BICRITICAL RATIONAL MAPS WITH A COMMON ITERATE

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ABSTRACT. Let f be a degree d bicritical rational map with critical point set $\mathcal{C}_f$ and critical value set $\mathcal{V}_f$. Using the group $\mathrm{Deck}(f^k)$ of deck transformations of f^k, we show that if g is a bicritical rational map which shares an iterate with f then $\mathcal{C}_f = \mathcal{C}_g$ and $\mathcal{V}_f = \mathcal{V}_g$. Using this, we show that if two bicritical rational maps of even degree d share an iterate then they share a second iterate, and both maps belong to the symmetry locus of degree d bicritical rational maps.

1. INTRODUCTION

For any integers $k, d \geq 2$, the k-fold iteration operator, $f \mapsto f^k$, on the set Rat_d consisting of all rational maps of degree d, is injective on the complement of a Zariski closed set ([19]). In this work, motivated by questions about matings of polynomials, we restrict our attention to *bicritical* rational maps. Our first main theorem is the following.

Theorem 1.1. *Let f and g be distinct bicritical rational maps and suppose there exists $k \in \mathbb{N}$ such that $f^k = g^k$. Then f and g have the same critical points and critical values.*

As demonstrated in Example 8.6, the converse to Theorem 1.1 does not hold.

In the even degree case, we show that (excluding the case where f and g are power maps), sharing *any* iterate is equivalent to sharing the second iterate.

Theorem 1.2. *Let f and g be distinct bicritical maps which are not power maps and of even degree d. If there exists k such that $f^k = g^k$, then $f^2 = g^2$. Furthermore, there exists an involution μ such $g = \mu \circ f = f \circ \mu$, and so f and g belong to the symmetry locus Σ_d.*

Work of Mike Zieve [22] gives a proof of Theorem 1.2 in the case $d = 2$. The proof technique is markedly different from those used in the present paper.

Theorem (Zieve [22])**.** *Let f and g be quadratic rational functions with a common iterate, and let n be the least positive integer for which $f^n = g^n$. If f and g are not power maps and $n > 1$, then $n = 2$.*

Additional work in progress by Luallen and Zieve ([21]) gives alternative proofs of Theorems 1.1 and 1.2. Their work has also obtained a number results on rational functions which share a common iterate.

To prove Theorems 1.1 and 1.2, we consider two different groups of "symmetries" of a rational map f. First, the well-known *symmetry group* or

automorphism group of a rational map f, $\mathrm{Aut}(f)$, is the group of all Möbius transformations τ that commute with f. The degree d *symmetry locus* is the set of degree d bicritical rational maps f such that $\mathrm{Aut}(f)$ is nontrivial. As shown in [12], when d is odd, the symmetry locus is a reducible variety, splitting into two "halves" with different dynamical behaviors, while when d is even, the symmetry locus is irreducible. Second, the group that we call the *deck group* of a rational map f, $\mathrm{Deck}(f)$, consists of all Möbius transformations τ such that $f \circ \tau = f$. The groups $\mathrm{Aut}(f)$, $\mathrm{Deck}(f)$, as well as other groups of symmetries, are studied by Pakovich in [14]. In particular, for a general rational map f, Pakovich considers the groups

$$\mathrm{Aut}_\infty(f) = \bigcup_{k=0}^\infty \mathrm{Aut}(f^k) \quad \text{and} \quad \mathrm{Deck}_\infty(f) = \bigcup_{k=0}^\infty \mathrm{Deck}(f^k).$$

and shows that, except for when f is a power map, these groups are finite. Furthermore, he provides methods which allow an explicit description of the groups in a number of cases.

We prove the following characterization of deck groups of iterates of bicritical rational maps.

Theorem 1.3. *Let f be a bicritical rational map and $k \in \mathbb{N}$. Then $\mathrm{Deck}(f^k)$ is either cyclic or dihedral. Furthermore, if the degree of f is odd, then $\mathrm{Deck}(f^k)$ is cyclic.*

If f is a bicritical rational map of degree d, $\mathrm{Deck}(f)$ contains the order-d elliptic rotation around the axis in hyperbolic 3-space whose endpoints are the critical points of f. Our strategy for detecting the critical points and values of f from the map f^k is to exploit the group structure of $\mathrm{Deck}(f)$ guaranteed by Theorem 1.3 and to distinguish the critical points of f from the set of all points in $\hat{\mathbb{C}}$ fixed by some nonidentity element of $\mathrm{Deck}(f)$.

The proof of this statement is much harder in the degree 2 case than in the seemingly more general case for bicritical maps of degree $d \geq 3$. The reason for this is that the degree 2 case is the only one where the degree coincides with the number of critical values. We obtain the following characterization of $\mathrm{Deck}(f^k)$ in the case that f quadratic.

Theorem 1.4. *If f is a quadratic rational map, then the possibilities for $\mathrm{Deck}(f^k)$ (up to isomorphism) are $\mathbb{Z}_{2^n}$ for $n \geq 1$, V_4 or D_8, the set of symmetries of a square. Furthermore, if f is not a power map then $|\mathrm{Deck}(f^k)| \leq 8$.*

Generalizations of Theorems 1.3 and 1.4 are given in [10], where a complete classification of the groups $\mathrm{Deck}(f^k)$ for bicritical rational maps is obtained. Since, for discrete subgroups of Rat_1, classification up to isomorphism is equivalent to classification up to conjugacy, the result of Theorem 1.4 actual holds in the seemingly more general setting of classification up to conjugacy.

The original motivation for this study was to understand and clarify the observation communicated to the authors by John Hubbard that the quadratic symmetry locus Σ_2 contains rational maps that can be viewed as variants of matings of quadratic polynomials in which the dynamics swap

which "hemisphere" a point belongs to. While sequels will explore this topic in greater detail, we offer the following provisional definition.

Definition 1.5. *Let F be a rational map of degree d and suppose there exist postcritically finite degree d polynomials f and g such that*

(i) $F^2 = (f \perp\!\!\!\perp g)^2$, *and*

(ii) $F \neq f \perp\!\!\!\perp g$,

where $f \perp\!\!\!\perp g$ denotes a rational map that is a geometric mating of f and g. Then we say F is a mixing *of f and g and write $F = f \times g$.*

An immediate consequence of Theorem 1.1 is that a mixing $f \times g$ of f and g and the corresponding geometric mating $f \perp\!\!\!\perp g$ have the same critical points and critical values. Theorem 1.2 implies that, in the even degree case, replacing the second iterates in Definition 1.5 with kth iterates, for $k \geq 2$, does not introduce any additional generality. Furthermore, it implies that if f and g have even degree and both geometric and mixed matings of f and g exist, then these matings live in the symmetry locus Σ_d.

Certain classes of quadratic rational maps which may be obtained by mixings can be described using the notion of two-sided invariant laminations, a concept which first appeared in the thesis of Ahmadi [5]. Indeed, Timorin [17], in the case of $\mathrm{Per}_2(0)$, the space of quadratic rational maps with a period two critical point, showed that maps in the external boundary of the connectedness locus of $\mathrm{Per}_2(0)$ can be modeled in terms of two-sided invariant laminations. Using this description, it may be possible to use the combinatorial model of Theorem B from [17] to show that the maps on this external boundary are indeed examples of mixings[1].

Other constructions or definitions that are conceptually related to mixings include Timorin's work on regluings ([18]), twisted matings ([3]), Meyer's anti-equators [11], and work in progress by Jung on quadratic anti-matings ([8])

The paper is organized as follows. In Section 2, we give some preliminary results about $\mathrm{Deck}(f)$ for general rational maps. In the following section, we then restrict our attention to $\mathrm{Deck}(f)$ where f is bicritical. In Section 4, we prove Theorem 1.1 for degree $d \geq 3$. We then turn our attention to iterates of quadratic rational maps, and in Section 5 we undertake a deeper analysis of the possibilities for $\mathrm{Deck}(f^k)$ when f is quadratic. This allows us, in Sections 6 and 7, to prove Theorem 1.1 for quadratics. The proof of Theorem 1.2 is given in Section 8. Finally, in the Appendix, we revisit the space Σ_2 and present some conjectural and computational observations about matings and mixings.

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[1]Note that in [17], what we consider to be mixings are called *anti-matings.* Compare [8].
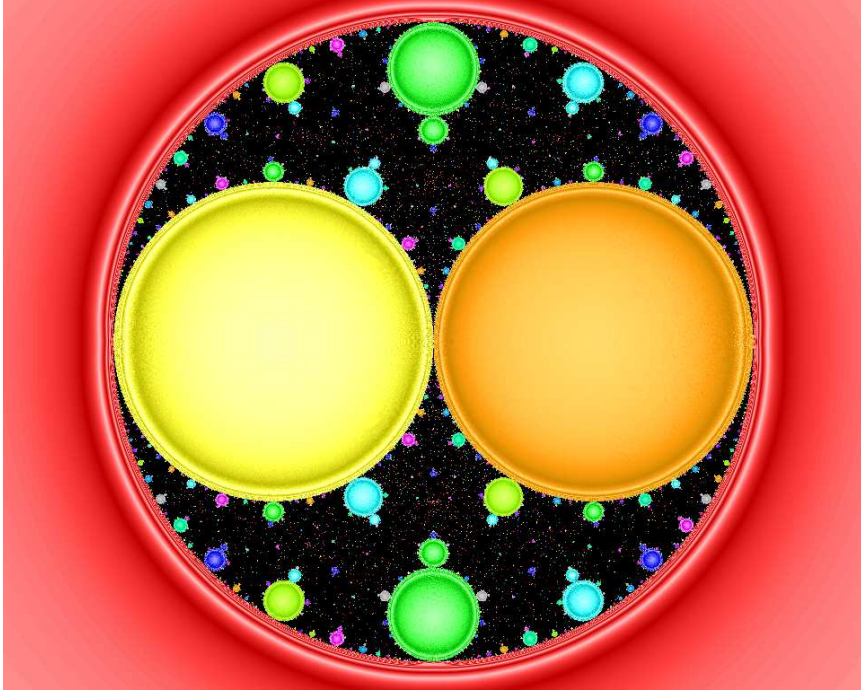
FIGURE 1. The symmetry locus Σ_2, up to conformal conjugacy, is parameterized by $f_c(z) = c\left(z + \frac{1}{z}\right)$ for $c \in \mathbb{C} - \{0\}$. The image shows the bifurcation locus. The outer red region contains maps for which both critical points are attracted to the same fixed point. The other hyperbolic components are colored according to the period of the map's attracting cycle(s). In particular, the large orange region consists of maps with two attracting fixed points; the large yellow region contains maps that have a period two attracting cycle which attracts both critical points. Note that the bifurcation locus has V_4 symmetry. One can easily check that $f_c^2 = f_{-c}^2$ and, setting $\mu(z) = -z$, $f_{-c} = \mu \circ f_c = f_c \circ \mu$.

2. The deck group of a rational map

Definition 2.1. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. The* deck group *of f is*

$$\mathrm{Deck}(f) := \{\tau \in \mathrm{Rat}_1 \mid f = f \circ \tau\}$$

and we say that an element τ of $\mathrm{Deck}(f)$ is a deck transformation *of f.*

W will call elements of the group $\mathrm{Aut}(f) := \{\tau \in \mathrm{Rat}_1 \mid f = \tau^{-1} \circ f \circ \tau\}$ *automorphisms* of f.

We will find the following notation for a fiber useful.

Definition 2.2. *For any rational map f on $\hat{\mathbb{C}}$ and $z \in \hat{\mathbb{C}}$, define the* fiber *of z with respect to f to be the set*

$$\rho_f(z) := \{w \in \hat{\mathbb{C}} \mid f(w) = z\}.$$

The following Proposition collects some elementary facts about the general deck group of a rational map.

