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BOUNDS FOR EXIT TIMES OF BROWNIAN MOTION AND THE FIRST DIRICHLET EIGENVALUE FOR THE LAPLACIAN

RODRIGO BAÑUELOS†, PHANUEL MARIANO⋆, AND JING WANG‡

Abstract. For domains in $\mathbb{R}^d$, $d \geq 2$, we prove universal upper and lower bounds on the product of the bottom of the spectrum for the Laplacian to the power $p > 0$ and the supremum over all starting points of the p-moments of the exit time of Brownian motion. It is shown that the lower bound is sharp for integer values of p and that for $p \geq 1$, the upper bound is asymptotically sharp as $d \to \infty$. For all $p > 0$, we prove the existence of an extremal domain among the class of domains that are convex and symmetric with respect to all coordinate axes. For this class of domains we conjecture that the cube is extremal.

Contents

1. Introduction and statements of main results

There is a large class of results often referred to as generalized isoperimetric inequalities that have wide interest in both the mathematics and physics community, see Polyá and Szego [57] and Bandle [4]. At the heart of these inequalities is the classical isoperimetric inequality which states that among all regions of fixed volume, surface area is minimized by balls. In spectral theory among the classical results is the celebrated Rayleigh-Faber-Krahn inequality which states that among all domains $D \subset \mathbb{R}^d$ having the same volume as a ball D^*,

$$\lambda_1(D) \geq \lambda_1(D^*), \tag{1.1}$$

where $\lambda_1(D)$ denotes the first Dirichlet eigenvalue for the Laplacian in D. Further, equality holds if and only if D is a ball. Without loss of generality we take D^* to be centered at the origin.

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On the other hand, it has also been known for many years that one can state many of these inequalities in terms of the exit time of Brownian motion from the domain D. This probabilistic connection provides new insights and raises new interesting questions on their validity for processes other than Brownian motion, such as Lévy processes. To illustrate, let B_t be a $d-$dimensional Brownian motion starting at the point $x \in D$ and let $\tau_D = \inf\{t > 0 \mid B_t \notin D\}$ be its first exit time from D. Using the symmetrization techniques for multiple integrals in [23, 44, 45] it follows that

$$\sup_{x \in D} \mathbb{P}_x(\tau_D > t) \leq \mathbb{P}_0(\tau_{D^*} > t), \tag{1.2}$$

for all $t > 0$. In particular, for any $p > 0$,

$$\sup_{x \in D} \mathbb{E}_x[\tau_D^p] \leq \mathbb{E}_0[\tau_{D^*}^p]. \tag{1.3}$$

Equality holds in these inequalities if and only if D is a ball. Inequality (1.1) follows from inequality (1.2) by taking into account the classical result that for any $D \subset \mathbb{R}^d$,

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}_x(\tau_D > t) = -\frac{\lambda_1(D)}{2}.$$

In a similar way, the classical isoperimetric inequality can be obtained from isoperimetric inequalities for exit times of Brownian motion using small time behavior. These are now classical results with many extensions and applications that can be found in [2, 6, 25, 60] and many other references given in these papers.

From the connections among (1.1), (1.2) and (1.3) we can already observe a competing relation between $\lambda_1(D)$ and $\sup_{x \in D} \mathbb{E}_x[\tau_D]$. Indeed fine connections between the two quantities have been investigated for many years by many authors. Consider the domain functional $G(D) = \lambda_1(D) \sup_{x \in D} \mathbb{E}_x[\tau_D]$. It was proven in [9] that for all simply connected domains D in $\mathbb{R}^2$,

$$2 \leq G(D) \leq \frac{7\zeta(3) j_0^2}{8} \approx 6.08. \tag{1.4}$$

In higher dimensional spaces, it is easy to show the lower bound $G(D) \geq 2$ for all domains $D \subset \mathbb{R}^d$ (see [9] and Section 4 below). In [17, 35] the authors independently show that 2 is in fact a sharp lower bound for all bounded domains. Many results have been devoted to obtaining the upper bound estimates for $G(D)$ (see [18], [34] and [61]). In particular the recent paper [61] improves the upper bound for all general domains to

$$\frac{d}{4} + \frac{\sqrt{d}}{2} \sqrt{5\left(1 + \frac{1}{4}\log 2\right)} + 2. \tag{1.5}$$

With the leading term $\frac{d}{4}$, this bound is asymptotically sharp as $d \to \infty$. However the question of proving a sharp upper bound is wide open even when restricted to special classes of domains such as planar simply connected or convex domains.

The main object of study of this paper is the shape functional

$$G_{p,d}(D) = \lambda_1^p(D) \sup_{x \in D} \mathbb{E}_x[\tau_D^p]. \tag{1.6}$$

Here $\lambda_1(D)$ is the first Dirichlet eigenvalue for the Laplacian in D. When the Laplacian has no discrete spectrum in D, then we take $\lambda_1(D)$ to be the **bottom of the spectrum** for the Laplacian which is given by $\lambda_1(D) = \inf_\phi \frac{\int_D |\nabla \phi|^2 dx}{\int_D \phi^2 dx}$ where $\phi \in H_0^1(D), \phi \neq 0$. The p-moments of the exit time $\mathbb{E}_x[\tau_D^p]$ can also be stated in terms of the torsion function. For $0 < p < \infty$, the $p-$*torsion moment*

function $u_p : D \to \bar{\mathbb{R}}_+$ is defined by

$$u_p(x) = \frac{1}{2^p \Gamma(p+1)} \mathbb{E}_x\left[\tau_D^p\right]. \tag{1.7}$$

Not to be confused with the $p-$torsion function related to the p-Laplacian [24, 32, 34]. When $k \in \mathbb{N}$, and $\sup_{x\in D} u_k(x) < \infty$, these functions are solutions to

$$\begin{cases} -\Delta u_1 = 1 & u_1 \in H_0^1(D) \\ -\Delta u_k = u_{k-1} & u_k \in H_0^1(D), k = 2, 3, \ldots \end{cases} \tag{1.8}$$

When $p = 1$, $u_1(x) = \frac{1}{2}\mathbb{E}_x[\tau_D]$ is the classical torsion function which has been extensively studied in the literature with applications to many areas of mathematics and mathematical physics. See for instance the classical works [4, 41, 57].

For general p, the literature in the study of exit time moments and their applications to different fields is extremely large by now. In the works [19–21, 48] Boudabra-Markowsky studied exit time moments, proved results on the location of their maximum and gave conditions for their finiteness. The k-torsion moment functions u_k have also been applied to the study of heat flow by McDonald-Meyers-Meyerson in [50–52]. In [28], de la Peña-McDonald provide an algorithm that produces uniform approximations of arbitrary continuous functions by exit time moments. We also point to the work of [38], where Hurtado-Markvorsen-Palmer use the L^1 norms of u_k to give an alternative characterization for $\lambda_1(D)$ on Riemannian manifolds. In the closely related papers [27, 33], the authors give upper bounds on $\lambda_1(D)$ using the L^1 norms of exit time moments on manifolds. Moreover, obtaining precise spectral bounds for the exit time in large dimensions has been of interest in other settings. For example in [54], Panzo obtains a spectral bound for the torsion function of symmetric stable processes that has the correct order of growth. For other applications to the study of exit time moments, torsional rigidity, stability, the study of minimal sub-manifolds, and optimal trapping of Brownian motion and gradient estimates, we refer the reader to [37, 40, 43, 46, 49].

Our main goals in this paper are to investigate sharp bounds for $G_{p,d}(D)$, $D \in \mathcal{X}$ where $\mathcal{X}$ contains all domains in $\mathbb{R}^d$ such that $\lambda_1(D) > 0$; and prove the existence of their extremals in the class of convex domains which are symmetric with respect to each coordinate axis.

For the rest of this paper we will work with the function $\mathbb{E}_x\left[\tau_D^p\right]$ and leave the trivial translation of the bounds for $u_p(x)$ to the interested reader. For a given class of domains $\mathcal{D}$, define

$$M_{p,d}(\mathcal{D}) = \sup_{D\in\mathcal{D}} G_{p,d}(D), \tag{1.9}$$

and

$$m_{p,d}(\mathcal{D}) = \inf_{D\in\mathcal{D}} G_{p,d}(D). \tag{1.10}$$

Our first main result provides a sharp asymptotic upper bound for $M_{p,d}(\mathcal{X})$ and a sharp lower for $m_{p,d}(\mathcal{X})$.

Theorem 1.1 (Sharp Lower and Asymptotic Upper Bounds in $\mathcal{X}$)**.** *For $p \geq 1$,*

$$\lim_{d\to\infty} \frac{M_{p,d}(\mathcal{X})}{d^p} = \frac{1}{4^p}. \tag{1.11}$$

Moreover, if $p > 0$ then

$$2^p \Gamma(p+1) \leq m_{p,d}(\mathcal{X}). \tag{1.12}$$

Furthermore, (1.12) *is sharp when p takes values in $\mathbb{N}$.*

The upper bound asymptotic (1.11) is accomplished by working on precise universal upper bounds for $G_{p,d}$ (see Section 2, more precisely, Theorems 2.1 and 3.1). The lower bound (1.12) has been independently obtained by Biswas-Lőrinczi in [15], but only for the restricted class of general convex domains. We prove the lower bound holds for any domain as long as the bottom of spectrum is positive. Moreover, our result proves sharpness for any integer p.

