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Article

Building Mathematics Learning through Inquiry Using Student-Generated Data: Lessons Learned from Plan-Do-Study-Act Cycles

Christopher R. Rakes [1,*], Angela Wesneski [2] and Rebecca Laws [3]

[1] Department of Education, University of Maryland Baltimore County, Baltimore, MD 21250, USA
[2] Curtis Bay Elementary, Baltimore City Public Schools, Baltimore, MD 21225, USA; arwesneski@bcps.k12.md.us
[3] Baltimore City College High School, Baltimore City Public Schools, Baltimore, MD 21218, USA; rilaws@bcps.k12.md.us
* Correspondence: rakes@umbc.edu

Abstract: This paper describes how plan-do-study-act cycles engaged a classroom mentor teacher and student teacher in a professional collaboration that resulted in two inquiry activities for high-school geometry classes. The PDSA cycles were carried out in four high school geometry classes, each with 30 to 35 students, in a mid-Atlantic urban school district in the U.S. The four geometry classes were co-taught by the second and third authors of this paper. The data consisted of classroom documents (e.g., activity prompts, tasks), classroom observations, student feedback about activities, and monthly PDSA reports. The PDSA cycles had a direct effect on the professional learning of the teachers. The resultant classroom activities used a data collection approach to engaging students in inquiry to learn about trigonometry functions and density. Student learning behaviors were noticeably improved during these activities compared with traditional mathematics instruction. We concluded that the data collection sequence provided an accessible entry point for students to begin scientific inquiry in mathematics. The process opened the conceptual space for students to develop curiosity about mathematical phenomena and to explore their own research questions. The use of culturally relevant topics was especially compelling to students, and the open-ended nature of these exploratory activities allowed students to see mathematics through their own cultural lenses.

Keywords: mathematics education; inquiry; student-generated data; improvement science; teacher classroom research

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1. Introduction

Teacher preparation programs are considered one of the most effective leverage points for long-term improvement in teacher performance and retention of productive teachers [1–3]. Yet, the reform-based practices promoted by universities seldom find their way into the secondary mathematics classroom, limiting the ability of these programs to transform the field. Several contributing factors have been identified to explain this discrepancy, such as lack of reform-teaching models, greater intellectual demands on teachers, and resistance to change [4]. Gainsburg [4] also noted that the demands of reform-based teaching are especially burdensome for new teachers.

Although research is scarce on how to effectively prepare new mathematics teachers [5,6], many aspects of effective professional development (PD) have been well studied (e.g., Desimone [7]; Loucks-Horsely et al. [8]). Structuring teacher preparation as initial professional development, consistent with Bangel et al. [9] and Pollock et al. [10], allows the preparation program to benefit from existing knowledge about effective PD. By "effective," we mean that the experience supports the development of teachers as professionals and results in significant improvements in classroom practice [11,12]. The inclusion of teachers in

PD design and the application of professional learning promotes their growth as professionals [13] and situates PD experiences within particular school and classroom contexts [14]. Other characteristics of effective PD are an emphasis on student learning and classroom practice, a focus on specific academic content, and sustained opportunities for teachers to collaborate and provide peer feedback [7,8,15]. The Professional Development: Research, Implementation, and Evaluation framework ("PrimeD;" [16,17]) was designed to synthesize research and theory about effective PD. In the present study, PrimeD was applied to a teacher preparation program to support the professional learning of new and experienced teachers simultaneously. Through iterative cycles of whole-group activities and classroom implementation, the connection between professional learning and classroom practice is made explicit. Plan-do-study-act (PDSA) cycles [18] provide an organizational structure to classroom implementation.

In this article, we present a case study of a teacher candidate and classroom mentor teacher (hereafter "mentor") who, through plan-do-study-act (PDSA) cycles [18], developed a series of reform-based lesson activities throughout the full-time student teaching semester. The overarching questions driving the project were:

1. How do PDSA cycles support pedagogical innovation in the classroom?
2. How can reform-based teaching be transferred from theoretical ideas to classroom practice during full-time student teaching?

The candidate and mentor were participants in a teacher preparation program guided by PrimeD and developed a series of reform-based lesson activities during the full-time student teaching semester. The experiences of the candidate and mentor provide insights into the dynamics and ramifications of framing teacher preparation as professional development through PrimeD.

2. Background

PrimeD structures professional learning through four phases: design, implementation, evaluation, and research. In Design Phase I, participants map out a challenge space that includes a mission, vision, goals, targets, and strategies. In Implementation Phase II, participants form a networked improvement community (NIC) and meet regularly as a group. Change ideas developed during NIC meetings are taken to the classroom using plan-do-study-act (PDSA) cycles. Participants return to the whole-group meetings with results from their PDSA cycles. Phase III Evaluation consists of both formative and summative feedback. In Research Phase IV, research about the PD program is conducted, and findings from PDSA cycles are generalized across contexts.

Using PrimeD to structure teacher preparation is a unique and comprehensive approach for examining how to translate learning from a preparation program into actual teaching practices in the field. A lack of coherence between theory and practice may explain why some teachers do not use the strategies learned in their preparation program in their classrooms [4,19]. The implementation of PrimeD [16,17] to structure teacher preparation directly addresses such incoherence by explicitly connecting a well-defined, commonly-agreed-upon challenge space to pedagogical strategies that are used in coursework and field experience settings and refined through an iterative improvement process.

2.1. The PrimeD Framework: A PD Framework for Teacher Preparation

The PrimeD framework was initially developed through a systematic review of the literature [20] and through the evaluation of a state-wide PD program [17]. PrimeD applies the principles of improvement science to professional learning [21–23]. The use of PrimeD situates teacher preparation as PD, consistent with Bangel et al. [9] and Pollock et al. [10]. PrimeD organizes PD into four phases that work in a cyclic nature and occur iteratively throughout a PD program (Figure 1).

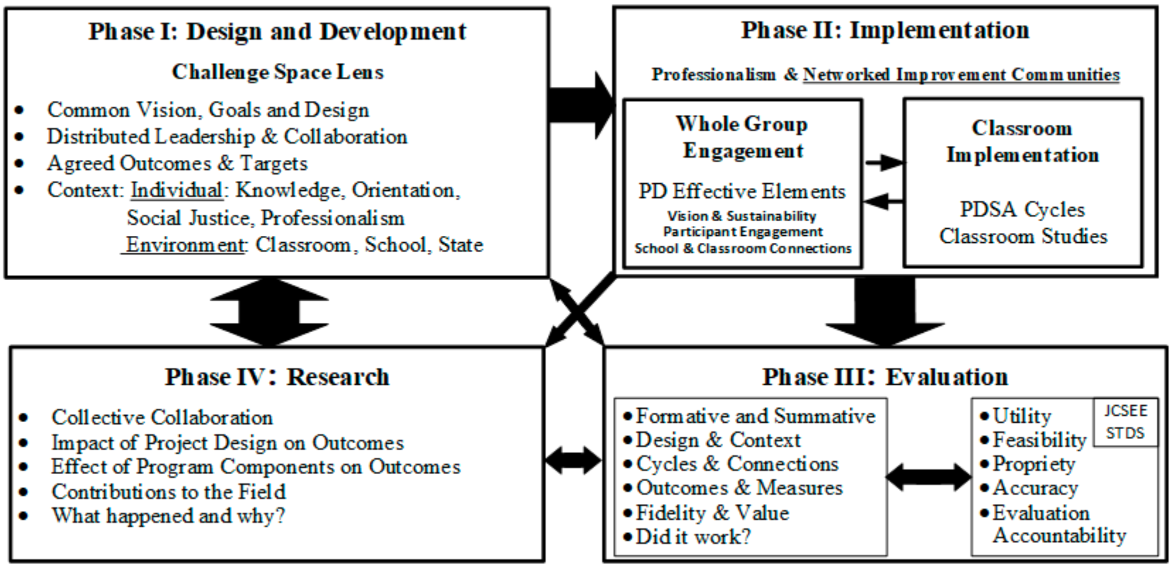

Figure 1. Condensed Model of the PrimeD Framework (adapted from [16]).