Proposition 2.3. *Let f be a rational map of degree $d \geq 1$.*
  (i) $\operatorname{Deck}(f)$ *is a group.*
  (ii) *For any $k \in \mathbb{N}$, $\operatorname{Deck}(f)$ is a subgroup of $\operatorname{Deck}(f^k)$.*
  (iii) *Conjugate rational maps have isomorphic deck groups.*
  (iv) *Fibers are preserved by elements of the deck group. More precisely, for any $\phi \in \operatorname{Deck}(f)$ and $z \in \hat{\mathbb{C}}$,*

$$\rho_f(z) = \phi(\rho_f(z)) = \phi^{-1}(\rho_f(z)).$$

  (v) *Local degrees under f are preserved by elements of the deck group. More precisely, denoting by $\deg_f(z)$ the local degree with which a point z maps forwards under f, we have that*

$$\deg_f(z) = \deg_f(\phi(z))$$

  for all $\phi \in \operatorname{Deck}(f)$ and $z \in \hat{\mathbb{C}}$.
  (vi) *The order of $\operatorname{Deck}(f)$ is at most d.*
  (vii) $\operatorname{Deck}(f)$ *is isomorphic to either a cyclic group, a dihedral group, A_4 (the symmetry group of the tetrahedron), S_4 (the symmetry group of the octahedron) or A_5 (the symmetry group of the icosahedron).*

Proof. Conclusions (i)-(v) are immediate from the definitions. The claim in (vi) follows from the uniqueness of lifts for covering spaces. Conclusion (vii) then follows from the well-known (see [9] for a reprint of the classical reference) fact that every finite group of Möbius transformations is isomorphic to a cyclic group, a dihedral group, A_4, S_4, or A_5. $\square$

We will sometimes refer to the groups A_4, S_4 and A_5 as the polyhedral groups. Note that in the above we consider that the Klein *Vierergruppe* V_4 is a dihedral group. Examples of rational maps exhibiting each of the possible types of deck groups are constructed in [7], where the term "half-symmetry" is used for what we call a deck transformation.

The following Lemma is classical.

Lemma 2.4. *Let f and g be bicritical rational maps such that $\mathcal{C}_f = \mathcal{C}_g$. Then $g = \nu \circ f$ for some Möbius transformation ν sending $\mathcal{V}_f$ to $\mathcal{V}_g$.*

Proof. Let c_1 and c_2 be the two (distinct) critical points of f and g and let a be an arbitrary point in $\hat{\mathbb{C}} \setminus \{c_1, c_2\}$. Note that $f(c_1)$, $f(c_2)$ and $f(a)$ are three distinct points in $\hat{\mathbb{C}}$ (because f is d-to-1, counted with multiplicity). Let $z : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the Möbius transformation satisfying

$$z \circ f(c_1) = 0, \quad z \circ f(c_2) = \infty \text{ and } z \circ f(a) = 1.$$

Similarly, $g(c_1)$, $g(c_2)$ and $g(a)$ are three distinct points in $\hat{\mathbb{C}}$. Let $w : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the Möbius transformation satisfying

$$w \circ g(c_1) = 0, \quad w \circ g(c_2) = \infty \text{ and } w \circ g(a) = 1.$$

Then the meromorphic functions $z \circ f$ and $w \circ g$ have d-fold zeroes at c_1 and d-fold poles at c_2. It follows that their quotient is constant. Since they coincide at a, they are equal. Set $\nu := w^{-1} \circ z : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Then, $g = \nu \circ f$. In addition,

$$\nu(\mathcal{V}_f) = \nu \circ f(\mathcal{C}_f) = g(\mathcal{C}_f) = g(\mathcal{C}_g) = \mathcal{V}_g.$$

$\square$

Remark 2.5. *The conclusion of Lemma 2.4 does not hold for general rational maps (see e.g. [6]), but it does hold for all polynomials. Indeed, given a finite collection of (not necessarily distinct) points $X = \{c_1, \ldots, c_k\}$ in the complex plane, one may construct a polynomial p for which the critical points are precisely the elements of X (with multiplicity of the critical point at c_i given by the number of times c_i appears in X) ([20]). The polynomial p is unique up to post-composition by a complex affine transformation.*

Applying Lemma 2.4 in the case where $g = f \circ \mu$ yields the following. Uniqueness follows from the surjectivity of f.

Lemma 2.6. *Let f be a bicritical rational map with critical point set $\mathcal{C}_f$. Then if μ is a Möbius transformation such that $\mu(\mathcal{C}_f) = \mathcal{C}_f$, then there exists a unique Möbius transformation ν such that $\nu \circ f = f \circ \mu$. Furthermore $\nu(\mathcal{V}_f) = \mathcal{V}_f$.*

Corollary 2.7. *Let f be a bicritical rational map and $\phi_k \in \text{Deck}(f^k)$ for some k. If $\phi_k(\mathcal{C}_f) = \mathcal{C}_f$, then there exists a unique $\phi_{k-1} \in \text{Deck}(f^{k-1})$ such that $f \circ \phi_k = \phi_{k-1} \circ f$. Moreover $\phi_{k-1}(\mathcal{V}_f) = \mathcal{V}_f$.*

Proof. Following Lemma 2.6, we only need to show that $\phi_{k-1} \in \text{Deck}(f^{k-1})$. To see this, consider the following diagram.

$$
\begin{array}{ccc}
\hat{\mathbb{C}} & \xrightarrow{\mu} & \hat{\mathbb{C}} \\
f \downarrow & & \downarrow f \\
\hat{\mathbb{C}} & \xrightarrow{\nu} & \hat{\mathbb{C}} \\
f^{k-1} \downarrow & & \downarrow f^{k-1} \\
\hat{\mathbb{C}} & \xrightarrow{\text{id}} & \hat{\mathbb{C}}
\end{array}
$$

The large outermost rectangle commutes since $\mu \in \text{Deck}(f^k)$. Therefore, the square in the bottom commutes as well. As a consequence, $\nu \in \text{Deck}(f^{k-1})$.

$\square$

3. The deck group of a bicritical rational map is cyclic or dihedral

We will mainly be concerned with the groups $\text{Deck}(f^k)$, where f is a degree d bicritical rational map. In this section we will show that the groups $\text{Deck}(f^k)$ cannot be polyhedral groups for bicritical maps.

Recall (see e.g [1]) that a Möbius transformation on $\hat{\mathbb{C}}$ may be extended to the unit ball model in hyperbolic 3-space $\mathbb{H}^3$. In this case the boundary of $\mathbb{H}^3$ is naturally identified with the Riemann sphere $\hat{\mathbb{C}}$. We then say that a Möbius transformation is *elliptic* if it has infinitely many fixed points in $\mathbb{H}^3$.

Lemma 3.1. *Let f be a bicritical rational map of degree $d \geq 2$. Then $\mathrm{Deck}(f)$ contains the elliptic Möbius transformation that is an order d-rotation around the axis (geodesic in $\mathbb{H}^3$) connecting the two critical points of f.*

Proof. As Milnor observes in [12], we can conjugate f by some Möbius transformation ϕ that sends the critical points of f to 0 and ∞; then $\phi^{-1} \circ f \circ \phi$ has the form

$$z \mapsto \frac{\alpha z^d + \beta}{\gamma z^d + \delta}$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Any map of this form is invariant under composition with the elliptic rotation $R(z) = z \exp(2\pi i / d)$, i.e.

$$\phi^{-1} \circ f \circ \phi(z) = \phi^{-1} \circ f \circ \phi(R_n(z)).$$

$\square$

Corollary 3.2. *Let f be a bicritical rational map of degree $d \geq 6$. Then for any $k \in \mathbb{N}$, $\mathrm{Deck}(f^k)$ is either cyclic or dihedral.*

Proof. By Lemma 3.1, $\mathrm{Deck}(f)$ contains an element of order d. Since A_4, S_4 and A_5 do not have any element of order ≥ 6, the claim follows from Proposition 2.3 part (vii). $\square$

We now turn our attention to the case where the degree is less than or equal to 5. The following definition will be useful.

Definition 3.3. *Let f be a bicritical rational map of degree d and let k be a natural number. The* degree partition *for a point $z \in \hat{\mathbb{C}}$ with respect to f^k is the ordered list of integers $\{a_{i,f^k}(z)\}_{i=0}^k$ where $a_{i,f^k}(z)$ is the number of points in the fiber $\rho_{f^k}(z)$ that map forward under f with local degree d^i.*

The following lemma is immediate from the definitions.

Lemma 3.4. *Let f be a bicritical rational map, $z \in \mathbb{C}$, $k \in \mathbb{N}$, and τ of $\mathrm{Deck}(f^k)$. Then for each nonzero element $a_{i,f^k}(z)$ of the degree partition, τ acts as a permutation on the set of points in the fiber $\rho_{f^k}(z)$ that map forward under f^k with local degree d^i.*

Lemma 3.5. *Let f be a bicritical rational map of degree d, let p be a prime number that does not divide d, and let k be any natural number. Suppose there exists some element τ of $\mathrm{Deck}(f^k)$ that has order p. Then for any point $z \in \hat{\mathbb{C}}$, there exists some element $a_{i,f^k}(z)$ of the degree partition that is not a multiple of p.*

Proof. Because both critical points of a bicritical, degree d rational map have local degree d, and the total degree of f^k is d^k, we immediately have that for any point $z \in \hat{\mathbb{C}}$,

$$d^k = \sum_{i=0}^k a_{i,f^k}(z) \cdot d^i.$$

If every $a_{i,f^k}(z)$ was a multiple of p, then the equation above would imply that d^k is a multiple of p, which is a contradiction. $\square$

The following is the key observation.

Proposition 3.6. *Let f be a bicritical rational map of degree d, and let p be a prime number that does not divide d. Then for all natural numbers k, the deck group $\mathrm{Deck}(f^k)$ has no element of order p.*

Proof. Suppose, for a contradiction, that some element τ of $\mathrm{Deck}(f^k)$ has order p. Consider any point $z \in \hat{\mathbb{C}}$. By Lemma 3.5, there exists some j such that $a_{j,f^k}(z)$ is not a multiple of p. By Lemma 3.4, τ acts as a permutation on the set, call it S, of the $a_{j,f^k}(z)$ many points in the fiber $\rho_{r^k}(z)$ whose local degree under f^k is $a_{j,f^k}(z)$. Since the group generated by τ, $\langle \tau \rangle$ is a cyclic group of prime order p, the Orbit Stabilizer Theorem gives that the cardinality of the orbit under $\langle \tau \rangle$ of any point in $\mathbb{C}$ equals either 1 or p. Since $|S| = a_{j,f^k}(z)$ is not divisible by p, it follows that S contains at least one point that is fixed by τ. Since the point z was arbitrary, this shows that every fiber contains at least one fixed point of τ. But since a non-identity Möbius transformation can have at most 3 fixed points, this is a contradiction. $\square$

Corollary 3.7. *Let f be a bicritical rational map of degree $d \leq 5$ and let $k \in \mathbb{N}$. Then $\mathrm{Deck}(f^k)$ is either cyclic or dihedral.*

Proof. By Proposition 2.3 part (vii), $\mathrm{Deck}(f^k)$ is either cyclic, dihedral, A_4, A_5 or S_4. Each of the polyhedral groups A_4, A_5 and S_4 have elements of order 2 and elements of order 3, and at least one of 2 and 3 does not divide d for each of choice of d in $\{2, \ldots, 5\}$. Thus, Proposition 3.6 implies $\mathrm{Deck}(f^k)$ cannot be isomorphic to any of the polyhedral groups. $\square$

We can now prove Theorem 1.3.

Proof of Theorem 1.3. Corollary 3.7 gives the result in the case that degree of f is ≤ 5, and Corollary 3.2 gives the result for degree ≥ 6. If the degree of f is odd, then by Proposition 3.6, $\mathrm{Deck}(f^k)$ cannot contain any elements of order 2 and so cannot be dihedral. $\square$

Remark 3.8. *The authors wish to thank the anonymous referee for pointing out that the arguments from Definition 3.3 up to Proposition 3.6 hold for a more general class of functions. Indeed, we only require that f be a rational map such that all critical points of f have local degree of the form d^k for some integer d. This in turn implies that the conclusion of Theorem 1.3 holds for this more general case, at least for $d \leq 5$. However, since this paper is focused on the behavior of bicritical rational maps, we only state the theorem for bicritical maps.*

4. DETECTING CRITICAL POINTS AND VALUES OF BICRITICAL MAPS OF DEGREE $d \geq 3$ FROM THEIR ITERATES

We begin with a definition.

Definition 4.1. *Let f be a rational map with critical point set $\mathcal{C}_f$ (respectively critical value set $\mathcal{V}_f$). We will say that we can* detect *the set $\mathcal{C}_f$ (respectively $\mathcal{V}_f$) from f^k if whenever $f^k = g^k$ for some bicritical rational map g we have $\mathcal{C}_f = \mathcal{C}_g$ (respectively $\mathcal{V}_F = \mathcal{V}_G$).*

The idea behind this definition is that knowledge of f^k provides enough information for us to be able to recover the sets $\mathcal{C}_f$ and $\mathcal{V}_f$. We will show that if f is a bicritical rational map and $k \geq 1$, then we can always detect $\mathcal{C}_f$ and $\mathcal{V}_f$ from f^k. In this section we prove the following.