Our second main result concerns the existence of extremals for classes of domains. Given the isoperimetric inequalities (1.1) and (1.3) (as well as other inequalities where balls are extremals in $\mathcal{X}$), one could speculate about the maximality of a ball B for these extremal problems. However it was pointed out in [9, pg. 599] that

$$\lambda_1(B)\sup_{x\in B}\mathbb{E}_x[\tau_B] < \lambda_1(\mathbb{T})\sup_{x\in\mathbb{T}}\mathbb{E}_x[\tau_{\mathbb{T}}], \tag{1.13}$$

where $\mathbb{T}$ is is the equilateral triangle. The existence of maximizers in the class of convex domains is proved in [35]. In the same paper the authors conjecture that when $d = 2$, the equilateral triangle $\mathbb{T}$ is an extremal for $M_{1,2}(\mathcal{X})$.

In this paper we are interested in the extremals, particularly their existence, for the shape functional $G_{p,d}$ among the class of bounded convex domains that are **doubly symmetric** (symmetric with respect to the both coordinate axes). There have been many interesting problems concerning the geometry of the Laplacian in such domains and substantial progress has been made. We refer the readers to some of this large literature [3, 8, 11, 13, 14, 30, 39, 55].

Definition 1.1. Let $\mathcal{C}$ be the class of bounded convex domains in $\mathbb{R}^d$, $d \geq 2$. Let $\mathcal{SC}$ be the subclass of domains in $\mathcal{C}$ that are symmetric with respect to each coordinate axis.

We obtain the following result.

Theorem 1.2 (Existence of extremals in $\mathcal{C}$ or $\mathcal{SC}$)**.** *For any $p > 0$, $d \geq 2$, the upper bounds $M_{p,d}(\mathcal{C})$ and $M_{p,d}(\mathcal{SC})$ admit extremals.*

Remark 1.3. The case for $p = 1$ and the class $\mathcal{C}$ is proved in [35]. Our proof of Theorem 1.2 depends on a key estimate (Lemma 5.2) which estimates the $p-$moment of the difference $(\tau_D - \tau_U)$, where $U \subset D \subset \mathbb{R}^d$ and D is a bounded Lipschitz domain. This in turn will allow us to show in Proposition 5.1 that $G_{p,d}$ is continuous with respect to the Hausdorff distance. This is quite different from the proof in the special case in [35, 36] which uses purely PDE techniques.

The paper is organized as follows. Upper bounds are contained in Sections 2,3 in Theorems 2.1 and 3.1. The proof of Theorem 1.1 is split into Sections 3 and 4. The proof of the asymptotic upper bound (1.11) is given in Section 3. The proof of the lower bound (1.12) is given in Section 4. In Section 5 we discuss the problems of finding extremal domains for $G_{p,d}$ restricted to various subclasses of domains. The proof of Theorem 1.2 is given in Sections 5.3 and 5.4. This section also contains a conjecture on the extremal domain for the class $\mathcal{SC}$, Conjecture 5.2.

2. Upper Bounds for $M_{p,d}(\mathcal{X})$

In this section we obtain some preliminary upper bound estimates that will allow us to prove the sharp asymptotic upper bound for $M_{p,d}(\mathcal{X})$, which will be done in Section 3.

Let $K_D(x, y, t)$ be the *Dirichlet heat kernel* for Δ_D in the domain D. The transition density p_D for Brownian motion killed upon leaving D is given by

$$p_D(x, y, t) = K_D(x, y, t/2),$$

as $\frac{1}{2}\Delta_D$ is the generator of Brownian motion. We can then write

$$\mathbb{E}\left[\tau_D^p\right] = p\int_0^\infty t^{p-1}\mathbb{P}_x\left(\tau_D > t\right)dt = p\int_0^\infty\int_D t^{p-1}p_D\left(x,y,t\right)dydt$$
$$= 2^p p\int_0^\infty\int_D s^{p-1}K_D\left(x,y,s\right)dyds.$$

We also recall the classical upper incomplete gamma function

$$\Gamma\left(s,x\right) = \int_x^\infty u^{s-1}e^{-u}du.$$

Theorem 2.1. *For any $p > 0$, we have*

$$M_{p,d}(\mathcal{X}) \leq 2^p\Gamma(p+1)C_1(d,p), \tag{2.1}$$

where

$$C_1\left(d,p\right) := \inf_{a>0,0<\epsilon<1}\left\{\frac{a^p}{2^p\Gamma\left(p+1\right)} + \frac{1}{\Gamma\left(p\right)}\frac{e^{d/4}\sqrt{2}}{\left(8d\right)^{d/4}}\sqrt{\frac{\Gamma\left(d\right)}{\Gamma\left(d/2\right)}}\left(1+\frac{1}{\sqrt{\epsilon}}\right)^{d/2}\frac{\Gamma\left(p,\left(1-\epsilon\right)a/2\right)}{\left(1-\epsilon\right)^p}\right\}. \tag{2.2}$$

Proof. For any $D\in\mathcal{X}$, since $\mathbb{E}\left[\tau_D^p\right] = \int_0^\infty t^{p-1}\mathbb{P}(\tau_D > t)dt$. We consider splitting the integral at the bottom of the spectrum λ_1 of the Dirichlet Laplacian. Precisely, for any $x\in D$ and $a>0$ we have

$$\mathbb{E}_x\left[\tau_D^p\right] = p\int_0^{a/\lambda_1} t^{p-1}\mathbb{P}_x\left(\tau_D > t\right)dt + p\int_{a/\lambda_1}^\infty t^{p-1}\mathbb{P}_x\left(\tau_D > t\right)dt$$
$$\leq \frac{a^p}{\lambda_1^p} + p\int_{a/\lambda_1}^\infty t^{p-1}\mathbb{P}_x\left(\tau_D > t\right)dt \tag{2.3}$$

Let $I = \int_{a/\lambda_1}^\infty t^{p-1}\mathbb{P}_x\left(\tau_D > t\right)dt$. The theorem is proved upon obtaining the estimate for I that we give in the next lemma. □

Lemma 2.2. *For any $x\in D$, $a>0$, we have*

$$I \leq 2^p\frac{e^{d/4}\sqrt{2}}{\left(8d\right)^{d/4}}\sqrt{\frac{\Gamma\left(d\right)}{\Gamma\left(d/2\right)}}\left(1+\frac{1}{\sqrt{\epsilon}}\right)^{d/2}\frac{\Gamma\left(p,\left(1-\epsilon\right)a/2\right)}{\left(1-\epsilon\right)^p\lambda_1^p}. \tag{2.4}$$

The proof of the above lemma relies on some improvement of Vogt's result in [61]. We split the major steps into the lemma and proposition below.

Lemma 2.3. *Let $D\subset\mathbb{R}^d$ be measurable, $\alpha>0$, and let L be a bounded operator on $L^2(D)$ satisfying*

$$\left\|e^{-\alpha\rho_w}Le^{\alpha\rho_w}\right\|_{2\to\infty} \leq 1 \tag{2.5}$$

for all $w\in D$, where $\rho_w\left(x\right) = \left|x-w\right|, w\in\mathbb{R}^d$. Then

$$\left\|L\right\|_{\infty\to\infty} \leq \frac{\sqrt{2}\pi^{d/4}}{\left(2\alpha\right)^{d/2}}\sqrt{\frac{\Gamma\left(d\right)}{\Gamma\left(d/2\right)}} \tag{2.6}$$

Proof. The proof is essentially the same as in [61, Proposition 2.5]. Note that $\|Lf\|_\infty = \sup_{w\in D}\|e^{-\alpha\rho_w}Lf\|_\infty$. Then we have

$$\|e^{-\alpha\rho_w}Lf\|_\infty \leq \|e^{-\alpha\rho_w}f\|_2 \leq \|e^{-\alpha\rho_w}\|_2\|f\|_\infty.$$

Let σ_{d-1} denote the surface measure of the unit sphere, then the conclusion follows from the estimate below.

$$\begin{aligned}\left\|e^{-\alpha\rho_w}\right\|_2^2 &= \int e^{-2\alpha|w-y|}\chi_D(y)dy \leq \int e^{-2\alpha|y|}dy \\ &= \sigma_{d-1}\int_0^\infty e^{-2\alpha r}r^{d-1}dr = \frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{\Gamma(d)}{(2\alpha)^d}.\end{aligned}$$

□

Proposition 2.1. For all $\epsilon \in (0,1]$, we have

$$\left\|e^{-t(-\Delta_D)}\right\|_{\infty\to\infty} \leq e^{d/4}\frac{\sqrt{2}}{(8d)^{d/4}}\sqrt{\frac{\Gamma(d)}{\Gamma(d/2)}}\left(1+\frac{1}{\sqrt{\epsilon}}\right)^{d/2}e^{-(1-\epsilon)\lambda_1 t},$$

for $t \geq 0$. In particular, for all $x \in D$ and $t \geq 0$,

$$\mathbb{P}_x(\tau_D > t) \leq e^{d/4}\frac{\sqrt{2}}{(8d)^{d/4}}\sqrt{\frac{\Gamma(d)}{\Gamma(d/2)}}\left(1+\frac{1}{\sqrt{\epsilon}}\right)^{d/2}e^{-(1-\epsilon)\frac{\lambda_1 t}{2}}. \tag{2.7}$$

