Noname manuscript No.
(will be inserted by the editor)

A Decomposable Resource Allocation Model with Generalized Overarching Protections

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the date of receipt and acceptance should be inserted later

Abstract This paper considers a defensive resource allocation problem in which a defender protects a set of assets either individually or collectively using overarching protections. An overarching protection refers to an option that protects multiple assets at the same time, e.g., emergency response, border security and counter intelligence. Most of the defensive resource allocation models with overarching protections assume that there is only one option that protects all targets. However, this may not be realistic considering that, for example, emergency response investment may cover only a certain region. In this paper, we develop a new resource allocation model to accommodate generalized overarching protections against intentional attacks. The model also considers multiple natural disaster types. We show that the proposed optimization model is a convex optimization problem and therefore can be solved to optimality in polynomial time. Furthermore, the overall country-level resource

This material is based upon work supported by the National Science Foundation (Grant No.1901721) and the United States National Institute of Justice (Grant No.2018-R2-CX-0011)

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allocation problem can be decomposed into smaller city-level subproblems, thus resulting in a more efficient algorithm. The numerical experiments demonstrate the performance of the proposed approach.

Keywords OR in Disaster Relief · Decision Analysis · Resource allocation · Decomposition · Homeland Security

1 Introduction

Defensive resource allocation to protect a set of valuable assets against terrorist attacks or natural disasters has been the subject of intensive studies [Bier et al., 2008, Brown et al., 2006, Hausken and Levitin, 2012, Seaberg et al., 2017, Zhuang and Bier, 2007]. One aspect of this problem is to strike a balance between protecting individual assets and the overarching protection options. Overarching protections refer to the options that protect multiple assets at the same time. For example, a country can allocate resources to protect its borders to reduce the potential damage from international terrorism. Similarly, expending resources on gathering information and intelligence to counter terrorism is another form of overarching protection.

Powell [2007], Haphuriwat and Bier [2011] conducted the early studies on the trade-off between individual target hardening and overarching protection. Powell investigated a model in which the defender has the option of allocating resources to harden the targets individually or to protect all of the targets via enhancing the border security. Haphuriwat and Bier introduced a model to allocate resources between target hardening and an overarching protection option covering all targets. They studied the effect of various factors on the relative desirability of each option. Golalikhani and Zhuang [2011] developed a model with a defender simultaneously protecting any subset of targets based on their functional similarity or geographical proximity. Hausken [2014] presented a two-period resource allocation game. In the first period, both players allocate their resources to engage in an overarching contest covering all of the targets. If the attacker wins the overarching contest, in the second period, the players decide on resource allocation to defend/attack individual targets. Hausken [2017] considered a system consisting of two components either in series or in parallel. In his model, the players can either allocate resources to special efforts to protect individual components or a general effort to protect both components in the system. The difference of this model from the existing ones with overarching protection is that, the existing models regard the overarching protection as an extra layer of protection that the attacker has to breach to have a successful attack. However, in his proposed model, there is only one protection layer and the special and general protection efforts operate additively to contribute to a single joint protection. Hausken [2019] investigated a similar system with two independent components.

There are a number of studies that considered systems consisting of logically linked components. For example, Levitin and Hausken [2012] examined individual and overarching protections for series and parallel systems. Hausken [2013] expanded this model to include heterogeneous unit protection costs. Levitin et al. [2013] introduced a model that generalizes the k-out-of-n system. In this model, the damage to the system depends on the number of destroyed elements as well as the unfulfilled demand. Levitin et al. [2014] developed a three-stage minimax game model with multiple overarching protections and a system consisting of identical elements. In this model, the defender decides the number of groups of targets to protect using overarching protections as well as the number of targets to protect individually within each group. Peng et al. [2014] considered the resource allocation problem to individual, overarching protection and replacement for a parallel system of heterogeneous components.

Most existing models in literature assume that there is only one overarching protection option that protects all of the targets. However, this may not be true in reality. For example, in case of emergency response, investment is not limited to only one option that covers the entire country. It is possible to make targeted investments that are focused on a city or an area inside a city. Moreover, investment in border security can be divided into different points of entry, each of which is expected to benefit areas that are closer to that particular point of entry. The only model with multiple overarching protections is proposed by Levitin et al. [2014]. However, this model assumes that the targets are identical and each overarching protection covers a fixed number of targets. Therefore, the overarching protection options are identical and the defender decides on how many times to use this option. In reality, the targets may not be identical and, depending on the subset of targets that are covered, various options for overarching protections may be available. To this end, we introduce overarching protection options that protect a subset of targets. We consider two types of overarching protections: country-level overarching protections and city-level overarching protections. Each country-level overarching protection option protects all of the assets in a set of cities. And each city-level overarching protection option in a city protects a subset of assets. Moreover, there are different types of natural disasters and the defender has to decide on how much to invest to protect against each disaster type in each city. Another consideration in this area is that the number of targets maybe very large and a practical resource allocation model needs to be scalable for problems of realistically large size. We show that our proposed resource allocation model is a convex optimization problem that can be solved in polynomial time. Moreover, we also demonstrate that the proposed model can be decomposed into smaller city-level subproblems. Using this observation, we develop an efficient decomposition approach to optimally solve the proposed resource allocation problem.

The rest of this paper is organized as follows. Section 2 introduces the proposed resource allocation model. Section 3 develops a solution approach based on decomposing the problem into city-level subproblems to solve the proposed model. Section 4 demonstrates numerical experiments to investigate the efficiency of the proposed algorithms and to gain insight into properties of the model. Finally, section 5 presents the main conclusions of the paper and future research ideas.