Theorem 4.2. *Fix a rational map F. If there exists a bicritical rational map f of degree ≥ 3 and $k \in \mathbb{N}$ such that $f^k = F$, then we can detect the sets $\mathcal{C}_f$ and $\mathcal{V}_f$ from F. Specifically, $\mathcal{C}(f)$ is the set of fixed points of any element of $\mathrm{Deck}(F)$ of order at least 3, and $\mathcal{V}_f = \{x \in \mathbb{C} \mid F^{-1}(x) \subseteq \mathcal{C}_F\}$.*

Before proceeding, we need a well-known result about commuting Möbius transformations (see e.g. [1], Theorem 4.3.6). We will use it throughout this paper.

Lemma 4.3. *Let ϕ and ψ be non-identity Möbius transformations with fixed point sets $\mathrm{Fix}(\phi)$ and $\mathrm{Fix}(\psi)$ respectively. Then the following are equivalent.*

 (i) $\phi \circ \psi = \psi \circ \phi$
 (ii) $\phi(\mathrm{Fix}(\psi)) = \mathrm{Fix}(\psi)$ *and* $\psi(\mathrm{Fix}(\phi)) = \mathrm{Fix}(\phi)$.
 (iii) *Either* $\mathrm{Fix}(\psi) = \mathrm{Fix}(\phi)$ *or* ϕ, ψ *and* $\phi \circ \psi$ *are involutions and* $\mathrm{Fix}(\phi) \cap \mathrm{Fix}(\psi) = \varnothing$.

Now we need a simple observation about finite cyclic groups of Möbius transformations.

Lemma 4.4. *Let G be a finite cyclic group of Möbius transformations. Then there exist two distinct points x_1 and x_2 in $\hat{\mathbb{C}}$ such that every nonidentity element of g is an elliptic rotation around the axis in $\mathbb{H}^3$ connecting x_1 and x_2.*

Proof. Recall that the only Möbius transformations of finite order are elliptic (see e.g [1]). By Lemma 4.3, two nonidentity Möbius transformations commute if and only if they have the same set of fixed points or are commuting involutions each interchanging the fixed points of the other. If $|G| = 2$, we are done. So suppose $|G| \geq 3$ and let $g \in G$ be an element of order ≥ 3; then since g commutes with every element of G, it must have the same set of fixed points as every nonidentity element of G. $\square$

The proof of Theorem 4.2 now follows from the following two lemmas.

Lemma 4.5. *Fix a rational map F. Suppose there exists at least one bicritical map f of degree $d \geq 3$ and integer $k \in \mathbb{N}$ such that $f^k = F$. Then all elements of $\mathrm{Deck}(F)$ of order ≥ 3 have the same set C of fixed points and $\mathcal{C}_f = C$.*

Proof. By Theorem 1.3, $\mathrm{Deck}(F)$ is cyclic or dihedral. By Lemma 3.1, the order d elliptic rotation around the axis connecting the two critical points of f is an element of $\mathrm{Deck}(f)$. Hence it is also an element of $\mathrm{Deck}(F)$ by Proposition 2.3 part (ii). But all elements of F that have order ≥ 3 are elliptic rotations that share the same set of fixed points by Lemma 4.4 Hence $\mathcal{C}(f)$ is the set of two fixed points of any element of $\mathrm{Deck}(F)$ of order at least 3. $\square$

Lemma 4.6. *Let f be a bicritical rational map of degree $d \geq 3$ and fix any integer $k \in \mathbb{N}$. Then $x \in \mathcal{V}_f$ if and only if $f^{-k}(x) \subseteq \mathcal{C}_{f^k}$.*

Proof. If $x \in \mathcal{V}_f$ then we have $f^{-k}(x) \in f^{-(k-1)}(\mathcal{C}_f) \subseteq \mathcal{C}_{f^{-k}}$.

Conversely suppose $x \notin \mathcal{V}_f$. We will inductively construct a sequence $x_0, x_1, \ldots, x_k$ with $x = x_0$ and such that $x_{i-1} = f(x_i)$ for each i and no x_i is a critical value of f. Since no x_i is a critical value of f, it follows that no x_i can be a critical point of f.

Since the degree of f is $d \geq 3$ and x_0 is not a critical value, there exists $x_1 \in f^{-1}(x_0)$ such that $x_1 \notin \mathcal{V}_f$. Inductively, suppose that x_i $(1 \leq i \leq k-1)$ is not a critical value. Then there exists $x_{i+1} \in f^{-1}(x_i)$ such that $x_{i+1} \notin \mathcal{V}_f$. Now consider x_k. It is clear that $x_k \in f^{-k}(x)$. Furthermore, we claim $x_k \notin \mathcal{C}_{f^k}$. The local degree of f^k at x_k is equal to the products of the local degrees of F at x_i. Since by construction the local degree of f at each x_i is equal to 1, we see that the local degree of f^k at x_k is 1. Thus x_k is not a critical point of f^k. $\square$

Proof of Theorem 4.2. The claim for $\mathcal{C}_f$ is contained in Lemma 4.5 and the claim for $\mathcal{V}_f$ is Lemma 4.6. $\square$

5. Deck groups of iterates of quadratic rational maps

Detecting $\mathcal{C}_f$ and $\mathcal{V}_f$ in the degree 2 case is more difficult than the general higher degree bicritical case. One reason is that the conclusion of Lemma 4.6 is not true in general for quadratic rational maps, due to what we call *critically coalescing* maps.

Definition 5.1. *We will say a quadratic rational map f is critically coalescing if the two critical values of f share a common image. In other words, denoting the critical values of f by v_1 and v_2, we have $f(v_1) = f(v_2)$.*

We will need the following observation, which will be refined later on.

Lemma 5.2. *Let f be a critically coalescing quadratic rational map. Then for all $k \geq 2$, $x \in \mathcal{V}_f \cup \{f(v_1) = f(v_2)\}$ if and only if $f^{-k}(x) \subseteq \mathcal{C}_{f^k}$.*

Proof. Let $\beta = f(v_1) = f(v_2)$. It is simple to see that if $x \in \mathcal{V}_f \cup \{\beta\}$ then $f^{-k}(x) \subseteq \mathcal{C}_{f^k}$. So suppose $x \notin \mathcal{V}_f \cup \{\beta\}$. We will construct a sequence $x_0, \ldots, x_k$ with $x = x_0$ and $f(x_j) = x_{j-1}$. So set $x = x_0$. In particular, since $x \neq \beta$, then neither element of $f^{-1}(x_0)$ is a critical value of f. Furthermore, at most one preimage can be equal to β. Thus there exists $x_1 \in f^{-1}(x_0)$ such that $x_1 \notin \mathcal{V}_f \cup \{\beta\}$. Now we can inductively find $x_2, x_3, \ldots, x_k$ such that $x_j \notin \mathcal{V}_f \cup \{\beta\}$ for all $j \leq k$ by the same reasoning. As with the proof of Lemma 4.6, we can conclude that $x_k \in f^{-k}(x)$ but $x_k \notin \mathcal{C}_{f^k}$. $\square$

In the case where f is a bicritical map of degree $d \geq 3$, we were able to detect the sets $\mathcal{C}_f$ and $\mathcal{V}_f$ by exploiting the facts that $\mathrm{Deck}(f^k)$ contains elements of order $d \geq 3$ and all such elements necessarily fixed the critical points of f pointwise. The case $d = 2$ and $\mathrm{Deck}(f^k) \cong V_4$ is harder because no deck group elements of order at least 3 exist, and elements of order $d = 2$ do not necessarily fix the critical points pointwise. The aim of the next three sections is to prove the following theorem, an analog to Lemma 4.5. The definition of *special pairs* used in case (ii)(a) is given in Definition 5.5.

Theorem 5.3. *Let f be a quadratic rational map. Then we can detect the critical points of f from f^k $(k > 1)$. Specifically:*

(i) *If f is not critically coalescing, then $\mathrm{Deck}(f^k)$ is cyclic. In particular, either*
    (a) *f is a power map and $\mathrm{Deck}(f^k)$ is isomorphic to $\mathbb{Z}_{2^k}$ so that the critical points of f are the fixed points of any element μ which generates $\mathrm{Deck}(f^k)$.*
    (b) *$\mathrm{Deck}(f^k) \cong \mathbb{Z}_2$, and the critical points of f are the fixed points of the unique non-identity element of $\mathrm{Deck}(f^k)$.*
(ii) *If f is critically coalescing and not conjugate to $z \mapsto \frac{z^2-1}{z^2+1}$ then $\mathrm{Deck}(f^k) \cong V_4$ for all $k \geq 2$. Furthermore:*
    (a) *If the forward orbit of the critical values does not contain a fixed point, then the image under f^k of the critical points of f is distinct from the image under f^k of the elements of the other special pairs of f.*
    (b) *If the forward orbit of the critical values does contain a fixed point α, then the critical points of f are the fixed points of μ, the unique element of $\mathrm{Deck}(f^k)$ for which $\mu(\alpha) = \beta$, where β is the unique element of $f^{-1}(\alpha)$ distinct from α.*
(iii) *If f is conjugate to $z \mapsto \frac{z^2-1}{z^2+1}$, then $\mathrm{Deck}(f^2) \cong V_4$ but $\mathrm{Deck}(f^k) \cong D_8$ for all $k \geq 3$ and, as in case (i), the critical points of f are the fixed points of any element of order 4 in $\mathrm{Deck}(f^k)$.*

We will prove Theorem 5.3 at the end of Section 7. Here, we prove that Theorem 5.3 implies Theorem 1.4.

Proof of Theorem 1.4. (Assuming Theorem 5.3) If f is not critically coalescing then, by Theorem 5.3, $\mathrm{Deck}(f^k)$ is cyclic of order 2^n for some n. Moreover, $\mathrm{Deck}(f^k) \cong \mathbb{Z}_2$ for all $k \geq 1$ if and only if f is neither a power map nor critically coalescing. If f is critically coalescing, then Theorem 5.3 asserts that $\mathrm{Deck}(f^k) \cong V_4$ for all $k \geq 2$, unless f is conjugate to $z \mapsto \frac{z^2-1}{z^2+1}$, in which case $\mathrm{Deck}(f^k) \cong D_8$ for all $k \geq 3$. $\square$

The proof of Theorem 5.3 requires studying the groups of deck transformations for iterates of quadratic rational maps; this is the goal of the present section. In the next two sections we will use the obtained results to detect the critical points of quadratic rational maps, thus enabling us to prove the theorem. Proposition 5.10 will show that $\mathrm{Deck}(f^k) \cong V_4$ precisely in the critically coalescing case.

Lemma 5.4. *Let $G \cong V_4$ be a group of Möbius transformations acting on $\hat{\mathbb{C}}$. Then there are precisely 6 points in $\hat{\mathbb{C}}$ that are fixed pointwise by some non-identity element of G; each of the three non-identity elements of G fixes a pair of these points.*

Proof. It is easy to construct a group $G \cong V_4$ of Möbius transformations that satisfies the conclusion. Then the fact that every $G \cong V_4$ satisfies the conclusion follows from the well-known fact (see e.g. [1]) that finite groups of Möbius transformations are isomorphic if and only if they are conjugate. $\square$

These pairs will play an important role in our strategy for detecting $\mathcal{C}_f$ and $\mathcal{V}_f$.

Definition 5.5. *For $G \cong V_4$ a group of Möbius transformations acting on $\hat{\mathbb{C}}$, the* special pairs *are the three pairs of points in $\hat{\mathbb{C}}$ defined by Lemma 5.4.*

We will give a characterization of the special pairs in Proposition 5.11.

5.1. **Characterizing when** $\mathrm{Deck}(f^k) \cong V_4$. In order to prove Theorem 5.3, we need to investigate exactly when we have $\mathrm{Deck}(f^k) \cong V_4$.

We begin with a few preliminary lemmas. We will use the notation $\mathrm{Fix}(\phi)$ to denote the set of fixed points of a map ϕ.

