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Bitangents to plane quartics via tropical geometry: rationality, A^1-enumeration, and real signed count

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BITANGENTS TO PLANE QUARTICS VIA TROPICAL GEOMETRY: RATIONALITY, $\mathbb{A}^1$-ENUMERATION, AND REAL SIGNED COUNT

HANNAH MARKWIG, SAM PAYNE, AND KRIS SHAW

Abstract. We explore extensions of tropical methods to arithmetic enumerative problems such as $\mathbb{A}^1$-enumeration with values in the Grothendieck-Witt ring and rationality over Henselian valued fields, using bitangents to plane quartics as a test case. We consider quartic curves over valued fields whose tropicalizations are smooth and satisfy a mild genericity condition. We then express obstructions to rationality of bitangents and their points of tangency in terms of twisting of edges of the tropicalization; the latter depends only on the tropicalization and the initial coefficients of the defining equation modulo squares. We also show that the GW-multiplicity of a tropical bitangent, i.e., the multiplicity with which its lifts contribute to the $\mathbb{A}^1$-enumeration of bitangents as defined by Larson and Vogt [22], can be computed from the tropicalization of the quartic together with the initial coefficients of the defining equation. As an application, we show that the four lifts of most tropical bitangent classes contribute $2\mathbb{H}$, twice the class of the hyperbolic plane, to the $\mathbb{A}^1$-enumeration.These results rely on a degeneration theorem relating the Grothendieck-Witt ring of a Henselian valued field to the Grothendieck-Witt ring of its residue field, in residue characteristic not equal to two.

Contents

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1. Introduction

Tropical geometry is well-known for its applications to complex and real enumerative geometry, especially for plane curves. Here, we explore extensions of tropical methods to arithmetic enumerative problems such as $\mathbb{A}^1$-enumeration with values in the Grothendieck-Witt ring, and rationality over Henselian valued fields, using bitangents to plane quartics as a test case.

If the tropicalization of a plane quartic is smooth (locally isomorphic to the tropicalization of a linear space) then it has exactly 7 deformation classes of tropical bitangents. These are in natural bijection with the odd tropical theta characteristics and each contains the tropicalization of 4 algebraic bitangents [5, 17, 25]. Cueto and the first author have given an exhaustive classification of the combinatorial types of these bitangent classes [6]. As an application, when the ground field K is real Puiseux series $\mathbb{R}\{\{t\}\}$ and the tropicalization satisfies the genericity constraints from [6, Remark 2.10], they showed that the number of K-rational bitangents tropicalizing into a bitangent class is either 0 or 4. In other words, roughly speaking, the obstruction to rationality over $\mathbb{R}\{\{t\}\}$ is independent of the choice of bitangent in a given bitangent class.

Throughout, let K be a Henselian valued field of residue characteristic not equal to 2, with 2-divisible value group, and with a section of the valuation $\sigma\colon \mathrm{val}(K^\times) \to K^\times$. Let k be the residue field of K. Let $Q \in K[x,y,z]$ be a homogeneous polynomial of degree 4 such that the projective plane algebraic curve $C = V(Q)$ is smooth and $\mathrm{Trop}(C)$ is tropically smooth. Assume furthermore that $\mathrm{Trop}(C)$ satisfies the genericity constraints from [6, Remark 2.10], the details of which we recall in Section 2.3.

Note that K could be the Puiseux series field $k\{\{t\}\}$ over any field k of characteristic not equal to 2. Such fields are commonly considered in tropical geometry. Other possibilities include mixed characteristic fields such as $\mathbb{Q}_p(p^{1/2^\infty}) = \bigcup_n \mathbb{Q}_p(p^{1/2^n})$, for $p \neq 2$. Our arguments are not sensitive to distinctions between pure and

mixed characteristic; any reader who prefers to do so may safely assume that $K = k\{\{t\}\}$ is a field of Puiseux series.

Lifting of tropical bitangents to general ground fields. Our work begins with the observation that the algebraic calculations used to prove the lifting result from [6] over the reals are only mildly sensitive to the ground field.

Theorem 1.1 (Theorem 3.3)**.** *Let $C = V(Q)$ be a quartic curve defined over K with* $\mathrm{Trop}(C)$ *a generic tropicalized quartic and suppose Λ is a liftable tropical bitangent to C. Then whether or not Λ lifts over K is determined by* $\mathrm{Trop}(C)$ *and the equivalence classes of initials of the coefficients of Q in $k^\times/(k^\times)^2$.*

When the residue field $k = \mathbb{R}$, then $\mathbb{R}^\times/(\mathbb{R}^\times)^2 \cong \mathbb{Z}/2\mathbb{Z}$. Knowing the tropicalization of C together with the equivalence classes of initials of the coefficients of Q modulo squares is precisely the information required to carry out Viro's patchworking [14, 35]. Motivated by this connection, we express obstructions to rationality in the case of arbitrary fields in terms of *twisting* of edges of the tropicalization, thus extending to general fields the concept of twisting coming from amoebas over the real numbers, see [3].

Theorem 1.2 (Theorem 3.14)**.** *The number of K-rational bitangents to C that tropicalize into each bitangent class of tropical bitangents is either 0 or 4.*

When K is $\mathbb{R}\{\{t\}\}$, Cueto and the first author showed that the points of tangency of a K-rational bitangent are K-rational [6, Theorem 7.1]. Again, we observe that this is only mildly sensitive to the ground field.

Theorem 1.3 (Corollary 3.15)**.** *Assume that 2 and 3 are squares in $k^\times$. Then the points of tangency of a K-rational bitangent to C are K-rational.*

Note that the hypotheses of Theorem 1.3 imply that the residue characteristic of K is also not equal to 3.

