Mathematical Biosciences 356 (2023) 108967

Contents lists available at ScienceDirect

Mathematical Biosciences

journal homepage: www.elsevier.com/locate/mbs

Original Research Article

Imperfect vaccine can yield multiple Nash equilibria in vaccination games

Ian B. Augsburger [a], Grace K. Galanthay [b], Jacob H. Tarosky [c], Jan Rychtář [d,*], Dewey Taylor [d]

[a] Department of Applied Mathematics and Statistics, Johns Hopkins University, Baltimore, MD 21218, USA
[b] Department of Mathematics and Computer Science, College of the Holy Cross, Worcester, MA 01610, USA
[c] Department of Mathematical Sciences, High Point University, High Point, NC 27268, USA
[d] Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, VA 23284, USA

ARTICLE INFO

Keywords:
Nash equilibrium
Imperfect vaccination
Multiple equilibria
Game theory
Golden ratio

ABSTRACT

As infectious diseases continue to threaten communities across the globe, people are faced with a choice to vaccinate, or not. Many factors influence this decision, such as the cost of the disease, the chance of contracting the disease, the population vaccination coverage, and the efficacy of the vaccine. While the vaccination games in which individuals decide whether to vaccinate or not based on their own interests are gaining in popularity in recent years, the vaccine imperfection has been an overlooked aspect so far. In this paper we investigate the effects of an imperfect vaccine on the outcomes of a vaccination game. We use a simple SIR compartmental model for the underlying model of disease transmission. We model the vaccine imperfection by adding vaccination at birth and maintain a possibility for the vaccinated individual to become infected. We derive explicit conditions for the existence of different Nash equilibria, the solutions of the vaccination game. The outcomes of the game depend on the complex interplay between disease transmission dynamics (the basic reproduction number), the relative cost of the infection, and the vaccine efficacy. We show that for diseases with relatively low basic reproduction numbers (smaller than about 2.62), there is a little difference between outcomes for perfect or imperfect vaccines and thus the simpler models assuming perfect vaccines are good enough. However, when the basic reproduction number is above 2.62, then, unlike in the case of a perfect vaccine, there can be multiple equilibria. Moreover, unless there is a mandatory vaccination policy in place that would push the vaccination coverage above the value of unstable Nash equilibrium, the population could eventually slip to the “do not vaccinate” state. Thus, for diseases that have relatively high basic reproduction numbers, the potential for the vaccine not being perfect should be explicitly considered in the models.

1. Introduction

Vaccination is one of the most effective ways to control and prevent the spread of infectious diseases [1]. As infectious diseases continue to threaten communities across the globe, people are faced with a choice to vaccinate, or not. Many factors influence this decision, such as the cost of the disease, the chance of contracting the disease, the population vaccination coverage, and the efficacy of the vaccine [2].

Individuals tend to choose a vaccination strategy that has the highest personal benefit, often without considering how their choice could lead to vaccination levels that are suboptimal for the entire population [3,4]. When the risk of infection and cost of the disease are high, it is optimal for an individual to vaccinate. However, as the proportion of vaccinated people in a population increases, the incentive for unvaccinated people to vaccinate decreases, leading to sub-optimal vaccination levels in the population [5].

Vaccination game theory gained popularity in recent years because it is useful in studying complex scenarios in which self-interested individuals take actions based on the decisions of the rest of the population [6]. As argued in [7], by incorporating human behavior, vaccination games provide more insight and better predictions than simpler compartmental ODE models. Vaccination game theory is thus an excellent tool for predicting vaccination coverage [8,9]. The game theory has been applied to modeling diseases such as smallpox [10,11], chickenpox [12], monkeypox [13,14], polio [5,15], measles [16], influenza [17], Ebola [18], COVID-19 [19–21], chikungunya [22], Hepatitis B [23,24], typhoid [25], and cholera [26].

The above models are all for perfect vaccines, i.e., individuals gain immunity once vaccinated. However, even the most effective vaccines are not perfect. The yellow fever vaccine has efficacy 99% [27], MMR vaccine is 97% effective against measles and rubella, and 88% effective against mumps [28], the chickenpox (varicella) vaccine is only 90% effective [29,30]. Imperfect vaccines have still provided many benefits, such as reducing the severity of symptoms (i.e. lowering the cost of

---

* Corresponding author.

E-mail addresses: iaugsbu1@jhu.edu (I.B. Augsburger), gkgala24@g.holycross.edu (G.K. Galanthay), jtarosky@highpoint.edu (J.H. Tarosky), rychtarj@vcu.edu (J. Rychtář), dttaylor2@vcu.edu (D. Taylor).

https://doi.org/10.1016/j.mbs.2023.108967
Received 8 August 2022; Received in revised form 13 December 2022; Accepted 7 January 2023

the disease), decreasing the rate of transmission, and reducing the prevalence of the disease in the population [1].

Imperfect vaccines in general have been studied both from mathematical [31] and biological [32] perspectives. Models have been developed for HIV/AIDS [33], SARS [34], malaria [35], cholera [36], tuberculosis [37], and COVID-19 [38]. Additionally, Magori and Park [39] studied the consequences of imperfect vaccines, Peng et al. [40] studied susceptible-infected-susceptible epidemic model with imperfect vaccination on dynamic contact networks, Abboubakar et al. [41] explored imperfect vaccines as a way to control arboviral diseases, and Arino and Milliken [42] related imperfect vaccines to the treatment of COVID-19.

There are two main mechanisms of imperfect vaccines: "all-or-nothing" and "leaky" vaccine [43]. An "all-or-nothing" vaccine offers full protection to a fraction of vaccinated individuals but does not offer any protection at all to the remainder of the vaccinated population. On the other hand, a "leaky" vaccine offers partial protection to every vaccinated individual. Both kind of mechanisms have been extensively modeled; see for example [44–46] for all-or-nothing vaccines and [47–50] for leaky vaccines.

From the vaccination game theory perspective, Wu et al. [51] was one of the first models to consider imperfect vaccines. They found thresholds for basic reproduction number to influence the outcomes. Kuga and Tanimoto [52] found out that even an imperfect vaccine is slightly better than a defense against contagion such as wearing a mask. Choi and Shim [20] incorporated imperfect vaccine to COVID-19 model and Augsburger et al. [14] considered imperfect smallpox vaccine in the prevention of monkeypox. The last two studies empirically found the existence of multiple equilibria but did not systematically study the conditions that yield to them.

In our study, we extend a simple SIRV model developed by Bauch and Earn [6]. We find steady states of the underlying disease transmission dynamics. We add the game-theoretical component and solve the resulting vaccination game. We derive explicit conditions for different Nash equilibria to exist. We show that there is a region of parameter values where multiple Nash equilibria can exist at the same time.

2. Model

2.1. Compartmental ODE model

We extend a simple SIRV model developed by Bauch and Earn [6] by adding the possibility of vaccinated individuals getting infected.

The population is divided into four groups: susceptible (S), vaccinated (V), infected (I), and recovered individuals (R). Individuals are born at a rate Λ. The proportion $\overline{p}$ of the newborns are vaccinated and enter the V compartment. The remaining proportion $1-\overline{p}$ enters the susceptible compartment. The susceptible individuals get infected at rate $\beta\frac{I}{N}$ where $N = S + I + R + V$ is the total population size. The vaccine is not perfect, and thus even vaccinated individuals can become infected. The vaccine efficacy is e, and thus the vaccinated individuals become infected at rate $(1-e)\beta\frac{I}{N}$. The infected individuals recover at rate γ and the recovered individuals gain lifelong immunity. All individuals die at a rate μ. The dynamics are summarized in Fig. 1. The notation is summarized in Table 1.

The model yields the following system of differential equations.

$$\frac{dS}{dt} = (1-\overline{p})\Lambda - \left(\mu + \beta\frac{I}{N}\right)S \tag{1}$$

$$\frac{dI}{dt} = \beta\frac{I}{N}\big(S + (1-e)V\big) - (\mu+\gamma)I \tag{2}$$

$$\frac{dR}{dt} = \gamma I - \mu R \tag{3}$$

$$\frac{dV}{dt} = \overline{p}\Lambda - \left(\mu + (1-e)\beta\frac{I}{N}\right)V \tag{4}$$

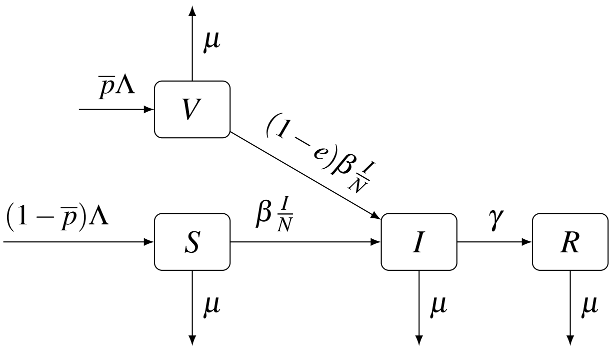


Fig. 1. Scheme of the SIRV model.

Table 1
Model notation.

Notation	Meaning
S	Susceptible individuals
I	Infected individuals
R	Recovered individuals
V	Vaccinated individuals
N	Total population size
μ	Natural death rate
Λ	Natural birth rate
β	Disease effective transmission rate
γ	Recovery rate
e	Vaccine efficacy
$\overline{p}$	Proportion of vaccinated newborns
C_D	Cost of "disease" relative to the cost of vaccination

As derived in Appendix A, the system has only one equilibrium $\mathscr{E}^* = (S^*, I^*, R^*, V^*)$ given as follows.

$$I^* = \begin{cases} 0, & \text{if } \mathscr{R}_0(1-e\overline{p}) \le 1,\\ \frac{\Lambda}{\beta}\big(\mathscr{R}_0(1-\overline{p})-1\big), & \text{if } e=1 \text{ and } \mathscr{R}_0(1-\overline{p}) > 1,\\ \frac{\Lambda}{\beta}(\mathscr{R}_0-1), & \text{if } \overline{p}=0 \text{ and } \mathscr{R}_0 > 1,\\ \frac{\Lambda}{\beta}\left(\mathscr{R}_0 - \frac{1}{1-e}\right), & \text{if } \overline{p}=1,\ e<1, \text{ and } \mathscr{R}_0(1-e) > 1,\\ \frac{\Lambda}{\beta}\,\frac{-b+\sqrt{b^2+4(1-e)c}}{2(1-e)}, & \text{if } e<1,\ \overline{p}\in(0,1) \text{ and } \mathscr{R}_0(1-e\overline{p}) > 1. \end{cases} \tag{5}$$

where

$$\mathscr{R}_0 = \frac{\beta}{\mu+\gamma} \tag{6}$$

is the basic reproduction number in the unvaccinated population and

$$b = 1 + (1-e)(1-\mathscr{R}_0) \tag{7}$$

$$c = \mathscr{R}_0(1-e\overline{p}) - 1. \tag{8}$$

Furthermore,

$$S^* = \frac{(1-\overline{p})N}{1+\frac{\beta}{\Lambda}I^*} \tag{9}$$

$$R^* = \frac{\gamma}{\mu}I^* \tag{10}$$

$$V^* = \frac{\overline{p}N}{1+\frac{(1-e)\beta}{\Lambda}I^*}. \tag{11}$$

We show the detailed calculations in the Appendix A.