Lemma 5.6. *Let f be a quadratic rational map so that $\mathrm{Deck}(f^k) \cong V_4$ for some iterate $k \in \mathbb{N}$. Let $\phi \in \mathrm{Deck}(f^k)$ be the generator of $\mathrm{Deck}(f)$. Then $\mathrm{Fix}(\phi) = \mathcal{C}_f$, and every element of $\mathrm{Deck}(f^k)$ maps the set $\mathrm{Fix}(\phi)$ to itself.*

Proof. Write $\mathrm{Deck}(f^k) = \{\eta, \psi, \phi, \mathrm{id}\}$ with $\mathrm{Deck}(f) = \{\phi, \mathrm{id}\}$. Since ϕ is a deck transformation of f, it preserves fibers of f (by Proposition 2.3). Because f is quadratic, the fiber over each critical value contains exactly one point (a critical point of f). Therefore, ϕ must fix each critical point of f. Since ϕ can have at most two fixed points, this implies $\mathrm{Fix}(\phi) = \mathcal{C}_f$. Write $\mathrm{Fix}(\phi) = \{c_1, c_2\}$, and consider how the elements ψ and η act on this set. Because $\mathrm{Deck}(f^k) \cong V_4$, we have $\psi = \eta \circ \phi$ and $\eta^2 = \psi^2 = \phi^2 = \mathrm{id}$. Hence,

$$x := \psi(c_1) = \eta(c_1), \quad \psi(x) = \eta(x) = c_1, \quad \text{and}$$
$$y := \psi(c_2) = \eta(c_2), \quad \psi(y) = \eta(y) = c_2.$$

So the involutions ψ and η coincide on the set $\{x, y, c_1, c_2\}$. Since η and ψ are distinct Möbius transformations, we must have $x, y \in \{c_1, c_2\}$. The points x and y are distinct, so there are two possibilities:

$$x = c_1 \text{ and } y = c_2 \qquad \text{or} \qquad x = c_2 \text{ and } y = c_1.$$

In the first case, $\mathrm{Fix}(\phi) = \mathrm{Fix}(\psi) = \mathrm{Fix}(\eta)$. But then ϕ, ψ, and η would be three nontrivial involutions with the same pair of fixed points, so they would all coincide, which is not possible. In the second case, c_1 and c_2 comprise a common 2-cycle for the elements η and ψ. In particular, we see that η and ψ map $\mathrm{Fix}(\phi)$ to itself. $\qquad\square$

Lemma 5.7. *Let f be a quadratic rational map. If there exists $k \in \mathbb{N}$ such that $\mathrm{Deck}(f^k) \cong V_4$, then the minimal such k is $k = 2$.*

Proof. Suppose that k is minimal so that $\mathrm{Deck}(f^k) \cong V_4$; note that $k > 1$. Write $\mathrm{Deck}(f^k) = \{\eta, \psi, \phi, \mathrm{id}\}$ with $\mathrm{Deck}(f) = \{\phi, \mathrm{id}\}$. By Lemma 5.6, η maps $\mathcal{C}_f$ to itself, so by Lemma 2.6, there is a Möbius transformation μ so that the top square in the following diagram commutes.

$$
\begin{array}{ccc}
\hat{\mathbb{C}} & \xrightarrow{\ \eta\ } & \hat{\mathbb{C}} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\
\hat{\mathbb{C}} & \xrightarrow{\ \mu\ } & \hat{\mathbb{C}} \\
{\scriptstyle f^{k-1}}\downarrow & & \downarrow{\scriptstyle f^{k-1}} \\
\hat{\mathbb{C}} & \xrightarrow{\ \mathrm{id}\ } & \hat{\mathbb{C}}
\end{array}
$$

As we saw above, this means that $\mu \in \mathrm{Deck}(f^{k-1})$. Note that since k is minimal with respect to the property that $\mathrm{Deck}(f^k) \cong V_4$, we must have $\mathrm{Deck}(f^{k-1}) = \mathrm{Deck}(f)$, so $\mu \in \mathrm{Deck}(f)$, and $f = f \circ \mu$. But then the outer rectangle commutes for $f^{k-1} = f$, or $k = 2$ as desired. $\square$

We now show that, with one (up to conjugacy) exception, if $\mathrm{Deck}(f^2) \cong V_4$ then $\mathrm{Deck}(f^k) \cong V_4$ for all $k \geq 2$.

Lemma 5.8. *Let f be a critically coalescing quadratic rational map. Then $\mathcal{V}_f \cup \mathcal{C}_f$ consists of 4 distinct points.*

Proof. Write $\mathcal{V}_f = \{v_1, v_2\}$ and $\mathcal{C}_f = \{c_1, c_2\}$. All bicritical rational maps satisfy $c_1 \neq c_2$ and $v_1 \neq v_2$ ([12]). Suppose, for a contradiction, that $c_1 = v_j$ for $j = 1$ or 2. Then $f(c_1) = f(v_j) = f(v_{3-j})$. But since c_1 maps forward with local degree 2, we see that v_j has at least three preimages (counting multiplicity) under f which is impossible since f is quadratic. $\square$

Lemma 5.9. *Let f be a quadratic rational map. Then $\mathrm{Deck}(f^2) \cong \mathbb{Z}_4$ if and only if f is a power map.*

Proof. It is clear that if f is a power map then $\mathrm{Deck}(f^2) \cong \mathbb{Z}_4$. So suppose $\mathrm{Deck}(f^2) \cong \mathbb{Z}_4$, and let $\phi \in \mathrm{Deck}(f^2)$ have order 4. We see that $\mathrm{Fix}(\phi) = \mathcal{C}_f = \{c_1, c_2\}$, and so by Lemma 2.6 there exists μ such that $\mu \circ f = f \circ \phi$, and $\mu(\mathcal{V}_f) = \mathcal{V}_f$. As in the argument of Lemma 5.7, we see that $\mu \in \mathrm{Deck}(f)$, and since ϕ is not an element of $\mathrm{Deck}(f)$, it follows that μ is the unique order 2 element of $\mathrm{Deck}(f)$. In particular, $\mathrm{Fix}(\mu) = \mathcal{C}_f$. Denoting $v_i = f(c_i)$, we observe that

$$v_i = f(c_i) = f \circ \phi(c_i) = \mu \circ f(c_i) = \mu(v_i)$$

and so $\mathcal{C}_f = \mathrm{Fix}(\mu) = \mathcal{V}_f$. Thus f is a power map. $\square$

It follows that if f is not a power map, then $\mathrm{Deck}(f^2)$ must be isomorphic to either $\mathbb{Z}_2$ or V_4. In either case, every non-identity element of $\mathrm{Deck}(f^2)$ has order 2.

Proposition 5.10. *Let f be a quadratic rational map that is not a power map. Then the following are equivalent:*
   (i) $\mathrm{Deck}(f^2) \cong V_4$,
   (ii) $\mathrm{Deck}(f^k) \cong V_4$ *for some* $k \in \mathbb{N}$,
   (iii) f *is critically coalescing.*

Proof. Lemma 5.7 gives (i) if and only if (ii). We will prove the equivalence of conditions (i) and (iii). Suppose $\mathrm{Deck}(f^2) \cong V_4$. Then $\mathrm{Deck}(f)$ is a cyclic group of order 2 generated by a rotation μ about the axis connecting the critical points of f (by Lemma 3.1). Consider any element $\tau \in \mathrm{Deck}(f^2) \setminus \mathrm{Deck}(f)$. We claim that the set $\mathcal{C}_f$ must be fixed by τ. Indeed, since V_4 is abelian, we see that τ commutes with μ, the unique non-identity element of $\mathrm{Deck}(f)$. Since $\mathrm{Fix}(\mu) = \mathcal{C}_f$, it follows from Lemma 4.3 that $\tau(\mathcal{C}_f) = \mathcal{C}_f$.

If τ fixed $\mathcal{C}_f$ pointwise, then τ would coincide with the generator of $\mathrm{Deck}(f)$, a contradiction. Hence τ must interchange the two points of $\mathcal{C}_f$. Then Proposition 2.3 part (iv) implies the two critical points of f belong to the same fiber under f^2, i.e. f is critically coalescing.

Now suppose f is critically coalescing. By Lemma 5.8, the critical points and values of f, which we will denote c_1, c_2 and v_1, v_2 respectively, are all

distinct. We may therefore normalize f so that $c_1 = 0$, $c_2 = \infty$ and $v_2 = 1$, which means that f belongs to the one-parameter family given by

$$(1) \qquad\qquad f_a(z) = \frac{z^2 - a}{z^2 + a}.$$

As the reader may verify, the maps $z \mapsto -z$ and $z \mapsto \frac{a}{z}$ belong to $\mathrm{Deck}(f^2)$ and they generate a subgroup isomorphic to V_4. Therefore, by Lemma 5.9 and Proposition 2.3 part (vii), $\mathrm{Deck}(f^2) \cong V_4$. $\qquad\square$

Proposition 5.11. *Let f be a critically coalescing quadratic rational map with critical points c_1 and c_2. Then the special pairs of f are the sets $\mathcal{C}_f$, $f^{-1}(c_1)$ and $f^{-1}(c_2)$.*

Proof. Denote $f^{-1}(c_1) = \{a_1, a_2\}$, $f^{-1}(c_2) = \{b_1, b_2\}$. From Proposition 5.10, we have $\mathrm{Deck}(f^2) \cong V_4$. We know that the elements of $\mathcal{C}_f$ are fixed by the unique non-identity element $\mu \in \mathrm{Deck}(f) \subset \mathrm{Deck}(f^2)$. Now observe that $f^{-2}(v_1) = \{a_1, a_2\}$. Since elements of $\mathrm{Deck}(f^2)$ preserve the fibers under f^2, we see that orbit of a_1 under the action of $\mathrm{Deck}(f^2)$ contains at most two elements. By the Orbit-Stabilizer Theorem, the stabilizer of a_1 must contain a non-identity element ν of $\mathrm{Deck}(f^k)$. In this case we must have $\nu(a_2) = a_2$, so the fixed points of ν are precisely the elements of $f^{-1}(c_1)$. The case for the elements of $f^{-1}(c_2)$ is similar. $\qquad\square$

We complete this subsection by showing that if f is a quadratic rational map such that $\mathrm{Deck}(f^k)$ is dihedral for some k, then f is critically coalescing. First we strengthen the result of Lemma 5.9.

Lemma 5.12. *Let f be a quadratic rational map. Then $\mathrm{Deck}(f^k)$ is a cyclic group of order greater than 2 if and only if f is a power map.*

Proof. It is clear that if f is a power map then $\mathrm{Deck}(f^k)$ is a cyclic group of order 2^k. Now suppose that k is minimal such that $\mathrm{Deck}(f^k)$ is a cyclic group of order 2^n for some $n > 1$. Let ν be a generator of $\mathrm{Deck}(f^k)$. Then $\mu = \nu^{2^{n-1}}$ has order 2 and so belongs to $\mathrm{Deck}(f)$. In particular we have $\mathrm{Fix}(\phi) = \mathcal{C}_f$ for all non-identity elements ϕ in $\mathrm{Deck}(f^k)$. Similarly, since $\mathrm{Deck}(f^j)$ is cyclic for all $1 \le j \le k$, we may repeatedly apply Corollary 2.7 to see that there exists $\phi_1 \in \mathrm{Deck}(f)$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\hat{\mathbb{C}} & \xrightarrow{\ \nu\ } & \hat{\mathbb{C}} \\
{\scriptstyle f^{k-1}}\downarrow & & \downarrow{\scriptstyle f^{k-1}} \\
\hat{\mathbb{C}} & \xrightarrow{\ \phi_1\ } & \hat{\mathbb{C}} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\
\hat{\mathbb{C}} & \xrightarrow{\ \mathrm{id}\ } & \hat{\mathbb{C}}
\end{array}
$$

We claim that $\phi_1 = \mu$. Otherwise, $\phi_1 = \mathrm{id}$, and so $\nu \in \mathrm{Deck}(f^{k-1})$, contradicting the minimality of k. Thus by Corollary 2.7 again, we see that μ must map $\mathcal{V}_f$ to itself. Let $\mathcal{V}_f = \{v_1, v_2\}$. If μ interchanges the elements of $\mathcal{V}_f$, then we would have $f(v_1) = f(v_2)$, and so by Proposition 5.10, $\mathrm{Deck}(f^k)$ would contain a subgroup isomorphic to V_4; a contradiction. So we see that μ must fix the elements of $\mathcal{V}_f$ pointwise. Since $\mu \in \mathrm{Deck}(f)$, the fixed points

of μ are also the critical points of f. Hence $\mathcal{C}_f = \mathcal{V}_f$, and f is a power map. $\qquad\square$

The following strengthens the result of Lemma 5.7.