The proofs of Theorems 1.2 and 1.3 involve solving systems of equations over the residue field to determine the initials of the coefficients of the defining equation of the bitangents and points of tangency in suitable local coordinates, and then lifting via Hensel's lemma. This is carried out case-by-case, according to the Cueto-Markwig classification from [6]. Details of the case analysis are provided in the Appendix.

Using the lifting theorems discussed above, we also prove that the liftability of tropical bitangents over different fields are related when the groups of the residue fields modulo squares are isomorphic, see Theorem 3.16 and Corollary 3.17.

The tropical approach to the $\mathbb{A}^1$-enumeration of bitangents to a plane quartic, and its foundations. Even when it is possible to count K-rational solutions to a geometric problem, the naive enumeration may not be invariant under deformations. Familiar examples include counts of real rational plane curves of

given degree and genus, passing through an appropriate number of specified points in general position. In such cases, there are often *refined* enumerative counts, such as Welschinger invariants for real rational plane curves, where objects are counted with multiplicities in such a way that the result is invariant under deformation. More recently, techniques from arithmetic geometry and $\mathbb{A}^1$-homotopy theory have produced more general $\mathbb{A}^1$-*enumerative invariants* taking values in Grothendieck-Witt rings of quadratic forms that specialize to classical enumerative invariants over $\mathbb{C}$, to Welschinger invariants and related real enumerative invariants over $\mathbb{R}$, and produce new invariants of interest over other fields [13, 18, 26]. For the $\mathbb{A}^1$-enumeration of the 27 lines on cubic surfaces and the 28 bitangents to plane quartics, see [19] and [22], respectively. We recall the relevant definitions in Section 4.

We write $t^v := \sigma(v)$ for $v \in \mathrm{val}(K^\times)$, where σ denotes the section of the valuation on our valued field K. For $A \in K$, we let $\mathrm{in}(A)$ be the image of $At^{-\mathrm{val}(A)}$ in the residue field and call it the initial of A. If K is the field of Puiseux series, $\mathrm{in}(A)$ is the leading coefficient of the series A.

Theorem 1.4 (Theorem 4.23)**.** *The $\mathbb{A}^1$-enumerative multiplicity of a bitangent L to C depends only on the tropicalization of C and on the initials of its coefficients modulo squares.*

Moreover, we give an effective algorithm for computing the $\mathbb{A}^1$-enumerative count of all bitangents to C in each of the 7 deformation classes, and hence the $\mathbb{A}^1$-enumerative count of all 28 bitangents, as elements of the Grothendieck-Witt ring $\mathrm{GW}(K)$ (see Remark 4.16).

Theorem 1.4 relies on the following observation relating the Grothendieck-Witt ring of K to that of its residue field. Recall our standing assumption that K is a Henselian valued field of residue characteristic not equal to 2, with 2-divisible value group, and with a section of the valuation $\sigma\colon \mathrm{val}(K^\times) \to K^\times$.

Theorem 1.5 (Theorem 4.7)**.** *Let K be a Henselian valued field of residue characteristic not equal to 2, with 2-divisible value group, and with a section of the valuation. Then there is an isomorphism of Grothendieck-Witt rings*

$$\mathrm{GW}(K) \xrightarrow{\sim} \mathrm{GW}(k), \quad \langle A \rangle \mapsto \langle \mathrm{in}(A) \rangle.$$

This isomorphism does not depend on the section σ.

The above theorem is a variant of Springer's theorem on the Witt group of a Henselian discretely valued field, as generalized to arbitrary value groups, e.g., in [9], in the special case where the value group is 2-divisible. Theorems of this form are a natural starting point for the application of tropical methods to $\mathbb{A}^1$-enumerative geometry. In forthcoming work, Jaramillo Puentes and Pauli define enriched tropical intersection multiplicities for 0-dimensional intersections of tropical hypersurfaces [34]. As a consequence they prove a tropical version of a

quadratically enriched Bézout theorem from [27] and also introduce an enriched Bernstein-Khovanskii-Kushnirenko theorem.

Remark 1.6. Our hypothesis that the value group $\mathrm{val}(K^\times)$ is 2-divisible is a simplifying assumption that is necessary for Theorems 1.4 and 1.5, as stated. Nevertheless, this framework, and all of the computations that we carry out for bitangents to plane quartics, can be extended to the case where K is Henselian of residue characteristic not equal to 2 with arbitrary value group, but then one must also keep track of the valuations of the coefficients, modulo valuations of squares. The calculations needed for our case analysis of bitangents are already sufficiently complicated when the value group is 2-divisible that we have chosen to compromise generality in this way. This work, along with that of Jaramillo Puentes and Pauli [34], demonstrates that tropical methods are relevant and useful to $\mathbb{A}^1$-enumerative geometry.

If an $\mathbb{A}^1$-enumerative multiplicity equals the hyperbolic plane $\mathbb{H} = \langle 1\rangle \oplus \langle -1\rangle$, then it contributes 2 to the corresponding complex count and 0 to the corresponding signed count over the real numbers. We prove that for many types of tropical bitangent classes in the classification, the $\mathbb{A}^1$-enumerative multiplicities of the four algebraic bitangents that tropicalize into this class add up to $2\mathbb{H}$. The exceptional cases for which this does not hold true are listed in Appendix A.3. To give an impression which tropical bitangent classes contribute $2\mathbb{H}$ to the $\mathbb{A}^1$-enumerative count, we list some sufficient conditions here. For the complete list in terms of the classification of tropical bitangent classes, see Theorem A.2.

