) dv dx + \int_0^T \iint_{\Omega\times\mathbb{R}^3} (\partial_t\phi + \nabla\phi\cdot\hat{v} + \mathfrak{F}_\pm\cdot\nabla_v\phi) f_\pm dv dx dt \\ &= \int_0^T \int_{\gamma_+} \phi f_\pm \hat{v}_3 dv dS_x + \underbrace{\int_0^T \int_{\gamma_-} \phi f_\pm \hat{v}_3 dv dS_x}_{(0.28)_{\text{BC}}}, \end{aligned} \tag{0.28}$$

and

$$\int_0^T \int_\Omega E \cdot \partial_t \Psi dxdt - \int_\Omega \Psi(0,x) \cdot E_0 dx$$
$$= -\int_0^T \int_\Omega (\nabla_x \times \Psi) \cdot Bdxdt + 4\pi \int_0^T \int_\Omega \Psi \cdot Jdxdt, \tag{0.29}$$
$$\int_0^T \int_\Omega B \cdot \partial_t \Phi dxdt + \int_\Omega \Phi(0,x) \cdot B_0 dx = \int_0^T \int_\Omega (\nabla_x \times \Phi) \cdot Edxdt, \tag{0.30}$$

and

$$\nabla \cdot E = 4\pi\rho,\ \nabla \cdot B = 0 \text{ in the sense of distributions in } (0,T) \times \Omega \times \mathbb{R}^3. \tag{0.31}$$

Here, the boundary term of (0.28) is determined by different boundary conditions:

$$(0.28)_{\mathrm{BC}} = \begin{cases} \int_0^T \int_{\gamma_-} \phi g_\pm \hat{v}_3\, dvdS_x, & \text{for the inflow BC (0.25)}, \\ \int_0^T \int_{\gamma_+} \left(-\frac{1}{2\pi T_w^2} \int_{u_3>0} e^{-\frac{|u|^2}{2T_w}} \phi(t,x,u) \hat{u}_3 du \right) \hat{v}_3 f_\pm \, dvdS_x, & \\ \quad \text{for the diffuse BC (0.26)}, & \\ \int_0^T \int_{\gamma_+} \phi(t,x,v_\parallel,-v_3) f_\pm \hat{v}_3\, dvdS_x, & \text{for the specular BC (0.27)}. \end{cases}$$

Now we state the main theorems.

Theorem 1 (inflow BC). *Suppose the initial datum $f_{0,\pm}$ satisfies, for some $\delta > 0$,*

$$\begin{aligned} &\| \langle v \rangle^{4+\delta} f_{0,\pm} \|_{L^\infty(\Omega\times\mathbb{R}^3)} + \| \langle v \rangle^{5+\delta} \nabla_{x_\parallel} f_{0,\pm} \|_{L^\infty(\Omega\times\mathbb{R}^3)} \\ &\quad + \| \langle v \rangle^{5+\delta} \alpha_\pm \partial_{x_3} f_{0,\pm} \|_{L^\infty(\Omega\times\mathbb{R}^3)} + \| \langle v \rangle^{5+\delta} \nabla_v f_{0,\pm} \|_{L^\infty(\Omega\times\mathbb{R}^3)} < \infty, \end{aligned} \tag{0.32}$$

and the inflow boundary datum $g_\pm$ satisfies

$$\begin{aligned} &\| \langle v \rangle^{5+\delta} \partial_t g_\pm \|_{L^\infty((0,\infty)\times\gamma_-)} + \| \langle v \rangle^{5+\delta} \nabla_{x_\parallel} g_\pm \|_{L^\infty(0,\infty)\times\gamma_-)} \\ &+ \| \langle v \rangle^{5+\delta} \nabla_v g_\pm \|_{L^\infty((0,\infty)\times\gamma_-)} < \infty. \end{aligned} \tag{0.33}$$

Moreover, E_0, B_0, g satisfies (0.16), and the compatibility conditions

$$\begin{aligned} \nabla \cdot E_0 = 4\pi\rho_0,\ \nabla \cdot B_0 = 0, &\qquad \text{in } \Omega, \\ E_{0,\parallel} = 0,\ B_{0,3} = 0, &\qquad \text{on } \partial\Omega, \end{aligned} \tag{0.34}$$

and

$$\|E_0\|_{C^2(\Omega)} + \|B_0\|_{C^2(\Omega)} < \infty. \tag{0.35}$$

Then there exists a unique solution $f_\pm(t,x,v)$, $E(t,x,v)$, $B(t,x,v)$ for $0 \le t \le T$ with $T \ll 1$ to RVM for the inflow BC (0.25) in the sense of Definition 2, such that,

$$\begin{aligned} \sup_{0\le t\le T} \Big(&\| \langle v \rangle^{4+\delta} \nabla_{x_\parallel} f_\pm(t) \|_{L^\infty(\Omega\times\mathbb{R}^3)} + \| \langle v \rangle^{5+\delta} \alpha_\pm \partial_{x_3} f_\pm(t) \|_{L^\infty(\Omega\times\mathbb{R}^3)} \\ &+ \| \langle v \rangle^{5+\delta} \nabla_v f_\pm(t) \|_{L^\infty(\Omega\times\mathbb{R}^3)} \Big) < \infty, \end{aligned} \tag{0.36}$$

and

$$\sup_{0\le t\le T} \left(\|\nabla_x E(t)\|_{L^\infty(\Omega)} + \|\nabla_x B(t)\|_{L^\infty(\Omega)} \right) < \infty. \tag{0.37}$$

Theorem 2 (diffuse BC). *Suppose $f_{0,\pm}$ satisfies (0.32), and E_0, B_0, g satisfy (0.16), (0.34), and (0.35). Then there exists a unique solution $f_\pm(t,x,v)$, $E(t,x,v)$, $B(t,x,v)$ for $0 \le t \le T$ with $T \ll 1$ to RVM for the diffuse BC (0.26) in the sense of Definition 2, such that both (0.36) and (0.37) hold.*

Theorem 3 (specular BC). *Suppose $f_{0,\pm}$ satisfies*

$$\|\langle v\rangle^{5+\delta} e^{\frac{C}{\sqrt{\alpha_\pm \langle v\rangle}}} \nabla_x f_{0,\pm}\|_\infty + \|\langle v\rangle^{5+\delta} e^{\frac{C}{\sqrt{\alpha_\pm \langle v\rangle}}} \nabla_v f_{0,\pm}\|_\infty < \infty, \tag{0.38}$$

for some $C > 0$. E_0, B_0 satisfy (0.16), (0.34), and (0.35). Then there exists a unique solution $f_\pm(t,x,v)$, $E(t,x,v)$, $B(t,x,v)$ for $0 \le t \le T$ with $T \ll 1$ to the RVM with system (0.27) in the sense of Definition 2, such that

$$\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta}\nabla_x f_\pm(t)\|_{L^\infty(\Omega\times\mathbb{R}^3)} + \|\langle v\rangle^{4+\delta}\nabla_v f_\pm(t)\|_{L^\infty(\Omega\times\mathbb{R}^3)}\Big) < \infty, \tag{0.39}$$

and (0.37) holds.

Remark 4. A large class of functional spaces satisfy the condition (0.32). Indeed any function, whose weak derivatives $\nabla_{x,v} f$ are bounded in $L^\infty(\Omega\times\mathbb{R}^3)$, and decays fast enough as $|v| \to \infty$, belongs to the space of (0.32). Actually $\partial_{x_3} f_{0,\pm}$ is allowed to be singular at the grazing set γ_0.

Remark 5. As far as the authors know, Theorem (1)–(3) provide the first unique solvability of the RVM system when the physical boundary has a global effect (cf. [34,64]). The time span T of existence depends on the size of the initial data f_0, E_0, B_0 and their derivatives, and c_1 in (0.16).

Remark 6. We prove the weighted regularity estimate using a Lagrangian approach of [21]. We take a direct differentiation to the Lagrangian solution along the generalized characteristics. The generalized characteristics depend on the boundary condition.

Remark 7. Here we require that the initial data f_0 to vanish exponentially fast towards the grazing set (0.39). This allows us to establish the regularity estimate for specular BC (0.39) with the $W^{1,\infty}$ field. We prove this theorem in Sect. 7.

Difficulties and key ingredients The problem in this paper is a coupled system of hyperbolic equations and kinetic Vlasov equation with characteristic boundary condition: the problem suffers *a loss of derivative of wave equation* (cf. [14]) and *the boundary singularity of Vlasov equation* (cf. [20,21]) at the same time. The key difficulty in the construction of a *unique* solutions of the RVM system with physical boundary conditions is a control of the nonlinear term $\mathfrak{F}_\pm \cdot \nabla_v f_\pm$. We overcome this difficulty by establishing a regularity estimate for both the electromagnetic field E and B, and the density distribution $f_\pm$ using the Glassey-Strauss representation. We detail several key difficulties along the road. In this section and the rest of the paper, for sake of simplicity, we will consider the one-species relativistic Vlasov–Maxwell-system since the analysis of the two-species case does not process essential difference from that of the one species case:

$$\begin{aligned} \partial_t f + \hat{v}\cdot\nabla_x f + \mathfrak{F}\cdot\nabla_v f &= 0, \text{ in } \mathbb{R}_+\times\Omega\times\mathbb{R}^3, \\ f(0,x,v) &= f_0(x,v), \text{ in } \Omega\times\mathbb{R}^3, \end{aligned} \tag{0.40}$$

We also set all the charge and mass of the plasma f equal to one, so here $\hat{v} = \frac{1}{\sqrt{1+|v|^2}}$, and

$$\mathfrak{F} = \left(E + E_{\text{ext}} + \frac{\hat{v}}{c} \times (B + B_{\text{ext}}) \right) - g\mathbf{e}_3, \tag{0.41}$$

and E, B satisfies the Maxwell equations (0.21), with

$$\rho = \int_{\mathbb{R}^3} f dv, \; J = \int_{\mathbb{R}^3} \hat{v} f dv. \tag{0.42}$$

•*Wave equation and the Neumann BC* From the Maxwell's equations (0.21), we have the inhomogenous wave equations for E and B:

$$\partial_t^2 E - \Delta_x E = -4\pi \nabla_x \rho - 4\pi \partial_t J, \qquad \text{in } \mathbb{R}_+ \times \Omega, \tag{0.43}$$

$$\partial_t^2 B - \Delta_x B = 4\pi \nabla_x \times J, \qquad \text{in } \mathbb{R}_+ \times \Omega, \tag{0.44}$$

with the boundary condition $E_1 = E_2 = 0$, $B_3 = 0$ on $\partial\Omega$ in (0.24) and the initial condition

$$\begin{aligned} E|_{t=0} &= E_0, \quad \partial_t E|_{t=0} = \partial_t E_0 := \nabla_x \times B_0 - 4\pi J|_{t=0}, \text{ in } \Omega, \\ B|_{t=0} &= B_0, \quad \partial_t B|_{t=0} = \partial_t B_0 := -\nabla_x \times E_0, \text{ in } \Omega. \end{aligned} \tag{0.45}$$

The boundary conditions of E_3, $B_\parallel$ components are not a priori given, which causes some trouble handling weak solutions based on the Glassey-Strauss representation. Of course, if the fields $E, B \in C^2(\Omega) \cap C^1(\bar{\Omega})$, and $\rho \in C^1(\Omega) \cap C(\bar{\Omega})$, then from the Maxwell's equations (0.21) and the perfect conductor boundary condition (0.24), we deduce the Neumann boundary condition

$$\partial_3 E_3 = 4\pi\rho, \partial_3 B_1 = 4\pi J_2, \; \partial_3 B_2 = -4\pi J_1 \quad \text{on } \mathbb{R}_+ \times \partial\Omega. \tag{0.46}$$

One of the main goals in this paper is to equip a solution space in which we can realize the Neumann BC (0.46) in a suitable sense and hence guarantee a unique solvability. Indeed we can justify the Neumann boundary condition (0.46) in a weak solution formulation testing against smooth test functions that do not vanish at the boundary $\partial\Omega$ in Lemma 1, and prove the uniqueness of weak solution. We then carefully show in Lemma 5 that, assuming the continuity equation

$$\nabla \cdot J + \partial_t \rho = 0,$$

and some compatibility conditions of the initial datum (0.34), the weak solution of wave equations with boundary conditions (0.43)–(0.46) is indeed a solution to the Maxwell equations. This equivalence allows us to solve the RVM system by looking for solutions to the wave equations with boundary conditions (0.43), (0.44), which is the first step of our analysis.

•*Glassey-Strauss representation in the half space* The wave equations (0.43), (0.44) suffer from the “loss of derivatives” of (E, B) with respect to the regularity of the source terms ρ and J. As Glassey mentions in his book [28], the key idea of the Glassey-Strauss representation is replacing the derivatives ∂_t, ∇_x by a geometric operator T in (0.48) and a kinetic transport operator S in (0.49):

$$\partial_t = \frac{S - \hat{v} \cdot T}{1 + \hat{v} \cdot \omega}, \quad \partial_i = \frac{\omega_i S}{1 + \hat{v} \cdot \omega} + \left(\delta_{ij} - \frac{\omega_i \hat{v}_j}{1 + \hat{v} \cdot \omega} \right) T_j, \tag{0.47}$$

while, for $\omega = \omega(x, y) = \frac{y-x}{|y-x|}$,

$$T_i := \partial_i - \omega_i \partial t, \tag{0.48}$$

$$S := \partial_t + \hat{v} \cdot \nabla_x. \tag{0.49}$$

Note that

$$T_j f(t - |y - x|, y, v) = \partial_{y_j}[f(t - |y - x|, y, v)], \tag{0.50}$$

which is a tangential derivative along the surface of a backward light cone [28]. On the other hand, the Vlasov equation (0.40) implies that

$$Sf = -\nabla_v \cdot [(E + E_{\text{ext}} + \hat{v} \times (B + B_{\text{ext}}) - g\mathbf{e}_3) f]. \tag{0.51}$$

Therefore, in [27,28], they can take off the derivatives T_j, S from f using the integration by parts within the Green’s formula of (0.43)–(0.44) by connecting the source terms to f via (0.42).

For our problem, we derive the Glassey-Strauss representation in the presence of a boundary. For $E_\parallel$ and B_3, to solve the wave equation with the Dirichlet boundary condition (0.24), we employ the odd extension of the initial data and the forcing term into the lower half space $\mathbb{R}^3_- := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\}$. Solving the whole space wave equation with the oddly extended data gives us the solution that satisfies (0.24). On the other hand, for E_3 and $B_\parallel$, we decompose the solution into two parts: one with the Neumann boundary condition of (0.46) and the zero forcing term and initial data, and the other part satisfying (0.43)–(0.45) with the zero Neumann boundary condition. For the first part, we find out the expression using the fundamental solution of the Helmholtz equation. And for the second part, we use the even extension to get the solution with the zero Neumann boundary condition.

For the expression in the lower half space, we introduce

$$\bar{\omega} = \left[\omega_1 \ \omega_2 \ -\omega_3 \right]^T, \tag{0.52}$$

and

$$\begin{aligned} \bar{T}_3 f &= -\partial_{y_3}[f(t - |y - x|, y_\parallel, -y_3, v)] = \partial_{y_3} f - \bar{\omega}_3 \partial_t f, \\ \bar{T}_i f &= \partial_{y_i}[f(t - |y - x|, y_\parallel, -y_3, v)] = \partial_{y_i} f - \bar{\omega}_i \partial_t f \ \text{ for } \ i = 1, 2. \end{aligned} \tag{0.53}$$

Then through direct calculation we obtain an explicit expressions of E and B by solving the wave equations (0.43)–(0.45) under the boundary condition (0.24) and (0.46) in Proposition 1 and Proposition 2 respectively.

•*Weighted $W^{1,\infty}$ estimate of f and the regularity of the fields* An intrinsic feature of the transport equation in domains with boundary is the singular behavior of its derivatives: the solution of a linear transport equation with physical boundaries is known to not have high regularity [21]. However, to get the unique solvability, one must control $\nabla_v f$ effectively. This in turn requires the control of spatial derivatives of the distribution function and the spatial derivatives of electromagnetic fields. But due to the characteristic boundary, f does not have high enough regularity to achieve the required regularity for E and B directly from the hyperbolic equations. We explain the ideas of the paper and the methods we use to overcome the difficulties in the rest of this section and the next.

Let's consider a solution of the RVM system with inflow boundary data (0.40)–(0.42), (0.25). The *characteristics (trajectory)* is determined by the Hamilton ODEs,

$$\begin{aligned} &\frac{d}{ds}X(s;t,x,v) = \hat{V}(s;t,x,v),\\ &\frac{d}{ds}V(s;t,x,v) = \mathfrak{F}(s, X(s;t,x,v), V(s;t,x,v)). \end{aligned} \tag{0.54}$$

We define *the backward exit time* $t_{\mathbf{b}}(t,x,v)$ as

$$t_{\mathbf{b}}(t,x,v) := \sup\{s \geq 0 : X(\tau;t,x,v) \in \Omega \quad \text{for all} \quad \tau \in (t-s,t)\}. \tag{0.55}$$

Furthermore, we define $x_{\mathbf{b}}(t,x,v) := X(t - t_{\mathbf{b}}(t,x,v);t,x,v)$, and $v_{\mathbf{b}}(t,x,v) := V(t - t_{\mathbf{b}}(t,x,v);t,x,v)$. We can solve the Vlasov equation (0.40) with the inflow boundary condition (0.25) as

$$\begin{aligned} f(t,x,v) = g(t &- t_{\mathbf{b}}(t,x,v), X(t - t_{\mathbf{b}}(t,x,v);t,x,v),\\ &V(t - t_{\mathbf{b}}(t,x,v);t,x,v)) \text{ for } t \geq t_{\mathbf{b}}(t,x,v). \end{aligned}$$

From some direct computations (see (5.8) and (5.9)), the derivatives of f have a bound in general as

$$\nabla_x f(t,x,v) \sim \nabla_x t_{\mathbf{b}}(t,x,v),$$

which can be further bounded from the direct computation of the characteristics (see (5.5)) as

$$\nabla_x t_{\mathbf{b}}(t,x,v) \lesssim \frac{1 + \sup_{0\leq s\leq t} \|\nabla_x \mathfrak{F}(s)\|_\infty}{\hat{v}_{\mathbf{b},3}}. \tag{0.56}$$

The formation of such singularity motivates us to introduce the following notion. As a first order approximation of $\hat{v}_{\mathbf{b},3}(t,x,v)$, we define the kinetic weight

$$\begin{aligned} \alpha(t,x_\parallel,x_3,v) &:= \sqrt{(x_3)^2 + (\hat{v}_3)^2 - 2\mathfrak{F}_3(t,x_\parallel,0,v)\frac{x_3}{\langle v\rangle}}\\ &= \sqrt{(x_3)^2 + (\hat{v}_3)^2 - 2\left(E_3(t,x_\parallel,0) + E_e + (\hat{v}\times B)_3(t,x_\parallel,0) - g\right)\frac{x_3}{\langle v\rangle}}. \end{aligned} \tag{0.57}$$

Note that $\alpha = \hat{v}_3$ on $\partial\Omega$. Crucially α is effectively invariant along the characteristics, thanks to the velocity lemma (Lemma 8). This allows us to prove an α-weighted bound on the derivatives of f, more specifically, we prove that for any $0 < \delta < 1$,

$$\langle v\rangle^{5+\delta}\alpha\nabla_x f(t) \in L^\infty(\Omega\times\mathbb{R}^3), \text{ for } 0 < t < T. \tag{0.58}$$

On the other hand, due to the generic singularity (0.56), to close the estimate we need to bound $\nabla_x E$, $\nabla_x B$ by $\langle v\rangle^{5+\delta}\alpha\nabla_x f$ in the generalized Glassey-Strauss representation. By taking the derivatives directly to the formulas of E and B, in Lemma 7 we achieve the bound

$$|\nabla_x E| + |\nabla_x B| \lesssim \text{“ initial data ”} \\ + \sup_{0\leq t\leq T} \|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t)\|_\infty \int_{\Omega\cap\{|x|<T\}}\int_{\mathbb{R}^3} \frac{1}{\langle v\rangle^{4+\delta}\alpha(t,x,v)} dvdx.$$

Then from the local-to-nonlocal estimate (Lemma 9), we derive

$$\int_{\mathbb{R}^3} \frac{1}{\langle v\rangle^{4+\delta}\alpha(t,x,v)} dv \lesssim \ln(1+\frac{1}{x_3}) \in L^1_{\text{loc}}(\Omega),$$

and we are able to close the estimate and conclude $E, B \in W^{1,\infty}((0,T)\times\Omega)$.

In the construction of solution, we study a sequence of solutions (f^ℓ, E^ℓ, B^ℓ) and pass the limit. To achieve a uniform estimate, we use a weight $\alpha^\ell(t,x,v)$, which is the same form of (0.57) with exchanging E, B to E^ℓ, B^ℓ. Since α^ℓ depends on E^ℓ, B^ℓ and hence f^ℓ, when passing the limit of the sequence $\{\alpha^{\ell-1}\partial_{x_3} f^\ell\}_{\ell=1}^\infty$, we need to verify that

$$\alpha^{\ell-1}\partial_{x_3} f^\ell \overset{*}{\rightharpoonup} \alpha\partial_{x_3} f \text{ in } L^\infty((0,T)\times\Omega\times\mathbb{R}^3). \tag{0.59}$$

Obviously this convergence is nontrivial since the norm itself is nonlinear. To obtain this, we observe that since we can bound ∇E^ℓ, ∇B^ℓ pointwisely, they have traces and a strong convergence

$$E^\ell|_{\partial\Omega} \to E|_{\partial\Omega},\ \ B^\ell|_{\partial\Omega} \to B|_{\partial\Omega}.$$

Thus we can prove that a strong convergence $\alpha^{\ell-1} \to \alpha$ in L^∞. On the other hand, using a positive lower bound of α^ℓ away from the grazing set, we obtain a upper uniform bound of $\partial_{x_3}\alpha^{\ell-1} - \partial_{x_3}\alpha$ locally. We then achieve the desired convergence (0.59) using uniform bound of f^ℓ.

Among other boundary conditions, we find that the specular boundary condition suffers most. Due to the lack of higher regularity of the fields (e.g. compare to [6] where the field is C^2), we can only derive an exponential-in-α singularity of the derivative of trajectory

$$\begin{aligned} |\partial_{\mathbf{e}} X_{\mathbf{cl}}(s;t,x,v)| &\leq C_1\langle v\rangle e^{\frac{C_1}{\sqrt{\alpha(t,x,v)\langle v\rangle}}}, \\ |\partial_{\mathbf{e}} V_{\mathbf{cl}}(s;t,x,v)| &\leq C_1\langle v\rangle e^{\frac{C_1}{\sqrt{\alpha(t,x,v)\langle v\rangle}}}. \end{aligned} \tag{0.60}$$

Clearly, such a strong singularity can be harmful in our analysis based on the Glassey-Strauss representation. We study the specular BC problem with great care. Details are presented in Sect. 7.

•*A Priori L^∞ estimate of $\nabla_v f$ and uniqueness* A simple Gronwall's inequality implies

$$\begin{aligned}\|\langle v\rangle^{4+\delta}(f-g)(t)\|_\infty &\lesssim \|\langle v\rangle^{4+\delta}(f-g)(0)\|_\infty\\ &\quad+\sup_{0\le t\le T}\|\langle v\rangle^{4+\delta}\nabla_v f(t)\|_\infty\int_0^t\|(E_f-E_g+B_f-B_g)(s)\|_\infty.\end{aligned}\tag{0.61}$$

For constructing a solution and proving its uniqueness, we establish an effective stability estimate of the difference of solutions $f-g$, and E_f-E_g, B_f-B_g. To control the nonlinear term of the equation of $f-g$, $(E_f-E_g+B_f-B_g)\cdot\nabla_v f$, we establish an estimate of $\nabla_v f$. From the Lagrangian view point along the characteristics (0.54), we have

$$\begin{aligned}&\nabla_v f(t,x,v)\\ &\quad\sim\nabla_x f_0(X(0;t,x,v),V(0;t,x,v))\cdot\nabla_v X(0;t,x,v)\\ &\qquad+\nabla_v f_0(X(0;t,x,v),V(0;t,x,v))\cdot\nabla_v V(0;t,x,v).\end{aligned}$$

Clearly effective control of $\nabla_x E$, $\nabla_x B$ is necessary. Now we crucially use our estimate of $\alpha\partial_{x_3} f$ to obtain bounds for $\nabla_x E$, $\nabla_x B$ in the Glassey-Strauss representation, which in turn gives the a priori L^∞ estimate of $\nabla_v f$. Using this $\nabla_v f$-bound and a pointwise bound from the Glassey-Strauss representation (Lemma 6)

$$\|E_{f-g}(t)\|_\infty+\|E_{f-g}(t)\|_\infty\lesssim\sup_{0\le s\le t}\|\langle v\rangle^{4+\delta}(f-g)(s)\|_\infty,$$

we achieve an L^∞ stability as $\sup_{0\le s\le t}\|\langle v\rangle^{4+\delta}(f-g)(s)\|_\infty\lesssim e^{Ct}\|\langle v\rangle^{4+\delta}(f-g)(0)\|_\infty$.

1. Uniqueness of the Maxwell Equations

In this section, we consider the uniqueness of solution to the Maxwell equations in $(0,T)\times\Omega$ in a presence of free charge:

$$\partial_t E=\nabla_x\times B-4\pi J,\tag{1.1}$$
$$\partial_t B=-\nabla_x\times E,\tag{1.2}$$
$$\nabla_x\cdot E=4\pi\rho,\tag{1.3}$$
$$\nabla_x\cdot B=0,\tag{1.4}$$

with initial condition:

$$E(0,x)=E_0(x),\ \ B(0,x)=B_0(x)\text{ in }\Omega,\tag{1.5}$$

and the perfect conductor boundary condition:

$$E_1=E_2=0\text{ on }\partial\Omega,\ \ B_3=0\text{ on }\partial\Omega.\tag{1.6}$$

It is worth to recall that the boundary conditions for E_3 and B_1, B_2 are not given originally.

Definition 3. For given E_0, B_0, ρ, j, we say functions

$$E(t,x), B(t,x) \in W^{1,\infty}((0,T)\times\Omega), \tag{1.7}$$

is a solution to the equations (1.1)–(1.6) if (1.1)–(1.4) holds almost everywhere in $(0,T)\times\Omega$, and (1.5), (1.6) holds in the sense of trace.

Remark 8. Traces of $W^{1,\infty}((0,T)\times\Omega)$ are well-defined in a classical sense since any uniformly continuous function in space and time can be extended up to $[0,T]\times\bar{\Omega}$.

The goal of this section is to prove the following uniqueness result:

Theorem 4. *Suppose $E_0(x), B_0(x) \in W^{1,p}(\Omega)$, and $\nabla_x\rho, \nabla_x J, \partial_t J \in L^\infty((0,T); L^p_{loc}(\Omega))$ for some $p>1$, and*

$$\nabla\cdot J = -\partial_t\rho. \tag{1.8}$$

Then a solution $E(t,x), B(t,x) \in W^{1,\infty}((0,T)\times\Omega)$ to the equations (1.1)–(1.6) in the sense of Definition 3 is unique.

The key of proof is to realize E_3, B_1, B_2 as weak solutions of inhomengenous wave equations with the Neumann boundary condition from a weak solution of Maxwell equations in the sense of Definition 3. For the general theory of hyperbolic equations with boundary, we refer to [33,40,42].

Definition 4. Given any $u_0, u_1 : \Omega\to\mathbb{R}$, $G:(0,T)\times\Omega\to\mathbb{R}$, and $g:(0,T)\times\partial\Omega\to\mathbb{R}$, we define a function $u(t)\in W^{1,p}(\Omega)$ for $t\in(0,T)$, $p>1$ to be a weak solution of the inhomengenous wave equation with Neumann boundary condition:

$$\begin{aligned} \partial_t^2 u - \Delta_x u = G, \quad -\partial_{x_3}u(t,x)|_{\partial\Omega} = g,\\ u(0,x)=u_0,\ \partial_t u(0,x) = u_1, \end{aligned} \tag{1.9}$$

if for any $\phi\in C_c^\infty([0,T)\times\bar{\Omega})$, we have

$$\begin{aligned} \langle u,\phi\rangle_N := &\int_\Omega (u_1(x)\phi(0,x) - u_0(x)\partial_t\phi(0,x))dx\\ &+\int_0^T\int_\Omega u(t,x)\left(\partial_t^2\phi(t,x)-\Delta_x\phi(t,x)\right)dxdt\\ &-\int_0^T\int_{\partial\Omega} u(t,x_\parallel,0)\partial_{x_3}\phi(t,x_\parallel,0)dx_\parallel dt\\ &+\int_0^T\int_{\partial\Omega} g(t,x_\parallel)\phi(t,x_\parallel,0)dx_\parallel dt - \int_0^T\int_\Omega G\phi\, dxdt = 0, \end{aligned} \tag{1.10}$$

and each terms in (1.10) are all bounded. Note that since $u(t)\in W^{1,p}(\Omega)$, it has a trace $u(t)|_{\partial\Omega}\in L^p(\partial\Omega)$. Also, note that $\text{supp}(\phi)\subset[0,T)\times\bar{\Omega}$ is compact, but $\phi|_{t=0}\neq 0$, and $\phi|_{\partial\Omega}\neq 0$ in general.

We also define a weak solution of the Dirichlet boundary problem:

$$\begin{aligned} \partial_t^2 u - \Delta_x u = G, \quad u(t,x)|_{\partial\Omega} = g,\\ u(0,x)=u_0,\ \partial_t u(0,x) = u_1, \end{aligned} \tag{1.11}$$

if for any $\phi \in C_c^\infty((0,T)\times\bar{\Omega})$, with $\phi|_{\partial\Omega}=0$, we have

$$\begin{aligned}\langle u,\phi\rangle_D &:= \int_\Omega (u_1(x)\phi(0,x)-u_0(x)\partial_t\phi(0,x))dx\\ &\quad+\int_0^T\int_\Omega u(t,x)\left(\partial_t^2\phi(t,x)-\Delta_x\phi(t,x)\right)dxdt \\ &\quad-\int_0^T\int_{\partial\Omega} g(t,x_\|)\partial_{x_3}\phi(t,x_\|,0)dx_\|dt-\int_0^T\int_\Omega G\phi\, dxdt=0,\end{aligned} \tag{1.12}$$

and each terms in (1.12) are all bounded.

Lemma 1. *Suppose $E(t,x)$, $B(t,x)\in W^{1,\infty}((0,T)\times\Omega)$ is a solution to the equations (1.1)–(1.6) in the sense of Definition 3, and $E_0(x)$, $B_0(x)\in W^{1,p}(\Omega)$, $\nabla_x\rho$, $\nabla_x J$, $\partial_t J\in L^\infty((0,T);L^p_{loc}(\Omega))$ for some $p>1$. Then E_1, E_2, B_3 solve the wave equation with the Dirichlet boundary condition (1.11) in the sense of (1.12) with*

$$\begin{aligned}&u_0=E_{0,i},\ u_1=\partial_t E_{0,i}:=(\nabla_x\times B)_i-4\pi J_{0,i},\\ &G=-4\pi\partial_{x_i}\rho-4\pi\partial_t J_i,\ g=0,\quad \textit{for } E_i,\ i=1,2,\end{aligned} \tag{1.13}$$

$$u_0=B_{0,3},\ u_1=\partial_t B_{0,3}:=-(\nabla_x\times E_0)_3,\ G=4\pi(\nabla_x\times J)_3,\ g=0,\quad \textit{for } B_3, \tag{1.14}$$

respectively.

Moreover, E_3, B_1, B_2 solve the wave equation with the Neumann boundary condition (1.9) in the sense of (1.10) with

$$\begin{aligned}&u_0=E_{0,3},\ u_1=\partial_t E_{0,3}:=-\partial_2 B_{0,1}+\partial_1 B_{0,2}-4\pi J_{0,3},\\ &G=-4\pi\partial_{x_3}\rho-4\pi\partial_t J_3,\ g=-4\pi\rho,\quad \textit{for } E_3,\end{aligned} \tag{1.15}$$

$$\begin{aligned}&u_0=B_{0,i},\ u_1=\partial_t B_{0,i}:=-(\nabla_x\times E_0)_i,\ G=4\pi(\nabla_x\times J)_i,\\ &g=(-1)^{i+1}4\pi J_{\underline{i}},\quad \textit{for } B_i,\ i=1,2,\end{aligned} \tag{1.16}$$

respectively.

Proof. We omit the proof. For a proof, see [7]. □

Next, we prove the uniqueness of wave equation with Neumann BC (1.9) and Dirichlet BC (1.11).

Lemma 2. *Suppose $u(t,x)$, $\tilde{u}(t,x)$ are weak solutions with the Neumann BC (1.9) with the same u_0, u_1, G, and g in the sense of weak formulation (1.10). Then $u(t,x)=\tilde{u}(t,x)$.*

Proof. It suffices to show that if u is the solution of (1.9) with

$$u_0=u_1=G=g=0, \tag{1.17}$$

in the sense of (1.10), then for any $\psi\in C_c^\infty((0,T)\times\Omega)$,

$$\int_0^T\int_\Omega u(t,x)\psi(t,x)dxdt=0. \tag{1.18}$$

Let $\tilde{\psi}(t, x) = \psi(T - t, x)$, then $\tilde{\psi} \in C_c^\infty((0, T) \times \Omega)$. We consider the function $\tilde{v}(t, x) : (0, T) \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$\begin{aligned} \tilde{v}(t, x) := \frac{1}{4\pi} \int_{B(x;t)\cap\{y_3>0\}} & \frac{\tilde{\psi}(t - |y - x|, y_\parallel, y_3)}{|y - x|} dy \\ + \frac{1}{4\pi} \int_{B(x;t)\cap\{y_3<0\}} & \frac{\tilde{\psi}(t - |y - x|, y_\parallel, -y_3)}{|y - x|} dy. \end{aligned} \tag{1.19}$$

Then the function $\tilde{v}(t, x)$ is a weak solution of the wave equation in $(0, T) \times \mathbb{R}^3$

$$\begin{aligned} (\partial_t^2 - \Delta_x)\tilde{v}(t, x) = \mathbf{1}_{x_3>0}\tilde{\psi}(t, x) + \mathbf{1}_{x_3<0}\tilde{\psi}(t, \bar{x}) \\ \tilde{v}(0, x) = 0, \ \ \partial_t \tilde{v}(0, x) = 0, \end{aligned} \tag{1.20}$$

where $x = (x_1, x_2, -x_3)$. And since $\psi \in C_c^\infty((0, T) \times \Omega)$, the function $\mathbf{1}_{x_3>0}\tilde{\psi}(t, x) + \mathbf{1}_{x_3<0}\tilde{\psi}(t, \bar{x})$ is smooth in $(0, T) \times \mathbb{R}^3$. Thus $\tilde{v}$ is smooth. Moreover, for some small $\delta > 0$,

$$\tilde{v}(s, x) = 0 \text{ for } s \in [0, \delta), \tag{1.21}$$

and a direct computation yields

$$\partial_{x_3} \tilde{v} = 0 \text{ on } (0, T) \times \partial\Omega. \tag{1.22}$$

Now, let $v(t, x) : [0, T) \times \bar{\Omega} \to \mathbb{R}^3$ be given by

$$v(t, x) = \tilde{v}(T - t, x). \tag{1.23}$$

Then $v(t, x)$ is smooth, and by (1.21), $v(t, x) \in C_c^\infty([0, T) \times \bar{\Omega})$. Moreover, from (1.20), (1.22),

$$\begin{aligned} (\partial_t^2 - \Delta_x)v(t, x) = \tilde{\psi}(T - t, x) \text{ in } (0, T) \times \Omega, \\ \partial_{x_3} v(t, x) = 0 \text{ on } (0, T) \times \partial\Omega. \end{aligned} \tag{1.24}$$

Now, since u is the solution of (1.9) with $u_0 = u_1 = G = g = 0$, and $v(t, x) \in C_c^\infty([0, T) \times \bar{\Omega})$, $\partial_{x_3} v|_{\partial\Omega} = 0$, from (1.10) we have

$$\begin{aligned} 0 = & \int_\Omega (u_1(x)v(0, x) - u_0(x)\partial_t v(0, x))dx \\ & + \int_0^T \int_\Omega u(t, x) \left(\partial_t^2 v(t, x) - \Delta_x v(t, x)\right) dxdt \\ & - \int_0^T \int_{\partial\Omega} u(t, x_\parallel)\partial_{x_3} v(t, x_\parallel, 0)dx_\parallel dt \\ & + \int_0^T \int_{\partial\Omega} g(t, x_\parallel)v(t, x_\parallel, 0)dx_\parallel dt - \int_0^T \int_\Omega Gv \, dxdt \\ = & \int_0^T \int_\Omega u(t, x)\tilde{\psi}(T - t, x)dxdt = \int_0^T \int_\Omega u(t, x)\psi(t, x)dxdt. \end{aligned} \tag{1.25}$$

Thus, we proved (1.18) and this conclude the lemma. □

We also prove a similar version of the lemma that will be used later.

Lemma 3. *Let $u:(0,T)\times\Omega\to\mathbb{R}$ be a function such that for any $\phi\in C_c^\infty([0,T)\times\bar{\Omega})$, with $\partial_{x_3}\phi|_{\partial\Omega}=0$,*

$$\int_0^T\int_\Omega u(\partial_t^2-\Delta_x)\phi dxdt=0, \tag{1.26}$$

then $u=0$.

Proof. Take any $\psi\in C_c^\infty((0,T)\times\Omega)$. Let $\tilde{\psi}(t,x)=\psi(T-t,x)$, and define $\tilde{v}(t,x)$ in the same way as (1.19). Then define $v(t,x)$ as in (1.23). Then, as showed in (1.19)–(1.24), $v(t,x)\in C_c^\infty([0,T)\times\bar{\Omega})$,

$$(\partial_t^2-\Delta_x)v(t,x)=\psi(t,x) \text{ in } (0,T)\times\Omega, \text{ and } \partial_{x_3}v|_{\partial\Omega}=0.$$

Therefore, from (1.26), we have $\int_0^T\int_\Omega u\psi dxdt=\int_0^T\int_\Omega u(\partial_t^2-\Delta_x)vdxdt=0$. Thus $u=0$. □

Lemma 4. *Suppose $u(t,x)$, $\tilde{u}(t,x)$ are weak solutions with the Dirichlet BC (1.11) with the same u_0, u_1, G, and g in the sense of weak formulation (1.12). Then $u(t,x)=\tilde{u}(t,x)$.*

Proof. It suffices to show that if u is the solution of (1.11) with

$$u_0=u_1=G=g=0, \tag{1.27}$$

in the sense of (1.12), then for any $\psi\in C_c^\infty((0,T)\times\Omega)$,

$$\int_0^T\int_\Omega u(t,x)\psi(t,x)dxdt=0. \tag{1.28}$$

Now, let $\tilde{\psi}(t,x)=\psi(T-t,x)$, and define the function $\tilde{w}(t,x):(0,T)\times\mathbb{R}^3\to\mathbb{R}^3$ as

$$\begin{aligned}\tilde{w}(t,x):=&\frac{1}{4\pi}\int_{B(x;t)\cap\{y_3>0\}}\frac{\tilde{\psi}(t-|y-x|,y_\parallel,y_3)}{|y-x|}dy\\&-\frac{1}{4\pi}\int_{B(x;t)\cap\{y_3<0\}}\frac{\tilde{\psi}(t-|y-x|,y_\parallel,-y_3)}{|y-x|}dy,\end{aligned} \tag{1.29}$$

and let $w(t,x):[0,T)\times\bar{\Omega}\to\mathbb{R}^3$ be given by

$$w(t,x)=\tilde{w}(T-t,x). \tag{1.30}$$

Then from direct computation and using the same argument as (1.19)–(1.24), we get $w(t,x)\in C_c^\infty([0,T)\times\bar{\Omega})$, and

$$\begin{aligned}(\partial_t^2-\Delta_x)w(t,x)=\tilde{\psi}(T-t,x)=\psi(t,x) \text{ in } (0,T)\times\Omega,\\ w(t,x)=0 \text{ on } (0,T)\times\partial\Omega.\end{aligned} \tag{1.31}$$

Therefore from (1.12) and (1.27), we have $0=\int_0^T\int_\Omega u(\partial_t^2-\Delta_x)wdxdt=\int_0^T\int_\Omega u\psi dxdt$. This proves (1.28). □

Next, we show that the Lipschitz solutions of the wave equations solves the Maxwell equations if the continuity equation conteq and some initial compatibility condition are satisfied.

Lemma 5. *Suppose* $E(t,x), B(t,x) \in W^{1,\infty}((0,T)\times\Omega)$, $\nabla_x\rho, \partial_t J, \nabla_x J \in L^\infty((0,T); L^p_{loc}(\Omega))$, *with*

$$\nabla\cdot J = -\partial_t\rho. \tag{1.32}$$

Assume

$$\begin{aligned}&E_1, E_2 \text{ solves } (1.11) \text{ with } (1.13), \text{ and } E_3 \text{ solves } (1.9) \text{ with } (1.15),\\ &B_3 \text{ solves } (1.11) \text{ with } (1.14), \text{ and } B_1, B_2 \text{ solves } (1.9) \text{ with } (1.16).\end{aligned} \tag{1.33}$$

Further we assume compatibility conditions

$$\nabla\cdot E_0 = 4\pi\rho_0,\ \nabla_x\cdot B_0 = 0, \quad in\ \ \Omega, \tag{1.34}$$

$$E_{0,1} = E_{0,2} = B_{0,3} = 0\ \ on\ \ \partial\Omega. \tag{1.35}$$

Then we have

$$\begin{aligned}&\partial_t E = \nabla_x\times B - 4\pi J,\ \nabla_x\cdot E = 4\pi\rho,\\ &\partial_t B = -\nabla_x\times E,\ \nabla_x\cdot B = 0.\end{aligned} \tag{1.36}$$

Proof. We provide the detailed proof of $\nabla\cdot E = 4\pi\rho$, and $\partial_t E = \nabla_x\times B - 4\pi J$. The proof of $\partial_t B = -\nabla_x\times E$, and $\nabla_x\cdot B = 0$ is similar.

In the view of Lemma 4, it suffices to show that for any $\phi(t,x)\in C_c^\infty([0,T)\times\bar\Omega)$ with $\phi|_{\partial\Omega} = 0$, we have

$$\int_0^T\int_\Omega(\nabla\cdot E - 4\pi\rho)(\partial_t^2 - \Delta_x)\phi\,dxdt = 0. \tag{1.37}$$

Now by direct computation with integration by parts, we have

$$\begin{aligned}&\int_0^T\int_\Omega(\nabla\cdot E - 4\pi\rho)(\partial_t^2-\Delta_x)\phi\,dxdt\\ &= \int_\Omega(-\partial_t E_0\cdot\nabla\phi(0,x) + E_0\cdot\nabla\partial_t\phi(0,x))dx + 4\pi\int_0^T\int_\Omega\nabla\cdot J\partial_t\phi\,dxdt\\ &\quad + 4\pi\int_\Omega J_0\cdot\nabla\phi(0,x)dx - 4\pi\int_0^T\int_\Omega\rho\partial_t^2\phi\,dxdt\end{aligned}$$

where in the second equality we've used (1.33).

Now, using $\nabla\cdot J = -\partial_t\rho$, (1.34), and integration by parts we obtain

$$\begin{aligned}&\int_0^T\int_\Omega(\nabla\cdot E - 4\pi\rho)(\partial_t^2-\Delta_x)\phi\,dxdt\\ &= \int_\Omega(-\partial_t E_0 + 4\pi J_0)\cdot\nabla\phi(0,x)dx - \int_\Omega(\nabla\cdot E_0 - 4\pi\rho_0)\partial_t\phi(0,x))dx\\ &= \int_\Omega -(\nabla_x\times B_0)\cdot\nabla\phi(0,x)dx = 0.\end{aligned}$$

This proves (1.37).

Next, let's show that $\partial_t E_1 = (\nabla_x \times B)_1 - 4\pi J_1$. It suffices to prove for any $\phi(t,x) \in C_c^\infty([0,T) \times \bar{\Omega})$ with $\phi|_{\partial\Omega} = 0$, we have

$$\int_0^T \int_\Omega (\partial_t E_1 - (\nabla_x \times B)_1 + 4\pi J_1)(\partial_t^2 - \Delta_x)\phi dxdt = 0. \tag{1.38}$$

Using (1.33) and integration by parts, we compute

$$\begin{aligned}
&\int_0^T \int_\Omega (\partial_t E_1 - (\nabla_x \times B)_1 + 4\pi J_1)(\partial_t^2 - \Delta_x)\phi dxdt \\
&= -\int_\Omega \partial_t E_{0,1}\partial_t\phi(0,x)dx + \int_\Omega 4\pi\rho_0\partial_1\phi(0,x)dx + \int_0^T \int_\Omega 4\pi\partial_t\rho\partial_1\phi dxdt \\
&\quad - \int_\Omega 4\pi J_{0,1}\partial_t\phi(0,x)dx + \int_\Omega E_{0,1}\Delta_x\phi(0,x)dx \\
&\quad - \int_0^T \int_\Omega 4\pi J_1\Delta_x\phi dxdt + \int_\Omega (B_{0,2}\partial_t\partial_3\phi - B_{0,3}\partial_t\partial_2\phi(0,x))dx \\
&\quad - \int_\Omega (-\partial_t B_{0,3}\partial_2\phi(0,x) + \partial_t B_{0,2}\partial_3\phi(0,x))dx \\
&\quad + \int_0^T \int_\Omega 4\pi(-J_2\partial_1\partial_2\phi + J_1\partial_2^2\phi - J_3\partial_1\partial_3\phi + J_1\partial_3^2\phi)dxdt \\
&\quad + \int_0^T \int_{\partial\Omega} 4\pi J_1\partial_3\phi dx_\parallel dt - \int_0^T \int_{\partial\Omega} 4\pi J_1\partial_3\phi dx_\parallel dt.
\end{aligned}$$

Then from (1.8) and integration by parts,

$$\begin{aligned}
&\int_0^T \int_\Omega 4\pi\partial_t\rho\partial_1\phi dxdt - \int_0^T \int_\Omega 4\pi J_1\Delta_x\phi dxdt \\
&\quad + \int_0^T \int_\Omega 4\pi(-J_2\partial_1\partial_2\phi + J_1\partial_2^2\phi - J_3\partial_1\partial_3\phi + J_1\partial_3^2\phi)dxdt \\
&= \int_0^T \int_\Omega 4\pi(J_1\partial_1^2\phi + J_2\partial_2\partial_1\phi + J_3\partial_3\partial_1\phi)dxdt - \int_0^T \int_\Omega 4\pi J_1\Delta_x\phi dxdt \\
&\quad + \int_0^T \int_\Omega 4\pi(-J_2\partial_1\partial_2\phi + J_1\partial_2^2\phi - J_3\partial_1\partial_3\phi + J_1\partial_3^2\phi)dxdt = 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_0^T \int_\Omega (\partial_t E_1 - (\nabla_x \times B)_1 + 4\pi J_1)(\partial_t^2 - \Delta_x)\phi dxdt \\
&= \int_\Omega ((-\partial_t E_{0,1} - 4\pi J_{0,1})\partial_t\phi(0,x) + B_{0,2}\partial_3\partial_t\phi(0,x) - B_{0,3}\partial_2\partial_t\phi(0,x))dx \\
&\quad + \int_\Omega (-4\pi\partial_1\rho_0\phi(0,x) + E_{0,1}\Delta_x\phi(0,x) + \partial_t B_{0,3}\partial_2\phi(0,x) - \partial_t B_{0,2}\partial_3\phi(0,x))dx.
\end{aligned} \tag{1.39}$$

From (1.33), we have $-\partial_t E_{0,1} - 4\pi J_{0,1} - \partial_3 B_{0,2} + \partial_2 B_{0,3} = 0$, and $\partial_t B_0 = -\nabla_x \times E_0$. Together with (1.34), (1.35), we use integration by parts to get

$$\begin{aligned}&\int_\Omega ((-\partial_t E_{0,1} - 4\pi J_{0,1})\partial_t\phi(0,x) + B_{0,2}\partial_3\partial_t\phi(0,x) - B_{0,3}\partial_2\partial_t\phi(0,x))dx \\ &\quad + \int_\Omega (-4\pi\partial_1\rho_0\phi(0,x) + E_{0,1}\Delta_x\phi(0,x) + \partial_t B_{0,3}\partial_2\phi(0,x) - \partial_t B_{0,2}\partial_3\phi(0,x))dx = 0.\end{aligned}$$

Thus, we conclude (1.38). And from the same argument we can show that $\partial_t E_2 = (\nabla_x \times B)_2 + 4\pi J_2$.