Proposition 5.13. *Let f be a quadratic rational map. Then if $\mathrm{Deck}(f^k)$ is dihedral for some k, then f is critically coalescing.*

Proof. Write $\mathrm{Deck}(f) = \{\mathrm{id}, \mu\}$ and suppose $k > 1$ is minimal such that $\mathrm{Deck}(f^k)$ is dihedral. Let Γ be a subgroup of $\mathrm{Deck}(f^k)$ such that $\Gamma \cong V_4$ and $\mathrm{Deck}(f) \subseteq \Gamma$. Write $\Gamma = \{\mathrm{id}, \mu, \alpha, \beta\}$. Since Γ is abelian and $\mathrm{Fix}(\mu) = \mathcal{C}_f$, then $\alpha(\mathcal{C}_f) = \beta(\mathcal{C}_f) = \mathcal{C}_f$. Thus, by Lemma 2.6 and the subsequent Corollary 2.7, there exists $\nu \in \mathrm{Deck}(f^{k-1})$ such that $\nu \circ f = f \circ \alpha$. Furthermore, $\nu(\mathcal{V}_f) = \mathcal{V}_f = \{v_1, v_2\}$, and since α is not an element of $\mathrm{Deck}(f)$, we have $\nu \neq \mathrm{id}$. By the assumption on the minimality of k, $\mathrm{Deck}(f^{k-1})$ must be cyclic, and so for any non-identity elements $\gamma \in \mathrm{Deck}(f^{k-1})$ we have $\mathrm{Fix}(\gamma) = \mathrm{Fix}(\mu) = \mathcal{C}_f$. Thus $\mathrm{Fix}(\nu) = \mathcal{C}_f$.

If ν fixes the elements of $\mathcal{V}_f$ pointwise, then we have $\mathcal{C}_f = \mathcal{V}_f$, and so f is a power map. But this is impossible, since $\mathrm{Deck}(f^k)$ is always cyclic for power maps. So ν must be an involution which exchanges the elements of $\mathcal{V}_f$. But since $\mathrm{Deck}(f^{k-1})$ is cyclic, it contains a unique involution, namely μ, and so $\nu = \mu$. Thus $\mu \in \mathrm{Deck}(f)$ interchanges the elements of $\mathcal{V}_f$, and so $f(v_1) = f(v_2)$. $\qquad\square$

5.2. Remarks on Critically Coalescing Quadratic Rational Maps.
Consider the family f_a (see Figure 2) from equation (1). The authors have not been able to find any reference to this family in the literature. Accordingly, we prove some preliminary results about this family here, and leave a more detailed investigation for future study. Note that the maps f_a and $f_{-1/a}$ are conjugate via the map $\phi(z) \mapsto -1/z$.
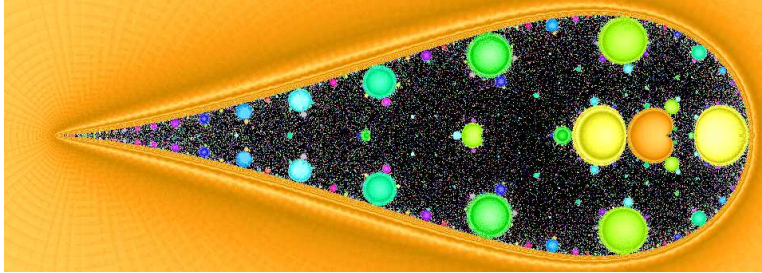

FIGURE 2. The parameter space of the family $f_a(z) = \frac{z^2-a}{z^2+a}$, characterized by the equivalent conditions that $\mathrm{Deck}(f^2) \cong V_4$ and $f(v_1) = f(v_2)$. The colors of the hyperbolic components represent the period of the attracting orbit of the map. For example, orange is period 1, yellow is period 2.

For the moment we show that for all $a \neq \pm 1$, we have $\mathrm{Deck}(f_a^k) \cong V_4$ for all $k \geq 2$. First we need a special case of a result of Pakovich ([14], Theorem 5.2). This result can also be obtained, using different techniques, by using the results of [10]. Given a rational map F, we define $\mathrm{Deck}_\infty(F) = \bigcup_{k=1}^{\infty} \mathrm{Deck}(F^k)$.

Proposition 5.14 ([14])**.** *Let F be a quadratic rational map. If $\sigma \in$ $\mathrm{Deck}_\infty(F)$ then $F \circ \sigma = \beta \circ F \circ \beta^{-1}$ for some Möbius map β.*

Proposition 5.15. *Let $f_a(z) = \frac{z^2-a}{z^2+a}$.*
  (i) *If $a = \pm 1$ then $\mathrm{Deck}(f^2) \cong V_4$ and $\mathrm{Deck}(f^k) \cong D_8$ for all $k \geq 3$.*
  (ii) *If $a \neq \pm 1$, then $\mathrm{Deck}(f_a^k) \cong V_4$ for all $k \geq 2$.*

Proof.
  (i) This was proven by Pakovich in [14]. Our argument to prove the second part is follows the method of Pakovich.
  (ii) We know that $\mathrm{Deck}(f_a^2) = \{\mathrm{id}, -z, a/z, -a/z\} \cong V_4$. Since we know that $\mathrm{Deck}(f_a^k)$ cannot be a polyhedral group, then if $\mathrm{Deck}(f_a^k)$ $(k > 2)$ is not isomorphic to V_4, it must be isomorphic to a dihedral group. Such a dihedral group must contain an element of order greater than 2 which, by Lemma 4.5, fixes the critical points 0 and ∞, and so this element must be of the form $\sigma(z) = cz$ for some $c \in \mathbb{C}^*$. By Proposition 5.14, we need to show that if $\sigma(z) = cz$ and

$$(2) \qquad\qquad f_a \circ \sigma = \beta \circ f_a \circ \beta^{-1}$$

for some Möbius map β then $c = \pm 1$. By equation (2), since both sides of the equation have the same critical points, any β satisfying the equation must be of the form $\beta(z) = dz^{\pm 1}$. If $\beta(z) = dz$, then (2) becomes

$$\frac{c^2z^2 - a}{c^2z^2 + a} = \frac{1}{d}\frac{d^2z^2 - a}{d^2z^2 + a}$$

which is solved by $c = \pm 1$ and $d = 1$. On the other hand, if $\beta = d/z$, then (2) now becomes

$$\frac{c^2z^2 - a}{c^2z^2 + a} = d\frac{ad^2 + z^2}{ad^2 - z^2},$$

and this is solved by $d = -1$ and $c = \pm ai$.

To complete the proof, observe that if $\sigma(z) = aiz$ is an element of $\mathrm{Deck}_\infty(F)$, then so is $\sigma^2(z) = -a^2z$. But the above computations show that this would mean $-a^2 \in \{1, -1, ai, -ai\}$. This yields the possibilities $a \in \{i, -i, 1, -1\}$. Since by assumption $a \neq \pm 1$ we are left with the case $a = \pm i$. But then we would have $ai = \mp 1$, means $\sigma(z) = \pm z$.

$\square$

We note that no hyperbolic map of the form f_a can be a mating (see e.g [13, 16]); a hyperbolic map which is a mating has to have disjoint critical orbits, and so the condition $f(v_1) = f(v_2)$ is incompatible with this requirement. However, there do exist matings in this family. For example, the map f_i is equal to the self-mating of the quadratic polynomial which is the landing point of the parameter ray of argument $1/4$ in the Mandelbrot set.

6. DETECTING CRITICAL POINTS AND VALUES OF NON-CRITICALLY COALESCING QUADRATIC MAPS

Before addressing the subtler critically coalescing (or, equivalently by Proposition 5.10, $\mathrm{Deck}(f^k) \cong V_4$) case, we briefly show that the techniques

of Section 4 can be used to detect $\mathcal{C}_f$ and $\mathcal{V}_f$ in the non-critically coalescing quadratic case. First we show we can detect the critical points of f.

Lemma 6.1. *Let f be a quadratic rational map which is not critically coalescing and $k \geq 1$. Then $\mathrm{Deck}(f^k)$ contains an element of order greater than 2 or $\mathrm{Deck}(f^k) = \mathrm{Deck}(f) \cong \mathbb{Z}_2$.*

Proof. Suppose that all elements of $\mathrm{Deck}(f^k)$ have order 1 or 2. If $\mathrm{Deck}(f^k) \cong V_4$ then by Proposition 5.10 f would be critically coalescing, which is a contradiction. Thus $\mathrm{Deck}(f^k) \cong \mathbb{Z}_2$. $\qquad\square$

Corollary 6.2. *Let f be a quadratic rational map which is not critically coalescing and $k \geq 1$. Then we can detect the critical points of f from $\mathcal{C}_f$.*

Proof. By the lemma, either $\mathrm{Deck}(f^k)$ contains an element of order greater than 2 or $\mathrm{Deck}(f^k) \cong \mathbb{Z}_2$. If $\mathrm{Deck}(f^k)$ contains an element μ of order greater than 2, then by Lemma 5.12 and Proposition 5.13, f is a power map and the critical points of f are fixed by every non-identity element of $\mathrm{Deck}(f^k)$. On the other hand if $\mathrm{Deck}(f^k) \cong \mathbb{Z}_2$ then $\mathrm{Deck}(f^k) = \mathrm{Deck}(f)$ and so the critical points of f are fixed by the unique non-identity element of $\mathrm{Deck}(f^k)$. $\quad\square$

We can also easily detect the critical values in the non-critically coalescing case.

Lemma 6.3. *Let f be a quadratic rational map that is not critically coalescing. Then for each $k \in \mathbb{N}$, $x \in \mathcal{V}_f$ if and only if $f^{-k}(x) \subseteq \mathcal{C}_{f^k}$.*

Proof. The argument proceeds similarly to that for Lemma 4.6. This time, we notice that since f is not critically coalescing, then for any $y \notin \mathcal{V}_f$, there exists an element $y' \in f^{-1}(y)$ which is not a critical value of f. Thus as before we may construct a sequence $x_0 = x, \dots, x_k$, so that $x_k \in f^{-k}(x)$ but x_k is not a critical point of f^k. $\qquad\square$

Lemma 6.3 has the following immediate corollary.

Corollary 6.4. *Fix a rational map F. If there exists a quadratic rational map f that is not critically coalescing such that $f^k = F$ for some $k \in \mathbb{N}$, then $\mathcal{V}_f = \{x \in \hat{\mathbb{C}} \mid F^{-1}(x) \subseteq \mathcal{C}_F\}$. In particular we can detect the critical values of f from f^k.*

7. Detecting Critical Points and Critical Values for Critically Coalescing quadratic rational maps

In this section we discuss how we may detect $\mathcal{C}_f$ and $\mathcal{V}_f$ in the case where $\mathrm{Deck}(f^k) \cong V_4$. Along with the results in the previous section, this will allow us to complete the proof of Theorem 5.3.

Suppose f is critically coalescing. By Proposition 5.11, there exist three special pairs

$$f^{-1}(c_1) := \{a_1, a_2\}, \quad f^{-1}(c_2) := \{b_1, b_2\}$$

and the true critical points $\mathcal{C}_f = \{c_1, c_2\}$, which are the fixed points of some non-identity element of $\mathrm{Deck}(f^k)$. *A priori* we cannot distinguish these pairs from one another, but we do know the true critical points are one of these pairs. We will show that a deeper analysis of f^k will allow us to differentiate

$\mathcal{C}_f = \{c_1, c_2\}$ from the other pairs, thus allowing us to detect $\mathcal{C}_f$ from f^k. The following Lemma is immediate.

Lemma 7.1. *Let f be a critically coalescing quadratic rational map and $k \geq 3$. Then*

$$f^k(a_1) = f^k(a_2) = f^k(b_1) = f^k(b_2).$$

Recall that given a rational map f with critical point set $\mathcal{C}_f$, the postcritical set of f is the set

$$P_f = \bigcup_{i=1}^{\infty} f^i(\mathcal{C}_f).$$

We assume in the following that $k > 1$ is fixed.

7.1. **The case where P_f does not contain a fixed point.**

Lemma 7.2. *Let f be a critically coalescing quadratic rational map and suppose that P_f does not contain a fixed point of f. Then $f^k(c_1) \neq f^k(a_1)$.*

Proof. Since the postcritical set of f does not contain a fixed point, we have for all $k \geq 3$ that

$$f^k(c_1) = f^k(f(a_1)) = f(f^k(a_1)) \neq f^k(a_1).$$

$\square$

Corollary 7.3. *Let f be a critically coalescing quadratic map and suppose that P_f does not contain a fixed point of f. Writing $F = f^k$, we can detect the critical points of f as follows. The group $\mathrm{Deck}(F)$ has three special pairs A, B and C, and each special pair maps to a single point under F. Of the points $F(A)$, $F(B)$ and $F(C)$, two must coincide, say $F(A) = F(B)$. Then $C = \mathcal{C}_f$.*

Proof. By Proposition 5.11, $\mathcal{C}_f$ is one of the special pairs of $\mathrm{Deck}(F)$. By Lemma 7.1, the two special pairs distinct from $\mathcal{C}_f$ have a common image under F, and by Lemma 7.2, this image is distinct from the point $F(\mathcal{C}_f)$.