Theorem 1.7 (Theorems 4.23 and A.2)**.** *Let S be a tropical bitangent class that is compact. Then the contribution of S to the $\mathbb{A}^1$-enumerative count of bitangents to C is $2\mathbb{H}$.*

We also show that the contributions of tropical bitangents classes to the $\mathbb{A}^1$-enumerative counts over different fields are related if the groups of the residue fields modulo squares are isomorphic, see Theorem 4.24 and Corollary 4.25.

The real signed count of bitangents to plane quartics. In [22], Larson and Vogt investigated the $\mathbb{A}^1$-enumeration of bitangents to a plane quartic, relative to a fixed line at infinity, and the resulting signed count over reals. Over the reals, they show that the signed count is equal to 4 if the real locus of the quartic does not meet the line at infinity. They also show that the signed count is nonnegative and conjectured that it is bounded above by 8.[1] The field of real Puiseux series $\mathbb{R}\{\{t\}\}$ also has its Grothendieck-Witt ring generated by the two elements $\langle 1\rangle$ and $\langle -1\rangle$. The $\mathbb{A}^1$-enumeration of bitangents can thus be expressed as $a\cdot\langle 1\rangle + b\cdot\langle -1\rangle$ for some a, b, and the signed count is $a - b$.

[1] Kummer and McKean have now announced a proof of this conjecture [20].

Theorem 1.8 (Theorem 5.2)**.** *Let $C = V(Q)$ be a quartic over $\mathbb{R}\{\{t\}\}$ whose tropicalization is smooth and generic. Then the signed count of bitangents of C is either 0, 2 or 4.*

As many, but not all, of the tropical bitangent classes contribute $2\mathbb{H}$ to the $\mathbb{A}^1$-enumeration of bitangents, we deduce that many tropical bitangents give a total contribution of 0 to the signed count when working over the reals. The total signed count of 2 and 4 can only be reached for those tropicalized quartics which admit exceptional tropical bitangent classes as listed in Appendix A.3. The strategy to prove Theorem 5.2 is to investigate how the dual motifs (see Definition 2.15) of such exceptional tropical bitangents can fit into the dual Newton subdivision of a tropicalized quartic and show that, at most, there can be either one that contributes 4, or one or two that contribute 2 to the signed count, with the rest contributing 0.

Organization of this paper. This paper is organized as follows. In Section 2, we discuss preliminaries. We fix our convention for the fields we study, introduce tropicalizations of plane quartics and discuss tropical bitangents and their bitangent classes.

In Section 3, we discuss obstructions to rationality. Section 3.1 reviews known techniques to solve so-called local lifting equations which provide local obstructions for a tropical bitangent to have a K-rational lift. In Section 3.2, we generalize the concept of twisted edges from the reals to more general fields. We also generalize to situations with so-called relative twisting, in which the relative behaviour of a tropical bitangent line and the tropicalized quartic play a role. We express obstructions to K-rationality in terms of twisting of edges. In Section 3.3, we sum up our results on K-rationality of tropical bitangent classes, and compare the behaviour over different fields.

Section 4 discusses aspects of $\mathbb{A}^1$-enumeration. Section 4.1 recalls the definition of the Grothendieck-Witt ring. Section 4.2 contains our result (Theorem 1.5 (4.7)) relating the Grothendieck-Witt ring of K to the Grothendieck-Witt ring of its residue field. This result lays the foundation for the infusion of tropical methods into the study of $\mathbb{A}^1$-enumerative geometry, and we apply it in Section 4.4 to investigate the $\mathbb{A}^1$-enumerative geometry of tropical bitangent classes to tropicalized quartics. Our main result here is Theorem 4.23, which sums up the statements of Theorems 1.4 and 1.7 in the introduction. Also in Section 4.4, we provide the methods to prove that many tropical bitangent classes contribute $2\mathbb{H}$ to the $\mathbb{A}^1$-enumeration of bitangents to quartics, and spell out the details for one case of the classification of tropical bitangent classes. The remaining cases are revisited in Appendix A.2. Before we turn to the $\mathbb{A}^1$-enumerative geometry of tropical bitangents, in Section 4.3, we review the results of [22] on the $\mathbb{A}^1$-enumeration of bitangents to quartics. Section 4.4 presents the results from the tropical approach

to the $\mathbb{A}^1$-enumeration of bitangents to quartics and Section 4.5 explores relations of the $\mathbb{A}^1$ types of lifts of tropical bitangents over different fields.

In Section 5, we consider tropical $\mathbb{A}^1$-enumeration of bitangents to quartics over real closed fields. We prove Theorem 1.8 (5.2) providing evidence in support of Larson and Vogt's conjecture on the possible values for the signed count of bitangents over the reals. We also point out that this result can be obtained without referring to the techniques of $\mathbb{A}^1$-enumeration. The signed count of bitangents to a real quartic has a geometric interpretation (see Section 4.3), and one can thus also use the method of Viro's patchworking to associate the correct sign to a tropical bitangent.

The Appendix contains details of the classification of tropical bitangents. Part A.1 spells out the details about all tropical bitangent classes and their dual motifs (see Definition 2.15) up to $\mathbb{S}_2$-symmetry. The original classification in [6] is up to $\mathbb{S}_3$-symmetry, but we have to break some of the symmetry to fix the line at infinity. Part A.2 adds the details to Theorem 4.23 (see Theorem A.2) by going through the classification and showing which bitangent classes precisely contribute $2\mathbb{H}$ to the $\mathbb{A}^1$-enumeration of bitangents to quartics. Part A.3 finally lists the exceptional tropical bitangent classes and their $\mathbb{A}^1$-enumerative multiplicity, which is given in terms of the initials of the coefficients of the defining equation of the quartic, as claimed in Theorem 1.4.