Proof. The proof is similar to Theorem 2.1 in [61]. Here we only sketch the key steps. Consider the operator $H = -\Delta_D - \lambda_1$, it is a self-adjoint operator in $L^2(D)$ with the bottom of spectrum $\lambda_1(H) = 0$. Clearly the heat kernel of e^{-tH} has the Gaussian upper bound

$$|K_t(x,y)| \leq e^{\lambda_1 t}\cdot\frac{1}{(4\pi t)^{d/4}}\exp\left(-\frac{|x-y|^2}{4t}\right)$$

for all $t > 0$ and a.e. $x, y \in D$. It then holds that (see proof of Theorem 2.1, page 43 in [61]) for any $\epsilon \in (0,1]$

$$\left\|e^{-\alpha\rho_w}e^{-tH}e^{\alpha\rho_w}\right\|_{2\to\infty} \leq (8\pi\epsilon t)^{-d/4}\left(1+\frac{1}{\beta}\right)^{d/4}e^{\lambda_1\epsilon t}e^{(1+\beta)\alpha^2\epsilon t+\alpha^2(1-\epsilon)t}. \tag{2.8}$$

Applying Lemma 2.3 to $L = e^{-tH}$ and using (2.8) we have that

$$\left\|e^{-tH}\right\|_{\infty\to\infty} \leq \frac{\sqrt{2}\pi^{d/4}}{(2\alpha)^{d/2}}\sqrt{\frac{\Gamma(d)}{\Gamma(d/2)}}\cdot(8\pi\epsilon t)^{-d/4}\left(1+\frac{1}{\beta}\right)^{d/4}e^{\lambda_1\epsilon t}e^{(1+\beta\epsilon)\alpha^2 t},$$

taking $\alpha^2 = \frac{d/4}{(1+\beta\epsilon)t}$ we obtain

$$\left\|e^{-tH}\right\|_{\infty\to\infty} \leq \sqrt{2}\pi^{d/4}\sqrt{\frac{\Gamma(d)}{\Gamma(d/2)}}(8\pi\epsilon t)^{-d/4}\left(\frac{(1+\beta\epsilon)t}{d}\left(1+\frac{1}{\beta}\right)\right)^{d/4}e^{\lambda_1\epsilon t+d/4}.$$

Optimizing the right hand side of the above inequality by taking $\beta = \epsilon^{-1/2}$ we have

$$\left\|e^{-tH}\right\|_{\infty\to\infty} \leq e^{d/4}\frac{\sqrt{2}}{(8d)^{d/4}}\sqrt{\frac{\Gamma(d)}{\Gamma(d/2)}}\cdot\left(1+\frac{1}{\sqrt{\epsilon}}\right)^{d/2}e^{\lambda_1\epsilon t}.$$

This then completes the proof. □

Proof of Lemma 2.2. By (2.7) we have

$$
\begin{aligned}
I &= \int_{a/\lambda_1}^{\infty} t^{p-1}\mathbb{P}_x\left(\tau_D > t\right) dt \\
&\leq e^{d/4}\frac{\sqrt{2}}{(8d)^{d/4}}\sqrt{\frac{\Gamma\left(d\right)}{\Gamma\left(d/2\right)}}\left(1+\frac{1}{\sqrt{\epsilon}}\right)^{d/2}\int_{a/\lambda_1}^{\infty} t^{p-1}e^{-(1-\epsilon)\frac{\lambda_1 t}{2}}dt \\
&= 2^p\frac{e^{d/4}\sqrt{2}}{(8d)^{d/4}}\sqrt{\frac{\Gamma\left(d\right)}{\Gamma\left(d/2\right)}}\left(1+\frac{1}{\sqrt{\epsilon}}\right)^{d/2}\frac{\Gamma\left(p,\left(1-\epsilon\right)a/2\right)}{\left(1-\epsilon\right)^p\lambda_1^p}.
\end{aligned}
\tag{2.9}
$$

□

Remark 2.4. For the interested reader, we remark the following numerical estimate as a consequence of Theorem 2.1. In particular, when $p=1$ one can obtain the following bound

$$
M_{1,2}(\mathcal{X}) \leq 2C_1(2,1) \leq 2f_{1,2}\left(1.65659, 0.173247\right) \leq 2\cdot\left(2.03785\right) = 4.0757.
$$

where

$$
f_{p,d}\left(a,\epsilon\right) := \frac{a^p}{2^p\Gamma\left(p+1\right)} + \frac{1}{\Gamma\left(p\right)}\frac{e^{d/4}\sqrt{2}}{(8d)^{d/4}}\sqrt{\frac{\Gamma\left(d\right)}{\Gamma\left(d/2\right)}}\left(1+\frac{1}{\sqrt{\epsilon}}\right)^{d/2}\frac{\Gamma\left(p,\left(1-\epsilon\right)a/2\right)}{\left(1-\epsilon\right)^p}.
$$

Moreover, we have the following corollary.

Corollary 2.2. We have

$$
M_{1,d}(\mathcal{X}) \leq \frac{d}{2}\frac{1}{y_d\left(1+\sqrt{y_d}\right)} =: 2C_3(d,1).
\tag{2.10}
$$

Here, $y = y_d \in (0,1)$ is the unique solution to

$$
-d + d\sqrt{y} + (4+4A_d)\,y + (2d)\,y\log\left(\left(1+1/\sqrt{y}\right)/2\right) = 0, \quad y\in(0,1),
\tag{2.11}
$$

where

$$
A_d = \log\left[\frac{2^{d/2}e^{d/4}\sqrt{2}}{(8d)^{d/4}}\sqrt{\frac{\Gamma\left(d\right)}{\Gamma\left(d/2\right)}}\right],
$$

and

$$
\lim_{d\to\infty} y_d = 1.
$$

Proof. From (2.2) we have

$$
C_1(d,1) := \inf_{x>0, 0<y<1} f(x,y)
$$

where $f:\mathbb{R}_+\times[0,1]\to\mathbb{R}_+$ is defined by

$$
f(x,y) = \frac{x}{2} + \frac{e^{d/4}\sqrt{2}}{(8d)^{d/4}}\sqrt{\frac{\Gamma\left(d\right)}{\Gamma\left(d/2\right)}}\left(1+\frac{1}{\sqrt{y}}\right)^{d/2}\frac{1}{(1-y)}e^{-(1-y)x/2}.
$$

Note that

$$
f_x\left(x,y\right) = \frac{1}{2} - \frac{1}{2}\frac{e^{d/4}\sqrt{2}}{(8d)^{d/4}}\sqrt{\frac{\Gamma\left(d\right)}{\Gamma\left(d/2\right)}}\left(1+\frac{1}{\sqrt{y}}\right)^{d/2}e^{-(1-y)x/2},
$$

we then obtain the minimizer of $f(\cdot,y)$ at

$$
x_y := \frac{2}{(1-y)}\log\left[\frac{e^{d/4}\sqrt{2}}{(8d)^{d/4}}\sqrt{\frac{\Gamma\left(d\right)}{\Gamma\left(d/2\right)}}\left(1+\frac{1}{\sqrt{y}}\right)^{d/2}\right].
$$

We are then led to minimize the one variable function

$$g(y) := f(x_y, y) = \frac{1}{1-y}\log\left[\frac{e^{d/4}\sqrt{2}}{(8d)^{d/4}}\sqrt{\frac{\Gamma(d)}{\Gamma(d/2)}}\left(1+\frac{1}{\sqrt{y}}\right)^{d/2}\right] + \frac{1}{(1-y)}$$
$$= \frac{\frac{d}{2}\log\left(\frac{1+1/\sqrt{y}}{2}\right) + 1 + A_d}{1-y}$$

where

$$A_d = \log\left[\frac{2^{d/2}e^{d/4}\sqrt{2}}{(8d)^{d/4}}\sqrt{\frac{\Gamma(d)}{\Gamma(d/2)}}\right].$$

Since

$$g'(y) = \frac{-d\left(1-\sqrt{y}\right) + \left((2d)\log\left(\frac{1+1/\sqrt{y}}{2}\right) + 4 + 4A_d\right)y}{4(1-y)^2 y}, \tag{2.12}$$

if we assume that y_d is a solution to $g'(y) = 0$, then

$$(1+A_d) = \frac{d}{4y_d}\left(1-\sqrt{y_d}\right) - \frac{d}{2}\log\left(\frac{1+1/\sqrt{y_d}}{2}\right). \tag{2.13}$$

Plugging (2.13) back in (2.12) we have that

$$g(y_d) = \frac{d}{4}\frac{1}{y_d\left(1+\sqrt{y_d}\right)},$$

hence we obtain (2.10). Next we show that (2.11) has a unique solution. Let $F_d : (0,1) \to \mathbb{R}$ be

$$F_d(y) = -\frac{d}{4} + \frac{d\sqrt{y}}{4} + y(1+A_d) + \frac{d}{2}y\log\left(\frac{1+1/\sqrt{y}}{2}\right).$$

We easily find that $\lim_{y\to 0} F_d(y) = -\frac{d}{4} < 0$, $F_d(1) = 1 + A_d > 0$ and $F_d'(y) > 0$. Therefore the conclusion follows. □

Remark 2.5. From the above corollary we can deduce that $\lim_{d\to\infty} y_d = 1$. First it can be easily shown that y_d exists (for instance see (3.3) in [61]). From (2.13) we have

$$\frac{(4+4A_d)\,y_d}{d} = 1 - \sqrt{y_d} - 2y_d\log\left(\frac{1+1/\sqrt{y_d}}{2}\right).$$

Taking $d \to \infty$ on both sides we then obtain $y_\infty = 1$. This limit coincides with the conclusion in [61], but the corollary is sharper comparing to [61] by providing an almost explicit expression for y_d.