Let p_{HI} be the minimal level of vaccination needed for achieving a herd immunity; specifically let $p_{HI} \in [0,1]$ be such that $\mathscr{R}_0(1-e\overline{p}) < 1$ for all $\overline{p} > p_{HI}$. With this interpretation, we are using $p_{HI} = 1$ as a shortcut for $\mathscr{R}_0(1-e\overline{p}) > 1$ for all $\overline{p} \in [0,1]$, i.e., if $p_{HI} = 1$, then the disease will not be eradicated even if the whole population gets vaccinated. We find

$$p_{HI} = \begin{cases} 0 & \text{if } \mathscr{R}_0 \le 1 \\ \frac{\beta-(\mu+\gamma)}{e\beta} & \text{if } 0 < 1 - \frac{1}{\mathscr{R}_0} \le e \le 1 \\ 1 & \text{if } 0 \le e < 1 - \frac{1}{\mathscr{R}_0}. \end{cases} \tag{12}$$

2.2. Game-theoretical framework

We extend the above ODE model by incorporating the game-theoretic component as done in [6].

A vaccination game is played by susceptible individuals. The individuals decide whether to vaccinate or stay unvaccinated. They are assumed to be rational, having complete information about the disease epidemic, and acting in their own interest. It means that they can evaluate the potential costs and benefits of their own action depending on the actions of others and they choose the action that maximizes the net payoffs (benefits minus costs). The key is that the payoff to the individual is a function that depends on the action of that individual and the actions of other players. The payoff incorporates the cost of the vaccination, the risk of getting infected, and the costs of the disease.

The cost of vaccination includes a combination of monetary costs (such as a copay or lost wages at work for taking time off to get vaccinated, etc.) and sometimes hard to quantify costs in terms of vaccine side-effects. For example, for COVID-19 vaccines, there is a variety of side-effects [53] that can range from relatively mild [54] to more serious [55]. The cost of disease is also a combination of monetary costs of medical expenses, potentially lost wages or productivity, and/or loss of quality of life. For simplicity, we will assume that the population is homogeneous and thus all individuals incur the same costs. With this, we can assume that the cost of vaccination is 1 and the cost of the disease, C_D, is expressed relative to the cost of the vaccination.

Assume that the whole population, aside from a single focal individual, adopts a strategy $\overline{p}$. If the population is large enough, the decision of a single individual will not have a significant impact on the steady state of the dynamics. Thus, in this population, the number of infectious individuals I^* is given by (5).

The probability that a susceptible unvaccinated individual contracts a disease in this is thus given by

$$\pi_{NV} = \frac{\beta \frac{I^*}{N}}{\beta \frac{I^*}{N} + \mu}. \tag{13}$$

Similarly, the probability that a vaccinated individual contracts the disease is

$$\pi_V = \frac{(1-e)\beta \frac{I^*}{N}}{(1-e)\beta \frac{I^*}{N} + \mu}. \tag{14}$$

As shown in Appendix A, I^* is a decreasing function of $\overline{p}$ whenever $I^* > 0$. Thus, π_{NV} and π_V are decreasing concave down functions on $[0, p_{HI}]$; see Fig. 2.

The incentive function $h(\overline{p})$ for an individual to vaccinate when the population vaccination coverage is $\overline{p}$ is given by a difference between the expected costs if not vaccinated and the cost if vaccinated (including the potential cost of contracting the disease). If not vaccinated, the individuals will contract the disease with probability π_{NV} and in that instance pay the cost C_D, the cost of the disease relative to the cost of the vaccine. If vaccinated, the individuals have to pay the unit vaccination cost and they can still contract the disease with probability π_V. Thus,

$$h(\overline{p}) = C_D(\pi_{NV} - \pi_V) - 1. \tag{15}$$

The solution of the vaccination game is called Nash equilibrium and it is the population vaccination coverage p_{NE} such that in this population no individual has an incentive to deviate from the population strategy. It means that

$$p_{NE} = \begin{cases} 0, & \text{if } h(0) < 0, \\ 1, & \text{if } h(1) > 0, \\ p \in (0,1), & \text{if } h(p) = 0. \end{cases} \tag{16}$$

We note that the above three options are, a priori, not mutually exclusive and we will indeed see later that multiple Nash equilibria can exist. At the same time, when $e = 1$, $\pi_V = 0$ and it was demonstrated already in [6] that there is a unique $p_{NE} \in [0, 1)$. When $e = 1$ and the vaccine is perfect, $\pi_V = 0$ and thus the shape of the incentive function is the same as the shape of π_{NV}. In particular, h is decreasing since π_{NV} is decreasing in $\overline{p}$. However, when $e < 1$, the incentive function h can be increasing for some values of $\overline{p}$ which yields multiple Nash equilibria; see Figs. 2 and 4.

Furthermore, the first two cases of (16) are convergently stable Nash equilibria (CSNE), i.e., regardless of what (nearby) strategy is adopted in the population, individuals should start to play strategies closer to the NE, and ultimately adopt the NE strategy [6]. The third case is CSNE when $h'(p) < 0$. A NE is CSNE if (1) when $\overline{p} > p_{NE}$ and $\overline{p} \in (0, 1)$ is close enough to p_{NE}, then $h(\overline{p}) < 0$, i.e., the individual prefers not to vaccinate (and thus the population vaccination strategy will decrease once enough individuals make the optimal choice), and (2) when $\overline{p} < p_{NE}$ and $\overline{p} \in (0, 1)$ is close enough to p_{NE}, then $h(\overline{p}) > 0$, i.e., the individual prefers to vaccinate (and the population vaccination strategy will increase once enough individuals make the optimal choice). In either case, the population coverage $\overline{p}$ will tend to p_{NE}. Finally, when $h(p) = 0$ and $h'(p) > 0$, then $p_{NE} = p$ is NE but not CSNE.

3. Analysis

To find and classify Nash equilibria, we have to find and classify roots of the incentive function h. Because the formulas for h are tedious, instead of studying the function h directly, we will study the auxiliary function

$$\widetilde{h}(x) = C_D\left(\frac{x}{x+1} - \frac{(1-e)x}{(1-e)x+1}\right) - 1 \tag{17}$$

and use the following Lemma to find the correspondence between the roots and signs of h and $\widetilde{h}$.

Lemma 1. *For every $\overline{p} \in [0, 1]$, $h(\overline{p}) = \widetilde{h}\left(\frac{\beta}{\Lambda} I^*(\overline{p})\right)$.*

Proof. By (15), (13), and (14),

$$\begin{aligned} h(\overline{p}) &= C_D(\pi_{NV} - \pi_V) - 1 && (18)\\ &= C_D\left(\frac{\beta \frac{I^*}{N}}{\beta \frac{I^*}{N} + \mu} - \frac{(1-e)\beta \frac{I^*}{N}}{(1-e)\beta \frac{I^*}{N} + \mu}\right) - 1 && (19)\\ &= C_D\left(\frac{\beta \frac{I^*}{N\mu}}{\beta \frac{I^*}{N\mu} + 1} - \frac{(1-e)\beta \frac{I^*}{N\mu}}{(1-e)\beta \frac{I^*}{N\mu} + 1}\right) - 1. && (20) \end{aligned}$$

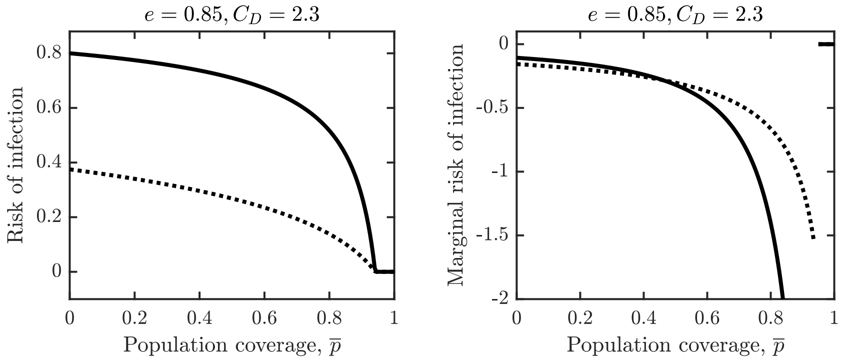
This concludes the proof because, as shown in Appendix B, $N = \frac{\Lambda}{\mu}$, i.e., $N\mu = \Lambda$. ☐

While we are primarily interested in the behavior of the incentive function $h(\overline{p})$ on $[0, 1]$, it will be easier to investigate the function $\widetilde{h}(x)$ on $[0, \infty)$.

Lemma 2. *For $x \in [0, \infty)$, the sign of $\widetilde{h}(x)$ is the opposite of the sign of*

$$q(x) = (1-e)x^2 + (2 - e(1 + C_D))x + 1. \tag{21}$$

In particular, the roots of $\widetilde{h}(x)$ are given by the roots of the function $q(x)$.



(a) (b)

Fig. 2. (a) Risks of infection if unvaccinated, π_{NV} (solid line), or if vaccinated, π_V (dotted line). (b) Marginal risks of infection if unvaccinated, $\frac{d\pi_{NV}}{dp}$ (solid line), or if vaccinated, $\frac{d\pi_V}{dp}$ (dotted line). Parameter values are $\Lambda = 5, \mu = 1, \gamma = 1, \beta = 10$, i.e., $\mathscr{R}_0 = 5$. While $\pi_V \le \pi_{NV}$ for all $\overline{p}$, the interesting behavior and multiple Nash equilibria are result of the fact that $\frac{d\pi_V}{dp}$ can be less than $\frac{d\pi_{NV}}{dp}$, i.e., the incentive function can be increasing when the vaccine is not perfect. Note that the marginal risk of infection is not continuous at $\overline{p} = p_{HI}$ with the value being very low for $\overline{p} < p_{HI}$ and 0 for $\overline{p} > p_{HI}$.

Proof.

$$\widetilde{h}(x) = C_D\left(\frac{x}{x+1} - \frac{(1-e)x}{(1-e)x+1}\right) - 1 = \frac{C_D xe}{(x+1)((1-e)x+1)} - 1 \tag{22}$$

$$= -\frac{(1-e)x^2 + (2-e(1+C_D))x + 1}{(x+1)((1-e)x+1)} = -\frac{q(x)}{(x+1)((1-e)x+1)}. \quad \square \tag{23}$$

Next, we show an intuitive fact that if $C_D < 1$, i.e., if the vaccine costs more than the disease, then nobody would vaccinate.

Lemma 3. *If $C_D < 1$, then $\widetilde{h}(x) < 0$ for all $x \in [0, \infty)$ and $h(\overline{p}) < 0$ for all $\overline{p} \in [0, 1]$.*

Proof. If $C_D < 1$, then, for any $x \in [0, \infty)$,

$$q(x) = (1-e)x^2 + (2-e(1+C_D))x + 1 \tag{24}$$

$$> (1-e)x^2 + (2-e(1+1))x + 1 \tag{25}$$

$$= (1-e)(x+1)^2 + e > 0. \tag{26}$$

The conclusions thus follow from Lemmas 2 and 1. ☐

For the rest of the analysis, we will thus assume $C_D > 1$ unless explicitly stated otherwise.

For $e = 1$, the behavior of $\widetilde{h}$ is easy to understand and it is summarized in the next Lemma.

Lemma 4. *If $e = 1$ (and $C_D > 1$), the function $\widetilde{h}$ is increasing on $[0, \infty)$ and the only root is given by*

$$x_{e=1} = \frac{1}{C_D - 1}. \tag{27}$$

Proof. When $e = 1$, by (17), $\widetilde{h}(x) = C_D \frac{x}{x+1} - 1$ which is an increasing function with the only root given by (27). ☐

Thus, unless explicitly stated otherwise, we will assume $e < 1$, i.e., we will be dealing with a quadratic function in (21).

The next Lemma quantifies the intuitive fact that if the vaccine efficacy e is too low, then there is no incentive to vaccinate.

Lemma 5. *If*

$$\frac{4C_D}{(1+C_D)^2} > e, \tag{28}$$

(and $e < 1$ and $C_D > 1$), $\widetilde{h}(x) < 0$ for $x \in [0, \infty)$ and $h(\overline{p}) < 0$ for $\overline{p} \in [0, 1]$.