2 Problem Description

A defender has a budget, say B, to allocate in order to protect cities in a country against both natural and man-made disasters. Each asset j in city i has a value V_{ij} that will be lost in case of a successful attack or a natural disaster. Against man-made attacks, the defender can either protect assets in cities individually or collectively through overarching protection options. Overarching protection refers to alternatives that lead to protecting more than one individual asset, e.g., border security, public health, emergency response, or intelligence. Two types of overarching protections exist: country-level overarching protections and city-level overarching protections. Each country-level overarching protection option o protects all of the assets in a set of cities Γ_o. On the other hand, each city-level overarching protection option l in city i protects a set of assets Λ_{il}. A single adversary is the perpetrator of a man-made disaster and chooses the asset with the highest expected damage to attack. If there are multiple assets with the highest expected damage, we assume that the adversary chooses one of them arbitrarily. In order to successfully destroy an asset, all of the protection measures need to be breached. There are different types of natural disasters and the defender decides how much to invest for protection against each disaster type in each city.

Model parameters are listed as follows:

- i : Index for cities, $i = 1,\dots,I$.
- j : Index for assets in city i, $j = 1,\dots,J_i$.
- k : Index for the type of natural disaster, $k = 1,\dots,K$.
- ρ : Probability of an intentional attack.
- ω_k : Probability of a type k natural disaster.
- B : Defenders budget.
- V_{ij} : Value of asset j in city i.
- x_i : Amount of resource allocated to protect city i against intentional attacks.
- x^H_{ij}: Amount of resource allocated to harden asset j in city i against intentional attacks.
- x^C_o : Amount of resource allocated to country-level overarching protection option o.
- x^N_{ik}: Amount of resource allocated to protect city i against natural disaster of type k.
- x^L_{il}: Amount of resource allocated to city-level overarching protection option l, in city i.
- Γ_o: Set of cities that are protected in country-level overarching protection option o.
- Ψ_i : Set of country-level overarching protection options that protect city i.
- Λ_{il} : Set of assets in city i that are protected through city-level overarching protection option l.
- Ω_{ij} : Set of city-level overarching protection options in city i that protect asset j.
- $f_i(x_i)$: Expected damage from a man-made attack in city i, given that budget level is x_i, and all of the country-level overarching protections are breached.
- $P^C_o(x^C_o)$: Probability of breaching country-level overarching protection option o.
- $P^L_{il}(x^L_{il})$: Probability of breaching city-level overarching protection in city i option l.
- $P^H_{ij}(x^H_{ij})$: Probability of breaching hardening protection in for asset j in city i.
- $P^N_{ik}(x^N_{ik})$: Probability of failure of protection against natural disaster type k in city i.

Using this notation, the resource allocation problem can be formulated as follows:

$$\min \rho \left[\max_{(i,j)} \left\{ V_{ij} P^H_{ij}(x^H_{ij}) \prod_{l \in \Omega_{ij}} P^L_{il}(x^L_{il}) \prod_{o \in \Psi_i} P^C_o(x^C_o) \right\} \right] + \sum_k \omega_k \sum_i P^N_{ik}(x^N_{ik}) \sum_j V_{ij} \quad (1)$$

$$\text{subject to} \sum_i \left[\sum_j \left(x^H_{ij} + \sum_{l \in \Omega_{ij}} x^L_{il} \right) + \sum_{o \in \Psi_i} x^C_o \right] + \sum_i \sum_k x^N_{ik} \le B, \quad (2)$$

$$x^H_{ij}, x^L_{il}, x^C_o, x^N_{ik} \ge 0, \ \forall\, k = 1,\dots,K, \ \ o \in \Psi_i, \ \ l \in \Omega_{ij} \ \text{ for } \ i = 1,\dots,I, \ \ j = 1,\dots,J. \quad (3)$$

In this formulation, the objective function is to minimize the expected damage from both man-made and natural disasters. The first term is the expected damage from man-made disasters. For each asset j in city i, the probability of a successful attack is

$$P_{ij}^H(x_{ij}^H)\prod_{l\in\Omega_{ij}}P_{il}^L(x_{il}^L)\prod_{o\in\Psi_i}P_o^C(x_o^C).$$

Therefore, the expected damage of an attack on asset j in city i is

$$V_{ij}P_{ij}^H(x_{ij}^H)\prod_{l\in\Omega_{ij}}P_{il}^L(x_{il}^L)\prod_{o\in\Psi_i}P_o^C(x_o^C).$$

The attacker chooses the asset that leads to the maximum expected damage. The second term in the objective function is the expected damage from natural disasters. We assume that natural disasters affect entire cities, thus investments to protect against them need to cover all assets in a city. Therefore, for each city i, the expected damage from a natural disaster of type k is equal to $\sum_i P_{ik}^N(x_{ik}^N)\sum_j V_{ij}$. Clearly, total investment is constrained by the budget.

The following lemma shows the conditions under which the above formulation is a convex optimization program.

Lemma 1 *If the success probability functions are log-convex, then the resource allocation problem is a convex optimization problem.*

Proof For each pair (i,j), the expression $P_{ij}^H(x_{ij}^H)\prod_{l\in\Omega_{ij}}P_{il}^L(x_{il}^L)\prod_{o\in\Psi_i}P_o^C(x_o^C)$ is a log-convex function. Note that log-convex functions are also convex. Thus,

$$\max_{(i,j)}\left\{V_{ij}P_{ij}^H(x_{ij}^H)\prod_{l\in\Omega_{ij}}P_{il}^L(x_{il}^L)\prod_{o\in\Psi_i}P_o^C(x_o^C)\right\},$$

is a point-wise maximum of a set of convex functions. This means that

$$\max_{(i,j)}\left\{V_{ij}P_{ij}^H(x_{ij}^H)\prod_{l\in\Omega_{ij}}P_{il}^L(x_{il}^L)\prod_{o\in\Psi_i}P_o^C(x_o^C)\right\},$$

is convex. Therefore, the objective function is a linear combination of convex functions, which is convex. Moreover, the constraint is linear. Therefore, the resource allocation problem (1)-(3) is to minimize a convex function with linear constraints. This completes the proof.