3. Sharp asymptotics for $M_{p,d}(\mathcal{X})$: proof (1.11) of Theorem 1.1

This section concerns the asymptotic estimates for $M_{p,d}(\mathcal{X})$ in high dimensions. First, we give an upper bound estimate of $M_{p,d}(\mathcal{X})$ by analyzing the variational problem in Theorem 2.1, which provides the correct leading order in d for all $p \geq 1$.

Theorem 3.1. For $p > 0$,

$$M_{p,d}(\mathcal{X}) \leq 2^p\left(\frac{d}{8} + c\sqrt{d} + 1 - \frac{1}{1-y_d}\right)^p C_2(d,p)$$

where

$$C_2(d,p) := 1 + p\int_1^\infty u^{p-1}e^{(1-u)\left[(1-y_d)\left(\frac{d}{8}+c\sqrt{d}+1\right)-1\right]}du,$$

and

$$c = \frac{1}{4}\sqrt{5\left(1+\frac{1}{4}\log 2\right)}, \quad \text{and} \quad y_d = \frac{1}{\left(1+\frac{16c}{5\sqrt{d}}\right)^2}. \tag{3.1}$$

The proof of Theorem 3.1 will require the following elementary estimate.

Lemma 3.1. *With c and y_d as in (3.1) we have*

$$\log\left[2^{\frac{1}{4}-\frac{d}{2}}\left(1+\frac{1}{\sqrt{y_d}}\right)^{d/2}\right]+1 \le (1-y_d)\left(\frac{d}{8}+c\sqrt{d}+1\right). \tag{3.2}$$

Proof. First note

$$\text{LHS} = \frac{1}{4}\log 2 + \frac{d}{2}\log\left(\frac{1+1/\sqrt{y_d}}{2}\right)+1.$$

Denote by $\gamma := \frac{8}{5}c$ and set $x = \frac{\gamma}{\sqrt{d}}$. We can easily check that $0<x<1$. Clearly $1+2x = \frac{1}{\sqrt{y_d}}$, and hence

$$\text{LHS} = \frac{5}{4}\gamma^2 + \frac{d}{2}\log\left(\frac{1+(1+2x)}{2}\right) = \frac{5}{4}x^2 d + \frac{d}{2}\log(1+x).$$

On the other hand

$$\text{RHS} = \left(1-\frac{1}{(1+2x)^2}\right)\frac{d}{8}\left(1+\frac{8c}{\sqrt{d}}+\frac{8}{d}\right) = \frac{x+x^2}{(1+2x)^2}\frac{d}{2}\left(1+5x+\frac{8}{\gamma^2}x^2\right).$$

Thus it suffices to show that for all $x\in(0,1)$,

$$\frac{5}{2}x^2+\log(1+x) \le \frac{x+x^2}{(1+2x)^2}\left(1+5x+\frac{8}{\gamma^2}x^2\right).$$

This can be shown by elementary calculus. See details in [61, page 46]. □

Proof of Theorem 3.1. Let

$$f(x,y) := x^p + 2^p\, p\, C_d\left(1+\frac{1}{\sqrt{y}}\right)^{d/2}\frac{1}{(1-y)^p}\Gamma\left(p,(1-y)\,x/2\right),$$

where $C_d = \frac{e^{d/4}\sqrt{2}}{(8d)^{d/4}}\sqrt{\frac{\Gamma(d)}{\Gamma(d/2)}}$. Then

$$2^p\Gamma(p+1)C_1(d,p) = \inf_{x>0,0<y<1} f(x,y).$$

First by letting

$$f_x(x,y) = px^{p-1}\left(1-C_d e^{-(1-y)x/2}\left(1+\frac{1}{\sqrt{y}}\right)^{d/2}\right) = 0$$

we obtain the critical point

$$x_y = \frac{2}{(1-y)}\log\left[C_d\left(1+\frac{1}{\sqrt{y}}\right)^{d/2}\right]. \tag{3.3}$$

Hence

$$\begin{aligned} f(x_y,y) = {} & \frac{2^p}{(1-y)^p}\left(\log\left[C_d\left(1+\frac{1}{\sqrt{y}}\right)^{d/2}\right]\right)^p \\ & + p2^pC_d\left(1+\frac{1}{\sqrt{y}}\right)^{d/2}\frac{1}{(1-y)^p}\Gamma\left(p,\log\left[C_d\left(1+\frac{1}{\sqrt{y}}\right)^{d/2}\right]\right). \end{aligned}$$

It is known that (for instance see [1, 6.1.18])

$$\frac{\Gamma(d)}{\Gamma(d/2)} = \frac{\Gamma(2(d/2))}{\Gamma(d/2)} \leq 2^{d-1/2}\left(\frac{d}{2e}\right)^{d/2},$$

hence we have

$$C_d \leq 2^{-d/2+1/4}. \tag{3.4}$$

Combining (3.2) and (3.4) we get

$$\begin{aligned}\log\left[C_d\left(1+\frac{1}{\sqrt{y_d}}\right)^{d/2}\right] &\leq \log\left[2^{\frac{1}{4}-\frac{d}{2}}\left(1+\frac{1}{\sqrt{y_d}}\right)^{d/2}\right] \\ &\leq (1-y_d)\left(\frac{d}{8}+c\sqrt{d}+1\right)-1. \end{aligned} \tag{3.5}$$

Using (3.5) in $f(x_y, y)$ we then obtain

$$f(x_{y_d}, y_d) \leq 2^p\left(\left(\frac{d}{8}+c\sqrt{d}+1\right)-\frac{1}{1-y_d}\right)^p + \frac{p2^p}{(1-y_d)^p}II, \tag{3.6}$$

where

$$II = C_d\left(1+\frac{1}{\sqrt{y_d}}\right)^{d/2}\Gamma\left(p, \log\left[C_d\left(1+\frac{1}{\sqrt{y_d}}\right)^{d/2}\right]\right).$$

Making the substitution $x = u\log\left[C_d\left(1+\frac{1}{\sqrt{y_d}}\right)^{d/2}\right]$ and plugging in (3.5) we have

$$\begin{aligned} II &= C_d\left(1+\frac{1}{\sqrt{y_d}}\right)^{d/2}\left(\log\left[C_d\left(1+\frac{1}{\sqrt{y_d}}\right)^{d/2}\right]\right)^p \\ &\times\left[\int_1^\infty u^{p-1}\left(C_d\left(1+\frac{1}{\sqrt{y_d}}\right)^{d/2}\right)^{-u}du\right] \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\leq C_d\left(1+\frac{1}{\sqrt{y_d}}\right)^{d/2}(1-y_d)^p\left(\left(\frac{d}{8}+c\sqrt{d}+1\right)-\frac{1}{(1-y_d)}\right)^p \\ &\times\left[\int_1^\infty u^{p-1}\left(C_d\left(1+\frac{1}{\sqrt{y_d}}\right)^{d/2}\right)^{-u}du\right]. \end{aligned} \tag{3.8}$$

Moreover, clearly from (3.5) we have

$$\left(C_d\left(1+\frac{1}{\sqrt{y_d}}\right)^{d/2}\right)^{-u} \leq e^{-u\left[(1-y_d)\left(\frac{d}{8}+c\sqrt{d}+1\right)-1\right]}. \tag{3.9}$$

Hence

$$II \leq (1-y_d)^p\left(\left(\frac{d}{8}+c\sqrt{d}+1\right)-\frac{1}{1-y_d}\right)^p\int_1^\infty u^{p-1}e^{(1-u)\left[(1-y_d)\left(\frac{d}{8}+c\sqrt{d}+1\right)-1\right]}du. \tag{3.10}$$

Using (3.10) in (3.6) we arrive at

$$f(x_{y_d}, y_d) \leq 2^p\left(\frac{d}{8}+c\sqrt{d}+1-\frac{1}{1-y_d}\right)^p\left[1+p\int_1^\infty u^{p-1}e^{(1-u)\left[(1-y_d)\left(\frac{d}{8}+c\sqrt{d}+1\right)-1\right]}du\right].$$

□

In the lemma below we show that our result is indeed sharp, by comparing to a unit ball.

Lemma 3.2. *Let $B(0,1) \subset \mathbb{R}^d$ be the unit ball centered at zero, then*

$$\left(\frac{d}{4}\right)^p \leq \lambda_1^p\left(B(0,1)\right) \sup_{x \in B(0,1)} \mathbb{E}_x\left[\tau_{B(0,1)}^p\right],$$

for $p \geq 1$.