$\square$

A similar argument allows us to detect the critical values of f in this case. We write $w = f(v_1) = f(v_2)$.
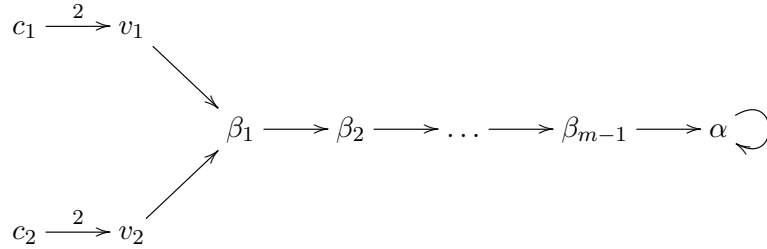
Lemma 7.4. *Let f be critically coalescing and suppose that P_f does not contain a fixed point of f. Then $f^k(v_1) \neq f^k(w)$.*

Proof. Since $w = f(v_1)$ we have $f^k(w) = f^k(f(v_1)) = f(f^k(v_1))$. Since $f^k(v_1)$ is not a fixed point of f, we see that $f^k(v_1) \neq f(f^k(v_1))$ and the result follows.

$\square$

Corollary 7.5. *Let f be a critically coalescing quadratic map and suppose that P_f does not contain a fixed point of f. Let $F = f^k$. We may detect $\mathcal{V}_f$ as follows. There are exactly three points satisfying $F^{-1}(z) \subseteq \mathcal{C}_F$; call them v, v' and v''. Two of these points, say v and v', have a common image under F, so that $F(v) = F(v')$, while $F(v'') \neq F(v)$. Then $\mathcal{V}_f = \{v, v'\}$.*

Proof. The claim that there are three points satisfying $F^{-1}(z) \subseteq \mathcal{C}_F$, and that two of them are the elements of $\mathcal{V}_f$ is contained in Lemma 5.2. Since f is critically coalescing, we must have $F(\mathcal{V}_f)$ is a singleton. However, by Lemma 7.4, the third point satisfying $F^{-1}(z) \subseteq \mathcal{C}_F$ must have an image distinct from $F(\mathcal{V}_f)$. $\qquad\square$

7.2. **The case where P_f contains a fixed point.** It only remains to show we can distinguish the true critical points of f when $\mathrm{Deck}(f^k) \cong V_4$ and P_f contains a fixed point. Our strategy is as follows. Since f is a quadratic rational map, we know that $\mathrm{Deck}(f) \cong \mathbb{Z}_2$. Furthermore, the two fixed points of the non-identity element μ of $\mathrm{Deck}(f)$ are the critical points of f. Thus, it suffices to distinguish μ from the other elements of $\mathrm{Deck}(f^k) \cong V_4$. Since $f(v_1) = f(v_2)$, the fixed point in P_f must be unique. Let α be this fixed point and let m be minimal such that $\alpha = f^m(v_1) = f^m(v_2)$. By taking iterates of f^k, if necessary, we may assume that $k \geq m$. The assumptions on f mean that all critical points of f^k are simple and that $|V_{f^k}| = m + 2$, with the following dynamics under f (we denote $\beta_j = f^j(v_1) = f^j(v_2)$ for $1 \leq j < m$).

$$c_1 \xrightarrow{\ 2\ } v_1 \searrow$$
$$\beta_1 \longrightarrow \beta_2 \longrightarrow \ldots \longrightarrow \beta_{m-1} \longrightarrow \alpha \circlearrowright$$
$$c_2 \xrightarrow{\ 2\ } v_2 \nearrow$$

The following Lemma and its Corollary show that it suffices to be able to detect the elements α and β_{m-1}.

Lemma 7.6. *Let f be a quadratic rational map such that $\mathrm{Deck}(f^k) \cong V_4$. If P_f contains a fixed point α, then the non-identity element μ of $\mathrm{Deck}(f)$ is the unique element of $\mathrm{Deck}(f^k)$ such that $\mu(\alpha) = \beta_{m-1}$.*
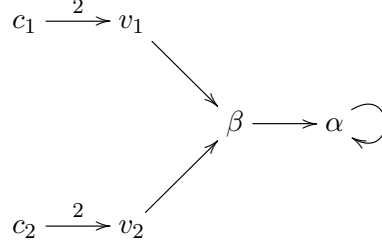
Proof. The assumption on f means that α is not an element of one of the special pairs of $\mathrm{Deck}(f^k)$. Thus the orbit of α under the action of $\mathrm{Deck}(f^k) \cong V_4$ consists of four elements, and each element in this orbit is the image of α for a unique element of $\mathrm{Deck}(f^k)$. Since $f^{-1}(\alpha) = \{\alpha, \beta_{m-1}\}$ and elements of the deck group are fiber-preserving, we see that if μ is the unique non-identity element of $\mathrm{Deck}(f)$ then $\mu(\alpha) = \beta_{m-1}$. $\qquad\square$

Corollary 7.7. *If we can detect the elements α and β_{m-1}, then we can detect $\mathcal{C}_f$ and $\mathcal{V}_f$.*

Proof. As noted above, if we know the points α and β_{m-1} we can recover the unique non-identity element μ of $\mathrm{Deck}(f)$ since it is characterized by the property that $\mu(\alpha) = \beta_{m-1}$. We can then detect the elements of $\mathcal{C}_f$, since they are precisely the fixed points of this element μ. To detect the critical values, we see that by virtue of Lemma 5.2, we can narrow down the options for $\mathcal{V}_f$ to the elements of the set $\{v_1, v_2, \beta_1\}$. But since $f(v_1) = f(v_2)$, we see that, using the deck transformation μ found above, $\mu(v_1) = v_2$ and

$\mu(v_2) = v_1$. This allows us to distinguish v_1 and v_2 from β_1, and so we can detect the set $\mathcal{V}_f$. □

When $m > 2$, it is possible to detect the point β_{m-1} using purely combinatorial arguments. However, the case $m = 2$, which we will deal with first, requires some further work. In this case, $P_f = \{v_1, v_2, \beta, \alpha\}$ and f is a quadratic Lattès map with the following critical portrait.

$$c_1 \xrightarrow{\ 2\ } v_1$$
$$\beta \longrightarrow \alpha \circlearrowright$$
$$c_2 \xrightarrow{\ 2\ } v_2$$

The following definition is standard.

Definition 7.8. *Let z_1, z_2, z_3 and z_4 be four distinct points in $\hat{\mathbb{C}}$. The cross-ratio of these four points is then*

$$[z_1 : z_2 : z_3 : z_4] = \frac{(z_4 - z_1)(z_2 - z_3)}{(z_3 - z_1)(z_2 - z_4)}.$$

In particular, we have $[0 : \infty : 1 : z_4] = z_4$. Furthermore, if μ is a Möbius transformation then

$$[z_1 : z_2 : z_3 : z_4] = [\mu(z_1) : \mu(z_2) : \mu(z_3) : \mu(z_4)].$$

Fixing z_1, z_2 and z_3, it follows that the cross-ratio $[z_1 : z_2 : z_3 : \beta] = z(\beta)$ where $z \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is the global coordinate which satisfies $z(z_1) = 0$, $z(z_2) = \infty$ and $z(z_3) = 1$.

Lemma 7.9. *Assume f is a quadratic rational map with critical values v_1 and v_2 satisfying $f(v_1) = f(v_2) = \beta$, $f(\beta) = \alpha$ and $f(\alpha) = \alpha$ with $\alpha \neq \beta$. Then,*

$$[v_1 : v_2 : \alpha : \beta] \in \left\{-1, 3 \pm 2\sqrt{2}\right\}.$$

Proof. Let us assume that the critical points of f are c_1 and c_2 with associated critical values $v_1 = f(c_1)$ and $v_2 = f(c_2)$. Let $w \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and $z \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the global coordinates defined by

$$w(c_1) = 0, \quad w(c_2) = \infty, \quad w(\alpha) = 1, \quad z(v_1) = 0, \quad z(v_2) = \infty \text{ and } z(\alpha) = 1.$$

Then, $z \circ f$ and w^2 both have a double zero at c_1 and a double pole at c_2, and both take the value 1 at α. It follows that $z \circ f = w^2$ (note that here we use w^2 to denote the square of w, not the second iterate, as is the case in the rest of the paper).

As a consequence, setting $\kappa = w(v_1)$, we get

$$\kappa^2 = w^2(v_1) = z \circ f(v_1) = z(\beta) = z \circ f(v_2) = w^2(v_2),$$

so that $w(v_2) = -\kappa$. Since w sends (v_1, v_2, α) to respectively $(\kappa, -\kappa, 1)$ and z sends (v_1, v_2, α) to respectively $(0, \infty, 1)$, we have that

$$z = \frac{(\kappa + 1)(\kappa - w)}{(\kappa - 1)(\kappa + w)}.$$

Thus

$$\kappa^2 = z(\beta) = \frac{(\kappa+1)(\kappa-w(\beta))}{(\kappa-1)(\kappa+w(\beta))} = \left(\frac{\kappa+1}{\kappa-1}\right)^2.$$

This forces

$$\kappa \in \left\{\pm i, 1 \pm \sqrt{2}\right\} \text{ and } z(\beta) = \kappa^2 \in \left\{-1, 3 \pm 2\sqrt{2}\right\}. \qquad \square$$

Lemma 7.10. *Let f be critically coalescing with $f(v_1) = f(v_2) = \beta$, $f(\beta) = \alpha$ and $f(\alpha) = \alpha$ with $\alpha \neq \beta$. If g is a rational map such that $f^k = g^k$ for some $k \geq 1$, then $\mathcal{V}_f = \mathcal{V}_g$ and $\mathcal{C}_f = \mathcal{C}_g$. In particular, we can detect $\mathcal{C}_f$ and $\mathcal{V}_f$ from f^k.*

Proof. By Corollary 7.7, we need to show we can recover the elements α and β. Firstly, by Lemma 5.2, we see that α is the unique element of P_f for which $\rho_{f^k}(\alpha)$ contains points which are not critical points of f^k. To recover β, again note that following Lemma 5.2, the set $\mathcal{V}_g$ consists of two points from the triple $S = \{v_1, v_2, \beta\}$. By Lemma 7.9, we know that

$$[v_1 : v_2 : \alpha : \beta] \in \left\{-1, 3 \pm 2\sqrt{2}\right\}.$$

If $\mathcal{V}_g \neq \mathcal{V}_f$, then either $\mathcal{V}_g = \{v_1, \beta\}$ or $\mathcal{V}_g = \{v_2, \beta\}$.

In the first case we would have

$$[\beta : v_1 : \alpha : v_2] \in \left\{\frac{1}{2}, \frac{1 \pm \sqrt{2}}{2}\right\}$$

and in the second case, we have

$$[\beta : v_2 : \alpha : v_1] = \left\{\frac{1}{2}, \frac{1 \pm \sqrt{2}}{2}\right\}.$$

In either case g would contradict the conclusion of Lemma 7.9. Thus $\mathcal{V}_f = \mathcal{V}_g$, and so we can recover β as the unique element of S which does not belong to $\mathcal{V}_f$. Thus we can detect $\mathcal{C}_f$ and $\mathcal{V}_f$. $\qquad \square$

Now assume that $m > 2$. We now show that we can detect the critical points of f from f^k.

Lemma 7.11. *If $k \leq m+1$ then we can detect the critical points and critical values of f from f^k.*

Proof. This is essentially the same as Corollaries 7.3 and 7.5, since in this case we have $f^k(c_1) \neq f^k(a_1)$ and $f^k(v_1) \neq f^k(\beta_1)$. $\qquad \square$

We remark that we don't actually need the above result, since P_f contains a fixed point, we may take n large enough so that $nk > m$, and then apply the analysis given below to the map f^{nk}.

To prove the case where $k \geq m > 2$, we first need to count the number of critical points in the fiber above each element of $P_f = V_{f^k}$. Using this notation, we have the following.

Lemma 7.12. *Let $k > m > 2$. Then:*

    (i) *there are exactly 2^{k-1} critical points in the fiber above z if and only if $z \in \{v_1, v_2, \beta_1\}$.*

(ii) *for $2 \leq j \leq m - 1$ there are exactly 2^{k-j} critical points in the fiber above β_j.*

(iii) *there are exactly $2^{k-(m-1)} - 2$ critical points above $\alpha = f^m(v_1)$.*

Proof.

(i) This first claim follows from Lemma 5.2.