3 A Decomposition Approach to Solve the Resource Allocation Problem

If functions $P_o^C(x_o^C)$, $P_{il}^L(x_{il}^L)$, and $P_{ij}^H(x_{ij}^H)$ are log-convex, then the resource allocation problem can be decomposed into smaller city-level resource allocation problems. The assumption of log-convexity is not very limiting and many of the existing functions in literature have this property Bier et al. [2008], Hao et al. [2009], Haphuriwat and Bier [2011], Wang and Bier [2011], Yolmeh and Baykal-Gürsoy [2019]. We can rewrite the defender's resource allocation problem as follows:

$$\min_{x_i, x_o^C, x_{ik}^N} \left[\left(\max_i \rho f_i(x_i) \right) \prod_{o \in \Psi_i} P_o^C(x_o^C) + \sum_k \omega_k \sum_i P_{ik}^N(x_{ik}^N) \sum_j V_{ij} \right] \quad (4)$$

$$\text{subject to} \quad \sum_i \left(x_i + \sum_{o \in \Psi_i} x_o^C \right) + \sum_i \sum_k x_{ik}^N \leq B, \quad (5)$$

$$x_i, x_o^C, x_{ik}^N \geq 0, \ \forall\, o \in \Psi_i, \ l \in \Omega_{ij} \ \text{ for } \ i = 1, \ldots, I. \quad (6)$$

In this formulation, $f_i(x_i)$ is the expected damage of a man-made attack in city i if x_i amount has been allocated to this city for its protection against intentional attacks and all country-level overarching protections have been breached. The value of $f_i(x_i)$ is obtained by solving the following city-level resource allocation problem against intentional attacks:

$$f_i(x_i) = \min_{x_{ij}^H, x_{il}^L} \quad \max_j V_{ij} P_{ij}^H(x_{ij}^H) \prod_{l \in \Omega_{ij}} P_{il}^L(x_{il}^L) \quad (7)$$

$$\text{subject to} \quad \sum_j x_{ij}^H + \sum_l x_{il}^L \leq x_i, \quad (8)$$

$$x_{ij}^H, x_{il}^L \geq 0, \ \forall\, l \in \Omega_{ij}, \text{ for } i = 1, \ldots, I, \ j = 1, \ldots, J. \quad (9)$$

We refer to the above optimization problem as the city-level subproblem and show that this problem, under the conditions of Lemma 1, is a convex optimization problem. First, we need the following lemma, which is adapted from [Boyd et al., 2004].

Lemma 2 *Let function $g_0 : \mathbb{R}^n \to \mathbb{R}$ be log-convex and $g_1, g_2, \ldots, g_h : \mathbb{R}^n \to \mathbb{R}$ be convex. Then, $f^*(\mathbf{x}) = \inf_{\mathbf{u}}\{g_0(\mathbf{u}) | \mathbf{u} \in D, g_i(\mathbf{u}) \leq x_i, i = 1, 2, \ldots, h\}$ is log-convex.*

Proof Let function $G(\mathbf{u}, \mathbf{x}) \equiv g_0(\mathbf{u})$ be defined in the domain, $\{(\mathbf{u}, \mathbf{x}) | \mathbf{u} \in D, g_i(\mathbf{u}) \leq x_i, i = 1, 2, \ldots, h\}$. Define the domain of $f^*(\mathbf{x})$ as $\textbf{dom}\, f^*(\mathbf{x}) = \{\mathbf{x} | (\mathbf{u}, \mathbf{x}) \in \textbf{dom}\, G$ for some $\mathbf{u} \in \mathbb{R}^n\}$. It is easy to see that the domain of $G(\mathbf{u}, \mathbf{x})$ is convex. By assumption, $G(\mathbf{u}, \mathbf{x})$ is log-convex. Next, we show that $f^*(\mathbf{x}) = \inf_{\mathbf{u}} G(\mathbf{u}, \mathbf{x})$ is log-convex. Consider two points (x_1) and

(x_2) both in **dom** f^*. For $\varepsilon > 0$ there are u_1 and u_2 in **dom** G such that $G(u_i, x_i) \le f^*(x_i) + \varepsilon$. We have:

$$
\begin{aligned}
f^*(\theta x_1 + (1-\theta)x_2) &= \inf_u G(\mathbf{u}, \theta x_1 + (1-\theta)x_2) \\
&\le G(\theta u_1 + (1-\theta)u_2, \theta x_1 + (1-\theta)x_2) \\
&\le G(u_1, x_1)^\theta G(u_2, x_2)^{(1-\theta)} \le (f^*(x_1) + \varepsilon)^\theta (f^*(x_2) + \varepsilon)^{(1-\theta)} \\
&\le f^*(x_1)^\theta f^*(x_2)^{(1-\theta)} + \delta(\varepsilon),
\end{aligned}
$$

with $\delta(\varepsilon)$ that converges to zero as ε goes to zero. Since this holds for any $\varepsilon > 0$, we have:

$$f^*(\theta x_1 + (1-\theta)x_2) \le f^*(x_1)^\theta f^*(x_2)^{(1-\theta)}.$$

This completes the proof.