2.1.1. Phase I Design and Development

Phase I is foundational to every other phase in PrimeD. It goes beyond simply planning PD (e.g., courses, seminars, and field experiences in a teacher preparation program). Stakeholders come together to map out a challenge space—an explicit description of needs, vision, goals, targets, and strategies for meeting the challenges being addressed by and faced within the program [21]. In teacher preparation, participants include university faculty, classroom teachers, field experience supervisors, and teacher candidates. The challenge space is more than a list of obstacles or difficulties; it embodies the program's call to action to improve professional practice (e.g., classroom teaching, professional learning, and leadership) and expresses a pragmatic vision of the potential for systemic and systematic change [16]. Each course, class session, and field experience should be purposeful and intentionally aligned with the challenge space. But perhaps more importantly in teacher preparation, structural supports are needed to bind course and field experiences together into a coherent system wholly focused on achieving particular goals and outcomes defined by the challenge space [4,19,21].

2.1.2. Phase II Implementation

A program using PrimeD as its framework intends to engage teachers and teacher candidates as professional partners. The role of PD providers is to engage participants collaboratively with research and tools to support professional decision-making. As Datnow and Stringfield [24] noted:

> The fundamental difference between an amateur and a professional in any field is not one of intelligence or willingness to work hard. Rather, it is that professionals are trained at accessing their own research field, and therefore are much less likely to spend time repeating the others' prior mistakes. Educational reforms seem to have a less-than-glorious tradition of replicating major aspects of previous failed efforts. (p. 197)

Network improvement communities (NICs) and plan-do-study-act (PDSA) cycles [18,21] are the primary components of PrimeD Phase II and provide participants with opportunities to direct their own professional learning and apply their learning to the classroom. An NIC focuses on a problem of practice and develops change ideas to address that problem in the field through PDSA cycles. A problem of practice addresses obstacles to learning in the classroom that are focused on instructional practices and are actionable, observable, and measurable. PDSA cycles are intended to be rapid, small-scale changes that build over time into measurable improvement at scale [21]. For example, a teacher may change the way new topics are introduced and may refine the strategy each class period of the day. One advantage of pursuing PDSA cycles in groups is that the same trial can be tried out by multiple teachers in multiple settings to provide a more comprehensive test of the strategies studied.

2.1.3. Phase III Evaluation

As professionals, teachers participate in establishing what is best practice [25]. Engaging candidates in evaluation lays a foundation for professionalism throughout their careers. Evaluation cycles in PrimeD include feedback mechanisms to the challenge space (arrow from Phase III to Phase I in Figure 1).

Participants engage in regular self-evaluation through the PDSA cycles and peer evaluation through small- and large-group presentations at NIC meetings. Facilitators observe discussions at the NIC meetings as a formative assessment. Two to three local and non-local evaluators observe NIC meetings and provide monthly feedback about the quality of the NIC meetings and alignment to PrimeD. This feedback is used by NIC planning teams to guide subsequent meetings. The planning teams consist of faculty and representatives from the participant groups (e.g., mentors and candidates).

2.1.4. Phase IV Research

Teachers regularly carry out action research in their classrooms [26] and seek out research that is directly applicable to the classroom [27]. Teachers may at the same time think of "research" as a hands-off activity with little connection to the classroom [18]. Methods such as design-based research are especially useful to support partnerships between researchers and practitioners with a goal of generating outcomes that are both practical and contribute to theory [26]. PrimeD recognizes that viewing research as a seamless component of PD adds access, richness, and complexity to the process and has been shown to improve professional learning outcomes for teachers (e.g., [28–31]).

Teachers ideally conduct research as a normal function of their practice; that is, they test and evaluate their approach to teaching every day, seeking causal explanations for outcomes they observe. But these types of efforts are often contextually limited. The connection between implementation (Phase II) and research (Phase IV) activities (one-way arrow in Figure 1) situates classroom research activities as a first step toward generalizing results to be useful for a larger audience. While implementing PD innovations in Phase II's PDSA cycles, teachers create research questions from their classrooms. Results are generalized in Phase IV, when they are shared with the larger group to be tried and vetted to determine what works and does not work for desired outcomes under various conditions and why [32]. The NIC may use a variety of approaches and designs to generalize results beyond specific classroom contexts.

The inclusion of Phase IV in teacher preparation indicates an intention to prepare candidates to engage in professional research as teachers. Through the NIC and PDSA cycles, candidates observe mentor activities, ask questions, engage collaboratively, and develop the necessary foundations for contributing to the knowledge base. Mentors, supervisors, and faculty help to hone candidates' professional judgment as they draw conclusions about their classroom research.

2.2. Reformed Teaching, Inquiry, and Constructivism

The mathematics teaching field has recognized for centuries the need to reform traditional teaching techniques to improve learners' conceptual and relational understanding, critical thinking, and reasoning (e.g., [33,34]). Traditional epistemology in U.S. mathematics classrooms views the teacher as an authority who conveys knowledge to students, who are largely viewed as blank slates [35,36]. Constructivism views learning as the construction of meaning by the learner rather than the passive reception of knowledge [37].

Piaget described the process of knowledge acquisition through a constructivist perspective. When students encounter new information that fits into their existing conceptual framework, the new information is assimilated (not requiring reconstruction of students' schemas/conceptual frameworks). For example, when a student believes that when two numbers are multiplied the product is always larger than the original two numbers, and every multiplication example they encounter results in larger products, their conceptual framework will be reinforced, leaving intact their belief about multiplicative structures. Sometimes, however, the information is recognized as not aligning to their current schema, requiring accommodation, in which case a restructuring of the schema is required to resolve the cognitive dissonance [38]. Teachers can encourage accommodation in mathematics by choosing tasks and activities within the range of their students' assimilation abilities but which have elements that introduce some degree of cognitive dissonance [38]. From the above example, students who encounter multiplication examples that result in products smaller than the original numbers must accommodate the new information when it does not fit their current understanding.