Proof. It is well known that $\lambda_1\left(B(0,1)\right) \geq \frac{d^2}{4}$ (for instance, see [42]). By a simple calculation we have that $\mathbb{E}_x\left[\tau_{B(0,1)}\right] = \frac{1-|x|^2}{d}$. Hence

$$\sup_{x \in B(0,1)} \mathbb{E}_x\left[\tau_{B(0,1)}\right] = \mathbb{E}_0\left[\tau_{B(0,1)}\right] = \frac{1}{d}.$$

By Jensen's inequality we have

$$\lambda_1^p\left(B(0,1)\right) \cdot \mathbb{E}_0\left[\tau_{B(0,1)}^p\right] \geq \lambda_1^p\left(B(0,1)\right) \cdot \left(\mathbb{E}_0\left[\tau_{B(0,1)}\right]\right)^p = \frac{d^p}{4^p}.$$

□

Proof of (1.11) of Theorem 1.1. From Theorem 3.1 we have

$$M_{p,d}(\mathcal{X}) \leq 2^p \left(\frac{d}{8} + c\sqrt{d} + 1 - \frac{1}{1-y_d}\right)^p C_2\left(d,p\right),$$

where

$$C_2(d,p) := 1 + p\int_1^\infty u^{p-1} e^{(1-u)\left[(1-y_d)\left(\frac{d}{8}+c\sqrt{d}+1\right)-1\right]} du,$$

and

$$y_d = \frac{1}{\left(1+\frac{16c}{5\sqrt{d}}\right)^2}, \quad c = \frac{1}{4}\sqrt{5\left(1+\frac{1}{4}\log 2\right)}.$$

First we claim that $\lim_{d\to\infty} C_2(d,p) = 1$. Note that

$$(1-y_d)\left(\frac{d}{8} + c\sqrt{d} + 1\right) - 1 \geq 1 + \frac{4c}{5}\sqrt{d} \to \infty$$

as $d \to \infty$. Hence when $u \geq 1$ we have

$$\lim_{d\to\infty} u^{p-1} e^{(1-u)\left[(1-y_d)\left(\frac{d}{8}+c\sqrt{d}+1\right)-1\right]} = 0,$$

Moreover, since

$$\int_1^\infty u^{p-1} e^{(1-u)\left[(1-y_d)\left(\frac{d}{8}+c\sqrt{d}+1\right)-1\right]} du \leq \int_1^\infty u^{p-1} e^{(1-u)} du \leq e\Gamma(p),$$

by the dominated convergence theorem we obtain that

$$\lim_{d\to\infty} \int_1^\infty u^{p-1} e^{(1-u)\left[(1-y_d)\left(\frac{d}{8}+c\sqrt{d}+1\right)-1\right]} du = 0. \tag{3.11}$$

It now follows readily that

$$\limsup_{d\to\infty} \frac{M_{p,d}(\mathcal{X})}{d^p} \leq \frac{1}{4^p}.$$

Together with Lemma 3.2 we then obtain that

$$\frac{1}{4^p} \leq \liminf_{d\to\infty} \frac{M_{p,d}(\mathcal{X})}{d^p},$$

and concludes the proof of the asymptotic bound (1.11) of Theorem 1.1.

□

4. Lower bound for $m_{p,d}(\mathcal{X})$: proof of (1.12) of Theorem 1.1

We remark that the lower bound in (1.12) for $p = 1$ has been known for many years, as mentioned in [9].

Proof of (1.12) *of Theorem 1.1.* We first prove the inequality. Let us assume for the moment that the domain D is bounded (or even just that it has finite volume). In this case we have a discrete spectrum with a complete set of eigenfunctions on $L^2(D)$ and the eigenfunction φ_1 corresponding to $\lambda_1(D)$ is in $L^\infty(D)$. For this, we refer the reader to [29]. Since

$$e^{-\lambda_1 t/2}\varphi_1(x) = \int_D p_D(x,y,t)\varphi_1(y)dy \tag{4.1}$$

integrating in time we find that

$$\begin{aligned}\varphi_1(x)\frac{2^p p}{\lambda_1^p(D)}\Gamma(p) &= \varphi_1(x)\int_0^\infty pt^{p-1}e^{-\lambda_1 t/2}dt \\ &= p\int_0^\infty\int_D t^{p-1}p_D(x,y,t)\varphi_1(y)dydt \\ &\le \sup_{y\in D}\varphi_1(y)\cdot\left(p\int_0^\infty\int_D t^{p-1}p_D(x,y,t)dydt\right) \\ &= \sup_{y\in D}\varphi_1(y)\,\mathbb{E}_x\left[\tau_D^p\right]. \qquad (4.3)\end{aligned}$$

Since $p\Gamma(p) = \Gamma(p+1)$, this gives the desired lower bound by taking a supremum over all $x \in D$.

To remove the boundedness assumption on D, let $r > 0$ and consider the open set $D \cap B(0,r)$ which is nonempty for large enough r. Since $D \cap B(0,r) \subset D$, we have $\mathbb{E}_x\left[\tau^p_{D\cap B(0,r)}\right] \le \mathbb{E}_x\left[\tau_D^p\right]$ and it follows that

$$\sup_{x\in D\cap B(0,r)}\mathbb{E}_x\left[\tau_D^p\right] \ge \sup_{x\in D\cap B(0,r)}\mathbb{E}_x\left[\tau^p_{D\cap B(0,r)}\right] \ge 2^p\Gamma(p+1)\cdot\left(\lambda_1\left(D\cap B(0,r)\right)\right)^{-p}.$$

Taking $r \to \infty$ completes the proof of the lower bound.

It remains to prove the sharpness of (1.12) for integers p. For any $d \ge 2$, it is shown in [17, Theorem 1] and [35, Theorem 3.3] that there exists a sequence of bounded domains $D_{\epsilon_n} \subset \mathbb{R}^d$ satisfying

$$2 \le \lambda_1(D_{\epsilon_n})\sup_{x\in D_{\epsilon_n}}\mathbb{E}_x[\tau_{D_{\epsilon_n}}] < 2+\epsilon_n, \tag{4.4}$$

where $\epsilon_n \to 0$, as $n \to \infty$. To finish, we need the following inequality whose proof we provide here for completeness. (See for example [7, Corollary 1] and [22, Lemma 18.1])

Lemma 4.1. *Let $D \subset \mathbb{R}^d$ be a domain satisfying $\sup_{x\in D}\mathbb{E}_x[\tau_D] < \infty$. Then for any $k \in \mathbb{N}$,*

$$\mathbb{E}_x\left[\tau_D^k\right] \le k!\left(\sup_{x\in D}\mathbb{E}_x[\tau_D]\right)^k, \quad x \in D.$$

Proof. By the Markov property and Fubini's theorem we have for any $a \ge 0$,

$$\begin{aligned}\int_a^\infty \mathbb{P}_x(\tau_D > t)\,dt &= \int_0^\infty \mathbb{P}_x(\tau_D > t+a)\,dt \\ &= \int_0^\infty \mathbb{E}_x\left[1_{(\tau_D>a)}\mathbb{P}_{X_a}(\tau_D > t)\right]dt = \mathbb{E}_x\left[1_{(\tau_D>a)}\mathbb{E}_{X_a}[\tau_D]\right] \\ &\le \left(\sup_{x\in D}\mathbb{E}_x[\tau_D]\right)\mathbb{P}_x(\tau_D > a). \qquad (4.5)\end{aligned}$$

Multiplying both sides by ka^{k-1} and integrating on a gives that

$$\int_0^\infty ka^{k-1}\int_a^\infty \mathbb{P}_x\left(\tau_D > t\right)dtda = \int_0^\infty t^k\mathbb{P}_x\left(\tau_D > t\right)dt = \frac{1}{k+1}\mathbb{E}_x\left[\tau_D^{k+1}\right],$$

and

$$\left(\sup_{x\in D}\mathbb{E}_x\left[\tau_D\right]\right)\int_0^\infty ka^{k-1}\mathbb{P}_x\left(\tau_D > a\right)da = \left(\sup_{x\in D}\mathbb{E}_x\left[\tau_D\right]\right)\mathbb{E}_x\left[\tau_D^k\right].$$

The desired inequality then follows by induction. □

Returning to the sharpness of inequality (1.12), fix $k \in \mathbb{N}$. Let D_{ϵ_n} be the domains satisfying (4.4). We claim that

$$\lambda_1\left(D_{\epsilon_n}\right)^k \cdot \sup_{x\in D_{\epsilon_n}}\mathbb{E}_x\left[\tau_{D_{\epsilon_n}}^k\right] \leq 2^k k! + 2k\cdot k!\epsilon_n + o(\epsilon_n)$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Indeed, from Lemma 4.1 and the estimate (4.4).

$$\begin{aligned}
\lambda_1\left(D_{\epsilon_n}\right)^k \cdot \sup_{x\in D_{\epsilon_n}}\mathbb{E}_x\left[\tau_{D_{\epsilon_n}}^k\right] &\leq \lambda_1\left(D_{\epsilon_n}\right)^k k!\left(\sup_{x\in D_{\epsilon_n}}\mathbb{E}_x\left[\tau_{D_{\epsilon_n}}\right]\right)^k \\
&= k!\left(\lambda_1\left(D_{\epsilon_n}\right)\cdot\sup_{x\in D_{\epsilon_n}}\mathbb{E}_x\left[\tau_{D_{\epsilon_n}}\right]\right)^k \leq k!\left(2+\epsilon_n\right)^k \\
&= 2^k k! + 2k\cdot k!\epsilon_n + o(\epsilon_n).
\end{aligned}$$

This proves the sharpness of the inequality (1.12) and completes the proof of the Theorem. □

It is reasonable to conjecture that under the same assumptions as in Lemma 4.1, the inequality

$$\mathbb{E}_x\left[\tau_D^p\right] \leq \Gamma\left(p+1\right)\left(\sup_{x\in D}\mathbb{E}_x\left[\tau_D\right]\right)^p \tag{4.6}$$

holds for any $p \geq 1$. This leads us to the following conjecture.