Next, let's prove $\partial_t E_3 = (\nabla_x \times B)_3 - 4\pi J_3$. In the view of Lemma 3, it suffices to show that for any $\psi \in C_c^\infty([0,T) \times \bar{\Omega})$ with $\partial_{x_3}\psi|_{\partial\Omega} = 0$, we have

$$\int_0^T \int_\Omega (\partial_t E_3 - (\nabla_x \times B)_3 + 4\pi J_3)(\partial_t^2 - \Delta_x)\psi\, dxdt = 0. \tag{1.40}$$

Using (1.33) and integration by parts, we compute

$$\begin{aligned}&\int_0^T \int_\Omega (\partial_t E_3 - (\nabla_x \times B)_3 + 4\pi J_3)(\partial_t^2 - \Delta_x)\psi\, dxdt \\ &= -\int_\Omega \partial_t E_{0,3}\partial_t\psi(0,x)dx + \int_0^T\int_\Omega 4\pi\partial_t\rho\partial_3\psi\, dxdt \\ &\quad + \int_\Omega 4\pi\rho_0\partial_3\psi(0,x)dx - \int_\Omega J_{0,3}\partial_t\psi(0,x)dx \\ &\quad + \int_\Omega E_{0,3}\Delta_x\psi(0,x)dx + \int_{\partial\Omega} 4\pi\rho_0\psi(0,x_\parallel)dx_\parallel \\ &\quad - \int_0^T\int_\Omega 4\pi J_3\Delta_x\psi\, dxdt + \int_\Omega (\partial_1 B_{0,2} - \partial_2 B_{0,1})\partial_t\psi(0,x)dx \\ &\quad + \int_\Omega (-\partial_t B_{0,1}\partial_2\psi(0,x) + \partial_t B_{0,2}\partial_1\psi(0,x))dx \\ &\quad + \int_0^T\int_\Omega 4\pi(J_3\partial_2^2\psi - J_2\partial_3\partial_2\psi - J_1\partial_3\partial_1\psi + J_3\partial_1^2\psi)dxdt.\end{aligned}$$

From (1.8) and integration by parts,

$$\begin{aligned}&\int_0^T\int_\Omega 4\pi\partial_t\rho\partial_3\psi\, dxdt - \int_0^T\int_\Omega 4\pi J_3\Delta_x\psi\, dxdt \\ &\quad + \int_0^T\int_\Omega 4\pi(J_3\partial_2^2\psi - J_2\partial_3\partial_2\psi - J_1\partial_3\partial_1\psi + J_3\partial_1^2\psi)dxdt \\ &= \int_0^T\int_\Omega 4\pi(J_1\partial_1\partial_3\psi + J_2\partial_2\partial_3\psi + J_3\partial_3^2\psi)dxdt - \int_0^T\int_\Omega 4\pi J_3\Delta_x\psi\, dxdt \\ &\quad + \int_0^T\int_\Omega 4\pi(J_3\partial_2^2\psi - J_2\partial_3\partial_2\psi - J_1\partial_3\partial_1\psi + J_3\partial_1^2\psi)dxdt = 0.\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_0^T \int_\Omega (\partial_t E_3 - (\nabla_x \times B)_3 + 4\pi J_3)(\partial_t^2 - \Delta_x)\psi \, dx dt \\
&\quad = \int_\Omega (-\partial_t E_{0,3} - 4\pi J_{0,3} + \partial_1 B_{0,2} - \partial_2 B_{0,1})\partial_t \psi(0,x) dx \\
&\qquad + \int_\Omega (-4\pi \partial_3 \rho_0 \psi(0,x) + E_{0,3}\Delta_x \psi(0,x) - \partial_t B_{0,1}\partial_2 \psi(0,x) + \partial_t B_{0,2}\partial_1 \psi(0,x)) dx.
\end{aligned} \tag{1.41}$$

From (1.33), we have $-\partial_t E_{0,3} - 4\pi J_{0,3} + \partial_1 B_{0,2} - \partial_2 B_{0,1} = 0$, and $\partial_t B_0 = -\nabla_x \times E_0$. Together with (1.34), (1.35), we use integration by parts to get

$$\begin{aligned}
&\int_\Omega (-\partial_t E_{0,3} - 4\pi J_{0,3} + \partial_1 B_{0,2} - \partial_2 B_{0,1})\partial_t \psi(0,x) dx \\
&\quad + \int_\Omega (-4\pi \partial_3 \rho_0 \psi(0,x) + E_{0,3}\Delta_x \psi(0,x) - \partial_t B_{0,1}\partial_2 \psi(0,x) + \partial_t B_{0,2}\partial_1 \psi(0,x)) dx \\
&\quad = 0.
\end{aligned}$$

This concludes (1.40). □

Now we are ready to prove Theorem 4.

Proof of Theorem 4. The proof is a direct consequence of previous lemmas. Let $(E, B) \in W^{1,\infty}((0,T) \times \Omega)$, and $(\tilde{E}, \tilde{B}) \in W^{1,\infty}((0,T) \times \Omega)$ be two solutions of (1.1)–(1.6).

We consider $E_1 - \tilde{E}_1$, $E_2 - \tilde{E}_2$, and $B_3 - \tilde{B}_3$. From Lemma 1, both E_i and $\tilde{E}_i$ satisfy (1.13) for $i = 1, 2$, and both B_3 and $\tilde{B}_3$ satisfy (1.14). Therefore, from Lemma 4, we have

$$E_1 = \tilde{E}_1, \;\; E_2 = \tilde{E}_2, \;\; B_3 = \tilde{B}_3. \tag{1.42}$$

And for $E_3 - \tilde{E}_3$, $B_1 - \tilde{B}_1$, and $B_2 - \tilde{B}_2$. From Lemma 1, we have both E_3 and $\tilde{E}_3$ satisfy (1.15), and both B_i and $\tilde{B}_i$ satisfy (1.16) for $i = 1, 2$. Therefore, from Lemma 2, we deduce that

$$E_3 = \tilde{E}_3, \;\; B_1 = \tilde{B}_1, \;\; B_2 = \tilde{B}_2. \tag{1.43}$$

Thus, we get $E = \tilde{E}$, $B = \tilde{B}$, and this concludes the uniqueness of the solution. Now this solution should solve the Maxwell equation by Lemma 5. □

2. Glassey-Strauss Representation of *E* and *B*

In this section, we give a representation of the field E and B by solving the wave equations (0.43), (0.44), under the boundary condition (0.24) and (0.46).

We first consider the electronic field E. The tangential component $E_\parallel = (E_1, E_2)$ satisfies

$$\begin{aligned}
\partial_t^2 E_\parallel - \Delta_x E_\parallel = G_\parallel &:= -4\pi \nabla_\parallel \rho - 4\pi \partial_t J_\parallel, \\
E_\parallel|_{t=0} = E_{0\parallel}, &\;\; \partial_t E_\parallel|_{t=0} = \partial_t E_{0\parallel}, \qquad (2.1)\\
E_\parallel = 0 &\quad \text{on } \partial\Omega. \qquad (2.2)
\end{aligned}$$

Define

$$\bar{x} = (x_\parallel, -x_3) \quad \text{for } x = (x_\parallel, x_3) = (x_1, x_2, x_3). \tag{2.3}$$

To solve the Dirichlet boundary condition, we employ the odd extension of the data: for $i = 1, 2$, and $x \in \mathbb{R}^3$,

$$\begin{aligned} G_i(t, x_\parallel, x_3) &= \mathbf{1}_{x_3>0} G_i(t, x) - \mathbf{1}_{x_3<0} G_i(t, \bar{x}), \\ E_{0i}(x_\parallel, x_3) &= \mathbf{1}_{x_3>0} E_{0i}(x) - \mathbf{1}_{x_3<0} E_{0i}(\bar{x}), \\ \partial_t E_{0i}(x_\parallel, x_3) &= \mathbf{1}_{x_3>0} \partial_t E_{0i}(x) - \mathbf{1}_{x_3<0} \partial_t E_{0i}(\bar{x}). \end{aligned} \tag{2.4}$$

Then the weak solution of $E_\parallel(t, x)$ to (2.1) with data (2.4) in the whole space $\mathbb{R}^3$ takes a form of, for $i = 1, 2$,

$$\begin{aligned} E_i(t, x) = &\frac{1}{4\pi t^2} \int_{\partial B(x;t)\cap\{y_3>0\}} (t\partial_t E_{0i}(y) + E_{0i}(y) + \nabla E_{0i}(y) \cdot (y - x))\, dS_y \\ &+ \frac{1}{4\pi t^2} \int_{\partial B(x;t)\cap\{y_3<0\}} \big(- t\partial_t E_{0i}(\bar{y}) - E_{0i}(\bar{y}) - \nabla E_{0i}(\bar{y}) \cdot (\bar{y} - \bar{x})\big) dS_y \\ &+ \frac{1}{4\pi} \int_{B(x;t)\cap\{y_3>0\}} \frac{G_i(t - |y - x|, y)}{|y - x|} dy \qquad (2.5) \\ &+ \frac{1}{4\pi} \int_{B(x;t)\cap\{y_3<0\}} \frac{-G_i(t - |y - x|, \bar{y})}{|y - x|} dy, \qquad (2.6) \end{aligned}$$

where $B(x, t) = \{y \in \mathbb{R}^3 : |x - y| < t\}$ and $\partial B(x, t) = \{y \in \mathbb{R}^3 : |x - y| = t\}$. The above form is then a (weak) solution of (2.1) and (2.2). Next, we express (2.5) and (2.6) using the Glassey-Strauss representation [27] in Ω. Define

$$\partial_t = \frac{S - \hat{v} \cdot T}{1 + \hat{v} \cdot \omega}, \quad \partial_i = \frac{\omega_i S}{1 + \hat{v} \cdot \omega} + \left(\delta_{ij} - \frac{\omega_i \hat{v}_j}{1 + \hat{v} \cdot \omega} \right) T_j, \tag{2.7}$$

while, for $\omega = \omega(x, y) = \frac{y-x}{|y-x|}$,

$$T_i := \partial_i - \omega_i \partial t, \tag{2.8}$$

$$S := \partial_t + \hat{v} \cdot \nabla_x. \tag{2.9}$$

Note that

$$T_j f(t - |y - x|, y, v) = \partial_{y_j} [f(t - |y - x|, y, v)], \tag{2.10}$$

and the Vlasov equation (0.40) implies that

$$Sf = -\nabla_v \cdot [(E + E_{\text{ext}} + \hat{v} \times (B + B_{\text{ext}}) - g\mathbf{e}_3) f]. \tag{2.11}$$

From (0.42) and (2.7),

$$\begin{aligned} (2.5) &= - \int_{B(x;t)\cap\{y_3>0\}} \frac{(\partial_i \rho + \partial_t J_i)(t - |y - x|, y)}{|y - x|} \mathrm{d}y \\ &= - \int_{B(x;t)\cap\{y_3>0\}} \int_{\mathbb{R}^3} (\partial_i f + \hat{v}_i \partial_t f)(t - |y - x|, y, v) \mathrm{d}v \frac{\mathrm{d}y}{|y - x|} \end{aligned}$$

$$
\begin{aligned}
&= -\int_{B(x;t)\cap\{y_3>0\}} \int_{\mathbb{R}^3} \frac{\omega_i + \hat{v}_i}{1+\hat{v}\cdot\omega} (Sf)(t-|y-x|, y, v) \mathrm{d}v \frac{\mathrm{d}y}{|y-x|} \\
&\quad - \int_{\substack{B(x;t)\\ \cap\{y_3>0\}}} \int_{\mathbb{R}^3} \left(\delta_{ij} - \frac{(\omega_i+\hat{v}_i)\hat{v}_j}{1+\hat{v}\cdot\omega}\right) T_j f(t-|y-x|, y, v) \mathrm{d}v \frac{\mathrm{d}y}{|y-x|}.
\end{aligned}
$$

Here, we followed the Einstein convention (when an index variable appears twice, it implies summation of that term over all the values of the index) and will do throughout this section.

Then replace $T_j f$ with (2.10) and apply the integration by parts to get the last term equals

$$
\begin{aligned}
&- \int_{\partial B(x;t)\cap\{y_3>0\}} \omega_j \left(\delta_{ij} - \frac{(\omega_i+\hat{v}_i)\hat{v}_j}{1+\hat{v}\cdot\omega}\right) f(0, y, v) \mathrm{d}v \frac{\mathrm{d}S_y}{|y-x|} \\
&+ \int_{B(x;t)\cap\{y_3=0\}} \int_{\mathbb{R}^3} \left(\delta_{i3} - \frac{(\omega_i+\hat{v}_i)\hat{v}_3}{1+\hat{v}\cdot\omega}\right) f(t-|y-x|, y_\parallel, 0, v) \mathrm{d}v \frac{\mathrm{d}y_\parallel}{|y-x|} \\
&+ \int_{\substack{B(x;t)\\ \cap\{y_3>0\}}} \int_{\mathbb{R}^3} \frac{(|\hat{v}|^2-1)(\hat{v}_i+\omega_i)}{(1+\hat{v}\cdot\omega)^2} f(t-|y-x|, y, v) \mathrm{d}v \frac{\mathrm{d}y}{|y-x|^2}.
\end{aligned}
\tag{2.12}
$$

where we have used that, from [27,28],

$$
\frac{\partial}{\partial y_j}\left[\frac{1}{|y-x|}\left(\delta_{ij} - \frac{(\omega_i+\hat{v}_i)\hat{v}_j}{1+\hat{v}\cdot\omega}\right)\right] = \frac{(|\hat{v}|^2-1)(\hat{v}_i+\omega_i)}{|y-x|^2(1+\hat{v}\cdot\omega)^2}.
$$

In order to express (2.6) in the lower half space we modify the idea of Glassey-Strauss slightly. Define

$$
\bar{\omega} = \begin{bmatrix}\omega_1 & \omega_2 & -\omega_3\end{bmatrix}^T. \tag{2.13}
$$

We use the same S of (2.9) but

$$
\begin{aligned}
\bar{T}_3 f &= -\partial_{y_3}[f(t-|y-x|, y_\parallel, -y_3, v)] = \partial_{y_3} f - \bar{\omega}_3 \partial_t f, \\
\bar{T}_i f &= \partial_{y_i}[f(t-|y-x|, y_\parallel, -y_3, v)] = \partial_{y_i} f - \bar{\omega}_i \partial_t f \ \text{ for } \ i = 1, 2.
\end{aligned}
\tag{2.14}
$$

Then we get

$$
\partial_t = \frac{S - \hat{v}\cdot\bar{T}}{1+\hat{v}\cdot\bar{\omega}}, \tag{2.15}
$$

$$
\partial_{y_i} = \bar{T}_i + \bar{\omega}_i \frac{S-\hat{v}\cdot\bar{T}}{1+\hat{v}\cdot\bar{\omega}} = \frac{\bar{\omega}_i S}{1+\hat{v}\cdot\bar{\omega}} + \bar{T}_i - \bar{\omega}_i \frac{\hat{v}\cdot\bar{T}}{1+\hat{v}\cdot\bar{\omega}}. \tag{2.16}
$$

Therefore, we derive

$$
\partial_i + \hat{v}_i \partial_t = \frac{\bar{\omega}_i + \hat{v}_i}{1+\hat{v}\cdot\bar{\omega}} S + \left(\delta_{ij} - \frac{\bar{\omega}_i\hat{v}_j + \hat{v}_i\hat{v}_j}{1+\hat{v}\cdot\bar{\omega}}\right)\bar{T}_j. \tag{2.17}
$$

Now we consider (2.6). From (2.17),

$$
(2.6) = \int_{B(x;t)\cap\{y_3<0\}} \int_{\mathbb{R}^3} (\partial_i f + \hat{v}_i \partial_t f)(t-|y-x|, \bar{y}, v) \mathrm{d}v \frac{\mathrm{d}y}{|y-x|}
$$

$$
\begin{aligned}
&= \int_{B(x;t)\cap\{y_3<0\}} \int_{\mathbb{R}^3} \frac{\bar{\omega}_i + \hat{v}_i}{1+\hat{v}\cdot\bar{\omega}} (Sf)(t-|y-x|,\bar{y},v)\mathrm{d}v \frac{\mathrm{d}y}{|y-x|} \\
&\quad + \int_{\substack{B(x;t)\\ \cap\{y_3<0\}}} \int_{\mathbb{R}^3} \Big(\delta_{ij} - \frac{\bar{\omega}_i\hat{v}_j + \hat{v}_i\hat{v}_j}{1+\hat{v}\cdot\bar{\omega}}\Big) \bar{T}_j f(t-|y-x|,\bar{y},v)\mathrm{d}v \frac{\mathrm{d}y}{|y-x|}.
\end{aligned}
$$

Applying (2.14) and the integration by parts, we derive that the last term equals

$$
\begin{aligned}
&\int_{\partial B(x;t)\cap\{y_3<0\}} \int_{\mathbb{R}^3} \bar{\omega}_j \Big(\delta_{ij} - \frac{\bar{\omega}_i\hat{v}_j + \hat{v}_i\hat{v}_j}{1+\hat{v}\cdot\bar{\omega}}\Big) f(0,\bar{y},v)\mathrm{d}v \frac{\mathrm{d}S_y}{|y-x|} \\
&+ \int_{B(x;t)\cap\{y_3=0\}} \int_{\mathbb{R}^3} \iota_3 \Big(\delta_{i3} - \frac{\bar{\omega}_i\hat{v}_3 + \hat{v}_i\hat{v}_3}{1+\hat{v}\cdot\bar{\omega}}\Big) f(t-|y-x|,y_\parallel,0,v)\mathrm{d}v \frac{\mathrm{d}y}{|y-x|} \\
&- \int_{B(x;t)\cap\{y_3<0\}} \int_{\mathbb{R}^3} \frac{(|\hat{v}|^2-1)(\hat{v}_i+\bar{\omega}_i)}{(1+\hat{v}\cdot\bar{\omega})^2} f(t-|y-x|,\bar{y},v)\mathrm{d}v \frac{\mathrm{d}y}{|y-x|^2},
\end{aligned}
\tag{2.18}
$$

where we have utilized the notation

$$
\iota_i = +1 \quad \text{for } i=1,2, \quad \iota_3 = -1, \tag{2.19}
$$

and the direct computation

$$
\iota_j \frac{\partial}{\partial y_j}\left[\frac{1}{|y-x|}\Big(\delta_{ij} - \frac{\iota_i\omega_i\hat{v}_j + \hat{v}_i\hat{v}_j}{1+\hat{v}\cdot\bar{\omega}}\Big)\right] = \frac{(|\hat{v}|^2-1)(\hat{v}_i+\bar{\omega}_i)}{|y-x|^2(1+\hat{v}\cdot\bar{\omega})^2}. \tag{2.20}
$$

Next, we consider the normal components of the Electronic field E_3. From (0.43), (0.45), and (0.46), we have

$$
\begin{aligned}
\partial_t^2 E_3 - \Delta_x E_3 &= G_3 := -4\pi\partial_3\rho - 4\pi\partial_t J_3, \\
E_3|_{t=0} &= E_{03},\ \partial_t E_3|_{t=0} = \partial_t E_{03}, && (2.21)\\
\partial_3 E_3 &= 4\pi\rho \quad \text{on} \quad \partial\Omega. && (2.22)
\end{aligned}
$$

It is convenient to decompose the solution into two parts: one with the Neumann boundary condition of (2.21) and the zero forcing term and initial data

$$
\begin{aligned}
&\partial_t^2 w - \Delta_x w = 0 && \text{in} \quad \Omega, \\
&w|_{t=0} = 0,\ \partial_t w|_{t=0} = 0 && \text{in} \quad \Omega, \\
&\partial_3 w = 4\pi\rho && \text{on} \quad \partial\Omega,
\end{aligned}
\tag{2.23}
$$

and the other part $\tilde{E}_3$ with the initial data of (2.21) and the zero Neumann boundary condition. We achieve it by the even extension trick. Recall $\bar{x}$ in (2.3). For $x \in \mathbb{R}^3$, define

$$
\begin{aligned}
G_3(t,x) &= \mathbf{1}_{x_3>0}G_3(t,x) + \mathbf{1}_{x_3<0}G_3(t,\bar{x}), \\
E_{03}(x) &= \mathbf{1}_{x_3>0}E_{03}(x) + \mathbf{1}_{x_3<0}E_{03}(\bar{x}), \\
\partial_t E_{03}(x) &= \mathbf{1}_{x_3>0}\partial_t E_{03}(x) + \mathbf{1}_{x_3<0}\partial_t E_{03}(\bar{x}).
\end{aligned}
\tag{2.24}
$$

The weak solution $\tilde{E}_3$ to (2.21) with the data (2.24) in the whole space $\mathbb{R}^3$ take a form of

$$\begin{aligned}\tilde{E}_3(t,x) = & \frac{1}{4\pi t^2}\int_{\substack{\partial B(x;t)\\ \cap\{y_3>0\}}} (t\partial_t E_{03}(y) + E_{03}(y) + \nabla E_{03}(y)\cdot(y-x))\,dS_y \\ & + \frac{1}{4\pi t^2}\int_{\substack{\partial B(x;t)\\ \cap\{y_3<0\}}} \big(t\partial_t E_{03}(\bar{y}) + E_{03}(\bar{y}) + \nabla E_{03}(\bar{y})\cdot(\bar{y}-\bar{x})\big)dS_y \\ & + \frac{1}{4\pi}\int_{\substack{B(x;t)\\ \cap\{y_3>0\}}} \frac{G_3(t-|y-x|, y)}{|y-x|}dy \qquad (2.25) \\ & + \frac{1}{4\pi}\int_{\substack{B(x;t)\\ \cap\{y_3<0\}}} \frac{G_3(t-|y-x|, \bar{y})}{|y-x|}dy. \qquad (2.26)\end{aligned}$$

Following the same argument to expand (2.5) and (2.6), we derive that

$$\begin{aligned}(2.25) = & -\int_{B(x;t)\cap\{y_3>0\}}\int_{\mathbb{R}^3} \frac{\omega_3+\hat{v}_3}{1+\hat{v}\cdot\omega}(Sf)(t-|y-x|, y, v)\mathrm{d}v\frac{\mathrm{d}y}{|y-x|} \\ & + \int_{B(x;t)\cap\{y_3>0\}}\int_{\mathbb{R}^3} \frac{(|\hat{v}|^2-1)(\hat{v}_3+\omega_3)}{(1+\hat{v}\cdot\omega)^2} f(t-|y-x|, y, v)\mathrm{d}v\frac{\mathrm{d}y}{|y-x|^2} \\ & - \int_{\partial B(x;t)\cap\{y_3>0\}} \omega_j\left(\delta_{3j} - \frac{(\omega_3+\hat{v}_3)\hat{v}_j}{1+\hat{v}\cdot\omega}\right) f(0,y,v)\mathrm{d}v\frac{\mathrm{d}S_y}{|y-x|} \\ & + \int_{B(x;t)\cap\{y_3=0\}}\int_{\mathbb{R}^3}\left(1-\frac{(\omega_3+\hat{v}_3)\hat{v}_3}{1+\hat{v}\cdot\omega}\right) f(t-|y-x|, y_\parallel, 0, v)\mathrm{d}v\frac{\mathrm{d}y_\parallel}{|y-x|},\end{aligned} \tag{2.27}$$

$$\begin{aligned}(2.26) = & -\int_{B(x;t)\cap\{y_3<0\}}\int_{\mathbb{R}^3} \frac{\bar{\omega}_3+\hat{v}_3}{1+\hat{v}\cdot\bar{\omega}}(Sf)(t-|y-x|, \bar{y}, v)\mathrm{d}v\frac{\mathrm{d}y}{|y-x|} \\ & + \int_{B(x;t)\cap\{y_3<0\}}\int_{\mathbb{R}^3} \frac{(|\hat{v}|^2-1)(\hat{v}_3+\bar{\omega}_3)}{(1+\hat{v}\cdot\bar{\omega})^2} f(t-|y-x|, \bar{y}, v)\mathrm{d}v\frac{\mathrm{d}y}{|y-x|^2} \\ & - \int_{\partial B(x;t)\cap\{y_3<0\}}\int_{\mathbb{R}^3} \bar{\omega}_j\left(\delta_{3j} - \frac{\bar{\omega}_3\hat{v}_j+\hat{v}_3\hat{v}_j}{1+\hat{v}\cdot\bar{\omega}}\right) f(0,\bar{y},v)\mathrm{d}v\frac{\mathrm{d}S_y}{|y-x|} \\ & + \int_{B(x;t)\cap\{y_3=0\}}\int_{\mathbb{R}^3}\left(1-\frac{(\bar{\omega}_3+\hat{v}_3)\hat{v}_3}{1+\hat{v}\cdot\bar{\omega}}\right) f(t-|y-x|, y_\parallel, 0, v)\mathrm{d}v\frac{\mathrm{d}y}{|y-x|}.\end{aligned} \tag{2.28}$$

Note that the weak derivative ∂_3 to the form of $\tilde{E}_3$ solves the linear wave equation (2.21) with oddly extended forcing term and the initial data in the sense of distributions. Thus it satisfies

$$\partial_3 \tilde{E}_3 = 0 \quad \text{on} \quad \partial\Omega. \tag{2.29}$$

Now for (2.23), using Laplace transformation and solving the corresponding Helmholtz equation with the Neumann boundary conditions (Lemma 4.1, 4.2 in [13]), we obtained that the solution w of (2.23) has the form:

$$w(t,x) = -2\int_{\sqrt{|y_\parallel - x_\parallel|^2 + x_3^2}<t} \frac{\rho(t-\sqrt{|y_\parallel - x_\parallel|^2+x_3^2}, y_\parallel)}{\sqrt{|y_\parallel - x_\parallel|^2 + x_3^2}} dy_\parallel. \tag{2.30}$$

Collecting the terms, we conclude the following formula:

Proposition 1.

$$E_i(t,x) = \frac{1}{4\pi t^2}\int_{\partial B(x;t)\cap\{y_3>0\}} \big(t\partial_t E_{0,i}(y_\parallel, y_3) + E_{0,i}(y_\parallel, y_3) + \nabla E_{0,i}(y_\parallel, y_3)\cdot(y-x)\big)\, dS_y \tag{2.31}$$

$$+\frac{1}{4\pi t^2}\int_{\partial B(x;t)\cap\{y_3<0\}} \iota_i\big(-t\partial_t E_i(0,y_\parallel,-y_3) - E_i(0,y_\parallel,-y_3)$$

$$-\nabla_\parallel E_i(0,y_\parallel,-y_3)\cdot(y_\parallel - x_\parallel) + \partial_3 E_i(0,y_\parallel,-y_3)\cdot(y_3-x_3)\big)dS_y \tag{2.32}$$

$$+\int_{B(x;t)\cap\{y_3>0\}}\int_{\mathbb{R}^3} \frac{(|\hat{v}|^2-1)(\hat{v}_i+\omega_i)}{|y-x|^2(1+\hat{v}\cdot\omega)^2} f(t-|y-x|, y, v)\mathrm{d}v\mathrm{d}y \tag{2.33}$$

$$-\int_{B(x;t)\cap\{y_3<0\}}\int_{\mathbb{R}^3} \iota_i\frac{(|\hat{v}|^2-1)(\hat{v}_i+\iota_i\omega_i)}{|y-x|^2(1+\hat{v}\cdot\omega^-)^2} f(t-|y-x|, y_\parallel, -y_3, v)\mathrm{d}v\mathrm{d}y \tag{2.34}$$

$$-\int_{B(x;t)\cap\{y_3>0\}}\int_{\mathbb{R}^3} \frac{\omega_i+\hat{v}_i}{1+\hat{v}\cdot\omega}(Sf)(t-|y-x|, y, v)\mathrm{d}v\frac{\mathrm{d}y}{|y-x|} \tag{2.35}$$

$$+\int_{B(x;t)\cap\{y_3<0\}}\int_{\mathbb{R}^3} \iota_i\frac{\iota_i\omega_i+\hat{v}_i}{1+\hat{v}\cdot\omega^-}(Sf)(t-|y-x|, y_\parallel, -y_3, v)\mathrm{d}v\frac{\mathrm{d}y}{|y-x|} \tag{2.36}$$

$$+\int_{B(x;t)\cap\{y_3=0\}}\int_{\mathbb{R}^3} \left(\delta_{i3} - \frac{(\omega_i+\hat{v}_i)\hat{v}_3}{1+\hat{v}\cdot\omega}\right) f(t-|y-x|, y_\parallel, 0, v)\mathrm{d}v\frac{\mathrm{d}y_\parallel}{|y-x|} \tag{2.37}$$

$$-\int_{B(x;t)\cap\{y_3=0\}}\int_{\mathbb{R}^3} \iota_i\left(\delta_{i3} - \frac{\iota_i\omega_i\hat{v}_3+\hat{v}_i\hat{v}_3}{1+\hat{v}\cdot\omega^-}\right) f(t-|y-x|, y_\parallel, 0, v)\mathrm{d}v\frac{\mathrm{d}y_\parallel}{|y-x|} \tag{2.38}$$

$$-\int_{\partial B(x;t)\cap\{y_3>0\}}\int_{\mathbb{R}^3} \sum_j \omega_j\left(\delta_{ij} - \frac{(\omega_i+\hat{v}_i)\hat{v}_j}{1+\hat{v}\cdot\omega}\right) f(0,y,v)\mathrm{d}v\frac{\mathrm{d}S_y}{|y-x|} \tag{2.39}$$

$$+\int_{\partial B(x;t)\cap\{y_3<0\}} \iota_i\int_{\mathbb{R}^3} \sum_j \iota_j\omega_j\left(\delta_{ij} - \frac{\iota_i\omega_i\hat{v}_j+\hat{v}_i\hat{v}_j}{1+\hat{v}\cdot\omega^-}\right) f(0,y_\parallel,-y_3,v)\mathrm{d}v\frac{\mathrm{d}S_y}{|y-x|} \tag{2.40}$$

$$-\delta_{i3}\int_{B(x;t)\cap\{y_3=0\}}\int_{\mathbb{R}^3} \frac{2f(t-|y-x|, y_\parallel, 0, v)}{|y-x|} dv dS_y. \tag{2.41}$$

Next, we solve for B. For B_1, B_2 we have, for $i = 1, 2$,

$$\begin{aligned} \partial_t^2 B_i - \Delta_x B_i &= 4\pi(\nabla_x\times J)_i := H_i \quad \text{in} \quad \Omega,\\ \partial_{x_3}B_1 = 4\pi J_2,\ \partial_{x_3}B_2 &= 4\pi J_1 \quad \text{on} \quad \partial\Omega,\\ B_i(0,x) = B_{0i},\ \partial_t B_i(0,x) &= \partial_t B_{0i} \quad \text{in} \quad \Omega. \end{aligned} \tag{2.42}$$

To solve (2.42) we write $B_i = \tilde{B}_i + B_{bi}$ with $\tilde{B}_i$ satisfies the wave equation in $(0,\infty)\times\mathbb{R}^3$ with even extension in x_3:

$$\begin{aligned} \partial_t^2\tilde{B}_i - \Delta_x\tilde{B}_i &= \mathbf{1}_{x_3>0}H_i(t,x) + \mathbf{1}_{x_3<0}H_i(t,\bar{x}),\\ \tilde{B}_i(0,x) &= \mathbf{1}_{x_3>0}B_{0i}(x) + \mathbf{1}_{x_3<0}B_{0i}(\bar{x}),\\ \partial_t\tilde{B}_i(0,x) &= \mathbf{1}_{x_3>0}\partial_t B_{0i}(x) + \mathbf{1}_{x_3<0}\partial_t B_{0i}(\bar{x}). \end{aligned} \tag{2.43}$$

And B_{bi} satisfies

$$\begin{aligned} \partial_t^2 B_{bi} - \Delta_x B_{bi} &= 0 \text{ in } \Omega,\\ B_{bi}(0,x) = 0,\ \partial_t B_{bi} &= 0 \text{ in } \Omega,\\ \partial_{x_3}B_{b1} = 4\pi J_2,\ \partial_{x_3}B_{b2} &= -4\pi J_1 \text{ on } \Omega. \end{aligned} \tag{2.44}$$

On the other hand, $B_3(t,x)$ satisfies

$$\begin{aligned} \partial_t^2 B_3 - \Delta_x B_3 &= 4\pi(\nabla_x \times J)_3 := H_3 \text{ in } \Omega, \\ B_3(0,x) = B_{03},\ \partial_t B_3(0,x) &= \partial_t B_{03} \text{ in } \Omega, \\ B_3 &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Using the odd extension in x_3:

$$\begin{aligned} H_3(t,x) &= \mathbf{1}_{x_3>0} H_3(t,x) - \mathbf{1}_{x_3<0} H_3(t,\bar{x}), \\ B_{03}(x) &= \mathbf{1}_{x_3>0} B_{03}(x) - \mathbf{1}_{x_3<0} B_{03}(\bar{x}), \\ \partial_t B_{03}(0,x) &= \mathbf{1}_{x_3>0} \partial_t B_{03}(x) - \mathbf{1}_{x_3<0} \partial_t B_{03}(\bar{x}), \end{aligned}$$

Using a similar argument as for the representation of E (we refer [13] for details), we obtain the following formula for B:

Proposition 2.

$$\begin{aligned} B_i(t,x) = \frac{1}{4\pi t^2} \int_{\partial B(x;t)\cap\{y_3>0\}} & \big(t\partial_t B_{0,i}(y_\parallel, y_3) + B_{0,i}(y_\parallel, y_3) \\ & + \nabla B_{0,i}(y_\parallel, y_3)\cdot(y-x)\big) dS_y \\ & + \frac{\iota_i}{4\pi t^2} \int_{\partial B(x;t)\cap\{y_3<0\}} \big(t\partial_t B_{0,i}(y_\parallel, -y_3) + B_{0,i}(y_\parallel, -y_3) \qquad (2.45)\\ & + \nabla_\parallel B_{0,i}(0, y_\parallel, -y_3)\cdot(y_\parallel - x_\parallel) - \partial_3 B_{0,i}(0, y_\parallel, -y_3)\cdot(y_3 - x_3)\big) dS_y \qquad (2.46)\\ & + \int_{B(x;t)\cap\{y_3>0\}} \int_{\mathbb{R}^3} \frac{(\omega\times\hat{v})_i \left(1-|\hat{v}|^2\right)}{(1+\hat{v}\cdot\omega)^2|y-x|^2} f(t-|y-x|, y_\parallel, y_3, v)\, dv dy \qquad (2.47)\\ & + \int_{B(x;t)\cap\{y_3<0\}} \int_{\mathbb{R}^3} \iota_i \frac{(\omega^-\times\hat{v})_i \left(1-|\hat{v}|^2\right)}{(1+\hat{v}\cdot\omega^-)^2|y-x|^2} f(t-|y-x|, y_\parallel, -y_3, v)\, dv dy \qquad (2.48)\\ & + \int_{B(x;t)\cap\{y_3>0\}} \int_{\mathbb{R}^3} \frac{(\omega\times\hat{v})_i}{1+\hat{v}\cdot\omega} Sf(t-|y-x|, y_\parallel, y_3, v)\, dv \frac{dy}{|y-x|} \qquad (2.49)\\ & + \int_{B(x;t)\cap\{y_3<0\}} \int_{\mathbb{R}^3} \iota_i \frac{(\omega^-\times\hat{v})_i}{1+\hat{v}\cdot\omega^-} Sf(t-|y-x|, y_\parallel, -y_3, v)\, dv \frac{dy}{|y-x|} \qquad (2.50)\\ & + \int_{B(x;t)\cap\{y_3=0\}} \int_{\mathbb{R}^3} \left(-(e_3\times\hat{v})_i + \frac{(\omega\times\hat{v})_i\hat{v}_3}{1+\hat{v}\cdot\omega}\right) f(t-|y-x|, y_\parallel, 0, v)\, dv \frac{dy_\parallel}{|y-x|} \\ & \qquad (2.51)\\ & + \int_{B(x;t)\cap\{y_3=0\}} \int_{\mathbb{R}^3} \iota_i \left(-(e_3\times\hat{v})_i + \frac{(\omega^-\times\hat{v})_i\hat{v}_3}{1+\hat{v}\cdot\omega^-}\right) f(t-|y-x|, y_\parallel, 0, v)\, dv \frac{dy_\parallel}{|y-x|} \\ & \qquad (2.52)\\ & + \int_{\partial B(x;t)\cap\{y_3>0\}} \int_{\mathbb{R}^3} \left(\frac{(\omega\times\hat{v})_i}{1+\hat{v}\cdot\omega}\right) f(0, y_\parallel, y_3, v)\, dv \frac{dS_y}{t} \qquad (2.53)\\ & + \int_{\partial B(x;t)\cap\{y_3<0\}} \int_{\mathbb{R}^3} \iota_i \left(\frac{(\omega^-\times\hat{v})_i}{1+\hat{v}\cdot\omega^-}\right) f(0, y_\parallel, -y_3, v)\, dv \frac{dS_y}{t} \qquad (2.54)\\ & + (-1)^i 2(1-\delta_{i3}) \int_{B(x;t)\cap\{y_3=0\}} \int_{\mathbb{R}^3} \frac{\hat{v}_{\underline{i}} f(t-|y-x|, y_\parallel, 0, v)}{|y-x|} dv dS_y. \qquad (2.55) \end{aligned}$$

3. Regularity Estimate of the Field

With the formula for E as in (2.31)–(2.41), and B as in (2.45)–(2.55), we have the estimate of the fields.

Lemma 6. *There exists a* $0 < T \ll 1$ *such that for any* $t \in [0, T]$*, we have*

$$\|E(t)\|_\infty + \|B(t)\|_\infty \lesssim \|E_0\|_\infty + \|B_0\|_\infty + t\left(\sup_{0\le t\le T} \|\langle v\rangle^{4+\delta} f(t)\|_\infty + \|E_0\|_{C^1} + \|B_0\|_{C^1}\right). \quad (3.1)$$

Proof. The proof is similar to (and easier than) the proof of the estimate of the derivatives of E and B in Lemma 7. We therefore omit the proof here for sake of avoiding redundancy. □

We focus on the proof of the estimate of the derivatives of the fields:

Lemma 7. *With the formula* $E(t, x)$ *as in (2.31)–(2.41), and* $B(t, x)$ *as in (2.45)–(2.55), there exists a* $T \ll 1$ *such that for any* $t \in [0, T]$*,*

$$\|\nabla_{x_\|} E(t)\|_\infty + \|\nabla_{x_\|} B(t)\|_\infty \lesssim \|E_0\|_{C^2} + \|B_0\|_{C^2} + \sup_{0\le t\le T}\left(\|\langle v\rangle^{4+\delta}\nabla_{x_\|} f(t)\|_\infty\right) + \sup_{0\le t\le T} \|\langle v\rangle^{4+\delta} f(t)\|_\infty, \quad (3.2)$$

$$\|\partial_{x_3} E(t)\|_\infty + \|\partial_{x_3} B(t)\|_\infty \lesssim \|E_0\|_{C^2} + \|B_0\|_{C^2} + \sup_{0\le t\le T}\left(\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t)\|_\infty + \|\langle v\rangle^{4+\delta}\nabla_{x_\|} f(t)\|_\infty\right) + \sup_{0\le t\le T} \|\langle v\rangle^{4+\delta} f(t)\|_\infty, \quad (3.3)$$

$$\|\partial_t E(t)\|_\infty + \|\partial_t B(t)\|_\infty \lesssim \|E_0\|_{C^2} + \|B_0\|_{C^2} + \sup_{0\le t\le T}\left(\|\langle v\rangle^{4+\delta}\nabla_{x_\|} f(t)\|_\infty + \|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t)\|_\infty\right) + \sup_{0\le t\le T} \|\langle v\rangle^{4+\delta} f(t)\|_\infty. \quad (3.4)$$

Proof. We take derivative to $\frac{\partial}{\partial x_k} E_i(t, x)$ in (2.31)–(2.41) and estimate each term.

First, by using the change of variables $z = y - x$ and spherical coordinate for z, we have

$$\begin{aligned}(2.31) &= \frac{1}{4\pi t^2}\int_{\{|z|=t, x_3+z_3>0\}} \left(t\partial_t E_{0,i}(x+z) + E_{0,i}(x+z) + \nabla E_{0,i}(x+z)\cdot z\right) dS_z \\ &= \frac{1}{4\pi t^2}\int_{\{t\cos\phi > -x_3\}}\int_0^{2\pi} \left(t\partial_t E_{0,i}(x+z) + E_{0,i}(x+z)\right. \\ &\quad \left. + \nabla E_{0,i}(x+z)\cdot z\right) t^2 \sin\phi \, d\theta d\phi.\end{aligned} \quad (3.5)$$

Thus

$$
\begin{aligned}
\frac{\partial}{\partial_{x_k}}(2.31) = \frac{1}{4\pi t^2}\int_{\{|z|=t, x_3+z_3>0\}} & \big(t\partial_{x_k}\partial_t E_{0,i}(x+z) + \partial_{x_k}E_{0,i}(x+z) \\
& + \nabla\partial_{x_k}E_{0,i}(x+z)\cdot z\big)\, dS_z \\
& + \frac{1}{4\pi t}\delta_{k3}\int_0^{2\pi} (t\partial_t E_{0,i}(x_\parallel + z_\parallel, 0) + E_{0,i}(x_\parallel + z_\parallel, 0) \\
& + \nabla E_{0,i}(x_\parallel + z_\parallel, 0)\cdot z) d\theta.
\end{aligned} \tag{3.6}
$$

For $i = 1, 2$, $E_{0,i}(x_\parallel + z_\parallel, 0) = 0$, thus we have

$$
\begin{aligned}
|\frac{\partial}{\partial_{x_k}}(2.31)_{i=1,2}| \lesssim & \; t\|\nabla_x\partial_t E_0\|_\infty + \|\nabla_x E_0\|_\infty + t\|\nabla_x^2 E_0\|_\infty \\
& + \|\partial_t E_0\|_\infty + \|\nabla_x E_0\|_\infty \lesssim \|E_0\|_{C^2}.
\end{aligned}
$$

And we apply the same estimate for $\frac{\partial}{\partial_{x_k}}(2.32)_{i=1,2}$ to obtain

$$
|\frac{\partial}{\partial_{x_k}}(2.31)_{i=1,2}| + |\frac{\partial}{\partial_{x_k}}(2.32)_{i=1,2}| \lesssim \|E_0\|_{C^2}.
$$

For $i = 3$, we use the cancellation for $\frac{\partial}{\partial_{x_k}}(2.31)_{i=3} + \frac{\partial}{\partial_{x_k}}(2.32)_{i=3}$ at $y_3 = 0$ to get

$$
\begin{aligned}
& \frac{\partial}{\partial_{x_k}}(2.31)_{i=3} + \frac{\partial}{\partial_{x_k}}(2.32)_{i=3} \\
& = \frac{1}{4\pi t^2}\int_{\{|z|=t, x_3+z_3>0\}} \big(t\partial_{x_k}\partial_t E_{0,3}(x+z) + \partial_{x_k}E_{0,3}(x+z) + \nabla\partial_{x_k}E_{0,3}(x+z)\cdot z\big)\, dS_z \\
& + \frac{1}{4\pi t^2}\int_{\{|z|=t, x_3+z_3<0\}} \iota_k \big(t\partial_{x_k}\partial_t E_{0,3}(x+z) + \partial_{x_k}E_{0,3}(x+z) \\
& + \nabla_\parallel\partial_{x_k}E_{0,3}(x+z)\cdot z_\parallel - \partial_{x_3}\partial_{x_k}E_{0,3}(x+z)\cdot z_3\big)\, dS_z \\
& + \frac{2}{4\pi t^2}\delta_{k3}\int_0^{2\pi} \big(\partial_{x_3}E_{0,3}(x_\parallel + z_\parallel, 0)\big)\frac{-x_3}{t} t^2 d\theta.
\end{aligned} \tag{3.7}
$$

Thus $|\frac{\partial}{\partial_{x_k}}(2.31)_{i=3}| + |\frac{\partial}{\partial_{x_k}}(2.32)_{i=3}| \lesssim \|E_0\|_{C^2}$, and therefore

$$
|\frac{\partial}{\partial_{x_k}}(2.31)| + |\frac{\partial}{\partial_{x_k}}(2.32)| \lesssim \|E_0\|_{C^2}. \tag{3.8}
$$

Next, using the change of variables $z = y - x$ we have

$$
\begin{aligned}
\frac{\partial}{\partial_{x_k}}(2.33) = & \underbrace{\int_{\{|y-x|<t\}\cap\{y_3>0\}}\int_{\mathbb{R}^3} \frac{(|\hat{v}|^2 - 1)(\hat{v}_i + \omega_i)}{|y-x|^2(1+\hat{v}\cdot\omega)^2}\partial_{x_k} f(t - |y-x|, y, v) \mathrm{d}v\mathrm{d}y}_{(3.9)_1} \\
& \underbrace{+ \delta_{k3}\int_{\{(|y_\parallel - x_\parallel|^2 + |x_3|^2)^{1/2}<t\}}\int_{\mathbb{R}^3} \frac{(|\hat{v}|^2 - 1)(\hat{v}_i + \omega_i)}{(|y_\parallel - x_\parallel|^2 + x_3^2)(1+\hat{v}\cdot\omega)^2} f(t - |y-x|, y_\parallel, 0, v)\mathrm{d}v\mathrm{d}y_\parallel}_{(3.9)_2}.
\end{aligned} \tag{3.9}
$$

From [30] we have

$$
\frac{1}{1+\hat{v}\cdot\omega} \leq \frac{2(1+|v|^2)}{1+|v\times\omega|^2} \leq 2(1+|v|^2), \text{ and } |\omega + \hat{v}|^2 \leq 2(1+\hat{v}\cdot\omega), \tag{3.10}
$$

and since $1-|\hat{v}|^2=\frac{1}{1+|v|^2}$,

$$|\frac{(|\hat{v}|^2-1)(\hat{v}_i+\omega_i)}{(1+\hat{v}\cdot\omega)^2}|\leq\frac{\sqrt{2}}{1+|v|^2}\frac{1}{(1+\hat{v}\cdot\omega)^{3/2}}\leq 4\sqrt{1+|v|^2}. \tag{3.11}$$

Using (3.11), we have for $k=1,2$,

$$\begin{aligned}&\int_{\{|y-x|<t\}\cap\{y_3>0\}}\int_{\mathbb{R}^3}\frac{(|\hat{v}|^2-1)(\hat{v}_i+\omega_i)}{|y-x|^2(1+\hat{v}\cdot\omega)^2}\partial_{x_k}f(t-|y-x|,y,v)\mathrm{d}v\mathrm{d}ya\\&\lesssim\sup_{0\leq t\leq T}\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel}f(t)\|_\infty\int_{\{|y-x|<t\}}\int_{\mathbb{R}^3}\frac{1}{|y-x|^2}(1+|v|)^{-3-\delta}\mathrm{d}v\mathrm{d}y\\&\lesssim\sup_{0\leq t\leq T}\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel}f(t)\|_\infty.\end{aligned} \tag{3.12}$$

For $k=3$, we have for any $1<p<\frac{3}{2}$, from (3.11), and Lemma 9 which will be proved in the next section,

$$\begin{aligned}&\int_{\{|y-x|<t\}\cap\{y_3>0\}}\int_{\mathbb{R}^3}\frac{(|\hat{v}|^2-1)(\hat{v}_i+\omega_i)}{|y-x|^2(1+\hat{v}\cdot\omega)^2}\partial_{x_3}f(t-|y-x|,y,v)\mathrm{d}v\mathrm{d}y\\&\lesssim\sup_{0\leq t\leq T}\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3}f(t)\|_\infty\int_{\{|y-x|<t\}\cap\{y_3>0\}}\int_{\mathbb{R}^3}\frac{1}{|y-x|^2}\frac{(1+|v|)^{-4-\delta}}{\alpha(t-|y-x|,y,v)}dvdy\\&\lesssim\sup_{0\leq t\leq T}\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3}f(t)\|_\infty.\end{aligned} \tag{3.13}$$

We leave the estimate of $(3.9)_2$ together with the estimate of ∂_{x_k} (2.37) later.