The constructivist perspective requires substantial shifts in traditional educational practice, such as decentering teacher authority, valuing social contexts, and emphasizing students' natural curiosity [37]. Reformed teaching is founded upon constructivist epistemology, including lesson pedagogy and a classroom culture that supports change [39]. Reformed teaching is typically inquiry-based, meaning that students engage in exploration and experimentation prior to a formal presentation. The National Research Council [40] summarized scientific inquiry through eight practices:

1. Asking questions;
2. Developing and using models;
3. Planning and carrying out investigations;
4. Analyzing and interpreting data;
5. Using mathematics and computational thinking;
6. Constructing explanations;
7. Engaging in argumentation from evidence;
8. Obtaining, evaluating, and communicating information. (p. 42)

The term "practice" is used to emphasize that students must simultaneously coordinate knowledge and skill [40]. The expectation for inquiry-based teaching is that students will themselves engage in the practices and not merely learn about them secondhand. By treating mathematics as a scientific endeavor, teachers promote the building of abstract knowledge from simpler, concrete experiences, and student explorations precede formal presentations. Students engage in predictions, hypotheses, and estimation as well as designing experiments to test their conjectures. Students engage in constructive criticism of one another's ideas [39]. These pedagogical approaches directly support constructivist views of learning by building new knowledge from pre-existing knowledge in learning communities and tapping into students' natural curiosity.

3. Methods

This classroom study followed PDSA cycles [18], which provided a structure for multiple classroom trials with refinements at each iteration. The trials were carried out in four high school geometry classes, each with 30 to 35 students in a mid-Atlantic urban school district in the U.S. The four geometry classes were co-taught by the second and third authors of this paper. By "co-taught," we mean that both teachers were involved in the design of the lessons. The teacher candidate led the enactment of the lessons with the mentor providing support, observing and taking notes, and providing feedback on both the design and enactment of the lessons.

3.1. PDSA Cycles in the NIC

The teachers in the present study were part of a networked improvement community (NIC). The overarching problem of practice was focused on how to improve mathematics teaching through inquiry. PDSA cycles provided the structure for participants to plan, enact, reflect, and refine a change idea (teaching strategy) that was decided upon during a monthly NIC meeting. The classroom artifacts, data and evidence, and participant reflections were brought back to the subsequent NIC meeting. The NIC then refined the overall strategies as a group based on the participant reports. The PDSA classroom research process mirrors design-based research in that the innovations and techniques evolve through each iteration.

The NIC met monthly throughout an entire school year. Participants developed their problem of practice during the fall semester (September through December) and tried out their initial change ideas. By the beginning of the spring semester (January through May), the change idea had been refined and was ready for more intensive try-outs. Participants completed at least one PDSA form each month, which represented a variable number of PDSA cycles. While lessons throughout the school year were affected by the PDSA cycles, the lessons presented in the present study represent the culmination of the teachers' reflections and refinements.

3.2. Data and Measures

Data consisted of classroom documents (e.g., activity prompts, tasks, assessments), classroom observations, student feedback about activities, and monthly PDSA reports. Student views were collected through classroom discussions, informal student interviews, and open-ended survey questions. Both the mentor and candidate took notes and observed student behaviors during lesson activities. Teacher views were collected through interviews and PDSA forms.

The degree to which the candidate's teaching improved in terms of reform-based teaching was measured through the formal observations of a field experience supervisor (not an author) and the mentor and scored on the Reformed Teaching Observation Protocol with equity-based performance descriptors (RTOP-E). With 25 indicators on the RTOP-E, each indicator is rated from 0 (no evidence) to 4 (fully reformed practice) for a possible total of 100 points. Level 2 performances are considered to be more traditional with some reformed elements, and Level 3 performances are considered to be more reformed with some traditional elements.

The RTOP-E was based on the RTOP+ [39,41,42] and explicitly incorporated the equitable teaching practices described in *Catalyzing Change in High School Mathematics* [43]. For example, Row 1 was revised to include students' cultural identity (RTOP-E new text italicized): "The instructional strategies and activities respected students' *cultural identity and* prior knowledge and the preconceptions inherent therein." Performance descriptors were revised to include expectations of equity explicitly, especially at Levels 3 and 4 of the rubric. For example, Row 1, Level 3, on the RTOP-E stated, "The teacher actively solicits student ideas *and cultural experiences*, and discussion of these ideas and *experiences* takes place throughout the lesson, but lesson direction is teacher determined". Level 4 stated, "The teacher actively solicits student ideas *and cultural experiences* and builds the lesson from these ideas *and experiences* as a starting point. The direction of the lesson is shaped by student ideas *and experiences*". The revisions were made by a team of mathematics and STEM faculty, then shared with an expert panel for feedback to enhance content validity. The RTOP-E indicators and performance descriptors were used in monthly NIC meetings to structure conversations about effective pedagogy. These conversations included scoring sample lesson videos on the RTOP-E and supported a common understanding of the measured constructs and performance descriptors (construct validity) and how to score the RTOP-E consistently (inter-rater reliability).

The supervisor and mentor independently scored three lessons, one near the end of the Phase I internship in November, one at the beginning of the Phase II internship in February, and one at the end of the Phase II internship in April. The supervisor and mentor scores were the same for 53/75 scores (70.6%) and were adjacent (a difference of 1) for 14/75 scores (18.7%), meaning that they were in agreement (exact or adjacent) for 89.3% of the indicators. The intraclass correlation (ICC) was 0.704, which was considered good based on Cicchetti's criteria [44]. For simplicity, the supervisor's scores were used in the present analysis.

4. Results

The NIC brought participants (mentors, teacher candidates, field experience supervisors, faculty, and alumni) together monthly to discuss the program challenge space, classroom change ideas, and strategies for implementing a change idea. The challenge space was developed by a team of participants and was organized by teacher knowledge, teacher orientation, teacher practice, and student outcomes (see Appendix A). Participants were invited to help to plan monthly NIC meetings and agree upon a focus within the challenge space. The NIC meetings focused primarily on the teacher practice category of the challenge space, especially using reform-based teaching practices [26] and the RTOP-E as a framework to discuss various challenge space goals.

Teachers in the NIC focused on the problem of practice of how to build connections between new mathematics content and students' pre-existing knowledge and experiences. One change idea that the mentor and teacher candidate explored was the use of an inquiry-based activity process to engage student pre-existing knowledge to build new understanding. As part of the "Plan" for PDSA cycles, and based on Watson [45] and Lamar and Boaler [46], it was hypothesized that a data collection inquiry process would facilitate student engagement in inquiry-based activities such as those described by Anderson et al. [47] and Engle and Conant [48]. It was also hypothesized that this type of engagement would improve learning of mathematics concepts that are typically taught procedurally at the high school level in the U.S. (for example, mathematical formulas and their proofs) [36]. By "engagement," we mean that students attempted at least one lesson activity, task, question, or problem.

4.1. Teaching Mathematics through Inquiry

The approach to inquiry in this setting began with student data collection and pattern analysis as a scaffolded entry to theoretical concepts. Through student discussions and debating of ideas, students were able to engage meaningfully with the material and continue developing their conceptual understanding.