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2. Preliminaries

2.1. **Field conventions.** Throughout the paper, let K be a Henselian valued field whose value group $\mathrm{val}(K^\times)$ is 2-divisible, and let σ be a section of the valuation,

i.e., a group homomorphism

$$\sigma\colon \operatorname{val}(K^\times) \to K^\times$$

such that $\operatorname{val} \circ \sigma$ is the identity. Let $R \subset K$ denote the valuation ring $\mathfrak{m} \subset R$ the maximal ideal, and $k = R/\mathfrak{m}$ the residue field. Throughout, we assume the characteristic of k is not 2.

Example 2.1. Such fields exist with any given residue field k; for instance, one could take the Puiseux series field $k\{\{t\}\}$, the generalized power series field $k(\!(t^{\mathbb{R}})\!)$, or the completion of either. The hypotheses are also satisfied by mixed characteristic fields such as $\mathbb{Q}_p(p^{1/2}, p^{1/4}, p^{1/8}, \ldots)$, where $p^{1/2}$ is a square root of p, $p^{1/4}$ is a square root of $p^{1/2}$, and so on; a section is given by $\sigma(v) = p^v$.

Definition 2.2. We write $\sigma(v) := t^v$, for $v \in \operatorname{val}(K^\times)$.

This notation intentionally emphasizes the analogy with generalized power series.

Definition 2.3 (Initials)**.** Let $A \in K^\times$. Then the initial of A, $\operatorname{in}(A)$, is the image of $At^{-\operatorname{val}(A)}$ in $k^\times$.

In case K is a generalized power series field, $\operatorname{in}(A)$ is the "initial coefficient."

2.2. **Tropicalizations of plane quartics.** Tropicalization can be viewed as degeneration of algebraic varieties defined over K. Here, we introduce it only for plane curves.

Definition 2.4 (Tropicalization)**.** Let $C \subset \mathbb{P}^2_K$ be a plane curve and assume that K is algebraically closed. The *tropicalization* is defined by taking $-\operatorname{val}$ componentwise to the torus points:

$$\operatorname{Trop}(C) := \overline{\{(-\operatorname{val} x, -\operatorname{val} y) \mid (x,y) \in C \cap (K^\times)^2\}},$$

where the bar denotes the Euclidean closure in $\mathbb{R}^2$.

If K is not algebraically closed, we apply minus valuation coordinatewise to all points of C in the algebraic closure of K to obtain the tropicalization.

Since we require our field to be Henselian, the valuation extends uniquely to the algebraic closure. Equivalently, $\operatorname{Trop}(C)$ is the image of the Berkovich analytification of $C \cap (\mathbb{G}_m^2)$ under coordinatewise valuation [33].

Remark 2.5. One can also define a compactification of $\mathbb{R}^2$ that is the tropicalization of $\mathbb{P}^2_K$, including its toric boundary, as in Section 3 of [33]. The definition above then extends naturally to take boundary points into account. A smooth quartic is never contained in the toric boundary, and hence meets the torus non-trivially. Consequently, this compact extended tropicalization equals the closure of the tropicalization of its points in the torus as defined above (see Lemma 3.1.1 [31]). We can therefore restrict attention to the tropicalization in the torus as described above.

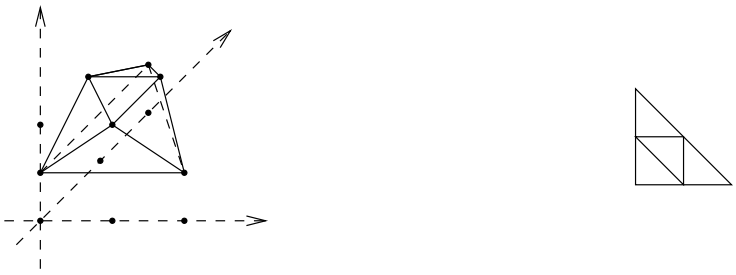

FIGURE 1. On the left, the set D for the polynomial Q in Example 2.9, on the right is the Newton subdivision Δ_Q.

Example 2.6 (Tropical line). Let $C = V(x + y + t)$. We can parametrize C as

$$\{(x, -x - t) \mid x \neq -t, x \in K^\times\}.$$

By considering the three cases $\mathrm{val}(x) > 1$, $\mathrm{val}(x) < 1$, and $\mathrm{val}(x) = 1$, where in the last case the situation $\mathrm{in}(x) = -1$ plays an important role, we can see that the tropicalization consists of three rays starting at the point $(-1, -1)$, one diagonal, one horizontal and one vertical.

Remark 2.7 (Tropical dual plane). By varying the coefficients in Example 2.6, one can see that the tropicalization of any line whose defining equation has three nonzero coefficients (what we are calling a *tropical line*) consists of three rays centered at a vertex whose coordinates are determined by the valuations of the coefficients. Thus, we can and do identify the *tropical dual plane* parametrizing tropical lines in $\mathbb{R}^2$ with the tropical plane $\mathbb{R}^2$ itself, where a tropical line is identified with its vertex. (Note, however, that this does not extend to an identification of the compact tropicalization of $\mathbb{P}^2$ with that of $(\mathbb{P}^2)^\vee$.)

Definition 2.8 (Newton subdivision)**.** Let $Q \in K[x, y, z]$ be a homogeneous polynomial of degree d, and denote the coefficient of the monomial $x^i y^j z^{d-i-j}$ by A_{ij}. Let $D \subset \mathbb{R}^2 \times \mathbb{R}$ be the convex hull of the set

$$\{((i, j), -\mathrm{val}(A_{ij})) \mid A_{ij} \neq 0\}.$$

Project the faces of D which can be seen from above down to $\mathbb{R}^2$. The images of these faces form a convex subdivision Δ_Q of the Newton polygon, called the *Newton subdivision.*

Example 2.9. Let $Q = t^{-1}x^2 + t^{-2}xy + t^{-1}y^2 + t^{-2}x + t^{-2}y + t^{-1}$. Figure 1 shows the set D on the left and the projection of its upper faces, i.e. its Newton subvision, on the right.