Based on this lemma for $h = 1$, the following corollary holds:

Corollary 1 *If $P_{ij}^H(x_{ij}^H)$ and $P_{il}^L(x_{il}^L)$ are log-convex, then $f_i(x_i)$ is also log-convex.*

Using Corollary 1, an iterative outer approximation method can be used to solve the decomposed problem. Given a set of points x_i^m for $m \in \Phi$, we can develop the following master problem:

$$\min_{x_i, x_o^C, x_{ik}^N} \quad z + \sum_k \omega_k \sum_i P_{ik}^N(x_{ik}^N) \sum_j V_{ij} \tag{10}$$

$$\text{subject to } \; z \ge \rho f_i(x_i^m) e^{\frac{f_i'(x_i^m)}{f_i(x_i^m)}(x_i - x_i^m)} \prod_{o \in \Psi_i} P_o^C(x_o^C), \quad \forall i = 1, \ldots, I, \; m \in \Phi \tag{11}$$

$$\sum_i x_i + \sum_{o \in \Psi_i} x_o^C + \sum_i \sum_k x_{ik}^N \le B, \tag{12}$$

$$x_i, x_o^C, x_{ik}^N, z \ge 0, \; \forall\, k = 1, \ldots, K, \; o \in \Psi_i, \text{ for } i = 1, \ldots, I. \tag{13}$$

In this formulation, $f_i'(x_i^m)$ is the first derivative of $f_i(x_i)$ with respect to x_i evaluated at $x_i = x_i^m$. Note that, because $f_i(x_i)$ is a log-convex function, the solution of this master problem gives a lower bound to the optimal solution of the resource allocation problem. We use the obtained x_i values to set $x_i^{M+1} = x_i$, $\Phi = \Phi \bigcup \{M+1\}$ and $M = M + 1$. We then use the $\mathbf{x}^M$ to solve the subproblems:

$$f_i(x_i^M) = \min_{x_{ij}^H, x_{il}^L} \quad \max_j V_{ij} P_{ij}^H(x_{ij}^H) \prod_{l \in \Omega_{ij}} P_{il}^L(x_{il}^L) \tag{14}$$

$$\text{subject to} \quad \sum_j x_{ij}^H + \sum_l x_{il}^L \le x_i^M, \tag{15}$$

$$x_{ij}^H, x_{il}^L \ge 0. \tag{16}$$

Note that, in the subproblem, x_i^M values are fixed and they are treated as parameters. Solving the subproblems gives an upper bound which can be computed as

$$UB = \max_i \rho f_i(x_i) \prod_{o \in \Psi_i} P_o^C(x_o^C) + \sum_k \omega_k \sum_i P_{ik}^N(x_{ik}^N) \sum_j V_{ij}.$$

We continue iteratively solving the master problem and the subproblems until the lower and upper bounds converge. Algorithm 1 provides the pseudo-code for the overall decomposition procedure. The algorithm starts by initializing $M = 0$ and $\Phi = \emptyset$. Then the master problem is solved to obtain the optimal solution as $\mathbf{x}^* = [x_i^*]$, $[x_o^{C*}]$ and $[x_{ik}^{N*}]$. The algorithm then sets the current lower bound LB as the optimal objective function obtained by solving the master problem. In the next step, the algorithm adds the current point $\mathbf{x}^*$ to the set of points x_i^m, $m \in \Phi$, and updates M and Φ. We then use $\mathbf{x}^M$ to solve the subproblems (14)-(16) and obtain $[x_{ij}^{H*}]$ and $[x_{il}^{L*}]$. In the next step, the algorithm uses the current solution to compute an upper bound on the optimal objective function. Subsequently, the current lower and upper bounds are compared to check if they are close enough. If the difference between the bounds is smaller than ε, then the algorithm terminates and the current solution is returned. Otherwise, we go back to line 2 to repeat the procedure until the bounds converge.

Algorithm 1: Pseudo-code for the overall decomposition algorithm

1 Initialize $M = 0$ and $\Phi = \emptyset$.
2 Solve master problem (10)-(13) to obtain $\mathbf{x}^* = [x_i^*]$, $[x_o^{C*}]$ and $[x_{ik}^{N*}]$.
3 Set the lower bound LB as the optimal objective function of the master problem.
4 Set $\mathbf{x}^{M+1} = [x_i^{M+1}] = \mathbf{x}^*$, $\Phi = \Phi \bigcup \{M+1\}$ and $M = M+1$.
5 Use $\mathbf{x}^M$ to solve the subproblems (14)-(16) to obtain $[x_{ij}^{H*}]$ and $[x_{il}^{L*}]$.
6 Compute the upper bound $UB = \max_i \rho f_i(x_i^*) \prod_{o \in \Psi_i} P_o^C(x_o^{C*}) + \sum_k \omega_k \sum_i P_{ik}^N(x_{ik}^{N*}) \sum_j V_{ij}$.
7 **if** $(UB - LB) \leq \varepsilon$ **then**
8   Return the current solution as the optimal solution of the problem.
9   Terminate the procedure.
10 **else**
11   Go to Line 2.
12 **end**

Remark 1 At every iteration of the decomposition algorithm, the values of $f_i(x_i^m)$ and $f_i'(x_i^m)$ give an aggregation of the asset-level data for each city i. Using these values, one can compare the cost effectiveness of investments in different cities with differing numbers of assets (and differing asset values). Specifically, at the current level of investments, a lower bound on the expected damage from man-made disasters in city i is given in the form of $C_i e^{-\lambda_i(x_i - x_i^m)}$, where $\lambda_i = -\frac{f_i'(x_i^m)}{f_i(x_i^m)}$ and $C_i = f_i(x_i^m)\prod_{o\in\Psi_i} P_o^C(x_o^C)$. This bound is tight at the current level of investments. Moreover, λ_i can be interpreted as the cost effectiveness of the new investments in city i. For example, if $\lambda_i = 0.01$, an extra unit of investment in city i will lead to a reduction of about 1% in the expected damage in city i.