While this process may seem fairly straightforward to those familiar with inquiry, many mathematics curriculum materials in the U.S. are not written in a way that supports student-led inquiry. Traditional mathematics teaching is not inquiry-driven, focusing instead on practicing procedures with a notable absence of mathematical reasoning [36,43,49–51]. In many ways, the teachers were "starting from scratch," determining how to adapt their curriculum to be a more robust learning experience for their students, especially those who struggled. The PDSA cycles provided them with a structure to organize their own learning of how to teach through experimentation, reflection, and adjustment. Table 1 presents an example plan developed during an NIC meeting.

Table 1. Example plan for PDSA cycles.

Prompt	Response
Challenge or goal of this PDSA cycle.	Collaboration with data collection
Context (e.g., grade level, course, topic)	10th grade, geometry, density
Expected duration of this PDSA cycle. (e.g., 10/15 min).	One lesson, modeling data collection will be at the beginning of the lesson
Change idea or strategy for meeting your challenge	Model data collection before the students collect their own data
Prediction(s)/hypotheses (What you think the change idea/strategy will accomplish?)	Modeling the data collection will show students how to complete procedures. Avoid confusion when starting the collaboration and data collection
Evidence to collect	Student work and student ability to complete data collection on their own/with minimal help from the teacher

The inquiry process developed through the PDSA cycles began with student data generation and development of a question about a phenomenon rather than procedures to be memorized, consistent with Anderson et al. [47]. Students gathered data, looked for patterns, drew conclusions, and discussed how to interpret the evidence. Student reflection was followed by reinforcement activities that helped students make connections between their exploration and mathematical procedures. This process addresses the tenets of Engle and Conant's productive disciplinary engagement [48]: using problems to engage students with content, giving students authority to investigate the problems, and facilitating their exploration with relevant resources and support. Lessons that use this process will engage students primarily in the Common Core Mathematics Practice #7, Look for and Make Use of Structure, but may also address Practice #4, Model with Mathematics [51]. The modeling of data collection shown in Table 1 was an important component that was added and refined during the PDSA cycles in response to student feedback. As cycles were completed, the teachers refined the change idea into a general process, shown in Figure 2.

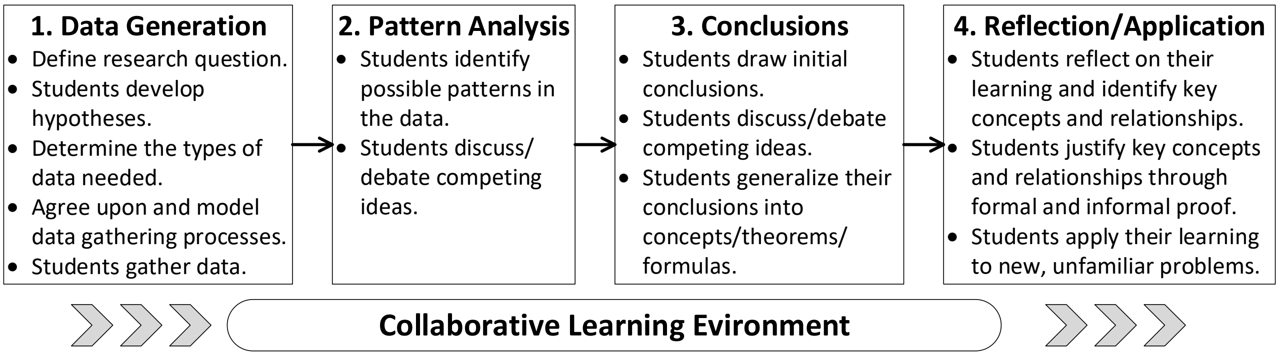

Figure 2. Inquiry learning process in a data collection context developed through PDSA cycles.

Two activities illustrate the inquiry process from Figure 2 and how the PDSA cycle process enhanced the lessons. The first activity, Inquiry into Trigonometric Ratios, focused on the development of deep connections between the various trigonometric functions. The second activity, Population Density, used a culturally relevant approach to developing conceptual understanding of density. Candidate and mentor reflections and notes, student feedback, and independent classroom observation notes were incorporated into the activity descriptions.

4.1.1. Example PDSA Lesson 1: Inquiry into Trigonometric Ratios

In this introductory lesson to trigonometric ratios, the objective was: *Students will be able to explain the relationship between sine and cosine of complementary angles verbally and algebraically*. We (mentor and student teacher) began the lesson by modeling a separate, simpler trigonometric concept with the goal of teaching students how to use the trigonometric functions on an online calculator. Students used the Desmos Graphing Calculator [52], which includes all six trigonometric functions. Student perceptions of the calculator component of the lesson, collected through a classroom survey, were mostly positive, with some students explicitly stating that they "liked using the calculators to solve problems". This whole-class introduction asked students to generate data by choosing angle values between 0 and 90 degrees then filling out the table in Figure 3.

Angle	Sin(A)	Cos(A)	Tan(A)	Sec(A)	Csc(A)	Cot(A)	1/Sec(A)	1/Csc(A)	1/Cot(A)

Figure 3. Introductory table for calculator exploration of trigonometric relationships.

After students completed the table, the teachers asked them to notice and wonder about the values they found [53]. Notice and Wonder is a method of open-ended pattern exploration that encourages students to look for whatever patterns they can find and ask questions about anything confusing. Students were able to identify that $sin(A) = \frac{1}{csc(A)}$, along with the other reciprocal identities. The class used these observations to write equations describing the relationships between all six trigonometric functions.

Once they had completed this introductory activity as a class, students were given another chart to use for data collection (Figure 4). One row of values was provided as an example for students to use as guidance.

Angle A	Sin(A)	Cos(A)	B=90 – A	Sin(B)	Cos(B)
51	0.777	0.629	49	0.629	0.777

Figure 4. Follow-up exploratory table to develop sine and cosine relationships.

We allowed students to choose whether to work independently or collaboratively. Most chose to work collaboratively. Based on task assessment and teacher reflections, some students struggled to understand the notation in the column titles. Many students began the activity by asking us what went in each column rather than interpreting the notation at the top of each column. Instead of simply answering these questions, we asked students to look at the notation and take an "educated guess" as to what we were looking for, then we asked guiding questions until they figured it out, for example, "What does the title of this column tell us to do? What does ''$\sin(A)$ mean? What is A?" Such productive struggle was embraced because it ended up helping them to build a stronger understanding of variable meaning and substitution. By the end of the activity, most students were referencing the notation and interpreting what each column required computationally, completing the charts without teacher support.

Students then engaged in another Notice and Wonder activity without teacher guidance [53] as a way to help students move deeper into pattern analysis (Step 2 in Figure 3). The candidate and mentor observed that students readily noticed columns that were identical in value. They also used the column headings to write equations describing the relationship between the sine and cosine of complementary angles, for example, noticing that $\sin(A) = \cos(B)$ when m∠A + m∠B = 90°. Based on classroom survey data, students found this part of the lesson intriguing; for example, students stated, "sin/cos = tan was very interesting to learn."