Proof. If (28) holds, then $4C_D > e(1+C_D)^2$. Thus, the discriminant for the quadratic function (21) given by

$$(2-e(1+C_D))^2 - 4(1-e) = 4 - 4e(1+C_D) + e^2(1+C_D)^2 - 4 + 4e \tag{29}$$

$$= -4eC_D + e^2(1+C_D)^2 \tag{30}$$

is negative. Since $q(0) = 1 > 0$, $q(x) > 0$ for all x. By Lemma 2, $\widetilde{h}(x) < 0$ for all $x \in [0, \infty)$. By Lemma 1, $h(\overline{p}) < 0$ for all $\overline{p} \in [0, 1]$. ☐

3.1. Condition for “do not vaccinate” to be NE and CSNE

First, we prove a general theorem that characterizes the existence of NE at 0. We note that when 0 is Nash equilibrium, it is automatically CSNE as discussed at the end of Section 2.2.

Theorem 1. *$\overline{p} = 0$ is Nash equilibrium if and only if*

$$e < \frac{\mathscr{R}_0^2}{(\mathscr{R}_0 - 1)\cdot(C_D + \mathscr{R}_0)}. \tag{31}$$

Proof. By (16), the strategy to not vaccinate is Nash equilibrium if and only if $h(0) < 0$. This is, by Lemma 1, equivalent to $\widetilde{h}\left(\frac{\beta}{\Lambda}I^*(0)\right) < 0$, i.e., by (5), to $\widetilde{h}(\mathscr{R}_0 - 1) < 0$. By Lemma 2, the last statement is equivalent to $q(\mathscr{R}_0 - 1) > 0$, i.e., to

$$0 < (1-e)(\mathscr{R}_0 - 1)^2 + (2-e(1+C_D))(\mathscr{R}_0 - 1) + 1. \tag{32}$$

The last statement is equivalent to (31). ☐

To fully understand the condition (31), we will use the following Lemma which investigates the behavior of the right-hand side of (31) as a function of $\mathscr{R}_0$ or as a function of C_D.

Lemma 6.

$$\frac{\mathscr{R}_0^2}{(\mathscr{R}_0 - 1)\cdot(C_D + \mathscr{R}_0)} \ge \frac{4C_D}{(C_D+1)^2}. \tag{33}$$

The equality in (33) happens if and only if one of the following equivalent statements is true:

1. *$\mathscr{R}_0 = \frac{2C_D}{C_D - 1} > 2$,*
2. *$\mathscr{R}_0 > 2$ and $C_D = 1 + \frac{2}{\mathscr{R}_0 - 2}$.*

Moreover, when the equality in (33) happens, then

$$\frac{4C_D}{(C_D+1)^2} = 1 - \frac{1}{(\mathscr{R}_0 - 1)^2}. \tag{34}$$

Proof. Consider C_D fixed for a moment and let us study $f(r) = \frac{r^2}{(r-1)\cdot(C_D+r)}$ as a function of $r \in [1,\infty)$. We have $f'(r) \lesseqgtr 0$ if and only if $r^2(C_D-1) - 2rC_D \lesseqgtr 0$. Thus, the function attains its minimum on $[1,\infty)$ when $r = \frac{2C_D}{C_D-1}$. Furthermore, $f\left(\frac{2C_D}{C_D-1}\right) = \frac{4C_D}{(C_D+1)^2}$. This concludes the proof of the first part.

Note that $\frac{2C_D}{C_D-1} > 2$ and thus the inequality (33) is always strict for $\mathscr{R}_0 \leq 2$.

When $2 < \mathscr{R}_0 = \frac{2C_D}{C_D-1}$, then $C_D = 1 + \frac{2}{\mathscr{R}_0-2}$ which concludes the proof of the second part.

Finally, if $C_D = 1 + \frac{2}{\mathscr{R}_0-2}$, then

$$\frac{4C_D}{(C_D+1)^2} = \frac{4\frac{\mathscr{R}_0}{\mathscr{R}_0-2}}{\left(\frac{2\mathscr{R}_0-2}{\mathscr{R}_0-2}\right)^2} = \frac{\mathscr{R}_0(\mathscr{R}_0-2)}{(\mathscr{R}_0-1)^2} = 1 - \frac{1}{(\mathscr{R}_0-1)^2}. \quad \square \tag{35}$$

3.2. Condition for “always vaccinate” to be NE and CSNE

First, we prove a general theorem that characterizes the existence of NE at 1. We note that when 1 is Nash equilibrium, it is automatically CSNE as discussed at the end of Section 2.2.

Theorem 2. *The following statements are equivalent.*

1. $\overline{p} = 1$ *is Nash equilibrium.*
2. C_D *is high enough, specifically*

$$C_D > \frac{\mathscr{R}_0^2(1-e)^2 - e}{e((1-e)\mathscr{R}_0 - 1)} - 1. \tag{36}$$

3. *The vaccine efficacy satisfies*

$$\frac{-B - \sqrt{B^2 - 4A\mathscr{R}_0}}{2A} < e < \frac{-B + \sqrt{B^2 - 4A\mathscr{R}_0}}{2A} \tag{37}$$

where

$$A = 1 + C_D + \mathscr{R}_0 \tag{38}$$

$$B = -2\mathscr{R}_0 - 1 - C_D\left(1 - \frac{1}{\mathscr{R}_0}\right) \tag{39}$$

Proof. By (16), the strategy to always vaccinate is NE if and only if $h(1) > 0$. This is, by Lemma 1, equivalent to $\widetilde{h}\left(\frac{\beta}{\Lambda}I^*(1)\right) > 0$, i.e., by (5), to $\widetilde{h}\left(\mathscr{R}_0 - \frac{1}{1-e}\right) > 0$. By Lemma 2, the last statement is equivalent to $q\left(\mathscr{R}_0 - \frac{1}{1-e}\right) < 0$, i.e., to

$$0 > (1-e)\left(\mathscr{R}_0 - \frac{1}{1-e}\right)^2 + (2 - e(1+C_D))\left(\mathscr{R}_0 - \frac{1}{1-e}\right) + 1 \tag{40}$$

$$= \mathscr{R}_0^2(1-e) - \frac{1}{1-e} - e(1+C_D)\mathscr{R}_0 + \frac{e(1+C_D)}{1-e} + 1, \tag{41}$$

which is equivalent to (36).

The equivalence of the last two statements can be shown by algebraic manipulations. $\square$

Note that as e approaches $1 - \frac{1}{\mathscr{R}_0}$ from below or 0 from above, the right hand side of (36) approaches ∞.

To better understand conditions on C_D and e under which $\overline{p} = 1$ can be NE, we will use the following two Lemmas.

Lemma 7. *The strategy to always vaccinate can be NE only if*

$$C_D > \frac{1}{\left(1 - \frac{1}{\sqrt{\mathscr{R}_0}}\right)^2}. \tag{42}$$

Proof. The condition (37) can be satisfied only if

$$0 < B^2 - 4A\mathscr{R}_0 = C_D^2\left(1 - \frac{1}{\mathscr{R}_0}\right)^2 - 2C_D\left(1 + \frac{1}{\mathscr{R}_0}\right) + 1. \tag{43}$$

This is possible only if either

$$C_D > \frac{2\left(1 + \frac{1}{\mathscr{R}_0}\right) + \sqrt{4\left(1 + \frac{1}{\mathscr{R}_0}\right)^2 - 4\left(1 - \frac{1}{\mathscr{R}_0}\right)^2}}{2\left(1 - \frac{1}{\mathscr{R}_0}\right)^2} = \frac{1 + \frac{1}{\mathscr{R}_0} + \sqrt{\frac{2}{\mathscr{R}_0}}}{\left(1 - \frac{1}{\mathscr{R}_0}\right)^2} \tag{44}$$

$$= \frac{1}{\left(1 - \frac{1}{\sqrt{\mathscr{R}_0}}\right)^2} \tag{45}$$

or

$$C_D < \frac{2\left(1 + \frac{1}{\mathscr{R}_0}\right) - \sqrt{4\left(1 + \frac{1}{\mathscr{R}_0}\right)^2 - 4\left(1 - \frac{1}{\mathscr{R}_0}\right)^2}}{2\left(1 - \frac{1}{\mathscr{R}_0}\right)^2} = \frac{1}{\left(1 + \frac{1}{\sqrt{\mathscr{R}_0}}\right)^2} < 1. \tag{46}$$

Since we need $C_D > 1$, $\overline{p} = 1$ can be NE only if this happens only if (42). $\square$

Lemma 8. *In the notation of Theorem 2,*

$$1 - \frac{1}{\mathscr{R}_0} > \frac{-B + \sqrt{B^2 - 4A\mathscr{R}_0}}{2A}. \tag{47}$$

and

$$\frac{4C_D}{(1+C_D)^2} \leq \frac{-B + \sqrt{B^2 - 4A\mathscr{R}_0}}{2A}. \tag{48}$$

Moreover, the equality in (48) happens if and only if

$$e = 1 - \left(\frac{1 + \sqrt{1 + 4\mathscr{R}_0}}{2\mathscr{R}_0}\right)^2, \text{ and} \tag{49}$$

$$C_D = \frac{2}{e} - 1 + 2\sqrt{\frac{1}{e^2} - \frac{1}{e}}. \tag{50}$$

Proof. To proof the first inequality, substitute $e = 1 - \frac{1}{\mathscr{R}_0}$ into (41). Since $1 - e = \mathscr{R}_0^{-1}$, we get

$$\mathscr{R}_0^2(\mathscr{R}_0^{-1}) - \mathscr{R}_0 - \left(1 - \frac{1}{\mathscr{R}_0}\right)(1 + C_D)\mathscr{R}_0 + \frac{\left(1 - \frac{1}{\mathscr{R}_0}\right)(1 + C_D)}{\mathscr{R}_0^{-1}} + 1 = 1 > 0. \tag{51}$$

Thus, as in the proof of Theorem 2, the inequalities in (37) cannot hold. Consequently, (47) must be true.

To prove the second part of the Lemma, substitute $e = \frac{4C_D}{(1+C_D)^2}$ into (40). Since $(2 - e(1+C_D))^2 = 2\sqrt{1-e}$, we get

$$(1-e)\left(\mathscr{R}_0 - \frac{1}{1-e}\right)^2 + (2 - e(1+C_D))\left(\mathscr{R}_0 - \frac{1}{1-e}\right) + 1$$

$$= \left(\sqrt{1-e}\left(\mathscr{R}_0 - \frac{1}{1-e}\right) + 1\right)^2 \geq 0. \tag{52}$$

Thus, as in the proof of Theorem 2, the inequalities in (37) must hold. Consequently, (48) must be true.

The equality in (52) (and, consequently, in (48)) happens if only if $\sqrt{1-e}\left(\mathscr{R}_0 - \frac{1}{1-e}\right) = -1$, i.e., when $e = 1 - \left(\frac{1+\sqrt{1+4\mathscr{R}_0}}{2\mathscr{R}_0}\right)^2$. Finally, solving $e = \frac{4C_D}{(C_D+1)^2}$ for C_D, yields

$$eC_D^2 + (2e - 4)C_D + e = 0. \tag{53}$$

The solutions are given by $\frac{4-2e\pm\sqrt{(2e-4)^2-4e^2}}{2e}$. Note that only the larger root is greater than 1 which yields (50). $\square$

3.3. Roots of the incentive function

In this section, we will be concerned with the roots of the functions h and $\widetilde{h}$. First let us state the theorems that gives the formulas for their roots.