Next, from the equation (0.40) and the definition of Sf, we have

$$Sf=-(E+E_{\text{ext}}+\hat{v}\times(B+B_{\text{ext}})-g\mathbf{e}_3)\cdot\nabla_v f.$$

From integration by parts in v and the fact that $\nabla_v\cdot(\hat{v}\times(B+B_{\text{ext}}))=0$,

$$\begin{aligned}(2.35)=&\int_{B(x;t)\cap\{y_3>0\}}\int_{\mathbb{R}^3}\mathcal{S}_i^E(v,\omega)\cdot(E+\hat{v}\times(B+B_{\text{ext}})-g\mathbf{e}_3)\\&f(t-|y-x|,y,v)\mathrm{d}v\frac{\mathrm{d}y}{|y-x|},\end{aligned} \tag{3.14}$$

where

$$\mathcal{S}_i^E(\omega,v)=\nabla_v\left(\frac{\omega_i+\hat{v}_i}{1+\hat{v}\cdot\omega}\right)=\frac{(e_i-\hat{v}_i\hat{v})(1+\hat{v}\cdot\omega)-(\omega_i+\hat{v}_i)(\omega-(\omega\cdot\hat{v})\hat{v})}{\langle v\rangle(1+\hat{v}\cdot\omega)^2}. \tag{3.15}$$

By writing

$$\omega-(\omega\cdot\hat{v})\hat{v}=\omega(1+\hat{v}\cdot\omega)-(\hat{v}\cdot\omega)(\omega+\hat{v}), \tag{3.16}$$

we have from (3.10),

$$|\mathcal{S}_i^E(\omega,\hat{v})|\leq|\frac{(e_i-\hat{v}_i\hat{v})}{\langle v\rangle(1+\hat{v}\cdot\omega)}|+|\frac{\omega(\omega_i+\hat{v}_i)}{\langle v\rangle(1+\hat{v}\cdot\omega)}|+|\frac{(\omega_i+\hat{v}_i)(\hat{v}\cdot\omega)(\omega+\hat{v})}{\langle v\rangle(1+\hat{v}\cdot\omega)^2}|$$

$$\leq 2\sqrt{1+|v|^2} + 2\sqrt{1+|v|^2} + +8\sqrt{1+|v|^2} = 12\sqrt{1+|v|^2}. \tag{3.17}$$

From (3.14), and using the change of variables $z = y - x$ and taking $\frac{\partial}{\partial_{x_k}}$ derivative to (2.35) we have

$$\begin{aligned}
\frac{\partial}{\partial_{x_k}}(2.35) &= \int_{B(x;t)\cap\{y_3>0\}} \int_{\mathbb{R}^3} \mathcal{S}_i^E(v,\omega) \cdot (\partial_{x_k} E + \hat{v} \times \partial_{x_k} B) f(t-|y-x|, y, v) \mathrm{d}v \frac{\mathrm{d}y}{|y-x|} \\
&+ \int_{B(x;t)\cap\{y_3>0\}} \int_{\mathbb{R}^3} \mathcal{S}_i^E(v,\omega) \cdot (E + \hat{v} \times (B + B_{\text{ext}}) \\
&- g\mathbf{e}_3) \partial_{x_k} f(t-|y-x|, y, v) \mathrm{d}v \frac{\mathrm{d}y}{|y-x|} \\
&+ \delta_{k3} \int_{B(x;t)\cap\{y_3=0\}} \int_{\mathbb{R}^3} \mathcal{S}_i^E(v,\omega) \cdot (E + \hat{v} \times (B + B_{\text{ext}}) \\
&- g\mathbf{e}_3) f(t-|y-x|, y_\parallel, 0, v) \mathrm{d}v \frac{\mathrm{d}y_\parallel}{|y-x|}.
\end{aligned} \tag{3.18}$$

Thus for $k = 1, 2$, from (3.17) we have,

$$\begin{aligned}
&|\int_{B(x;t)\cap\{y_3>0\}} \int_{\mathbb{R}^3} \mathcal{S}_i^E(v,\omega) \cdot (\partial_{x_k} E + \hat{v} \times \partial_{x_k} B) f(t-|y-x|, y, v) \mathrm{d}v \frac{\mathrm{d}y}{|y-x|}| \\
&\lesssim t^2 \sup_{0\leq t\leq T} \|(1+|v|^{4+\delta}) f(t)\|_\infty \left(\sup_{0\leq t\leq T} \|\nabla_{x_\parallel} E(t)\|_\infty + \sup_{0\leq t\leq T} \|\nabla_{x_\parallel} B(t)\|_\infty \right).
\end{aligned} \tag{3.19}$$

Similarly, for $k = 3$,

$$\begin{aligned}
&|\int_{B(x;t)\cap\{y_3>0\}} \int_{\mathbb{R}^3} \mathcal{S}_i^E(v,\omega) \cdot (\partial_{x_k} E + \hat{v} \times \partial_{x_3} B) f(t-|y-x|, y, v) \mathrm{d}v \frac{\mathrm{d}y}{|y-x|}| \\
&\lesssim t^2 \sup_{0\leq t\leq T} \|(1+|v|^{4+\delta}) f(t)\|_\infty \left(\sup_{0\leq t\leq T} \|\partial_{x_3} E(t)\|_\infty + \sup_{0\leq t\leq T} \|\partial_{x_3} B(t)\|_\infty \right),
\end{aligned} \tag{3.20}$$

for any $0 < \delta \ll 1$. Thus for $t \ll 1$, the terms from (3.19), (3.20) will be absorbed into the LHS. From (3.17), we have for $k = 1, 2$,

$$\begin{aligned}
&|\int_{B(x;t)\cap\{y_3>0\}} \int_{\mathbb{R}^3} \mathcal{S}_i^E(v,\omega) \cdot (E + E_{\text{ext}} + \hat{v} \times (B + B_{\text{ext}}) \\
&\quad - g\mathbf{e}_3) \partial_{x_k} f(t-|y-x|, y, v) \mathrm{d}v \frac{\mathrm{d}y}{|y-x|}| \\
&\lesssim \left(\sup_{0\leq t\leq T} \|E(t)\|_\infty + \sup_{0\leq t\leq T} \|B(t)\|_\infty + |B_e| + g \right) \sup_{0\leq t\leq T} \left(\|\langle v\rangle^{4+\delta} \nabla_{x_\parallel} f(t)\|_\infty \right).
\end{aligned} \tag{3.21}$$

And, for $k = 3$,

$$|\int_{B(x;t)\cap\{y_3>0\}} \int_{\mathbb{R}^3} \mathcal{S}_i^E(v,\omega) \cdot (E + E_{\text{ext}} + \hat{v} \times (B + B_{\text{ext}})$$

$$
\begin{aligned}
&-g\mathbf{e}_3)\partial_{x_3} f(t-|y-x|,y,v)\mathrm{d}v\frac{\mathrm{d}y}{|y-x|}|\\
&\lesssim \left(\sup_{0\le t\le T}\|E(t)\|_\infty+\sup_{0\le t\le T}\|B(t)\|_\infty+|B_e|+E_e+g\right)\sup_{0\le t\le T}\left(\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t)\|_\infty\right)\\
&\quad\times\int_{B(x;t)\cap\{y_3>0\}}\int_{\mathbb{R}^3}\frac{1}{|y-x|}\left(\frac{\langle v\rangle^{-4-\delta}}{\alpha(t-|y-x|,y,v)}\right)dvdy\\
&\lesssim \left(\sup_{0\le t\le T}\|E(t)\|_\infty+\sup_{0\le t\le T}\|B(t)\|_\infty\right)\sup_{0\le t\le T}\left(\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t)\|_\infty\right).
\end{aligned}\tag{3.22}
$$

And

$$
\begin{aligned}
&|\int_{B(x;t)\cap\{y_3=0\}}\int_{\mathbb{R}^3}\mathcal{S}_i^E(v,\omega)\cdot(E+E_{\text{ext}}+\hat{v}\times(B+B_{\text{ext}})\\
&\quad-g\mathbf{e}_3)f(t-|y-x|,y_\parallel,0,v)\mathrm{d}v\frac{\mathrm{d}y_\parallel}{|y-x|}|\\
&\lesssim \left(\sup_{0\le t\le T}\|E(t)\|_\infty+\sup_{0\le t\le T}\|B(t)\|_\infty+|B_e|+E_e+g\right)\sup_{0\le t\le T}\|\langle v\rangle^{4+\delta} f(t)\|_\infty\\
&\quad\times\int_{\{(|y_\parallel-x_\parallel|^2+|x_3|^2)<t^2\}}\frac{1}{(|y_\parallel-x_\parallel|^2+x_3^2)^{1/2}}dy_\parallel\\
&\lesssim \left(\sup_{0\le t\le T}\|E(t)\|_\infty+\sup_{0\le t\le T}\|B(t)\|_\infty+|B_e|+E_e+g\right)\sup_{0\le t\le T}\|(1+|v|^{4+\delta}) f(t)\|_\infty.
\end{aligned}\tag{3.23}
$$

Thus combining (3.18), (3.19), (3.20), (3.21), (3.22), (3.23), we get

$$
\begin{aligned}
&|\nabla_{x_\parallel}(2.35)|\lesssim t^2\left(\sup_{0\le t\le T}\|\nabla_{x_\parallel}E(t)\|_\infty+\sup_{0\le t\le T}\|\nabla_{x_\parallel}B(t)\|_\infty\right)+\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_\infty\\
&|\frac{\partial}{\partial_{x_3}}(2.35)|\lesssim t^2\left(\sup_{0\le t\le T}\|\partial_{x_3}E(t)\|_\infty+\sup_{0\le t\le T}\|\partial_{x_3}B(t)\|_\infty\right)\\
&\qquad+\sup_{0\le t\le T}\left(\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t)\|_\infty\right)+\sup_{0\le t\le T}\|(1+|v|^{4+\delta}) f(t)\|_\infty.
\end{aligned}
$$

By the same argument we get the same estimate for $|\frac{\partial}{\partial_{x_k}}(2.36)|$. Therefore

$$
\begin{aligned}
&|\nabla_{x_\parallel}(2.35)|+|\nabla_{x_\parallel}(2.36)|\lesssim t^2\left(\sup_{0\le t\le T}\|\nabla_{x_\parallel}E(t)\|_\infty+\sup_{0\le t\le T}\|\nabla_{x_\parallel}B(t)\|_\infty\right)+\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_\infty\\
&|\frac{\partial}{\partial_{x_3}}(2.35)|+|\frac{\partial}{\partial_{x_3}}(2.36)|\lesssim t^2\left(\sup_{0\le t\le T}\|\partial_{x_3}E(t)\|_\infty+\sup_{0\le t\le T}\|\partial_{x_3}B(t)\|_\infty\right)\\
&\qquad+\sup_{0\le t\le T}\left(\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t)\|_\infty\right)+\sup_{0\le t\le T}\|(1+|v|^{4+\delta}) f(t)\|_\infty.
\end{aligned}\tag{3.24}
$$

Next, using the change of variables $z_\parallel=y_\parallel-x_\parallel$ we have

$$
\frac{\partial}{\partial_{x_k}}(2.37)
$$

$$
\begin{aligned}
&= \frac{\partial}{\partial_{x_k}} \left(\int_{\sqrt{t^2-|z_\parallel|^2} > x_3} \int_{\mathbb{R}^3} \left(\delta_{i3} - \frac{(\omega_i + \hat{v}_i)\hat{v}_3}{1 + \hat{v} \cdot \omega} \right) \right. \\
&\left. f(t - (|z_\parallel|^2 + x_3^2)^{1/2}, z_\parallel + x_\parallel, 0, v) \mathrm{d}v \frac{\mathrm{d}z_\parallel}{(|z_\parallel|^2 + x_3^2)^{1/2}} \right) \\
&= (1 - \delta_{k3}) \int_{\sqrt{t^2-|z_\parallel|^2} > x_3} \int_{\mathbb{R}^3} \left(\delta_{i3} - \frac{(\omega_i + \hat{v}_i)\hat{v}_3}{1 + \hat{v} \cdot \omega} \right) \partial_{x_k} \\
&f(t - (|z_\parallel|^2 + x_3^2)^{1/2}, z_\parallel + x_\parallel, 0, v) \mathrm{d}v \frac{\mathrm{d}z_\parallel}{(|z_\parallel|^2 + x_3^2)^{1/2}} \\
&\quad + \delta_{k3} \int_{\sqrt{t^2-|z_\parallel|^2} > x_3} \int_{\mathbb{R}^3} \partial_{x_3} \left(\frac{1}{(|z_\parallel|^2 + x_3^2)^{1/2}} \left(\delta_{i3} - \frac{(\omega_i + \hat{v}_i)\hat{v}_3}{1 + \hat{v} \cdot \omega} \right) \right) \\
&f(t - (|z_\parallel|^2 + x_3^2)^{1/2}, z_\parallel + x_\parallel, 0, v) \mathrm{d}v \ \mathrm{d}z_\parallel \\
&\quad + \delta_{k3} \int_{\sqrt{t^2-|z_\parallel|^2} > x_3} \int_{\mathbb{R}^3} \left(\delta_{i3} - \frac{(\omega_i + \hat{v}_i)\hat{v}_3}{1 + \hat{v} \cdot \omega} \right) \partial_t \\
&f(t - (|z_\parallel|^2 + x_3^2)^{1/2}, z_\parallel + x_\parallel, 0, v) \mathrm{d}v \frac{-x_3}{(|z_\parallel|^2 + x_3^2)} \mathrm{d}z_\parallel \\
&\quad - \delta_{k3} \int_{\sqrt{t^2-|z_\parallel|^2} = x_3} \int_{\mathbb{R}^3} \left(\delta_{i3} \frac{(\omega_i + \hat{v}_i)\hat{v}_3}{1 + \hat{v} \cdot \omega} \right) \\
&f(0, z_\parallel + x_\parallel, 0, v) \mathrm{d}v \frac{x_3}{\sqrt{t^2 - x_3^2}} \left(\frac{z_\parallel}{\sqrt{t^2 - x^2}} \cdot \frac{z_\parallel}{|z_\parallel|} \right) \frac{\mathrm{d}S_{z_\parallel}}{t}.
\end{aligned} \tag{3.25}
$$

The first term is only contribute as the tangential derivative, from (3.10),

$$
|\frac{(\omega_i + \hat{v}_i)}{1 + \hat{v} \cdot \omega}| \leq 2\sqrt{1 + |v|^2}. \tag{3.26}
$$

Thus from (3.26),

$$
\begin{aligned}
&|(1 - \delta_{k3}) \int_{\sqrt{t^2-|z_\parallel|^2} > x_3} \int_{\mathbb{R}^3} \left(\delta_{i3} - \frac{(\omega_i + \hat{v}_i)\hat{v}_3}{1 + \hat{v} \cdot \omega} \right) \partial_{x_k} \\
&\quad f(t - (|z_\parallel| + x_3^2)^{1/2}, z_\parallel + x_\parallel, 0, v) \mathrm{d}v \frac{\mathrm{d}z_\parallel}{(|z_\parallel|^2 + x_3^2)^{1/2}}| \\
&\quad \lesssim \sup_{0 \leq t \leq T} \|(1 + |v|^{4+\delta}) \nabla_{x_\parallel} f(t)\|_\infty \int_{r^2 + x_3^2 < t^2} \frac{r}{(r^2 + x_3^2)^{1/2}} dr \\
&\quad \lesssim \sup_{0 \leq t \leq T} \|(1 + |v|^{4+\delta}) \nabla_{x_\parallel} f(t)\|_\infty.
\end{aligned} \tag{3.27}
$$

For the second term, recall $(3.9)_2$ using the identity [13]

$$
\frac{(|\hat{v}|^2 - 1)(\hat{v}_i + \omega_i)}{(|y_\parallel - x_\parallel|^2 + x_3^2)(1 + \hat{v} \cdot \omega)^2} = \sum_{j=1}^{3} \frac{\partial}{\partial y_j} \left[\frac{1}{|y - x|} \left(\delta_{ij} - \frac{(\omega_i + \hat{v}_i)\hat{v}_j}{1 + \hat{v} \cdot \omega} \right) \right] \Bigg|_{y_3 = 0},
$$

we have

$$
\begin{aligned}
&|(3.9)_2 + \int_{\sqrt{t^2-|z_\parallel|^2}>x_3} \int_{\mathbb{R}^3} \partial_{x_3} \left(\frac{1}{(|z_\parallel|^2 + x_3^2)^{1/2}} \left(\delta_{i3} - \frac{(\omega_i + \hat{v}_i)\hat{v}_3}{1+\hat{v}\cdot\omega} \right) \right) \\
&f(t-(|z_\parallel|^2 + x_3^2)^{1/2}, z_\parallel + x_\parallel, 0, v)\mathrm{d}v\,\mathrm{d}z_\parallel| \\
&= |\delta_{k3} \int_{\{(|y_\parallel - x_\parallel|^2 + |x_3|^2)^{1/2} < t\}} \int_{\mathbb{R}^3} \sum_{j=1}^{2} \frac{\partial}{\partial y_j} \left[\frac{1}{(|y_\parallel - x_\parallel|^2 + x_3^2)^{1/2}} \left(\delta_{ij} - \frac{(\omega_i + \hat{v}_i)\hat{v}_j}{1+\hat{v}\cdot\omega} \right) \right] \\
&\times f(t-(|y_\parallel - x_\parallel|^2 + x_3^2)^{1/2}, y_\parallel, 0, v)\mathrm{d}v\mathrm{d}y_\parallel|,
\end{aligned} \tag{3.28}
$$

where we've used the cancellation $\frac{\partial}{\partial y_3}\left[\frac{1}{|y-x|}\left(\delta_{i3} - \frac{(\omega_i+\hat{v}_i)\hat{v}_j}{1+\hat{v}\cdot\omega}\right]\right)\Big|_{y_3=0} = -\partial_{x_3}\left(\frac{1}{(|y_\parallel - x_\parallel|^2 + x_3^2)^{1/2}}\left(\delta_{i3} - \frac{(\omega_i+\hat{v}_i)\hat{v}_3}{1+\hat{v}\cdot\omega}\right)\right)$. Thus from integration by parts and (3.10),

$$
\begin{aligned}
&|\delta_{k3} \int_{\{(|y_\parallel - x_\parallel|^2 + |x_3|^2)^{1/2} < t\}} \int_{\mathbb{R}^3} \sum_{j=1}^{2} \frac{\partial}{\partial y_j} \left[\frac{1}{|y-x|} \left(\delta_{ij} - \frac{(\omega_i + \hat{v}_i)\hat{v}_j}{1+\hat{v}\cdot\omega} \right) \right] \\
&f(t-|y-x|, y, v)\mathrm{d}v\mathrm{d}y_\parallel| \\
&\lesssim | \int_{\{(|y_\parallel - x_\parallel|^2 + |x_3|^2)^{1/2} < t\}} \left[\frac{||y_\parallel - x_\parallel|}{(|y_\parallel - x_\parallel|^2 + x_3^2)} \left(\delta_{ij} - \frac{(\omega_i + \hat{v}_i)\hat{v}_j}{1+\hat{v}\cdot\omega} \right) \right] \partial_t \\
&f(t-|y-x|, y, v)\mathrm{d}v\mathrm{d}y_\parallel| \\
&\quad + \sup_{0\le t\le T} \|(1+|v|^{4+\delta})\nabla_{x_\parallel} f(t)\|_\infty + \|(1+|v|^{4+\delta}) f_0\|_\infty.
\end{aligned} \tag{3.29}
$$

Now from (0.40), we write $\partial_t f = -\hat{v}\cdot\nabla_x f - (E + E_{\mathrm{ext}} + \hat{v}\times(B + B_{\mathrm{ext}}) - g\mathbf{e}_3)\cdot\nabla_v f$. Then using $\alpha = \hat{v}_3$ on $\partial\Omega$, integration by parts in v, and that $\nabla_v\cdot(\hat{v}\times B) = 0$, we get

$$
\begin{aligned}
&| \int_{\sqrt{t^2-|z_\parallel|^2}>x_3} \int_{\mathbb{R}^3} \frac{(\omega_i + \hat{v}_i)\hat{v}_3}{1+\hat{v}\cdot\omega} \partial_t f(t-(|z_\parallel| + x_3^2)^{1/2}, z_\parallel + x_\parallel, 0, v)\mathrm{d}v \frac{x_3}{(|z_\parallel|^2 + x_3^2)}\mathrm{d}z_\parallel| \\
&\lesssim \sup_{0\le t\le T} \left(\|(1+|v|^{5+\delta})\alpha\partial_{x_3} f(t)\|_\infty + \|(1+|v|^{4+\delta})\nabla_{x_\parallel} f(t)\|_\infty \right) \\
&+ \sup_{0\le t\le T} \|(1+|v|^{4+\delta}) f(t)\|_\infty,
\end{aligned} \tag{3.30}
$$

where we've used from (3.15), (3.17), and (3.26) that

$$
\left| \nabla_v \left(\frac{(\omega_i + \hat{v}_i)\hat{v}_3}{1+\hat{v}\cdot\omega} \right) \right| = \left| \mathcal{S}_i^E(\omega, \hat{v})\hat{v}_3 + \frac{(\omega_i + \hat{v}_i)}{1+\hat{v}\cdot\omega}\nabla_v \hat{v}_3 \right| \le 14\sqrt{1+|v|^2}.
$$

By the same argument we have

$$
\begin{aligned}
&| \int_{\{(|y_\parallel - x_\parallel|^2 + |x_3|^2)^{1/2} < t\}} \left[\frac{||y_\parallel - x_\parallel|}{(|y_\parallel - x_\parallel|^2 + x_3^2)} \left(\delta_{ij} - \frac{(\omega_i + \hat{v}_i)\hat{v}_j}{1+\hat{v}\cdot\omega} \right) \right] \partial_t f(t-|y-x|, y, v)\mathrm{d}v\mathrm{d}y_\parallel| \\
&\lesssim \sup_{0\le t\le T} \left(\|(1+|v|^{5+\delta})\alpha\partial_{x_3} f(t)\|_\infty + \|(1+|v|^{4+\delta})\nabla_{x_\parallel} f(t)\|_\infty \right) \\
&+ \sup_{0\le t\le T} \|(1+|v|^{4+\delta}) f(t)\|_\infty.
\end{aligned} \tag{3.31}
$$

We also have

$$
\begin{aligned}
&|\int_{\sqrt{t^2-|z_\parallel|^2}=x_3}\int_{\mathbb{R}^3}\frac{(\omega_i+\hat{v}_i)\hat{v}_3}{1+\hat{v}\cdot\omega}f(0,z_\parallel+x_\parallel,0,v)\mathrm{d}v\frac{x_3}{\sqrt{t^2-x_3^2}}\left(\frac{z_\parallel}{\sqrt{t^2-x_3^2}}\cdot\frac{z_\parallel}{|z_\parallel|}\right)\frac{\mathrm{d}S_{z_\parallel}}{t}| \\
&\lesssim \|\langle v\rangle^{4+\delta}f_0\|_\infty\int_{|z_\parallel|=\sqrt{t^2-x_3^2}}\int_{\mathbb{R}^3}(1+|v|^{3+\delta})^{-1}\frac{x_3}{\sqrt{t^2-x_3^2}}dv\frac{\mathrm{d}S_{z_\parallel}}{t} \\
&\lesssim \|\langle v\rangle^{4+\delta}f_0\|_\infty\left(\frac{x_3}{t}\right)\lesssim \sup_{0\le t\le T}\|\langle v\rangle^{4+\delta}f(t)\|_\infty.
\end{aligned}
\tag{3.32}
$$

Thus from (3.25), (3.27), (3.28), (3.29), (3.30), (3.31), (3.32), and together with (3.9)–(3.13), we have

$$
\begin{aligned}
&|\nabla_{x_\parallel}(2.33)+\nabla_{x_\parallel}(2.37)|\lesssim \sup_{0\le t\le T}\left\{\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel}f(t)\|_\infty\right\}, \\
&|\frac{\partial}{\partial_{x_3}}(2.33)+\frac{\partial}{\partial_{x_3}}(2.37)| \\
&\quad\lesssim \sup_{0\le t\le T}\left\{\|\langle v\rangle^{4+\delta}f(t)\|_\infty+\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3}f(t)\|_\infty+\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel}f(t)\|_\infty\right\}.
\end{aligned}
$$

By the same argument we get the same estimate for $\frac{\partial}{\partial_{x_k}}(2.34)+\frac{\partial}{\partial_{x_k}}(2.38)$. Thus

$$
\begin{aligned}
&|\nabla_{x_\parallel}(2.33)+\nabla_{x_\parallel}(2.37)|+|\nabla_{x_\parallel}(2.34)+\nabla_{x_\parallel}(2.38)|\lesssim \sup_{0\le t\le T}\left\{\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel}f(t)\|_\infty\right\}, \\
&|\frac{\partial}{\partial_{x_3}}(2.33)+\frac{\partial}{\partial_{x_3}}(2.37)|+|\frac{\partial}{\partial_{x_3}}(2.34)+\frac{\partial}{\partial_{x_3}}(2.38)| \\
&\lesssim \sup_{0\le t\le T}\left\{\|\langle v\rangle^{4+\delta}f(t)\|_\infty+\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3}f(t)\|_\infty+\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel}f(t)\|_\infty\right\}.
\end{aligned}
\tag{3.33}
$$

Next, by using the change of variables $z=y-x$ and spherical coordinate for z, we have

$$
(2.39)=\int_{t\cos\phi>-x_3}\int_0^{2\pi}\sum_j\omega_j\left(\delta_{ij}-\frac{(\omega_i+\hat{v}_i)\hat{v}_j}{1+\hat{v}\cdot\omega}\right)f(0,z+x,v)\mathrm{d}v\frac{t^2\sin\phi\,\mathrm{d}\theta\mathrm{d}\phi}{t}. \tag{3.34}
$$

Thus

$$
\begin{aligned}
&\frac{\partial}{\partial_{x_k}}(2.39) \\
&=-\int_{t\cos\phi>-x_3}\int_0^{2\pi}\sum_j\omega_j\left(\delta_{ij}-\frac{(\omega_i+\hat{v}_i)\hat{v}_j}{1+\hat{v}\cdot\omega}\right)\partial_{x_k}f(0,z+x,v)(t\sin\phi)\,\mathrm{d}v\mathrm{d}\theta\mathrm{d}\phi \\
&\quad-\delta_{k3}\int_0^{2\pi}\sum_j\omega_j\left(\delta_{ij}-\frac{(\omega_i+\hat{v}_i)\hat{v}_j}{1+\hat{v}\cdot\omega}\right)f(0,z_\parallel+x_\parallel,0,v)\frac{-1}{\sqrt{1-(\frac{x_3}{t})^2}}\frac{-1}{t}\left(t\sqrt{1-\left(\frac{x_3}{t}\right)^2}\right)\mathrm{d}v\mathrm{d}\theta.
\end{aligned}
\tag{3.35}
$$

So from (3.26), for $k=1,2$,

$$
\begin{aligned}
&|\int_{t\cos\phi>-x_3}\int_0^{2\pi}\sum_j\omega_j\left(\delta_{ij}-\frac{(\omega_i+\hat{v}_i)\hat{v}_j}{1+\hat{v}\cdot\omega}\right)\partial_{x_k}f(0,z+x,v)(t\sin\phi)\,\mathrm{d}v\mathrm{d}\theta\mathrm{d}\phi| \\
&\lesssim \|\langle v\rangle^{4+\delta}\nabla_{x_\parallel}f_0\|_\infty\int_{t\cos\phi>-x_3}\int_0^{2\pi}\int_{\mathbb{R}^3}\langle v\rangle^{-4-\delta}(t\sin\phi)\,\mathrm{d}v\mathrm{d}\theta\mathrm{d}\phi\lesssim \|\langle v\rangle^{4+\delta}\nabla_{x_\parallel}f_0\|_\infty.
\end{aligned}
\tag{3.36}
$$

And for $k = 3$,

$$
\begin{aligned}
&|\int_{t\cos\phi > -x_3} \int_0^{2\pi} \sum_j \omega_j \left(\delta_{ij} - \frac{(\omega_i + \hat{v}_i)\hat{v}_j}{1 + \hat{v}\cdot\omega} \right) \partial_{x_3} f(0, z + x, v)(t \sin\phi)\, \mathrm{d}v \mathrm{d}\theta \mathrm{d}\phi| \\
&\lesssim \|\langle v\rangle^{5+\delta} \alpha \partial_{x_3} f_0\|_\infty \int_0^{t+x_3} |\ln(s)| ds \lesssim \|\langle v\rangle^{5+\delta} \alpha f_0\|_\infty,
\end{aligned}
\tag{3.37}
$$

and

$$
\begin{aligned}
&|\int_0^{2\pi} \sum_j \omega_j \left(\delta_{ij} - \frac{(\omega_i + \hat{v}_i)\hat{v}_j}{1 + \hat{v}\cdot\omega} \right) f(0, z_\| + x_\|, 0, v) \frac{-1}{\sqrt{1 - \left(\frac{x_3}{t}\right)^2}} \frac{-1}{t} \left(t\sqrt{1 - \left(\frac{x_3}{t}\right)^2} \right) \mathrm{d}v \mathrm{d}\theta| \\
&= |\int_0^{2\pi} \sum_j \omega_j \left(\delta_{ij} - \frac{(\omega_i + \hat{v}_i)\hat{v}_j}{1 + \hat{v}\cdot\omega} \right) f(0, z_\| + x_\|, 0, v)\, \mathrm{d}v \mathrm{d}\theta| \lesssim \|\langle v\rangle^{4+\delta} f(0)\|_\infty.
\end{aligned}
\tag{3.38}
$$

Therefore, we have

$$
|\nabla_{x_\|}(2.39)| \lesssim \|\langle v\rangle^{4+\delta} \nabla_{x_\|} f_0\|_\infty, \ |\frac{\partial}{\partial_{x_3}}(2.39)| \lesssim \|\langle v\rangle^{5+\delta} \alpha \partial_{x_3} f_0\|_\infty + \|\langle v\rangle^{4+\delta} f(0)\|_\infty.
$$

And by the same argument we have the same estimate for $\frac{\partial}{\partial_{x_k}}(2.40)$. Thus

$$
\begin{aligned}
&|\nabla_{x_\|}(2.39)| + |\nabla_{x_\|}(2.40)| \lesssim \|\langle v\rangle^{4+\delta} \nabla_{x_\|} f_0\|_\infty, \\
&|\frac{\partial}{\partial_{x_3}}(2.39)| + |\frac{\partial}{\partial_{x_3}}(2.40)| \lesssim \|\langle v\rangle^{5+\delta} \alpha \partial_{x_3} f_0\|_\infty + \|\langle v\rangle^{4+\delta} f(0)\|_\infty.
\end{aligned}
\tag{3.39}
$$

Finally, we estimate $\frac{\partial}{\partial x_k}(2.41)$. We have

$$
\begin{aligned}
|\frac{\partial}{\partial x_k}(2.41)| \lesssim (1 - \delta_{k3}) & \int_{\sqrt{t^2 - |z_\||^2} > x_3} \int_{\mathbb{R}^3} |\partial_{x_k} f(t - (|z_\||^2 + x_3^2)^{1/2}, z_\| \\
& + x_\|, 0, v)| \mathrm{d}v \frac{\mathrm{d}z_\|}{(|z_\||^2 + x_3^2)^{1/2}} \\
& + \delta_{k3} \int_{\sqrt{t^2 - |z_\||^2} > x_3} \int_{\mathbb{R}^3} |f(t - (|z_\||^2 + x_3^2)^{1/2}, z_\| \\
& + x_\|, 0, v) \mathrm{d}v \ \partial_{x_3} \left(\frac{1}{(|z_\||^2 + x_3^2)^{1/2}} \right) |\mathrm{d}z_\| \\
& + \delta_{k3} \int_{\sqrt{t^2 - |z_\||^2} > x_3} \int_{\mathbb{R}^3} |\partial_t f(t - (|z_\||^2 + x_3^2)^{1/2}, z_\| \\
& + x_\|, 0, v) \mathrm{d}v \frac{-x_3}{(|z_\||^2 + x_3^2)} |\mathrm{d}z_\| \\
& - \delta_{k3} \int_0^{2\pi} \int_{\mathbb{R}^3} |\frac{|f(0, z_\| + x_\|, 0, v)|}{t} \mathrm{d}v \frac{x_3}{\sqrt{t^2 - x_3^2}} \sqrt{t^2 - x_3^2} d\theta.
\end{aligned}
$$

Similar to the estimate in (3.25)–(3.32), we get

$$
\begin{aligned}
&|\nabla_{x_\parallel}(2.41)| \lesssim \sup_{0\le t\le T}\left\{\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_\infty\right\},\\
&|\frac{\partial}{\partial x_3}(2.41)| \lesssim \sup_{0\le t\le T}\left\{\|\langle v\rangle^{4+\delta} f(t)\|_\infty + \|\langle v\rangle^{5+\delta}\alpha\nabla_x f(t)\|_\infty + \|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_\infty\right\}.
\end{aligned}
\tag{3.40}
$$

Next, we estimate the ∂_{x_k} derivatives to B. Using the same argument as in (3.5)–(3.8), we get

$$
|\frac{\partial}{\partial_{x_k}}(2.45)| + |\frac{\partial}{\partial_{x_k}}(2.46)| \lesssim \|E_0\|_{C^2}. \tag{3.41}
$$

Next, From (3.10) we have

$$
\begin{aligned}
|\frac{(\omega\times\hat{v})\left(1-|\hat{v}|^2\right)}{(1+\hat{v}\cdot\omega)^2}| &\le 2|\frac{\omega\times v}{(1+|v|^2)^{3/2}(1+\hat{v}\cdot\omega)^2}|\\
&\le 8\frac{|\omega\times v|\sqrt{1+|v|^2}}{(1+|\omega\times v|^2)^2} \le 8\sqrt{1+|v|^2}.
\end{aligned}
\tag{3.42}
$$

Following the same argument as in (3.9)–(3.13), and (3.25)–(3.33), we obtain

$$
\begin{aligned}
&|\nabla_{x_\parallel}(2.47) + \nabla_{x_\parallel}(2.51)| + |\nabla_{x_\parallel}(2.48) + \nabla_{x_\parallel}(2.52)| \lesssim \sup_{0\le t\le T}\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_\infty\\
&|\frac{\partial}{\partial_{x_3}}(2.47) + \frac{\partial}{\partial_{x_3}}(2.51)| + |\frac{\partial}{\partial_{x_3}}(2.48) + \frac{\partial}{\partial_{x_3}}(2.52)|\\
&\lesssim \sup_{0\le t\le T}\|(1+|v|^{5+\delta})\alpha\partial_{x_3} f(t)\|_\infty + \sup_{0\le t\le T}\|\langle v\rangle^{4+\delta} f(t)\|_\infty|.
\end{aligned}
\tag{3.43}
$$

Next, from (3.44) we have

$$
\begin{aligned}
\frac{\partial}{\partial x_k}(2.49) = \frac{\partial}{\partial x_k}\Bigg(\int_{B(x;t)\cap\{y_3>0\}}\int_{\mathbb{R}^3} &\mathcal{S}_i^B(v,\omega)\cdot(E+E_{\text{ext}}+\hat{v}\\
&\times(B+B_{\text{ext}}) - g\mathbf{e}_3) f(t-|y-x|,y,v)\mathrm{d}v\frac{\mathrm{d}y}{|y-x|}\Bigg),
\end{aligned}
\tag{3.44}
$$

where by direct calculation we have

$$
\mathcal{S}_i^B(v,\omega) = \frac{\nabla_v[(\omega\times v)_i]}{\sqrt{1+|v|^2}(1+\hat{v}\cdot\omega)} + \frac{(\omega\times v)_i(\hat{v}+\omega)}{(\sqrt{1+|v|^2}(1+\hat{v}\cdot\omega))^2}. \tag{3.45}
$$

Then applying the same argument as in (3.18)–(3.24), we obtain

$$
\begin{aligned}
|\nabla_{x_\parallel}(2.49)| + |\nabla_{x_\parallel}(2.50)| &\lesssim t^2\left(\sup_{0\le t\le T}\|\nabla_{x_\parallel}E(t)\|_\infty + \sup_{0\le t\le T}\|\nabla_{x_\parallel}B(t)\|_\infty\right)\\
&\quad + \|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_\infty\\
|\frac{\partial}{\partial_{x_3}}(2.49)| + |\frac{\partial}{\partial_{x_3}}(2.50)| &\lesssim t^2\left(\sup_{0\le t\le T}\|\partial_{x_3}E(t)\|_\infty + \sup_{0\le t\le T}\|\partial_{x_3}B(t)\|_\infty\right)\\
&\quad + \sup_{0\le t\le T}\left(\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t)\|_\infty\right) + \sup_{0\le t\le T}\|(1+|v|^{4+\delta}) f(t)\|_\infty.
\end{aligned}
\tag{3.46}
$$

Next, using the same argument as in (3.34)–(3.39), we have

$$
\begin{aligned}
&|\nabla_{x_\|}(2.53)| + |\nabla_{x_\|}(2.54)| \lesssim \|\langle v\rangle^{4+\delta}\nabla_{x_\|} f_0\|_\infty,\\
&|\frac{\partial}{\partial_{x_3}}(2.53)| + |\frac{\partial}{\partial_{x_3}}(2.54)| \lesssim \|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f_0\|_\infty + \|\langle v\rangle^{4+\delta} f(0)\|_\infty.
\end{aligned}
\tag{3.47}
$$

Finally, similar (3.40), we have

$$
\begin{aligned}
|\nabla_{x_\|}(2.55)| &\lesssim \sup_{0\le t\le T}\left\{\|\langle v\rangle^{4+\delta}\nabla_{x_\|} f(t)\|_\infty\right\},\\
|\frac{\partial}{\partial x_3}(2.55)| &\lesssim \sup_{0\le t\le T}\Big\{\|\langle v\rangle^{4+\delta} f(t)\|_\infty(1+\sup_{0\le t\le T}(\|E(t)\|_\infty+\|B(t)\|_\infty))\\
&\quad + \|\langle v\rangle^{5+\delta}\alpha\nabla_x f(t)\|_\infty + \|\langle v\rangle^{4+\delta}\nabla_{x_\|} f(t)\|_\infty\Big\}.
\end{aligned}
\tag{3.48}
$$

Collecting (3.9), (3.24), (3.33), (3.39), and (3.40), and (3.8)–(3.48), and letting $T \ll 1$, we get

$$
\begin{aligned}
\|\nabla_{x_\|}E\|_\infty + \|\nabla_{x_\|}B\|_\infty &\lesssim \|E_0\|_{C^2} + \|B_0\|_{C^2} + \sup_{0\le t\le T}\langle v\rangle^{4+\delta}\nabla_{x_\|} f(t)\|_\infty.\\
\|\partial_{x_3}E\|_\infty + \|\partial_{x_3}B\|_\infty &\lesssim \|E_0\|_{C^2} + \|B_0\|_{C^2}\\
&\quad + \sup_{0\le t\le T}\|\left(\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t)\|_\infty + \langle v\rangle^{4+\delta}\nabla_{x_\|} f(t)\|_\infty\right)\\
&\quad + \sup_{0\le t\le T}\|\langle v\rangle^{4+\delta} f(t)\|_\infty.
\end{aligned}
\tag{3.49}
$$

This concludes (3.2) and (3.3).

For $\partial_t E$, and $\partial_t B$, from the Maxwell equations (0.21), we have $\|\partial_t E\|_\infty \le \|\nabla_x B\|_\infty + \|\langle v\rangle^{4+\delta} f\|_\infty$, $\|\partial_t B\|_\infty \le \|\nabla_x E\|_\infty$. So from (3.2), (3.3), we get (3.4). □

4. Estimates on Trajectories

We have the following crucial lemma [38]:

Lemma 8 (Velocity lemma). *Let α be defined as in (0.57). Suppose*

$$
\begin{aligned}
\sup_{0\le t\le T}\Big(&\|E(t)\|_\infty + \|B(t)\|_\infty\\
&+ \|\partial_t E_3(t)\|_\infty + \|\partial_t(\hat{v}\times B)_3(t)\|_\infty\\
&+ \|\nabla_x E_3(t)\|_\infty + \|\nabla_x(\hat{v}\times B)_3(t)\|_\infty\Big) + g + B_e < C.
\end{aligned}
\tag{4.1}
$$

And for all $t, x_\|$,

$$
g - E_e - E_3(t, x_\|, 0) - (\hat{v}\times B)_3(t, x_\|, 0) > c_0 \text{ for some } c_0 > 0. \tag{4.2}
$$

Then for any $(t, x, v) \in (0, T)\times\Omega\times\mathbb{R}^3$, with the trajectory $X(s; t, x, v)$ and $V(s; t, x, v)$ satisfies (0.54),

$$
e^{-10\frac{C}{c_0}|t-s|}\alpha(t, x, v) \le \alpha(s, X(s; t, x, v), V(s; t, x, v)) \le e^{10\frac{C}{c_0}|t-s|}\alpha(t, x, v). \tag{4.3}
$$

Proof. Note that

$$\frac{\partial \hat{v}_3}{\partial v_3} = \frac{1}{\langle v\rangle} - \frac{(\hat{v}_3)^2}{\langle v\rangle},\ \nabla_v\left(\frac{1}{\langle v\rangle}\right) = -\frac{\hat{v}}{\langle v\rangle^2}.$$

By direct computation,

$$\begin{aligned}
&[\partial_t + \hat{v}\cdot\nabla_x + \mathfrak{F}\cdot\nabla_v](\alpha^2)\\
&\quad = -2\frac{\hat{v}_3}{\langle v\rangle}\mathfrak{F}_3(t,x_\parallel,0,v) + 2\frac{\hat{v}_3}{\langle v\rangle}\mathfrak{F}_3(t,x,v) \qquad (4.4)\\
&\qquad + 2\left(\partial_t\mathfrak{F}_3(t,x_\parallel,0,v)\right)\frac{x_3}{\langle v\rangle} + 2\hat{v}_3x_3 - (2\hat{v}_\parallel\cdot\nabla_{x_\parallel}\mathfrak{F}_3(t,x_\parallel,0,v))\frac{x_3}{\langle v\rangle}\\
&\qquad - 2\frac{(\hat{v}_3)^3}{\langle v\rangle}\mathfrak{F}_3(t,x,v) - 2(\hat{v}_3)^2\frac{\hat{v}_\parallel}{\langle v\rangle}\cdot\mathfrak{F}_\parallel(t,x,v) - 2\frac{x_3}{\langle v\rangle}\mathfrak{F}(t,x,v)\cdot\nabla_v\mathfrak{F}_3(t,x_\parallel,0,v)\\
&\qquad + 2x_3\frac{\hat{v}}{\langle v\rangle^2}\cdot\mathfrak{F}(t,x,v)\mathfrak{F}_3(t,x_\parallel,0,v). \qquad (4.5)
\end{aligned}$$

Using the fundamental theorem of calculus

$$(4.4) = 2\frac{\hat{v}_3}{\langle v\rangle}\left(\int_0^{x_3}\partial_{x_3}\mathfrak{F}_3(t,x_\parallel,s,v)ds\right).$$

Since $\mathfrak{F}_3 = E_3 + E_e + \hat{v}_1B_2 - \hat{v}_2B_1 - g$, and since $\hat{v}\cdot\mathfrak{F} = \hat{v}\cdot(E + E_{\text{ext}} + \hat{v}\times(B_3 + B_{\text{ext}}) - g\mathbf{e}_3) = \hat{v}\cdot E - (g - E_e)\hat{v}_3$, we have

$$\begin{aligned}
&[\partial_t + \hat{v}\cdot\nabla_x + \mathfrak{F}\cdot\nabla_v](\alpha^2)\\
&\quad = 2\frac{\hat{v}_3}{\langle v\rangle}\left(\int_0^{x_3}\partial_{x_3}E_3(t,x_\parallel,s) + \hat{v}_1\partial_{x_3}B_2(t,x_\parallel,s) - \hat{v}_2\partial_{x_3}B_1(t,x_\parallel,s)ds\right)\\
&\qquad + 2\frac{x_3}{\langle v\rangle}\left((\partial_t - \hat{v}_\parallel\cdot\nabla_{x_\parallel})\left(E_3(t,x_\parallel,0) + \hat{v}_1B_2(t,x_\parallel,0) - \hat{v}_2B_1(t,x_\parallel,0)\right)\right) + 2\hat{v}_3x_3\\
&\qquad - 2\frac{(\hat{v}_3)^2}{\langle v\rangle}\left(\hat{v}\cdot E(t,x,v) - (g - E_e)\hat{v}_3\right)\\
&\qquad - 2\frac{x_3}{\langle v\rangle}\left(E + E_{\text{ext}} + (\hat{v}\times(B + B_{\text{ext}}) - g\mathbf{e}_3)\right)\cdot\nabla_v(\hat{v}_1B_2(t,x_\parallel,0) - \hat{v}_2B_1(t,x_\parallel,0)) \qquad (4.6)\\
&\qquad + 2x_3\frac{\hat{v}\cdot E - (g - E_e)\hat{v}_3}{\langle v\rangle^2}\left(E_3(t,x_\parallel,0) + \hat{v}_1B_2(t,x_\parallel,0) - \hat{v}_2B_1(t,x_\parallel,0) - g\right)\\
&\qquad \underbrace{-2\frac{x_3}{\langle v\rangle}(\hat{v}\times B_{\text{ext}})\cdot\nabla_v(\hat{v}_1B_2(t,x_\parallel,0) - \hat{v}_2B_1(t,x_\parallel,0))}_{(4.6)_1}.
\end{aligned}$$

Now from the assumptions (4.1) and (4.2), all the terms on the RHS of (4.6) except $(4.6)_1$ can be bounded by

$$\frac{C_1}{c_0}\left(c_0\frac{x_3}{\langle v\rangle} + (x_3)^2 + (\hat{v}_3)^2\right), \qquad (4.7)$$

where $C_1 = \sup_{0\leq t\leq T}\Big(\|E(t)\|_\infty + \|B(t)\|_\infty + \|E_3(t)\|_{W^{1,\infty}} + \|B_1(t)\|_{W^{1,\infty}} + \|B_2(t)\|_{W^{1,\infty}}\Big) + E_e + g$. And from direct computation,

$$\begin{aligned}
(4.6)_1 &= -2B_e\frac{x_3}{\langle v\rangle^2}\begin{bmatrix}\hat v_2\\ -\hat v_1\\ 0\end{bmatrix}\cdot\left(\begin{bmatrix}1-\hat v_1^2\\ -\hat v_1\hat v_2\\ -\hat v_1\hat v_3\end{bmatrix}B_2(t,x_\parallel,0) - \begin{bmatrix}-\hat v_2\hat v_1\\ 1-\hat v_2^2\\ -\hat v_2\hat v_3\end{bmatrix}B_1(t,x_\parallel,0)\right)\\
&= -2B_e\frac{x_3}{\langle v\rangle^2}\left(\hat v_2B_2(t,x_\parallel,0) + \hat v_1B_1(t,x_\parallel,0)\right),
\end{aligned}$$

thus from (4.2)

$$|(4.6)_1| \leq \frac{B_e}{c_0}c_0\frac{x_3}{\langle v\rangle}. \tag{4.8}$$

Combining (4.7) and (4.8), we get

$$|[\partial_t + \hat v\cdot\nabla_x + \mathfrak{F}\cdot\nabla_v](\alpha^2)| \leq 10(\frac{C_1+B_e}{c_0})c_0\frac{x_3}{\langle v\rangle} + 8C_1\left((x_3)^2 + (\hat v_3)^2\right). \tag{4.9}$$

From the expression of α in (0.57) and the assumption (4.2), this yields

$$|[\partial_t + \hat v\cdot\nabla_x + \mathfrak{F}\cdot\nabla_v](\alpha^2)| \leq 20(\frac{C_1+B_e}{c_0})\alpha^2, \tag{4.10}$$

Thus along the characteristics, by the Gröwall's inequality we get

$$\begin{aligned}
e^{-20(\frac{C_1+B_e}{c_0})|t-s|}\alpha^2(t,x,v) &\leq \alpha^2(s,X(s;t,x,v),V(s;t,x,v))\\
&\leq e^{20(\frac{C_1+B_e}{c_0})|t-s|}\alpha^2(t,x,v).
\end{aligned} \tag{4.11}$$

Taking square root we get (4.3). □

Lemma 9. *Let α be defined as in (0.57). Then for any $(t,x)\in[0,T)\times\Omega$, we have*

$$\int_{\mathbb{R}^3}\frac{\mathbf{1}_{|v|\leq M}}{\alpha(t,x,v)}dv \leq 4M^3\ln\left(1+\frac{1}{x_3}\right), \tag{4.12}$$

$$\int_{\mathbb{R}^3}\frac{1}{1+|v|^{4+\delta}}\frac{1}{\alpha(t,x,v)}dv \leq C_\delta\ln\left(1+\frac{1}{x_3}\right). \tag{4.13}$$

Proof. From (0.57) we have

$$\begin{aligned}
\int_{\mathbb{R}^3}\frac{\mathbf{1}_{|v|\leq M}}{\alpha(t,x,v)}dv &= \int_{|v|\leq M}\Big((x_3)^2 + (\hat v_3)^2 - 2\big(E_3(t,x_\parallel,0)\\
&\quad + E_e + (\hat v\times B)_3(t,x_\parallel,0) - g\big)\frac{x_3}{\langle v\rangle}\Big)^{-1/2}dv\\
&\leq \int_{|v|\leq M}\frac{2}{x_3+\frac{|v_3|}{\langle v\rangle}}dv \leq \int_{|v_3|\leq M}\frac{2M^2}{x_3+\frac{|v_3|}{M}}dv_3\\
&= 4M^3\ln\left(x_3+\frac{v_3}{M}\right)\Big|_0^M = 4M^3\ln\left(1+\frac{1}{x_3}\right).
\end{aligned}$$

Now, for (4.13), we have

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{1}{1+|v|^{4+\delta}} \frac{1}{\alpha(t,x,v)} dv &\le \int_{\mathbb{R}^3} \frac{1}{1+|v|^{4+\delta}} \frac{2}{x_3 + \frac{|v_3|}{\langle v\rangle}} dv \\ &\lesssim \int_{\mathbb{R}^3} \frac{1}{\left(1+|v|^{3+\delta}\right)(x_3+|v_3|)} dv.\end{aligned} \tag{4.14}$$

Using the spherical coordinate $v = (r,\theta,\phi)$ we have $dv = r^2 \sin\phi\, dr d\theta d\phi$, $v_3 = r\cos\phi$, and

$$\begin{aligned}&\int_{\mathbb{R}^3} \frac{1}{\left(1+|v|^{3+\delta}\right)(x_3+|v_3|)} dv = 4\pi \int_0^\infty \int_0^{\pi/2} \frac{r^2\sin\phi}{(1+r^{3+\delta})(x_3+r\cos\phi)} d\phi dr \\ &= 4\pi \int_0^1 \int_0^{\pi/2} \frac{r^2\sin\phi}{(1+r^{3+\delta})(x_3+r\cos\phi)} d\phi dr \\ &+ 4\pi \int_1^\infty \int_0^{\pi/2} \frac{r^2\sin\phi}{(1+r^{3+\delta})(x_3+r\cos\phi)} d\phi dr.\end{aligned} \tag{4.15}$$

Using change of variables $r\cos\phi = u$, $-r\sin\phi\, d\phi = du$, we have

$$\begin{aligned}&\int_0^1 \int_0^{\pi/2} \frac{r^2\sin\phi}{(1+r^{3+\delta})(x_3+r\cos\phi)} d\phi dr \le \int_0^1 \int_0^{\pi/2} \frac{r^2\sin\phi}{x_3+r\cos\phi} d\phi dr \\ &= \int_0^1 \int_0^r \frac{r}{x_3+u} du dr = \int_0^1 r\ln\left(1+\frac{r}{x_3}\right) dr < \ln\left(1+\frac{1}{x_3}\right).\end{aligned} \tag{4.16}$$

And using change of variables $\cos\phi = u$, $-\sin\phi\, d\phi = du$, we have

$$\begin{aligned}&\int_1^\infty \int_0^{\pi/2} \frac{r^2\sin\phi}{(1+r^{3+\delta})(x_3+r\cos\phi)} d\phi dr \le C_\delta \int_0^{\pi/2} \frac{\sin\phi}{x_3+\cos\phi} d\phi \\ &= C_\delta \int_0^1 \frac{1}{x_3+u} du = C_\delta \ln\left(1+\frac{1}{x_3}\right).\end{aligned} \tag{4.17}$$

Combining (4.14), (4.15), (4.16), and (4.17) we conclude (4.13). □

We have the following estimate on the backward exit time $t_{\mathbf{b}}$ for the trajectory.