Theorem 2.10 (Duality theorem)**.** *For $C = V(Q)$, the tropical curve* $\mathrm{Trop}(C)$ *is the* 1*-skeleton of a subdivision of* $\mathbb{R}^2$ *that is dual to the Newton subdivision Δ_Q, i.e., there is a natural inclusion reversing bijection of faces between these two subdivisions.*

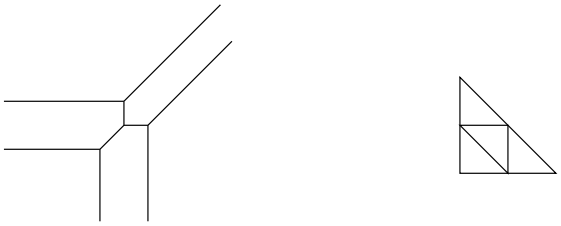

FIGURE 2. The tropicalization of the curve from Example 2.9 and the Newton subdivision Δ_Q.

A proof can be found, for example, in [28], Proposition 3.11. This duality is illustrated in Figure 2.

The tropicalization of a plane curve is called a *tropical plane curve*. From duality to a Newton subdivision, we conclude that tropical curves satisfy the *balancing condition*: if we weight each edge by the lattice length of its dual edge, then the weighted sum of the primitive edge generators around a vertex is 0.

A tropical plane curve is *smooth* of degree d if it is dual to a unimodular triangulation of the triangle with vertices $(0,0),(0,d),(d,0)$. Here, unimodular means that it is subdivided into d^2 triangles each having area $\frac{1}{2}$.

2.3. **Bitangents to tropicalized quartics.** Suppose that two tropical plane curves Γ_1, Γ_2 intersect transversally, i.e. the intersection is a finite set of points each of which is contained in the interior of an edge of both Γ_1 and Γ_2. Let $P \in \Gamma_1 \cap \Gamma_2$. Choose weighted direction vectors u_1, u_2 for edges of Γ_1, Γ_2 emanating from P. Then the *intersection multiplicity* of the two curves at P is $\Gamma_1 \cdot \Gamma_2 \mid_P = |\det(u_1, u_2)|$, and $\Gamma_1 \cdot \Gamma_2 = \sum_{P \in \Gamma_1 \cap \Gamma_2} \Gamma_1 \cdot \Gamma_2 \mid_P$. The balancing condition ensures that this definition does not depend on the choice of direction vectors. If the curves do not intersect transversally, we use the *stable* intersection: choose a direction v so that Γ_1 and $\Gamma_2 + \epsilon \cdot v$ intersect transversally whenever ϵ is small enough. Then the stable intersection is

$$\lim_{\epsilon \to 0} \Gamma_1 \cdot (\Gamma_2 + \epsilon \cdot v).$$

When the intersection is not transverse, the number of intersection points, counted with multiplicity, in the preimage of a connected component of the intersection of the tropicalizations is the sum of the multiplicities of the points of the stable tropical intersection that lie in this component, see [32], Theorem 6.4.

Definition 2.11 (Bitangent)**.** We say that a tropical line Λ is *bitangent* to a tropicalized quartic $\mathrm{Trop}(C)$ in $\mathbb{R}^2$ if Λ and $\mathrm{Trop}(C)$ intersect in either

- one connected component with total intersection multiplicity 4, or
- two connected components each with total intersection multiplicity 2.

In the first case, the intersection can be a point, or a segment of an edge of Γ containing a vertex, or three edges adjacent to the same vertex (in which case

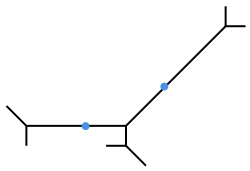

FIGURE 3. A local picture of a tropicalized quartic $\mathrm{Trop}(C)$ together with the tropicalization of the two tangency points of a bitangent ℓ to C. Here the line ℓ is a lift of a tropical line Λ whose vertex coincides with the center vertex of $\mathrm{Trop}(C)$. The intersection of Λ and $\mathrm{Trop}(C)$ is a single connected component with tropical intersection multiplicity 4. The tropical line does not appear in the figure.

— if liftable, see Definition 2.12 — the two tangency points tropicalize to the midpoints of the two segments we obtain by subtracting the shortest edge length from the others; see Figure 3).

In the second case of Definition 2.11, the intersection can be a point, or a segment of an edge of Γ (in which case — if liftable, see Definition 2.12 — the tangency point tropicalizes to the midpoint of the segment).

For a bitangent Λ to $\mathrm{Trop}(C)$, we call a connected component of the intersection $\Lambda \cap \mathrm{Trop}(C)$ a *tropical tangency component*.

It is often the case that a tropicalized quartic $\mathrm{Trop}(C)$ in $\mathbb{R}^2$ admits infinitely many tropical bitangents. A natural question to ask is: which of the infinitely many tropical bitangent lines are tropicalizations of bitangent lines of C?

Definition 2.12 (Lift of a bitangent)**.** Let C be a quartic defined over K. Let Λ be a tropical line which is tropically bitangent to $\mathrm{Trop}(C)$. A bitangent line L of C is called a *lift* of Λ if $\mathrm{Trop}(L) = \Lambda$. If such a lift exists, we say that Λ is *liftable*. If in addition the lift is defined over K, we say that Λ is *liftable over* K.