Theorem 3. *If* $C_D > 1$ *and* $\frac{4C_D}{(1+C_D)^2} \leq e$ *and* $e < 1$*, then there are two roots of* $\widetilde{h}(x)$ *given by*

$$x_{1,2} = \frac{e(1+C_D) - 2 \pm \sqrt{(e(1+C_D)-2)^2 - 4(1-e)}}{2(1-e)}. \tag{54}$$

Moreover, the solutions of $h(\overline{p}) = 0$ *are given by*

$$p_{1,2} = \frac{\mathscr{R}_0 - 1 - (1-e)x_{1,2}^2 - bx_{1,2}}{e\mathscr{R}_0}, \tag{55}$$

where

$$b = 1 + (1-e)(1-\mathscr{R}_0). \tag{56}$$

Also, $x_1 > x_2 > 0$ *and* $p_1 < p_2$*.*

Proof. As in the proof of Lemma 5 under the assumptions of this Theorem, the discriminant of $q(x)$ is positive. Thus, $q(x)$ has two real roots given by (54). By Lemma 2, the roots of $\widetilde{h}$ are given by the same formulas.

Moreover,

$$e(1+C_D) - 2 \geq \frac{4C_D}{1+C_D} - 2 = 2\frac{C_D - 1}{1+C_D} > 0. \tag{57}$$

Thus, $0 < x_2 < x_1$. To study the roots of h, recall (see also Appendix A) that I^* solves

$$(1-e)\left(\frac{\beta}{\Lambda}I\right)^2 + b\left(\frac{\beta}{\Lambda}I\right) - \mathscr{R}_0(1-e\overline{p}) + 1 = 0. \tag{58}$$

Thus, if $x = \frac{\beta}{\Lambda}I$ is a root of $\widetilde{h}(x)$, it solves

$$(1-e)x^2 + bx - \mathscr{R}_0(1-e\overline{p}) + 1 = 0. \tag{59}$$

Consequently, the two roots of h are given by (55). Moreover, since $x_1 > x_2 > 0$, $p_2 > p_1$. ☐

For the completeness, we will find the root of h even in the case of $e = 1$.

Lemma 9. *If* $e = 1$ *and* $C_D > 1$*, there is only one solution of* $h(\overline{p}) = 0$ *given by*

$$p_{e=1} = 1 - \frac{C_D}{\mathscr{R}_0(C_D - 1)}. \tag{60}$$

Proof. By Lemma 4, there is only one root of $\widetilde{h}$ given by $x_{e=1} = \frac{1}{C_D - 1}$. As in the proof of Theorem 3, the relationship between the roots of h and $\widetilde{h}$ is given by (59). For $e = 1$, this simplifies to

$$x_{e=1} - \mathscr{R}_0(1 - p_{e=1}) + 1 = 0. \tag{61}$$

Thus,

$$p_{e=1} = 1 - \frac{x_{e=1} + 1}{\mathscr{R}_0} = 1 - \frac{C_D}{\mathscr{R}_0(C_D - 1)}. \quad \square \tag{62}$$

While the above Theorem 3 gives formulas for the roots of h, it does not guarantee that the roots will be in $[0,1]$. For example, note that $p_{e=1}$ is in $[0,1]$, or, more precisely, in $[0, 1 - \frac{1}{\mathscr{R}_0})$ only when $C_D \geq 1 - \frac{1}{\mathscr{R}_0}$. When $C_D < 1 - \frac{1}{\mathscr{R}_0}$, then $p_{e=1} < 0$, which means that there is no root of $h(\overline{p})$ in $[0,1]$. Similarly, we need to find conditions on when $p_{1,2} \in [0,1]$.

We will proceed by first determining the intervals on which $\widetilde{h}$ is increasing and decreasing.

Lemma 10. *The function* $\widetilde{h}(x)$ *is increasing on* $[0, x_{\max})$ *and decreasing on* $(x_{\max}, \infty)$ *where*

$$x_{\max} = \frac{1}{\sqrt{1-e}}. \tag{63}$$

Proof. The statement easily follows from the fact that

$$\widetilde{h}(x)' = C_D e \frac{1-(1-e)x^2}{(x+1)^2((1-e)x+1)^2}. \quad \square \tag{64}$$

Table 2
Possible NE and CSNE depending on the signs of $h(0)$, $h(p_{\max})$ and $h(1)$. We ignore non-generic cases (such as $h(0) = 0$) which can happen only for a negligible set of parameter values. This yields 8 potential combinations, but since h attains its maximum on $[0,1]$ at $p_{\max}$, three of these combinations are not possible.

Sign of			Possible NE	Possible CSNE
$h(0)$	$h(p_{\max})$	$h(1)$		
−	−	−	0	0
−	−	+	Impossible combination	
−	+	−	0, p_1, p_2	0, p_2
−	+	+	0, p_1, 1	0, 1
+	−	−	Impossible combination	
+	−	+	Impossible combination	
+	+	−	p_2	p_2
+	+	+	1	1

Now, we will determine the intervals on which h is increasing and decreasing and use the intermediate value theorem to determine the location of the roots of h.

Theorem 4. *The function* h *is increasing on* $[0, p_{\max})$ *and decreasing on* $(p_{\max}, 1]$ *where*

$$p_{\max} = \begin{cases} 0 & \text{if } 1 - \frac{1}{(\mathscr{R}_0-1)^2} \leq e \leq 1, \\ \frac{\mathscr{R}_0 - 1 - (1-e)x_{\max}^2 - bx_{\max}}{e\mathscr{R}_0} & \text{if } 1 - \left(\frac{1+\sqrt{1+4\mathscr{R}_0}}{2\mathscr{R}_0}\right)^2 < e < 1 - \frac{1}{(\mathscr{R}_0-1)^2}, \\ 1 & \text{if } 0 \leq e \leq 1 - \left(\frac{1+\sqrt{1+4\mathscr{R}_0}}{2\mathscr{R}_0}\right)^2. \end{cases} \tag{65}$$

The location of roots p_1 *and* p_2 *can be determined by the signs of* h *at* 0, $p_{\max}$ *and* 1 *as shown in* Table 2*.*

Proof. Similarly to deriving $p_{1,2}$ by (55) from (59), the expression $\frac{\mathscr{R}_0 - 1 - (1-e)x_{\max}^2 - bx_{\max}}{e\mathscr{R}_0}$ is a good candidate for $p_{\max}$. However, to make sure that $(1-e)^{-1/2} = x_{\max} = \frac{\beta}{\Lambda}I^*(p_{\max})$ for $p_{\max} \in [0,1]$, we need $x_{\max}$ between $\frac{\beta}{\Lambda}I^*|_{\overline{p}=1} = \max\left\{0, \mathscr{R}_0 - \frac{1}{1-e}\right\}$ and $\frac{\beta}{\Lambda}I^*|_{\overline{p}=0} = \mathscr{R}_0 - 1$.

In particular, when $(1-e)^{-1/2} > \mathscr{R}_0 - 1$, i.e., when $e \geq 1 - \frac{1}{(\mathscr{R}_0-1)^2}$, then $x_{\max}$ is too large to be of the form $\frac{\beta}{\Lambda}I^*(\overline{p})$ for any $\overline{p} \in (0,1]$ and thus $h(\overline{p})$ is decreasing for all $\overline{p} \in (0,1)$.

Similarly, when $\mathscr{R}_0 > \frac{1}{1-e}$ and $(1-e)^{-1/2} < \mathscr{R}_0 - (1-e)^{-1}$, i.e., when $e < 1 - \frac{1}{\mathscr{R}_0}$ and $e \leq 1 - \left(\frac{1+\sqrt{1+4\mathscr{R}_0}}{2\mathscr{R}_0}\right)^2$, then $x_{\max}$ is too small to be of the form $\frac{\beta}{\Lambda}I^*(\overline{p})$ for any $\overline{p} \in (0,1]$ and thus $h(\overline{p})$ is increasing for all $\overline{p} \in (0,1)$. Note that $\left(\frac{1+\sqrt{1+4\mathscr{R}_0}}{2\mathscr{R}_0}\right)^2 > \frac{1}{\mathscr{R}_0}$, and so if $e < 1 - \left(\frac{1+\sqrt{1+4\mathscr{R}_0}}{2\mathscr{R}_0}\right)^2$ then $e < 1 - \frac{1}{\mathscr{R}_0}$.

The last statement of the Theorem follows from the intermediate value theorem.

Finally, as a remark, observe that $p_{\max} \leq p_{HI}$. ☐

3.4. Summary of the analysis

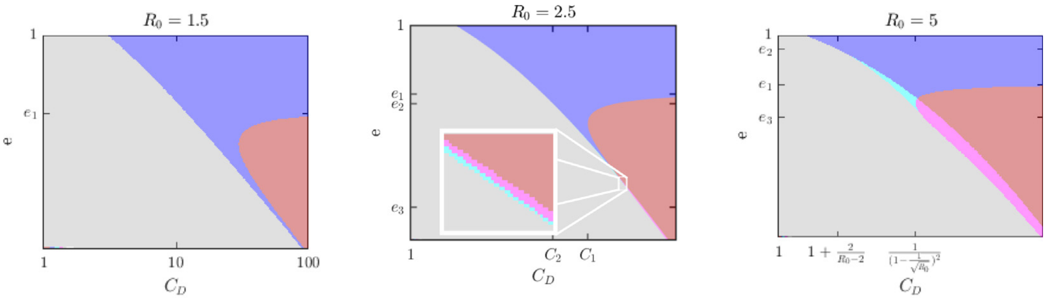
There are three natural thresholds

$$e_1 = 1 - \frac{1}{\mathscr{R}_0}, \tag{66}$$

$$e_2 = 1 - \frac{1}{(\mathscr{R}_0 - 1)^2}, \tag{67}$$

$$e_3 = 1 - \left(\frac{1 + \sqrt{1 + 4\mathscr{R}_0}}{2\mathscr{R}_0}\right)^2, \tag{68}$$

for the vaccine efficacy e that depend only on the value of $\mathscr{R}_0$. The threshold e_1 follows from the formula for p_{HI}; the thresholds e_2 and



(a) (b) (c)

Fig. 3. Illustration of different patterns of NE as they depend on C_D and e. For better contrast, the figures have a log scale on x and y axis. In all figures, $\Lambda = 5, \mu = 1, \gamma = 1$. (a) $\beta = 3$ and thus $\mathscr{R}_0 = 1.5$, (b) $\beta = 5$ and thus $\mathscr{R}_0 = 2.5$, (c) $\beta = 10$ and thus $\mathscr{R}_0 = 10$. Gray: 0 is the only NE. Red: 1 is the only NE. Blue: The only NE is $p_2 \in (0,1)$ given by (55). Light blue: three different NE 0, p_1 and p_2. Magenta: three different NE 0, p_1 and 1. In (b), the light blue and magenta regions are present but tiny and thus shown in zoom.

e_3 follow from the formula (65) for $p_{\max}$. The thresholds create three natural thresholds for $\mathscr{R}_0$. When $\mathscr{R}_0 > 1$, $e_1 > 0$. When $\mathscr{R}_0 > 2$, $e_2 > 0$ and $e_3 > 0$. When $\mathscr{R}_0 > \frac{3+\sqrt{5}}{2} = 1 + \varphi$ where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, then $e_2 > e_1$. Furthermore, there are four natural boundaries for e that depend on C_D and $\mathscr{R}_0$ given by

$$f_1(C_D) = \frac{\mathscr{R}_0^2}{(\mathscr{R}_0 - 1)\cdot(C_D + \mathscr{R}_0)}, \tag{69}$$

$$f_2(C_D) = \frac{-B + \sqrt{B^2 - 4A\mathscr{R}_0}}{2A}, \tag{70}$$

$$f_3(C_D) = \frac{-B - \sqrt{B^2 - 4A\mathscr{R}_0}}{2A}, \tag{71}$$

$$f_4(C_D) = \frac{4C_D}{(1 + C_D)^2}, \tag{72}$$

where A and B are given by (38) and (39). The function f_1 arises from Theorem 1, f_2 and f_3 arise from Theorem 2 and f_4 arises from Lemma 5.