Lemma 10. *Let $(t,x,v) \in (0,T)\times\Omega\times\mathbb{R}^3$, and the trajectory $X(s;t,x,v)$ and $V(s;t,x,v)$ satisfies (0.54). Extending $E(t) = E_0$, $B(t) = B_0$ for $t < 0$. Suppose for all t,x,v,*

$$g - E_e - E_3(t,x_\parallel,0) - (\hat v \times B)_3(t,x_\parallel,0) > c_0, \tag{4.18}$$

then there exists a C depending on T, g, B_e, $\|E\|_{W^{1,\infty}((0,T)\times\Omega)}$, $\|B\|_{W^{1,\infty}((0,T)\times\Omega)}$ such that

$$\frac{t_{\mathbf{b}}(t,x,v)}{\sup_{t-t_{\mathbf{b}}<s<t}\sqrt{1+|V(s)|^2}} \le \frac{C}{c_0}\hat v_{\mathbf{b},3}. \tag{4.19}$$

If $t - t_{\mathbf{b}}(t,x,v) \le 0$, then

$$\frac{t}{\sup_{0\le s\le t}\sqrt{1+|V(s)|^2}} \le \frac{C}{c_0}\alpha(0,X(0),V(0)). \tag{4.20}$$

Proof. We first prove (4.19). For any $(t, x, v) \in (0, T) \times \Omega \times \mathbb{R}^3$ with $t - t_{\mathbf{b}}(t, x, v) > 0$, we have

$$v_{\mathbf{b},3} - v_3 = \int_{t-t_{\mathbf{b}}}^{t} -\mathfrak{F}_3(s, X(s; t, x, v), V(s; t, x, v))ds.$$

From (4.18) this implies

$$\int_{t-t_{\mathbf{b}}}^{t} c_0 ds < \int_{t-t_{\mathbf{b}}}^{t} -\mathfrak{F}_3(s, X(s; t, x, v), V(s; t, x, v))ds \leq |v_{\mathbf{b},3}| + |v_3| \quad (4.21)$$

On the other hand, from (4.3),

$$\begin{aligned} |v_{\mathbf{b},3}| + |v_3| = \langle v_{\mathbf{b}}\rangle|\hat{v}_{\mathbf{b},3}| + \langle v\rangle|\hat{v}_3| &\leq \sup_{t-t_{\mathbf{b}}<s<t} \langle V(s)\rangle \left(\hat{v}_{\mathbf{b},3} + \alpha(t, x, v)\right) \\ &< C \sup_{t-t_{\mathbf{b}}<s<t} \langle V(s)\rangle \hat{v}_{\mathbf{b},3}. \end{aligned} \quad (4.22)$$

Combining (4.21) and (4.22) we get $t_{\mathbf{b}} c_0 < C \sup_{t-t_{\mathbf{b}}<s<t} \langle V(s)\rangle \hat{v}_{\mathbf{b},3}$. This implies (4.19).

For $t - t_{\mathbf{b}}(t, x, v) \leq 0$, using the same argument we have,

$$c_0 t < \int_0^t -\mathfrak{F}_3(s, X(s), V(s))ds \leq |V_3(0)| + |v_3|$$

and from (4.3),

$$\begin{aligned} |V_3(0)| + |v_3| &\leq \sup_{t-t_{\mathbf{b}}<s<t} \langle V(s)\rangle \left(\alpha(0, X(0), V(0)) + \alpha(t, x, v)\right) \\ &< C \sup_{t-t_{\mathbf{b}}<s<t} \langle V(s)\rangle \alpha(0, X(0), V(0)), \end{aligned}$$

thus we get $c_0 t < C \sup_{t-t_{\mathbf{b}}<s<t} \langle V(s)\rangle \alpha(0, X(0), V(0))$, and this yields (4.20). □

Lemma 11. *Suppose*

$$\sup_{0\leq t\leq T} \|\nabla_x E(t)\|_\infty + \sup_{0\leq t\leq T} \|\nabla_x B(t)\|_\infty < \infty.$$

Then for any $s, t \in (0, T)$, we have

$$\begin{aligned} &|\partial_{x_i} X(s; t, x, v)| \lesssim e^{C_1|t-s|}, \ |\partial_{x_i} V(s; t, x, v)| \lesssim (t - s)e^{C_1|t-s|}, \\ &|\partial_{v_i} X(s; t, x, v)| \lesssim \frac{|t - s|}{\langle V(s)\rangle} e^{C_1|t-s|}, \ |\partial_{v_i} V(s; t, x, v)| \lesssim e^{C_1|t-s|}, \end{aligned} \quad (4.23)$$

and for $i \neq j$,

$$|\partial_{x_i} X_j(s; t, x, v)| \lesssim e^{C_1|t-s|} \frac{|t - s|^2}{\langle V(s)\rangle}. \quad (4.24)$$

where $C_1 = \left(\sup_{0\leq t\leq T} (\|\nabla_x E(t)\|_\infty + \|\nabla_x B(t)\|_\infty + \|E(t)\|_\infty + \|B(t)\|_\infty) + g + |B_e|\right)^2$.

Proof. The expressions of $X(s;t,x,v)$ and $V(s;t,x,v)$ are

$$\begin{aligned} X(s;t,x,v) &= x-(t-s)\hat{v}+\int_s^t\int_\tau^t \hat{\mathfrak{F}}(\tau',X(\tau'),V(\tau'))\mathrm{d}\tau'\mathrm{d}\tau, \\ V(s;t,x,v) &= v-\int_s^t \mathfrak{F}(\tau,X(\tau),V(\tau))\mathrm{d}\tau. \end{aligned} \tag{4.25}$$

We denote

$$\begin{aligned} \frac{d}{ds}\hat{V}(s) &= \frac{1}{\sqrt{1+|V(s)|^2}}\frac{d}{ds}V(s)-\frac{1}{\sqrt{1+|V(s)|^2}}\hat{V}(s)\cdot\frac{d}{ds}V(s)\hat{V}(s) \\ &= \frac{1}{\sqrt{1+|V(s)|^2}}\mathfrak{F}(s,X(s),V(s)) \\ &\quad -\frac{1}{\sqrt{1+|V(s)|^2}}\hat{V}(s)\cdot\mathfrak{F}(s,X(s),V(s))\hat{V}(s) \\ &:= \hat{\mathfrak{F}}(s,X(s),V(s)). \end{aligned} \tag{4.26}$$

By direct computation we get

$$\begin{aligned} \partial_{x_i}X(s;t,x,v) &= e_i+\int_s^t\int_\tau^t[\nabla_x\hat{\mathfrak{F}}(\tau')\cdot\partial_{x_i}X(\tau')+\nabla_v\hat{\mathfrak{F}}(\tau')\cdot\partial_{x_i}V(\tau')]\mathrm{d}\tau'\mathrm{d}\tau, \\ \partial_{v_i}X(s;t,x,v) &= -(t-s)\partial_{v_i}\hat{v}+\int_s^t\int_\tau^t[\nabla_x\hat{\mathfrak{F}}(\tau')\cdot\partial_{v_i}X(\tau')+\nabla_v\hat{\mathfrak{F}}(\tau')\cdot\partial_{v_i}V(\tau')]\mathrm{d}\tau'\mathrm{d}\tau, \\ \partial_{x_i}V(s;t,x,v) &= -\int_s^t[\nabla_x\mathfrak{F}(\tau')\cdot\partial_{x_i}X(\tau)+\nabla_v\mathfrak{F}(\tau')\cdot\partial_{x_i}V(\tau)]\mathrm{d}\tau \\ \partial_{v_i}V(s;t,x,v) &= e_i-\int_s^t[\nabla_x\mathfrak{F}(\tau')\cdot\partial_{v_i}X(\tau)+\nabla_v\mathfrak{F}(\tau')\cdot\partial_{v_i}V(\tau)]\mathrm{d}\tau. \end{aligned} \tag{4.27}$$

Since $\mathfrak{F}=E+E_{\mathrm{ext}}+\hat{v}\times(B+B_{\mathrm{ext}})-g\mathbf{e}_3$, we have

$$|\nabla_x\mathfrak{F}|\le|\nabla_xE|+|\nabla_xB|. \tag{4.28}$$

And since $\partial_{v_i}\hat{v}=\frac{e_i}{\langle v\rangle}-\frac{\hat{v}_i\hat{v}}{\langle v\rangle}$,

$$|\nabla_v\mathfrak{F}|\lesssim\frac{1}{\sqrt{1+|v|^2}}(|B|+|B_e|). \tag{4.29}$$

From the expression of $\hat{\mathfrak{F}}$ in (4.26), we have from (4.28),

$$|\nabla_x\hat{\mathfrak{F}}|\le\frac{2}{\sqrt{1+|v|^2}}|\nabla_x\mathfrak{F}|\lesssim\frac{1}{\sqrt{1+|v|^2}}\left(|\nabla_xE|+|\nabla_xB|\right), \tag{4.30}$$

and from (4.29),

$$\begin{aligned} |\nabla_v\hat{\mathfrak{F}}| &\le \left|\nabla_v(\frac{1}{\sqrt{1+|v|^2}})(\mathfrak{F}-\hat{v}\cdot\mathfrak{F}\cdot\hat{v})\right|+\left|\frac{1}{\sqrt{1+|v|^2}}\left(\nabla_v\mathfrak{F}-\nabla_v(\hat{v}\cdot\mathfrak{F}\cdot\hat{v})\right)\right| \\ &\lesssim \frac{1}{1+|v|^2}|\mathfrak{F}|+\frac{1}{\sqrt{1+|v|^2}}|\nabla_v\mathfrak{F}| \\ &\lesssim \frac{1}{1+|v|^2}\left(|E|+|B|+g+|B_e|\right) \end{aligned} \tag{4.31}$$

From (4.27) and Fubini's theorem,

$$\begin{aligned}\partial_{x_i} X(s) &= e_i + \int_s^t \int_s^{\tau'} [\nabla_x \hat{\mathfrak{F}}(\tau') \cdot \partial_{x_i} X(\tau') + \nabla_v \hat{\mathfrak{F}}(\tau') \cdot \partial_{x_i} V(\tau')] \mathrm{d}\tau \mathrm{d}\tau' \\ &= e_i + \int_s^t (\tau' - s)[\nabla_x \hat{\mathfrak{F}}(\tau') \cdot \partial_{x_i} X(\tau') + \nabla_v \hat{\mathfrak{F}}(\tau') \cdot \partial_{x_i} V(\tau')] \mathrm{d}\tau'.\end{aligned}$$

Therefore, from (4.30) and (4.31) we have

$$\begin{aligned}|\partial_{x_i} X(s)| &\leq 1 + (t-s) \int_s^t \Big(|\nabla_x \hat{\mathfrak{F}}(\tau)| |\partial_{x_i} X(\tau)| + |\nabla_v \hat{\mathfrak{F}}(\tau)| |\partial_{x_i} V(\tau)| \Big) \mathrm{d}\tau \\ &\lesssim 1 + (t-s) \left(\|\nabla_x E\|_\infty + \|\nabla_x B\|_\infty \right) \int_s^t \frac{1}{\sqrt{1+|V(\tau)|^2}} |\partial_{x_i} X(\tau)| d\tau \\ &\quad + (t-s) \left(\|E\|_\infty + \|B\|_\infty + g + |B_e| \right) \int_s^t \frac{1}{1+|V(\tau)|^2} |\partial_{x_i} V(\tau)| d\tau.\end{aligned} \tag{4.32}$$

Thus

$$\begin{aligned}&\langle V(s) \rangle |\partial_{x_i} X(s)| \\ &\lesssim \langle V(s) \rangle + (t-s) \left(\|\nabla_x E\|_\infty + \|\nabla_x B\|_\infty \right) \frac{\sup_{0 \leq s \leq t} \langle V(s) \rangle}{\inf_{0 \leq s \leq t} \langle V(s) \rangle} \\ &\int_s^t \frac{1}{\sqrt{1+|V(\tau)|^2}} \langle V(\tau) \rangle |\partial_{x_i} X(\tau)| d\tau \\ &\quad + (t-s) \left(\|E\|_\infty + \|B\|_\infty + g + |B_e| \right) \frac{\sup_{0 \leq s \leq t} \langle V(s) \rangle}{\inf_{0 \leq s \leq t} \langle V(s) \rangle} \\ &\int_s^t \frac{1}{\sqrt{1+|V(\tau)|^2}} |\partial_{x_i} V(\tau)| d\tau \\ &\lesssim \langle V(s) \rangle + C_1 \int_s^t \frac{1}{\sqrt{1+|V(\tau)|^2}} \Big(\langle V(\tau) \rangle |\partial_{x_i} X(\tau)| + |\partial_{x_i} V(\tau)| \Big) d\tau,\end{aligned} \tag{4.33}$$

where $C_1 = \big(\sup_{0 \leq t \leq T} \left(\|\nabla_x E(t)\|_\infty + \|\nabla_x B(t)\|_\infty + \|E(t)\|_\infty + \|B(t)\|_\infty \right) + g + |B_e| \big)^2$. From (4.27), (4.28), and (4.29),

$$\begin{aligned}|\partial_{x_i} V(s)| &\lesssim \left(\|\nabla_x E\|_\infty + \|\nabla_x B\|_\infty \right) \int_s^t \frac{1}{\sqrt{1+|V(\tau)|^2}} \langle V(\tau) \rangle |\partial_{x_i} X(\tau)| d\tau \\ &\quad + \left(\|B\|_\infty + |B_e| \right) \int_s^t \frac{1}{\sqrt{1+|V(\tau)|^2}} |\partial_{x_i} V(\tau)| \mathrm{d}\tau \\ &\lesssim C_1 \int_s^t \frac{1}{\sqrt{1+|V(\tau)|^2}} \Big(\langle V(\tau) \rangle |\partial_{x_i} X(\tau)| + |\partial_{x_i} V(\tau)| \Big) d\tau.\end{aligned} \tag{4.34}$$

Combine (4.33) and (4.34) we have

$$\langle V(s) \rangle |\partial_{x_i} X(s)| + |\partial_{x_i} V(s)| \lesssim \langle V(s) \rangle$$

$$+C_1 \int_s^t \frac{1}{\sqrt{1+|V(\tau)|^2}} \left(\langle V(\tau)\rangle |\partial_{x_i} X(\tau)| + |\partial_{x_i} V(\tau)| \right) d\tau. \tag{4.35}$$

So from Gronwall's inequality,

$$\langle V(s)\rangle |\partial_{x_i} X(s)| + |\partial_{x_i} V(s)| \lesssim \langle V(s)\rangle e^{C_1 \int_s^t \frac{1}{\sqrt{1+|V(\tau)|^2}} d\tau} \lesssim e^{C_1|t-s|}\langle V(s)\rangle, \tag{4.36}$$

Next, using the same argument as (4.28)–(4.35), from (4.27) we get

$$\langle V(s)\rangle |\partial_{v_i} X(s)| + |\partial_{v_i} V(s)| \lesssim 1 + C_1 \int_s^t \frac{1}{\sqrt{1+|V(\tau)|^2}} \left(\langle V(\tau)\rangle |\partial_{x_i} X(\tau)| + |\partial_{x_i} V(\tau)| \right) d\tau.$$

Again by Gronwall's inequality,

$$\langle V(s)\rangle |\partial_{v_i} X(s)| + |\partial_{v_i} V(s)| \lesssim e^{C_1|t-s|}. \tag{4.37}$$

Thus,

$$|\partial_{v_i} X(s;t,x,v)| \lesssim \frac{e^{C_1|t-s|}}{\langle V(s)\rangle},\ |\partial_{v_i} V(s;t,x,v)| \lesssim e^{C_1|t-s|}. \tag{4.38}$$

Now plug (4.36), (4.38) back to (4.27) and using (4.30), (4.31), and (4.38), we have

$$|\partial_{v_i} X(s;t,x,v)| \lesssim \frac{|t-s|}{\langle v\rangle} + e^{C_1|t-s|} \int_s^t \int_\tau^t \frac{1}{\langle V(\tau')\rangle^2} d\tau' d\tau \lesssim \frac{|t-s|}{\langle V(s)\rangle} e^{C_1|t-s|}, \tag{4.39}$$

$$|\partial_{x_i} V(s;t,x,v)| \lesssim e^{C_1|t-s|} \int_s^t \frac{1}{\langle V(\tau)\rangle} \langle V(\tau)\rangle \mathrm{d}\tau \lesssim |t-s| e^{C_1|t-s|} \tag{4.40}$$

From (4.36), (4.38), and (4.39) we conclude (4.23). Finally, for $i \neq j$, from (4.27), (4.30), and (4.31),

$$|\partial_{x_i} X_j(s;t,x,v)| \lesssim e^{C_1|t-s|} \int_s^t \int_\tau^t \frac{1}{\langle V(\tau')\rangle} d\tau' d\tau \lesssim e^{C_1|t-s|} \frac{|t-s|^2}{\langle V(s)\rangle}.$$

□

5. $W^{1,\infty}$ Estimate of Inflow Problem

In this section, we prove an a priori estimate for the inflow problem (0.40), (0.25). From (0.54), we have

$$f(t,x,v) = \mathbf{1}_{t_{\mathbf{b}} \geq t} f(0, X(0), V(0)) + \mathbf{1}_{t_{\mathbf{b}} < t} g(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v_{\mathbf{b}}). \tag{5.1}$$

From (0.54), we have

$$X_i(s;t,x,v) = x_i - (t-s)\hat{v}_i + \int_s^t \int_\tau^t \hat{\mathfrak{F}}_i(\tau', X(\tau'), V(\tau')) \mathrm{d}\tau' \mathrm{d}\tau. \tag{5.2}$$

Set $s = t - t_{\mathbf{b}}$ so that $X_3(t - t_{\mathbf{b}}; t, x, v) = 0$. Then

$$t_{\mathbf{b}} \hat{v}_3 = x_3 + \int_{t-t_{\mathbf{b}}}^t \int_\tau^t \hat{\mathfrak{F}}_3(\tau') \mathrm{d}\tau' \mathrm{d}\tau \tag{5.3}$$

By taking derivatives we obtain

$$\partial_x x_3 + \int_{t-t_{\mathbf{b}}}^{t}\int_{\tau}^{t} \partial_x[\hat{\mathfrak{F}}_3(\tau', X(\tau'), V(\tau'))]\mathrm{d}\tau'\mathrm{d}\tau = \Big(\hat{v}_3 - \int_{t-t_{\mathbf{b}}}^{t} \hat{\mathfrak{F}}_3(\tau')\mathrm{d}\tau'\Big)\partial_x t_{\mathbf{b}} \\ = \hat{v_{\mathbf{b}}}_{,3}\partial_x t_{\mathbf{b}}, \tag{5.4}$$

and hence

$$\begin{aligned}\partial_{x_i} t_{\mathbf{b}} &= \frac{1}{\hat{v}_{\mathbf{b},3}}\left\{\partial_{x_i} x_3 + \int_{t-t_{\mathbf{b}}}^{t}\int_{\tau}^{t} \partial_{x_i}[\hat{\mathfrak{F}}_3(\tau', X(\tau'), V(\tau'))]\mathrm{d}\tau'\mathrm{d}\tau\right\}\\ &= \frac{1}{\hat{v}_{\mathbf{b},3}}\left\{\partial_{x_i} x_3 + \int_{t-t_{\mathbf{b}}}^{t}\int_{\tau}^{t} [\nabla_x\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{x_i}X(\tau') + \nabla_v\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{x_i}V(\tau')]\mathrm{d}\tau'\mathrm{d}\tau\right\}.\end{aligned} \tag{5.5}$$

Similarly,

$$\partial_{v_i} x_3 - t_{\mathbf{b}}\partial_{v_i}\hat{v}_3 + \int_{t-t_{\mathbf{b}}}^{t}\int_{\tau}^{t} \partial_{v_i}[\hat{\mathfrak{F}}_3(\tau', X(\tau'), V(\tau'))]\mathrm{d}\tau'\mathrm{d}\tau = \hat{v}_{\mathbf{b},3}\partial_{v_i} t_{\mathbf{b}}, \tag{5.6}$$

Thus

$$\begin{aligned}\partial_{v_i} t_{\mathbf{b}} &= \frac{1}{\hat{v}_{\mathbf{b},3}}\left\{-t_{\mathbf{b}}\partial_{v_i}\hat{v}_3 + \int_{t-t_{\mathbf{b}}}^{t}\int_{\tau}^{t} \partial_{v_i}[\hat{\mathfrak{F}}_3(\tau', X(\tau'), V(\tau'))]\mathrm{d}\tau'\mathrm{d}\tau\right\}\\ &= \frac{1}{\hat{v}_{\mathbf{b},3}}\left\{-t_{\mathbf{b}}\partial_{v_i}\hat{v}_3 + \int_{t-t_{\mathbf{b}}}^{t}\int_{\tau}^{t} [\nabla_x\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{v_i}X(\tau') + \nabla_v\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{v_i}V(\tau')]\mathrm{d}\tau'\mathrm{d}\tau\right\}.\end{aligned} \tag{5.7}$$

And we have

$$\begin{aligned}\partial_{x_i} x_{\mathbf{b}} &= e_i - (\partial_{x_i} t_{\mathbf{b}})\hat{v_{\mathbf{b}}} + \int_{t-t_{\mathbf{b}}}^{t}\int_{\tau}^{t} [\nabla_x\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{x_i}X(\tau') + \nabla_v\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{x_i}V(\tau')]\mathrm{d}\tau'\mathrm{d}\tau\\ \partial_{x_i} v_{\mathbf{b}} &= -(\partial_{x_i} t_{\mathbf{b}})\mathfrak{F}(t-t_{\mathbf{b}}, x_{\mathbf{b}}, t_{\mathbf{b}}) - \int_{t-t_{\mathbf{b}}}^{t} [\nabla_x\mathfrak{F}_3(\tau)\cdot\partial_{x_i}X(\tau) + \nabla_v\mathfrak{F}_3(\tau)\cdot\partial_{x_i}V(\tau)]\mathrm{d}\tau\\ \partial_{v_i} x_{\mathbf{b}} &= -(\partial_{v_i}\hat{v})t_{\mathbf{b}} - (\partial_{v_i} t_{\mathbf{b}})\hat{v_{\mathbf{b}}} + \int_{t-t_{\mathbf{b}}}^{t}\int_{\tau}^{t} [\nabla_x\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{v_i}X(\tau') + \nabla_v\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{v_i}V(\tau')]\mathrm{d}\tau'\mathrm{d}\tau\\ \partial_{v_i} v_{\mathbf{b}} &= e_i - (\partial_{v_i} t_{\mathbf{b}})\mathfrak{F}(t-t_{\mathbf{b}}, x_{\mathbf{b}}, t_{\mathbf{b}}) - \int_{t-t_{\mathbf{b}}}^{t} [\nabla_x\mathfrak{F}_3(\tau)\cdot\partial_{v_i}X(\tau) + \nabla_v\mathfrak{F}_3(\tau)\cdot\partial_{v_i}V(\tau)]\mathrm{d}\tau.\end{aligned} \tag{5.8}$$

We have the following calculations for the derivatives of f in (5.1).

$$\begin{aligned}&\partial_{x_i} f(t,x,v)\\ &\quad= \mathbf{1}_{t_{\mathbf{b}}>t}\{\nabla_x f_0(X(0), V(0))\cdot\partial_{x_i}X(0)\\ &\quad+ \nabla_v f_0(X(0), V(0))\cdot\partial_{x_i}V(0)\}\\ &\qquad+ \mathbf{1}_{t_{\mathbf{b}}<t}\{-\partial_t g(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v_{\mathbf{b}})\partial_{x_i} t_{\mathbf{b}} + \nabla_x g(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v_{\mathbf{b}})\partial_{x_i} x_{\mathbf{b}}\\ &\quad+ \nabla_v g(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v_{\mathbf{b}})\partial_{x_i} v_{\mathbf{b}}\}\\ &\quad= \mathbf{1}_{t_{\mathbf{b}}>t}\{\nabla_x f_0(X(0), V(0))\cdot\partial_{x_i}X(0)\\ &\quad+ \nabla_v f_0(X(0), V(0))\cdot\partial_{x_i}V(0)\}\end{aligned}$$

$$
\begin{aligned}
&+ \mathbf{1}_{t_{\mathbf{b}}<t}\Bigg(- \partial_t g(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v_{\mathbf{b}}) \frac{1}{\hat{v_{\mathbf{b}}}_{,3}} \\
&\left\{ \partial_{x_i} x_3 + \int_{t-t_{\mathbf{b}}}^{t} \int_{\tau}^{t} [\nabla_x \hat{\mathfrak{F}}_3(\tau') \cdot \partial_{x_i} X(\tau') + \nabla_v \hat{\mathfrak{F}}_3(\tau') \cdot \partial_{x_i} V(\tau')] \mathrm{d}\tau' \mathrm{d}\tau \right\} \\
&+ \nabla_x g(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v_{\mathbf{b}}) \cdot \\
&\left\{ e_i - (\partial_{x_i} t_{\mathbf{b}}) \hat{v_{\mathbf{b}}} + \int_{t-t_{\mathbf{b}}}^{t} \int_{\tau}^{t} [\nabla_x \hat{\mathfrak{F}}_3(\tau') \cdot \partial_{x_i} X(\tau') + \nabla_v \hat{\mathfrak{F}}_3(\tau') \cdot \partial_{x_i} V(\tau')] \mathrm{d}\tau' \mathrm{d}\tau \right\} \\
&+ \nabla_v g(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v_{\mathbf{b}}) \cdot \\
&\Bigg\{ -(\partial_{x_i} t_{\mathbf{b}}) \mathfrak{F}(t - t_{\mathbf{b}}, x_{\mathbf{b}}, t_{\mathbf{b}}) - \int_{t-t_{\mathbf{b}}}^{t} [\nabla_x \mathfrak{F}_3(\tau) \cdot \partial_{v_i} X(\tau) \\
&+ \nabla_v \mathfrak{F}_3(\tau) \cdot \partial_{v_i} V(\tau)] \mathrm{d}\tau \Bigg\} \Bigg),
\end{aligned} \tag{5.9}
$$

and

$$
\begin{aligned}
&\partial_{v_i} f(t, x, v) \\
&\quad = \mathbf{1}_{t_{\mathbf{b}}>t} \{ \nabla_x f_0(X(0), V(0)) \cdot \partial_{v_i} X(0) + \nabla_v f_0(X(0), V(0)) \cdot \partial_{v_i} V(0) \} \\
&\quad + \mathbf{1}_{t_{\mathbf{b}}<t} \{ -\partial_t g(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v_{\mathbf{b}}) \partial_{v_i} t_{\mathbf{b}} \\
&\quad + \nabla_x g(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v_{\mathbf{b}}) \partial_{v_i} x_{\mathbf{b}} + \nabla_v g(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v_{\mathbf{b}}) \partial_{v_i} v_{\mathbf{b}} \} \\
&\quad = \mathbf{1}_{t_{\mathbf{b}}>t} \{ \nabla_x f_0(X(0), V(0)) \cdot \partial_{v_i} X(0) \\
&\quad + \nabla_v f_0(X(0), V(0)) \cdot \partial_{v_i} V(0) \} \\
&\quad + \mathbf{1}_{t_{\mathbf{b}}<t} \Bigg(- \partial_t g(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v_{\mathbf{b}}) \frac{1}{\hat{v_{\mathbf{b}}}_{,3}} \Bigg\{ - t_{\mathbf{b}} \partial_{v_i} \hat{v}_3 + \int_{t-t_{\mathbf{b}}}^{t} \int_{\tau}^{t} [\nabla_x \hat{\mathfrak{F}}_3(\tau') \cdot \partial_{v_i} X(\tau') \\
&\quad + \nabla_v \hat{\mathfrak{F}}_3(\tau') \cdot \partial_{v_i} V(\tau')] \mathrm{d}\tau' \mathrm{d}\tau \Bigg\} \\
&\quad + \nabla_x g(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v_{\mathbf{b}}) \cdot \Bigg\{ -(\partial_{v_i} \hat{v}) t_{\mathbf{b}} - (\partial_{v_i} t_{\mathbf{b}}) \hat{v_{\mathbf{b}}} + + \int_{t-t_{\mathbf{b}}}^{t} \int_{\tau}^{t} [\nabla_x \hat{\mathfrak{F}}_3(\tau') \cdot \partial_{v_i} X(\tau') \\
&\quad + \nabla_v \hat{\mathfrak{F}}_3(\tau') \cdot \partial_{v_i} V(\tau')] \mathrm{d}\tau' \mathrm{d}\tau \Bigg\} \\
&\quad + \nabla_v g(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v_{\mathbf{b}}) \cdot \Bigg\{ e_i - (\partial_{v_i} t_{\mathbf{b}}) \mathfrak{f}(t - t_{\mathbf{b}}, x_{\mathbf{b}}, t_{\mathbf{b}}) - \int_{t-t_{\mathbf{b}}}^{t} [\nabla_x \mathfrak{F}_3(\tau) \cdot \partial_{v_i} X(\tau) \\
&\quad + \nabla_v \mathfrak{F}_3(\tau) \cdot \partial_{v_i} V(\tau)] \mathrm{d}\tau \Bigg\} \Bigg)
\end{aligned} \tag{5.10}
$$

Proposition 3. *Let (f, E, B) be a solution of (0.40), (0.25), (0.21). Suppose the fields satisfies (4.18), and*

$$
\sup_{0 \le t \le T} (\|\nabla_x E(t)\|_\infty + \|\nabla_x B(t)\|_\infty) < \infty.
$$

And assume that for $\delta > 0$,

$$
\begin{aligned}
\|\langle v \rangle^{5+\delta} \nabla_{x_\parallel} f_0\|_\infty + \|\langle v \rangle^{5+\delta} \alpha \partial_{x_3} f_0\|_\infty + \|\langle v \rangle^{5+\delta} \nabla_v f_0\|_\infty < \infty, \\
\|\langle v \rangle^{5+\delta} \partial_t g\|_\infty + \|\langle v \rangle^{5+\delta} \nabla_{x_\parallel} g\|_\infty + \|\langle v \rangle^{5+\delta} \nabla_v g\|_\infty < \infty.
\end{aligned}
$$

then for $0 < T \ll 1$ *small enough, we have*

$$\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_\infty + \|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t)\|_\infty + \|\langle v\rangle^{5+\delta}\nabla_v f(t)\|_\infty\Big) < \infty. \tag{5.11}$$

Proof. For notational simplicity, we assume that the lower order terms of E and B are smaller than the higher order terms:

$$\begin{aligned}\sup_{0\le t\le T}\|E(t)\|_\infty + g + |B_e| \lesssim \sup_{0\le t\le T}\|\nabla_x E(t)\|_\infty,\\ \sup_{0\le t\le T}\|B(t)\|_\infty + g + |B_e| \lesssim \sup_{0\le t\le T}\|\nabla_x B(t)\|_\infty.\end{aligned} \tag{5.12}$$

From (5.9), (5.10), we have

$$\begin{aligned}|\partial_{x_i} f(t,x,v)| \le\ & \mathbf{1}_{t_{\mathbf{b}}>t}\{|\nabla_x f_0(X(0),V(0))||\partial_{x_i}X(0)| + |\nabla_v f_0(X(0),V(0))||\partial_{x_i}V(0)|\}\\ &+\mathbf{1}_{t_{\mathbf{b}}<t}\Bigg(|\partial_t g(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\,\Bigg|\frac{1}{\hat{v}_{\mathbf{b},3}}\\ &\Big\{\partial_{x_i}x_3 + \int_{t-t_{\mathbf{b}}}^t\int_\tau^t [\nabla_x\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{x_i}X(\tau')\\ &+\nabla_v\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{x_i}V(\tau')]\mathrm{d}\tau'\mathrm{d}\tau\Big\}\Bigg|\\ &+|\nabla_x g(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\,\Bigg|e_i-(\partial_{x_i}t_{\mathbf{b}})\hat{v_{\mathbf{b}}}\\ &+\int_{t-t_{\mathbf{b}}}^t\int_\tau^t[\nabla_x\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{x_i}X(\tau')+\nabla_v\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{x_i}V(\tau')]\mathrm{d}\tau'\mathrm{d}\tau\Bigg|\\ &+|\nabla_v g(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\,\Bigg|-(\partial_{x_i}t_{\mathbf{b}})\mathfrak{F}(t-t_{\mathbf{b}},x_{\mathbf{b}},t_{\mathbf{b}})\\ &-\int_{t-t_{\mathbf{b}}}^t[\nabla_x\mathfrak{F}_3(\tau)\cdot\partial_{x_i}X(\tau)+\nabla_v\mathfrak{F}_3(\tau)\cdot\partial_{x_i}V(\tau)]\mathrm{d}\tau\Bigg|\Bigg).\end{aligned} \tag{5.13}$$

$$\begin{aligned}|\partial_{v_i} f(t,x,v)| \le\ & \mathbf{1}_{t_{\mathbf{b}}>t}\{|\nabla_x f_0(X(0),V(0))||\partial_{v_i}X(0)| + |\nabla_v f_0(X(0),V(0))||\partial_{v_i}V(0)|\}\\ &+\mathbf{1}_{t_{\mathbf{b}}<t}\Bigg(|\partial_t g(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\,\Bigg|\frac{1}{\hat{v}_{\mathbf{b},3}}\Big\{-t_{\mathbf{b}}\partial_{v_i}\hat{v}_3 + \int_{t-t_{\mathbf{b}}}^t\int_\tau^t[\nabla_x\hat{\mathfrak{F}}_3(\tau')\\ &\cdot\partial_{v_i}X(\tau')+\nabla_v\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{v_i}V(\tau')]\mathrm{d}\tau'\mathrm{d}\tau\Big\}\Bigg|\\ &+|\nabla_x g(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\,\Bigg|-(\partial_{v_i}\hat{v})t_{\mathbf{b}}-(\partial_{v_i}t_{\mathbf{b}})\hat{v_{\mathbf{b}}}\\ &+\int_{t-t_{\mathbf{b}}}^t\int_\tau^t[\nabla_x\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{v_i}X(\tau')+\nabla_v\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{v_i}V(\tau')]\mathrm{d}\tau'\mathrm{d}\tau\Bigg|\\ &+|\nabla_v g(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\,\Bigg|e_i-(\partial_{v_i}t_{\mathbf{b}})\mathfrak{F}(t-t_{\mathbf{b}},x_{\mathbf{b}},t_{\mathbf{b}})\\ &-\int_{t-t_{\mathbf{b}}}^t[\nabla_x\mathfrak{F}_3(\tau)\cdot\partial_{v_i}X(\tau)+\nabla_v\mathfrak{F}_3(\tau)\cdot\partial_{v_i}V(\tau)]\mathrm{d}\tau\Bigg|\Bigg).\end{aligned} \tag{5.14}$$

From (4.30), (4.31), (4.23), (5.5), and (5.7), we have

$$|\partial_{x_i}t_{\mathbf{b}}| \le \frac{1}{\hat{v}_{\mathbf{b},3}}\Big\{|\partial_{x_i}x_3| + \int_{t-t_{\mathbf{b}}}^t\int_\tau^t|\nabla_x\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{x_i}X(\tau')+\nabla_v\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{x_i}V(\tau')|\mathrm{d}\tau'\mathrm{d}\tau\Big\}$$

$$
\begin{aligned}
&\leq \frac{1}{\hat{v}_{\mathbf{b},3}}\left\{|\partial_{x_i}x_3| + \int_{t-t_{\mathbf{b}}}^{t}\int_{\tau}^{t}|\nabla_x\hat{\mathfrak{F}}_3(\tau')||\partial_{x_i}X(\tau')| + |\nabla_v\hat{\mathfrak{F}}_3(\tau')||\partial_{x_i}V(\tau')|\mathrm{d}\tau'\mathrm{d}\tau\right\}\\
&\lesssim \frac{1}{\hat{v}_{\mathbf{b},3}}\left(|\partial_{x_i}x_3| + \frac{Ct_{\mathbf{b}}^2}{\langle v\rangle}(\|\nabla_x E\|_{L^\infty_{t,x}} + \|\nabla_x B\|_{L^\infty_{t,x}})\right), \qquad (5.15)
\end{aligned}
$$

$$
\begin{aligned}
|\partial_{v_i}t_{\mathbf{b}}| &\leq \frac{1}{\hat{v}_{\mathbf{b},3}}\left\{|t_{\mathbf{b}}\partial_{v_i}\hat{v}_3| + \int_{t-t_{\mathbf{b}}}^{t}\int_{\tau}^{t}|\nabla_x\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{v_i}X(\tau') + \nabla_v\hat{\mathfrak{F}}_3(\tau')\cdot\partial_{v_i}V(\tau')|\mathrm{d}\tau'\mathrm{d}\tau\right\}\\
&\leq \frac{1}{\hat{v}_{\mathbf{b},3}}\left\{|t_{\mathbf{b}}\partial_{v_i}\hat{v}_3| + \int_{t-t_{\mathbf{b}}}^{t}\int_{\tau}^{t}|\nabla_x\hat{\mathfrak{F}}_3(\tau')||\partial_{v_i}X(\tau')| + |\nabla_v\hat{\mathfrak{F}}_3(\tau')||\partial_{v_i}V(\tau')|\mathrm{d}\tau'\mathrm{d}\tau\right\}\\
&\lesssim \frac{1}{\hat{v}_{\mathbf{b},3}}\left(\frac{t_{\mathbf{b}}}{\langle v\rangle} + \frac{Ct_{\mathbf{b}}^2}{\langle v\rangle^2}(\|\nabla_x E\|_{L^\infty_{t,x}} + \|\nabla_x B\|_{L^\infty_{t,x}})\right). \qquad (5.16)
\end{aligned}
$$

Thus from (5.13) and (5.15), for $i = 1, 2$,

$$
\begin{aligned}
&|\langle v\rangle^{4+\delta}\partial_{x_i}f(t,x,v)|\\
&\quad\leq \langle v\rangle^{4+\delta}\big(|\nabla_{x_\parallel}f_0(X(0),V(0))||\partial_{x_i}X_\parallel(0)| + |\partial_{x_3}f_0(X(0),V(0))||\partial_{x_i}X_3(0)|\\
&\quad+ |\nabla_v f_0(X(0),V(0))||\nabla_x V(0)|\big) \qquad (5.17)\\
&\quad+ \langle v\rangle^{4+\delta}|\partial_t g(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\frac{Ct_{\mathbf{b}}^2}{\hat{v}_{\mathbf{b},3}\langle v\rangle}(\|\nabla_x E\|_{L^\infty_{t,x}} + \|\nabla_x B\|_{L^\infty_{t,x}}) \qquad (5.18)\\
&\quad+ \langle v\rangle^{4+\delta}|\nabla_x g(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\left(1 + \left(\frac{1}{\hat{v}_{\mathbf{b},3}}+1\right)\frac{Ct_{\mathbf{b}}^2}{\langle v\rangle}(\|\nabla_x E\|_{L^\infty_{t,x}} + \|\nabla_x B\|_{L^\infty_{t,x}})\right)\\
&\quad+ \langle v\rangle^{4+\delta}|\nabla_v g(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|C\left(\frac{t_{\mathbf{b}}^2}{\hat{v}_{\mathbf{b},3}\langle v\rangle}|\mathfrak{F}(t-t_{\mathbf{b}},x_{\mathbf{b}},t_{\mathbf{b}})| + t_{\mathbf{b}}\right)\\
&\quad\times(\|\nabla_x E\|_{L^\infty_{t,x}} + \|\nabla_x B\|_{L^\infty_{t,x}}). \qquad (5.19)
\end{aligned}
$$

From (4.23), (4.27), and (4.20), we have

$$
\begin{aligned}
|(5.17)| \leq\; & C\langle v\rangle^{4+\delta}\left(|\nabla_{x_\parallel}f_0(X(0),V(0))| + t|\partial_{x_3}f_0(X(0),V(0))|\right.\\
&\left.+(1+|v|)|\nabla_v f_0(X(0),V(0))|\right) \qquad (5.20)\\
&\leq (C^2+1)\left(\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3}f_0\|_{L^\infty_x} + \|\langle v\rangle^{4+\delta}\nabla_{x_\parallel}f_0\|_{L^\infty_x} + \|\langle v\rangle^{5+\delta}\nabla_v f_0\|_{L^\infty_x}\right).
\end{aligned}
$$

And

$$
\begin{aligned}
&|(5.18)| + |(5.19)| + |(5.19)|\\
&\quad\leq (C + t_{\mathbf{b}}C(\|\nabla_x E\|_{L^\infty_{t,x}} + \|\nabla_x B\|_{L^\infty_{t,x}})) \qquad (5.21)\\
&\quad\left(\|\langle v\rangle^{4+\delta}\partial_t g\|_{L^\infty_{t,x}} + \|\langle v\rangle^{4+\delta}\nabla_x g\|_{L^\infty_{t,x}} + \|\langle v\rangle^{4+\delta}\nabla_v g\|_{L^\infty_{t,x}}\right).
\end{aligned}
$$

And from (5.13) and (5.15), for $i = 3$,

$$
\begin{aligned}
&|\langle v\rangle^{5+\delta}\alpha\partial_{x_3}f(t,x,v)|\\
&\quad\leq \langle v\rangle^{5+\delta}\alpha(t,x,v)\big(|\nabla_{x_\parallel}f_0(X(0),V(0))||\partial_{x_3}X_\parallel(0)| + |\partial_{x_3}f_0(X(0),V(0))||\partial_{x_3}X_3(0)|\\
&\quad+ |\nabla_v f_0(X(0),V(0))||\nabla_x V(0)|\big) \qquad (5.22)
\end{aligned}
$$

$$+ \langle v\rangle^{5+\delta}|\partial_t g(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\frac{\alpha(t,x,v)}{\hat{v}_{\mathbf{b},3}}(1+\frac{Ct_{\mathbf{b}}^2}{\langle v\rangle}(\|\nabla_x E\|_{L^\infty_{t,x}}+\|\nabla_x B\|_{L^\infty_{t,x}})) \tag{5.23}$$

$$+ \langle v\rangle^{5+\delta}|\nabla_x g(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\frac{\alpha(t,x,v)}{\hat{v}_{\mathbf{b},3}}(1+\frac{Ct_{\mathbf{b}}^2}{\langle v\rangle}(\|\nabla_x E\|_{L^\infty_{t,x}}+\|\nabla_x B\|_{L^\infty_{t,x}})) \tag{5.24}$$

$$\begin{aligned}&+ \langle v\rangle^{5+\delta}|\nabla_v g(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\frac{\alpha(t,x,v)}{\hat{v}_{\mathbf{b},3}}|\mathfrak{F}(t-t_{\mathbf{b}},x_{\mathbf{b}},t_{\mathbf{b}})|\\&\times(1+t_{\mathbf{b}}C(\|\nabla_x E\|_{L^\infty_{t,x}}+\|\nabla_x B\|_{L^\infty_{t,x}})).\end{aligned} \tag{5.25}$$

Now from (4.23) and the velocity lemma (4.3),

$$|(5.22)|\le C\left(\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3}f_0\|_{L^\infty_x}+\|\langle v\rangle^{5+\delta}\nabla_{x_\parallel}f_0\|_{L^\infty_x}+\|\langle v\rangle^{5+\delta}\alpha\nabla_v f(0)\|_{L^\infty_x}\right), \tag{5.26}$$

and

$$\begin{aligned}&|(5.23)|+|(5.24)|+|(5.25)|\\&\quad\le (C+t_{\mathbf{b}}C(\|\nabla_x E\|_{L^\infty_{t,x}}+\|\nabla_x B\|_{L^\infty_{t,x}}))\\&\quad\left(\|\langle v\rangle^{5+\delta}\partial_t g\|_{L^\infty_{t,x}}+\|\langle v\rangle^{5+\delta}\nabla_x g\|_{L^\infty_{t,x}}+\|\langle v\rangle^{5+\delta}\nabla_v g\|_{L^\infty_{t,x}}\right).\end{aligned} \tag{5.27}$$

Also, similarly from (5.14) and (5.16),

$$\begin{aligned}&|\langle v\rangle^{5+\delta}\partial_{v_i}f(t,x,v)|\\&\quad\le \langle v\rangle^{5+\delta}\big(|\nabla_{x_\parallel}f_0(X(0),V(0))||\partial_{v_i}X_\parallel(0)|+|\partial_{x_3}f_0(X(0),V(0))||\partial_{v_i}X_3(0)|\\&\qquad+|\nabla_v f_0(X(0),V(0))||\nabla_{v_i}V(0)|\big)\\&\qquad+\langle v\rangle^{5+\delta}|\partial_t g(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\frac{t_{\mathbf{b}}}{\hat{v}_{\mathbf{b},3}\langle v\rangle}\times(1+\frac{t_{\mathbf{b}}}{\langle v\rangle}C(\|\nabla_x E\|_{L^\infty_{t,x}}+\|\nabla_x B\|_{L^\infty_{t,x}}))\\&\qquad+\langle v\rangle^{5+\delta}|\nabla_x g(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\\&\qquad\left(\frac{t_{\mathbf{b}}}{\langle v\rangle}+\left(\frac{1}{\hat{v}_{\mathbf{b},3}}+1\right)\frac{t_{\mathbf{b}}^2}{\langle v\rangle}C(\|\nabla_x E\|_{L^\infty_{t,x}}+\|\nabla_x B\|_{L^\infty_{t,x}})\right)\\&\qquad+\langle v\rangle^{5+\delta}|\nabla_v g(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\\&\qquad\left((1+\frac{t_{\mathbf{b}}}{\hat{v}_{\mathbf{b},3}\langle v\rangle}|\mathfrak{F}(t-t_{\mathbf{b}},x_{\mathbf{b}},t_{\mathbf{b}})|+t_{\mathbf{b}}C(\|\nabla_x E\|_{L^\infty_{t,x}}+\|\nabla_x B\|_{L^\infty_{t,x}})\right)\\&\quad\le (C_2^2+1)\left(\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3}f_0\|_{L^\infty_x}+\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel}f_0\|_{L^\infty_x}+\|\langle v\rangle^{5+\delta}\nabla_v f_0\|_{L^\infty_x}\right)\\&\qquad+(C+t_{\mathbf{b}}C(\|\nabla_x E\|_{L^\infty_{t,x}}+\|\nabla_x B\|_{L^\infty_{t,x}}))\\&\qquad\times\left(\|\langle v\rangle^{5+\delta}\partial_t g\|_{L^\infty_{t,x}}+\|\langle v\rangle^{5+\delta}\nabla_x g\|_{L^\infty_{t,x}}+\|\langle v\rangle^{5+\delta}\nabla_v g\|_{L^\infty_{t,x}}\right).\end{aligned} \tag{5.28}$$

Now from (3.3) we have

$$\begin{aligned}&\|\nabla_x E\|_{L^\infty_{t,x}}+\|\nabla_x B\|_{L^\infty_{t,x}}\\&\quad\lesssim\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel}f(t,x,v)\|_{L^\infty_{t,x}}+\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3}f(t,x,v)\|_{L^\infty_{t,x}}+C,\end{aligned}$$

thus combining (5.20), (5.21), (5.26), (5.27), and (5.28), and by choosing $0 < T \ll 1$ small enough we have

$$\begin{aligned}&\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t,x,v)\|_{L^\infty_{t,x}} + \|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t,x,v)\|_{L^\infty_{t,x}} + \|\langle v\rangle^{5+\delta}\nabla_v f(t,x,v)\|_{L^\infty_{t,x}}\\ &\quad\le 2\left(\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f_0\|_{L^\infty_x} + \|\langle v\rangle^{5+\delta}\nabla_{x_\parallel} f_0\|_{L^\infty_x} + \|\langle v\rangle^{5+\delta}\nabla_v f_0\|_{L^\infty_x}\right)\\ &\quad\quad + C_1\left(\|\langle v\rangle^{5+\delta}\partial_t g\|_{L^\infty_{t,x}} + \|\langle v\rangle^{5+\delta}\nabla_x g\|_{L^\infty_{t,x}} + \|\langle v\rangle^{5+\delta}\nabla_v g\|_{L^\infty_{t,x}}\right)\\ &\quad\quad + \frac{1}{2}\left(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t,x,v)\|_{L^\infty_{t,x}} + \|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t,x,v)\|_{L^\infty_{t,x}}\right) < \infty.\end{aligned} \tag{5.29}$$

This conclude (5.11). □

We state and prove a variation of Ukai's trace theorem in [1,35,62].