(ii) We proceed by induction on k. For $k = m + 1$ it is easy to verify the claim, so assume that for some $k \geq m + 1$ the statement holds. Observe that the fiber over β_j under f^{k+1} is the union of the fibers over the elements of $f^{-1}(\beta_j)$ under f^k and that $f^{-1}(\beta_j) = \{\beta_{j-1}, \zeta\}$ where $\zeta \notin P_f$. Thus by the observation the fiber over β_j under f^{k+1} is equal to the union of the fibers over β_{j-1} and ζ under f^k. Since ζ is not postcritical, there are no critical points in the fiber over it. Therefore the critical points in the fiber over β_j under f^{k+1} are precisely those over f^k. The claim follows by the inductive hypothesis.

(iii) To get the last claim, we can use the fact that the total number of critical points for f^k is $2^{k+1} - 2$. Summing the number of critical points (which are all simple) from the first two cases, we see there are exactly $2^{k-(m-1)} - 2$ critical points unaccounted for. These must lie in the fiber over α.

$\square$

We are now able to prove the following.

Proposition 7.13. *If $k > m > 2$, we can detect the critical points and critical values of f from f^k.*

Proof. Using Lemma 7.12, we can pick out the elements α and β_{m-1} in P_f by looking at the number of critical points in the fiber above each point of P_f. Hence by Corollary 7.7, we can detect $\mathcal{C}_f$ and $\mathcal{V}_f$. $\square$

We are now ready to prove Theorem 5.3.

Proof of Theorem 5.3. The claims in part (i) follow from Lemma 5.12, Proposition 5.13 and Corollary 6.2. Parts (ii) and (iii) follow from the combination of Propositions 5.10, 5.15, 7.13, along with Lemmas 7.10, 7.11 and Corollary 7.3. $\square$

This also completes the proof of Theorem 1.4.

8. Bicritical rational maps with shared iterates

We begin this section with a proof of Theorem 1.1.

Proof of Theorem 1.1. Proposition 4.2 gives the result in the case that f and g have degree $d \geq 3$. Now consider the case $d = 2$. In the non-critically coalescing case, f^k uniquely determines $\mathcal{V}_f$ by Lemma 6.3 and uniquely determines $\mathcal{C}_f$ by Corollary 6.2. In the critically coalescing case, the fact that f^k uniquely determines $\mathcal{C}_f$ and $\mathcal{V}_f$ follows from a combination of Lemma 7.10, Lemma 7.11 and Proposition 7.13. $\square$

We remark that the converse to Theorem 1.1 does not hold. For an example, consider $f(z) = z^2$ and $g(z) = -z^2$. Observe that $\mathcal{C}_f = \mathcal{C}_g = \mathcal{V}_f = \mathcal{V}_g = \{0, \infty\}$. However, for $k \geq 1$ we have $f^k(1) = 1$, but $g^k(1) = -1 \neq 1$ and so $f^k \neq g^k$. An example where f and g are not power maps is given below Theorem 8.4. We will use Theorem 1.1 to help us prove Theorem 1.2.

Lemma 8.1. *Suppose f is surjective and $f^k = (\mu \circ f)^k$ for $k \in \mathbb{N}$ and μ a Möbius transformation. Then*

(i) $f^k = (f \circ \mu)^k$, *and*
(ii) $f^k \circ \mu = \mu \circ f^k$

Proof. Since f is surjective, we can cancel one f from the right side of
$$f^k = (\mu \circ f)^k = (\mu \circ f)^{k-1} \circ \mu \circ f$$
to obtain

(3) $$f^{k-1} = (\mu \circ f)^{k-1} \circ \mu.$$

Therefore, postcomposing both sides with f yields part (i),
$$f^k = f \circ (\mu \circ f)^{k-1} \circ \mu = (f \circ \mu)^k.$$

Next,
$$\mu^{-1} \circ f^k \circ \mu = \mu^{-1} \circ (\mu \circ f)^k \circ \mu = (f \circ \mu)^k = f^k,$$
with the leftmost equality due to the assumption $f^k = (\mu \circ f)^k$ and the rightmost equality due to part (i). □

Lemma 8.2. *Let f and g be bicritical rational maps, neither of which is a power map, such that $f^k = g^k$ for some $k \in \mathbb{N}$. Then either $f = g$ or there exists a Möbius involution μ such that*

(i) $g = \mu \circ f$, *and*
(ii) *μ fixes both $\mathcal{C}_f$ and $\mathcal{V}_f$ as sets.*

Proof. By Theorem 1.1, $\mathcal{C}_f = \mathcal{C}_g$ and $\mathcal{V}_f = \mathcal{V}_g$. Therefore Lemma 2.4 guarantees that there exists a Möbius transformation μ such that $g = \mu \circ f$ and μ fixes $\mathcal{V}_f = \mathcal{V}_g$ as a set, and so μ^2 fixes the elements of $\mathcal{V}_f$ pointwise.

We have $f^k = g^k = (\mu \circ f)^k$ by assumption, so by Lemma 8.1 part (i) we also have $f^k = (f \circ \mu)^k$. Note that $f \circ \mu$ is a bicritical rational map, so Theorem 1.1 implies the leftmost equality of
$$\mathcal{C}_f = \mathcal{C}_{f \circ \mu} = \mu^{-1}(\mathcal{C}_f).$$

Therefore μ also fixes $\mathcal{C}_f$ as a set and so μ^2 fixes $\mathcal{C}_f$ pointwise. To see that μ is an involution, note that since f is assumed to not be a power map, at least one point of $\mathcal{C}_f$ is not in $\mathcal{V}_f$. Thus μ^2 fixes pointwise at least three distinct points (the elements of $\mathcal{C}_f \cup \mathcal{V}_f$), and so $\mu^2 = \text{Id}$. □

Lemma 8.3. *Let f and g be bicritical rational maps such that*

(i) *neither f nor g is a power map,*
(ii) $f^k = g^k$ *for some $k \in \mathbb{N}$,*
(iii) *the degree of f and g is even,*
(iv) $g = \mu \circ f$ *for some nonidentity Möbius transformation μ that fixes both $\mathcal{C}_f$ and $\mathcal{V}_f$ as sets.*

Then μ transposes the elements of $\mathcal{V}_f$ and transposes the elements of $\mathcal{C}_f$.

Note the assumption of even degree in Lemma 8.3.

Proof. If μ fixes the elements of $\mathcal{C}_f$ and $\mathcal{V}_f$ pointwise, then μ is the identity. First suppose that μ fixes $\mathcal{V}_f$ pointwise, but transposes the elements of $\mathcal{C}_f$. By Lemma 8.1, we have $f \circ \mu = \mu \circ f^k$ and so for each $i = 1, 2$ we have

$$f^k(v_i) = f^k(\mu(v_i)) = \mu(f^k(v_i)),$$

which means $f^k(v_i)$ is a fixed point of μ. Thus $f^k(v_i) \in \mathcal{V}_f = \{v_1, v_2\}$ and thus f is postcritically finite. We claim that for $i = 1, 2$ we have $f^{2k}(c_i) = c_i$. To see this, we split into cases.

- *Case 1.* $f^k(v_1) = v_1$. In this case we must also have $f^k(c_1) = c_1$. But then

$$c_2 = \mu(c_1) = \mu(f^k(c_1)) = f^k(\mu(c_1)) = f^k(c_2).$$

  Thus since $f^k(c_2) = c_2$ we have $f^k(v_2) = v_2$.
- *Case 2.* $f^k(v_1) = v_2$. Then we must have $f^k(c_1) = c_2$, and then a similar computation to the above gives $f^k(c_2) = c_1$.

This proves the claim. Now note that f^{2k} has $d^{2k} + 1$ fixed points (counting multiplicity). However, if f^k had repeated fixed points, then f^k would have a parabolic fixed point, and so could not be postcritically finite. But this contradicts the fact that f is postcritically finite. Thus f^{2k} has exactly $d^{2k} + 1$ fixed points.

To complete the argument, note that since μ commutes with f^{2k}, it must permute the $d^{2k} + 1$ fixed points of f^{2k}. But since μ is an involution, all points must have period 1 or 2 under μ. By assumption, v_1 and v_2 are fixed under μ. However, since $d^{2k} + 1 - 2 = d^{2k} - 1$ is odd, there must be another fixed point of f^{2k} which is fixed by μ. But then μ has three fixed points, and so must be the identity. This is a contradiction.

One can prove the case where μ is an involution which fixes $\mathcal{C}_f$ pointwise and transposes the elements of $\mathcal{V}_f$ in a similar way to the above. However, a quicker argument is as follows. In this case, since μ is the unique involution such that $\mathrm{Fix}(\mu) = \mathcal{C}_f$, we know that μ must belong to the deck group of f. Hence $f \circ \mu = f$. But then we have $\mu \circ f^k = f^k \circ \mu = f^k$, which is true if and only if μ is the identity. Once again we have obtained a contradiction. $\square$

Theorem 8.4. *If f and g are bicritical rational maps of even degree, and neither f nor g is a power map, and $f^k = g^k$ for some $k \in \mathbb{N}$, then $f^2 = g^2$.*

Proof. By Lemma 8.2, either $f = g$ or $g = \mu \circ f$ for some Möbius involution μ that fixes both $\mathcal{C}_f$ and $\mathcal{V}_f$ as sets. If $f = g$ we are done, so assume the latter.

By Lemma 8.1,

$$(4) \qquad\qquad \mu \circ f^k \circ \mu = \mu \circ \mu \circ f^k = f^k.$$

So $f^k = (\mu \circ f \circ \mu)^k$. Theorem 1.1 gives $\mathcal{C}_f = \mathcal{C}_{\mu \circ f \circ \mu}$ and $\mathcal{V}_f = \mathcal{V}_{\mu \circ f \circ \mu}$.

Since f is not a power map, $\mu \circ f \circ \mu$ is also not a power map. Then Lemma 8.2 gives that either $f = \mu \circ f \circ \mu$ or there exists a Möbius involution ν such that

$$(5) \qquad\qquad f = \nu \circ \mu \circ f \circ \mu$$

and ν fixes $\mathcal{C}_f$ and $\mathcal{V}_f$ as sets. If $f = \mu \circ f \circ \mu$ we are done (since then $\mu \circ f \circ \mu \circ f = f^2$), so we assume that such a ν exists and $\nu \neq \mathrm{id}$. We will show that $\nu = \mu$, which, by (5), gives $f = f \circ \mu$ and so $\mu \in \mathrm{Deck}(f)$. However, all elements of $\mathrm{Deck}(f)$ must fix pointwise the elements of $\mathcal{C}_f$; this contradicts Lemma 8.4, which asserts that μ interchanges the elements of $\mathcal{C}_f$.

Equation (5) implies

$$f = \nu \circ \mu \circ (\nu \circ \mu \circ f \circ \mu) \circ \mu = (\nu \circ \mu)^2 \circ f.$$

Since f is surjective, this implies $(\nu \circ \mu)^2 = \mathrm{Id}$, and hence $\nu \circ \mu = \mu \circ \nu$.

Note that if x is a fixed point of ν, then $\mu(x)$ is a fixed point of $\mu \circ \nu \circ \mu^{-1} = \nu \circ \mu \circ \mu^{-1} = \nu$, i.e. μ sends fixed points of ν to fixed points of ν. Hence μ fixes setwise the set of fixed points of ν; similarly, ν fixes setwise the set of fixed points of μ. So either the fixed points of ν and μ coincide, or ν and μ interchange each other's fixed points.

If $\mathrm{Fix}(\mu) = \mathrm{Fix}(\nu)$, then since μ and ν are both involutions we would have $\nu = \mu$.

The other possibility is that μ and ν interchange each other's fixed points. In this case, since by Lemma 8.3 we have $\mu(c_i) = c_{3-i}$, we have

$$v_i = f(c_i) = \nu \circ \mu \circ f \circ \mu(c_i) = \nu \circ \mu \circ f(c_{3-i}) = \nu \circ \mu(v_{3-i}) = \nu(v_i),$$

and so ν must fix the elements of $\mathcal{V}_f$ pointwise. Since ν is an involution which fixes $\mathcal{C}_f$ as a set, we see that $\nu(c_i) = c_{3-i}$ for $i = 1, 2$. But then $\nu \circ \mu$ is the involution fixing $\mathcal{C}_f$ pointwise, and so, since f is an even degree bicritical rational map, we have $\mu \circ \nu = \nu \circ \mu \in \mathrm{Deck}(f)$.