The functions f_i create natural thresholds for C_D given by

$$C_1 = \frac{1}{\left(1 - \frac{1}{\sqrt{\mathscr{R}_0}}\right)^2}, \tag{73}$$

$$C_2 = 1 + \frac{2}{\mathscr{R}_0 - 2}, \tag{74}$$

$$C_3 = \frac{2}{e_3} - 1 + 2\sqrt{\frac{1}{e_3^2} - \frac{1}{e_3}}. \tag{75}$$

The thresholds arise from Lemmas 6–8.

The main findings can be summarized as follows.

- The disease can be eliminated by vaccination only if $e > e_1$. This follows from the work in Section 2.
- By Theorem 1, "do not vaccinate" is NE (and CSNE) if and only if $e < f_1(C_D)$.
- By Theorem 2, "always vaccinate" is NE (and CSNE) if and only if $f_3(C_D) < e < f_2(C_D)$.
- By Lemma 7, the functions f_2 and f_3 exist only if $C_D \geq C_1$. Moreover, $f_2(C_D) \geq f_3(C_D)$ for all $C_D \geq C_1$ with the equality happening exactly for $C_D = C_1$.
- By Lemma 8, $f_2(C_D) < e_1$ for all $C_D \geq 1$.
- By Lemma 6, $f_1(C_D) \geq f_4(C_D)$ for all $C_D \geq 1$ with equality happening exactly at $C_D = C_2$. We have $f_1(C_2) = e_2$. In particular, $f_1(C_D) > f_4(C_D)$ for all $C_D \geq 1$ when $\mathscr{R}_0 \in (1,2)$.
- By simple algebra, when $\mathscr{R}_0 \in (1,2)$, then $e_1 \in (0,1)$ but $e_3 < e_2 < 0$. It follows from (65) that the incentive function $h(\overline{p})$ attains its maximum on $[0,1]$ at $\overline{p} = 0$. Thus, if $h(0) \leq 0$, then $h(1) \leq 0$. Consequently, $f_1(C_D) < f_3(C_D)$ for all $C_D \geq C_1$. A formula-based proof is also shown in Appendix B.

4. Results

The patterns of Nash equilibria depend on a complex interplay between the disease basic reproduction number, $\mathscr{R}_0$, the cost of the disease relative to the cost of the vaccine, C_D, and the vaccine efficacy, e. The patterns are illustrated in Fig. 3 and specific conditions on when different combination equilibria exist are given in Table 3.

Below and in Table 4, we summarize how the outcome depends on $\mathscr{R}_0$.

When $\mathscr{R}_0 < 1$, then the vaccine efficacy thresholds e_i, given in (66)–(68), are not in $[0,1]$. In particular, the disease will not become endemic in the population. Thus "do not vaccinate" is the only Nash equilibrium (which is also CSNE).

When $1 < \mathscr{R}_0 < 2$, then the incentive function h is decreasing in $[0,1]$ and there are only three possible outcomes.

1. When $e < f_1(C_D)$, then "do not vaccinate" is the only CSNE.
2. When $f_3(C_D) < e < f_2(C_D)$, then "always vaccinate" is the only CSNE.
3. Otherwise, there is a unique CSNE between 0 and 1.

The cases correspond to (1) $h(0) < 0$, (2) $h(1) > 0$, and (3) $h(0) > 0$ and $h(1) < 0$. When the vaccine is effective enough to eliminate the disease if enough people vaccinate (i.e., if $e > e_1$, or equivalently, $p_{HI} < 1$), there are only two options: either $p_{NE} = 0$ if $e_1 < e < f_1(C_D)$ or $p_{NE} \in (0, p_{HI})$ for $e_1 < f_1(C_D) < e$. In the latter case, as C_D grows, p_{NE} is getting closer to p_{HI}.

If $2 < \mathscr{R}_0 < 1 + \varphi$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, then all $e_i \in (0,1)$ and $e_3 < e_2 < e_1$. The situation is more complex than in the case when $\mathscr{R}_0 < 2$. We can have multiple Nash equilibria as illustrated in Fig. 3(b). However, the regions where multiple equilibria occur are relatively small. Moreover, the multiple equilibria occur only for vaccines that are not very effective; specifically only if $e < 1 - \frac{1}{(\mathscr{R}_0 - 1)^2} = e_2 < e_1 = 1 - \frac{1}{\mathscr{R}_0}$, i.e., if the vaccine cannot eliminate the disease. For the vaccine that is efficient enough to eliminate the disease ($e > e_1$, i.e., $p_{HI} < 1$), there are only two possibilities: either $p_{NE} = 0$ if $e_1 < e < f_1(C_D)$ or $p_{NE} \in (0, p_{HI})$ for $e_1 < f_1(C_D) < e$.

If $1 + \varphi < \mathscr{R}_0$, then all $e_i \in (0,1)$ and $e_3 < e_1 < e_2$. While the situation is similar to the case $2 < \mathscr{R}_0 < 1 + \varphi$ in a sense that there are no new combinations of the NE, there is now a qualitative difference since multiple equilibria exist even if $e > e_1$. Specifically, when $f_4(C_D) < e < f_1(C_D)$ and $C_D < C_2$, then there are three NE, $0 < p_1 < p_2$. 0 and p_2 are CSNE while p_1 is NE but not CSNE.

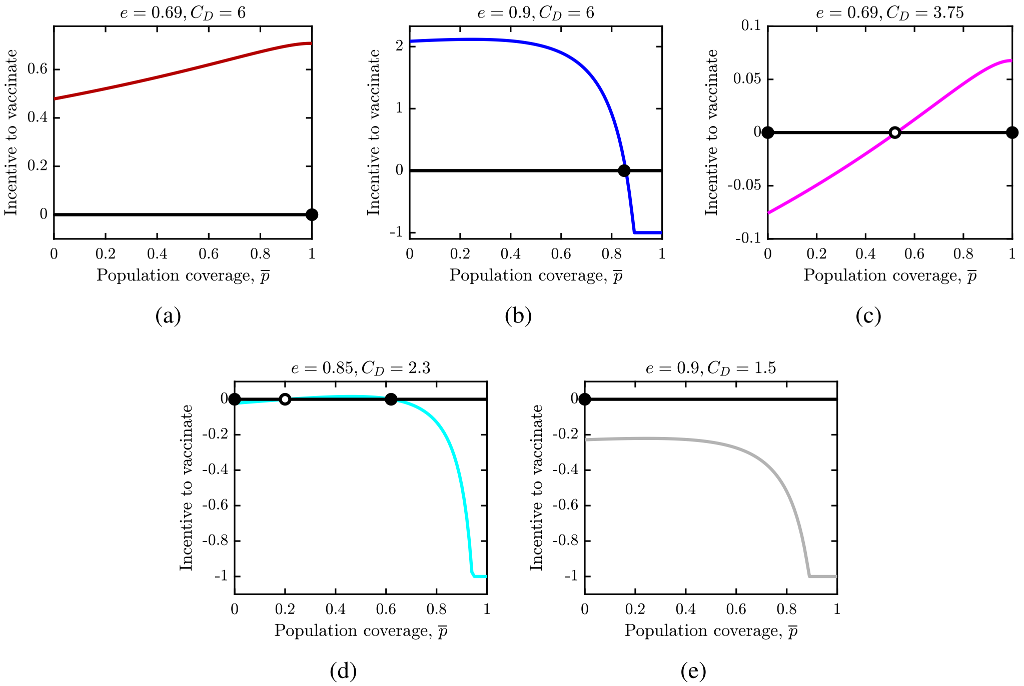
The Nash equilibria are illustrated in Fig. 5. We see that as the cost of the disease increases, the value of the CSNE increases as well and approaches the value needed for herd immunity. However, there is a region of C_D for which three Nash equilibria exist with a third Nash equilibrium between the two CSNE. This unstable Nash equilibrium is decreasing in C_D; although it also exists only for a relatively small region of C_D and only when $\mathscr{R}_0 > 2$.

Table 3
All possible combinations of Nash equilibria and conditions on when they occur. The CSNE “do not vaccinate” can occur under three distinct sets of conditions.

Type of equilibria	Conditions on h	Equivalent condition(s) on e
CSNE at 0	$p_{\max} = 0,\ h(0) < 0$ or $p_{\max} \in (0,1),\ h(p_{\max}) < 0$ or $p_{\max} = 1,\ h(1) < 0$	$e \geq e_2,\ e < f_1(C_D)$ or $e_3 < e < e_2,\ e < f_4(C_D)$ or $e \leq e_3,\ (e < f_3(C_D)$ or $e > f_2(C_D))$
CSNE at 0 NE at $p_1 \in (0, p_{\max})$ and CSNE at 1	$h(0) < 0$ and $h(p_{\max}) > 0$ and $h(1) > 0$	$e < f_1(C_D)$ and $e > f_4(C_D)$ and $f_3(C_D) < e < f_2(C_D)$
CSNE at 0 and NE at $p_1 \in (0, p_{\max})$ and CSNE at p_2	$h(0) < 0$ and $p_{\max} \in (0,1),\ h(p_{\max}) > 0$ and $h(1) < 0$	$e < f_1(C_D)$ and $e > f_4(C_D)$ and $(e < f_3(C_D)$ or $e > f_2(C_D))$
CSNE at p_2	$h(0) > 0,\ h(1) < 0$	$e > f_1(C_D)$ and $(e < f_3(C_D)$ or $e > f_2(C_D))$
CSNE at 1	$h(0) > 0,\ h(1) > 0$	$e > f_1(C_D),\ f_3(C_D) < e < f_2(C_D)$

Table 4
Patterns of NE and CSNE as they depend on $\mathscr{R}_0$. In the last two lines, for the multiple equilibria to exist even when the vaccine can eliminate the disease ($e > e_1$), we need $\mathscr{R}_0 > 1 + \varphi$.

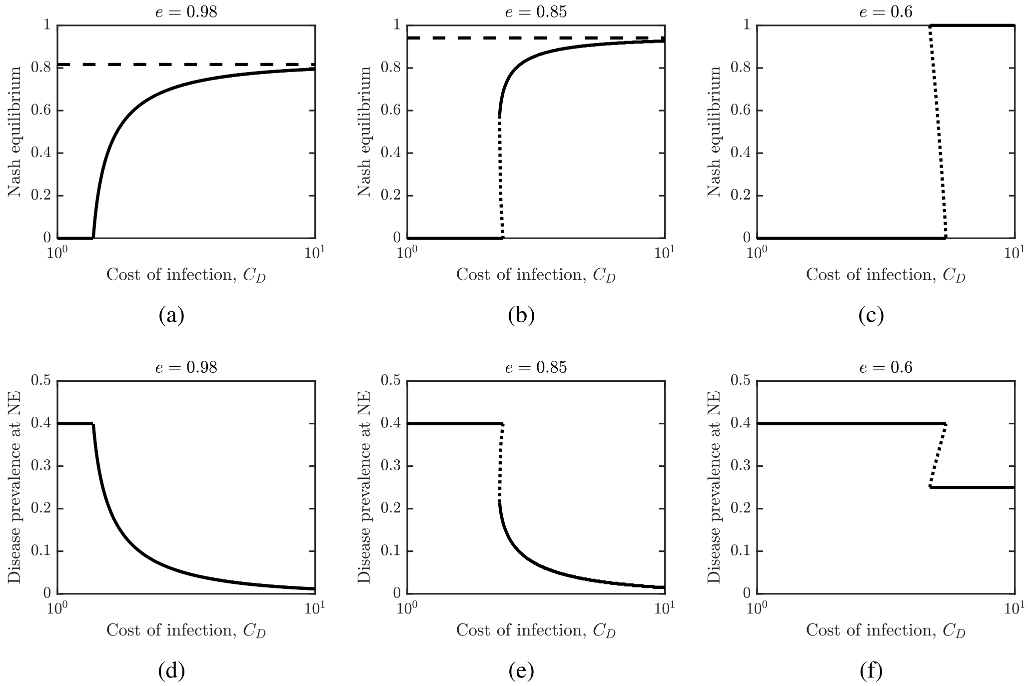
Conditions on $\mathscr{R}_0$	Additional conditions	Type of equilibria
$\mathscr{R}_0 < 1$	none	CSNE at 0
$1 < \mathscr{R}_0 < 2$	$e < \frac{\mathscr{R}_0^2}{(\mathscr{R}_0-1)\cdot(C_D+\mathscr{R}_0)}$	CSNE at 0
	$f_3(C_D) < e < f_2(C_D)$	CSNE at 1
	$e < \frac{\mathscr{R}_0^2}{(\mathscr{R}_0-1)\cdot(C_D+\mathscr{R}_0)}$ but not $f_3(C_D) < e < f_2(C_D)$	CSNE at p_2
$2 < \mathscr{R}_0$	$f_1(C_D) < e$ but not $f_3(C_D) < e < f_2(C_D)$	CSNE at p_2
	$e_2 < e < f_1(C_D)$	CSNE at 0
	$e_3 < e < e_2$ and $e < f_4(C_D)$	CSNE at 0
	$e < e_3$ and $e < f_3(C_D)$	CSNE at 0
	$f_1(C_D) < e$ and $f_3(C_D) < e < f_2(C_D)$	CSNE at 1
	$f_4(C_D) < e < f_1(C_D)$ and $f_3(C_D) < e < f_2(C_D)$	CSNE at 0 and 1, NE at p_1
	$f_4(C_D) < e < f_1(C_D)$ but not $f_3(C_D) < e < f_2(C_D)$	CSNE at 0 and p_2, NE at p_1