Lemma 12. *Suppose* $f \in L^\infty((0,T)\times\Omega\times\mathbb{R}^3)$*, and* $\mathfrak{F} \in W^{1,\infty}((0,T)\times\mathbb{R}^3)$ *satisfy*

$$\partial_t f + \hat{v}\cdot\nabla_x f + \mathfrak{F}\cdot\nabla_v f = h \in L^\infty((0,T)\times\Omega\times\mathbb{R}^3). \tag{5.30}$$

Then $f \in L^\infty((0,T)\times(\gamma\backslash\gamma_0))$*, and*

$$\sup_{0\le t\le T}\|f(t)\|_{L^\infty(\gamma\backslash\gamma_0)} \le \sup_{0\le t\le T}\|f(t)\|_{L^\infty(\Omega\times\mathbb{R}^3)}. \tag{5.31}$$

Proof. Denote the characteristics $X(s;t,x,v)$, $V(s;t,x,v)$ which solves

$$\begin{aligned}&\frac{d}{ds}X(s;t,x,v) = \hat{V}(s;t,x,v),\\ &\frac{d}{ds}V(s;t,x,v) = \mathfrak{F}(s, X(s;t,x,v), V(s;t,x,v)),\end{aligned} \tag{5.32}$$

and $X(t;t,x,v) = x$, $V(t;t,x,v) = v$. Then since $\mathfrak{F} \in W^{1,\infty}((0,T)\times\mathbb{R}^3)$, the characteristics (5.32) is Hölder continuous. From (5.30), for almost every $(t,x,v) \in (0,T)\times\gamma_+$, and $\max\{0, t - t_{\mathbf{b}}(t,x,v)\}$,

$$f(t,x,v) = f(s, X(s;t,x,v), V(s;t,x,v)) + \int_s^t h(\tau; X(\tau;t,x,v), V(\tau;t,x,v))d\tau.$$

Thus

$$\sup_{0<t<T}\|f(t)\|_{L^\infty(\gamma_+)} \le \sup_{0<t<T}\|f(t)\|_{L^\infty(\Omega\times\mathbb{R}^3)} + (t-s)\|h\|_{L^\infty((0,T)\times\Omega\times\mathbb{R}^3)}.$$

Since $t - s > 0$ can be arbitrarily small, we have

$$\sup_{0<t<T}\|f(t)\|_{L^\infty(\gamma_+)} \le \sup_{0<t<T}\|f(t)\|_{L^\infty(\Omega\times\mathbb{R}^3)}.$$

Now, for $(x,v) \in \gamma_-$ and $s \in (t, \max\{T, t_{\mathbf{f}}(t,x,v)\})$, we have

$$f(t,x,v) = f(s, X(s;t,x,v), V(s;t,x,v)) - \int_t^s h(\tau; X(\tau;t,x,v), V(\tau;t,x,v))d\tau.$$

Using the same argument we get

$$\sup_{0<t<T}\|f(t)\|_{L^\infty(\gamma_-)} \le \sup_{0<t<T}\|f(t)\|_{L^\infty(\Omega\times\mathbb{R}^3)}.$$

This proves (5.31). □

Next, we prove a trace theorem for the derivatives of f.

Lemma 13. *Let (f, E, B) be a solution of (0.40), (0.25), (0.21). Suppose*

$$\sup_{0\le t\le T} (\|\nabla_x E(t)\|_\infty + \|\nabla_x B(t)\|_\infty) < \infty, \tag{5.33}$$

$$\|\langle v\rangle^{5+\delta}\partial_t g\|_\infty + \|\langle v\rangle^{5+\delta}\nabla_{x_\parallel} g\|_\infty + \|\langle v\rangle^{5+\delta}\nabla_v g\|_\infty < \infty, \tag{5.34}$$

and

$$\sup_{0\le t\le T} \Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_\infty + \|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t)\|_\infty + \|\langle v\rangle^{5+\delta}\nabla_v f(t)\|_\infty\Big) < \infty. \tag{5.35}$$

Then

$$\begin{aligned}\sup_{0\le t\le T} &\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_{L^\infty(\gamma\backslash\gamma_0)} + \|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t,x,v)\|_{L^\infty(\gamma\backslash\gamma_0)} \\ &+\|\langle v\rangle^{5+\delta}\nabla_v f(t)\|_{L^\infty(\gamma\backslash\gamma_0)}\Big) < \infty \end{aligned}\tag{5.36}$$

Proof. The proof uses similar argument as Ukai's proof of a trace theorem in [62]. Next, notice that for $\partial_{\mathbf{e}} \in \{\nabla_{x_\parallel}, \nabla_v\}$, and $p = 4+\delta, 5+\delta$, we have

$$\begin{aligned}&\partial_t(\langle v\rangle^p\partial_{\mathbf{e}} f) + \hat{v}\cdot\nabla_x(\langle v\rangle^p\partial_{\mathbf{e}} f) + \mathfrak{F}\cdot\nabla_v(\langle v\rangle^p\partial_{\mathbf{e}} f) \\ &\quad = -\langle v\rangle^p\partial_{\mathbf{e}}\hat{v}\cdot\nabla_x f - \langle v\rangle^p\partial_{\mathbf{e}}\mathfrak{F}\cdot\nabla_v f - \mathfrak{F}\cdot\nabla_v(\langle v\rangle^p)\partial_{\mathbf{e}} f.\end{aligned}\tag{5.37}$$

Then for almost every $(x, v) \in \gamma_+$, and $s \in (\max\{0, t - t_{\mathbf{b}}(t,x,v)\}, t)$, we have

$$\begin{aligned}\langle v\rangle^p\partial_{\mathbf{e}} f(t,x,v) &= \langle V(s;t,x,v)\rangle^p\partial_{\mathbf{e}} f(s, X(s;t,x,v), V(s;t,x,v)) \\ &\quad + \int_s^t \Big(-\langle v\rangle^p\partial_{\mathbf{e}}\hat{v}\cdot\nabla_x f - \langle v\rangle^p\partial_{\mathbf{e}}\mathfrak{F}\cdot\nabla_v f \\ &\quad -\mathfrak{F}\cdot\nabla_v(\langle v\rangle^p)\partial_{\mathbf{e}} f\Big)(\tau, X(\tau;t,x,v), V(\tau;t,x,v))d\tau.\end{aligned}\tag{5.38}$$

Thus, from (5.33) and (5.35),

$$\begin{aligned}&|\langle v\rangle^p\partial_{\mathbf{e}} f(t,x,v)| \\ &\le \sup_{0\le s\le t}\|\langle v\rangle^p\partial_{\mathbf{e}} f(s)\|_\infty + \frac{C(t-s)}{\alpha(t,x,v)}\bigg(\big(\sup_{0\le t\le T}(\|\nabla_x E(t)\|_\infty + \|\nabla_x B(t)\|_\infty) + g + |B_e|\big) \\ &\quad\times\sup_{0\le s\le t}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_\infty + \|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t)\|_\infty + \|\langle v\rangle^{5+\delta}\nabla_v f(t)\|_\infty\Big)\bigg),\end{aligned}\tag{5.39}$$

since we can choose s close enough to t such that

$$\begin{aligned}&\frac{C(t-s)}{\alpha(t,x,v)}\bigg(\big(\sup_{0\le t\le T}(\|\nabla_x E(t)\|_\infty + \|\nabla_x B(t)\|_\infty) + g + |B_e|\big)\sup_{0\le s\le t} \\ &\quad\times\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_\infty + \|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t)\|_\infty + \|\langle v\rangle^{5+\delta}\nabla_v f(t)\|_\infty\Big)\bigg) < \varepsilon \ll 1,\end{aligned}$$

we have

$$|\langle v\rangle^p \partial_{\mathbf{e}} f(t,x,v)| \le \sup_{0\le s\le t} \|\langle v\rangle^p \partial_{\mathbf{e}} f(s)\|_\infty + \varepsilon,$$

for any $\varepsilon > 0$. Therefore, we get

$$\begin{aligned} \sup_{0\le t\le T} \|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_{L^\infty(\gamma_+)} &\le \sup_{0\le t\le T} \|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_\infty, \\ \sup_{0\le t\le T} \|\langle v\rangle^{5+\delta}\nabla_v f(t)\|_{L^\infty(\gamma_+)} &\le \sup_{0\le t\le T} \|\langle v\rangle^{5+\delta}\nabla_v f(t)\|_\infty. \end{aligned} \tag{5.40}$$

Similarly, since

$$\begin{aligned} &(\partial_t + \hat{v}\cdot\nabla_x + \mathfrak{F}\cdot\nabla_v)(\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f) \\ &\quad = -\langle v\rangle^{5+\delta}\alpha\partial_{x_3}\mathfrak{F}\cdot\nabla_v f - \mathfrak{F}\cdot\nabla_v(\langle v\rangle^{5+\delta})\alpha\partial_{x_3} f \\ &\quad - [(\partial_t + \hat{v}\cdot\nabla_x + \mathfrak{F}\cdot\nabla_v)\alpha]\langle v\rangle^{5+\delta}\partial_{x_3} f \\ &\quad =: G_\alpha(t,x,v). \end{aligned}$$

Then for almost every $(x,v)\in\gamma_+$, and $s\in(\max\{0, t-t_{\mathbf{b}}(t,x,v)\}, t)$, we have

$$\begin{aligned} &\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t,x,v) \\ &\quad = \langle V(s;t,x,v)\rangle^{5+\delta}\alpha\partial_{x_3} f(s, X(s;t,x,v), V(s;t,x,v)) \\ &\qquad + \int_s^t G_\alpha(\tau, X(\tau;t,x,v), V(\tau;t,x,v))d\tau. \end{aligned} \tag{5.41}$$

Since

$$\begin{aligned} |G_\alpha(t,x,v)| &\le C\Big(\big((\sup_{0\le t\le T}\|\nabla_x E(t)\|_\infty + \|\nabla_x B(t)\|_\infty) + g + |B_e|\big) \\ &\times \sup_{0\le s\le t}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_\infty + \|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t)\|_\infty + \|\langle v\rangle^{5+\delta}\nabla_v f(t)\|_\infty\Big)\Big), \end{aligned}$$

using the same argument as (5.39)–(5.40), we obtain

$$\sup_{0\le t\le T}\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t,x,v)\|_{L^\infty(\gamma_+)} \le \sup_{0\le t\le T}\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t,x,v)\|_\infty. \tag{5.42}$$

Now, for $(x,v)\in\gamma_-$, and any $s\in(t, \max\{T, t_{\mathbf{f}}(t,x,v)\})$ we have the same formula (5.38) and (5.41) for $\langle v\rangle^p\partial\mathbf{e} f$ and $\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f$ respectively. Therefore by the same argument, we get

$$\begin{aligned} \sup_{0\le t\le T} \|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_{L^\infty(\gamma_-)} &\le \sup_{0\le t\le T} \|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_\infty, \\ \sup_{0\le t\le T} \|\langle v\rangle^{5+\delta}\nabla_v f(t)\|_{L^\infty(\gamma_-)} &\le \sup_{0\le t\le T} \|\langle v\rangle^{5+\delta}\nabla_v f(t)\|_\infty, \\ \sup_{0\le t\le T}\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t,x,v)\|_{L^\infty(\gamma_-)} &\le \sup_{0\le t\le T}\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t,x,v)\|_\infty. \end{aligned} \tag{5.43}$$

Combining (5.40), (5.42), and (5.43), we conclude (5.36). □

Next, we sketch the proof of local existence of the solution with inflow boundary condition (Theorem 1). Since the argument is similar to (and easier than) the proof of Theorem 2, we skip the details here to avoid redundancy.

Sketch of proof of Theorem 1. We recursively define a sequence of functions:

$$f^0(t,x,v) = f_0(x,v),\ E^0(t,x) = E_0(t,x),\ B^0(t,x) = B_0(x).$$

For $\ell \geq 1$, let f^ℓ be the solution of

$$\begin{aligned}\partial_t f^\ell + \hat{v}\cdot\nabla_x f^\ell + \mathfrak{F}^{\ell-1}\cdot\nabla_v f^\ell = 0,\ \text{where}\\ \mathfrak{F}^{\ell-1} &= E^{\ell-1} + E_{\text{ext}} + \hat{v}\times(B^{\ell-1}+B_{\text{ext}}) - g\mathbf{e}_3,\\ f^\ell(0,x,v) &= f_0(x,v),\\ f^\ell(t,x,v)|_{\gamma_-} &= g(t,x,v).\end{aligned} \tag{5.44}$$

Let $\rho^\ell = \int_{\mathbb{R}^3} f^\ell dv$, $j^\ell = \int_{\mathbb{R}^3}\hat{v} f^\ell dv$. Let

$$E^\ell = (2.31)+\cdots+(2.41),\ B^\ell = (2.45)+\cdots+(2.55),\ \text{with } f \text{ changes to } f^\ell. \tag{5.45}$$

$$\alpha^\ell(t,x,v) = \sqrt{(x_3)^2+(\hat{v}_3)^2 - 2\left(E_3^\ell(t,x_\parallel,0)+E_e+(\hat{v}\times B^\ell)_3(t,x_\parallel,0)-g\right)\frac{x_3}{\langle v\rangle}}. \tag{5.46}$$

Then similar to the argument in Lemma 14, Lemma 15, there exits M_1, M_2, M_3, M_4, and c_0, such that for $0 < T \ll 1$,

$$\begin{aligned}&\sup_\ell\sup_{0\le t\le T}\left(\|\langle v\rangle^{4+\delta} f^\ell(t)\|_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)}\right) < M_1,\\ &\sup_\ell\sup_{0\le t\le T}\left(\|E^\ell(t)\|_\infty + \|B^\ell(t)\|_\infty\right) + |B_e| + E_e + g < M_2,\\ &\inf_\ell\inf_{t,x_\parallel}\left(g - E_e - E_3^\ell(t,x_\parallel,0) - (\hat{v}\times B^\ell)_3(t,x_\parallel,0)\right) > c_0.\end{aligned} \tag{5.47}$$

$$\begin{aligned}&\sup_\ell\sup_{0\le t\le T}\left(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f^\ell(t)\|_\infty + \|\langle v\rangle^{5+\delta}\alpha^{\ell-1}\partial_{x_3} f^\ell(t)\|_\infty + \|\langle v\rangle^{4+\delta}\nabla_v f^\ell(t)\|_\infty\right)\\ &+\sup_\ell\sup_{0\le t\le T}\left(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f^\ell(t)\|_{L^\infty(\gamma\backslash\gamma_0)} + \|\langle v\rangle^{5+\delta}\alpha^{\ell-1}\partial_{x_3} f^\ell(t)\|_{L^\infty(\gamma\backslash\gamma_0)}\right.\\ &\left.+\|\langle v\rangle^{4+\delta}\nabla_v f^\ell(t)\|_{L^\infty(\gamma\backslash\gamma_0)}\right) < M_3,\\ &\sup_\ell\sup_{0\le t\le T}\left(\|\partial_t E^\ell(t)\|_\infty + \|\partial_t B^\ell(t)\|_\infty + \|\nabla_x E^\ell(t)\|_\infty + \|\nabla_x B^\ell(t)\|_\infty\right) < M_4.\end{aligned} \tag{5.48}$$

Similar to the argument in Lemma 17, there exists functions (f, E, B) with $\langle v\rangle^{4+\delta} f(t,x,v) \in L^\infty((0,T); L^\infty(\bar{\Omega}\times\mathbb{R}^3))$, and $(E, B) \in L^\infty((0,T); L^\infty(\Omega)\cap L^\infty(\partial\Omega))$, such that as $\ell\to\infty$,

$$\sup_{0\le t\le T}\left(\|E^\ell(t) - E(t)\|_{L^\infty(\Omega)} + \|E^\ell(t) - E(t)\|_{L^\infty(\partial\Omega)}\right.$$

$$+\|B^{\ell}(t)-B(t)\|_{L^\infty(\Omega)}+\|B^{\ell}(t)-B(t)\|_{L^\infty(\partial\Omega)}\Big)\to 0, \tag{5.49}$$

and

$$\sup_{0\le t\le T}\|\langle v\rangle^{4+\ell}f^{\ell}(t)-\langle v\rangle^{4+\delta}f(t)\|_{L^\infty(\bar\Omega\times\mathbb{R}^3)}\to 0. \tag{5.50}$$

Moreover, (f,E,B) is a (weak) solution of the system (0.40)–(0.42), and (0.25). And the solution (f,E,B) obtained in Lemma 17 satisfies

$$\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel}f(t)\|_\infty+\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3}f(t)\|_\infty+\|\langle v\rangle^{4+\delta}\nabla_v f(t)\|_\infty<\infty, \tag{5.51}$$

and

$$\|\partial_t E(t)\|_\infty+\|\partial_t B(t)\|_\infty+\|\nabla_x E(t)\|_\infty+\|\nabla_x B(t)\|_\infty<\infty. \tag{5.52}$$

Finally, using the argument as in Lemma 19, we can show that the solutions of the RVM system (0.40)–(0.42), (0.25) is unique. □

6. Diffuse BC

In (0.26), we denote $\mu=\frac{1}{(2\pi)^{3/2}}e^{-\frac{|v|^2}{2}}$. And let the constant c_μ be such that $c_\mu\int_{v_3>0}\hat v_3\,\mu(v)dv=1$. We first prove an a priori estimate for diffuse BC.

Proposition 4. *Let (f,E,B) be a solution of (0.40)–(0.42), (0.26). Suppose the fields satisfies (4.2), and*

$$\sup_{0\le t\le T}(\|\nabla_x E(t)\|_\infty+\|\nabla_x B(t)\|_\infty)<\infty. \tag{6.1}$$

Assume that for $\delta>0$, $\langle v\rangle^{4+\delta}\nabla_{x_\parallel}f$, $\langle v\rangle^{5+\delta}\alpha\partial_{x_3}f$, $\langle v\rangle^{5+\delta}\nabla_v f\in L^\infty((0,T)\times\Omega\times\mathbb{R}^3)$, then for $0<T\ll 1$, there exists a $C>0$ such that

$$\begin{aligned}&\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel}f(t)\|_\infty+\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3}f(t)\|_\infty+\|\langle v\rangle^{5+\delta}\nabla_v f(t)\|_\infty\Big)\\&\quad+\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel}f(t)\|_{L^\infty(\gamma\setminus\gamma_0)}\\&\quad+\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3}f(t)\|_{L^\infty(\gamma\setminus\gamma_0)}+\|\langle v\rangle^{5+\delta}\nabla_v f(t)\|_{L^\infty(\gamma\setminus\gamma_0)}\Big)\\&<C\Big(\|\langle v\rangle^{5+\delta}\nabla_{x_\parallel}f_0\|_\infty+\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3}f_0\|_\infty+\|\langle v\rangle^{5+\delta}\nabla_v f_0\|_\infty\Big).\end{aligned} \tag{6.2}$$

Proof. For any $(t,x,v)\in(0,T)\times\Omega\times\mathbb{R}^3$, from (5.13) and (5.14), we have

$$\begin{aligned}&|\partial_{x_i}f(t,x,v)|\\&\lesssim \mathbf{1}_{t_{\mathbf{b}}>t}\{|\nabla_x f_0(X(0),V(0))||\partial_{x_i}X(0)|+|\nabla_v f_0(X(0),V(0))||\partial_{x_i}V(0)|\}\\&\quad+\mathbf{1}_{t_{\mathbf{b}}<t}\Big(|\partial_t f(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\frac{\delta_{i3}+t_{\mathbf{b}}}{\hat v_{\mathbf{b},3}}\\&\quad+|\nabla_{x_\parallel}f(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\frac{\delta_{i3}+t_{\mathbf{b}}}{\hat v_{\mathbf{b},3}}+|\nabla_v f(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\left(\frac{\delta_{i3}+t_{\mathbf{b}}}{\hat v_{\mathbf{b},3}}+\langle v\rangle\right)\Big),\end{aligned} \tag{6.3}$$

and

$$
\begin{aligned}
&|\partial_{v_i} f(t,x,v)| \\
&\quad\lesssim \mathbf{1}_{t_{\mathbf{b}}>t}\{|\nabla_x f_0(X(0),V(0))||\partial_{v_i} X(0)| + |\nabla_v f_0(X(0),V(0))||\partial_{v_i} V(0)|\} \\
&\qquad + \mathbf{1}_{t_{\mathbf{b}}<t}\Bigg(|\partial_t f(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\frac{t_{\mathbf{b}}}{\hat{v}_{\mathbf{b},3}} \\
&\quad + |\nabla_{x_\parallel} f(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\frac{t_{\mathbf{b}}}{\hat{v}_{\mathbf{b},3}} + |\nabla_v f(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}})|\left(\frac{t_{\mathbf{b}}}{\hat{v}_{\mathbf{b},3}} + \langle v\rangle\right)\Bigg).
\end{aligned}
\tag{6.4}
$$

Now, using the boundary condition (0.26) and equation (0.40), we have

$$
\begin{aligned}
\nabla_{x_\parallel} f(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}}) &= c_\mu \mu(v_{\mathbf{b}}) \int_{u_3<0} -\nabla_{x_\parallel} f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)\hat{u}_3 du, \\
\nabla_v f(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}}) &= c_\mu v_{\mathbf{b}} \mu(v_{\mathbf{b}}) \int_{u_3<0} f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)\hat{u}_3 du, \\
\partial_t f(t-t_{\mathbf{b}},x_{\mathbf{b}},v_{\mathbf{b}}) &= c_\mu \mu(v_{\mathbf{b}}) \int_{u_3<0} \hat{u}\cdot\nabla_x f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)\hat{u}_3 + f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)\mathfrak{F}\cdot \\
&\qquad \nabla_u(\hat{u}_3) du.
\end{aligned}
$$

Therefore, for $i=1,2$,

$$
\begin{aligned}
&|\partial_{x_i} f(t,x,v)| \\
&\quad\lesssim \mathbf{1}_{t_{\mathbf{b}}>t}\{|\nabla_x f_0(X(0),V(0))||\partial_{x_i} X(0)| + |\nabla_v f_0(X(0),V(0))||\partial_{x_i} V(0)|\} \\
&\qquad + \mathbf{1}_{t_{\mathbf{b}}<t}\left(c_\mu \mu(v_{\mathbf{b}})\langle v\rangle^2 \int_{u_3<0} \big(|\nabla_x f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)|\hat{u}_3 + f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)\big) du\right) \\
&\quad\lesssim \mathbf{1}_{t_{\mathbf{b}}>t}\{|\nabla_{x_\parallel} f_0(X(0),V(0))||\partial_{x_i} X_\parallel(0)| + |\partial_{x_3} f_0(X(0),V(0))||\nabla_{x_\parallel} X_3(0)| \\
&\qquad + |\nabla_v f_0(X(0),V(0))||\partial_{x_i} V(0)|\} \\
&\qquad + \mathbf{1}_{t_{\mathbf{b}}<t}\left(c_\mu \mu(v_{\mathbf{b}})\langle v\rangle^2 \int_{u_3<0} \big(|\nabla_x f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)|\hat{u}_3 + f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)\big) du\right),
\end{aligned}
\tag{6.5}
$$

where we've used (4.19). And for $i=3$, we have

$$
\begin{aligned}
&|\partial_{x_3} f(t,x,v)| \\
&\quad\lesssim \mathbf{1}_{t_{\mathbf{b}}>t}\{|\nabla_{x_\parallel} f_0(X(0),V(0))||\partial_{x_3} X_\parallel(0)| + |\partial_{x_3} f_0(X(0),V(0))||\nabla_{x_3} X_3(0)| \\
&\qquad + |\nabla_v f_0(X(0),V(0))||\partial_{x_i} V(0)|\} \\
&\qquad + \mathbf{1}_{t_{\mathbf{b}}<t}\left(c_\mu \frac{1}{\hat{v}_{\mathbf{b},3}} \mu(v_{\mathbf{b}})\langle v\rangle^2 \int_{u_3<0} \big(|\nabla_x f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)|\hat{u}_3 + f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)\big) du\right).
\end{aligned}
\tag{6.6}
$$

Also,

$$
\begin{aligned}
&|\partial_{v_i} f(t,x,v)| \\
&\quad\lesssim \mathbf{1}_{t_{\mathbf{b}}>t}\{|\nabla_{x_\parallel} f_0(X(0),V(0))||\partial_{v_i} X_\parallel(0)| + |\partial_{x_3} f_0(X(0),V(0))||\nabla_{v_i} X_3(0)| \\
&\qquad + |\nabla_v f_0(X(0),V(0))||\partial_{v_i} V(0)|\} \\
&\qquad + \mathbf{1}_{t_{\mathbf{b}}<t}\left(c_\mu \mu(v_{\mathbf{b}})|v_{\mathbf{b}}|\langle v\rangle^2 \int_{u_3<0} \big(|\nabla_x f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)|\hat{u}_3 + f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)\big) du\right).
\end{aligned}
\tag{6.7}
$$

Let $(x,v)\notin\gamma_0$ and $(t^0,x^0,v^0)=(t,x,v)$. For the characteristic

$$\begin{aligned}\frac{d}{ds}X(s;t,x,v)&=\hat{V}(s;t,x,v),\\ \frac{d}{ds}V(s;t,x,v)&=\mathfrak{F}(s,X(s;t,x,v),V(s;t,x,v)),\end{aligned}\tag{6.8}$$

we define the stochastic (diffuse) cycles as

$$\begin{aligned}&t^1=t-t_{\mathbf{b}}(t,x,v),\ x^1=x_{\mathbf{b}}(t,x,v)=X(t-t_{\mathbf{b}}(t,x,v);t,x,v),\\ &v_b^0=V(t-t_{\mathbf{b}}(t,x,v);t,x,v)=v_{\mathbf{b}}(t,x,v),\end{aligned}\tag{6.9}$$

and $v^1\in\mathbb{R}^3$ with $n(x^1)\cdot v^1>0$. For $l\geq 1$, define

$$\begin{aligned}t^{l+1}&=t^l-t_{\mathbf{b}}(t^l,x^l,v^l),\,x^{l+1}=x_{\mathbf{b}}(t^l,x^l,v^l),\\ v_b^l&=v_{\mathbf{b}}(t^l,x^l,v^l),\end{aligned}\tag{6.10}$$

and $v^{l+1}\in\mathbb{R}^3$ with $n(x^{l+1})\cdot v^{l+1}>0$. Also, define

$$X^l(s)=X(s;t^l,x^l,v^l),\ V^l(s)=V(s;t^l,x^l,v^l),\tag{6.11}$$

so $X(s)=X^0(s)$, $V(s)=V^0(s)$.

Expanding $\nabla_x f(t^1,x^1,v^1)+f(t^1,x^1,v^1)$ in (6.6) again, we get for $i=1,2$,

$$\begin{aligned}&|\partial_{x_i}f(t,x,v)|\\ &\lesssim \mathbf{1}_{t^1<0}\{|\nabla_x f_0(X(0),V(0))||\partial_{x_i}X(0)|+|\nabla_v f_0(X(0),V(0))||\partial_{x_i}V(0)|\}\\ &\quad+\mathbf{1}_{t^2<0<t^1}\Bigg\{c_\mu\mu(v_{\mathbf{b}})\langle v\rangle^2\int_{v_3^1<0}\Big(\Big(|\nabla_x f(0,X^1(0),V^1(0))+\langle v^1\rangle|\nabla_v f(0,X^1(0),V^1(0))|\Big)|\hat{v}_3^1\\ &\quad+f(0,X^1(0),V^1(0))\Big)dv^1\Bigg\}\\ &\quad+\mathbf{1}_{t^2>0}\Bigg(c_\mu\mu(v_{\mathbf{b}})\langle v\rangle^2\int_{v_3^1<0}\Bigg(c_\mu\mu(v_{\mathbf{b}}^1)\langle v_1\rangle^2\frac{\hat{v}_3^1}{\hat{v}_{\mathbf{b},3}^1}\int_{v_3^2<0}|\Big(\nabla_{x_\parallel}f(t^2,x^2,v^2)|\hat{v}_3^2\\ &\quad+f(t^2,x^2,v^2)\Big)dv^2\Bigg)dv^1\Bigg).\end{aligned}$$

Keep doing the expansion we get for $\ell>1$,

$$\begin{aligned}&|\nabla_{x_\parallel}f(t,x,v)|\\ &\lesssim\mathbf{1}_{t^1<0}\{|\nabla_{x_\parallel}f_0(X(0),V(0))||\nabla_{x_\parallel}X_\parallel(0)|+|\partial_{x_3}f_0(X(0),V(0))||\nabla_{x_\parallel}X_3(0)|\\ &\quad+|\nabla_v f_0(X(0),V(0))||\nabla_{x_\parallel}V(0)|\}\\ &\quad+\mu(v_{\mathbf{b}})\langle v\rangle^2\int_{\prod_{j=1}^{l-1}\mathcal{V}_j}\sum_{i=1}^{l-1}\mathbf{1}_{\{t^{i+1}<0<t^i\}}\\ &\Bigg(\Big(|\nabla_x f(0,X^i(0),V^i(0))|+\langle v^i\rangle|\nabla_v f(0,X^i(0),V^i(0))|\Big)\hat{v}_3^i\\ &\quad+f(0,X^i(0),V^i(0))\Bigg)d\Sigma_i^{l-1}\\ &\quad+\mu(v_{\mathbf{b}})\langle v\rangle^2\int_{\prod_{j=1}^{l-1}\mathcal{V}_j}\mathbf{1}_{\{t^l>0\}}\int_{\mathcal{V}_l}\Big(|\nabla_{x_\parallel}f(t^l,x^l,v^l)|\hat{v}_3^l+f(t^l,x^l,v^l)\Big)dv^l d\Sigma_{l-1}^{l-1},\end{aligned}\tag{6.12}$$

where $\mathcal{V}_j = \{v^j \in \mathbb{R}^3 : v_3^J < 0\}$, and

$$d\Sigma_i^{l-1} = \{\prod_{j=i+1}^{l-1} \mu(v^j) c_\mu |\hat{v}_3^j| dv^j\}\{\prod_{j=1}^{i-1} c_\mu \mu(v_{\mathbf{b}}^j)\langle v^j\rangle^2 \frac{\hat{v}_3^J}{\hat{v}_{\mathbf{b},3}^j} dv^j\},$$

where c_μ is the constant that $c_\mu \int_{\mathbb{R}^3} \mu(v^j)|\hat{v}_3^j| dv^j = 1$. Similarly, we get

$$\begin{aligned}
&|\partial_{x_3} f(t,x,v)| \\
&\quad \lesssim \mathbf{1}_{t^1<0}\{|\nabla_{x_\parallel} f_0(X(0),V(0))||\partial_{x_3} X_\parallel(0)| + |\partial_{x_3} f_0(X(0),V(0))||\partial_{x_3} X_3(0)| \\
&\qquad + |\nabla_v f_0(X(0),V(0))||\partial_{x_3} V(0)|\} \\
&\qquad + \frac{\mu(v_{\mathbf{b}})}{\hat{v}_{\mathbf{b},3}}\langle v\rangle^2 \int_{\prod_{j=1}^{l-1}\mathcal{V}_j} \sum_{i=1}^{l-1} \mathbf{1}_{\{t^{i+1}<0<t^i\}} \\
&\quad \bigg(\Big(|\nabla_x f(0,X^i(0),V^i(0))| + \langle v^i\rangle |\nabla_v f(0,X^i(0),V^i(0))| \Big) \hat{v}_3^i \\
&\qquad + f(0,X^i(0),V^i(0)) \bigg) d\Sigma_i^{l-1} \\
&\qquad + \frac{\mu(v_{\mathbf{b}})}{\hat{v}_{\mathbf{b},3}}\langle v\rangle^2 \int_{\prod_{j=1}^{l-1}\mathcal{V}_j} \mathbf{1}_{\{t^l>0\}} \int_{\mathcal{V}_l} \Big(|\nabla_x f(t^l,x^l,v^\ell)|\hat{v}_3^\ell + f(t^\ell,x^\ell,v^\ell) \Big) dv^l d\Sigma_{l-1}^{l-1},
\end{aligned} \tag{6.13}$$

and

$$\begin{aligned}
&|\nabla_v f(t,x,v)| \\
&\quad \lesssim \mathbf{1}_{t^1<0}\{|\nabla_{x_\parallel} f_0(X(0),V(0))||\nabla_v X_\parallel(0)| \\
&\qquad + |\partial_{x_3} f_0(X(0),V(0))||\nabla_v X_3(0)| + |\nabla_v f_0(X(0),V(0))||\nabla_v V(0)|\} \\
&\qquad + \mu(v_{\mathbf{b}})|v_{\mathbf{b}}|\langle v\rangle^2 \int_{\prod_{j=1}^{l-1}\mathcal{V}_j} \sum_{i=1}^{l-1} \mathbf{1}_{\{t^{i+1}<0<t^i\}} \\
&\quad \bigg(\Big(|\nabla_x f(0,X^i(0),V^i(0))| + \langle v^i\rangle |\nabla_v f(0,X^i(0),V^i(0))| \Big) \hat{v}_3^i \\
&\qquad + f(0,X^i(0),V^i(0)) \bigg) d\Sigma_i^{l-1} \\
&\qquad + \mu(v_{\mathbf{b}})|v_{\mathbf{b}}|\langle v\rangle^2 \int_{\prod_{j=1}^{l-1}\mathcal{V}_j} \mathbf{1}_{\{t^l>0\}} \int_{\mathcal{V}_l} \Big(|\nabla_x f(t^l,x^l,v^l)|\hat{v}_3^l + f(t^l,x^l,v^l) \Big) dv^l d\Sigma_{l-1}^{l-1},
\end{aligned} \tag{6.14}$$

Next, we claim that there exists $l_0 \gg 1$ such that for $l \geq l_0$, we have

$$\int_{\prod_{j=1}^{l-1}\mathcal{V}_j} \mathbf{1}_{\{t^l(t,x,v,v^1,\ldots,v^{l-1})>0\}} \, d\,\Sigma_{l-1}^{l-1} \lesssim \left(\frac{1}{2}\right)^l. \tag{6.15}$$

Since

$$\begin{aligned}
&|v_{\mathbf{b}}^j|^2 \lesssim |v^j|^2 + (t^j - t^{j+1})^2(\|E\|_\infty^2 + \|B\|_\infty^2), \ \langle v_{\mathbf{b}}^j\rangle \\
&\lesssim \langle v^j\rangle + (t^j - t^{j+1})(\|E\|_\infty + \|B\|_\infty),
\end{aligned}$$

and using (4.3), we have for some fixed constant $C_0 > 0$,

$$d\,\Sigma_{l-1}^{l-1} \le (C_0)^l \prod_{j=1}^{l-1} \sqrt{\mu(v_j)}\langle v^j\rangle^2 dv^j.$$

Choose a sufficiently small $\delta = \delta(C_0) > 0$. Define

$$\mathcal{V}_j^\delta = \{v^j \in \mathcal{V}_j : v_3^J \ge \delta\},$$

where we have $\int_{\mathcal{V}_j\setminus\mathcal{V}_j^\delta} C_0\sqrt{\mu(v_j)}\langle v^j\rangle^2 dv^j \lesssim \delta$.

On the other hand if $v^j \in \mathcal{V}_j^\delta$, we have from (5.2)

$$\begin{aligned}|(t^j - t^{j+1})\hat{v}_3^j| &= |\int_{t^{j+1}}^{t^j}\int_s^{t^j} \hat{\mathfrak{F}}_3(\tau, X^j(\tau), V^j(\tau))\mathrm{d}\tau\mathrm{d}s| \\ &\le \int_{t^{j+1}}^{t^j}\int_s^{t^j} \frac{2g}{\langle V^j(\tau)\rangle} d\tau ds \le \frac{(t^j - t^{j+1})^2 g}{\min_{0\le\tau\le T}\langle V^j(\tau)\rangle}.\end{aligned}$$

Thus

$$|t^j - t^{j+1}| \ge \frac{v_3^J}{g}\frac{\min_{0\le\tau\le T}\langle V^j(\tau)\rangle}{\max_{0\le\tau\le T}\langle V^j(\tau)\rangle} \gtrsim v_3^J \ge \delta.$$

Now if $t^l \ge 0$ then there are at most $\left[\frac{C_\Omega}{\delta}\right] + 1$ numbers of $v^m \in \mathcal{V}_m^\delta$ for $1 \le m \le l-1$. Equivalently there are at least $l - 2 - \left[\frac{C_\Omega}{\delta}\right]$ numbers of $v^{m_i} \in \mathcal{V}_{m_i}\setminus\mathcal{V}_{m_i}^\delta$. Therefore we have:

$$\begin{aligned}&\int_{\prod_{j=1}^{l-1}\mathcal{V}_j} \mathbf{1}_{\{t^l(t,x,v,v^1,\dots,v^{l-1})>0\}}\, d\,\Sigma_{l-1}^{l-1} \\ &\le \sum_{m=1}^{\left[\frac{C_\Omega}{\delta}\right]+1} \int_{\left\{\begin{array}{l}\text{there are exactly } m \text{ of } v^{m_i} \in \mathcal{V}_{m_i}^\delta \\ \text{and } l-1-m \text{ of } v^{m_i} \in \mathcal{V}_{m_i}\setminus\mathcal{V}_{m_i}^\delta\end{array}\right\}} \prod_{j=1}^{l-1} C_0\sqrt{\mu(v_j)}\langle v^j\rangle^2 dv^j \\ &\le \sum_{m=1}^{\left[\frac{C_\Omega}{\delta}\right]+1} \binom{l-1}{m}\left\{\int_{\mathcal{V}} C_0\sqrt{\mu(v)}\langle v\rangle^2 dv\right\}^m \left\{\int_{\mathcal{V}\setminus\mathcal{V}^\delta} C_0\sqrt{\mu(v)}\langle v\rangle^2 dv\right\}^{l-1-m} \\ &\le \left(\left[\frac{C_\Omega}{\delta}\right]+1\right)(l-1)^{\left[\frac{C_\Omega}{\delta}\right]+1}(\delta)^{l-2-\left[\frac{C_\Omega}{\delta}\right]}\left\{\int_{\mathcal{V}} C_0\sqrt{\mu(v)}\langle v\rangle^2 dv\right\}^{\left[\frac{C_\Omega}{\delta}\right]+1} \\ &\le C\delta^{l/2} \le C(\frac{1}{2})^l,\end{aligned} \tag{6.16}$$

if $l \gg 1$, say $l = 2\left(\left[\frac{C_\Omega}{\delta}\right]+1\right)^2$.

Therefore, from (4.20), (4.23), (4.27), (6.12),(6.15), and (6.16) we have

$$
\begin{aligned}
&\langle v\rangle^{4+\delta}|\nabla_{x_\parallel} f(t,x,v)| \\
&\quad\lesssim \langle v\rangle^{4+\delta}\{|\nabla_{x_\parallel} f_0(X(0),V(0))| + t|\partial_{x_3} f_0(X(0),V(0))| + \langle v\rangle|\nabla_v f_0(X(0),V(0))|\} \\
&\qquad + \langle v\rangle^{4+\delta}\mu(v_{\mathbf{b}})\langle v\rangle^2 \int_{\prod_{j=1}^{l-1}\mathcal{V}_j} \sum_{i=1}^{l-1} \mathbf{1}_{\{t^{i+1}<0<t^i\}} \Big(\Big(|\nabla_x f(0,X^i(0),V^i(0))| \\
&\qquad + \langle v^i\rangle|\nabla_v f(0,X^i(0),V^i(0))| \Big) \hat{v}_3^i + f(0,X^i(0),V^i(0)) \Big) d\Sigma_i^{l-1} \\
&\qquad + \langle v\rangle^{4+\delta}\mu(v_{\mathbf{b}})\langle v\rangle^2 \int_{\prod_{j=1}^{l-1}\mathcal{V}_j} \mathbf{1}_{\{t^l>0\}} \int_{\mathcal{V}_l} \left(|\nabla_{x_\parallel} f(t^l,x^l,v^l)|\hat{v}_3^l + f(t^l,x^l,v^l) \right) dv^l d\Sigma_{l-1}^{l-1} \\
&\quad\lesssim C_l \left(\|(1+|v|^{4+\delta})\nabla_{x_\parallel} f_0\|_\infty + \|(1+|v|^{4+\delta})\alpha\partial_{x_3} f_0\|_\infty + \|(1+|v|^{5+\delta})\nabla_v f_0\|_\infty \right) \\
&\qquad + C\left(\frac{1}{2}\right)^l \sup_{0\le t\le T} \|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_\infty,
\end{aligned}
\tag{6.17}
$$

and similarly,

$$
\begin{aligned}
&\langle v\rangle^{5+\delta}|\alpha\partial_{x_3} f(t,x,v)| \\
&\quad\lesssim \langle v\rangle^{5+\delta}\alpha(t,x,v)\{|\nabla_{x_\parallel} f_0(X(0),V(0))| \\
&\quad + |\partial_{x_3} f_0(X(0),V(0))| + \langle v\rangle|\nabla_v f_0(X(0),V(0))|\} \\
&\qquad + \langle v\rangle^{5+\delta}\frac{\alpha(t,x,v)\mu(v_{\mathbf{b}})}{\hat{v}_{\mathbf{b},3}}\langle v\rangle^2 \int_{\prod_{j=1}^{l-1}\mathcal{V}_j} \sum_{i=1}^{l-1} \mathbf{1}_{\{t^{i+1}<0<t^i\}} \\
&\quad \Big(\Big(|\nabla_x f(0,X^i(0),V^i(0))| + \langle v^i\rangle|\nabla_v f(0,X^i(0),V^i(0))| \Big) \hat{v}_3^i \\
&\qquad + f(0,X^i(0),V^i(0)) \Big) d\Sigma_i^{l-1} \\
&\qquad + \langle v\rangle^{5+\delta}\frac{\alpha(t,x,v)\mu(v_{\mathbf{b}})}{\hat{v}_{\mathbf{b},3}}\langle v\rangle^2 \int_{\prod_{j=1}^{l-1}\mathcal{V}_j} \mathbf{1}_{\{t^l>0\}} \\
&\quad \int_{\mathcal{V}_l} \left(|\nabla_x f(t^l,x^l,v^l)|\hat{v}_3^l + f(t^l,x^l,v^l) \right) dv^l d\Sigma_{l-1}^{l-1} \\
&\quad\lesssim C_l \left(\|(1+|v|^{5+\delta})\nabla_{x_\parallel} f_0\|_\infty + \|(1+|v|^{5+\delta})\alpha\partial_{x_3} f_0\|_\infty + \|(1+|v|^{5+\delta})\nabla_v f_0\|_\infty \right) \\
&\qquad + C\left(\frac{1}{2}\right)^l \sup_{0\le t\le T} \|\langle v\rangle^{5+\delta}\alpha\nabla_x f(t)\|_\infty,
\end{aligned}
\tag{6.18}
$$

and

$$
\begin{aligned}
&\langle v\rangle^{5+\delta}|\nabla_v f(t,x,v)| \\
&\quad\lesssim C_l \left(\|(1+|v|^{4+\delta})\nabla_{x_\parallel} f_0\|_\infty + \|(1+|v|^{4+\delta})\alpha\partial_{x_3} f_0\|_\infty + \|(1+|v|^{5+\delta})\nabla_v f_0\|_\infty \right) \\
&\qquad + C\left(\frac{1}{2}\right)^l \sup_{0\le t\le T} \|\langle v\rangle^{5+\delta}\alpha\nabla_x f(t)\|_\infty,
\end{aligned}
\tag{6.19}
$$

where we've used (4.3). Adding (6.12), (6.18), and (6.19) and choosing $l \gg 1$, we get for a large $C > 0$,

$$\begin{aligned}&\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_\infty + \|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t)\|_\infty + \|\langle v\rangle^{5+\delta}\nabla_v f(t)\|_\infty\Big)\\ &\quad < C\Big(\|\langle v\rangle^{5+\delta}\nabla_{x_\parallel} f_0\|_\infty + \|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f_0\|_\infty + \|\langle v\rangle^{5+\delta}\nabla_v f_0\|_\infty\Big).\end{aligned} \tag{6.20}$$

Next, using the same argument in Lemma 13, we obtain

$$\begin{aligned}&\sup_{0\le t\le T}\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_{L^\infty(\gamma\setminus\gamma_0)} \le \sup_{0\le t\le T}\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f(t)\|_\infty,\\ &\sup_{0\le t\le T}\|\langle v\rangle^{5+\delta}\nabla_v f(t)\|_{L^\infty(\gamma\setminus\gamma_0)} \le \sup_{0\le t\le T}\|\langle v\rangle^{5+\delta}\nabla_v f(t)\|_\infty,\\ &\sup_{0\le t\le T}\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t,x,v)\|_{L^\infty(\gamma\setminus\gamma_0)} \le \sup_{0\le t\le T}\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f(t,x,v)\|_\infty.\end{aligned} \tag{6.21}$$

Together (6.20), we conclude (6.2). □

In order to construct a solution to the system (0.40)–(0.42), (0.26), we define a sequence of functions:

$$f^0(t,x,v) = f_0(x,v),\ \ E^0(t,x) = E_0(t,x),\ \ B^0(t,x) = B_0(x).$$

For $\ell \ge 1$, let f^ℓ be the solution of

$$\begin{aligned}\partial_t f^\ell + \hat v\cdot\nabla_x f^\ell + \mathfrak{F}^{\ell-1}\cdot\nabla_v f^\ell &= 0,\ \text{ where } \mathfrak{F}^{\ell-1}\\ &= E^{\ell-1} + E_{\text{ext}} + \hat v\times(B^{\ell-1}+B_{\text{ext}}) - g\mathbf{e}_3,\\ f^\ell(0,x,v) &= f_0(x,v),\\ f^\ell(t,x,v)|_{\gamma_-} &= c_\mu\mu(v)\int_{u_3<0} -f^{\ell-1}(t,x,u)\hat u_3\,du.\end{aligned} \tag{6.22}$$

Let $\rho^\ell = \int_{\mathbb{R}^3} f^\ell dv$, $j^\ell = \int_{\mathbb{R}^3}\hat v f^\ell dv$. Let

$$E^\ell = (2.31)+\cdots+(2.41),\ \ B^\ell = (2.45)+\cdots+(2.55),\ \text{ with } f \text{ changes to } f^\ell. \tag{6.23}$$

And let

$$\mathfrak{F}^\ell = E^\ell + E_{\text{ext}} - \hat v\times(B^\ell+B_{\text{ext}}) - g\mathbf{e}_3. \tag{6.24}$$

We prove several uniform-in-ℓ bounds for the sequence before passing the limit.