Student feedback on the survey was mostly positive, and students were able to meet the lesson objective. As part of the Act step in the PDSA cycle, we considered ways to improve the lesson going forward. Several students on the survey noted that the numerical analyses were difficult, with statements such as "I didn't like all the numbers", "I didn't like looking at all the numbers and getting mixed up", and "The numbers being different, sometimes it confused me, thinking I was wrong". We realized upon reflection that this data collection method was too separate from the tangible work we had been doing with triangles in class prior to the lesson. We also noted that, in an introductory lesson to trigonometric ratios, greater emphasis needed to be placed on the reference angle. In subsequent lessons, students struggled to transfer their learning from this lesson to the triangle contexts, which supported our analysis of the lesson.

Upon reflection during a PDSA cycle, it was determined that an exploration that includes a visual representation offers a way to enhance these kinds of connections and emphases with students. For example, a Geogebra app such as the one shown in Figure 5 allows students to discover that, regardless of the size or orientation of the triangle, the ratios of the side lengths stay constant.

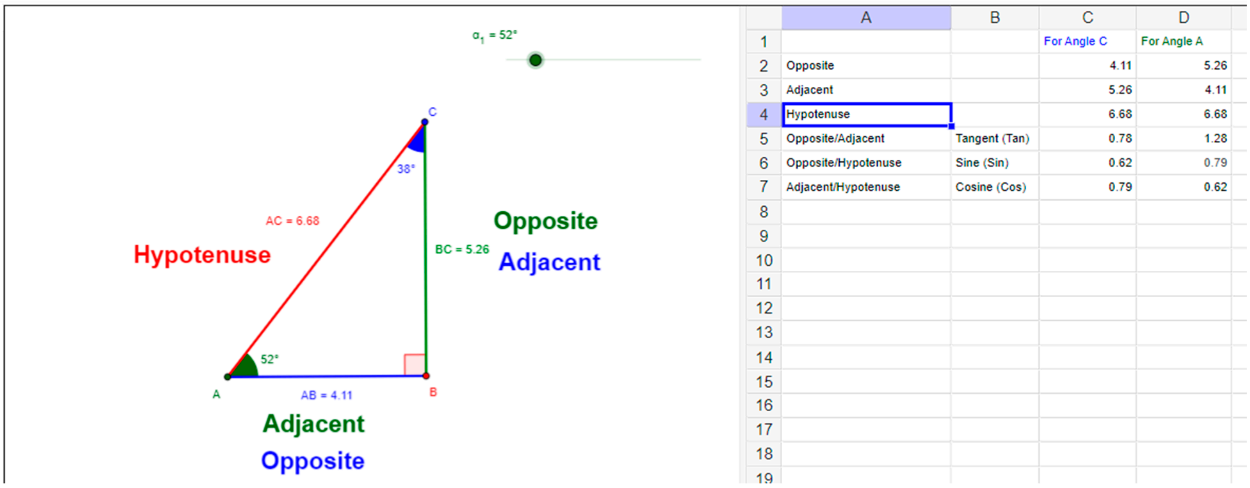

Figure 5. Screenshot of the Trigonometric Ratio Geogebra app [54].

The slider a1 defines the measure of ∠A. Points A and B can move to change the side lengths but not the measures of the angles. Point C is fixed to maintain the right angle at Point B. As Point A and Point B are moved, the side lengths change, but the angle measures and ratios of the side lengths remain constant. With the slider, students can discover that even a slight change to the angle measure changes the ratios of the side lengths. The color coding of the segments and text in the image helps reinforce how the labels of opposite and adjacent are specific to the angle of interest. The app includes sample questions that teachers can use to guide students through the exploration process to discover the one-to-one relationship between angle measures and trigonometric ratios. For example, students can move Point A to several new positions, as shown in Figure 6.

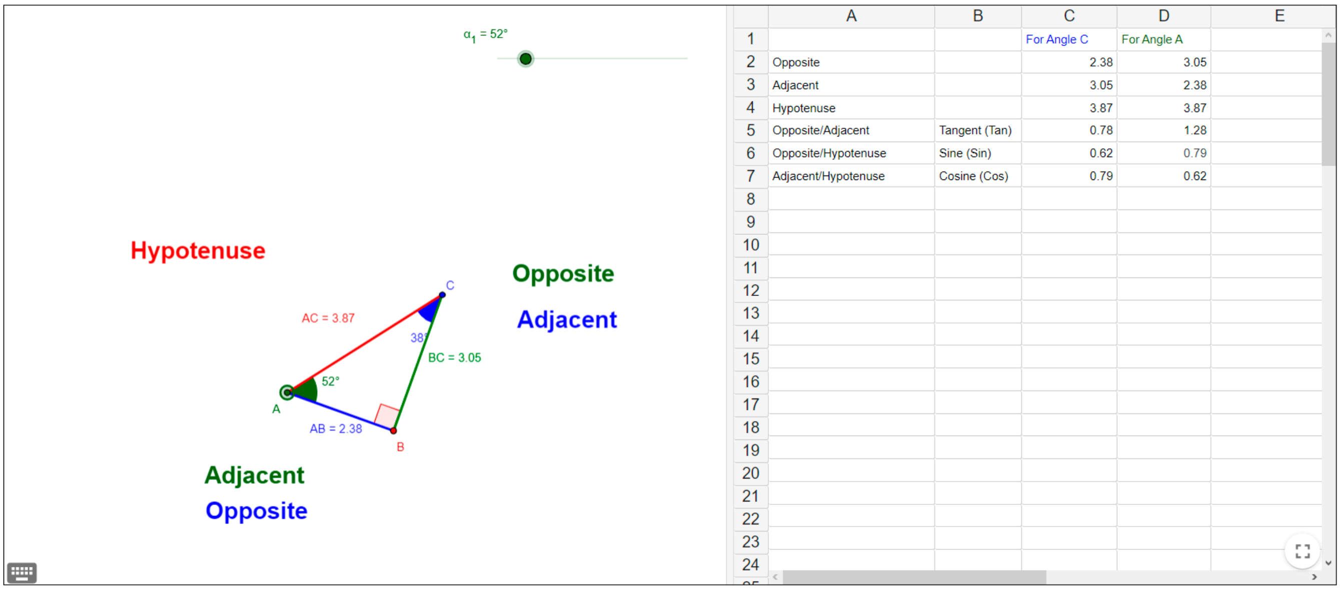

Figure 6. Moving Point A in Figure 4 shows different side lengths but constant trigonometric ratios [54].

By contrast, any change in the angle measure will change the trigonometric ratios, as shown in Figure 7. The sample questions provide a pre-planned experiment. Once students are already familiar with experimentation, the activity can be modified to allow them to design their own experiment.