Now observe that

$$
\begin{aligned}
(\mu \circ f)^k &= g^k \\
&= f^k \\
&= (\nu \circ \mu \circ f \circ \mu)^k && \text{(by (5))} \\
&= \nu \circ (\mu \circ f \circ (\mu \circ \nu))^{k-1} \circ (\mu \circ f \circ \mu) \\
&= \nu \circ (\mu \circ f)^k \circ \mu \\
&= (\nu \circ \mu) \circ (f \circ \mu)^k \\
&= (\nu \circ \mu) \circ (\mu \circ f)^k. && \text{(by Lemma 8.1)}
\end{aligned}
$$

Hence cancelling the surjective function $(\mu \circ f)^k$ on the right, we get $\nu \circ \mu = \mathrm{id}$. Since μ and ν are involutions, we conclude that $\nu = \mu$. $\qquad\square$

We remark that the conclusion of Theorem 8.4 is not true in the odd degree case.

Example 8.5. Let $f(z) = \frac{z^3 - 1}{z^3 + 1}$ and $g(z) = -f(z)$. It is easy to see that $\mathcal{C}_f = \mathcal{C}_g = \{0, \infty\}$ and $\mathcal{V}_f = \mathcal{V}_g = \{-1, 1\}$. The critical portrait for f is

$$0 \underset{}{\overset{2}{\rightleftarrows}} 1 \longrightarrow \infty \overset{2}{\longrightarrow} -1$$

and the critical portrait for g is

$$0 \overset{2}{\underset{}{\rightleftarrows}} -1 \qquad \infty \overset{2}{\underset{}{\rightleftarrows}} 1 \ .$$

Since $f(f(0)) \neq g(g(0))$, we see that $f^2 \neq g^2$. On the other hand, a direct computation shows that $f^4 = g^4$.

As promised, we also include an example to show that the converse of Theorem 1.1 does not hold, even if we exclude counterexamples which are power maps.

Example 8.6. Here we provide an example of bicritical rational maps f and g such that $\mathcal{C}_f = \mathcal{C}_g$ and $\mathcal{V}_f = \mathcal{V}_g$ but f and g do not share an iterate. Let $f(z) = \frac{2(z^2-1)}{16z^2-1}$ and $g(z) = \frac{z^2-16}{8(z^2-1)}$. Then we have $\mathcal{C}_f = \mathcal{C}_g = \{0, \infty\}$ and $\mathcal{V}_f = \mathcal{V}_g = \left\{\frac{1}{8}, 2\right\}$. However, a quick computation yields

$$f^2(z) = \frac{2(84z^4 - 8z^2 - 1)}{64z^4 + 32z^2 - 21}$$

whereas

$$g^2(z) = \frac{341z^4 - 672z^2 + 256}{8(21z^4 - 32z^2 - 64)}.$$

Since $f^2 \neq g^2$ it follows from Theorem 8.4 that $f^k \neq g^k$ for all $k \geq 1$.

In Example 8.6 above, we have $\mathcal{C}_f = \mathcal{C}_g$, and so by Lemma 2.4 there exists $\mu(z) = \frac{17}{8} - \frac{1}{4z}$ such that $g = \mu \circ f$. Furthermore, $\mathrm{Fix}(\mu) = \mathcal{V}_f = \mathcal{V}_g$. However, since μ is not an involution, it follows from Lemma 8.2 that we cannot have $f^k = g^k$ for some $k \geq 1$. Indeed, this provides a recipe for constructing such counterexamples: given a bicritical map f, let μ be a Möbius transformation such that $\mathrm{Fix}(\mu) = \mathcal{V}_f$ but μ is not an involution. Then Lemma 2.4 holds for f and $g = \mu \circ f$, but by Lemma 8.2 we cannot have $f^k = g^k$.

Our current results allow us to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. The first claim is precisely that of Theorem 8.4. Using the fact that $f^2 = g^2$, the second claim then follows from Lemmas 8.1 and 8.2. □

Appendix A. Symmetry Locus and Mixing

A motivation for the present work is to lay the foundations for an investigation of the structure of the symmetry locus Σ_d in terms of mixings and matings of polynomials. Recall our provisional definition (Definition 1.5) that a degree d rational map F is a *mixing*[2] of postcritically finite degree d polynomials f and g if $F^2 = (f \perp\!\!\!\perp g)^2$ and $F \neq f \perp\!\!\!\perp g$ for some for geometric mating $f \perp\!\!\!\perp g$ of f and g. (See [13, 16] for definitions and background on matings). This section contains mainly conjectures and observations obtained from looking at computer pictures. We hope to give a more rigorous treatment of these ideas in a later work.

The notion of the mixing of two polynomials seems to be very rich. For simplicity, we restrict the present discussion to the degree 2 case. Recall that the symmetry locus in degree 2, Σ_2, may be parameterised by c via the map $f_c(z) = c(z + 1/z)$. Such a map has critical points at -1 and 1. It is not hard to see that there are many matings in the space Σ_2. Indeed, it

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[2]Another name for this construction could be the anti-mating. However, we avoid this terminology to avoid confusion with the work of Jung [8].

can be shown that if f is a postcritically finite quadratic polynomial, then if $f \perp\!\!\!\perp f$ is not obstructed (equivalently, f does not belong to the 1/2-limb of the Mandelbrot set) then the mating $F = f \perp\!\!\!\perp f$ belongs to Σ_2. However, there exist matings in Σ_2 which are not self-matings, as we show below.

We give a number of examples of mixings and their corresponding matings. Claims in these examples are given without proof, but may be verified by the assiduous reader. We include images showing the Julia sets, with arrows indicating the critical orbits of the maps.

Example A.1. When $c \approx 0.221274 + 0.48342i$, the map f_c is the self-mating of Douady's rabbit. Since f_c is a mating and f_c is a hyperbolic map, the forward orbits of the critical points -1 and 1 are disjoint. Indeed,
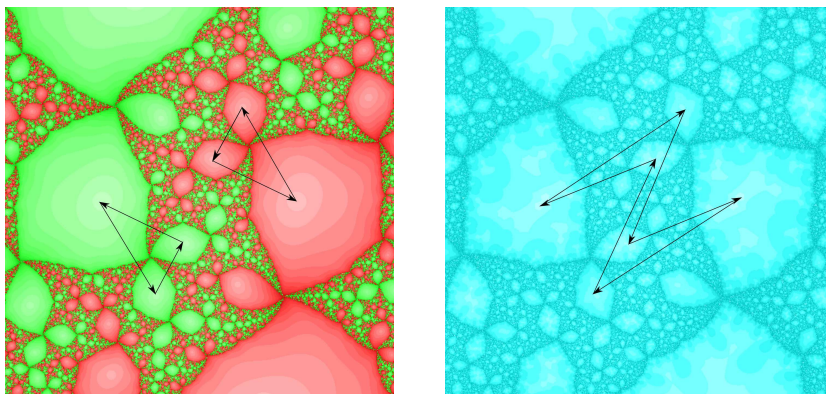

FIGURE 3. The Julia sets for the self-mating and self-mixing of Douady's rabbit.

both critical points belong to a period 3 superattracting cycle. The map f_{-c} is also a hyperbolic map, but it is not a mating since $f^3(-1) = 1$ and $f^3(1) = -1$, so the two critical points belong to the same period 6 superattracting cycle. Accordingly, we say that f_{-c} is the self-mixing of Douady's rabbit; see Figure 3.

Example A.2. There exist matings in Σ_2 which are not self-matings. For a particular example, take $c \approx -0.471274 - 0.813859i$. This is the mating of Douady's rabbit with the airplane polynomial (or, equivalently, the mating of the airplane polynomial with Douady's rabbit, since these maps are equal by the results of [15]). As with the previous example, the two critical points belong to disjoint period 3 superattracting orbits. However, for the map f_{-c}, the two critical points belong to the same period 6 superattracting cycle. Thus f_{-c} is the mixing of Douady's rabbit and the airplane, see Figure 4.

Example A.3. It is possible to be a mixing and a mating. Let $c \approx 0.661848i$. Then f_c is the self-mating of Kokopelli. On the other hand, f_{-c} is the self-mating of co-Kokopelli. Accordingly, we see that f_c is the self-mixing of co-Kokopelli and f_{-c} is the self-mixing of Kokopelli, see Figure 5. This example also shows that the critical orbits in a mixing may be disjoint.
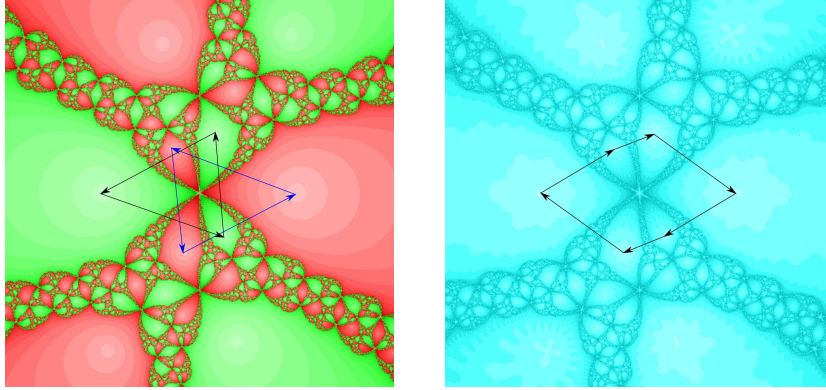
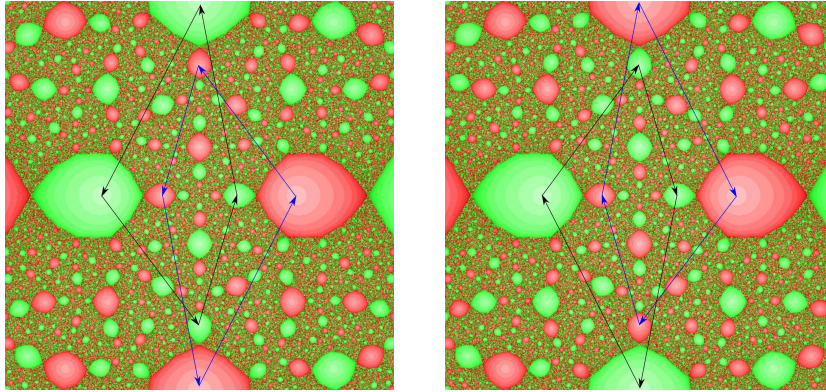FIGURE 4. The Julia sets for the mating and mixing of Douady's rabbit with the airplane.

FIGURE 5. The self-mating and self-mixing of Kokopelli.

Example A.4. A rather neat example is the following. Let $c \approx 0.501604 + 0.531587i$. Then f_c is the self-mating of the 1/4-rabbit. Thus f_{-c} is the self-mixing of the 1/4-rabbit. However, it was shown by Rees that the map f_{-c} is a shared mating[3]: it is the mating of the double basilica with Kokopelli and the mating of co-Kokopelli with the Airbus polynomial, see Figure 6. We then may state that f_c is a *shared mixing*, being a mixing of the double basilica with Kokopelli and the mixing of co-Kokopelli with the Airbus polynomial.

We end with a number of questions about mixings, which we hope will be the subject of future work.

Question 1. Is there a way of constructing a mixing in an analogous way to the topological mating of the formal mating of two polynomials? If so, for which pairs of polynomials is this construction well-defined? What are the obstructions?

Question 2. In [4], Meyer observed that when F is a degree d rational map with $J(F) = \hat{\mathbb{C}}$, it was sometimes possible to find an *anti-equator*; a simple

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[3]An excellent video exhibiting this shared mating can be found on Chéritat's website: `https://www.math.univ-toulouse.fr/~cheritat/MatMovies/ReesSharedExample/`
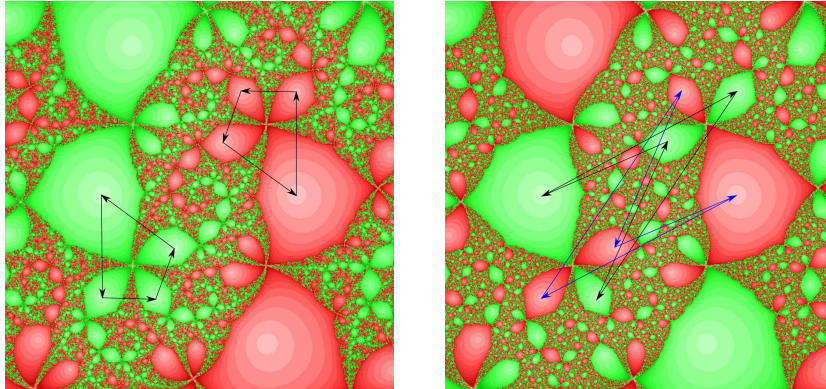
FIGURE 6. The self-mating of the 1/4-rabbit is a shared mixing!.

closed curve which maps (isotopically) onto itself as a d-fold cover in an orientation-reversing way. He asked if it were possible to characterize such "matings". Could these matings observed by Meyer in fact be mixings?

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