We declare that two tropical bitangent lines are equivalent if we can move one tropical line to the other (connecting their corresponding points in the tropical dual plane) while maintaining bitangency, i.e., the equivalence classes are the connected components of the locus of tropical bitangents in the tropical dual plane. For tropicalized quartics, this is equivalent to saying that the tropical bitangent lines correspond to the same theta characteristic in the tropical Jacobian [2, Definition 3.8].

When studying tropical bitangents and their lifts, we restrict to *generic* tropicalized quartics (see Remark 2.17 and [25], 3.3). This ensures that the tropicalized quartic is smooth, that the tropical tangency components are not contained on the same ray, and that when the intersection $\mathrm{Trop}(C) \cap \Lambda$ consists of a vertex and three ray segments for a liftable tropical bitangent Λ, then the tropicalization of tangency points are not on the vertex.

Definition 2.13 (Bitangent class)**.** A connected component of the set of tropical bitangents lines to a tropicalized quartic $\mathrm{Trop}(C)$ in the tropical dual plane (see Remark 2.7) is called a *bitangent class* of $\mathrm{Trop}(C)$.

By [2], a smooth tropicalized quartic has precisely 7 bitangent classes. The statement can be generalized for non-smooth tropicalized quartics, in which case bitangent classes must be counted with a suitable multiplicity [24].

By [6], a bitangent class is a polyhedral complex. First, we equip it with the coarsest polyhedral complex structure that is needed (for all the shapes in the classification, such a coarsest polyhedral complex structure exists). By duality (see Theorem 2.10), $\mathbb{R}^2$ obtains a polyhedral complex structure with cells the vertices and edges of $\mathrm{Trop}(C)$, and the connected components of $\mathbb{R}^2 \setminus \mathrm{Trop}(C)$. We use this to refine the polyhedral complex structure of a bitangent class:

Definition 2.14 (Shape of a bitangent class)**.** The *shape* of a bitangent class is the polyhedral complex structure obtained by refining the bitangent class with the polyhedral complex structure of $\mathbb{R}^2$ given by $\mathrm{Trop}(C)$ (where the containment of a polyhedral cell in $\mathrm{Trop}(C)$ is encoded using colors in Figure 4). We identify the tropical dual plane with the original $\mathbb{R}^2$ to make this refinement.

Shapes of bitangent classes are classified up to $\mathbb{S}_3$-symmetry in [6]; see especially Figure 6 in *loc. cit.* There are 41 such shapes, denoted with letters (A), (B), ... or double letters (EE) etc. For convenience of the reader, we reproduce this figure here; see Figure 4.

Definition 2.15 (Dual motif, [12])**.** Let $\mathrm{Trop}(C)$ be a smooth tropicalized quartic and B one of its seven bitangent classes. Let Λ be a tropical bitangent in B. The *dual motif* of the bitangent class B is the set of all triangles and edges in the Newton subdivision whose dual vertices resp. edges intersect a bitangent in B.

For an example, see Figure 6. All tropical bitangents in the bitangent class considered there share the same tropical tangency component on the lower right, the intersection of the vertical segment. Dual to this edge, with its two end vertices, are the two triangles on the right in the dual Newton subdivision. The other tropical tangency component varies for the tropical bitangents in B. It is either on an edge, or an end vertex of the same edge. Accordingly, we obtain another pair of triangles on the left in the dual Newton subdivision for the dual motif. If, as in Figure 6, all tropical bitangents in a bitangent class have two disjoint tropical tangency components, we can divide the dual motif into two parts, one for each tropical tangency component.

Example 2.16. Figure 5 shows a tropicalized quartic and a tropical bitangent line. The tropical line intersects the tropicalized quartic in two tropical tangency components. For the right tropical tangency component, we have to use stable intersection to check that it is a tangency. The tropicalization of a tangency point for any lift is the midpoint of the segment of intersection. We can see that we