Conjecture 4.1. The lower bounds (1.12) is sharp for any $p \geq 1$.

5. Extremal Domains for $M_{p,d}(\mathcal{C})$

The main goal of this section is to prove that the shape functional

$$G_{p,d}\left(D\right) = \lambda_1^p\left(D\right)\sup_{x\in D}\mathbb{E}_x\left[\tau_D^p\right]$$

admits a maximizer in the class of bounded convex domains in $\mathbb{R}^d$ or in the class of convex domains which are symmetric with respect to each coordinate axis.

5.1. **Motivation and preliminary discussions.** While balls appear to be extremals for several isoperimetric type inequalities (including the classical ones (1.1) and (1.3)), surprisingly they are not extremals for $M_{p,d}(\mathcal{X})$. For instance it is observed in [9, pg. 599] that

$$\lambda_1\left(B\right)\sup_{x\in B}\mathbb{E}_x\left[\tau_B\right] < \lambda_1\left(\mathbb{T}\right)\sup_{x\in\mathbb{T}}\mathbb{E}_x\left[\tau_\mathbb{T}\right], \tag{5.1}$$

where $\mathbb{T}$ is is the equilateral triangle (see also [35, Corollary 3.7]). Moreover it was conjectured in [35] that no extremal domain exists over the class of all domains.

Therefore it is reasonable, when looking for extremals, to restrict the class of domains. When restricted to the class of convex domains, Payne showed in [56] that

$$m_{1,d}(\mathcal{C}) = \frac{\pi^2}{4}. \tag{5.2}$$

From this it follows trivially that the minimizer domain over convex domains is given by the infinite slab $S_d = \mathbb{R}^{d-1} \times (-1, 1)$. The existence of extremal for $M_{1,d}(\mathcal{C})$ is proved in [35]. The authors further conjectured that when $d = 2$, the equilateral triangle $\mathbb{T}$ is an extremal. That is,

$$M_{1,2}(\mathcal{C}) = \lambda_1(\mathbb{T}) \sup_{x\in\mathbb{T}} \mathbb{E}_x[\tau_{\mathbb{T}}]. \tag{5.3}$$

5.2. **Motivation, symmetric convex domains.** Another class of domains that is worth investigation is the class of doubly symmetric planar domains $\mathcal{SC}$.

Regarding the question of extremals, it is not hard to see the ball fails again in the class $\mathcal{SC}$. In fact, with brief computations below we can show that

$$\lambda_1(B) \sup_{x\in B} \mathbb{E}_x[\tau_B] < \lambda_1(Q_2) \sup_{x\in Q_2} \mathbb{E}_x[\tau_{Q_2}]. \tag{5.4}$$

First note that in both cases

$$\sup_{x\in B} \mathbb{E}_x[\tau_B] = \mathbb{E}_{(0,0)}[\tau_B]$$

and

$$\sup_{x\in Q_2} \mathbb{E}_x[\tau_{Q_2}] = \mathbb{E}_{(0,0)}[\tau_{Q_2}].$$

Furthermore,

$$\lambda_1(B)\mathbb{E}_{(0,0)}[\tau_B] = \frac{j_0^2}{2} \approx 2.8916, \tag{5.5}$$

where j_0 is the first positive root of the first Bessel function.

On the other hand, $\lambda_1(Q_2) = \frac{\pi^2}{2}$ and by independence,

$$\mathbb{P}_{(0,0)}(\tau_{Q_2} > t) = \mathbb{P}_0(\tau_I > t)\mathbb{P}_0(\tau_I > t),$$

where $I = (-1, 1)$. The eigenfunction expansion for the heat kernel for the interval I (see [26, 47]) leads to the formula

$$\mathbb{E}_{(0,0)}[\tau_{Q_2}] = \left[1 - \frac{32}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \operatorname{sech}\left[\left(n + \frac{1}{2}\right)\pi\right]\right]. \tag{5.6}$$

Thus

$$\begin{aligned} \lambda_1(Q_2)\mathbb{E}_0[\tau_{Q_2}] &= \frac{\pi^2}{2}\left[1 - \frac{32}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \operatorname{sech}\left[\left(n + \frac{1}{2}\right)\pi\right]\right] \\ &\approx 2.90843 \end{aligned} \tag{5.7}$$

which verifies (5.4).

Given conjecture (5.3), it is reasonable to conjecture that the extremal for $M_{1,2}(\mathcal{SC})$, if exists, is given by the square $Q_2 := \{(x, y), |x| < 1, |y| < 1\}$. In the next two sections we will focus on proving the existence of the extremals for both classes $\mathcal{C}$ and $\mathcal{SC}$.

5.3. **Preliminary results.** For any given domain $D \subset \mathbb{R}^d$, define $\mathcal{M}_p(D) = \sup_{x\in D} \mathbb{E}_x[\tau_D^p]$ so that our function from (1.6) becomes

$$G_{p,d}(D) = \lambda_1^p(D) \sup_{x\in D} \mathbb{E}_x[\tau_D^p] = \lambda_1^p(D)\mathcal{M}_p(D).$$

Recall that for any two compact sets $K_1, K_2 \subset \mathbb{R}^d$ we define the *Hausdorff distance* $d_{\mathcal{H}}$ by

$$d_{\mathcal{H}}(K_1, K_2) = \max\left\{\sup_{x\in K_1} d(x, K_2), \sup_{y\in K_2} d(K_1, y)\right\},$$

where $d(\cdot,\cdot)$ denotes the Euclidean distance in $\mathbb{R}^d$. Therefore for any bounded open sets $A, B \subset \mathbb{R}^d$ we have that

$$d_{\mathcal{H}}(A,B) = \max\left\{ \sup_{x\in B\setminus A} \inf_{y\in\partial B} d(x,y), \sup_{x\in A\setminus B} \inf_{y\in\partial A} d(x,y) \right\}. \tag{5.8}$$

This definition is given by [36, Corollary 2.2.13]. In the sequel we use the fact that inclusion is stable under convergence with respect to $d_{\mathcal{H}}$. That is, take sets $U_n \subset D_n \subset \mathbb{R}^d$ for all n. If $U_n \to U$ with respect to $d_{\mathcal{H}}$ and $D_n \to D$ with respect to $d_{\mathcal{H}}$, then $U \subset D$.

Lemma 5.1. *If a sequence $\{D_n\}_{n=1}^{\infty}$ in $\mathcal{SC}$ converges to a set $D \in \mathcal{C}$ with respect to the Hausdorff metric, then $D \in \mathcal{SC}$.*

Proof. Note that D is open. Take any $x = (x_1, x_2, \ldots, x_d) \in D$, then $x \in D_n$ for n large enough. Since D_n is symmetric then $(-x_1, \ldots, x_d), (x_1, -x_2, \ldots, x_d), \ldots, (x_1, x_2, \ldots, -x_d) \in D_n$ for n large enough. Since inclusion is stable under limits of the Hausdorff distance then

$$(-x_1, \ldots, x_d), (x_1, -x_2, \ldots, x_d), \ldots, (x_1, x_2, \ldots, -x_d) \in D$$

as well. This shows D is symmetric with respect to all axes. Convexity is well know. □

We will need the following key estimates on the $p-$moments of exit times in order to prove that M_p is continuous in the class $\mathcal{SC}$ and $\mathcal{C}$.

Lemma 5.2. *Suppose $U \subset D \subset \mathbb{R}^d$, where D is a bounded Lipschitz domain and U is a domain.*

(i) If $p \geq 1$, then

$$\sup_{x\in D} \mathbb{E}_x\left[(\tau_D - \tau_U)^p\right] \leq C_{p,D} \sup_{x\in D\setminus U} \left(d(x,\partial D)\right)^{\beta}. \tag{5.9}$$

(ii) If $0 < p < 1$, then

$$\sup_{x\in D} \mathbb{E}_x\left[(\tau_D - \tau_U)^p\right] \leq C_{\beta,D} \sup_{x\in D\setminus U} \left(d(x,\partial D)\right)^{\beta p}. \tag{5.10}$$

Here $\beta > 0$ depends on the Lipschitz character of the domain.