Remark 2 Note that a similar decomposition approach can be developed for the case in which, instead of using a budget constraint, investment costs are added to the objective function.

4 Numerical Experiments

In this section, we perform computational experiments to investigate the efficiency of the proposed algorithm and to gain insight into the properties of the game. The algorithms are coded in GAMS and the IPOPT (Interior Point OPTimizer) solver has been used to solve the NLPs. The computational experiments are performed on a computer with 2.6 GH processor and 32 GB of RAM. Throughout this section, unless mentioned otherwise, we use the following parameter values. Similar to [Haphuriwat and Bier, 2011], power-law functions represent the success probability of an attack and the failure probability of protection against natural hazards. Specifically, assume $P_{ij}^H(x_{ij}^H) \equiv \left(\frac{\alpha_{ij}^H}{\alpha_{ij}^H + x_{ij}^H}\right)^{\kappa_{ij}^H}$, where α_{ij}^H and κ_{ij}^H are positive-valued parameters that determine the cost effectiveness of defensive investment. Similarly, let $P_{il}^L(x_{il}^L) \equiv \left(\frac{\alpha_{il}^L}{\alpha_{il}^L + x_{il}^L}\right)^{\kappa_{il}^L}$, $P_o^C(x_o^C) \equiv \left(\frac{\alpha_o^C}{\alpha_o^C + x_o^C}\right)^{\kappa_o^C}$, and $P_{ik}^N(x_{ik}^N) \equiv \left(\frac{\alpha_{ik}^N}{\alpha_{ik}^N + x_{ik}^N}\right)^{\kappa_{ik}^N}$. In addition, assume $\kappa_{ij}^H = \kappa_{il}^L = \kappa_o^C = \kappa_{ik}^N = 7$, $\alpha_{ij}^H = 0.01$, $\alpha_{il}^L = 0.1$, and $\alpha_o^C = \alpha_{ik}^N = 1$. The acceptable gap of the optimum objective function value, ε, is set as 0.001. Thus, in all experiments, the run time represents the time it takes the algorithm to reach a gap of less than or equal to ε. Furthermore, assume that all types of disasters are equally likely to happen with a probability of 0.001.

In the first experiment, we compare the performance of the decomposition approach with directly solving the mathematical model. We generate the instances for these experiments randomly. Asset values are uniform random variables in the range $[43, 56]$. This range includes the minimum and maximum risk scores given in the case study by Haphuriwat and Bier [2011]. We use $L1$ and $L2$ to denote the number of country-level and city-level

Table 1: Average run times (in seconds) of the decomposition approach (DA) and the direct optimization (DO) of the mathematical model

I	$J = 200$		$J = 250$		$J = 300$		Mean	
	DO	DA	DO	DA	DO	DA	DO	DA
100	71.79	**30.49**	79.73	**38.33**	97.19	**46.74**	82.91	**38.52**
150	151.76	**48.03**	153.11	**58.42**	190.24	**72.19**	165.04	**59.55**
200	189.37	**67.79**	258.04	**81.06**	264.86	**97.50**	237.42	**82.12**
Mean	137.64	**48.77**	163.63	**59.27**	184.10	**72.14**	161.79	**60.06**

overarching protections, respectively. For all possible combinations of $I \in \{100, 150, 200\}$, $J \in \{200, 250, 300\}$, $K, L1, L2 \in \{10, 15, 20\}$, we generated an instance of the problem to obtain a data set of 243 problem instances. We then used our proposed decomposition approach as well as the direct optimization approach to solve all of these problem instances. Table 1 exhibits the average run times for different number of cities (I) and number of assets in each city ($J_i = J, \forall i = 1, \dots, I$). The columns DA and DO show the average run times for the decomposition approach and the direct optimization method, respectively. As seen in this table, the decomposition approach performs significantly better than the direct optimization approach. This table also reveals that the run times for both DA and DO increase as the number of cities increases. Moreover, the run times also increase as the number of assets inside each city increases.

Table 2: Average run times (in seconds) of the decomposition approach (DA) and the direct optimization (DO) of the mathematical model

I	$K = 10$		$K = 15$		$K = 20$		Mean	
	DO	DA	DO	DA	DO	DA	DO	DA
100	83.42	**38.80**	84.77	**38.60**	80.53	**38.16**	82.91	**38.52**
150	149.47	**59.24**	194.48	**59.25**	151.17	**60.16**	165.04	**59.55**
200	240.14	**81.28**	237.35	**82.90**	234.77	**82.18**	237.42	**82.12**
Mean	157.68	**59.77**	172.20	**60.25**	155.49	**60.16**	161.79	**60.06**

Table 2 shows the average run times for different number of cities and number of natural disasters (K). The columns DA and DO present the average run times for the decomposition approach and the direct optimization method, respectively. The decomposition approach performs significantly better than the direct optimization approach. The run times for DO increase as the number of natural disaster types increases. However, the number of natural disaster types does not seem to influence the run times of DA.

Table 3: Comparison of the decomposition approach with the mathematical model

L1	L2=10		L2=15		L2=20		Mean	
	DO	DA	DO	DA	DO	DA	DO	DA
10	123.90	**48.09**	148.76	**62.69**	176.83	**74.43**	149.83	**61.74**
15	129.58	**46.12**	152.57	**61.59**	180.66	**71.84**	154.27	**59.85**
20	178.34	**45.45**	165.37	**60.47**	200.09	**69.90**	181.27	**58.60**
Mean	143.94	**46.55**	155.57	**61.58**	185.86	**72.05**	161.79	**60.06**

Table 3 exhibits the average run times for different number of country-level (L1) and city-level overarching protections (L2). The columns DA and DO show the average run times for the decomposition approach and the direct optimization approach, respectively. The decomposition approach performs significantly better than direct optimization of the mathematical model. In general, the run times for both DA and DO increases as the number of city level overarching protections increases. Increasing the number of country-level overarching protections, leads to an increase in the run times of DO but decreases the run times of DA.