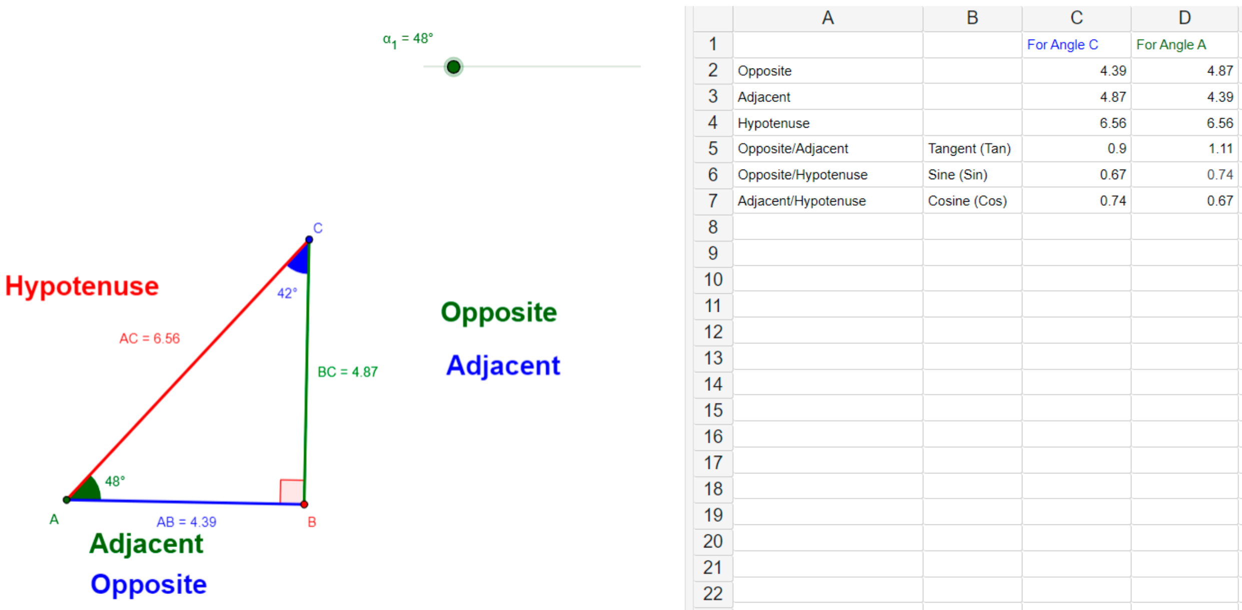

Figure 7. Changing the slider changes the measures of Angle A and Angle C and the trigonometric ratios [54].

Through such an exploration, students are able to make predictions/hypotheses about the triangles and side length ratios and conduct an experiment. This type of applet could be used as an extension of the original exploration but would be more powerful if it were used to create the initial data for the calculator activity (Columns 1–4 in Figure 3). Once students are able to connect the triangle ratios to the values for sine, cosine, and tangent, they will likely be more ready to investigate the secant, cosecant, and cotangent functions (and their relationships to sine, cosine, and tangent) with a calculator.

4.1.2. Example PDSA Lesson 2: Exploration of Population Density

Building from the trigonometry lesson, our PDSA change idea of using student inquiry through data exploration was applied to a geometry lesson about density. Density as a concept is often taught conceptually, usually with a real-world tie to physics that is difficult for some students to grasp when learning about density for the first time. Rather than introducing density through a physics application, we chose to take a more general approach, describing density as simply "an amount of stuff in an amount of space." This approach opened doors for us to explore density in a multitude of ways that included, but was not restricted to, an amount of mass in a specific volume. Using population density as an entry point, students collected data (Step 1 in Figure 2) and used their data to design and conduct a research investigation.

We began the lesson by prompting students to recall previous examples of density that we had worked with in class. We also recalled the framing:

$$\text{Density} = \frac{\text{stuff}}{\text{space}} \tag{1}$$

With some prompting, students were able to generate an equation for population density:

$$\text{Population Density} = \frac{\text{Population}}{\text{SquareMiles}} \tag{2}$$

Based on the PDSA cycle reflections from the trigonometry lesson, we included teacher modeling of the data collection process by finding the population density of Baltimore City, where our school is located. As a class, we used an online search engine to find recent data on the population of Baltimore and the land area of Baltimore, and then we substituted this into our equation to find the population density.

We asked students to reflect individually on why population density might be important to them or other members of the community and pose a question to explore. Having students pose research questions and design their own investigations moved our overall change idea deeper into reformed teaching from the trigonometry lesson, from a class exploration of a phenomenon to a student-led, open-ended inquiry (as described by Sawada et al. [39]). This trajectory was purposeful in the evolution of our change idea: as Blair [55] noted, teachers may restrict an inquiry's activity in the hope of engaging the whole class. Rather than incorporating other inquiry pathways later in the trigonometry lesson (more consistent with Blair's un-planning process), we opted to instead incorporate more inquiry pathways in the density lesson. This approach allowed us to continue enhancing our ability to conduct inquiry-based lessons without falling behind in the required district curriculum.

Most student research questions in this lesson filled the sentence frame "How does population density affect ______________?" Topics chosen by students included police interactions, number of schools, and commute times. Students then researched on the internet to find the population density of three locations. Ideally, these locations spanned different geographic areas, including a city, suburb, and rural area. They also found a data point related to their research question. They used their data to fill out the prompts shown in Figure 8.

Lastly, students completed a reflection question in which they answered their original research question based on the data they collected. Most students reflected that they had enjoyed this lesson more than usual, and some students displayed a deep interest in their research questions. This topic sparked interest from students that had typically had trouble focusing in class. Students enjoyed picking their own research questions as well as collecting their own data. Some students struggled to find data, which led to some discussion on how to research and what questions to type into an online search engine to find the data we are looking for.

City: _______________

Population: _____________

Land Area: ______________

Population Density: __________________

Data for your impact question:____________

Suburb: _______________

Population: _____________

Land Area: ______________

Population Density: __________________

Data for your impact question:_____________

Rural town: _______________

Population: _____________

Land Area: ______________

Population Density: __________________

Data for your impact question:_____________

Figure 8. Example population density student data information form.

4.2. Participation in Lessons Affected by PDSA Change Ideas

The PDSA cycle outcomes reported here were conducted in four high school geometry classes. Student participation in the lessons was generally higher in the PDSA-affected lessons ("PDSA lessons" hereafter) than in comparable lessons before and after. Participation was operationalized as either engagement (at least one activity, task, question, or problem) or full participation (completing all independent work). For comparability, lessons were chosen that required independent work using an online district platform (ImagineMath [56]) intended to increase student accessibility and participation in the lessons. Table 2 provides an example of the numbers and percents of students who participated during one PDSA lesson compared with ImagineMath Lessons, before and after.

Table 2. Numbers and percents of students that participated in PDSA and comparison lessons.

		PDSA Lesson (Inquiry through Trigonometric Ratios)		Comparison Lessons: No (and Percent) That Engaged in Any Work (at Least One Question)		
Class	**No. Students**	**No. (and Percent) That Completed Independent Work**	**No. (and Percent) That Engaged in Any Work (at Least One Question)**	**Lesson 1 before PDSA Lessons**	**Lesson 2 after PDSA Lessons**	**Lesson 3 after PDSA Lessons**
1	22	8 (36.4)	13 (59.1)	2 (9.1)	2 (9.1)	3 (13.6)
2	27	18 (66.7)	21 (77.8)	9 (33.3)	12 (44.4)	18 (66.7)
3	29	15 (51.7)	16 (55.2)	3 (10.3)	10 (34.5)	5 (17.2)
4	31	19 (61.3)	20 (64.5)	3 (9.7)	12 (38.7)	15 (48.4)

Based on task assessments, participation was generally higher in the example PDSA lesson. The candidate and mentor compared participation rates for individual students and found that students with consistently low participation rates had higher participation rates in the PDSA lessons. While there are many potential contributing factors to participation rates in a lesson, student feedback on the classroom survey was also quite positive for the approach of examining data. For example, multiple students stated in a class survey, "I liked picking my own numbers." Students considered the lessons to be more accessible.