Lemma 14. *Suppose f_0 satisfies (0.32), E_0, B_0 satisfy (0.16), (0.35), then there exits M_1, M_2, and c_0, such that for $0 < T \ll 1$,*

$$\begin{aligned}\sup_\ell\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta} f^\ell(t)\|_{L^\infty(\bar\Omega\times\mathbb{R}^3)}\Big) &< M_1,\\ \sup_\ell\sup_{0\le t\le T}\Big(\|E^\ell(t)\|_\infty + \|B^\ell(t)\|_\infty\Big) + |B_e| + E_e + g &< M_2,\\ \inf_\ell\inf_{t,x_\parallel}\Big(g - E_e - E_3^\ell(t,x_\parallel,0) - (\hat v\times B^\ell)_3(t,x_\parallel,0)\Big) &> c_0.\end{aligned} \tag{6.25}$$

Proof. Let $\ell \geq 1$. By induction hypothesis we assume that

$$\begin{aligned} \sup_{0\leq i\leq \ell}\sup_{0\leq t\leq T}\Big(\|\langle v\rangle^{4+\delta} f^{\ell-i}(t)\|_{L^\infty(\bar\Omega\times\mathbb{R}^3)}\Big) &< M_1, \\ \sup_{0\leq i\leq \ell}\sup_{0\leq t\leq T}\Big(\|E^{\ell-i}(t)\|_\infty + \|B^{\ell-i}(t)\|_\infty\Big) + |B_e| + E_e + g &< M_2. \end{aligned} \tag{6.26}$$

Denote the characteristics (X^ℓ, V^ℓ) which solves

$$\begin{aligned} &\frac{d}{ds}X^\ell(s;t,x,v) = \hat{V}^\ell(s;t,x,v), \\ &\frac{d}{ds}V^\ell(s;t,x,v) = \mathfrak{F}^\ell(s, X^\ell(s;t,x,v), V^\ell(s;t,x,v)). \end{aligned} \tag{6.27}$$

We define the stochastic cycles:

$$\begin{aligned} t_1^\ell(t,x,v) &:= \sup\{s < t : X^\ell(s;t,x,v) \in \partial\Omega\}, \\ x_1^\ell(t,x,v) &:= X^\ell(t_1^\ell(t,x,v); t,x,v), \\ t_2^{\ell-1}(t,x,v,v_1) &:= \sup\{s < t_1^\ell : X^{\ell-1}(s; t_1^\ell(t,x,v), x_1^\ell(t,x,v), v_1) \in \partial\Omega\}, \\ x_2^{\ell-1}(t,x,v,v_1) &:= X^{\ell-1}(t_2^{\ell-1}(t,x,v,v_1); t_1^\ell(t,x,v), x_1^\ell(t,x,v), v_1), \end{aligned} \tag{6.28}$$

and inductively

$$\begin{aligned} &t_k^{\ell-(k-1)}(t,x,v,v_1,\ldots,v_{k-1}) \\ &\quad := \sup\Big\{s < t_{k-1}^{\ell-(k-2)} : X^{\ell-1}(s; t_{k-1}^{\ell-(k-2)}, x_{k-1}^{\ell-(k-2)}, v_{k-1}) \in \partial\Omega\Big\}, \\ &x_k^{\ell-(k-1)}(t,x,v,v_1,\ldots,v_{k-1}) \\ &\quad := X^{\ell-(k-2)}(t_k^{\ell-(k-1)}; t_{k-1}^{\ell-(k-2)}, x_{k-1}^{\ell-(k-2)}, v_{k-1}). \end{aligned} \tag{6.29}$$

Here,

$$\begin{aligned} t_i^{\ell-(i-1)} &:= t_i^{\ell-(i-1)}(t,x,v,v_1,\ldots,v_{i-1}), \\ x_i^{\ell-(i-1)} &:= x_i^{\ell-(i-1)}(t,x,v,v_1,\ldots,v_{i-1}). \end{aligned}$$

First, we note that for any $t_{i+1}^{\ell-i} \leq s < t_i^{\ell-(i-1)}$, since

$$V^{\ell-i}(s; t_i^{\ell-(i-1)}, x_i^{\ell-(i-1)}, v_i) = v_i - \int_s^{t_i^{\ell-(i-1)}} \mathfrak{F}^{\ell-i}(\tau, X^{\ell-i}(\tau), V^{\ell-i}(\tau))d\tau,$$

from (6.26), we have

$$\begin{aligned} &|v_i| - (t_i^{\ell-(i-1)} - t_{i+1}^{\ell-i})M_2 \leq |V^{\ell-i}(s; t_i^{\ell-(i-1)}, x_i^{\ell-(i-1)}, v_i)| \\ &\quad \leq |v_i| + (t_i^{\ell-(i-1)} - t_{i+1}^{\ell-i})M_2. \end{aligned}$$

Thus

$$\Big(1 + (t_i^{\ell-(i-1)} - t_{i+1}^{\ell-i})M_2\Big)^{-1}\langle v\rangle \leq \langle V^{\ell-i}(s; t_i^{\ell-(i-1)}, x_i^{\ell-(i-1)}, v_i)\rangle$$

$$\leq \left(1 + (t_i^{\ell-(i-1)} - t_{i+1}^{\ell-i})M_2\right)\langle v\rangle. \tag{6.30}$$

From (6.22) we have for any $(t, x, v) \in (0, T) \times \bar{\Omega} \times \mathbb{R}^3$,

$$\begin{aligned} f^{\ell+1}(t, x, v) = \mathbf{1}_{t_1^\ell \leq 0} f^{\ell+1}(0, X^\ell(0), V^\ell(0)) \\ + \mathbf{1}_{t_1^\ell \geq 0} f^{\ell+1}(t_1^\ell, X^\ell(t_1^\ell; t, x, v), V^\ell(t_1^\ell; t, x, v)) \\ = \mathbf{1}_{t_1^\ell \leq 0} f^{\ell+1}(0, X^\ell(0), V^\ell(0)) \\ - \mathbf{1}_{t_1^\ell \geq 0} c_\mu \mu(V^\ell(t_1^\ell)) \int_{v_{1,3}<0} f^\ell(t_1^\ell, x_1^\ell, v_1)\hat{v}_{1,3} \mathrm{d}v_1. \end{aligned} \tag{6.31}$$

And (6.30) gives

$$\begin{aligned} &\langle v\rangle^{4+\delta}|f^{\ell+1}(t, x, v)| \\ &\leq \mathbf{1}_{t_1^\ell \leq 0}(1 + T(M_2 + g))^{4+\delta}|\langle V^\ell(0)\rangle^{4+\delta} f^{\ell+1}(0, X^\ell(0), V^\ell(0))| \\ &\quad + \mathbf{1}_{t_1^\ell \geq 0} c_\mu (1 + T(M_2 + g))^{4+\delta}|\langle V^\ell(t_1^\ell)\rangle^{4+\delta} \mu(V^\ell(t_1^\ell)) \\ &\int_{v_{1,3}<0} \langle v_1\rangle^{4+\delta} f^\ell(t_1^\ell, x_1^\ell, v_1)\frac{\hat{v}_{1,3}}{\langle v_1\rangle^{4+\delta}}\mathrm{d}v_1|. \end{aligned} \tag{6.32}$$

Then inductively, we obtain

$$\begin{aligned} &\langle v\rangle^{4+\delta}|f^{\ell+1}(t, x, v)| \leq \mathbf{1}_{t_1^\ell \leq 0}(1 + t(M_2 + g))^{4+\delta}||\langle V^\ell(0)\rangle^{4+\delta} f^{\ell+1}(0, X^\ell(0), V^\ell(0))| \\ &+ (1 + T(M_2 + g))^{4+\delta} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{i=1}^{k-1} \mathbf{1}_{\{t_{i+1}^{\ell-i} \leq 0 < t_i^{\ell-(i-1)}\}} |\langle V^{\ell-i}(0; v_i)\rangle^{4+\delta} f^{\ell-(i-1)} \\ &\times (0, X^{\ell-i}(0; v_i), V^{\ell-i}(0; v_i))|\, d\Sigma_i^{k-1} + (1 + T(M_2 + g))^{4+\delta} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{t_k^{\ell-(k-1)}>0\}} \\ &\int_{\mathcal{V}_k} |f^{\ell-k}(t_k^{\ell-(k-1)}, x_k^{\ell-(k-1)}, v_k)|dv_k d\Sigma_{k-1}^{k-1}, \end{aligned} \tag{6.33}$$

where

$$\begin{aligned} X^{\ell-i}(0; v_i) &= X^{\ell-i}(0; t_i^{\ell-(i-1)}, x_i^{\ell-(i-1)}, v_i), \\ V^{\ell-i}(0; v_i) &= V^{\ell-i}(0; t_i^{\ell-(i-1)}, x_i^{\ell-(i-1)}, v_i), \end{aligned}$$

$\mathcal{V}_j = \{v_j \in \mathbb{R}^3 : v_{j,3} < 0\}$, and

$$\begin{aligned} d\Sigma_i^{k-1} = \{\prod_{j=1}^{i-1} c_\mu (1 + T(M_2 + g))^{4+\delta} \mu(V^{\ell-(j-1)}(t_j^{\ell-(j-1)})) \\ \frac{\hat{v}_{j,3}\langle V^{\ell-(j-1)}(t_j^{\ell-(j-1)})\rangle^{4+\delta}}{\hat{V}_3^{\ell-(j-1)}(t_j^{\ell-(j-1)})\langle v_j\rangle^{4+\delta}} dv_j\}\{\prod_{j=i+1}^{k-1} \mu(v_j)c_\mu|\hat{v}_{j,3}|dv_j\}. \end{aligned} \tag{6.34}$$

From the same argument as in (6.15)–(6.16), we get there exists $k_0 \gg 1$ such that for $k \geq k_0$,

$$\int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \mathbf{1}_{\{t_k^{\ell-(k-1)}>0\}} d\Sigma_{k-1}^{k-1} \leq \left(\frac{1}{2}\right)^k. \tag{6.35}$$

Thus, from (6.33), (6.35), we have

$$\begin{aligned}
&\sup_{0\leq t\leq T} \|\langle v\rangle^{4+\delta} f^{\ell+1}(t)\|_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} \\
&\quad \leq k(1+TM_2)^{4+\delta}\|\langle v\rangle^{4+\delta} f_0\|_\infty \\
&\qquad + (1+T(M_2+g))^{4+\delta}\left(\frac{1}{2}\right)^k \sup_{0\leq t\leq T}\|\langle v\rangle^{4+\delta} f^{\ell-i}(t)\|_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)}.
\end{aligned} \tag{6.36}$$

By choosing $M_1 \gg 1$ and then $T \ll 1$, we get

$$\sup_{0\leq t\leq T} \|\langle v\rangle^{4+\delta} f^{\ell+1}(t)\|_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} < M_1. \tag{6.37}$$

Now from (6.23) and (0.16),

$$\begin{aligned}
&\sup_{0\leq t\leq T}\|E^{\ell+1}(t)\|_\infty + \sup_{0\leq t\leq T}\|B^{\ell+1}(t)\|_\infty \\
&\quad \leq C(\|E_0\|_\infty + \|B_0\|_\infty) + CT(\|E_0\|_{C^1} + \|B_0\|_{C^1}) \\
&\qquad + CT\sup_{0\leq t\leq T}\|\langle v\rangle^{4+\delta} f^\ell(t)\|_\infty\left(1+T(\sup_{0\leq t\leq T}\left(\|E^\ell(t)\|_\infty + \|B^\ell(t)\|_\infty\right) + g + |B_e|)\right) \\
&\quad \leq C(\|E_0\|_{C^1} + \|B_0\|_{C^1}) + CTM_1\left(1+T(M_2+g+|B_e|)\right).
\end{aligned} \tag{6.38}$$

Letting $M_2 = (C+1)(\|E_0\|_{C^1} + \|B_0\|_{C^1}) + |B_e| + E_e + g$ and $T \ll 1$, we get

$$\sup_{0\leq t\leq T}\|E^{\ell+1}(t)\|_\infty + \sup_{0\leq t\leq T}\|B^{\ell+1}(t)\|_\infty + |B_e| + E_e + g < M_2. \tag{6.39}$$

Next, from (0.16) and the proof of Lemma 6, by letting $c_0 = \frac{c_1}{2}$ in (0.16), we get

$$\begin{aligned}
\inf_{t,x_\parallel}\left(g - E_e - E_3^{\ell+1}(t,x_\parallel,0) - (\hat{v}\times B^{\ell+1})_3(t,x_\parallel,0)\right) > \quad & 2c_0 \\
- CTM_1\left(1+T(M_2+g+|B_e|)\right). &
\end{aligned}$$

By choosing $T \ll 1$ small enough, we have

$$\inf_{t,x_\parallel}\left(g - E_e - E_3^{\ell+1}(t,x_\parallel,0) - (\hat{v}\times B^{\ell+1})_3(t,x_\parallel,0)\right) > c_0. \tag{6.40}$$

Thus we conclude (6.25) by induction. □

Next, we consider the derivative of the sequences. Define α^ℓ as in (5.46). We have the following estimate.

Lemma 15. *Suppose f_0 satisfies (0.32), E_0, B_0 satisfy (0.35), then there exits M_3, M_4 such that for* $0 < T \ll 1$,

$$
\begin{aligned}
&\sup_{\ell}\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f^\ell(t)\|_\infty + \|\langle v\rangle^{5+\delta}\alpha^{\ell-1}\partial_{x_3} f^\ell(t)\|_\infty + \|\langle v\rangle^{4+\delta}\nabla_v f^\ell(t)\|_\infty\Big)\\
&\quad+\sup_{\ell}\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f^\ell(t)\|_{L^\infty(\gamma\backslash\gamma_0)} + \|\langle v\rangle^{5+\delta}\alpha^{\ell-1}\partial_{x_3} f^\ell(t)\|_{L^\infty(\gamma\backslash\gamma_0)}\\
&\quad+\|\langle v\rangle^{4+\delta}\nabla_v f^\ell(t)\|_{L^\infty(\gamma\backslash\gamma_0)}\Big) < M_3,\\
&\quad\sup_{\ell}\sup_{0\le t\le T}\Big(\|\partial_t E^\ell(t)\|_\infty + \|\partial_t B^\ell(t)\|_\infty + \|\nabla_x E^\ell(t)\|_\infty + \|\nabla_x B^\ell(t)\|_\infty\Big) < M_4.
\end{aligned}
\tag{6.41}
$$

Proof. The proof is essentially the same as the proof of Proposition 4. The only difference is that instead of using the stochastic cycles (6.8)–(6.11) that flows under fixed $E(t,x)$, $B(t,x)$, we use the (6.28)–(6.29) that flows with a different $E^\ell(t,x)$, $B^\ell(t,x)$ after each bounce.

From the uniform estimate (6.25), and from the velocity lemma (Lemma 8), we have for some $C > 0$,

$$
\begin{aligned}
e^{-C|t-s|}\alpha^\ell(t,x,v) &\le \alpha^\ell(s, X^\ell(s;t,x,v), V^\ell(s;t,x,v))\\
&\le e^{C|t-s|}\alpha^\ell(t,x,v), \quad \text{for all } \ell.
\end{aligned}
\tag{6.42}
$$

Therefore, following the same proof of Proposition 4 we get

$$
\begin{aligned}
&\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f^{\ell+1}(t)\|_\infty + \|\langle v\rangle^{5+\delta}\alpha^{\ell}\partial_{x_3} f^{\ell+1}(t)\|_\infty + \|\langle v\rangle^{5+\delta}\nabla_v f^{\ell+1}(t)\|_\infty\Big)\\
&\quad+\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f^\ell(t)\|_{L^\infty(\gamma\backslash\gamma_0)} + \|\langle v\rangle^{5+\delta}\alpha^{\ell-1}\partial_{x_3} f^\ell(t)\|_{L^\infty(\gamma\backslash\gamma_0)}\\
&\quad+\|\langle v\rangle^{4+\delta}\nabla_v f^\ell(t)\|_{L^\infty(\gamma\backslash\gamma_0)}\Big)\\
&\le C_k\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f_0\|_\infty + \|\|\langle v\rangle^{4+\delta}\alpha\partial_{x_3} f_0\|_\infty + \|\|\langle v\rangle^{5+\delta}\nabla_v f_0\|_\infty\Big)\\
&\quad+C\left(\frac{1}{2}\right)^k\sup_{0\le i\le\ell}\Bigg(\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f^i(t)\|_\infty\\
&\quad+\|\langle v\rangle^{5+\delta}\alpha\partial_{x_3} f^i(t)\|_\infty + \|\langle v\rangle^{5+\delta}\nabla_v f^i(t)\|_\infty\Big)\\
&\quad+\sup_{0\le i\le\ell}\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f^i(t)\|_{L^\infty(\gamma\backslash\gamma_0)} + \|\langle v\rangle^{5+\delta}\alpha^{i-1}\partial_{x_3} f^i(t)\|_{L^\infty(\gamma\backslash\gamma_0)}\\
&\quad+\|\langle v\rangle^{4+\delta}\nabla_v f^i(t)\|_{L^\infty(\gamma\backslash\gamma_0)}\Big)\Bigg).
\end{aligned}
\tag{6.43}
$$

Thus, by choosing $k \gg 1$ and $M_3 \gg 1$, we conclude

$$
\begin{aligned}
&\sup_{\ell}\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f^\ell(t)\|_\infty + \|\langle v\rangle^{5+\delta}\alpha^{\ell-1}\partial_{x_3} f^\ell(t)\|_\infty + \|\langle v\rangle^{4+\delta}\nabla_v f^\ell(t)\|_\infty\Big)\\
&\quad+\sup_{\ell}\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f^\ell(t)\|_{L^\infty(\gamma\backslash\gamma_0)} + \|\langle v\rangle^{5+\delta}\alpha^{\ell-1}\partial_{x_3} f^\ell(t)\|_{L^\infty(\gamma\backslash\gamma_0)}\\
&\quad+\|\langle v\rangle^{4+\delta}\nabla_v f^\ell(t)\|_{L^\infty(\gamma\backslash\gamma_0)}\Big) < M_3.
\end{aligned}
\tag{6.44}
$$

From this, we use the same argument to get (3.49) in the proof of Lemma 7 and obtain

$$
\begin{aligned}
&\sup_{0\le t\le T}\Big(\|\partial_t E^{\ell+1}(t)\|_\infty+\|\partial_t B^{\ell+1}(t)\|_\infty+\|\nabla_x E^{\ell+1}(t)\|_\infty+\|\nabla_x B^{\ell+1}(t)\|_\infty\Big)\\
&\quad\le TC\sup_{1\le i\le \ell}\sup_{0\le t\le T}\Big(\|\partial_t E^{i}(t)\|_\infty+\|\partial_t B^{i}(t)\|_\infty+\|\nabla_x E^{i}(t)\|_\infty+\|\nabla_x B^{i}(t)\|_\infty\Big)\\
&\quad C\left(\|E_0\|_{C^2}+\|B_0\|_{C_2}\right)+C\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f^{\ell+1}(t)\|_\infty+\|\langle v\rangle^{\ell+\delta}\alpha^n\partial_{x_3} f^{\ell+1}(t)\|_\infty\Big)\\
&\qquad+C\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f^{\ell}(t)\|_{L^\infty(\gamma\backslash\gamma_0)}+\|\langle v\rangle^{5+\delta}\alpha^{\ell-1}\partial_{x_3} f^{\ell}(t)\|_{L^\infty(\gamma\backslash\gamma_0)}\Big)\\
&\qquad+C\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta} f^{\ell+1}(t)\|_\infty+\|E^{\ell+1}(t)\|_\infty+\|B^{\ell+1}(t)\|_\infty\Big).
\end{aligned}
$$

From (6.25) and (6.44), this gives

$$
\begin{aligned}
&\sup_{\ell}\sup_{0\le t\le T}\Big(\|\partial_t E^{\ell}(t)\|_\infty+\|\partial_t B^{\ell}(t)\|_\infty+\|\nabla_x E^{\ell}(t)\|_\infty+\|\nabla_x B^{\ell}(t)\|_\infty\Big)\\
&\quad\le TC\sup_{\ell}\sup_{0\le t\le T}\Big(\|\partial_t E^{\ell}(t)\|_\infty+\|\partial_t B^{\ell}(t)\|_\infty+\|\nabla_x E^{\ell}(t)\|_\infty+\|\nabla_x B^{\ell}(t)\|_\infty\Big)\\
&\qquad+C\left(\|E_0\|_{C^2}+\|B_0\|_{C_2}\right)+C(M_1+M_2+M_3).
\end{aligned}
$$

Therefore, by choosing $M_4\gg 1$ and $T\ll 1$, we get

$$
\sup_{\ell}\sup_{0\le t\le T}\Big(\|\partial_t E^{\ell}(t)\|_\infty+\|\partial_t B^{\ell}(t)\|_\infty+\|\nabla_x E^{\ell}(t)\|_\infty+\|\nabla_x B^{\ell}(t)\|_\infty\Big)<M_4. \tag{6.45}
$$

Together with (6.44), we conclude (6.41). □

We have the following trace properties for E^ℓ and B^ℓ:

Lemma 16. *Suppose E^ℓ, B^ℓ satisfies (6.41). Then for any $\ell\ge 1$, $0<t<T$,*

$$
E^\ell(t,\cdot,0)\in L^\infty(\partial\Omega),\ \ B^\ell(t,\cdot,0)\in L^\infty(\partial\Omega). \tag{6.46}
$$

Proof. From (6.41) we have

$$
E^\ell(t,x)\in W^{1,\infty}((0,T)\times\Omega),\ \ B^\ell(t,x)\in W^{1,\infty}((0,T)\times\Omega),
$$

in particular, from the Morrey's inequality, $E(t)$, $B(t)$ are Lipschitz continuous on Ω. Now pick any $x_\parallel\in\mathbb{R}^2$, and $0<x_3<1$, from the fundamental theorem of calculus, we have

$$
\begin{aligned}
E^\ell(t,x_\parallel,0)&=E^\ell(t,x_\parallel,x_3)-\int_0^{x_3}\partial_3 E^\ell(t,x_\parallel,y)dy,\\
B^\ell(t,x_\parallel,0)&=B^\ell(t,x_\parallel,x_3)-\int_0^{x_3}\partial_3 B^\ell(t,x_\parallel,y)dy.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|E^\ell(t,\cdot,0)\|_{L^\infty(\partial\Omega)}&\le\|E^\ell(t)\|_{L^\infty(\Omega)}+x_3\|\partial_3 E^\ell(t)\|_{L^\infty(\Omega)},\\
\|B^\ell(t,\cdot,0)\|_{L^\infty(\partial\Omega)}&\le\|B^\ell(t)\|_{L^\infty(\Omega)}+x_3\|\partial_3 B^\ell(t)\|_{L^\infty(\Omega)},
\end{aligned}
$$

for any $x_3 > 0$. Thus

$$\|E^\ell(t,\cdot,0)\|_{L^\infty(\partial\Omega)} \le \|E^\ell(t)\|_{L^\infty(\Omega)} < \infty, \ \|B^\ell(t,\cdot,0)\|_{L^\infty(\partial\Omega)} \le \|B^\ell(t)\|_{L^\infty(\Omega)} < \infty,$$

and we conclude (6.46). □

Next, we prove the strong convergence of the sequence f^ℓ.

Lemma 17. *Suppose f_0 satisfies (0.32), E_0, B_0 satisfy (0.16), (0.35). There exists functions (f, E, B) with $\langle v\rangle^{4+\delta} f(t,x,v) \in L^\infty((0,T); L^\infty(\bar\Omega \times \mathbb{R}^3))$, and $(E,B) \in L^\infty((0,T); L^\infty(\Omega) \cap L^\infty(\partial\Omega))$, such that as $\ell \to \infty$,*

$$\sup_{0\le t\le T}\Big(\|E^\ell(t) - E(t)\|_{L^\infty(\Omega)} + \|E^\ell(t) - E(t)\|_{L^\infty(\partial\Omega)} + \|B^\ell(t) - B(t)\|_{L^\infty(\Omega)} + \|B^\ell(t) - B(t)\|_{L^\infty(\partial\Omega)}\Big) \to 0, \tag{6.47}$$

and

$$\sup_{0\le t\le T} \|\langle v\rangle^{4+\ell} f^\ell(t) - \langle v\rangle^{4+\delta} f(t)\|_{L^\infty(\bar\Omega\times\mathbb{R}^3)} \to 0. \tag{6.48}$$

Moreover, (f, E, B) is a (weak) solution of the system (0.40)–(0.42), and (0.26).

Proof. Let $m > n \ge 1$. Note that $f^m - f^n$ satisfies $(f^m - f^n)|_{t=0} = 0$ and

$$(f^m - f^n)|_{\gamma_-} = c_\mu \mu \int_{\gamma_+} (f^{m-1} - f^{n-1})(t,x,u)\hat u_3 du.$$

The equation for $f^m - f^n$ is

$$\partial_t(f^m - f^n) + \hat v\cdot\nabla_x(f^m - f^n) + \mathfrak{F}^{m-1}\cdot\nabla_v(f^m - f^n) = -(\mathfrak{F}^{m-1} - \mathfrak{F}^{n-1})\cdot\nabla_v f^n.$$

Thus, for any $(t,x,v) \in (0,T)\times\bar\Omega\times\mathbb{R}^3$, using (6.30), we get

$$\begin{aligned}
&|\langle v\rangle^{4+\delta}(f^m - f^n)(t,x,v)|\\
&\le C_1 \int_{\max\{t_1^{m-1},0\}}^t |\langle V^{m-1}(s)\rangle^{4+\delta}(\mathfrak{F}^{m-1} - \mathfrak{F}^{n-1})\cdot\nabla_v f^n)(s, X^{m-1}(s), V^{m-1}(s))|ds\\
&\quad + \mathbf{1}_{t_1^{m-1}>0} C_1 c_\mu V^{m-1}(t_1^m)\mu(V^m(t_1^m)) \int_{v_{1,3}<0} |(f^{m-1} - f^{n-1})(t_1^{m-1}, x_1^{m-1}, v_1)\hat v_{1,3}| dv_1,
\end{aligned}$$

where $C_1 = (1 + T(M_2 + g))^{4+\delta}$. Doing this inductively, we obtain

$$\begin{aligned}
&|\langle v\rangle^{4+\delta}(f^m - f^n)(t,x,v)|\\
&\le C_1 \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{i=1}^{k-1} \int_{\max\{t_i^{m-i},0\}}^{t_{i-1}^{m-(i-1)}} \mathbf{1}_{\{t_i^{m-i}\le 0 < t_{i-1}^{m-(i-1)}\}}\\
&\quad \times |\langle V^{m-i}(s)\rangle^{4+\delta}(\mathfrak{F}^{m-i} - \mathfrak{F}^{n-i})\cdot\nabla_v f^{n-(i-1)})(s, X^{m-i}(s), V^{m-i}(s))| ds d\Sigma_i^{k-1}\\
&\quad + C_1 \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{t_{k-1}^{m-(k-1)}>0\}} \int_{\mathcal{V}_k} |(f^{m-k} - f^{n-k})(t_k^{m-(k-1)}, x_k^{m-(k-1)}, v_k)| dv_k d\Sigma_{k-1}^{k-1}.
\end{aligned} \tag{6.49}$$

Where, $\mathcal{V}_j$ and Σ_i^{k-1} are in (6.34). Then from (6.35) and (6.41), by fixing $k \gg 1$,we get

$$\begin{aligned} \|\langle v\rangle^{4+\delta} f^m(t) - \langle v\rangle^{4+\delta} f^n)(t)\|_\infty &\le C_k C_1 \left(\sup_\ell \sup_{0\le s\le t} \|\langle v\rangle^{4+\delta} \nabla_v f^\ell(s)\|_\infty\right) \\ &\quad \int_0^t \sup_{1\le i\le k} \|\mathfrak{F}^{m-i}(s) - \mathfrak{F}^{n-i}(s)\|_\infty ds \\ &\le C_2 \int_0^t \sup_{1\le i\le k} \|\mathfrak{F}^{m-i}(s) - \mathfrak{F}^{n-i}(s)\|_\infty ds, \end{aligned} \tag{6.50}$$

where $C_2 = C_k C_1 M_3$.

Now, from (6.24) and using the same argument as Lemma 6 with (6.50), we have

$$\begin{aligned} \|\mathfrak{F}^{m-i}(s) - \mathfrak{F}^{n-i}(s)\|_\infty &\le \|E^{n-i}(s) - E^{m-i}(s)\|_\infty + \|B^{n-i}(s) - B^{m-i}(s)\|_\infty \\ &\le C\left(\sup_{0\le s'\le s} \|\langle v\rangle^{4+\delta}(f^{n-i} - f^{m-i})(s')\|_\infty \right. \\ &\quad \left. + \int_0^s \|\mathfrak{F}^{m-i-1}(s') - \mathfrak{F}^{n-i-1}(s')\|_\infty ds'\right) \\ &\le C \int_0^s \sup_{1\le i_1\le k} \|\mathfrak{F}^{m-i-i_1}(s') - \mathfrak{F}^{n-i-i_1}(s')\|_\infty ds' \\ &\le C \int_0^s \sup_{1\le i\le 2k} \Big(\|E^{m-i}(s') - E^{n-i}(s')\|_\infty \\ &\quad + \|B^{m-i}(s') - B^{n-i}(s')\|_\infty\Big) ds'. \end{aligned} \tag{6.51}$$

Iteration of (6.51) and using (6.25) yields

$$\begin{aligned} &\|E^m(t) - E^n(t)\|_\infty + \|B^m(t) - B^n(t)\|_\infty \\ &\le C^2 \int_0^t \int_0^s \sup_{1\le i\le 2k} \Big(\|E^{m-i}(s') - E^{n-i}(s')\|_\infty + \|B^{m-i}(s') - B^{n-i}(s')\|_\infty\Big) ds' ds \\ &= C^2 \int_0^t \tau \sup_{1\le i\le 2k} \Big(\|E^{m-i}(\tau) - E^{n-i}(\tau)\|_\infty + \|B^{m-i}(\tau) - B^{n-i}(\tau)\|_\infty\Big) d\tau \\ &\le C^l \int_0^t \frac{\tau^{l-1}}{(l-1)!} \sup_{1\le i\le lk} \Big(\|E^{m-i}(\tau) - E^{n-i}(\tau)\|_\infty + \|B^{m-i}(\tau) - B^{n-i}(\tau)\|_\infty\Big) d\tau \\ &\le M_2 \frac{C^l t^l}{l!}. \end{aligned}$$

Thus the sequences E^ℓ, B^ℓ are Cauchy in $L^\infty((0,T)\times\Omega)$, moreover, from Lemma 16, $E^\ell, B^\ell \in L^\infty([0,T]\times\partial\Omega)$. Therefore, there exists functions $E, B \in L^\infty((0,T); L^\infty(\Omega)\cap L^\infty(\partial\Omega))$, such that

$$E^\ell \to E,\ B^\ell \to B \text{ in } L^\infty((0,T)\times\Omega)\cap L^\infty((0,T)\times\partial\Omega). \tag{6.52}$$

This proves (6.47). Also, from (6.50), (6.51),

$$\|\langle v\rangle^{4+\delta} f^m(t) - \langle v\rangle^{4+\delta} f^n)(t)\|_{L^\infty((0,T)\times\bar\Omega)} \le M_2 \frac{C^{l-1} t^{l-1}}{(l-2)!},$$

therefore we get (6.48).

Now, take any $\phi(t,x,v) \in C_c^\infty([0,T)\times\bar{\Omega}\times\mathbb{R}^3$ with $\operatorname{supp}\phi \subset \{[0,T)\times\bar{\Omega}\times\mathbb{R}^3\}\backslash\{(0\times\gamma)\cup(0,T)\times\gamma_0\}$, from (6.22), we have

$$\begin{aligned}\int_{\Omega\times\mathbb{R}^3} f_0\phi(0)dvdt + \int_0^T\int_{\Omega\times\mathbb{R}^3} f^\ell\left(\partial_t\phi+\hat{v}\cdot\nabla_x\phi+\mathfrak{F}^{\ell-1}\cdot\nabla_v\phi\right)dvdxdt\\ = \int_0^T\int_{\gamma_+}\phi f^\ell\hat{v}_3dvdS_x + \int_0^T\int_{\gamma_+}\left(-c_\mu\int_{u_3>0}\mu(u)\phi(t,x,u)\hat{u}_3du\right)\hat{v}_3 f^\ell\,dvdS_x.\end{aligned}\tag{6.53}$$

Because of the strong convergence (6.47), (6.48), we have that as $\ell\to\infty$, each term in (6.53) goes to the corresponding terms with f^ℓ replaced by f and $\mathfrak{F}^\ell$ replaced by $\mathfrak{F}$. Therefore we conclude that (f,E,B) satisfy (0.28).

Next, from Propositions 1 and 2, we have that E^ℓ and B^ℓ are (weak) solutions to the wave equations with the initial data, boundary condition and forcing term with ρ, J changed to ρ^ℓ, J^ℓ. Then from (5.48) and Lemma 5, we have

$$\begin{aligned}&\partial_t E^\ell = \nabla_x\times B^\ell - 4\pi J^\ell,\ \nabla_x\cdot E^\ell = 4\pi\rho^\ell,\\ &\partial_t B^\ell = -\nabla_x\times E^\ell,\ \nabla_x\cdot B^\ell = 0,\end{aligned}\tag{6.54}$$

with

$$E_1^\ell = E_2^\ell = 0,\ B_3^\ell = 0,\quad \text{on }\ \partial\Omega.\tag{6.55}$$

Clearly, (0.31) is satisfied. Now, for any test functions $\Psi(t,x)\in C_c^\infty([0,T)\times\bar{\Omega};\mathbb{R}^3)$, $\Phi(t,x)\in C_c^\infty([0,T)\times\Omega;\mathbb{R}^3)$, from (6.54) and (6.55), we have

$$\begin{aligned}&\int_0^T\int_\Omega E^\ell\cdot\partial_t\Psi dxdt - \int_\Omega\Psi(0,x)\cdot E_0dx\\ &\quad = -\int_0^T\int_\Omega(\nabla_x\times\Psi)\cdot B^\ell dxdt + 4\pi\int_0^T\int_\Omega\Psi\cdot J^\ell dxdt,\end{aligned}\tag{6.56}$$

$$\begin{aligned}&\int_0^T\int_\Omega B^\ell\cdot\partial_t\Phi dxdt + \int_\Omega\Phi(0,x)\cdot B_0dx\\ &\quad = \int_0^T\int_\Omega(\nabla_x\times\Phi)\cdot E^\ell dxdt,\end{aligned}\tag{6.57}$$

Then from the strong convergence (5.50), (5.49), we can pass $\ell\to\infty$ and deduce that each term in (6.56) and (6.57) converges to the corresponding term with E^ℓ, B^ℓ, J^ℓ replace by E, B, and J respectively. Therefore, (f,E,B) satisfy (0.29) and (0.30), and we conclude that (f,E,B) is a (weak) solution of the RVM system (0.40)–(0.42) with diffuse BC (0.26). □

In the next lemma, we consider the regularity of the solution.

Lemma 18. *Let $\alpha(t,x,v)$ be defined as in (0.57). The solution (f,E,B) obtained in Lemma 17 satisfies*

$$\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel}f(t)\|_\infty + \|\langle v\rangle^{5+\delta}\alpha\partial_{x_3}f(t)\|_\infty + \|\langle v\rangle^{4+\delta}\nabla_v f(t)\|_\infty < \infty,\tag{6.58}$$

and

$$\|\partial_t E(t)\|_\infty + \|\partial_t B(t)\|_\infty + \|\nabla_x E(t)\|_\infty + \|\nabla_x B(t)\|_\infty < \infty.\tag{6.59}$$

Proof. From the L^∞ strong convergence (6.47), and the uniform-in-ℓ bound (6.41), we can pass the limit up to subsequence if necessary and get the weak$-*$ convergence

$$\begin{aligned}
&\partial_t E^\ell \overset{*}{\rightharpoonup} \partial_t E,\ \nabla_x E^\ell \overset{*}{\rightharpoonup} \nabla_x E,\\
&\partial_t B^\ell \overset{*}{\rightharpoonup} \partial_t B,\ \nabla_x B^\ell \overset{*}{\rightharpoonup} \nabla_x B \text{ in } L^\infty((0,T)\times\Omega),
\end{aligned} \tag{6.60}$$

and

$$\begin{aligned}
&\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f^\ell \overset{*}{\rightharpoonup} \langle v\rangle^{4+\delta}\nabla_{x_\parallel} f,\\
&\langle v\rangle^{4+\delta}\nabla_v f^\ell \overset{*}{\rightharpoonup} \langle v\rangle^{4+\delta}\nabla_v f \text{ in } L^\infty((0,T)\times\Omega\times\mathbb{R}^3).
\end{aligned} \tag{6.61}$$

We also claim

$$\langle v\rangle^{5+\delta}\alpha^{\ell-1}\partial_{x_3} f^\ell \overset{*}{\rightharpoonup} \langle v\rangle^{5+\delta}\alpha\partial_{x_3} f \text{ in } L^\infty((0,T)\times\Omega\times\mathbb{R}^3). \tag{6.62}$$

For any test function $\phi \in C_c^\infty((0,T)\times\Omega\times\mathbb{R}^3)$, we have

$$\int_0^t \iint_{\Omega\times\mathbb{R}^3} (\langle v\rangle^{5+\delta}\alpha^{\ell-1}\partial_{x_3} f^\ell - \langle v\rangle^{5+\delta}\alpha\partial_{x_3} f)\phi dvdxdt$$

$$= -\int_0^t \iint_{\Omega\times\mathbb{R}^3} (\langle v\rangle^{5+\delta}\alpha^{\ell-1} f^\ell - \langle v\rangle^{5+\delta}\alpha f)\partial_{x_3}\phi dvdxdt \tag{6.63}$$

$$-\int_0^t \iint_{\Omega\times\mathbb{R}^3} (\langle v\rangle^{5+\delta}\partial_{x_3}\alpha^{\ell-1} f^\ell - \langle v\rangle^{5+\delta}\partial_{x_3}\alpha^{\ell-1} f)\phi dvdxdt \tag{6.64}$$

$$-\int_0^t \iint_{\Omega\times\mathbb{R}^3} (\langle v\rangle^{5+\delta}\partial_{x_3}\alpha^{\ell-1} f - \langle v\rangle^{5+\delta}\partial_{x_3}\alpha f)\phi dvdxdt. \tag{6.65}$$

From (5.46) and (5.49) we have

$$\|\alpha^\ell - \alpha\|_{L^\infty((0,T)\times\Omega\times\mathbb{R}^3)} \to 0 \text{ as } \ell\to\infty. \tag{6.66}$$

Thus, together with (5.50), we have (6.63) $\to 0$ as $n\to\infty$. Next, note that

$$\partial_{x_3}\alpha^\ell = \frac{x_3 - \left(E_3^\ell(t,x_\parallel,0) + E_e + (\hat{v}\times B^\ell)_3(t,x_\parallel,0) - g\right)\frac{1}{\langle v\rangle}}{\alpha^\ell}.$$

For $(t,x,v)\in \operatorname{supp}\phi$, $x_3 > 0$ for some $c>0$. Thus, $\frac{1}{\alpha^\ell(t,x,v)} < \frac{1}{c}$, so

$$|\partial_{x_3}\alpha^\ell \mathbf{1}_{\operatorname{supp}(\phi)}(t,x,v)| < \frac{CM_2}{c}$$

From (5.50), this yields

$$|(6.64)| \le \frac{CM_2}{c}\int_0^t \iint_{\Omega\times\mathbb{R}^3} |\langle v\rangle^{5+\delta}(f^\ell - f)\phi| dvdxdt \to 0.$$

For (6.65), since

$$
\begin{aligned}
&\partial_{x_3}\alpha^{\ell} - \partial_{x_3}\alpha \\
&= \frac{x_3 - \left(E_3^{\ell}(t,x_{\parallel},0) + E_e + (\hat{v}\times B^{\ell})_3(t,x_{\parallel},0) - g\right)\frac{1}{\langle v\rangle}}{\alpha^{\ell}} \\
&\quad - \frac{x_3 - \left(E_3(t,x_{\parallel},0) + E_e + (\hat{v}\times B)_3(t,x_{\parallel},0) - g\right)\frac{1}{\langle v\rangle}}{\alpha} \\
&= \frac{-\left((E^{\ell}-E)_3(t,x_{\parallel},0) + (\hat{v}\times(B^{\ell}-B))_3(t,x_{\parallel},0)\right)\frac{1}{\langle v\rangle}}{\alpha^{\ell}} + \\
&\quad + \left(x_3 - \left(E_3(t,x_{\parallel},0) + E_e + (\hat{v}\times B)_3(t,x_{\parallel},0) - g\right)\right)\frac{1}{\langle v\rangle}\frac{\alpha-\alpha^{\ell}}{\alpha^{\ell}\alpha}.
\end{aligned}
$$

Again, for $(t,x,v)\in \operatorname{supp}\phi$, $x_3 > 0$ for some $c > 0$. Thus $\frac{1}{\alpha^{\ell}(t,x,v)} < \frac{1}{c}$, $\frac{1}{\alpha(t,x,v)} < \frac{1}{c}$. So from (5.49), (6.66), we have

$$
\begin{aligned}
&\|(\partial_{x_3}\alpha^{\ell} - \partial_{x_3}\alpha)\mathbf{1}_{\operatorname{supp}(\phi)}(t,x,v)\|_{L^\infty((0,T)\times\Omega\times\mathbb{R}^3)} \\
&\quad \le \frac{1}{c}\sup_{0\le t\le T}\left(\|E(t)-E^{\ell}(t)\|_\infty + \|B(t)-B^{\ell}(t)\|_\infty\right) \\
&\qquad + \frac{CM_2}{c^2}\|\alpha^{\ell}-\alpha\|_{L^\infty((0,T)\times\Omega\times\mathbb{R}^3)} \to 0 \text{ as } \ell\to\infty.
\end{aligned}
\tag{6.67}
$$

Thus, we have (6.65) $\to 0$, and this gives (6.62).

Therefore, from using the weak lower semi-continuity of the weak-$*$ convergence (6.60), (6.61), and the uniform-in-ℓ bound (6.41), we conclude (6.58), (6.59). □

Next, we prove the uniqueness of the solutions of the RVM system (0.40)–(0.42), (0.26).

Lemma 19. *Suppose (f, E_f, B_f) and (g, E_g, B_g) are solutions to the VM system (0.40)–(0.42), (0.26) with $f(0) = g(0)$, $E_f(0) = E_g(0)$, $B_f(0) = B_g(0)$, and that*

$$
\begin{aligned}
&E_f, B_f, E_g, B_g \in W^{1,\infty}((0,T)\times\Omega),\ \nabla_x\rho_f, \nabla_x J_f, \partial_t J_f, \nabla_x\rho_g, \\
&\nabla_x J_g, \partial_t J_g \in L^\infty((0,T); L^p_{loc}(\Omega)) \text{ for some } p > 1.
\end{aligned}
$$

And

$$
\sup_{0<t<T}\|\langle v\rangle^{5+\delta}\nabla_v f(t)\|_\infty < \infty,\ \sup_{0<t<T}\|\langle v\rangle^{5+\delta}\nabla_v g(t)\|_\infty < \infty. \tag{6.68}
$$

Then $f = g$, $E_f = E_g$, $B_f = B_g$.

Proof. The difference function $f - g$ satisfies

$$
\begin{aligned}
(\partial_t + \hat{v}\cdot\nabla_x + \mathfrak{F}_f\cdot\nabla_v)(f-g) &= (\mathfrak{F}_g - \mathfrak{F}_f)\cdot\nabla_v g \\
(f-g)(0) = 0,\ (f-g)|_{\gamma_-} & \\
&= c_\mu\mu(v)\int_{u_3<0} -(f-g)(t,x,u)\hat{u}_3\,du,
\end{aligned}
\tag{6.69}
$$

where

$$\mathfrak{F}_f = E_f + E_{\text{ext}} + \hat{v} \times (B_f + B_{\text{ext}}) - g\mathbf{e}_3, \; \mathfrak{F}_g = E_g + E_{\text{ext}} + \hat{v} \times (B_g + B_{\text{ext}}) - g\mathbf{e}_3,$$

so

$$\mathfrak{F}_g - \mathfrak{F}_f = E_f - E_g + \hat{v} \times (B_f - B_g). \tag{6.70}$$

From Lemma 1 we have $E_{f,1} - E_{g,1}$, $E_{f,2} - E_{g,2}$, $B_{f,3} - B_{g,3}$ solve the wave equation with the Dirichlet boundary condition (1.11) in the sense of (1.12) with

$$\begin{aligned} &u_0 = 0, \; u_1 = 0, \; G = -4\pi \partial_{x_i}(\rho_f - \rho_g) - 4\pi \partial_t (J_{f,i} - J_{g,i}), \\ &\; g = 0, \quad \text{for } E_{f,i} - E_{g,i}, i = 1, 2, \end{aligned} \tag{6.71}$$

$$u_0 = 0, \; u_1 = 0, \; G = 4\pi(\nabla_x \times (J_f - J_g))_3, \; g = 0, \quad \text{for } B_{f,3} - B_{g,3}, \tag{6.72}$$

respectively. And $E_{f,3} - E_{g,3}$, $B_{f,1} - B_{g,1}$, $B_{f,2} - B_{g,2}$ solve the wave equation with the Neumann boundary condition (1.9) in the sense of (1.10) with

$$\begin{aligned} &u_0 = 0, \; u_1 = 0, \; G = -4\pi \partial_{x_3}(\rho_f - \rho_g) - 4\pi \partial_t (J_{f,3} - J_{g,3}), \\ &\; g = -4\pi(\rho_f - \rho_g), \quad \text{for } E_{f,3} - E_{g,3}, \end{aligned} \tag{6.73}$$

$$\begin{aligned} &u_0 = 0, \; u_1 = 0, \; G = 4\pi(\nabla_x \times (J_f - J_g))_i, \\ &\; g = (-1)^{i+1} 4\pi (J_{f,\underline{i}} - J_{g,\underline{i}}), \quad \text{for } B_{f,i} - B_{j,i}, \; i = 1, 2, \end{aligned} \tag{6.74}$$

respectively. Therefore, from Lemmas 2 and 4, we know that $E_f - E_g$ and $B_f - B_g$ would have the form of

$$\begin{aligned} E_f - E_g = (2.31) + \cdots + (2.41), \; B_f - B_g = (2.45) + \cdots + (2.55), \\ \text{with } E_0, B_0 \text{ changes to } 0, \text{ and } f \text{ changes to } f - g. \end{aligned} \tag{6.75}$$

Now consider the characteristics

$$\begin{aligned} &\dot{X}_f(s; t, x, v) = \hat{V}_f(s; t, x, v), \\ &\dot{V}_f(s; t, x, v) = \mathfrak{F}_f(s, X_f(s; t, x, v), V_f(s; t, x, v)). \end{aligned}$$

Then from (6.69), same as (6.49), we obtain

$$\begin{aligned} |\langle v \rangle^{4+\delta}(f - g)(t, x, v)| &\le C_1 \int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \sum_{i=1}^{k-1} \int_{\max\{t_i, 0\}}^{t_{i-1}} \mathbf{1}_{\{t_i \le 0 < t_{i-1}\}} \\ &\times |\langle V_f(s) \rangle^{4+\delta} (\mathfrak{F}_g - \mathfrak{F}_f) \cdot \nabla_v f)(s, X_f(s), V_f(s))| ds d\Sigma_i^{k-1} \\ &+ C_1 \int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \mathbf{1}_{\{t^{m-(k-1)} > 0\}} \int_{\mathcal{V}_k} |(f - g)(t_k, x_k, v_k)| dv_k d\Sigma_{k-1}^{k-1}. \end{aligned} \tag{6.76}$$

So using (6.16) and (6.35), we have

$$\sup_{0 \le s \le t} \|\langle v \rangle^{5+\delta}(f - g)(s)\|_\infty \le C \int_0^t \|(\mathfrak{F}_g - \mathfrak{F}_f)(s)\|_\infty \|\langle v \rangle^{5+\delta} \nabla_v g(s)\|_\infty ds. \tag{6.77}$$

Now, from (6.75) and the estimate in Lemma 6, we have

$$\begin{aligned}\|(\mathfrak{F}_g - \mathfrak{F}_f)(s)\|_\infty &\leq \|(E_f - E_g)(s)\|_\infty + \|(B_f - B_g)(s)\|_\infty \\ &\leq C \sup_{0\leq s'\leq s} \|\langle v\rangle^{5+\delta}(f-g)(s')\|_\infty,\end{aligned} \tag{6.78}$$

and from the assumption (6.68), $\sup_{0\leq s\leq t}\|(1+|v|^{5+\delta})\nabla_v g(s)\|_\infty < C$. Therefore from (6.77) and (7.91), we have

$$\sup_{0\leq s\leq t}\|\langle v\rangle^{5+\delta}(f-g)(s)\|_\infty \leq C'\int_0^t \sup_{0\leq s'\leq s}\|\langle v\rangle^{5+\delta}(f-g)(s')\|_\infty ds. \tag{6.79}$$

Therefore from Gronwall

$$\sup_{0\leq s'\leq t}\|\langle v\rangle^{5+\delta}(f-g)(s')\|_\infty \leq e^{C't}\|\langle v\rangle^{5+\delta}(f-g)(0)\|_\infty = 0.$$

Therefore we conclude that the solutions to (0.40)–(0.42), (0.26) is unique. □

proof of Theorem 2. Using the sequence f^ℓ, E^ℓ, B^ℓ constructed in (6.22), (6.23), we have from Lemma 17 that the limit (f, E, B) is a solution to the VM system (0.40)–(0.42), (0.26). This proves the existence. From Lemma 18, we have the regularity estimate (0.36), (0.37). And from Lemma 19, we conclude the uniqueness. □

7. Specular BC

In this section we consider the solution f of the Vlasov–Maxwell system (0.40) satisfies the specular reflection boundary condition (0.27). We have the following a priori estimate for f.

Proposition 5. *Let (f, E, B) be a solution of (0.40)–(0.42), (0.27). Suppose the fields satisfies (4.18), and*

$$\sup_{0\leq t\leq T}(\|\nabla_x E(t)\|_\infty + \|\nabla_x B(t)\|_\infty) < \infty. \tag{7.1}$$

Assume that for some $\delta > 0$, and some $C > 0$ such that

$$\|\langle v\rangle^{5+\delta}e^{\frac{C}{\sqrt{\alpha\langle v\rangle}}}\nabla_x f_0\|_\infty + \|\langle v\rangle^{5+\delta}e^{\frac{C}{\sqrt{\alpha\langle v\rangle}}}\nabla_v f_0\|_\infty < \infty,$$

then there exists a $0 < T \ll 1$ small enough such that

$$\begin{aligned}&\sup_{0\leq t\leq T}\left(\|\langle v\rangle^{4+\delta}\nabla_x f(t)\|_\infty + \|\langle v\rangle^{4+\delta}\nabla_v f(t)\|_\infty\right) \\ &\quad + \sup_{0\leq t\leq T}\left(\|\langle v\rangle^{4+\delta}\nabla_x f(t)\|_{L^\infty(\gamma\backslash\gamma_0)} + \|\langle v\rangle^{4+\delta}\nabla_v f(t)\|_{L^\infty(\gamma\backslash\gamma_0)}\right) < \infty.\end{aligned} \tag{7.2}$$

Let $(t,x,v)\in(0,T)\times\bar{\Omega}\times\mathbb{R}^3$. Recall the definition of $t_{\mathbf{b}}(t,x,v)$, $x_{\mathbf{b}}(t,x,v)$, $v_{\mathbf{b}}(t,x,v)$ in (0.55). Now let $(t^0,x^0,v^0)=(t,x,v)$. We define the specular cycles, for $\ell\geq 0$,

$$(t^{\ell+1},x^{\ell+1},v^{\ell+1})=(t^{\ell}-t_{\mathbf{b}}(t^{\ell},x^{\ell},v^{\ell}),x_{\mathbf{b}}(t^{\ell},x^{\ell},v^{\ell}),v_{\mathbf{b}}(t^{\ell},x^{\ell},v^{\ell})-2v_{\mathbf{b},3}(t^{\ell},x^{\ell},v^{\ell})\mathbf{e}_3).$$

And we define the generalized characteristics for the specular BC as

$$\begin{aligned} X_{\mathbf{cl}}(s;t,x,v)&=\sum_{\ell}\mathbf{1}_{[t^{\ell+1},t^{\ell})}(s)X(s;t^{\ell},x^{\ell},v^{\ell}),\quad V_{\mathbf{cl}}(s;t,x,v)\\ &=\sum_{\ell}\mathbf{1}_{[t^{\ell+1},t^{\ell})}(s)V(s;t^{\ell},x^{\ell},v^{\ell}). \end{aligned}\tag{7.3}$$

The key to prove Proposition 5 is the following estimate for the derivative of the characteristics under the specular reflection.