In our next experiment, we study the effect of some of the model parameters on the optimal resource allocation. For this experiment, we assume that all assets have the same valuations, i.e., $V_{ij} = 1$. We consider 5 randomly generated country-level overarching protection options. We also assume that, for each city, there is only one city-level overarching protection option, and it covers all of the assets inside the city. Figure 1 demonstrates the effect of number of assets per city on the optimal allocation of resources. In this figure, C and L refer to the portion of the budget that has been assigned to country-level and city-level overarching protection options, respectively. Moreover, H refers to the proportion of the budget that has been assigned to individual target hardening. According to this figure, as J increases, the resource amount allocated to individual target hardening decreases and the resources are shifted toward the overarching protection types. This is in line with exist-

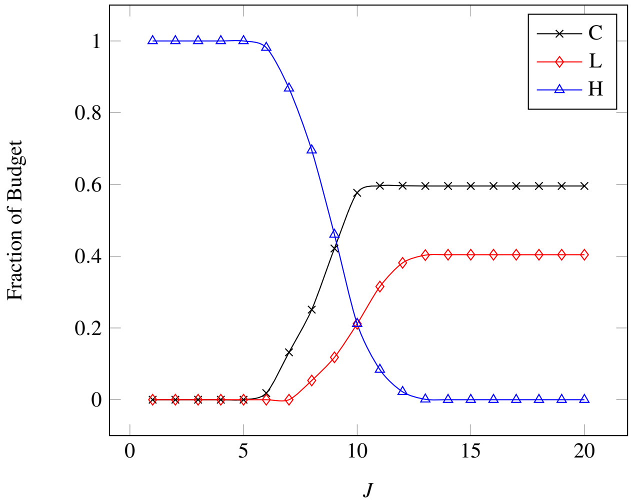

Fig. 1: The effect of number of assets per city on the optimal resource allocation

ing observations in the literature. Moreover, as J increases, the optimal resource allocation levels converge and after a certain point, the optimal allocation of resources does not change.

Figure 2 displays the effect of number of cities on the optimal allocation of resources. In this figure, the optimal resource level for country-level overarching protection options increases as I increases. However, as I increases, the amount of resource allocated to city-level overarching protection options decreases. Moreover, the effect of I on the amount of resource allocated to target hardening options is not monotonic. Specifically, as I increases, the amount of resource allocated to target hardening options increases at first, then decreases.

The effect of I on the resource allocated to different protection options depends on the cost efficiency of these options. For example, Figure 3 shows the effect of number of cities on the optimal allocation of resources, for the case with $\alpha_{ij}^{L} = 0.075$ parameter. This is a slight change from Figure 2, in which we had $\alpha_{ij}^{L} = 0.1$. As seen in this figure, the effect of I on the amount of resource allocated to city-level overarching protections and target hardening is different from Figure 2. This highlights the importance of having accurate estimates of the parameters that determine the cost efficiency of the protection options. Figure 4 exhibits the effect of α_{ij}^{H} on the optimal resource allocation. As seen in this figure, as α_{ij}^{H} increases, the amount of resource allocated to harden individual targets decreases while amounts allocated to city-level and country-level overarching options both increase. This is due to the

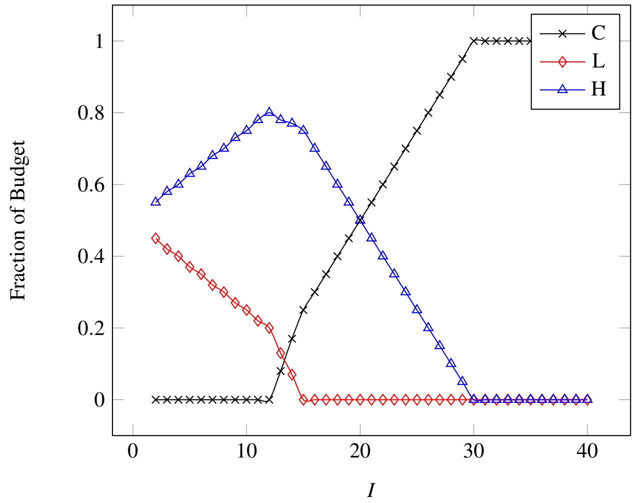

Fig. 2: The effect of number of cities on the optimal resource allocation for $\alpha_{ij}^{L} = 0.1$

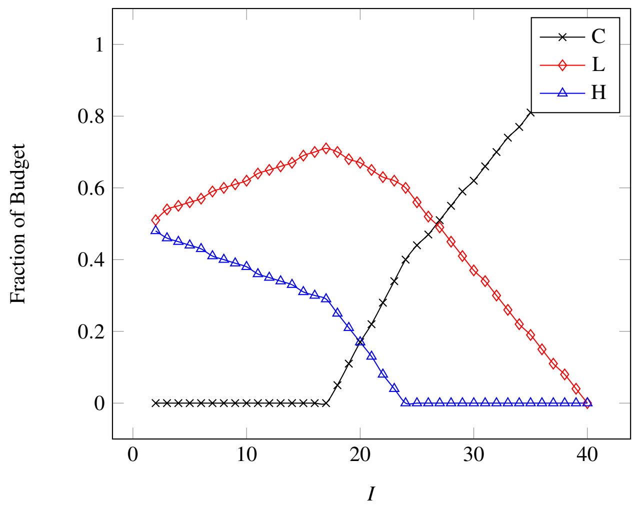

Fig. 3: The effect of number of cities on the optimal resource allocation for $\alpha_{ij}^{L} = 0.075$

fact that, as α_{ij}^{H} increases, the cost efficiency of target hardening decreases. Therefore, allocating resources to other protection options becomes more appealing. Figure 5 displays the effect of α_{o}^{C} on the optimal resource allocation. As α_{o}^{C} increases, the amount of resources allocated to country-level overarching options decreases and the amount allocated to other protection options increases. This is due to the fact that, as α_{o}^{C} increases, the cost efficiency