Proof. Take $x \in U$. By the strong Markov property we have for any $p > 0$,

$$\begin{aligned} \mathbb{E}_x\left[(\tau_D - \tau_U)^p\right] &= \mathbb{E}_x\left[\mathbb{E}_{B_{\tau_U}}\left[\tau_D^p\right]\right] \\ &\leq \sup_{x\in\partial U} \mathbb{E}_x\left[\tau_D^p\right]. \end{aligned} \tag{5.11}$$

Under the assumption that D is a bounded Lipschitz domain, it follows that D is intrinsic ultracontractive (IU). That is, for any $\eta > 0$, there is a $t_0 = t_0(\eta, D) > 0$ such that for all $t > t_0$ and all $x, y \in D$

$$(1-\eta)\, e^{-\lambda_1(D)t}\varphi_1(x)\varphi_1(y) \leq K_D(x,y,t) \leq (1+\eta)\, e^{-\lambda_1(D)t}\varphi_1(x)\varphi_1(y) \tag{5.12}$$

where φ_1 is the ground state eigenfunction for D. In fact, (IU) holds for a wider class of domains (beyond Lipschitz) and wider class of diffusion. It has been extensively studied in the literature with many different applications. We refer the reader to [29] and [5] for some of the first results on this topic that include the Lipschitz domains case. Writing

$$H_D(x,y) = \int_0^{\infty} K_D(x,y,t)dt$$

for the Green's function for D, it follows trivially that for all IU domains D, $H_D(x,y) \geq C_D\varphi_1(x)\varphi_1(y)$, uniformly on $x, y \in D$. Integrating over D we see that

$$\mathbb{E}_x\left[\tau_D\right] \geq C_D\varphi_1(x). \tag{5.13}$$

Take $\eta = 1/2$. Let us first assume $p > 1$. Applying (5.12) we have for all $x \in D$,

$$\begin{aligned}
\mathbb{E}_x\left[\tau_D^p\right] &= p\int_0^\infty t^{p-1}\mathbb{P}_x\left(\tau_D > t\right)dt \\
&= p\int_0^{t_0} t^{p-1}\mathbb{P}_x\left(\tau_D > t\right)dt + p\int_{t_0}^\infty t^{p-1}\int_D K_D(x,y,t/2)dydt \\
&\le p\, t_0^{p-1}\mathbb{E}_x\left[\tau_D\right] + \frac{3}{2}\varphi_1(x)\, p\int_{t_0}^\infty t^{p-1}\int_D e^{-\lambda_1(D)t/2}\varphi_1(y)dydt \\
&\le C_1\mathbb{E}_x\left[\tau_D\right] + C_2\varphi_1(x).
\end{aligned}$$

where C_1, C_2 are constants that depend on p and D. Taking into account (5.13) we then obtain that

$$\mathbb{E}_x\left[\tau_D^p\right] \le C_{p,D}\mathbb{E}_x\left[\tau_D\right]$$

for some constant $C_{p,D}$ that only depend on p and D. Thus

$$\begin{aligned}
\sup_{x\in U}\mathbb{E}_x\left[(\tau_D - \tau_U)^p\right] &\le C_{p,D}\sup_{x\in\partial U}\mathbb{E}_x\left[\tau_D\right] \\
&\le C_{p,D}\sup_{x\in D\setminus U}\mathbb{E}_x\left[\tau_D\right] \qquad (5.14)
\end{aligned}$$

On the other hand, for $x \in D\setminus U$, we have $\mathbb{P}_x(\tau_U > 0) = 0$, then

$$\begin{aligned}
\sup_{x\in D\setminus U}\mathbb{E}_x\left[(\tau_D - \tau_U)^p\right] &= \sup_{x\in D\setminus U}\mathbb{E}_x\left[\tau_D^p\right] \\
&\le C_{p,D}\sup_{x\in D\setminus U}\mathbb{E}_x\left[\tau_D\right]. \qquad (5.15)
\end{aligned}$$

Recall the fact that for a bounded Lipshitz domains, $\mathbb{E}_x\left[\tau_D\right] \le C_D\left(d\left(x,\partial D\right)\right)^\beta$ where $\beta > 0$ depends on the Lipszhitz character of the domain. For the proof of the case $d = 2$, which extends to any $d \ge 2$, see [31, Proposition 2.3] or the remark in [10, pg 199]. This proves the case $p \ge 1$ in (i).

If $0 < p \le 1$, then Jensen's inequality gives that $\mathbb{E}_x\left[\tau_D^p\right] \le \left(\mathbb{E}_x\left[\tau_D\right]\right)^p$ and (ii) follows from (5.11) and and (5.15). □

Proposition 5.1 (Continuity of $\mathcal{M}_p$). For any $p > 0$, the functional $\mathcal{M}_p\left(D\right)$ is continuous in the class $\mathcal{C}$ or $\mathcal{SC}$ with respect to the Hausdorff metric.

Proof. Fix $p > 0$. We first prove $\mathcal{M}_p$ is continuous in the class $\mathcal{SC}$. Showing $\mathcal{M}_p$ is continuous in the class $\mathcal{C}$ is done similarly. Let $\{D_n\} \in \mathcal{SC}$ such that $D_n \to D \in \mathcal{SC}$ as $n \to \infty$ with respect to the Hausdorff metric. We show $\mathcal{M}_p\left(D_n\right) \to \mathcal{M}_p\left(D\right)$ as $n \to \infty$.

There exists a sequence $\{t_n\} \subset \mathbb{R}_+$ such that $t_n \to 1$ and $t_n D_n \subset D$ for every n. By monotonicity of exit times we have for all $x \in D$ almost surely that

$$\tau_{t_n D_n} \le \tau_D. \qquad (5.16)$$

If $0 < p < 1$, using the elementary inequality $a^p - b^p \le (a-b)^p$ whenever $0 < b \le a$, we have that $\mathbb{E}_x\left[\tau_D^p\right] \le \mathbb{E}_x\left[\tau_{t_n D_n}^p\right] + \mathbb{E}_x\left[(\tau_D - \tau_{t_n D_n})^p\right]$ for all $x \in D$. By Lemma 5.2 (ii) and (5.8) we have that

$$\begin{aligned}
\mathcal{M}_p\left(D\right) &\le \mathcal{M}_p\left(t_n D_n\right) + \mathbb{E}_x\left[(\tau_D - \tau_{t_n D_n})^p\right] \\
&\le \mathcal{M}_p\left(t_n D\right) + C_{\beta,D}\sup_{x\in D\setminus t_n D}\left(d\left(x,\partial D\right)\right)^{\beta p} \\
&\le \mathcal{M}_p\left(t_n D\right) + C_{\beta,D}\left(d_{\mathcal{H}}\left(D, t_n D\right)\right)^{\beta p} \qquad (5.17)
\end{aligned}$$

where the constant $C_{\beta,D}$ depends only on D.

For $p \geq 1$, using the elementary inequality $x^p - y^p \leq p x^{p-1}(x-y)$ whenever $0 < y \leq x$, we have that

$$\begin{aligned} \mathbb{E}_x\left[\tau_D^p - \tau_{t_n D_n}^p\right] &\leq p\mathbb{E}_x\left[\tau_D^{p-1}\left(\tau_D - \tau_{t_n D_n}\right)\right] \\ &\leq p\left(\mathbb{E}_x\left[\tau_D^p\right]\right)^{(p-1)/p}\left(\mathbb{E}_x\left[\left(\tau_D - \tau_{t_n D_n}\right)^p\right]\right)^{1/p}. \end{aligned} \tag{5.18}$$

Again by Lemma 5.2 (i), we have,

$$\begin{aligned} \sup_{x\in D}\mathbb{E}_x\left[\left(\tau_D - \tau_{t_n D_n}\right)^p\right] &\leq C_{p,D}\sup_{x\in D\setminus t_n D_n} d(x,\partial D)^\beta \\ &\leq C_{p,D}\, d_{\mathcal{H}}(D, t_n D_n)^\beta \end{aligned}$$

so that

$$\sup_{x\in D}\mathbb{E}_x\left[\tau_D^p - \tau_{t_n D_n}^p\right] \leq C_{p,D}^{1/p}\, p\, \mathcal{M}_p(D)^{(p-1)/p}\, d_{\mathcal{H}}(D, t_n D_n)^{\beta/p}. \tag{5.19}$$

Thus using (5.19) we have

$$\mathcal{M}_p(D) \leq \mathcal{M}_p(t_n D_n) + C_{p,D}^{1/p}\, p\, \mathcal{M}_p(D)^{(p-1)/p}\, d_{\mathcal{H}}(D, t_n D_n)^{\beta/p}. \tag{5.20}$$

Together with (5.17) we then conclude that there exist constants $C_{p,D}, C_{\beta,D} > 0$ such that

$$\begin{aligned} \mathcal{M}_p(D) \leq \mathcal{M}_p(t_n D_n) &+ C_{\beta,D}\left(d_{\mathcal{H}}(D, t_n D_n)\right)^p 1_{(p<1)} \\ &+ C_{p,D}^{1/p}\, p\, \mathcal{M}_p(D)^{(p-1)/p}\, d_{\mathcal{H}}(D, t_n D_n)^{\beta/p} 1_{(p\geq 1)} \end{aligned} \tag{5.21}$$

Combining (5.16), (5.21) and the fact that $\mathcal{M}_p(t_n D_n) = t_n^{2p}\mathcal{M}_p(D_n)$ gives the desired result. □

5.4. **Proof of Theorem 1.2 and a conjecture on the extremal.** We may finally prove our main result of this section.

Proof of Theorem 1.2. Fix $p > 0$. We consider the class of symmetric bounded convex domains $\mathcal{SC}$. The proof is the same for $\mathcal{C}$. Let $M_{p,d}(\mathcal{SC}) = \sup_{D\in\mathcal{SC}} G_{p,d}(D)$ and pick $\{D_n\} \subset \mathcal{SC}$ such that

$$\lim_{n\to\infty} G_{p,d}(D_n) = M_{p,d}(\mathcal{SC}).$$

By scaling we may assume the domains D_n are all contained in a fixed compact set K. By the Blaschke selection Theorem, there is a subsequence $\{D_{n_k}\} \subset \mathcal{SC}$ such that $D_{n_k} \to D \in \mathcal{SC}$ with respect to $d_{\mathcal{H}}$. By Lemma 5.1, we know that $D \in SC$. We can rename this subsequence D_n. By Equations (3.2) and (3.3) of [35, page 12] we know that D has a non-empty interior. By Proposition 5.1, $\mathcal{M}_p$ is continuous with respect to the Hausdorff metric in the class $\mathcal{SC}$ and $\lambda_1(D)$ is also well known to be continuous with respect to $d_{\mathcal{H}}$ (see [36]). Thus

$$G_{p,d}(D) = \lim_{n\to\infty} G_{p,d}(D_n) = M_{p,d}(\mathcal{SC}),$$

as needed. □

With the existence of extremals guaranteed for all dimension and all $0 < p < \infty$, we have the following.