(a) (b) (c)

(d) (e)

Fig. 4. The incentive function for regions from Fig. 3 with corresponding colors. The full dots correspond to CSNE, the empty circle corresponds to NE that is not convergent stable. e and C_D varies as shown, the values of other parameters are $\Lambda = 5, \mu = 1, \gamma = 1$, and $\beta = 10$ (i.e., $\mathscr{R}_0 = 5$). (a) Red for no roots of h (1 is CSNE). (b) Blue for one root of h (the root is CSNE). (c) Magenta for one root of h (0 is CSNE, 1 is CSNE, the root is p_{NE} but not CSNE). (d) Light blue for two roots of h (0 is CSNE, mid p_{NE} is not CSNE and the largest root is CSNE). (e) Gray for no roots of h (0 is CSNE).



(a) (b) (c)

(d) (e) (f)

Fig. 5. (a)–(c) Nash equilibria (CSNE — solid line, NE that is not CSNE — dotted line, the coverage needed for herd immunity dashed line) and (d)–(f) the price of anarchy, i.e., in this case the disease prevalence in the population using the NE vaccination coverage. The vaccine effectiveness, e, varies as shown and the values of other parameters are $\Lambda = 5, \mu = 1, \gamma = 1$, and $\beta = 10$ (i.e., $\mathscr{R}_0 = 5$). In (a) and (d), the situation is almost as if the vaccine is 100% effective. As C_D increases, the Nash equilibrium approaches the coverage needed for herd immunity. In (b) and (e), we can see multiple equilibria; but again as C_D increases, the Nash equilibrium approaches p_{HI}. (c) and (f) exhibits multiple equilibria as well, but this time the vaccine effectiveness is so low that the herd immunity cannot be achieved.

Fig. 5 also illustrates the price of anarchy. We measured the price as the disease prevalence at the Nash equilibrium, i.e., I^*/N where

$$\frac{I^*(p_{NE})}{N} = \begin{cases} \frac{\mu}{\beta}(\mathscr{R}_0 - 1), & \text{if } p_{NE} = 0, \\ \frac{\mu}{\beta}\left(\mathscr{R}_0 - \frac{1}{1-e}\right), & \text{if } p_{NE} = 1, \\ \frac{\mu}{\beta} \cdot \frac{e(1+C_D)-2 \pm \sqrt{(e(1+C_D)-2)^2 - 4(1-e)}}{2(1-e)}, & \text{if } p_{NE} \in (0,1). \end{cases} \quad (76)$$

As expected, if e is large ($e > \max(e_1, e_2)$), or if $\mathscr{R}_0 < 2$, then the price of anarchy decreases with the cost of the disease and for vaccines that can eradicate the disease, the price of anarchy approaches 0 as C_D grows. Also, the price of anarchy exhibits the same sort of phenomenon as the Nash equilibria. When the multiple equilibria exist, the price grows with C_D at the unstable NE.

When the vaccine is perfect, the risk of infection if unvaccinated is a decreasing function while the risk of infection if vaccinated is 0. When the vaccine is not perfect, both risk functions are decreasing. However, the marginal risk of infection can differ for vaccinated and unvaccinated individuals. This causes the possibility for the incentive function to be increasing on the whole or part of $[0,1]$ as demonstrated in Fig. 4. And, as a result, there can be multiple Nash equilibria.

5. Conclusions and discussion

We investigated the effects of an imperfect vaccine on the outcomes of a vaccination game in which individuals decide whether to vaccinate or not based on their own interests. We used a simple SIR compartmental model for the underlying model of disease transmission. We modeled the vaccine imperfection by adding the compartment with vaccinated individuals from which there is still a possibility to become infected after encountering an infectious individual. We determined explicit conditions for the existence of different Nash equilibria, the solutions of the vaccination game. Unlike in the case of a perfect vaccine, we have seen that there can be multiple equilibria. The outcomes of the game depend on the complex interplay between disease transmission dynamics (the basic reproduction number), the relative cost of the infection, and the vaccine efficacy.

When the vaccine is effective enough to eliminate the disease ($e > 1 - \frac{1}{\mathscr{R}_0}$), and the basic reproduction number is low ($\mathscr{R}_0 < 1 + \varphi \approx 2.62$), then there is very little difference between perfect or imperfect vaccines. Individuals will choose to vaccinate only if the cost of the disease is above a certain threshold and the Nash equilibrium vaccination level will increase with the cost of the infection. Consequently, the simpler models that assume that a vaccine is perfect are good enough approximations.

However, when the basic reproduction number is larger than $1+\varphi \approx 2.62$, then there is a region where multiple Nash equilibria exist. Specifically, we need the vaccine efficacy to be less than $1 - \frac{1}{(\mathscr{R}_0-1)^2}$ and the cost of the infection roughly between $1 + \frac{2}{\mathscr{R}_0 - 2}$ and $(1 - \mathscr{R}_0^{-0.5})^{-2}$. Under these conditions, there can be three Nash equilibria at the same time: "do not vaccinate" is convergently stable Nash equilibrium (CSNE), there is another CSNE with a relatively large value of vaccination, but there is also a Nash equilibrium between the two and this third one is not convergently stable. The existence of multiple Nash equilibria has important implications for the public health officials. Unless there is a mandatory vaccination policy in place that would push the vaccination coverage above the value of unstable Nash equilibrium, the population could eventually slip to the "do not vaccinate" state. Thus, for diseases that have relatively high basic reproduction number, the potential for the vaccine not being perfect should be taken into proper consideration.

Most of the previous vaccination game theory models assumed that the vaccine is perfect. As a consequence, the models predicted only a single Nash equilibrium. Wu et al. [51] explicitly studied imperfect vaccines and used Fermi's updating rule for imitation of the vaccination behavior. They discovered a similar threshold for the basic reproduction number, but their game always had a single Nash equilibrium as if the vaccine was perfect. Choi and Shim [20] studied an imperfect vaccine for COVID-19 prevention. The underlying COVID-19 disease dynamics was more complicated than the simple SIRV dynamics considered here and the vaccination was not done at birth. However, Choi and Shim [20] also observed the multiple Nash equilibria for certain ranges of parameter values. A similar result was obtained by Augsburger et al. [14] who considered a SVEIR model of imperfect smallpox vaccine against monkeypox. Unlike our model which assumed vaccination at birth, the models considered in [14,20] considered vaccination at any stage prior to infection exposure. While Augsburger et al. [14] did not try to systematically quantify the conditions on NE and CSNE patterns, their result are remarkably similar to the results presented here; specifically the existence of multiple equilibria was observed for relatively large $\mathscr{R}_0$. This suggests that the findings of this paper are robust and do not significantly depend on the underlying disease transmission dynamics.

Our research can be expanded in several ways as discussed below.

First, we modeled the vaccine imperfection as a “leaky” vaccine rather than “all-or-nothing” vaccine. The calculations would change slightly for the “all-or-nothing” vaccine; the factor $1-\overline{p}$ for the proportion of unvaccinated individuals would have to be replaced by $(1-e\overline{p}) = (1-\overline{p})+\overline{p}(1-e)$. Here $1-\overline{p}$ corresponds to unvaccinated individuals and $\overline{p}(1-e)$ corresponds to vaccinated individuals with a (complete) vaccine failure. While we believe that even this kind of vaccine imperfection would yield multiple Nash equilibria of the vaccination game, it is important for the modelers to distinguish between the two kinds [43].

Second, we assumed homogeneous population in which all individuals perceive the cost of the disease and the cost of the vaccine in the same way. However, individuals may have different underlying conditions which can make them more vulnerable to a particular disease such as COVID-19 [56] causing the perceived cost of disease to vary in the population. Similarly, in countries with limited healthcare, individuals may live at various distances from the vaccination sites, creating a heterogeneity in the cost of the vaccination. This assumption of heterogeneous costs could significantly alter our results. In the present (homogeneous) model, if the cost of the disease is smaller than the cost of vaccination, individuals do not vaccinate. However, in the heterogeneous model, even if the average cost of disease is smaller than the average cost of vaccination, there can still be a non-negligible proportion of population who perceive the cost of disease to be significantly larger than the cost of vaccine and opt to vaccinate. Thus, there will be an inherent heterogeneity in the vaccination decisions and it yet remains to be seen whether multiple equilibria will exist even in this more realistic scenario.

Finally, as discussed in [39], there are many kinds of vaccine imperfections. We focused on the case when the vaccine reduces the transmission rate. However, the vaccine can also reduce the proportion of exposed hosts becoming infectious or the length of the infectious period in vaccinated individuals. We anticipate that multiple Nash equilibria will occur even in the latter two cases, but they deserve more detailed and explicit investigation.

CRediT authorship contribution statement

Ian B. Augsburger: Writing – original draft, Writing – review & editing, Software, Validation, Formal analysis, Data curation, Visualization. **Grace K. Galanthay:** Writing – original draft, Writing – review & editing, Software, Validation, Formal analysis, Data curation, Visualization. **Jacob H. Tarosky:** Writing – original draft, Writing – review & editing, Software, Validation, Formal analysis, Data curation, Visualization. **Jan Rychtář:** Writing – original draft, Writing – review & editing, Software, Methodology, Supervision, Conceptualization, Resources. **Dewey Taylor:** Writing – original draft, Writing – review & editing, Methodology, Supervision, Conceptualization, Formal analysis, Validation, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

We would like to thanks the anonymous reviewers who helped to improve the quality and clarity of the presentation.

Funding

I.B.A., G.K.G. and J.H.T. were supported by the VCU REU program in mathematics funded by the National Science Foundation, United States grant number DMS1950015 awarded to D.T. The funders had no role in study design, data collection and analysis, decision to publish, or preparation of the manuscript.