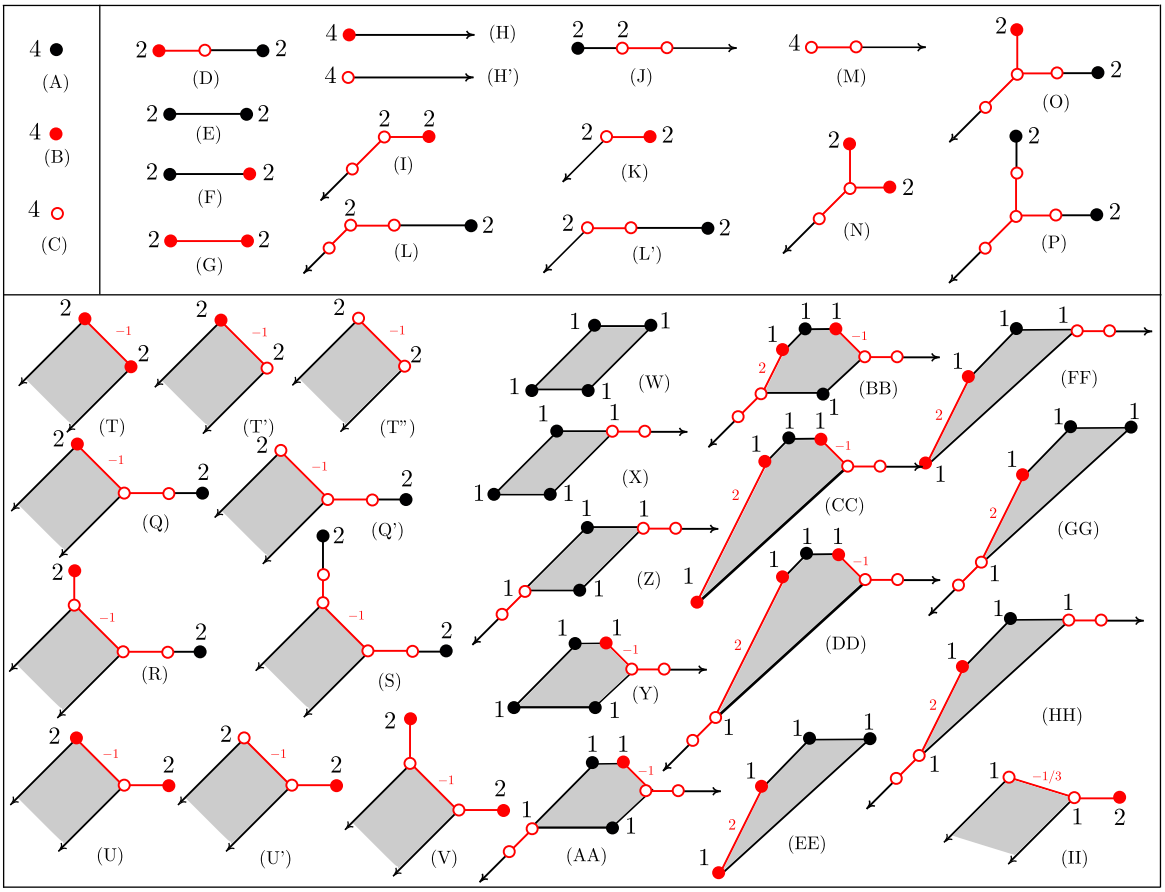

FIGURE 4. Taken from [6]: Orbit representatives of all 41 shapes of bitangent classes to Γ, grouped by the dimension of its maximal cells. The numbers above each vertex indicate lifting multiplicities over the complex numbers, whereas the red ones above edges indicate slopes. The black cells of each bitangent class miss Γ, whereas the red ones lie on it. The unfilled dots are vertices of Γ.

can move the vertex of the tropical bitangent upwards or downwards, until we hit vertices of the tropicalized quartic, maintaining the bitangency. The bitangent class is therefore a line segment as depicted in Figure 6. As it is disjoint from $\mathrm{Trop}(C)$, its shape is also just a segment. This is a shape of type (E) in the classification of [6]. In [25], it is shown that, if C is any algebraic quartic with this tropicalization, exactly 2 of the 28 bitangent lines to C tropicalize to the tropical line with vertex the upper red point, exactly 2 to the one with vertex the lower red point, and none to a point in the interior of the red segment (see Theorem 2.18). Figure 6 also depicts the dual motifs for the two tropical tangency components. Circled in red is the vertex dual to the connected component of $\mathbb{R}^2$ in which the bitangent class is contained.

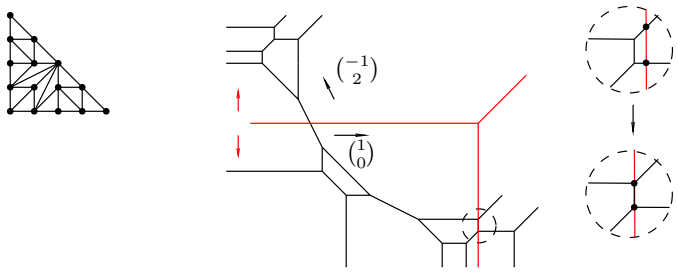

FIGURE 5. A tropicalized quartic, with its dual Newton subdivision, and a tropical bitangent line. We can move the vertex of the tropical bitangent upwards or downwards maintaining the bitangency.

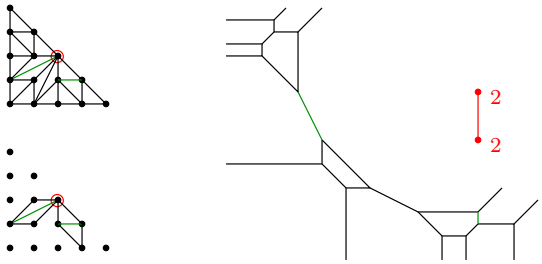

FIGURE 6. The bitangent class and dual motif of the tropical bitangent to the tropicalized quartic depicted in Figure 5.

Remark 2.17 (Genericity of tropicalized quartics). Compared to [25], we require an additional genericity assumption on our tropicalized quartics: we subdivide the cone in the secondary fan corresponding to the unimodular triangulation according to the types of tropical bitangent classes that can occur and require our tropicalized quartic to correspond to a point in the interior of a cone in this subdivision. This subdivision of the secondary fan is computed in [11]. Put differently, we require the edge lengths of $\mathrm{Trop}(C)$ to be generic in the sense that no unexpected alignment of vertices happens, see Figure 7. This picture shows a local piece of a tropicalized quartic on the left. For a generic element in the corresponding cone of the secondary fan, we expect the two lengths l_1 and l_2 to be different. If these lengths are equal, as depicted on the right, the two lower vertices align.

We call a bitangent shape that occurs for such a generic tropicalized quartic a *generic bitangent shape*.

The following statements concern lifts of tropical bitangent classes over $\mathbb{C}\{\{t\}\}$ and $\mathbb{R}\{\{t\}\}$.