Lemma 20. *For any $(t,x,v)\in(0,T)\times\bar{\Omega}\times\mathbb{R}^3$, and $0\leq s\leq t$, let $\partial_{\mathbf{e}}\in\{\nabla_x,\nabla_v\}$, then for some $C_1\gg 1$, we have*

$$\begin{aligned} |\partial_{\mathbf{e}}X_{\mathbf{cl}}(s;t,x,v)|&\leq C_1\langle v\rangle e^{\frac{C_1}{\sqrt{\alpha(t,x,v)\langle v\rangle}}},\\ |\partial_{\mathbf{e}}V_{\mathbf{cl}}(s;t,x,v)|&\leq C_1\langle v\rangle e^{\frac{C_1}{\sqrt{\alpha(t,x,v)\langle v\rangle}}}. \end{aligned}\tag{7.4}$$

Proof. We need to estimate along the bounces:

$$\begin{aligned} &\frac{\partial(X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v))}{\partial(x,v)}\\ &=\underbrace{\frac{\partial(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))}{\partial(t^{\ell_*},x^{\ell_*}_{\|_{\ell_*}},v^{\ell_*}_{3_{\ell_*}},v^{\ell_*}_{\|_{\ell_*}})}}_{\text{from the last bounce to the } s-\text{plane}}\times\underbrace{\prod_{\ell=1}^{\ell_*}\frac{\partial(t^{\ell+1},x^{\ell+1}_{\|},v^{\ell+1}_{3},v^{\ell+1}_{\|})}{\partial(t^{\ell},x^{\ell}_{\|},v^{\ell}_{3},v^{\ell}_{\|})}}_{\text{intermediate groups}}\\ &\times\underbrace{\frac{\partial(t^1,x^1_{\|_1},v^1_{3_1},v^1_{\|_1})}{\partial(x,v)}}_{\text{from the } t-\text{plane to the first bounce}}. \end{aligned}\tag{7.5}$$

We first find out the matrix of derivatives in the intermediate groups from the ℓ-th bounce to the $(\ell+1)$-th bounce:

$$J^{\ell+1}_{\ell}:=\frac{\partial(t^{\ell+1},x^{\ell+1}_{\|},v^{\ell+1}_{3},v^{\ell+1}_{\|})}{\partial(t^{\ell},x^{\ell}_{\|},v^{\ell}_{3},v^{\ell}_{\|})}.\tag{7.6}$$

We have

$$(t^{\ell}-t^{\ell+1})\hat{v}^{\ell}_3=\int_{t^{\ell+1}}^{t^{\ell}}\int_s^{t^{\ell}}\hat{\mathfrak{F}}_3(\tau)d\tau ds.\tag{7.7}$$

Taking $\frac{\partial}{\partial t^{\ell}}$(7.7) and direct computation gives

$$\frac{\partial t^{\ell+1}}{\partial t^{\ell}}=1-\frac{1}{\hat{v}^{\ell+1}_3}\int_{t^{\ell+1}}^{t^{\ell}}\int_s^{t^{\ell}}\left(\frac{\partial\hat{\mathfrak{F}}_3(\tau)}{\partial_{\tau}}+\frac{\partial\hat{\mathfrak{F}}_3(\tau)}{\partial_{t^{\ell}}}\right)d\tau ds.\tag{7.8}$$

Taking $\frac{\partial}{\partial t^\ell}$ derivative to

$$x_{\parallel}^{\ell+1} = x_{\parallel}^{\ell} - (t^\ell - t^{\ell+1})\hat{v}_{\parallel}^{\ell} + \int_{t^{\ell+1}}^{t^\ell} \int_s^{t^\ell} \hat{\mathfrak{F}}_{\parallel}(\tau) d\tau ds, \tag{7.9}$$

we get

$$\begin{aligned} \frac{\partial x_{\parallel}^{\ell+1}}{\partial t^\ell} &= \int_{t^{\ell+1}}^{t^\ell} \int_s^{t^\ell} \left(\frac{\partial \hat{\mathfrak{F}}_{\parallel}(\tau)}{\partial_\tau} + \frac{\partial \hat{\mathfrak{F}}_{\parallel}(\tau)}{\partial_{t^\ell}} \right) d\tau ds \\ &\quad - \frac{\hat{v}_{\parallel}^{\ell+1}}{\hat{v}_3^{\ell+1}} \int_{t^{\ell+1}}^{t^\ell} \int_s^{t^\ell} \left(\frac{\partial \hat{\mathfrak{F}}_3(\tau)}{\partial_\tau} + \frac{\partial \hat{\mathfrak{F}}_3(\tau)}{\partial_{t^\ell}} \right) d\tau ds. \end{aligned} \tag{7.10}$$

And taking $\frac{\partial}{\partial t^\ell}$ derivative to

$$v_{\parallel}^{\ell+1} = v_{\parallel}^{\ell} - \int_{t^{\ell+1}}^{t^\ell} \mathfrak{F}_{\parallel}(s) ds, \tag{7.11}$$

we get

$$\begin{aligned} \frac{\partial v_{\parallel}^{\ell+1}}{\partial t^\ell} &= -\frac{\mathfrak{F}_{\parallel}(t^{\ell+1})}{\hat{v}_3^{\ell+1}} \int_{t^{\ell+1}}^{t^\ell} \int_s^{t^\ell} \left(\frac{\partial \hat{\mathfrak{F}}_3(\tau)}{\partial_\tau} + \frac{\partial \hat{\mathfrak{F}}_3(\tau)}{\partial_{t^\ell}} \right) d\tau ds \\ &\quad - \int_{t^{\ell+1}}^{t^\ell} \left(\frac{\partial \mathfrak{F}_{\parallel}(s)}{\partial_s} + \frac{\partial \mathfrak{F}_{\parallel}(s)}{\partial_{t^\ell}} \right) ds. \end{aligned} \tag{7.12}$$

Similarly, taking taking $\frac{\partial}{\partial t^\ell}$ derivative to

$$v_3^{\ell+1} = -v_3^{\ell} - \int_{t^{\ell+1}}^{t^\ell} \mathfrak{F}_3(s) ds, \tag{7.13}$$

we get

$$\begin{aligned} \frac{\partial v_3^{\ell+1}}{\partial t^\ell} &= -\frac{\mathfrak{F}_3(t^{\ell+1})}{\hat{v}_3^{\ell+1}} \int_{t^{\ell+1}}^{t^\ell} \int_s^{t^\ell} \left(\frac{\partial \hat{\mathfrak{F}}_3(\tau)}{\partial_\tau} + \frac{\partial \hat{\mathfrak{F}}_3(\tau)}{\partial_{t^\ell}} \right) d\tau ds \\ &\quad - \int_{t^{\ell+1}}^{t^\ell} \left(\frac{\partial \mathfrak{F}_3(s)}{\partial_s} + \frac{\partial \mathfrak{F}_3(s)}{\partial_{t^\ell}} \right) ds. \end{aligned} \tag{7.14}$$

Now let's calculate the matrix $J_\ell^{\ell+1}$ in (7.6). Taking $\partial_{\mathbf{e}} \in \{\partial_{x_{\parallel}^{\ell}}, \partial_{v_3^\ell}, \partial_{v_{\parallel}^{\ell}}\}$ derivatives to (7.7) we get

$$\begin{aligned} \frac{\partial t^{\ell+1}}{\partial x_{\parallel}^{\ell}} &= -\frac{1}{\hat{v}_3^{\ell+1}} \int_{t^{\ell+1}}^{t^\ell} \int_s^{t^\ell} \frac{\partial \hat{\mathfrak{F}}_3(\tau)}{\partial_{x_{\parallel}^{\ell}}} d\tau ds, \\ \frac{\partial t^{\ell+1}}{\partial v_3^{\ell}} &= \frac{t^\ell - t^{\ell+1}}{\hat{v}_3^{\ell+1}} \frac{\partial \hat{v}_3^{\ell}}{\partial v_3^{\ell}} + \frac{1}{\hat{v}_3^{\ell+1}} \int_{t^{\ell+1}}^{t^\ell} \int_s^{t^\ell} \frac{\partial \hat{\mathfrak{F}}_3(\tau)}{\partial_{v_3^{\ell}}} d\tau ds \end{aligned}$$

$$
\begin{aligned}
&= \frac{(t^{\ell}-t^{\ell+1})}{\hat{v}_3^{\ell+1}} \frac{(1-(\hat{v}_3^{\ell})^2)}{\langle v^{\ell}\rangle} + \frac{1}{\hat{v}_3^{\ell+1}} \int_{t^{\ell+1}}^{t^{\ell}} \int_s^{t^{\ell}} \frac{\partial \hat{\mathfrak{F}}_3(\tau)}{\partial_{v_3^{\ell}}} d\tau ds, \\
\frac{\partial t^{\ell+1}}{\partial v_{\|}^{\ell}} &= -\frac{1}{\hat{v}_3^{\ell+1}} \int_{t^{\ell+1}}^{t^{\ell}} \int_s^{t^{\ell}} \frac{\partial \hat{\mathfrak{F}}_3(\tau)}{\partial_{v_{\|}^{\ell}}} d\tau ds.
\end{aligned}
\tag{7.15}
$$

Taking $\partial_{\mathbf{e}} \in \{\partial_{x_{\|}^{\ell}}, \partial_{v_3^{\ell}}, \partial_{v_{\|}^{\ell}}\}$ derivatives to (7.9) we get

$$
\begin{aligned}
\frac{\partial x_{\|}^{\ell+1}}{\partial x_{\|}{}^{\ell}} &= \mathbf{Id}_{2,2} + \frac{\partial t^{\ell+1}}{\partial x_{\|}^{\ell}} \hat{v}_{\|}^{\ell+1} + \int_{t^{\ell+1}}^{t^{\ell}} \int_s^{t^{\ell}} \frac{\partial \hat{\mathfrak{F}}_{\|}(\tau)}{\partial_{x_{\|}^{\ell}}} d\tau ds, \\
\frac{\partial x_{\|}^{\ell+1}}{\partial v_3{}^{\ell}} &= \frac{\partial t^{\ell+1}}{\partial v_3^{\ell}} \hat{v}_{\|}^{\ell+1} + \int_{t^{\ell+1}}^{t^{\ell}} \int_s^{t^{\ell}} \frac{\partial \hat{\mathfrak{F}}_{\|}(\tau)}{\partial_{v_3^{\ell}}} d\tau ds, \\
\frac{\partial x_{\|}^{\ell+1}}{\partial v_{\|}{}^{\ell}} &= \frac{\partial t^{\ell+1}}{\partial v_{\|}^{\ell}} \hat{v}_{\|}^{\ell+1} - (t^{\ell}-t^{\ell+1}) \frac{1-(\hat{v}_{\|}^{\ell})^2}{\langle v^{\ell}\rangle} + \int_{t^{\ell+1}}^{t^{\ell}} \int_s^{t^{\ell}} \frac{\partial \hat{\mathfrak{F}}_{\|}(\tau)}{\partial_{v_{\|}^{\ell}}} d\tau ds.
\end{aligned}
\tag{7.16}
$$

Taking $\partial_{\mathbf{e}} \in \{\partial_{x_{\|}^{\ell}}, \partial_{v_3^{\ell}}, \partial_{v_{\|}^{\ell}}\}$ derivatives to (7.11) we get

$$
\begin{aligned}
\frac{\partial v_{\|}^{\ell+1}}{\partial x_{\|}{}^{\ell}} &= \frac{\partial t^{\ell+1}}{\partial x_{\|}^{\ell}} \mathfrak{F}_{\|}(t^{\ell+1}) - \int_{t^{\ell+1}}^{t^{\ell}} \frac{\partial \mathfrak{F}_{\|}(s)}{\partial_{x_{\|}^{\ell}}} ds, \\
\frac{\partial v_{\|}^{\ell+1}}{\partial v_3{}^{\ell}} &= \frac{\partial t^{\ell+1}}{\partial v_3^{\ell}} \mathfrak{F}_{\|}(t^{\ell+1}) - \int_{t^{\ell+1}}^{t^{\ell}} \frac{\partial \mathfrak{F}_{\|}(s)}{\partial_{v_3^{\ell}}} ds, \\
\frac{\partial v_{\|}^{\ell+1}}{\partial v_{\|}{}^{\ell}} &= \mathbf{Id}_{2,2} + \frac{\partial t^{\ell+1}}{\partial v_{\|}^{\ell}} \mathfrak{F}_{\|}(t^{\ell+1}) - \int_{t^{\ell+1}}^{t^{\ell}} \frac{\partial \mathfrak{F}_{\|}(s)}{\partial_{v_{\|}^{\ell}}} ds.
\end{aligned}
\tag{7.17}
$$

And finally, taking $\partial_{\mathbf{e}} \in \{\partial_{x_{\|}^{\ell}}, \partial_{v_3^{\ell}}, \partial_{v_{\|}^{\ell}}\}$ derivatives to (7.13) we get

$$
\begin{aligned}
\frac{\partial v_3^{\ell+1}}{\partial x_{\|}{}^{\ell}} &= \frac{\partial t^{\ell+1}}{\partial x_{\|}^{\ell}} \mathfrak{F}_3(t^{\ell+1}) - \int_{t^{\ell+1}}^{t^{\ell}} \frac{\partial \mathfrak{F}_3(s)}{\partial_{x_{\|}^{\ell}}} ds, \\
\frac{\partial v_3^{\ell+1}}{\partial v_3{}^{\ell}} &= -1 - \frac{\partial t^{\ell+1}}{\partial v_3^{\ell}} \mathfrak{F}_3(t^{\ell+1}) + \int_{t^{\ell+1}}^{t^{\ell}} \frac{\partial \mathfrak{F}_3(s)}{\partial_{v_3^{\ell}}} ds, \\
\frac{\partial v_3^{\ell+1}}{\partial v_{\|}{}^{\ell}} &= \frac{\partial t^{\ell+1}}{\partial v_{\|}^{\ell}} \mathfrak{F}_3(t^{\ell+1}) - \int_{t^{\ell+1}}^{t^{\ell}} \frac{\partial \mathfrak{F}_3(s)}{\partial_{v_{\|}^{\ell}}} ds.
\end{aligned}
\tag{7.18}
$$

For the estimate, from (7.8) and that $|\frac{\partial \hat{\mathfrak{F}}_3(\tau)}{\partial_\tau} + \frac{\partial \hat{\mathfrak{F}}_3(\tau)}{\partial_{t^{\ell}}}| \lesssim \frac{1}{\langle v\rangle}$, we obtain

$$
\begin{aligned}
|\frac{\partial t^{\ell+1}}{\partial t^{\ell}}| \quad &\leq 1 + M \tfrac{|t^{\ell}-t^{\ell+1}|^2}{\hat{v}_3^{\ell+1}\langle v\rangle}, |\tfrac{\partial x_{\|}^{\ell+1}}{\partial t^{\ell}}| \leq M \tfrac{|t^{\ell}-t^{\ell+1}|^2}{\hat{v}_3^{\ell+1}\langle v\rangle}, \\
|\frac{\partial v_{\|}^{\ell+1}}{\partial t^{\ell}}| &\leq M(\tfrac{|t^{\ell}-t^{\ell+1}|^2}{\hat{v}_3^{\ell+1}\langle v\rangle} + \tfrac{|t^{\ell}-t^{\ell+1}|}{\langle v\rangle}), |\tfrac{\partial v_3^{\ell+1}}{\partial t^{\ell}}| \leq M(\tfrac{|t^{\ell}-t^{\ell+1}|^2}{\hat{v}_3^{\ell+1}\langle v\rangle} + \tfrac{|t^{\ell}-t^{\ell+1}|}{\langle v\rangle}).
\end{aligned}
\tag{7.19}
$$

Next, we estimate $\frac{\partial \hat{\mathfrak{F}}(\tau)}{\partial_{x_\parallel^\ell}}, \frac{\partial \hat{\mathfrak{F}}(\tau)}{\partial_{v^\ell}}, \frac{\partial \mathfrak{F}(\tau)}{\partial_{x_\parallel^\ell}}, \frac{\partial \mathfrak{F}(\tau)}{\partial_{v^\ell}}$. Since

$$\frac{\partial \hat{\mathfrak{F}}(\tau)}{\partial_{x_\parallel^\ell}} = \nabla_{x_\parallel} \hat{\mathfrak{F}}(\tau) \cdot \partial_{x_\parallel^\ell} X_\parallel(\tau) + \partial_{x_3} \hat{\mathfrak{F}}(\tau) \cdot \partial_{x_\parallel^\ell} X_3(\tau) + \nabla_v \hat{\mathfrak{F}}(\tau) \cdot \partial_{x_\parallel^\ell} V(\tau).$$

From (4.23), (4.24), (4.30), (4.31), we have

$$|\frac{\partial \hat{\mathfrak{F}}(\tau)}{\partial_{x_\parallel^\ell}}| \lesssim \frac{1}{\langle V(\tau)\rangle^2}|t^\ell - t^{\ell+1}| + \frac{1}{\langle V(\tau)\rangle}.$$

And,

$$|\frac{\partial \hat{\mathfrak{F}}(\tau)}{\partial_{v^\ell}}| = |\nabla_x \hat{\mathfrak{F}}(\tau) \cdot \partial_{v^\ell} X(\tau) + \nabla_v \hat{\mathfrak{F}}(\tau) \cdot \partial_{v^\ell} V(\tau)| \lesssim \frac{|t^\ell - t^{\ell+1}|}{\langle V(\tau)\rangle^2} + \frac{1}{\langle V(\tau)\rangle^2}.$$

Similarly, from (4.23), (4.24), (4.28) and (4.29),

$$\begin{aligned} |\frac{\partial \mathfrak{F}(\tau)}{\partial_{x_\parallel^\ell}}| &= |\nabla_{x_\parallel} \mathfrak{F}(\tau) \cdot \partial_{x_\parallel^\ell} X_\parallel(\tau) + \partial_{x_3} \mathfrak{F}(\tau) \cdot \partial_{x_\parallel^\ell} X_3(\tau) + \nabla_v \mathfrak{F}(\tau) \cdot \partial_{x_\parallel^\ell} V(\tau)| \\ &\lesssim \frac{1}{\langle V(\tau)\rangle}|t^\ell - t^{\ell+1}| + 1, \\ |\frac{\partial \mathfrak{F}(\tau)}{\partial_{v^\ell}}| &= |\nabla_x \mathfrak{F}(\tau) \cdot \partial_{v^\ell} X(\tau) + \nabla_v \mathfrak{F}(\tau) \cdot \partial_{v^\ell} V(\tau)| \lesssim \frac{|t^\ell - t^{\ell+1}|}{\langle V(\tau)\rangle} + \frac{1}{\langle V(\tau)\rangle}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\int_{t^{\ell+1}}^{t^\ell}\int_s^{t^\ell} |\frac{\partial \hat{\mathfrak{F}}(\tau)}{\partial_{x_\parallel^\ell}}| d\tau ds \lesssim \frac{|t^\ell - t^{\ell+1}|^2}{\langle v\rangle}, \quad \int_{t^{\ell+1}}^{t^\ell}\int_s^{t^\ell} |\frac{\partial \hat{\mathfrak{F}}(\tau)}{\partial_{v^\ell}}| d\tau ds \lesssim \frac{|t^\ell - t^{\ell+1}|^2}{\langle v\rangle^2}, \\ &\int_{t^{\ell+1}}^{t^\ell} |\frac{\partial \mathfrak{F}(\tau)}{\partial_{x_\parallel^\ell}}| ds \lesssim |t^\ell - t^{\ell+1}|, \quad \int_{t^{\ell+1}}^{t^\ell} |\frac{\partial \mathfrak{F}(s)}{\partial_{v^\ell}}| ds \lesssim \frac{|t^\ell - t^{\ell+1}|}{\langle v\rangle}. \end{aligned} \tag{7.20}$$

Thus, from (7.15) and (7.20), we obtain

$$|\frac{\partial t^{\ell+1}}{\partial x_\parallel^\ell}| \le M \frac{|t^\ell - t^{\ell+1}|^2}{\hat{v}_3^{\ell+1}\langle v\rangle}, |\frac{\partial t^{\ell+1}}{\partial v_3^\ell}| \le M \frac{|t^\ell - t^{\ell+1}|}{\hat{v}_3^{\ell+1}\langle v\rangle}, |\frac{\partial t^{\ell+1}}{\partial v_\parallel^\ell}| \le M \frac{|t^\ell - t^{\ell+1}|^2}{\hat{v}_3^{\ell+1}\langle v\rangle^2}. \tag{7.21}$$

From (7.16) and (7.20),

$$\begin{aligned} &|\frac{\partial x_\parallel^{\ell+1}}{\partial x_\parallel^\ell}| \le 1 + M \frac{|t^\ell - t^{\ell+1}|^2}{\hat{v}_3^{\ell+1}\langle v\rangle}, |\frac{\partial x_\parallel^{\ell+1}}{\partial v_3^\ell}| \le M \frac{|t^\ell - t^{\ell+1}|}{\hat{v}_3^{\ell+1}\langle v\rangle}, |\frac{\partial x_\parallel^{\ell+1}}{\partial v_\parallel^\ell}| \\ &\le M\left(\frac{|t^\ell - t^{\ell+1}|^2}{\hat{v}_3^{\ell+1}\langle v\rangle^2} + \frac{|t^\ell - t^{\ell+1}|}{\langle v\rangle}\right). \end{aligned} \tag{7.22}$$

From (7.17) and (7.20),

$$
\begin{aligned}
&|\frac{\partial v_{\parallel}^{\ell+1}}{\partial x_{\parallel}^{\ell}}| \leq M\left(\frac{|t^{\ell}-t^{\ell+1}|^2}{\hat{v}_3^{\ell+1}\langle v\rangle}+|t^{\ell}-t^{\ell+1}|\right), |\frac{\partial v_{\parallel}^{\ell+1}}{\partial v_3^{\ell}}| \leq M\left(\frac{|t^{\ell}-t^{\ell+1}|}{\hat{v}_3^{\ell+1}\langle v\rangle}+\frac{|t^{\ell}-t^{\ell+1}|}{\langle v\rangle}\right),\\
&|\frac{\partial v_{\parallel}^{\ell+1}}{\partial v_{\parallel}^{\ell}}| \leq 1+M\left(\frac{|t^{\ell}-t^{\ell+1}|^2}{\hat{v}_3^{\ell+1}\langle v\rangle^2}+\frac{|t^{\ell}-t^{\ell+1}|}{\langle v\rangle}\right).
\end{aligned}
\tag{7.23}
$$

From (7.18) and (7.20),

$$
|\frac{\partial v_3^{\ell+1}}{\partial x_{\parallel}^{\ell}}| \leq M\left(\frac{|t^{\ell}-t^{\ell+1}|^2}{\hat{v}_3^{\ell+1}\langle v\rangle}+|t^{\ell}-t^{\ell+1}|\right), |\frac{\partial v_3^{\ell+1}}{\partial v_{\parallel}^{\ell}}| \leq M\left(\frac{|t^{\ell}-t^{\ell+1}|^2}{\hat{v}_3^{\ell+1}\langle v\rangle^2}+\frac{|t^{\ell}-t^{\ell+1}|}{\langle v\rangle}\right).
\tag{7.24}
$$

Now we estimate $|\frac{\partial v_3^{\ell+1}}{\partial v_3^{\ell}}|$. Notice that since

$$
0=(t^{\ell}-t^{\ell+1})\hat{v}_3^{\ell+1}+\frac{(t^{\ell}-t^{\ell+1})^2}{2}\hat{\mathfrak{F}}_3(t^{\ell+1})+\int_{t^{\ell+1}}^{t^{\ell}}\int_{t^{\ell+1}}^{s}\int_{t^{\ell+1}}^{\tau}\frac{d}{d\tau'}\hat{\mathfrak{F}}_3(\tau')d\tau' d\tau ds,
\tag{7.25}
$$

and from (4.30), (4.31),

$$
|\frac{d}{d\tau'}\hat{\mathfrak{F}}_3(\tau')| \leq \frac{M}{\langle v\rangle}
$$

where M depends on $\|\nabla_x E\|_\infty$, $\|\nabla_x B\|_\infty$, and g. Thus

$$
1+\frac{(t^{\ell}-t^{\ell+1})}{2\hat{v}_3^{\ell+1}}\hat{\mathfrak{F}}_3(t^{\ell+1})=M\frac{|t^{\ell}-t^{\ell+1}|^2}{\hat{v}_3^{\ell+1}\langle v\rangle}.
\tag{7.26}
$$

From (4.26),

$$
\hat{\mathfrak{F}}_3(t^{\ell+1})=\frac{1-|\hat{v}_3^{\ell+1}|^2}{\langle v^{\ell+1}\rangle}\mathfrak{F}_3(t^{\ell+1})+O_{\mathfrak{F}}(1)\frac{\hat{v}_3^{\ell+1}}{\langle v^{\ell+1}\rangle}.
$$

Therefore,

$$
|1+\frac{(t^{\ell}-t^{\ell+1})}{2\hat{v}_3^{\ell+1}}\frac{1-|\hat{v}_3^{\ell+1}|^2}{\langle v^{\ell+1}\rangle}\mathfrak{F}_3(t^{\ell+1})| \leq M\left(\frac{|t^{\ell}-t^{\ell+1}|^2}{\hat{v}_3^{\ell+1}\langle v\rangle}+|t^{\ell}-t^{\ell+1}|\right).
\tag{7.27}
$$

Also, notice that

$$
\begin{aligned}
|\frac{(1-(\hat{v}_3^{\ell})^2)}{\langle v^{\ell}\rangle}-\frac{(1-(\hat{v}_3^{\ell+1})^2)}{\langle v^{\ell+1}\rangle}| &= |\int_{t^{\ell+1}}^{t^{\ell}}\frac{d}{ds}\left(\frac{(1-(\hat{V}_3^{\ell}(s))^2)}{\langle V^{\ell}(s)\rangle}\right)ds|\\
&\leq M\frac{|t^{\ell}-t^{\ell+1}|}{\langle v\rangle^2}.
\end{aligned}
\tag{7.28}
$$

Therefore from (7.20), (7.20), (7.27), and (7.28), we have

$$\begin{aligned}\frac{\partial v_3^{\ell+1}}{\partial {v_3}^{\ell}} &= -1 - \frac{\partial t^{\ell+1}}{\partial v_3^{\ell}}\mathfrak{F}_3(t^{\ell+1}) + \int_{t^{\ell+1}}^{t^{\ell}} \frac{\partial \mathfrak{F}_3(s)}{\partial_{v_3^{\ell}}} ds \\ &= 1 + M\left(\frac{|t^{\ell} - t^{\ell+1}|^2}{\hat{v}_3^{\ell+1}\langle v\rangle^2} + \frac{|t^{\ell} - t^{\ell+1}|}{\langle v\rangle}\right).\end{aligned} \tag{7.29}$$

Finally, from (4.19), (4.3),

$$|t^{\ell} - t^{\ell+1}| \lesssim \hat{v}_3^{\ell+1}\langle v\rangle \lesssim \alpha(t,x,v)\langle v\rangle,$$

and from (7.25),

$$|t^{\ell} - t^{\ell+1}|\hat{v}_3^{\ell+1} = -\int_{t^{\ell+1}}^{t^{\ell}}\int_{t^{\ell+1}}^{s} \hat{\mathfrak{F}}_3(\tau)d\tau ds \lesssim \frac{|t^{\ell} - t^{\ell+1}|^2 g}{\langle v\rangle},$$

so by (4.3),

$$\alpha(t,x,v)\langle v\rangle \lesssim \hat{v}_3^{\ell+1}\langle v\rangle \lesssim |t^{\ell} - t^{\ell+1}|.$$

Therefore, we have for $\ell \geq 1$, there exists $c, C > 0$ depending on T, $\|E\|_\infty$, $\|B\|_\infty$, and g such that

$$c\alpha(t,x,v)\langle v\rangle \leq |t^{\ell} - t^{\ell+1}| \leq C\alpha(t,x,v)\langle v\rangle. \tag{7.30}$$

Put together the above estimates and using (7.30), we have for some $M > 0$,

$$\begin{aligned} J_{\ell}^{\ell+1} &\leq \left[\begin{array}{c|cc|c|cc} 1+M\alpha\langle v\rangle & M\alpha\langle v\rangle & M\alpha\langle v\rangle & M & M\alpha\langle v\rangle & M\alpha\langle v\rangle \\ \hline M\alpha\langle v\rangle & 1+M\alpha\langle v\rangle & M\alpha\langle v\rangle & M & M\alpha\langle v\rangle & M\alpha\langle v\rangle \\ M\alpha\langle v\rangle & M\alpha\langle v\rangle & 1+M\alpha\langle v\rangle & M & M\alpha\langle v\rangle & M\alpha\langle v\rangle \\ \hline M\alpha\langle v\rangle & M\alpha\langle v\rangle & M\alpha\langle v\rangle & 1+M\alpha\langle v\rangle & M\alpha\langle v\rangle & M\alpha\langle v\rangle \\ \hline M\alpha\langle v\rangle & M\alpha\langle v\rangle & M\alpha\langle v\rangle & M & 1+M\alpha\langle v\rangle & M\alpha\langle v\rangle \\ M\alpha\langle v\rangle & M\alpha\langle v\rangle & M\alpha\langle v\rangle & M & M\alpha\langle v\rangle & 1+M\alpha\langle v\rangle \end{array}\right] \\ &:= J(\alpha\langle v\rangle). \end{aligned} \tag{7.31}$$

From diagonalization, we get

$$J(\alpha\langle v\rangle) = \mathcal{P}\Lambda\mathcal{P}^{-1},$$

where

$$\begin{aligned}\Lambda = \mathrm{diag}\Big[&1, 1, 1, 1, 1 + M\left(\sqrt{\alpha\langle v\rangle(4\alpha\langle v\rangle + 5)} + 3\alpha\langle v\rangle\right), 1 \\ &+M\left(-\sqrt{\alpha\langle v\rangle(4\alpha\langle v\rangle + 5)} + 3\alpha\langle v\rangle\right)\Big],\end{aligned}$$

and

$$\mathcal{P} = \begin{bmatrix} -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \sqrt{\alpha\langle v\rangle(4\alpha\langle v\rangle+5)} - 2\alpha\langle v\rangle & -\sqrt{\alpha\langle v\rangle(4\alpha\langle v\rangle+5)} - 2\alpha\langle v\rangle \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

$$\mathcal{P}^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{4}{5} & -\frac{1}{5} & 0 & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{4}{5} & 0 & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & 0 & \frac{4}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & 0 & -\frac{1}{5} & \frac{4}{5} \\ a & a & a & b & a & a \\ -a & -a & -a & -b & -a & -a \end{bmatrix},$$

where

$$a = \frac{2\alpha\sqrt{\langle v\rangle} + \sqrt{\alpha(4\alpha\langle v\rangle+5)}}{10\sqrt{\alpha(4\alpha\langle v\rangle+5)}},\ b = \frac{1}{2\sqrt{\alpha\langle v\rangle(4\alpha\langle v\rangle+5)}}.$$

Now from (7.30), the number of bounces

$$\ell^*(0; t, x, v) \le \frac{T}{c\alpha(t,x,v)\langle v\rangle}. \tag{7.32}$$

Therefore,

$$\prod_{\ell=1}^{\ell^*(0;t,x,v)} J_\ell^{\ell+1} \le J^{\ell^*(0;t,x,v)} \le \widetilde{\mathcal{P}}\widetilde{\Lambda}^{\ell^*}\widetilde{\mathcal{P}^{-1}} \le (1+2M\sqrt{\alpha\langle v\rangle})^{\frac{1}{c\alpha\langle v\rangle}}\widetilde{\mathcal{P}\mathcal{P}^{-1}}, \tag{7.33}$$

where we use the notation: for a matrix A, the entries of a matrix $\widetilde{A}$ are absolute values of the entries of A, i.e. $(\widetilde{A})_{ij} = |(A)_{ij}|$. From

$$(1+2M\sqrt{\alpha\langle v\rangle})^{\frac{1}{c\alpha\langle v\rangle}} = \left(\left(1+2M\sqrt{\alpha\langle v\rangle}\right)^{\frac{1}{2M\sqrt{\alpha\langle v\rangle}}}\right)^{\frac{2M}{c\sqrt{\alpha\langle v\rangle}}} \le e^{\frac{2M}{c\sqrt{\alpha\langle v\rangle}}},$$

and that $\left(\widetilde{\mathcal{P}\mathcal{P}^{-1}}\right)_{ij} \le \frac{M}{\sqrt{\alpha\langle v\rangle}}$, we get

$$\left(\prod_{\ell=1}^{\ell^*(0;t,x,v)} J_\ell^{\ell+1}\right)_{ij} \le e^{\frac{2M}{c\sqrt{\alpha\langle v\rangle}}}\left(\widetilde{\mathcal{P}\mathcal{P}^{-1}}\right)_{ij} \le \frac{M}{\sqrt{\alpha\langle v\rangle}}e^{\frac{2M}{c\sqrt{\alpha\langle v\rangle}}}. \tag{7.34}$$

Next, we estimate $\frac{\partial(X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))}{\partial(t^{\ell*}, x_\parallel^{\ell*}, v_3^{\ell*}, v_\parallel^{\ell*})}$ and $\frac{\partial(t^1, x_\parallel^1, v_3^1, v_\parallel^1)}{\partial(x,v)}$. From

$$X(s; t^{\ell^*}, x^{\ell^*}, v^{\ell^*}) = x^{\ell^*} - (t^{\ell^*} - s)\hat{v}^{\ell^*} + \int_s^{t^{\ell^*}}\int_\tau^{t^{\ell^*}} \hat{\mathfrak{F}}(\tau', X(\tau'), V(\tau'))\mathrm{d}\tau'\mathrm{d}\tau,$$

$$V(s; t^{\ell^*}, x^{\ell^*}, v^{\ell^*}) = v^{\ell^*} - \int_s^{t^{\ell^*}} \mathfrak{F}(\tau, X(\tau), V(\tau))\mathrm{d}\tau,$$

we have

$$\frac{\partial X(s)}{\partial t^{\ell *}} = -\hat{V}(s) + \int_s^{t^{\ell *}} \int_\tau^{t^{\ell *}} \frac{\partial \hat{\mathfrak{F}}(\tau')}{\partial t^{\ell *}} \mathrm{d}\tau' \mathrm{d}\tau,$$
$$\frac{\partial V(s)}{\partial t^{\ell *}} = -\mathfrak{F}(t^{\ell *}) - \int_s^{t^{\ell *}} \frac{\partial \mathfrak{F}(\tau)}{\partial t^{\ell *}} \mathrm{d}\tau.$$

Therefore, $|\frac{\partial X(s)}{\partial t^{\ell *}}| \lesssim |\hat{V}(s)| \lesssim 1$, and $\frac{\partial V(s)}{\partial t^{\ell *}} \lesssim 1$. Combine this with (4.23), we have

$$\frac{\partial (X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))}{\partial (t^{\ell_*}, x_\parallel^{\ell_*}, v_3^{\ell_*}, v_\parallel^{\ell_*})} \le \begin{bmatrix} M & M & \frac{|t^{\ell *} - s|}{\langle v \rangle} \\ M & |t^{\ell *} - s| & M \end{bmatrix}. \tag{7.35}$$

Lastly, since $t^1 = t_{\mathbf{b}}(t,x,v)$, $(x_\parallel^1, 0) = x_{\mathbf{b}}(t,x,v)$, $(v_3^1, v_\parallel^1) = v_{\mathbf{b}}(t,x,v)$, from (5.5) and (5.7),

$$|\partial_x t^1| \lesssim \frac{1}{\hat{v}_3^1} \lesssim \frac{1}{\alpha}, \ |\partial_v t^1| \lesssim \frac{t^1}{\hat{v}_3^1 \langle v \rangle} \lesssim M.$$

And thus from (5.8), we have

$$\frac{\partial (t^1, x_\parallel^1, v_3^1, v_\parallel^1)}{\partial (x,v)} \le \begin{bmatrix} \frac{M}{\alpha} & M \\ 1 + \frac{M}{\alpha} & M \\ \frac{M}{\alpha} & M \end{bmatrix}. \tag{7.36}$$

Finally, combining (7.34), (7.35), and (7.36), we get for some $C_1 = C_1(M, c)$,

$$\left| \left(\frac{\partial (X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v))}{\partial (x,v)} \right)_{ij} \right| \le \frac{36 M^3}{\alpha^{3/2} \sqrt{\langle v \rangle}} e^{\frac{2M}{c\sqrt{\alpha \langle v \rangle}}} \le C_1 \langle v \rangle e^{\frac{C_1}{\sqrt{\alpha \langle v \rangle}}}. \tag{7.37}$$

□

Proposition 5 comes as a consequence of the lemma.

proof of Proposition 5. For any $(t,x,v) \in (0,T) \times \bar{\Omega} \times \mathbb{R}^3$, let $\partial_{\mathbf{e}} \in \{\nabla_x, \nabla_v\}$, then

$$\begin{aligned} \partial_{\mathbf{e}} f(t,x,v) &= \partial_{\mathbf{e}} (f(0, X_{\mathbf{cl}}(0;t,x,v), V_{\mathbf{cl}}(0;t,x,v)) \\ &= \nabla_x f_0 \cdot \partial_{\mathbf{e}} X_{\mathbf{cl}}(0;t,x,v) + \nabla_v f_0 \cdot \partial_{\mathbf{e}} V_{\mathbf{cl}}(0;t,x,v). \end{aligned} \tag{7.38}$$

Now, from (4.3) and (7.4), write $X(0) = X_{\mathbf{cl}}(0;t,x,v)$, and $V(0) = V_{\mathbf{cl}}(0;t,x,v)$, we have

$$\begin{aligned} &\langle v \rangle^{4+\delta} |\partial_{\mathbf{e}} f(t,x,v)| \\ &\le C_1 |\nabla_x f(0, X(0), V(0))| \langle v \rangle^{5+\delta} e^{\frac{C_1}{\sqrt{\alpha(t,x,v)\langle v \rangle}}} \\ &+ C_1 |\nabla_v f(0, X(0), V(0))| \langle v \rangle^{5+\delta} e^{\frac{C_1}{\sqrt{\alpha(t,x,v)\langle v \rangle}}} \\ &\le C |\nabla_x f(0, X(0), V(0)) \langle | V(0) \rangle^{5+\delta} e^{\frac{C}{\sqrt{\alpha(0,X(0),V(0))\langle V(0) \rangle}}} \\ &\quad + C |\nabla_v f(0, X(0), V(0)) \langle V(0) \rangle^{5+\delta} e^{\frac{C}{\sqrt{\alpha(0,X(0),V(0))\langle V(0) \rangle}}} \\ &\le C \left(\| \langle v \rangle^{5+\delta} e^{\frac{C}{\sqrt{\alpha \langle v \rangle}}} \nabla_x f_0 \|_\infty + \| \langle v \rangle^{5+\delta} e^{\frac{C}{\sqrt{\alpha \langle v \rangle}}} \nabla_v f_0 \|_\infty \right). \end{aligned} \tag{7.39}$$

Next, using the same argument in Lemma 13, we obtain

$$\begin{aligned}
\sup_{0\le t\le T} \|\langle v\rangle^{4+\delta}\nabla_x f(t)\|_{L^\infty(\gamma\setminus\gamma_0)} &\le \sup_{0\le t\le T} \|\langle v\rangle^{4+\delta}\nabla_x f(t)\|_\infty,\\
\sup_{0\le t\le T} \|\langle v\rangle^{4+\delta}\nabla_v f(t)\|_{L^\infty(\gamma\setminus\gamma_0)} &\le \sup_{0\le t\le T} \|\langle v\rangle^{4+\delta}\nabla_v f(t)\|_\infty.
\end{aligned} \tag{7.40}$$

Combining with (7.39), we conclude (7.2). $\square$

We consider the sequence of functions:

$$f^0(t,x,v) = f_0(x,v),\ E^0(t,x) = E_0(t,x),\ B^0(t,x) = B_0(x).$$

For $\ell \ge 1$, let f^ℓ be the solution of

$$\begin{aligned}
\partial_t f^\ell + \hat v\cdot\nabla_x f^\ell + \mathfrak{F}^{\ell-1}\cdot\nabla_v f^\ell = 0,\ \text{where } \mathfrak{F}^{\ell-1}&\\
= E^{\ell-1} + E_{\text{ext}} + \hat v\times(B^{\ell-1}+B_{\text{ext}}) - g\mathbf{e}_3&,\\
f^\ell(0,x,v) = f_0(x,v)&,\\
f^\ell(t,x,v)|_{\gamma_-} = f^{\ell-1}(t,x,v_\parallel,-v_3)&.
\end{aligned} \tag{7.41}$$

Let $\rho^\ell = \int_{\mathbb{R}^3} f^\ell dv$, $J^\ell = \int_{\mathbb{R}^3} \hat v f^\ell dv$. Let

$$E^\ell = (2.31)+\cdots+(2.41),\ B^\ell = (2.45)+\cdots+(2.55),\ \text{with } f \text{ changes to } f^\ell \tag{7.42}$$

And let

$$\mathfrak{F}^\ell = E^\ell + E_{\text{ext}} - \hat v\times(B^\ell + B_{\text{ext}}) - g\mathbf{e}_3. \tag{7.43}$$

We prove several uniform-in-ℓ bounds for the sequence before passing the limit.

Lemma 21. *Suppose f_0 satisfies (0.32), E_0, B_0 satisfy (0.16), (0.35), then there exits M_1, M_2 such that for $0<T\ll 1$,*

$$\begin{aligned}
\sup_\ell \sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta} f^\ell(t)\|_{L^\infty(\bar\Omega\times\mathbb{R}^3)}\Big) &< M_1,\\
\sup_\ell \sup_{0\le t\le T}\Big(\|E^\ell(t)\|_\infty + \|B^\ell(t)\|_\infty\Big) + |B_e| + E_e + g &< M_2,\\
\inf_\ell \inf_{t,x_\parallel}\Big(g - E_e - E_3^\ell(t,x_\parallel,0) - (\hat v\times B^\ell)_3(t,x_\parallel,0)\Big) &> c_0.
\end{aligned} \tag{7.44}$$

Proof. By induction hypothesis we assume that

$$\begin{aligned}
\sup_{0\le i\le\ell} \sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta} f^i(t)\|_{L^\infty(\bar\Omega\times\mathbb{R}^3)}\Big) &< M_1,\\
\sup_{0\le i\le\ell} \sup_{0\le t\le T}\Big(\|E^i(t)\|_\infty + \|B^i(t)\|_\infty\Big) + |B_e| + E_e + g &< M_2.
\end{aligned} \tag{7.45}$$

Denote the characteristics (X^ℓ, V^ℓ) which solves

$$\begin{aligned}
\frac{d}{ds}X^\ell(s;t,x,v) &= \hat V^\ell(s;t,x,v),\\
\frac{d}{ds}V^\ell(s;t,x,v) &= \mathfrak{F}^\ell(s, X^\ell(s;t,x,v), V^\ell(s;t,x,v)).
\end{aligned} \tag{7.46}$$

And define the specular cycles:

$$
\begin{aligned}
t_1^{\ell}(t,x,v) &:= \sup\{s \geq 0 : X^{\ell}(\tau;t,x,v) \in \Omega \text{ for all } \tau \in (t-s,t)\},\\
x_1^{\ell}(t,x,v) &:= X^{\ell}(t_1^{\ell}(t,x,v);t,x,v),\\
v_1^{\ell}(t,x,v) &:= V^{\ell}(t_1^{\ell}(t,x,v);t,x,v) - 2V_3^{\ell}(t_1^{\ell}(t,x,v);t,x,v)\mathbf{e}_3
\end{aligned}
\tag{7.47}
$$

and inductively for $k \geq 2$,

$$
\begin{aligned}
t_k^{\ell-(k-1)}(t,x,v) &:= \sup\big\{s \geq 0 : X^{\ell-(k-1)}(\tau; t_{k-1}^{\ell-(k-2)}, x_{k-1}^{\ell-(k-2)}, v_{k-1}^{\ell-(k-2)})\\
&\qquad \in \Omega \text{ for all } \tau \in (t-s,t)\big\},\\
x_k^{\ell-(k-1)}(t,x,v) &:= X^{\ell-(k-1)}(t_k^{\ell-(k-1)}; t_{k-1}^{\ell-(k-2)}, x_{k-1}^{\ell-(k-2)}, v_{k-1}^{\ell-(k-2)}),\\
&\quad v_k^{\ell-(k-1)}(t,x,v)\\
&:= V^{\ell-(k-1)}(t_k^{\ell-(k-1)}; t_{k-1}^{\ell-(k-2)}, x_{k-1}^{\ell-(k-2)}, v_{k-1}^{\ell-(k-2)})\\
&\qquad - 2V_3^{\ell-(k-1)}(t_k^{\ell-(k-1)}; t_{k-1}^{\ell-(k-2)}, x_{k-1}^{\ell-(k-2)}, v_{k-1}^{\ell-(k-2)})\mathbf{e}_3.
\end{aligned}
\tag{7.48}
$$

And we define the generalized characteristics for the specular BC as

$$
\begin{aligned}
X_{\mathbf{cl}}^{\ell}(s;t,x,v) = \ &\mathbf{1}_{[t_1^{\ell}(t,x,v),t)}(s) X^{\ell}(s;t,x,v)\\
&+ \sum_{k\geq 1} \mathbf{1}_{[t_{k+1}^{\ell-k}, t_k^{\ell-(k-1)})}(s) X^{\ell-k}(s; t_k^{\ell-(k-1)}, x_k^{\ell-(k-1)}, v_k^{\ell-(k-1)}),\\
V_{\mathbf{cl}}^{\ell}(s;t,x,v) = \ &\mathbf{1}_{[t_1^{\ell}(t,x,v),t)}(s) V^{\ell}(s;t,x,v)\\
&+ \sum_{k\geq 1} \mathbf{1}_{[t_{k+1}^{\ell-k}, t_k^{\ell-(k-1)})}(s) V^{\ell-k}(s; t_k^{\ell-(k-1)}, x_k^{\ell-(k-1)}, v_k^{\ell-(k-1)}).
\end{aligned}
\tag{7.49}
$$

From (7.41) and (7.47), for any $(t,x,v) \in (0,T) \times \bar{\Omega} \times \mathbb{R}^3$, let k be such that $t_{k+1}^{\ell-k}(t,x,v) \leq 0 < t_k^{\ell-(k-1)}(t,x,v)$, then we have

$$
\begin{aligned}
f^{\ell+1}(t,x,v) &= f^{\ell-(k-1)}\left(0, X^{\ell-k}\left(0; t_k^{\ell-(k-1)}, x_k^{\ell-(k-1)}, v_k^{\ell-(k-1)}\right),\right.\\
&\qquad \left. V^{\ell-k}\left(0; t_k^{\ell-(k-1)}, x_k^{\ell-(k-1)}, v_k^{\ell-(k-1)}\right)\right)\\
&= f_0\left(X^{\ell-k}\left(0; t_k^{\ell-(k-1)}, x_k^{\ell-(k-1)}, v_k^{\ell-(k-1)}\right),\right.\\
&\qquad \left. V^{\ell-k}\left(0; t_k^{\ell-(k-1)}, x_k^{\ell-(k-1)}, v_k^{\ell-(k-1)}\right)\right).
\end{aligned}
\tag{7.50}
$$

Thus

$$
\langle v\rangle^{4+\delta} f^{\ell+1}(t,x,v) \leq \frac{\langle v\rangle^{4+\delta}}{\left(V^{\ell-k}\left(0; t_k^{\ell-(k-1)}, x_k^{\ell-(k-1)}, v_k^{\ell-(k-1)}\right)\right)^{4+\delta}} \|\langle v\rangle^{4+\delta} f_0\|_{\infty}.
\tag{7.51}
$$

Now, since

$$
\begin{aligned}
&|v - V^{\ell-k}\left(0; t_k^{\ell-(k-1)}, x_k^{\ell-(k-1)}, v_k^{\ell-(k-1)}\right)| \\
&= \left| v - v_1^\ell + \sum_{i=1}^{k-1}\left(v_i^{\ell-(i-1)} - v_{i+1}^{\ell-i}\right) + v_k^{\ell-(k-1)} - V^{\ell-k}\left(0; t_k^{\ell-(k-1)}, x_k^{\ell-(k-1)}, v_k^{\ell-(k-1)}\right)\right| \\
&\leq \int_{t_1^\ell}^t \|\mathfrak{F}^\ell(s)\|_\infty ds + \sum_{i=1}^{k-1}\int_{t_{i+1}^{\ell-i}}^{t_i^{\ell-(i-1)}} \|\mathfrak{F}^{\ell-i}(s)\|_\infty ds + \int_0^{t_k^{\ell-(k-1)}} \|\mathfrak{F}^{\ell-k}(s)\|_\infty ds \\
&\leq \int_0^t \sup_{0\leq i\leq l} \|\mathfrak{F}^i(s)\|_\infty ds.
\end{aligned}
$$

From (7.45) we have

$$
|v| \leq \left|V^{\ell-k}\left(0; t_k^{\ell-(k-1)}, x_k^{\ell-(k-1)}, v_k^{\ell-(k-1)}\right)\right| + tM_2,
$$

and this yields

$$
\frac{\langle v\rangle}{\langle V^{\ell-k}\left(0; t_k^{\ell-(k-1)}, x_k^{\ell-(k-1)}, v_k^{\ell-(k-1)}\right)\rangle} < 1 + tM_2. \tag{7.52}
$$

Thus (7.51) gives

$$
\sup_{0\leq t\leq T} \|\langle v\rangle^{4+\delta} f^{\ell+1}(t)\|_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} < (1 + T(g + M_2))\|\langle v\rangle^{4+\delta} f_0\|_\infty < M_1, \tag{7.53}
$$

for T small enough. Now from (7.42) and (0.16), using the same argument as (6.38)–(6.40), we get

$$
\sup_{0\leq t\leq T} (\|E^{\ell+1}(t)\|_\infty + \|B^{\ell+1}(t)\|_\infty) + |B_e| + E_e + g < M_2, \tag{7.54}
$$

$$
\inf_{t,x_\|}\left(g - E_e - E_3^{\ell+1}(t, x_\|, 0) - (\hat{v}\times B^{\ell+1})_3(t, x_\|, 0)\right) > c_0. \tag{7.55}
$$

Thus we conclude (6.25) by induction.