Appendix A. Calculations of the equilibria of the dynamics

The equilibria of the dynamics (1)–(4) are obtained by solving the following system of algebraic equations.

$$0 = (1-\overline{p})\Lambda - \left(\mu + \beta\frac{I}{N}\right)S \tag{A.1}$$

$$0 = \beta\frac{I}{N}\left(S + (1-e)V\right) - (\mu+\gamma)I \tag{A.2}$$

$$0 = \gamma I - \mu R \tag{A.3}$$

$$0 = \overline{p}\Lambda - \left(\mu + (1-e)\beta\frac{I}{N}\right)V. \tag{A.4}$$

By adding (A.1)–(A.4), we get

$$N = \frac{\Lambda}{\mu}. \tag{A.5}$$

By (A.2), we have that either $I = 0$ or

$$0 = \frac{\beta}{N}\left(S + (1-e)V\right) - (\mu+\gamma). \tag{A.6}$$

If $I = 0$, then, by (A.3), $R = 0$. By (A.1), $S = (1-\overline{p})N$ and by (A.4) $V = \overline{p}N$. Thus, the disease-free equilibrium $\mathscr{E}^0 = (S^0, I^0, R^0, V^0)$ is given by $((1-\overline{p})N, 0, 0, \overline{p}N)$.

We find the effective reproduction number, i.e., the number of secondary infections from a single infected individual in an otherwise healthy population, using the procedure outlined in [57]. The only compartment carrying the infection is I. The inflow of new infections is given by $F = \beta((1-\overline{p})+(1-e)\overline{p})$ and the outflow is given by $V = -(\mu+\gamma)$.

$$\mathscr{R}(\overline{p}) = \varrho(FV^{-1}) = \frac{\beta}{\mu+\gamma}(1-e\overline{p}). \tag{A.7}$$

Note that

$$\mathscr{R}(\overline{p}) = \mathscr{R}_0(1-e\overline{p}) \tag{A.8}$$

where $\mathscr{R}_0 = \frac{\beta}{\mu+\gamma}$ is the basic reproduction number in the unvaccinated population.

The DFE is locally asymptotically stable if $\mathscr{R}(\overline{p}) < 1$ and the endemic equilibrium is stable if $\mathscr{R}(\overline{p}) > 1$ [57].

Now, assume $I > 0$. By (A.3),

$$R = \frac{\gamma}{\mu}I. \tag{A.9}$$

By (A.1) and (A.4),

$$S = \frac{(1-\overline{p})\Lambda}{\mu + \frac{\beta}{N}I} \tag{A.10}$$

$$V = \frac{\overline{p}\Lambda}{\mu + \frac{(1-e)\beta}{N}I}. \tag{A.11}$$

Plugging in (A.10) and (A.11) into (A.6), we get

$$1 = \frac{\mathscr{R}_0}{N}\left(\frac{(1-\overline{p})N}{1+\frac{\beta}{\Lambda}I} + (1-e)\frac{\overline{p}N}{1+(1-e)\frac{\beta}{\Lambda}I}\right). \tag{A.12}$$

When $e = 1$, it follows from (A.12) that

$$I^* = \frac{\Lambda}{\beta}(\mathscr{R}_0 - 1). \tag{A.13}$$

When $\overline{p} = 0$, it follows from (A.12) that

$$I^* = \frac{\Lambda}{\beta}(\mathscr{R}_0 - 1). \tag{A.14}$$

Similarly, when $\overline{p} = 1$, it follows from (A.12) that

$$I^* = \frac{\Lambda}{\beta}(\mathscr{R}_0 - \frac{1}{1-e}). \tag{A.15}$$

From now, we will assume that $\overline{p} \in (0,1)$. After several algebraic steps, the (A.12) yields

$$(1-e)\left(\frac{\beta}{\Lambda}I\right)^2 + [1+(1-e)(1-\mathscr{R}_0)]\left(\frac{\beta}{\Lambda}I\right) - \mathscr{R}_0(1-e\overline{p}) + 1 = 0. \tag{A.16}$$

This means that

$$I^* = \frac{\Lambda}{\beta}\frac{-b+\sqrt{b^2+4(1-e)c}}{2(1-e)} \tag{A.17}$$

where

$$b = 1 + (1-e)(1-\mathscr{R}_0) \tag{A.18}$$

$$c = \mathscr{R}_0(1-e\overline{p}) - 1. \tag{A.19}$$

Once I^* is found, S^*, R^*, V^* are given by

$$S^* = \frac{(1-\overline{p})N}{1+\frac{\beta}{\Lambda}I^*} \tag{A.20}$$

$$R^* = \frac{\gamma}{\mu}I^* \tag{A.21}$$

$$V^* = \frac{\overline{p}N}{1+\frac{(1-e)\beta}{\Lambda}I^*}. \tag{A.22}$$

We conclude the section by showing that I^* is always non-increasing in $\overline{p}$ and it is decreasing whenever $I^* > 0$. By (A.8) and (A.17), when $e < 1$, $\overline{p} \in (0,1)$ and $\mathscr{R}_0(1-e\overline{p}) > 1$, we get

$$\frac{\partial I^*}{\partial \overline{p}} = \left(\frac{\partial I^*}{\partial c}\right)\left(\frac{\partial c}{\partial \overline{p}}\right) \tag{A.23}$$

$$= \left(\frac{\Lambda}{2\beta(1-e)}\frac{4(1-e)}{2\sqrt{b^2+4(1-e)c}}\right)(-\mathscr{R}_0 e) \tag{A.24}$$

$$= -\frac{\mathscr{R}_0 e}{\sqrt{b^2+4(1-e)c}} < 0. \tag{A.25}$$

Similarly, when $e = 1$,

$$\frac{\partial I^*}{\partial \overline{p}} = -\frac{\Lambda}{\beta}\mathscr{R}_0 < 0. \tag{A.26}$$

Appendix B. Formula based proof

Theorem. *$f_1(C_D) < f_3(C_D)$ when $C_D > 4$ and $\mathscr{R}_0 \in [\mathscr{R}_{exists}, \mathscr{R}_{int})$ where $\mathscr{R}_{exists} = \frac{C_D+2C_D^{3/2}+C_D^2}{(C_D-1)^2}$ and $\mathscr{R}_{int} = \frac{2C_D}{C_D-2}$.*

1. To begin, we must know where f_3 exists, or when the discriminant of (37) is 0. We find for $\mathscr{R}_0 > \mathscr{R}_{exists} = \frac{C_D+2C_D^{3/2}+C_D^2}{(C_D-1)^2} > 1$ for $C_D > 1$, the discriminant is ≥ 0.
2. Next to understand when $f_3 > f_1$ at $\mathscr{R}_{exists}$, we evaluate $f_3(\mathscr{R}_{exists}, C_D)$ and $f_1(\mathscr{R}_{exists}, C_D)$, to which we find $f_3(\mathscr{R}_{exists}, C_D) = \frac{\sqrt{C_D}}{1-\sqrt{C_D}+C_D}$ and $f_1(\mathscr{R}_{exists}, C_D) = \frac{C_D}{-2+6\sqrt{C_D}-5C_D+2C_D^{3/2}}$. We find $f_3(\mathscr{R}_{exists}, C_D) > f_1(\mathscr{R}_{exists}, C_D)$ for $C_D > 4$, otherwise for $1 < C_D < 4$ we have $f_3(\mathscr{R}_{exists}, C_D) < f_1(\mathscr{R}_{exists}, C_D)$.
3. To finish the proof we need to show there is only one point of intersection for $C_D > 4$ and $\mathscr{R}_0 > 1$ and that it is greater than $\mathscr{R}_{exists}$. To do this we can solve for the roots of the following function $f_{int}(\mathscr{R}_0, C_D) = f_1(\mathscr{R}_0, C_D) - f_3(\mathscr{R}_0, C_D)$. Solving for the roots and simplifying, we get the following cubic:

   $$\mathscr{R}_0\left((C_D-2)\mathscr{R}_0^2 + (C_D^2 - 2 - 3C_D)\mathscr{R}_0 - 2C_D(CD+1)\right) = 0 \tag{B.1}$$

   Then, factoring we get:

   $$(\mathscr{R}_0 - 0)(\mathscr{R}_0 - (-1-C_D))(\mathscr{R}_0 - (\frac{2C_D}{C_D-2})) = 0. \tag{B.2}$$

   We see that at most three intersections exist at the roots, however we can immediately disregard roots at $\mathscr{R}_0 = 0$ and $\mathscr{R}_0 = -1-C_D$ as $\mathscr{R}_0$ must be > 1 for $C_D > 1$, under our assumptions in the Analysis section. So, only one intersection exists for $C_D > 4$ and this intersection is unique to f_1 and f_3 by the Lemma below, which we may denote as $\mathscr{R}_{int} = \frac{2C_D}{C_D-2} > 2$. Finally we have that $\mathscr{R}_{exists}(C_D) < \mathscr{R}_{int}(C_D)$. We get this using the fact that $\mathscr{R}_{exists}(4) = \mathscr{R}_{int}(4)$ and $\frac{\partial \mathscr{R}_{exists}}{\partial C_D} < \frac{\partial \mathscr{R}_{int}}{\partial C_D} < 0$, point-wise for $C_D > 4$.

So we have shown that $f_3 > f_1$ at $\mathscr{R}_{exists}$ and there exists at most one point of intersection for which $\mathscr{R}_{exists} < \mathscr{R}_{int}$ when $C_D > 4$. Thus we have $f_3 > f_1$ for $\mathscr{R}_0 \in [\mathscr{R}_{exists}, \mathscr{R}_{int})$ for $C_D > 4$.

Lemma. *For $4 > C_D > 2$, $\mathscr{R}_{int}$ is the intersection between f_1 and f_2 and for $C_D > 4$, $\mathscr{R}_{int}$ is the intersection between f_1 and f_3*

To show $\mathscr{R}_{int}$ is between f_1 and f_2, for $4 > C_D > 2$, we first establish $0 < \mathscr{R}_{min_{f_1}} = \frac{2C_D}{C_D-1} < \mathscr{R}_{exist} = \frac{C_D+2C_D^{3/2}+C_D^2}{1-2C_D+C_D^2}$ for $1 < C_D < 9$. Since $\mathscr{R}_{min_{f_1}}$ is the unique minimum point for $\mathscr{R}_0 > 1$, f_1 is increasing for $\mathscr{R}_0 > \mathscr{R}_{exists}$. We then find $f_1(\mathscr{R}_{exists}) > f_2(\mathscr{R}_{exists})$ when $1 < C_D < 4$. Thus if $f_1(\mathscr{R}_{exists}) > f_2(\mathscr{R}_{exists})$ for $C_D < 4$, and f_1 is increasing for all $\mathscr{R}_0 > \mathscr{R}_{exists} > 0$, then there must be a strictly positive intersection between f_1 and f_2 or no strictly positive intersection at all. But $\mathscr{R}_{int} > 2 > 0$ only for $C_D > 2$, so we have for $4 > C_D > 2$, $\mathscr{R}_{int}$ is the intersection between f_1 and f_2. Finally for $C_D < 2$, we have $\mathscr{R}_{int} < 0$, so there does not exist a strictly positive intersection point and so $f_1 > f_2 > f_3$ always. This allows us to conclude that for $C_D > 4$, $\mathscr{R}_{int} > 0$ must be the intersection between f_1 and f_3.

Appendix C. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.mbs.2023.108967.

References

[1] V.V. Ganusov, R. Antia, Imperfect vaccines and the evolution of pathogens causing acute infections in vertebrates, Evolution 60 (5) (2006) 957–969.

[2] X. Chen, F. Fu, Imperfect vaccine and hysteresis, Proc. R. Soc. B 286 (1894) (2019) 20182406.

[3] E. Maskin, Nash equilibrium and welfare optimality, Rev. Econom. Stud. 66 (1) (1999) 23–38.

[4] T.C. Reluga, C.T. Bauch, A.P. Galvani, Evolving public perceptions and stability in vaccine uptake, Math. Biosci. 204 (2) (2006) 185–198.