4.3. Teacher Candidate Growth in Reformed Teaching

Lessons led by the teacher candidate were scored three times by a field experience supervisor. The supervisor's observations provided an independent measure of her ability to use reform-based teaching methods. The lesson observed at Time 1 was an exploration of rotational symmetry. The lesson observed at Time 2 was an introduction to identifying trigonometric ratios on a right triangle, a precursor to the trigonometric pattern exploration described in Section 4.1.1 above. The lesson observed at Time 3 was the density lesson described in Section 4.1.2 above. As shown in Figure 9, most of the growth occurred during the Phase II Internship, which is the full-time student teaching semester.

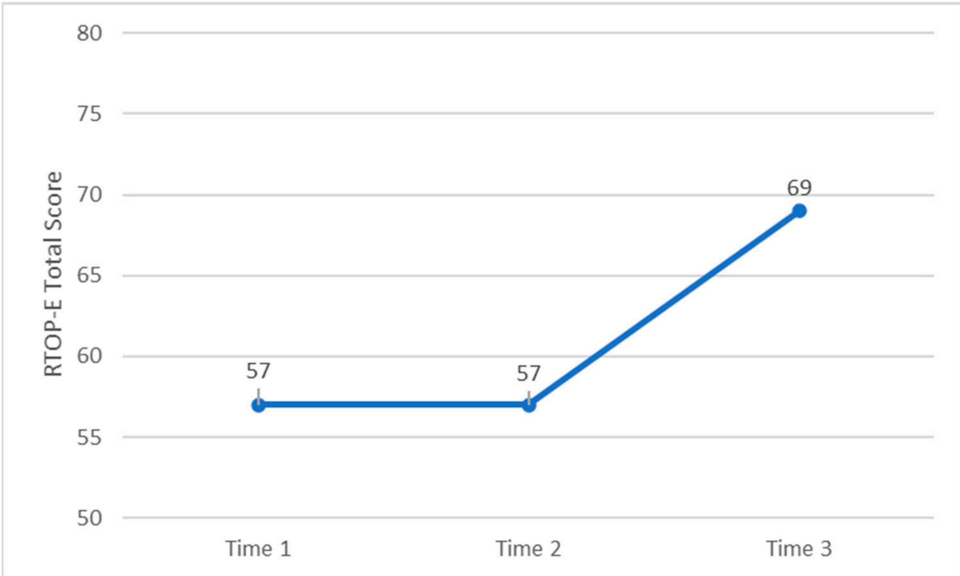

Figure 9. Teacher candidate RTOP-E scores. Time 1 = Phase I internship, mid-November. Time 2 = Beginning of Phase II internship, late January. Time 3 = Second half of Phase II, late March.

This growth trend is consistent with prior cohorts [42] and comparable to the cohort means at each time point. A repeated-measures ANOVA (RM-ANOVA) was used to analyze differences across time. Mauchly's test of sphericity indicated that the assumption of sphericity was not violated, $W(\text{df} = 2) = 0.921$, $p = 0.386$. The RM-ANOVA showed that the growth was significant, $F(2, 48) = 3.197$, $p = 0.05$.

The supervisor noted at Time 1 that the lesson included multiple opportunities for student collaboration and discussion within groups. He noted, "[the candidate] comfortably discussed the activity with various groups and visited all of the groups during the period." Issues to be addressed focused on classroom management issues and equitable access to technology used in the lesson and equitable participation of students in whole-class discussions.

At Time 2, the supervisor noted that the lesson acknowledged students' cultural perspectives. Students were given opportunities to lead discussions. Issues to be addressed focused on suggestions for multiple ways to represent and clarify the trigonometric reference angle.

At Time 3, the supervisor remarked on several strengths of the lesson, especially its cultural relevance: "the ability to make mathematics relevant to students and show how it can be used to plan for living in various environments, urban and suburban. Relating mathematics to different content areas such as urban planning. Allowing students to use research in the development of mathematical concepts." Issues to be addressed included planning for students to write a summative paragraph of their findings and to share their work in class.

5. Discussion and Conclusions

The present study focused on how PDSA cycles support pedagogical innovation in the classroom (Research Question 1) and how reform-based teaching can be transferred from theory into practice during full-time student teaching (Research Question 2). The PDSA cycles provided a structure for improving pedagogy in subsequent lessons. The results showed that the integration of candidate and mentor observations and reflections, student feedback, and independent observation feedback provided the data needed for the candidate to improve the way inquiry was used in the lessons. The RTOP-E scores demonstrated that measurable growth in the candidate's use of reform-based teaching was observed by the supervisor and mentor.

The PDSA cycles for this project focused on engaging students in pre-exploration and collaborative discussions. Student participation was noticeably improved during these activities compared with traditional mathematics instruction. For example, the candidate and mentor observed that students who frequently gave up on exploratory activities instead engaged in productive struggle. The strong connections to their local community in the density project led to student excitement about the mathematics, expressed to the candidate and mentor through the classroom survey and informal class discussions. As noted in Section 4.1.2, students led their own investigations by posing their own questions and designing their own experiments. The supervisor noticed that students led more of the classroom discussions in this lesson compared with prior observed lessons. The willingness of students to take on more responsibilities during the trigonometry activities (e.g., leading classroom discussions) surprised the teachers and spurred them to give more responsibility for the learning to the students in the density lesson (e.g., picking their own research questions).

Based on the improved participation rates, we concluded that the student data collection sequence provided an accessible entry point to begin scientific inquiry in mathematics. The process provided an opportunity for students to develop curiosity about mathematical phenomena and to explore their own research questions. Such open-ended opportunities are sometimes described as "opening the conceptual space" (e.g., Niesser et al. [57]) because they allow students to understand the content in multiple ways and through multiple perspectives rather than through a narrow interpretation provided by a lecturer. Such an approach allows students to analyze conceptions that are partially correct and determine whether such conceptions are valid in various contexts [58]. In the present study, students analyzed data to determine the extent to which the patterns they noticed held true. According to survey results, the use of culturally relevant topics was especially compelling to students, and the open-ended nature of these exploratory activities allowed students to see mathematics through their own cultural lenses.

As shown in Table 1, the PDSA cycles provided a structure in which both mentor and candidate could direct their own professional learning. The NIC meetings provided a monthly forum in which the mentor and candidate explored mathematics pedagogy with a community of educators, planned strategies for improving their classroom practice, and received feedback on their change ideas. Between NIC meetings, the mentor and candidate completed multiple PDSA cycles and wrote up the results and reflections on a PDSA form. Both mentor and candidate found that the PDSA cycles provided structure to their daily reflections, especially the explicit focus on planning to collect data about outcomes resulting from the change idea. We concluded that the PDSA process opened up communication between the teacher candidate and the mentor to mutually support their professional learning as they enacted, studied, and refined their pedagogical change ideas.