□

Next, we define $\alpha^\ell(t, x, v)$ in the same way as in (5.46). Then we have

Lemma 22. *Suppose f_0 satisfies (0.38), E_0, B_0 satisfy (0.16), (0.35), then there exits M_3, M_4 such that for $0 < T \ll 1$,*

$$
\begin{aligned}
&\sup_\ell \sup_{0\leq t\leq T}\left(\|\langle v\rangle^{4+\delta}\nabla_x f^\ell(t)\|_\infty + \|\langle v\rangle^{4+\delta}\nabla_v f^\ell(t)\|_\infty\right) \\
&\quad + \sup_\ell \sup_{0\leq t\leq T}\left(\|\langle v\rangle^{4+\delta}\nabla_x f^\ell(t)\|_{L^\infty(\gamma\backslash\gamma_0)} + \|\langle v\rangle^{4+\delta}\nabla_v f^\ell(t)\|_{L^\infty(\gamma\backslash\gamma_0)}\right) < M_3, \\
&\quad \sup_\ell \sup_{0\leq t\leq T}\left(\|\partial_t E^\ell(t)\|_\infty + \|\partial_t B^\ell(t)\|_\infty + \|\nabla_x E^\ell(t)\|_\infty + \|\nabla_x B^\ell(t)\|_\infty\right) < M_4.
\end{aligned} \tag{7.56}
$$

And with the M_2 as in Lemma 21,

$$
\sup_\ell \sup_{0\leq t\leq T}\left(\|E^\ell(t, x)\|_{L^\infty(\partial\Omega)} + \|B^\ell(t, x)\|_{L^\infty(\partial\Omega)}\right) < M_2. \tag{7.57}
$$

Proof. Let $\partial_{\mathbf{e}} \in \{\nabla_x, \nabla_v\}$. From (7.50) we have

$$\partial_{\mathbf{e}} f^{\ell+1}(t,x,v) = \nabla_x f_0 \cdot \partial_{\mathbf{e}} X^{\ell-k}\left(0; t_k^{\ell-(k-1)}(t,x,v), x_k^{\ell-(k-1)}(t,x,v), v_k^{\ell-(k-1)}(t,x,v)\right) \\ + \nabla_v f_0 \cdot \partial_{\mathbf{e}} V^{\ell-k}\left(0; t_k^{\ell-(k-1)}(t,x,v), x_k^{\ell-(k-1)}(t,x,v), v_k^{\ell-(k-1)}(t,x,v)\right) \tag{7.58}$$

Then from (7.44), we can use essentially the same argument as the proof of Lemma 20 to get

$$\left|\left(\frac{\partial(X^{\ell-k}(0; t_k^{\ell-(k-1)}, x_k^{\ell-(k-1)}, v_k^{\ell-(k-1)}), V^{\ell-k}(0; t_k^{\ell-(k-1)}, x_k^{\ell-(k-1)}, v_k^{\ell-(k-1)})}{\partial(x,v)}\right)_{ij}\right| \\ \leq C_1 \langle V^{\ell-k}(0)\rangle \exp\left(\frac{C_1}{\sqrt{\alpha^{\ell-k}(0, X^{\ell-k}(0), V^{\ell-k}(0))\langle V^{\ell-k}(0)\rangle}}\right), \tag{7.59}$$

where

$$\begin{aligned} X^{\ell-k}(0) &= X^{\ell-k}(0; t_k^{\ell-(k-1)}, x_k^{\ell-(k-1)}, v_k^{\ell-(k-1)}),\ V^{\ell-k}(0) \\ &= V^{\ell-k}(0; t_k^{\ell-(k-1)}, x_k^{\ell-(k-1)}, v_k^{\ell-(k-1)}), \\ t_k^{\ell-(k-1)} &= t_k^{\ell-(k-1)}(t,x,v),\ x_k^{\ell-(k-1)} \\ &= x_k^{\ell-(k-1)}(t,x,v),\ v_k^{\ell-(k-1)} = v_k^{\ell-(k-1)}(t,x,v), \end{aligned}$$

and C_1 depends on M_1, M_2. Therefore, from (7.58) and (7.59),

$$\langle v\rangle^{4+\delta}\partial_{\mathbf{e}} f^{\ell+1}(t,x,v) \leq C_1 \frac{\langle v\rangle^{4+\delta}}{\langle V^{\ell-k}(0)\rangle^{4+\delta}} \\ \times \left(\|\langle v\rangle^{5+\delta} e^{\frac{C}{\sqrt{\alpha\langle v\rangle}}}\nabla_x f_0\|_\infty + \|\langle v\rangle^{5+\delta} e^{\frac{C}{\sqrt{\alpha\langle v\rangle}}}\nabla_v f_0\|_\infty\right).$$

And using (7.52), we conclude

$$\sup_\ell \sup_{0\leq t\leq T} \|\langle v\rangle^{4+\delta}\partial_{\mathbf{e}} f^{\ell+1}(t)\|_\infty \\ \leq C\left(\|\langle v\rangle^{5+\delta} e^{\frac{C}{\sqrt{\alpha\langle v\rangle}}}\nabla_x f_0\|_\infty + \|\langle v\rangle^{5+\delta} e^{\frac{C}{\sqrt{\alpha\langle v\rangle}}}\nabla_v f_0\|_\infty\right) < M_3. \tag{7.60}$$

Then, from the same argument as in Lemma 13, we get

$$\sup_\ell \sup_{0\leq t\leq T} \|\langle v\rangle^{4+\delta}\partial_{\mathbf{e}} f^{\ell+1}(t)\|_{L^\infty(\gamma\setminus\gamma_0)} \leq \sup_\ell \sup_{0\leq t\leq T} \|\langle v\rangle^{4+\delta}\partial_{\mathbf{e}} f^{\ell+1}(t)\|_\infty < M_3. \tag{7.61}$$

From this, we use the same argument to get (3.49) in the proof of Lemma 7 and obtain

$$\begin{aligned}
&\sup_{0\le t\le T}\Big(\|\partial_t E^{\ell+1}(t)\|_\infty+\|\partial_t B^{\ell+1}(t)\|_\infty+\|\nabla_x E^{\ell+1}(t)\|_\infty+\|\nabla_x B^{\ell+1}(t)\|_\infty\Big)\\
&\le TC\sup_{1\le i\le\ell}\sup_{0\le t\le T}\Big(\|\partial_t E^{i}(t)\|_\infty+\|\partial_t B^{i}(t)\|_\infty+\|\nabla_x E^{i}(t)\|_\infty+\|\nabla_x B^{i}(t)\|_\infty\Big)\\
&C\Big(\|E_0\|_{C^2}+\|B_0\|_{C_2}\Big)+C\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f^{\ell+1}(t)\|_\infty+\|\langle v\rangle^{\ell+\delta}\alpha^{n}\partial_{x_3} f^{\ell+1}(t)\|_\infty\Big)\\
&\quad+C\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta}\nabla_{x_\parallel} f^{\ell}(t)\|_{L^\infty(\gamma\backslash\gamma_0)}+\|\langle v\rangle^{5+\delta}\alpha^{\ell-1}\partial_{x_3} f^{\ell}(t)\|_{L^\infty(\gamma\backslash\gamma_0)}\Big)\\
&\quad+C\sup_{0\le t\le T}\Big(\|\langle v\rangle^{4+\delta} f^{\ell+1}(t)\|_\infty+\|E^{\ell+1}(t)\|_\infty+\|B^{\ell+1}(t)\|_\infty\Big).
\end{aligned}$$

From (7.44) and (7.60), this gives

$$\begin{aligned}
&\sup_{\ell}\sup_{0\le t\le T}\Big(\|\partial_t E^{\ell}(t)\|_\infty+\|\partial_t B^{\ell}(t)\|_\infty+\|\nabla_x E^{\ell}(t)\|_\infty+\|\nabla_x B^{\ell}(t)\|_\infty\Big)\\
&\le TC\sup_{\ell}\sup_{0\le t\le T}\Big(\|\partial_t E^{\ell}(t)\|_\infty+\|\partial_t B^{\ell}(t)\|_\infty+\|\nabla_x E^{\ell}(t)\|_\infty+\|\nabla_x B^{\ell}(t)\|_\infty\Big)\\
&\quad+C\Big(\|E_0\|_{C^2}+\|B_0\|_{C_2}\Big)+C(M_1+M_2+M_3).
\end{aligned}$$

Therefore, by choosing $M_4\gg 1$ and $T\ll 1$, we get

$$\sup_{\ell}\sup_{0\le t\le T}\Big(\|\partial_t E^{\ell}(t)\|_\infty+\|\partial_t B^{\ell}(t)\|_\infty+\|\nabla_x E^{\ell}(t)\|_\infty+\|\nabla_x B^{\ell}(t)\|_\infty\Big)<M_4. \tag{7.62}$$

Together with (7.60) and (7.61), we conclude (7.56).

Now, from (7.62), we use the same argument as in Lemma 16 to conclude that for any $0<t<T$,

$$\begin{aligned}
&E^{\ell}(t,x)|_{\partial\Omega}\in L^\infty(\partial\Omega),\ \ B^{\ell}(t,x)|_{\partial\Omega}\in L^\infty(\partial\Omega),\\
&\sup_{0\le t\le T}\Big(\|E^{\ell}(t,x)\|_{L^\infty(\partial\Omega)}+\|B^{\ell}(t,x)\|_{L^\infty(\partial\Omega)}\Big)<\sup_{0\le t\le T}\Big(\|E^{\ell}(t,x)\|_\infty+\|B^{\ell}(t,x)\|_\infty\Big)<M_2.
\end{aligned}$$

This proves (7.57). □

Next, we prove a pointwise convergence result for f^ℓ.

Lemma 23. *Suppose f_0 satisfies (0.38), E_0, B_0 satisfy (0.16), (0.35). Then there exists a function f such that $f^\ell\to f$ pointwise almost everywhere on $(0,T)\times(\bar\Omega\times\mathbb{R}^3\backslash\gamma_0)$.*

Proof. Fix any $(t,x,v)\in(0,T)\times(\bar\Omega\times\mathbb{R}^3\backslash\gamma_0)$, then it suffices to show that $\{f^\ell(t,x,v)\}_{\ell=1}^\infty$ is a Cauchy sequence. Fix $\varepsilon>0$, and let $n>m\ge N_0$. Note that f^m-f^n satisfies $(f^m-f^n)|_{t=0}=0$ and

$$(f^m-f^n)(t,x,v)|_{\gamma_-}=(f^{m-1}-f^{n-1})(t,x,v_\parallel,-v_3).$$

The equation for f^m-f^n is

$$\partial_t(f^m-f^n)+\hat v\cdot\nabla_x(f^m-f^n)+\mathfrak{F}^{m-1}\cdot\nabla_v(f^m-f^n)$$

$$= -(\mathfrak{F}^{m-1} - \mathfrak{F}^{n-1}) \cdot \nabla_v f^n. \tag{7.63}$$

Thus, for any $(t, x, v) \in (0, T) \times (\bar{\Omega} \times \mathbb{R}^3 \backslash \gamma_0)$, there is a $k \geq 0$ such that $t_k^{m-k}(t, x, v) \leq 0 < t_{k-1}^{m-(k-1)}(t, x, v)$, and we have from (7.63),

$$\begin{aligned}
&(f^m - f^n)(t, x, v) \\
&= \int_{t_1^{m-1}}^{t} \left(-(\mathfrak{F}^{m-1} - \mathfrak{F}^{n-1}) \cdot \nabla_v f^n\right)(s, X^{m-1}(s), V^{m-1}(s))ds \\
&\quad + \sum_{i=1}^{k-2} \int_{t_{i+1}^{m-(i+1)}}^{t_i^{m-i}} \left(-(\mathfrak{F}^{m-1-i} - \mathfrak{F}^{n-1-i}) \cdot \nabla_v f^{n-i}\right) \\
&(s, X^{m-1-i}(s), V^{m-1-i}(s))ds \\
&\quad + \int_0^{t_{k-1}^{m-(k-1)}} \left(-(\mathfrak{F}^{m-1-(k-1)} - \mathfrak{F}^{n-1-(k-1)}) \cdot \nabla_v f^{n-(k-1)}\right) \\
&(s, X^{m-1-(k-1)}(s), V^{m-1-(k-1)}(s))ds.
\end{aligned} \tag{7.64}$$

Together with (7.52), this implies

$$\begin{aligned}
\langle v\rangle^{4+\delta}|(f^m - f^n)(t, x, v)| \leq C_1 &\left(\sup_{\ell} \sup_{0\leq s\leq t} \|\langle v\rangle^{4+\delta} \nabla_v f^\ell(s)\|_\infty\right) \\
&\int_0^t \sup_{1\leq i\leq k-1} \|\mathfrak{F}^{m-i}(s) - \mathfrak{F}^{n-i}(s)\|_\infty ds
\end{aligned} \tag{7.65}$$

where $C_1 = (1 + T(M_2 + g))^{4+\delta}$. Note that since $(x, v) \notin \gamma_0$, $\alpha(t, x, v) > 0$. And from (7.30) and (7.32), we have

$$k \leq \frac{T}{c\alpha(t, x, v)\langle v\rangle},$$

where c depends on M_2 and g. Now, for some small $\delta' > 0$, we write

$$\begin{aligned}
\mathfrak{F}^{m-i} - \mathfrak{F}^{n-i} &= E_{f^{m-i}} - E_{f^{n-i}} + B_{f^{m-i}} - B_{f^{n-i}} \\
&= E_{f^{m-i} - f^{n-i}} + B_{f^{m-i} - f^{n-i}} \\
&= E_{\mathbf{1}_{\{|v_3|>\delta'\}}(f^{m-i} - f^{n-i})} \\
&\quad + E_{\mathbf{1}_{\{|v_3|\leq\delta'\}}(f^{m-i} - f^{n-i})} + B_{\mathbf{1}_{\{|v_3|>\delta'\}}(f^{m-i} - f^{n-i})} + B_{\mathbf{1}_{\{|v_3|\leq\delta'\}}(f^{m-i} - f^{n-i})},
\end{aligned}$$

where

$$\begin{aligned}
&E_{\mathbf{1}_{\{|v_3|>\delta'\}}(f^{m-i} - f^{n-i})} = (2.31) + \cdots + (2.41), \quad \text{with } f \text{ changes to } \mathbf{1}_{|v_3|>\delta'}(f^{m-i} - f^{n-i}), \\
&B_{\mathbf{1}_{\{|v_3|>\delta'\}}(f^{m-i} - f^{n-i})} = (2.45) + \cdots + (2.55), \quad \text{with } f \text{ changes to } \mathbf{1}_{|v_3|>\delta'}(f^{m-i} - f^{n-i}), \\
&E_{\mathbf{1}_{\{|v_3|\leq\delta'\}}(f^{m-i} - f^{n-i})} = (2.31) + \cdots + (2.41), \quad \text{with } f \text{ changes to } \mathbf{1}_{|v_3|\leq\delta'}(f^{m-i} - f^{n-i}), \\
&B_{\mathbf{1}_{\{|v_3|\leq\delta'\}}(f^{m-i} - f^{n-i})} = (2.45) + \cdots + (2.55), \quad \text{with } f \text{ changes to } \mathbf{1}_{|v_3|\leq\delta'}(f^{m-i} - f^{n-i}).
\end{aligned} \tag{7.66}$$

Now, using the estimate in Lemma 6 and that

$$\int_{|v_3|<\delta'} \frac{1}{\langle v\rangle^{3+\delta}} dv \le C(\delta')^{\delta},$$

we have

$$\begin{aligned} &\|E_{\mathbf{1}_{\{|v_3|\le\delta'\}}(f^{m-i}-f^{n-i})}(s)\|_\infty + \|B_{\mathbf{1}_{\{|v_3|\le\delta'\}}(f^{m-i}-f^{n-i})}(s)\|_\infty \\ &\quad\le C(\delta')^{\delta} \sup_{0\le s'\le s} \|\langle v\rangle^{4+\delta}(f^{m-i}-f^{n-i})(s')\|_\infty. \end{aligned} \tag{7.67}$$

And

$$\begin{aligned} &\|E_{\mathbf{1}_{\{|v_3|>\delta'\}}(f^{m-i}-f^{n-i})}(s)\|_\infty + \|B_{\mathbf{1}_{\{|v_3|>\delta'\}}(f^{m-i}-f^{n-i})}(s)\|_\infty \\ &\quad\le C \sup_{0\le s'\le s} \|\mathbf{1}_{\{|v_3|>\delta'\}}\langle v\rangle^{4+\delta}(f^{m-i}-f^{n-i})(s')\|_\infty. \end{aligned} \tag{7.68}$$

So from (7.65), (7.67), and (7.68),

$$\begin{aligned} &\langle v\rangle^{4+\delta}|(f^m-f^n)(t,x,v)| \le CC_1M_3 \\ &\int_0^t \left(\sup_{1\le i\le k-1}\sup_{0\le s'\le s} \|\mathbf{1}_{\{|v_3|>\delta'\}}\langle v\rangle^{4+\delta}(f^{m-i}-f^{n-i})(s')\|_\infty + M_1(\delta')^{\delta}\right) ds. \end{aligned} \tag{7.69}$$

Now, let j be such that $t_j^{m-i-j}(s',x,v)\le 0 < t_{j-1}^{m-i-(j-1)}(s',x,v)$. Then if $|v_3|>\delta'$, from (7.32),

$$j \le \frac{T}{c\alpha(s',x,v)\langle v\rangle} \le \frac{T}{c\delta'} := k'.$$

So same as (7.65), this gives

$$\begin{aligned} &\sup_{1\le i\le k-1}\sup_{0\le s'\le s} \|\mathbf{1}_{\{|v_3|>\delta'\}}\langle v\rangle^{4+\delta}(f^{m-i}-f^{n-i})(s')\|_\infty \\ &\quad\le C_1M_3\int_0^s \sup_{1\le i'\le k'-1}\sup_{1\le i\le k-1} \|\mathfrak{F}^{m-i-i'}(s')-\mathfrak{F}^{n-i-i'}(s')\|_\infty ds'. \end{aligned}$$

Using the same split (7.66), like (7.67) and (7.68), we thus get

$$\begin{aligned} &\sup_{1\le i\le k-1}\sup_{0\le s'\le s} \|\mathbf{1}_{\{|v_3|>\delta'\}}\langle v\rangle^{4+\delta}(f^{m-i}-f^{n-i})(s')\|_\infty \\ &\quad\le CC_1M_3\int_0^s \sup_{2\le i\le k+k'}\sup_{0\le s''\le s'} \Big(\|\mathbf{1}_{\{|v_3|>\delta'\}}\langle v\rangle^{4+\delta}(f^{m-i}-f^{n-i})(s'')\|_\infty \\ &+M_1(\delta')^{\delta}\Big)\, ds'. \end{aligned} \tag{7.70}$$

Plug (7.70) into (7.68) yields

$$\begin{aligned} &\langle v\rangle^{4+\delta}|(f^m-f^n)(t,x,v)| \\ &\quad\le (CC_1M_3)^2\int_0^t\int_0^s \sup_{2\le i\le k+k'}\sup_{0\le s''\le s'} \|\mathbf{1}_{\{|v_3|>\delta'\}}\langle v\rangle^{4+\delta}(f^{m-i}-f^{n-i})(s'')\|_\infty ds'ds \\ &\quad\quad + M_1(\delta')^{\delta}\left(CC_1M_3t + \frac{(CC_1M_3t)^2}{2}\right). \end{aligned}$$

Iteration of the above gives

$$
\begin{aligned}
&\langle v\rangle^{4+\delta}|(f^m-f^n)(t,x,v)|\\
&\quad\leq (CC_1M_3)^l\frac{t^l}{l!}\sup_{2\leq i\leq k+lk'}\sup_{0\leq s\leq t}\|\mathbf{1}_{\{|v_3|>\delta'\}}\langle v\rangle^{4+\delta}(f^{m-i}-f^{n-i})(s)\|_\infty\\
&\qquad+M_1(\delta')^\delta\sum_{i=1}^{l}\frac{(CC_1M_3t)^i}{i!}.
\end{aligned}
$$

Now, by choosing δ' small enough we have

$$
M_1(\delta')^\delta\sum_{i=1}^{l}\frac{(CC_1M_3t)^i}{i!}<M_1(\delta')^\delta e^{CC_1M_3t}<\frac{\varepsilon}{2}.
$$

And choosing l large enough such that

$$
(CC_1M_3)^l\frac{t^l}{l!}\sup_{2\leq i\leq k+lk'}\sup_{0\leq s\leq t}\|\mathbf{1}_{\{|v_3|>\delta'\}}\langle v\rangle^{4+\delta}(f^{m-i}-f^{n-i})(s)\|_\infty<\frac{\varepsilon}{2}.
$$

Finally choose N_0 large enough such that $N_0>k+lk'$, we get for $n,m>N_0$,

$$
\langle v\rangle^{4+\delta}|(f^m-f^n)(t,x,v)|<\varepsilon.
$$

Therefore the sequence $\{f^\ell(t,x,v)\}_{\ell=1}^\infty$ is Cauchy, and this proves the lemma. $\square$

Lemma 24. *Suppose f_0 satisfies (0.38), E_0, B_0 satisfy (0.16), (0.35). Then for $0<T\ll 1$, there exists functions (f,E,B) with $\langle v\rangle^{4+\delta}f(t,x,v)\in L^\infty((0,T);L^\infty(\bar\Omega\times\mathbb{R}^3))$, and $(E,B)\in L^\infty((0,T);L^\infty(\Omega)\cap L^\infty(\partial\Omega))$, such that (f,E,B) is a (weak) solution of the system (0.40)–(0.42), (0.27). Moreover,*

$$
\|\langle v\rangle^{4+\delta}\nabla_x f(t)\|_\infty+\|\langle v\rangle^{4+\delta}\nabla_v f(t)\|_\infty<\infty, \tag{7.71}
$$

$$
\|\partial_t E(t)\|_\infty+\|\partial_t B(t)\|_\infty+\|\nabla_x E(t)\|_\infty+\|\nabla_x B(t)\|_\infty<\infty. \tag{7.72}
$$

Proof. From the uniform-in-ℓ bound (7.44), we can pass the limit up to subsequence if necessary and get the weak$-*$ convergence

$$
\langle v\rangle^{4+\delta}f^\ell\overset{*}{\rightharpoonup}\langle v\rangle^{4+\delta}f \text{ in } L^\infty((0,T)\times\Omega\times\mathbb{R}^3)\cap L^\infty((0,T)\times\gamma), \tag{7.73}
$$

$$
E^\ell\overset{*}{\rightharpoonup}E,\ B^\ell\overset{*}{\rightharpoonup}B \text{ in } L^\infty((0,T)\times\Omega)\cap L^\infty((0,T)\times\partial\Omega). \tag{7.74}
$$

for some (f,E,B). Then from (7.56) we also have

$$
\begin{aligned}
&\partial_t E^\ell\overset{*}{\rightharpoonup}\partial_t E,\ \nabla_x E^\ell\overset{*}{\rightharpoonup}\nabla_x E,\ \partial_t B^\ell\\
&\quad\overset{*}{\rightharpoonup}\partial_t B,\ \nabla_x B^\ell\overset{*}{\rightharpoonup}\nabla_x B \text{ in } L^\infty((0,T)\times\Omega),
\end{aligned} \tag{7.75}
$$

$$
\begin{aligned}
&\langle v\rangle^{4+\delta}\nabla_x f^\ell\overset{*}{\rightharpoonup}\langle v\rangle^{4+\delta}\nabla_x f,\ \langle v\rangle^{4+\delta}\nabla_v f^\ell\\
&\quad\overset{*}{\rightharpoonup}\langle v\rangle^{4+\delta}\nabla_v f \text{ in } L^\infty((0,T)\times\Omega\times\mathbb{R}^3).
\end{aligned} \tag{7.76}
$$

Now it left to show that (f, E, B) is a solution to the system (0.40)–(0.42), (0.27). Take any $\phi(t,x,v) \in C_c^\infty([0,T) \times \bar{\Omega} \times \mathbb{R}^3$ with supp $\phi \subset \{[0,T) \times \bar{\Omega} \times \mathbb{R}^3\} \backslash \{\{0\} \times \gamma \cup (0,T) \times \gamma_0\}$, from (7.41), we have

$$\begin{aligned}\int_{\Omega\times\mathbb{R}^3} f_0\phi(0)dvdt &+ \int_0^T \int_{\Omega\times\mathbb{R}^3} f^\ell(\partial_t\phi + \hat{v}\cdot\nabla_x\phi + \mathfrak{F}^{\ell-1}\cdot\nabla_v\phi)dvdxdt \\ &= \int_0^T\int_{\gamma_+} \phi f^\ell \hat{v}_3 dvdS_x + \int_0^T\int_{\gamma_+} \phi(t,x,v_\parallel,-v_3) f^\ell \hat{v}_3\, dvdS_x.\end{aligned} \tag{7.77}$$

Because of (7.73) and (7.74), we have

$$\begin{aligned}&\int_0^T\int_{\Omega\times\mathbb{R}^3} f^\ell(\partial_t\phi + \hat{v}\cdot\nabla_x\phi)dvdxdt + \int_0^T\int_{\gamma_+}\phi f^\ell\hat{v}_3 dvdS_x \\ &+ \int_0^T\int_{\gamma_+}\phi(t,x,v_\parallel,-v_3) f^\ell_\pm \hat{v}_3\, dvdS_x \\ &\to \int_0^T\int_{\Omega\times\mathbb{R}^3} f(\partial_t\phi + \hat{v}\cdot\nabla_x\phi)dvdxdt + \int_0^T\int_{\gamma_+}\phi f\hat{v}_3 dvdS_x \\ &+ \int_0^T\int_{\gamma_+}\phi(t,x,v_\parallel,-v_3) f\hat{v}_3\, dvdS_x\end{aligned} \tag{7.78}$$

as $\ell \to \infty$. As for the term $\int_0^T\int_{\Omega\times\mathbb{R}^3} f^\ell\mathfrak{F}^{\ell-1}\cdot\nabla_v\phi dvdxdt$, since

$$\begin{aligned}&\int_0^T\int_{\Omega\times\mathbb{R}^3}(f^\ell\mathfrak{F}^{\ell-1} - f\mathfrak{F})\cdot\nabla_v\phi dvdxdt \\ &= \int_0^T\int_{\Omega\times\mathbb{R}^3}(f^\ell - f)\mathfrak{F}^{\ell-1}\cdot\nabla_v\phi dvdxdt + \int_0^T\int_{\Omega\times\mathbb{R}^3} f(\mathfrak{F}^{\ell-1} - \mathfrak{F})\cdot\nabla_v\phi dvdxdt\end{aligned} \tag{7.79}$$

From (7.74), we have $\int_0^T\int_{\Omega\times\mathbb{R}^3} f(\mathfrak{F}^{\ell-1} - \mathfrak{F})\cdot\nabla_v\phi dvdxdt \to 0$ as $\ell \to \infty$.

Now, let supp$(\phi) = D$. From Lemma 23, $\mathbf{1}_D(t,x,v)f^\ell(t,x,v)$ converges to $\mathbf{1}_D(t,x,v)f(t,x,v)$ pointwise almost everywhere. And from (7.44), $|\mathbf{1}_D herefore, fromthe dominated (t,x,v)f^\ell(t,x,v)| \le \mathbf{1}_D(t,x,v)M_1$. Therefore, from the dominated convergence theorem, we have

$$\int_0^T\int_{\Omega\times\mathbb{R}^3}|\mathbf{1}_D(f - f^\ell)|dvdxdt \to 0 \text{ as } \ell\to\infty.$$

Thus

$$\begin{aligned}&\int_0^T\int_{\Omega\times\mathbb{R}^3}(f - f^\ell)\mathfrak{F}^{\ell-1}\cdot\nabla_v\phi dvdxdt \\ &\le \sup_{0\le t\le T}\|\mathfrak{F}^{\ell-1}(t)\|_\infty \sup_{0\le t\le T}\|\nabla_v\phi(t)\|_\infty \\ &\quad\int_0^T\int_{\Omega\times\mathbb{R}^3}|\mathbf{1}_D(f - f^\ell)|dvdxdt \to 0 \text{ as } \ell\to\infty.\end{aligned} \tag{7.80}$$

Put together (7.77)–(7.80), we deduce that (f, E, B) satisfy (0.28).

Next, using the same argument as in (6.54)–(6.57), we get (f, E, B) satisfy (0.29) and (0.30). Therefore, we conclude that (f, E, B) is a (weak) solution of the RVM system (0.40)–(0.42), with specular BC (0.27).

Finally, from using the weak lower semi-continuity of the weak-$*$ convergence (7.75), (7.76), and the uniform-in-ℓ bound (7.56), we conclude (7.71), (7.72). $\square$

Lastly, we prove the uniqueness.

Lemma 25. *Suppose (f, E_f, B_f) and (g, E_g, B_g) are solutions to the RVM system (0.40)–(0.42), (0.27) with $f(0) = g(0)$, $E_f(0) = E_g(0)$, $B_f(0) = B_g(0)$, and that*

$$\begin{aligned}E_f, B_f, E_g, B_g \in W^{1,\infty}((0,T)\times\Omega),\ \nabla_x\rho_f, \nabla_x J_f, \partial_t J_f, \nabla_x\rho_g, \nabla_x J_g,\\ \partial_t J_g \in L^\infty((0,T); L^p_{loc}(\Omega))\ \text{for some } p>1.\end{aligned}$$

And

$$\sup_{0<t<T}\|\langle v\rangle^{4+\delta}\nabla_v f(t)\|_\infty < \infty,\ \sup_{0<t<T}\|\langle v\rangle^{4+\delta}\nabla_v g(t)\|_\infty < \infty. \tag{7.81}$$

Then $f = g$, $E_f = E_g$, $B_f = B_g$.

Proof. The difference function $f - g$ satisfies

$$\begin{aligned}(\partial_t + \hat{v}\cdot\nabla_x + \mathfrak{F}_f\cdot\nabla_v)(f-g) &= (\mathfrak{F}_g - \mathfrak{F}_f)\cdot\nabla_v g\\ (f-g)(0) &= 0,\ (f-g)(t,x,v)|_{\gamma_-}\\ &= (f-g)(t,x,v_\parallel,-v_3),\end{aligned} \tag{7.82}$$

where

$$\mathfrak{F}_f = E_f + E_{\text{ext}} + \hat{v}\times(B_f + B_{\text{ext}}) - g\mathbf{e}_3,\ \mathfrak{F}_g = E_g + E_{\text{ext}} + \hat{v}\times(B_g + B_{\text{ext}}) - g\mathbf{e}_3,$$

so

$$\mathfrak{F}_g - \mathfrak{F}_f = E_f - E_g + \hat{v}\times(B_f - B_g). \tag{7.83}$$

From Lemma 1 we have $E_{f,1} - E_{g,1}$, $E_{f,2} - E_{g,2}$, $B_{f,3} - B_{g,3}$ solve the wave equation with the Dirichlet boundary condition (1.11) in the sense of (1.12) with

$$\begin{aligned}&u_0 = 0,\ u_1 = 0,\ G = -4\pi\partial_{x_i}(\rho_f - \rho_g) - 4\pi\partial_t(J_{f,i} - J_{g,i}),\\ &\ g = 0,\quad \text{for } E_{f,i} - E_{g,i},\ i = 1,2,\end{aligned} \tag{7.84}$$

$$u_0 = 0,\ u_1 = 0,\ G = 4\pi(\nabla_x\times(J_f - J_g))_3,\ g = 0,\quad \text{for } B_{f,3} - B_{g,3}, \tag{7.85}$$

respectively. And $E_{f,3} - E_{g,3}$, $B_{f,1} - B_{g,1}$, $B_{f,2} - B_{g,2}$ solve the wave equation with the Neumann boundary condition (1.9) in the sense of (1.10) with

$$\begin{aligned}&u_0 = 0,\ u_1 = 0,\ G = -4\pi\partial_{x_3}(\rho_f - \rho_g) - 4\pi\partial_t(J_{f,3} - J_{g,3}),\\ &\ g = -4\pi(\rho_f - \rho_g),\ \text{ for } E_{f,3} - E_{g,3},\end{aligned} \tag{7.86}$$

$$\begin{aligned}&u_0 = 0,\ u_1 = 0,\ G = 4\pi(\nabla_x\times(J_f - J_g))_i,\\ &\ g = (-1)^{i+1}4\pi(J_{f,\underline{i}} - J_{g,\underline{i}}),\quad \text{for } B_{f,i} - B_{j,i},\ i = 1,2,\end{aligned} \tag{7.87}$$

respectively. Therefore, from Lemmas 2 and 4, we know that $E_f - E_g$ and $B_f - B_g$ would have the form of

$$\begin{aligned} E_f - E_g = (2.31) + \cdots + (2.41),\ \ B_f - B_g = (2.45) + \cdots + (2.55), \\ \text{with } E_0,\, B_0 \text{ changes to } 0, \text{ and } f \text{ changes to } f-g. \end{aligned} \tag{7.88}$$

Now consider the characteristics

$$\begin{aligned} \dot{X}_f(s;t,x,v) &= \hat{V}_f(s;t,x,v), \\ \dot{V}_f(s;t,x,v) &= \mathfrak{F}_f(s, X_f(s;t,x,v), V_f(s;t,x,v)). \end{aligned}$$

Then from (7.82), same as (7.64), we obtain

$$\begin{aligned} (f-g)(t,x,v) = &\int_{t_1}^{t} \left((\mathfrak{F}_g - \mathfrak{F}_f)\cdot \nabla_v g\right)(s, \dot{X}(s), \dot{V}(s))ds \\ &+ \sum_{i=1}^{k-2} \int_{t_{i+1}}^{t_i} \Big((\mathfrak{F}_g - \mathfrak{F}_f)\cdot \nabla_v g\Big)(s, \cdot X(s), \cdot V(s))ds \\ &+ \int_0^{t_{k-1}} \Big((\mathfrak{F}_g - \mathfrak{F}_f)\cdot \nabla_v g\Big)(s, \cdot X(s), \cdot V(s))ds. \end{aligned} \tag{7.89}$$

So using (7.52), (7.81), we have

$$\begin{aligned} &\sup_{0\le s\le t} \|\langle v\rangle^{4+\delta}(f-g)(s)\|_\infty \\ &\quad\le \sup_{0\le t<T} \|\langle v\rangle^{4+\delta}\nabla_v g(t)\|_\infty \int_0^t \sup_{0\le s'\le s} \|(\mathfrak{F}_g - \mathfrak{F}_f)(s')\|_\infty ds \\ &\le C\int_0^t \sup_{0\le s'\le s} \|(\mathfrak{F}_g - \mathfrak{F}_f)(s')\|_\infty ds. \end{aligned} \tag{7.90}$$

Now, from (6.75) and the estimate in Lemma 6, we have

$$\begin{aligned} \sup_{0\le s'\le s} \|(\mathfrak{F}_g - \mathfrak{F}_f)(s')\|_\infty &\le \sup_{0\le s'\le s} \|(E_f - E_g)(s')\|_\infty + \sup_{0\le s'\le s} \|(B_f - B_g)(s')\|_\infty \\ &\le C \sup_{0\le s'\le s} \|\langle v\rangle^{4+\delta}(f-g)(s')\|_\infty, \end{aligned} \tag{7.91}$$

Therefore from (7.90) and (7.91), we have

$$\sup_{0\le s\le t} \|\langle v\rangle^{4+\delta}(f-g)(s)\|_\infty \le C'\int_0^t \sup_{0\le s'\le s} \|\langle v\rangle^{4+\delta}(f-g)(s')\|_\infty ds. \tag{7.92}$$

Therefore from Gronwall

$$\sup_{0\le s'\le t} \|\langle v\rangle^{4+\delta}(f-g)(s')\|_\infty \le e^{C't}\|\langle v\rangle^{4+\delta}(f-g)(0)\|_\infty = 0.$$

Therefore we conclude that the solutions to (0.40)–(0.42), (0.27) is unique. □

We conclude the section by proving Theorem 3.

proof of Theorem 3. Using the sequence f^ℓ, E^ℓ, B^ℓ constructed in (7.41), (7.42), we have from Lemma 25 that the limit (f, E, B) is a solution to the RVM system (0.40)–(0.42), (0.26), and it satisfies the regularity estimate (0.39), (0.37). And from Lemma 25, we conclude the uniqueness. □

Acknowledgements This project is supported in part by National Science Foundation under Grant No. 1900923 and 2047681 (NSF-CAREER). CK was supported by Brain Pool program funded by the Ministry of Science and ICT through the National Research Foundation of Korea (2021H1D3A2A01039047), and he thanks Professor Seung Yeal Ha for the kind hospitality of the BP program.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

References

1. Beals, R., Protopopescu, V.: Abstract time-dependent transport equations. J. Math. Anal. Appl. **121**, 370–405 (1987)
2. Burbidge, E.M., Burbidge, G.R., Fowler, W.A., Hoyle, F.: Synthesis of the elements in stars. Rev. Mod. Phys. **29**, 547 (1957)
3. Bronold, F., Fehske, H.: Kinetic modeling of the electronic response of a dielectric plasma-facing solid. J. Phys. D: Appl. Phys. **50**, 294003 (2017)
4. Cao, Y.: Regularity of Boltzmann equation with external fields in convex domains of diffuse reflection. SIAM J. Math. Anal. **51**(4), 3195–3275 (2019)
5. Cao, Y.: A note on two species collisional plasma in bounded domains. Kinet. Relat. Models **12**(6), 1359–1429 (2019)
6. Cao, Y.: Rarefied gas dynamics with external fields under specular reflection boundary condition. Commun. Math. Sci. **20**(8), 2133–2206 (2022)
7. Cao, Y., Kim, C.: Arvix version 1 of "Lipschitz continuous solutions of the Vlasov-Maxwell systems with a conductor boundary condition", arXiv:2203.01615
8. Cao, Y., Kim, C.: On some recent progress in the Vlasov–Poisson–Boltzmann system with diffuse reflection boundary. In: Salvarani, F. (ed.) Recent Advances in Kinetic Equations and Applications. Springer INdAM Series, vol. 48. Springer, Cham (2021)
9. Cao, Y., Kim, C., Lee, D.: Global strong solutions of the Vlasov–Poisson–Boltzmann system in bounded domains. Arch. Ration. Mech. Anal. **233**(3), 1027–1130 (2019)
10. Chen, H., Kim, C., Li, Q.: Local well-posedness of Vlasov–Poisson–Boltzmann equation with generalized diffuse boundary condition. J. Stat. Phys. **179**(2), 535–631 (2020)
11. Chamberlain, J.W.: Interplanetary gas. III. A hydrodynamic model of the corona. Astrophys. J. **133**, 675 (1961)
12. Cheng, Y., Gamba, I.M., Li, F., Morrison, P.J.: Discontinuous Galerkin methods for the Vlasov–Maxwell equations. SIAM J. Numer. Anal. **52**(2), 1017–1049 (2014)
13. Cao, Y., Kim, C.: Glassey-Strauss representation of Vlasov–Maxwell systems in a Half Space. Kinet. Relat. Models. (2021). https://doi.org/10.3934/krm.2021034
14. Cooper, J., Strauss, W.: The initial boundary problem for the Maxwell equations in the presence of a moving body. SIAM J. Math. Anal. **16**, 1165–1179 (1985)
15. Dantas, R.M.A., Peña-Benitez, F., Roy, B., et al.: Magnetotransport in multi-Weyl semimetals: a kinetic theory approach. J. High Energy Phys. **2018**, 69 (2018). https://doi.org/10.1007/JHEP12(2018)069

16. DiPerna, R.J., Lions, P.-L.: Global weak solutions of Vlasov–Maxwell systems. Commun. Pure Appl. Math. **42**(6), 729–757 (1989)
17. Filbet, F., Sonnendrücker, E., Bertrand, P.: Conservative numerical schemes for the Vlasov equation. J. Comput. Phys. **172**(1), 166–187 (2001)
18. Guo, Y.: Global weak solutions of the Vlasov–Maxwell system with boundary conditions. Commun. Math. Phys. **154**(2), 245–263 (1993)
19. Guo, Y.: Regularity for the Vlasov equations in a half-space. Indiana Univ. Math. J. **43**(1), 255–320 (1994)
20. Guo, Y.: Singular solutions of the Vlasov–Maxwell system on a half line. Arch. Ration. Mech. Anal. **131**(3), 241–304 (1995)
21. Guo, Y., Kim, C., Tonon, D., Trescases, A.: Regularity of the Boltzmann equation in convex domains. Invent. Math. **207**, 115–290 (2017)
22. Guo, Y., Kim, C., Tonon, D., Trescases, A.: BV-regularity of the Boltzmann equation in Non-convex domains. Arch. Ration. Mech. Anal. **220**, 1045–1093 (2016)
23. Gerard-Varet, D., Prestipino, M.: Formal derivation and stability analysis of boundary layer models in MHD. Z. Angew. Math. Phys. **68**(3), 76 (2017)
24. Glassey, R.T., Schaeffer, J.: The “two and one-half-dimensional” relativistic Vlasov–Maxwell system. Commun. Math. Phys. **185**(2), 257–284 (1997)
25. Glassey, R.T., Schaeffer, J.: The relativistic Vlasov–Maxwell system in two space dimensions. I. Arch. Ration. Mech. Anal. **141**(4), 331–354 (1998)
26. Glassey, R.T., Schaeffer, J.: The relativistic Vlasov–Maxwell system in two space dimensions. II. Arch. Ration. Mech. Anal. **141**(4), 355–374 (1998)
27. Glassey, R.T., Strauss, W.A.: Singularity formation in a collisionless plasma could occur only at high velocities. Arch. Ration. Mech. Anal. **92**(1), 59–90 (1986)
28. Glassey, R.T.: The Cauchy problem in kinetic theory. (English summary) Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, xii+241 pp. ISBN: 0-89871-367-6 (1996)
29. Glassey, R.T., Strauss, W.A.: Absence of shocks in an initially dilute collisionless plasma. Commun. Math. Phys. **113**(2), 191–208 (1987)
30. Glassey, R.T., Strauss, W.A.: High velocity particles in a collisionless plasma. Math. Methods Appl. Sci. **9**(1), 46–52 (1987)
31. Glassey, R.T., Strauss, W.A.: Large velocities in the relativistic Vlasov–Maxwell equations. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **36**(3), 615–627 (1989)
32. Hwang, H.-J., Velazquez, J.: Global existence for the Vlasov–Poisson system in bounded domains. Arch. Ration. Mech. Anal. **195**, 763–796 (2010)
33. Ikawa, M.: Decay of solutions of the wave equation in the exterior of two convex obstacles. Osaka Math. J. **19**(3), 459–509 (1982)
34. Jang, J.W., Strain, R.M., Wong, T.K.: Magnetic confinement for the 2D axisymmetric relativistic Vlasov–Maxwell system in an annulus. Kinet. Relat. Models (2021). https://doi.org/10.3934/krm.2021039
35. Jin, J., Kim, C.: Damping of kinetic transport equation with diffuse boundary condition. SIAM J. Math. Anal. **54**(5), 5524–5550 (2022)
36. Klainerman, S., Staffilani, G.: A new approach to study the Vlasov–Maxwell system. Commun. Pure Appl. Anal. **1**(1), 103–125 (2002)
37. Kim, C.: Formation and propagation of discontinuity for Boltzmann equation in non-convex domains. Commun. Math. Phys **308**, 641–701 (2011)
38. Kim, C., Lee, D.: The Boltzmann equation with specular boundary condition in convex domains. Commun. Pure Appl. Math. **71**, 411–504 (2018)
39. Kim, C., Lee, D.: Hölder Regularity of the Boltzmann equation past an obstacle (submitted). arXiv:2111.07558
40. Leoni, G.: A first course in Sobolev spaces. Second edition. Graduate Studies in Mathematics, vol. 181. American Mathematical Society, Providence, RI, xxii+734 pp. (2017) ISBN: 978-1-4704-2921-8
41. Lemaire, J.: Half a century of kinetic solar wind models. AIP Conference Proceedings, vol. 1216. no. 1. American Institute of Physics (2010)
42. Lions, J.-L., Magenes, E.: Non-homogeneous boundary value problems and applications. vol. II. Translated from the French by P. Kenneth. Die Grundlehren der mathematischen Wissenschaften, Band 182. Springer, New York-Heidelberg, xi+242 pp (1972)
43. Lemaire, J., Scherer, M.: Kinetic models of the solar wind. J. Geophys. Res. **76**(31), 7479–7490 (1971)
44. Luk, J., Strain, R.M.: A new continuation criterion for the relativistic Vlasov–Maxwell system Comm. Math. Phys. **331**(3), 1005–1027 (2014)
45. Luk, J., Strain, R.M.: Strichartz estimates and moment bounds for the relativistic Vlasov–Maxwell system. Arch. Ration. Mech. Anal. **219**(1), 445–552 (2016)
46. Lin, Z., Strauss, W.A.: Linear stability and instability of relativistic Vlasov–Maxwell systems. Commun. Pure Appl. Math. **60**(5), 724–787 (2007)

47. Lin, Z., Strauss, W.A.: Nonlinear stability and instability of relativistic Vlasov–Maxwell systems. Commun. Pure Appl. Math. **60**(6), 789–837 (2007)
48. Lin, Z., Strauss, W.A.: A sharp stability criterion for the Vlasov–Maxwell system. Invent. Math. **173**(3), 497–546 (2008)
49. Liu, C., Xie, F., Yang, T.: MHD boundary layers theory in Sobolev spaces without monotonicity I: Well-posedness theory. Commun. Pure Appl. Math. **72**(1), 63–121 (2019)
50. Maxwell, J.C.: On the dynamical theory of gases. Philos. Trans. R. Soc. Lond. **157**, 49–88 (1866)
51. Matus, P., Dantas, R., Moessner, R., Surówka, P.: Skin effect as a probe of transport regimes in Weyl semimetals, arXiv:2111.11810
52. Nguyen, T.T., Nguyen, T.V., Strauss, W.A.: Global magnetic confinement for the 1.5D Vlasov–Maxwell system. Kinet. Relat. Models **8**(1), 153–168 (2015)
53. Nguyen, T.T., Strauss, W.A.: Stability analysis of collisionless plasmas with specularly reflecting boundary. SIAM J. Math. Anal. **45**(2), 777–808 (2013)
54. Nguyen, T.T., Strauss, W.A.: Linear stability analysis of a hot plasma in a solid torus. Arch. Ration. Mech. Anal. **211**(2), 619–672 (2014)
55. Ola, P., Paivarinta, L., Somersalo, E.: An inverse boundary value problem in electrodynamics. Duke Math. J. **70**(3), 617–653 (1993)
56. Pannekoek, A.: Ionization in stellar atmospheres. Bull. Astron. Inst. Neth. **1**, 107 (1922)
57. Pierrard, V., Lamy, H., Lemaire, J.: Exospheric distributions of minor ions in the solar wind. J. Geophys. Res. **109**, A02118 (2004)
58. Pierrard, V., Pieters, M.: Coronal heating and solar wind acceleration for electrons, protons, and minor ions obtained from kinetic models based on kappa distributions. J. Geophys. Res. Space Phys. **119**, 9441–9455 (2014)
59. Rein, G.: Existence of stationary, collisionless plasmas in bounded domains. Math. Methods Appl. Sci. **15**(5), 365–374 (1992)
60. Rosseland, S.: Electrical State of a Star. Mon. Not. R. Astron. Soc. **84**, 720 (1924)
61. Spitz, M.: Regularity theory for nonautonomous Maxwell equations with perfectly conducting boundary conditions. J. Math. Anal. Appl. **506**(1), 125646 (2022)
62. Ukai, S.: Solutions of the Boltzmann equation. Patterns and waves. Stud. Math. Appl. **18**, 37–96 (1986)
63. Zhang, K.Z.: Linear stability analysis of the relativistic Vlasov–Maxwell system in an axisymmetric domain. SIAM J. Math. Anal. **51**(6), 4683–4723 (2019)
64. Zhang, K.Z.: Magnetic confinement at a boundary approximates specular reflection. Math. Methods Appl. Sci. **44**(7), 5705–5738 (2021)