Conjecture 5.2 (Conjecture for $M_{p,d}(\mathcal{SC})$). With the supremum taken over all domains in $\mathcal{SC}$, we have

$$M_{p,d}(\mathcal{SC}) = \lambda_1^p(Q_d)\mathbb{E}_0[\tau_{Q_d}^p],$$

where

$$Q_d = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : |x_i| < 1\},$$

denotes the unit cube in $\mathbb{R}^d$.

5.5. **Remarks on conjectures; rectangles, triangles, and ellipses.**

Remark 5.3 (Rectangles). Conjecture 5.2 in general seems to be nontrivial. In fact, even the simplest case of rectangles does not seem obvious. More precisely, let $\mathbf{a} = (a_1, a_2, \dots, a_d)$, where $a_k > 0$ for all k. Set $R_{\mathbf{a}} = \{x = (x_1, x_2, \dots, x_d) : |x_k| < a_k, k = 1, \dots, d\}$. (We call $R_{\mathbf{a}}$ a rectangle.) Denote the origin in $\mathbb{R}^d$ by $\mathbf{0}$. In this case we would want to show that for all $\mathbf{a} \in \mathbb{R}^d$,

$$\lambda_1^p(R_{\mathbf{a}})\mathbb{E}_{\mathbf{0}}\left[\tau_{R_{\mathbf{a}}}^p\right] \leq \lambda_1^p(Q_d)\mathbb{E}_{\mathbf{0}}\left[\tau_{Q_d}^p\right] \tag{5.22}$$

with equality only when $R_{\mathbf{a}} = Q_d$. Since the eigenvalues of both $R_{\mathbf{a}}$ and Q_d are explicit and the components of the Brownian motion are independent, the inequality (5.22) can be stated in several different forms. Here is one. Let $I_{a_k} = (-a_k, a_k)$ and recall that $I = (-1, 1)$. Then (5.22) is equivalent to

$$\left(\sum_{k=1}^{d} \frac{1}{a_k^2}\right)^p \int_0^\infty p\, t^{p-1} \prod_{k=1}^{d} \mathbb{P}_0(\tau_{I_{a_k}} > t)dt \leq d^p \int_0^\infty p\, t^{p-1} \left(\mathbb{P}_0(\tau_I > t)\right)^d dt. \tag{5.23}$$

Using the fact that $\mathbb{P}_0\left(\tau_{I_{a_k}} > t\right) = \mathbb{P}_0\left(\tau_I > \frac{t}{a_k^2}\right)$ we may even assume that

$$a_1 = 1 < a_2 < \cdots < a_d.$$

Using the fact that we know the heat kernel for an interval in terms of the eigenfunctions expansion (all which are explicitly given), the inequality has a rather appealing form. Let us look at the case $d = 2$ and $p = 1$. Then (5.23) is equivalent to

$$\begin{aligned} &\left(1 + a^2\right)\left[1 - \frac{32}{\pi^3}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \operatorname{sech}\left[\left(n + \frac{1}{2}\right)\frac{\pi}{a}\right]\right], \\ &\leq 2\left[1 - \frac{32}{\pi^3}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \operatorname{sech}\left[\left(n + \frac{1}{2}\right)\pi\right]\right], \end{aligned} \tag{5.24}$$

for all $a > 1$.

Unfortunately, despite its simplicity and all its possible formulations, we have not been able to fully verify (5.22) for all rectangles even in the case $d = 2$ and $p = 1$.

Remark 5.4 (Triangles). It may be of interest to mention as well that, to the best of our knowledge, the special case of Conjecture (5.3) for triangles does not seem to have been proven:

$$\lambda_1(T) \sup_{x \in T} \mathbb{E}_x[\tau_T] \leq \lambda_1(\mathbb{T}) \sup_{x \in \mathbb{T}} \mathbb{E}_x[\tau_{\mathbb{T}}], \tag{5.25}$$

for all triangles T, where $\mathbb{T}$ is the equilateral triangle. Furthermore, equality holds only when $T = \mathbb{T}$. As pointed out in [35, Corollary 3.7], with explicit expressions for $\mathbb{E}_x[\tau_{\mathbb{T}}]$ and $\lambda_1(\mathbb{T})$, we have

$$\lambda_1(\mathbb{T}) \sup_{x \in \mathbb{T}} \mathbb{E}_x[\tau_{\mathbb{T}}] = \frac{8\pi^2}{27} \approx 2.9243.$$

Combining this with (5.5) and (5.7), we see that

$$\lambda_1(B) \sup_{x \in B} \mathbb{E}_x[\tau_B] < \lambda_1(Q_2) \sup_{x \in Q_2} \mathbb{E}_x[\tau_{Q_2}] < \lambda_1(\mathbb{T}) \sup_{x \in \mathbb{T}} \mathbb{E}_x[\tau_{\mathbb{T}}]. \tag{5.26}$$

For any convex domain $D \subset \mathbb{R}^2$ with finite inradius R_D (supremum of radii of all disc contained in D), it holds that

$$\frac{1}{2}R_D^2 \leq \sup_{x \in D} \mathbb{E}_x[\tau_D] \leq \sup_{x \in S} \mathbb{E}_x[\tau_S] = R_D^2, \tag{5.27}$$

where $S \subset \mathbb{R}^d$ is the infinite strip of inradius R_D. The left hand side inequality is trivial by domain monotonicity of the exit time. For the second inequality, we refer the reader to [59]. For a different proof, which extends to all moments, see [12]. In [16], it is proved that

$$\lambda_1(T)R_T^2 \leq \lambda_1(\mathbb{T})R_{\mathbb{T}}^2 = \frac{4\pi^2}{9}, \tag{5.28}$$

with equality only when T is the equilateral triangle $\mathbb{T}$. For a different proof of (5.28) which uses dissymmetrization techniques, see [58]

Although the inequalities (5.25) and (5.28) are in fact quite different and one does not imply the other, the validity of one lends credibility to the validity of the other. One can also see, for example, that with (5.27) inequality (5.28) gives (5.25) with a factor of 2 on the right hand side.

Remark 5.5 (Ellipses). As a final remark we point out that for $p = 1$, both conjectures (5.3) and 5.2 hold for ellipses. In fact, the following stronger statement holds. Let

$$E_{a,b} := \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\right\}.$$

Then, with B the unit disc in $\mathbb{R}^2$,

$$\frac{\pi^2}{4} \leq \lambda_1(E_{a,b})\mathbb{E}_{(0,0)}[\tau_{E_{(a,b)}}] \leq \lambda_1(B)\mathbb{E}_{(0,0)}[\tau_B] = \frac{j_0^2}{2}, \tag{5.29}$$

To prove this inequality, it suffices to show that

$$\frac{\pi^2}{4}\left(\frac{a^2+b^2}{a^2b^2}\right) \leq \lambda_1(E_{a,b}) \leq \frac{j_0^2}{2}\left(\frac{a^2+b^2}{a^2b^2}\right). \tag{5.30}$$

Assuming for the moment the validity of (5.30), observe that since it is easy to check that

$$\mathbb{E}_{(x,y)}\left[\tau_{E_{a,b}}\right] = \frac{a^2b^2 - b^2x^2 - a^2y^2}{(a^2+b^2)},$$

by showing that the right hand side satisfies $\frac{1}{2}\Delta u = -1$ with zero boundary conditions, we have

$$\mathbb{E}_{(0,0)}\left[\tau_{E_{a,b}}\right] = \frac{a^2b^2}{a^2+b^2}$$

Thus the right hand side of (5.30) implies the right hand side of (5.29).

The left hand side of (5.30) is trivial by domain monotonicity. Since $E_{a,b} \subset (-a, a) \times (-b, b)$, it follows immediately that

$$\lambda_1(E_{a,b}) \geq \lambda_1((-a, a) \times (-b, b)) = \frac{\pi^2}{4}\left(\frac{a^2+b^2}{a^2b^2}\right).$$

The right hand side inequality in (5.30) is due to Polyá and Szegö and can be found in [57, pg. 98]. Their proof is based on the technique known as conformal transplantation. To do so, one can use a test function $\varphi(x, y)$ with $\varphi\,|_{\partial E_{a,b}} = 0$ which is an obvious modification of the eigenfunction for the disc and plug it into the Rayleigh quotient. Such function is given by

$$\varphi(x, y) = J_0\left(j_0\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}\right),$$

where J_0 is the first Bessel function and j_0 is its first positive root. See [57] for details.

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Department of Mathematics, Purdue University, West Lafayette, IN 47907, U.S.A.
Email address: banuelos@purdue.edu

Department of Mathematics, Union College, Schenectady, NY 12308, U.S.A.
Email address: marianop@union.edu

Department of Mathematics, Purdue University, West Lafayette, IN 47907, U.S.A.
Email address: jingwang@purdue.edu