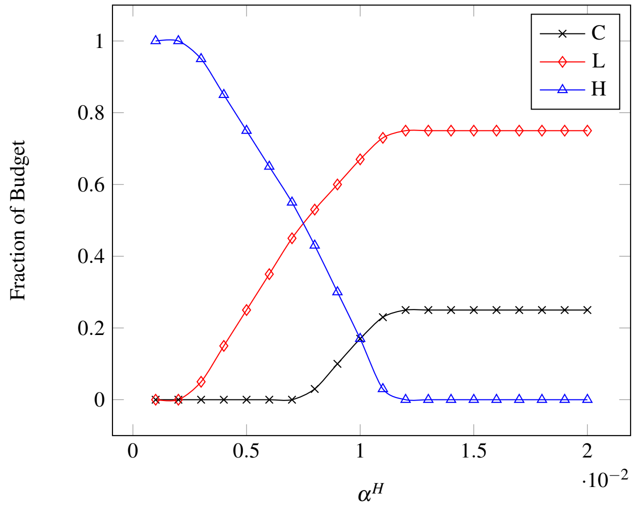

Fig. 4: The effect of α_{ij}^H on the optimal resource allocation

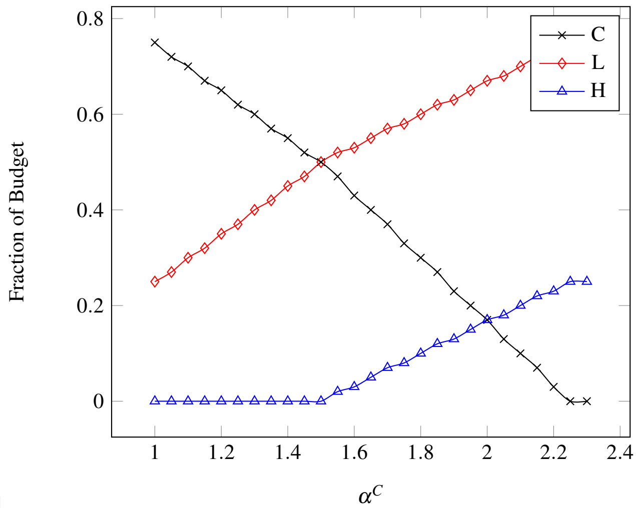

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Fig. 5: The effect of α_o^C on the optimal resource allocation

of country-level overarching options decreases. Therefore, allocating resources to other protection options becomes more appealing.

In many realistic situations, the cities are not identical and they differ in the number and valuation of their assets. In such cases, an interesting question to address is how to compare the cost effectiveness of investments in different cities. In this experiment, we highlight the ability of the decomposition approach to aggregate asset-level data to compare cities in terms of cost effectiveness. For this experiment, we ignore the natural disasters and assume that there are no overarching protection options. Therefore, all of the available resources will be assigned to individual target hardening against intentional attacks. Given a budget of 100 units, we consider two cities; the first one has 10 assets with the values given as $(V_{1,1}, V_{1,2}, \ldots, V_{1,10}) = (10, 10, 5, 8, 10, 10, 7, 5, 10, 10)$. Moreover, we have $(\alpha^H_{1,1}, \alpha^H_{1,2}, \ldots, \alpha^H_{1,10}) = (1, 3, 4, 2, 4, 1, 5, 3, 4, 3)$. The other city has only one asset with $V_{2,1} = 5$ and $\alpha^H_{2,1} = 2$. Using the decomposition approach we solve this instance of the problem and obtain the expected damage from an intentional attack as 1.94 units under the optimal resource allocation policy. The optimal policy is to assign 96.84 units to the first city and 3.16 units to the second city. At the optimal solution, we have $f_1(x_1) = f_2(x_2) = 1.94$, $f_1'(x_1) = -0.0153$ and $f_2'(x_2) = -0.3761$. These values aggregate the asset-level data and enable us to compare these two cities in terms of the current expected damage due to an attack and the cost effectiveness of new investments. Both cities have the same level of expected damage from an attack, $f_1(x_1) = f_2(x_2) = 1.94$.

The decomposition approach also gives us an idea about the cost effectiveness of investment in each city. Because the first city has more assets, which in general have higher values than the asset in the second city, we expect protecting the first city to be more costly than protecting the second city. In other words, for each unit of extra investment, we expect the rate of reduction in the expected damage for the first city to be smaller than for the second city. However, quantifying the difference is not a trivial task. The decomposition approach offers a way to address this issue. Specifically, the values $f_1'(x_1) = -0.0153$ and $f_2'(x_2) = -0.3761$ give us an idea about the cost effectiveness of the two cities for new investments. Based on these numbers, protecting the first city is roughly 24 times more costly than protecting the second city.

5 Conclusions and Future Research

This paper introduces a new decomposable resource allocation model for protection against both man-made and natural disasters. The model accommodates multiple types of natural disasters and offers more flexibility in modeling overarching protections to cover multiple targets. We show that the proposed model is a convex optimization problem and can be solved in polynomial time. This means that the model can scale well to solve resource allocation problems of realistic size. Furthermore, we develop a decomposition approach to break the country-level resource allocation problem into smaller city-level problems. The results of our numerical experiments demonstrate the efficiency of the proposed decomposition approach. The experiments further investigate the effect of different parameters on the optimal resource allocation policy.