The mentor and supervisor both noticed in their observations that the teacher candidate developed stronger classroom communication skills; for example, the ability to clearly convey mathematical ideas to students (e.g., using multiple representations as noted at

Time 2 observation). The mentor found that she became more acclimated to shifting responsibility for thinking to her students, which was also shown in the supervisor feedback. She also noticed an increase in the number of lessons developed by the candidate that focused on discovery, exploration, and inquiry rather than processes and procedures outside the formally observed lessons.

The PrimeD framework guided the teacher preparation program, structuring the program challenge space and directing the process for developing pedagogical change ideas as a professional community and testing those ideas in specific classroom contexts. PrimeD is recommended as one way to structure teacher preparation to facilitate professional learning. The results of the present case study provide encouraging results for teachers to use data collection and inquiry activities to frame mathematics as a vibrant, interesting, and relevant scientific endeavor.

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Appendix A

Appendix A.1. Secondary Mathematics Challenge Space

Development of the challenge space is ongoing and includes input from classroom mentor teachers, mathematics coaches and supervisors, and teacher candidates and program completers. The goals of the program are viewed through four constructs: teacher knowledge, teacher orientation (e.g., attitudes, beliefs, self-efficacy), teacher practice, and student outcomes. Teacher knowledge and orientation influence each other and inform teacher practice. Teacher practice includes reflection on student outcomes, thereby reinforcing or refining teacher knowledge and orientation and informing the program (i.e., a feedback loop).

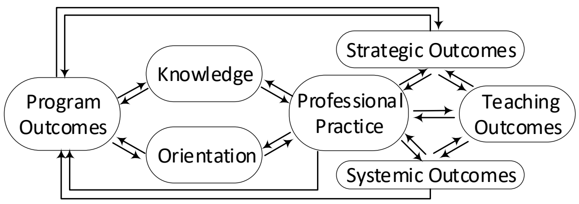

Figure A1. Model of outcome relationships in a teacher education program [42]. Note: “Student” is used to refer exclusively to children in PreK-12 classrooms; “Teacher” refers to teacher candidates as well as PreK-12 classroom teachers.

Appendix A.2. Vision

The mathematics teacher preparation program is designed to help candidates explore, enact, and insist upon equitable teaching practices to support robust mathematics learning communities. Learning communities are defined as collaborative groups that are pursuing common goals for mathematics learning experiences. Learning communities that are safe, humanizing, collaborative, and culturally aware empower participants to direct their engagement in scientific inquiry and the examination of diverse ideas and perspectives. Candidates enter the field ready to improve the quality of mathematics teaching and learning for each and every student in their classrooms, schools, and districts and to become societal change agents in the field.

Appendix A.3. Goals

Teacher Knowledge

- Subject Matter Knowledge. Teachers have a robust knowledge of mathematics, understanding how concepts and procedures are interrelated and how to frame mathematics knowledge in a meaningful way to help students learn (Mathematics Knowledge for Teaching).
- Pedagogical Content Knowledge. Teachers develop robust pedagogical knowledge to support deep mathematics learning in their classrooms, including the use of tools for teaching mathematics (Knowledge for Teaching Mathematics).
- Knowledge of Orientation. Teachers understand and respect the relevance of the affect of each member of a learning community (e.g., attitudes, culture, beliefs, values, confidence, and anxiety) in learning mathematics.
- Knowledge of Discernment. Teachers understand that discernment encompasses the connections between cognition, metacognition, and learning and decision-making processes. Knowledge of discernment includes understanding developmental processes and the socio-emotional and sociocultural components of learning.
- Knowledge of Individual Context. Teachers understand that learning and decision-making processes take place within the context of the intersectionality of social categories.
- Knowledge of Environmental Context. Teachers understand the importance of building an inclusive and equitable environment to support a robust learning community.

Appendix A.4. Teacher Orientation

Orientation plays an important role in how teachers approach the profession individually as well as in collaboration with students, colleagues, schools, and the community. Orientation includes, but is not limited to, constructs such as attitudes, perceptions, self-efficacy, beliefs, confidence, self-concept, motivation, value of mathematics, interest in mathematics, enjoyment of mathematics, enjoyment of teaching, usefulness of mathematics, mathematics goals, professional goals, attributions of success/failure, mathematics anxiety, professional anxiety, professional dispositions, commitment to lifelong learning, and perceptions of power and agency.

These orientations can be about a wide range of topics, including, but not limited to, mathematics, teaching and learning, assessment, students, socio-cultures, families and caregivers, collaboration, the profession, and schools and districts.

Teachers examine orientation as an ongoing part of their growth and learning to ensure that all aspects of the profession are approached through a productive lens. Teachers are willing to change their views when appropriate.

Appendix A.5. Teacher Practice

- Culture. Teachers establish a culture of access and equity through classroom structures and culturally relevant pedagogy to support each and every student in learning and participating in mathematics deeply. These classroom structures empower students to value diverse perspectives by elevating their voices, providing leadership opportunities, and developing a strong learning community. Teachers model vulnerability, viewing mistakes as learning opportunities. Varied approaches are visible and valued.

- Active Engagement. Teachers actively engage students in learning mathematics and/or science with meaning.
- Conceptual Understanding. Teachers explicitly foster, model, and insist upon conceptual understanding and coherence for all learners at all levels as a primary means for promoting procedural understanding in mathematics. Teachers insist that all teaching activities and learning experiences embrace the development of conceptual understanding as the fundamental core of learning and form the foundation for peer discussions.
- Connections. Teachers structure lessons through a phenomena-first approach, recognizing that authentic contexts are the foundation of the lesson and frame the content to be learned. Contexts are not simply enrichment that happens after the "real" lesson if at all.
- Reasoning. Inquiry-based projects are incorporated in every unit. Quantitative reasoning is modeled as scientific inquiry (claim, evidence, rationale).
- Questioning. Questioning is purposefully crafted to foster higher-order thinking and alternative modes of thinking about mathematics. Teachers pose questions of their students and encourage their students to ask deep, rich questions about their mathematical reasoning and that of their peers.
- Assessment. The ability to provide students feedback through formative (ongoing) and summative (reflective) assessment is differentiated from and valued more than grades. Assessments are ongoing, are aligned to standards, and (in)form teacher practice. Teachers understand that assessment can take many forms including formative (ongoing) and summative (reflective) assessment. Teachers incorporate a variety of assessments to ensure that each and every student has an opportunity to express their current understanding, including, but not limited to, observations, student-to-student and student-to-teacher dialogue, projects, performance tasks, interviews, portfolios, presentations, exit slips, and dynamic technology-based activities. Teachers recognize that understanding develops over time and leverage opportunities to reassess throughout the learning process.

Appendix A.6. Student Outcomes

Teachers assess and reflect upon a wide range of student outcomes to inform their practice, such as social and emotional well-being, persistence, goal setting, achievement, thinking/reasoning/explaining, orientation, cognition and meta-cognition, and learning behaviors.

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