[5] T.C. Reluga, A.P. Galvani, A general approach for population games with application to vaccination, Math. Biosci. 230 (2) (2011) 67–78.

[6] C.T. Bauch, D.J. Earn, Vaccination and the theory of games, Proc. Natl. Acad. Sci. 101 (36) (2004) 13391–13394.

[7] Z. Wang, C.T. Bauch, S. Bhattacharyya, A. d'Onofrio, P. Manfredi, M. Perc, N. Perra, M. Salathé, D. Zhao, Statistical physics of vaccination, Phys. Rep. 664 (2016) 1–113.

[8] S.L. Chang, M. Piraveenan, P. Pattison, M. Prokopenko, Game theoretic modelling of infectious disease dynamics and intervention methods: a review, J. Biol. Dyn. 14 (1) (2020) 57–89.

[9] F. Verelst, L. Willem, P. Beutels, Behavioural change models for infectious disease transmission: a systematic review (2010–2015), J. R. Soc. Interface 13 (125) (2016) 20160820.

[10] C.T. Bauch, A.P. Galvani, D.J. Earn, Group interest versus self-interest in smallpox vaccination policy, Proc. Natl. Acad. Sci. 100 (18) (2003) 10564–10567.

[11] C. Molina, D.J. Earn, Game theory of pre-emptive vaccination before bioterrorism or accidental release of smallpox, J. R. Soc. Interface 12 (107) (2015) 20141387.

[12] J. Liu, B.F. Kochin, Y.I. Tekle, A.P. Galvani, Epidemiological game-theory dynamics of chickenpox vaccination in the USA and Israel, J. R. Soc. Interface 9 (66) (2012) 68–76.

[13] S.V. Bankuru, S. Kossol, W. Hou, P. Mahmoudi, J. Rychtář, D. Taylor, A game-theoretic model of monkeypox to assess vaccination strategies, PeerJ 8 (2020) e9272.

[14] I.B. Augsburger, G.K. Galanthay, J.H. Tarosky, J. Rychtář, D. Taylor, Voluntary vaccination may not stop monkeypox outbreak: a game-theoretic model, PLOS Negl. Trop. Dis. 16 (12) (2022) e0010970.

[15] E. Cheng, N. Gambhirrao, R. Patel, A. Zhowandai, J. Rychtář, D. Taylor, A game-theoretical analysis of Poliomyelitis vaccination, J. Theoret. Biol. 499 (2020) 110298.

[16] E. Shim, J.J. Grefenstette, S.M. Albert, B.E. Cakouros, D.S. Burke, A game dynamic model for vaccine skeptics and vaccine believers: measles as an example, J. Theoret. Biol. 295 (2012) 194–203.

[17] E. Shim, G.B. Chapman, J.P. Townsend, A.P. Galvani, The influence of altruism on influenza vaccination decisions, J. R. Soc. Interface 9 (74) (2012) 2234–2243.

[18] A. Brettin, R. Rossi-Goldthorpe, K. Weishaar, I.V. Erovenko, Ebola could be eradicated through voluntary vaccination, Royal Soc. Open Sci. 5 (1) (2018) 171591.

[19] F.B. Agusto, I.V. Erovenko, A. Fulk, Q. Abu-Saymeh, D. Romero-Alvarez, J. Ponce, S. Sindi, O. Ortega, J.M. Saint Onge, A.T. Peterson, To isolate or not to isolate: The impact of changing behavior on COVID-19 transmission, BMC Publ. Health 22 (1) (2022) 1–20.

[20] W. Choi, E. Shim, Optimal strategies for social distancing and testing to control COVID-19, J. Theoret. Biol. 512 (2021) 110568.

[21] M. Piraveenan, S. Sawleshwarkar, M. Walsh, I. Zablotska, S. Bhattacharyya, H.H. Farooqui, T. Bhatnagar, A. Karan, M. Murhekar, S. Zodpey, Optimal governance and implementation of vaccination programmes to contain the COVID-19 pandemic, R. Soc. Open Sci. 8 (6) (2021) 210429.

[22] S.R.M. Klein, A.O. Foster, D.A. Feagins, J.T. Rowell, I.V. Erovenko, Optimal voluntary and mandatory insect repellent usage and emigration strategies to control the chikungunya outbreak on Reunion Island, PeerJ 8 (2020) e10151.

[23] K. Scheckelhoff, A. Ejaz, I.V. Erovenko, J. Rychtář, D. Taylor, Optimal voluntary vaccination of adults and adolescents can help eradicate hepatitis B in China, Games 12 (4) (2021) 82.

[24] A. Chouhan, S. Maiwand, M. Ngo, V. Putalapattu, J. Rychtář, D. Taylor, Game-theoretical model of retroactive Hepatitis B vaccination in China, Bull. Math. Biol. 82 (6) (2020) 1–18.

[25] C.B. Acosta-Alonzo, I.V. Erovenko, A. Lancaster, H. Oh, J. Rychtář, D. Taylor, High endemic levels of typhoid fever in rural areas of Ghana may stem from optimal voluntary vaccination behaviour, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 476 (2241) (2020) 20200354.

[26] J. Kobe, N. Pritchard, Z. Short, I.V. Erovenko, J. Rychtář, J.T. Rowell, A game-theoretic model of cholera with optimal personal protection strategies, Bull. Math. Biol. 80 (10) (2018) 2580–2599.

[27] WHO, Yellow fever, 2019, https://www.who.int/news-room/fact-sheets/detail/yellow-fever. (Accessed June 30, 2022).

[28] CDC, Measles, Mumps, and Rubella (MMR) Vaccination: What Everyone Should Know, 2021, https://www.cdc.gov/vaccines/vpd/mmr/public/index.html. (Accessed June 30, 2022).

[29] R. Baxter, P. Ray, T.N. Tran, S. Black, H.R. Shinefield, P.M. Coplan, E. Lewis, B. Fireman, P. Saddier, Long-term effectiveness of varicella vaccine: a 14-year, prospective cohort study, Pediatrics 131 (5) (2013) e1389–e1396.

[30] D. Guris, A.O. Jumaan, L. Mascola, B.M. Watson, J.X. Zhang, S.S. Chaves, P. Gargiullo, D. Perella, R. Civen, J.F. Seward, Changing varicella epidemiology in active surveillance sites—United States, 1995–2005, J. Infect. Dis. 197 (Suppl. 2) (2008) S71–S75.

[31] J. Mohammed-Awel, E. Numfor, R. Zhao, S. Lenhart, A new mathematical model studying imperfect vaccination: Optimal control analysis, J. Math. Anal. Appl. 500 (2) (2021) 125132.

[32] S. Gandon, M.J. Mackinnon, S. Nee, A.F. Read, Imperfect vaccines and the evolution of pathogen virulence, Nature 414 (6865) (2001) 751–756.

[33] A. Gumel, C.C. McCluskey, P. van den Driessche, Mathematical study of a staged-progression HIV model with imperfect vaccine, Bull. Math. Biol. 68 (8) (2006) 2105–2128.

[34] A.B. Gumel, C.C. McCluskey, J. Watmough, An SVEIR model for assessing potential impact of an imperfect anti-SARS vaccine, Math. Biosci. Eng. 3 (3) (2006) 485.

[35] M.I. Teboh-Ewungkem, C.N. Podder, A.B. Gumel, Mathematical study of the role of gametocytes and an imperfect vaccine on malaria transmission dynamics, Bull. Math. Biol. 72 (1) (2010) 63–93.

[36] M.A. Safi, D.Y. Melesse, A.B. Gumel, Dynamics analysis of a multi-strain cholera model with an imperfect vaccine, Bull. Math. Biol. 75 (7) (2013) 1104–1137.

[37] A. Egonmwan, D. Okuonghae, Mathematical analysis of a tuberculosis model with imperfect vaccine, Int. J. Biomath. 12 (07) (2019) 1950073.

[38] E.A. Iboi, C.N. Ngonghala, A.B. Gumel, Will an imperfect vaccine curtail the COVID-19 pandemic in the US? Infect. Dis. Model. 5 (2020) 510–524.

[39] K. Magori, A.W. Park, The evolutionary consequences of alternative types of imperfect vaccines, J. Math. Biol. 68 (4) (2014) 969–987.

[40] X.-L. Peng, X.-J. Xu, M. Small, X. Fu, Z. Jin, Prevention of infectious diseases by public vaccination and individual protection, J. Math. Biol. 73 (6) (2016) 1561–1594.

[41] H. Abboubakar, J.C. Kamgang, L.N. Nkamba, D. Tieudjo, Bifurcation thresholds and optimal control in transmission dynamics of arboviral diseases, J. Math. Biol. 76 (1) (2018) 379–427.

[42] J. Arino, E. Milliken, Bistability in deterministic and stochastic SLIAR-type models with imperfect and waning vaccine protection, J. Math. Biol. 84 (7) (2022) 1–31.

[43] E. Shim, A.P. Galvani, Distinguishing vaccine efficacy and effectiveness, Vaccine 30 (47) (2012) 6700–6705.

[44] I.M. Longini Jr., M.E. Halloran, M. Haber, Estimation of vaccine efficacy from epidemics of acute infectious agents under vaccine-related heterogeneity, Math. Biosci. 117 (1–2) (1993) 271–281.

[45] F.G. Ball, O.D. Lyne, Optimal vaccination policies for stochastic epidemics among a population of households, Math. Biosci. 177 (2002) 333–354.

[46] F. Ball, T. Britton, An epidemic model with infector and exposure dependent severity, Math. Biosci. 218 (2) (2009) 105–120.

[47] C.M. Kribs-Zaleta, J.X. Velasco-Hernández, A simple vaccination model with multiple endemic states, Math. Biosci. 164 (2) (2000) 183–201.

[48] E. Shim, A note on epidemic models with infective immigrants and vaccination, Math. Biosci. Eng. 3 (3) (2006) 557.

[49] E. Shim, Z. Feng, M. Martcheva, C. Castillo-Chavez, An age-structured epidemic model of rotavirus with vaccination, J. Math. Biol. 53 (4) (2006) 719–746.

[50] E. Shim, A.P. Galvani, Impact of transmission dynamics on the cost-effectiveness of rotavirus vaccination, Vaccine 27 (30) (2009) 4025–4030.

[51] B. Wu, F. Fu, L. Wang, Imperfect vaccine aggravates the long-standing dilemma of voluntary vaccination, PLoS One 6 (6) (2011) e20577.

[52] K. Kuga, J. Tanimoto, Which is more effective for suppressing an infectious disease: imperfect vaccination or defense against contagion? J. Stat. Mech. Theory Exp. 2018 (2) (2018) 023407.

[53] A.L. Beatty, N.D. Peyser, X.E. Butcher, J.M. Cocohoba, F. Lin, J.E. Olgin, M.J. Pletcher, G.M. Marcus, Analysis of COVID-19 vaccine type and adverse effects following vaccination, JAMA Netw. Open 4 (12) (2021) e2140364.

[54] A. Riad, A. Pokorná, S. Attia, J. Klugarová, M. Koščík, M. Klugar, Prevalence of COVID-19 vaccine side effects among healthcare workers in the Czech Republic, J. Clin. Med. 10 (7) (2021) 1428.

[55] G. Ciccarese, F. Drago, S. Boldrin, M. Pattaro, A. Parodi, Sudden onset of vitiligo after COVID-19 vaccine, Dermatol. Therapy (2022).

[56] A. Clark, M. Jit, C. Warren-Gash, B. Guthrie, H.H. Wang, S.W. Mercer, C. Sanderson, M. McKee, C. Troeger, K.L. Ong, et al., Global, regional, and national estimates of the population at increased risk of severe COVID-19 due to underlying health conditions in 2020: a modelling study, Lancet Global Health 8 (8) (2020) e1003–e1017.

[57] P. van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Math. Biosci. 180 (2002) 29–48.