This paper addresses a gap in the literature of resource allocation problems to develop a decomposable model that allows more flexibility in modeling protection options. It also develops an efficient solution approach to solve problems of larger size. However, there are many ideas to extend the current research and obtain even more realistic models. One way to extend the current model is to accommodate parameter uncertainty. Our numerical results show the effect of various model parameters on the optimal resource allocation. The model seems to be particularly sensitive to the parameters that determine the cost efficiency of the protection options. These parameters are often not known with certainty. Therefore, developing a model that addresses the uncertainty of these parameters is a possible avenue for future research. Another possible extension is to consider all-hazard protection options. Some of the protection alternatives may protect against both terrorism and natural disasters; e.g., hardening a bridge. Using a similar argument to the proof of Lemma 1, we can show that the resource allocation model still remains a convex optimization problem after adding all-hazard protection options. We plan to develop a decomposition approach for the problem with the addition of all-hazard protection options. Another extension is to incorporate discrete decision variables into the model in case protection decisions are not continuous and it is more appropriate to model them as binary or integer variables. The addition of discrete decision variables into the model will make the problem a mixed integer nonlinear program (MINLP). We expect the resulting MINLP to be amenable to outer approximation approaches similar to the one proposed in [Fletcher and Leyffer, 1994]. Lastly, an all encompassing model may accommodate multiple planning periods.

Extending the proposed solution approach to solve these models is another inviting topic to be explored. For example, the addition of binary or integer variables to the model leads to mixed integer non-linear models that can be addressed using outer approximation or decomposition approaches. Similarly, if a scenario-based stochastic programming approach is used to address parameter uncertainties, then the model can be solved using decomposition approaches. Because our proposed model is a convex optimization problem, we expect the extensions of this model to also have the same property. Therefore, we expect the future extensions of the model to be amenable to decomposition and outer approximation solution approaches.

References

V. M. Bier, N. Haphuriwat, J. Menoyo, R. Zimmerman, and A. M. Culpen. Optimal resource allocation for defense of targets based on differing measures of attractiveness. *Risk Analysis: An International Journal*, 28(3):763–770, 2008.

S. Boyd, S. P. Boyd, and L. Vandenberghe. *Convex optimization*. Cambridge university press, 2004. Ex. 5.32.

G. Brown, M. Carlyle, J. Salmerón, and K. Wood. Defending critical infrastructure. *Interfaces*, 36(6):530–544, 2006.

R. Fletcher and S. Leyffer. Solving mixed integer nonlinear programs by outer approximation. *Mathematical programming*, 66(1-3):327–349, 1994.

M. Golalikhani and J. Zhuang. Modeling arbitrary layers of continuous-level defenses in facing with strategic attackers. *Risk Analysis: An International Journal*, 31(4):533–547, 2011.

M. Hao, S. Jin, and J. Zhuang. Robustness of optimal defensive resource allocations in the face of less fully rational attacker. In *IIE Annual Conference. Proceedings*, page 886. Institute of Industrial and Systems Engineers (IISE), 2009.

N. Haphuriwat and V. M. Bier. Trade-offs between target hardening and overarching protection. *European Journal of Operational Research*, 213(1):320–328, 2011.

K. Hausken. Combined series and parallel systems subject to individual versus overarching defense and attack. *Asia-Pacific Journal of Operational Research*, 30(02):1250056, 2013.

K. Hausken. Individual versus overarching protection and attack of assets. *Central European Journal of Operations Research*, 22(1):89–112, 2014.

K. Hausken. Special versus general protection and attack of parallel and series components. *Reliability Engineering & System Safety*, 165:239–256, 2017.

K. Hausken. Special versus general protection and attack of two assets. *Operations Research and Decisions*, 29(4):53–93, 2019.

K. Hausken and G. Levitin. Review of systems defense and attack models. *International Journal of Performability Engineering*, 8(4):355–366, 2012.

G. Levitin and K. Hausken. Individual versus overarching protection against strategic attacks. *Journal of the Operational Research Society*, 63(7):969–981, 2012.

G. Levitin, K. Hausken, and Y. Dai. Individual vs. overarching protection for minimizing the expected damage caused by an attack. *Reliability Engineering & System Safety*, 119: 117–125, 2013.

G. Levitin, K. Hausken, and Y. Dai. Optimal defense with variable number of overarching and individual protections. *Reliability Engineering & System Safety*, 123:81–90, 2014.

R. Peng, L. Guo, G. Levitin, H. Mo, and W. Wang. Maintenance versus individual and overarching protections for parallel systems. *Quality Technology & Quantitative Management*, 11(3):353–362, 2014.

R. Powell. Defending against terrorist attacks with limited resources. *American Political Science Review*, 101(3):527–541, 2007.

D. Seaberg, L. Devine, and J. Zhuang. A review of game theory applications in natural disaster management research. *Natural Hazards*, 89(3):1461–1483, 2017.

C. Wang and V. M. Bier. Target-hardening decisions based on uncertain multiattribute terrorist utility. *Decision Analysis*, 8(4):286–302, 2011.

A. Yolmeh and M. Baykal-Gürsoy. Two-stage invest–defend game: Balancing strategic and operational decisions. *Decision Analysis*, 16(1):46–66, 2019.

J. Zhuang and V. M. Bier. Balancing terrorism and natural disastersdefensive strategy with endogenous attacker effort. *Operations Research*, 55(5):976–991, 2007.