Theorem 2.18 ([5, 25])**.** *Each bitangent class of a generic tropicalized quartic has* 4 *lifts over* $\mathbb{C}\{\{t\}\}$*. More precisely, there are either* 4 *tropical lines in the*

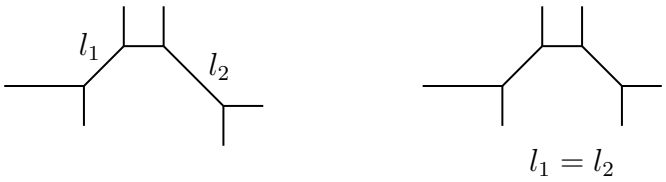

FIGURE 7. Local pictures of tropicalized quartics. An unexpected alignment of two vertices is depicted on the right. For a generic quartic, as shown on the left, the two lengths l_1 and l_2 are different.

bitangent class which each lifts once, or 2 *which each lift twice, or* 3 *of which one lifts twice, or* 1 *which lifts four times.*

Theorem 2.19 ([6])**.** *Each bitangent class of a generic tropicalized quartic has either* 0 *or* 4 *lifts over* $\mathbb{R}\{\{t\}\}$*, i.e. the possible obstruction for real lifting is the same for all representatives of a given tropical bitangent class that have a complex lift.*

These results are based on the classification of shapes of bitangent classes in [6] up to $\mathbb{S}_3$-symmetry. When considering GW-multiplicities as in Section 4, we fix the line $\{z = 0\}$ and do not have $\mathbb{S}_3$-symmetry for that reason, only $\mathbb{S}_2$-symmetry for exchanging the variables x and y.

In Appendix A.1, we classify tuples of shapes of generic bitangent classes together with their dual motifs up to $\mathbb{S}_2$-symmetry exchanging x and y. The classification is built on the classification in [6]. It restricts the cases studied there to generic shapes, considers $\mathbb{S}_3/\mathbb{S}_2$-orbits, and pairs up with dual motifs. In Appendix A.1, we depict all dual motifs together with local pictures of the tropicalized tangency points of the liftable tropical bitangents in the bitangent class are given.

Example 2.20. Consider the plane quartic $C = V(Q)$ for

$$\begin{aligned} Q(x,y) =& t^{36}x^4 + t^{18}x^3y + t^2x^2y^2 + t^{18}xy^3 + t^{36}y^4 + t^{23}x^3 + t^6x^2y + t^6xy^2 \\ &+ t^{23}y^3 + t^{12}x^2 + xy + t^{12}y^2 + t^2x + t^2y + 1 \end{aligned}$$

over the field of Puiseux series $\mathbb{C}\{\{t\}\}$.

Its tropicalization $\mathrm{Trop}(C)$ together with the seven bitangent classes (in red) is depicted in Figure 8. The liftable tropical bitangents are depicted as red dots. If a bitangent class has four red dots, each lifts once, if it has two red dots, each lifts twice and a single dot lifts to four bitangents.

3. Lifting tropical bitangents and tropical points of tangencies

3.1. **Local lifting equations and their solutions.** We review the local lifting techniques for tropical bitangent lines to tropicalized plane quartics from [6, 25].

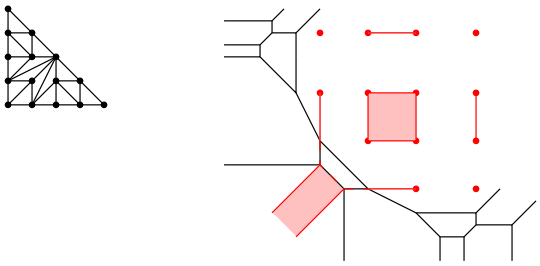

FIGURE 8. A tropicalized quartic with its seven bitangent classes.

As before, we let $C = V(Q)$ be a quartic and denote the coefficients of the defining polynomial Q by A_{ij}. For a point $p \in \mathrm{Trop}(C)$, let $\mathrm{in}_p(Q)$ denote the sum of the (dehomogenized) terms $\mathrm{in}(A_{ij})x^iy^j \in k[x,y]$ for which the maximum $\max\{-\mathrm{val}(A_{ij}) + ip_x + jp_y\}$ is attained. We call it the *initial form* of Q at p. The initial form can be be read off from the dual Newton subdivision: if p is contained in the interior of a region of $\mathbb{R}^2 \setminus C$, then the initial form is given precisely by the term of Q corresponding to this region. If p is in the interior of an edge, we use all terms that correspond to points of the dual edge. If p is a vertex, we use all terms that correspond to points of the dual polygon.

Definition 3.1 (Local equations for tropicalized tangency points)**.** Let T be a connected component of the intersection of $\mathrm{Trop}(C)$ with a tropical bitangent Λ, and assume a lift of Λ is given by an equation of the form $\ell = y + M + Nx$, where M and N are unknowns. Let $A(T) = \bigcup_{p\in T} \mathrm{supp}\{\mathrm{in}_p Q\}$ be the union of the supports of the initial forms of Q for all $p \in T$. We let Q_T be the restriction of Q to the terms appearing in $A(T)$, and call it the *local equation for C at T*.

Analogously, we let $B(T)$ be the union of the supports of the initial forms of ℓ for all $p \in T$, and ℓ_T, the restriction of ℓ to $B(T)$, the local equation of ℓ.

We let $W_T = \frac{\partial Q_T}{\partial x} \cdot \frac{\partial \ell_T}{\partial y} - \frac{\partial Q_T}{\partial y} \cdot \frac{\partial \ell_T}{\partial x}$ be the local version of the Wronskian.

Notice that vanishing of the Wronskian $W = \frac{\partial Q}{\partial x} \cdot \frac{\partial \ell}{\partial y} - \frac{\partial Q}{\partial y} \cdot \frac{\partial \ell}{\partial x}$ implies that the gradients of $V(Q)$ and $V(\ell)$ coincide, i.e. that the line $V(\ell)$ is tangent to $V(Q)$ at (x,y).

If $p \in T$ is the tropicalization of a tangency point, we solve the *local lifting equations*

$$Q_T(x,y) = 0, \quad \ell_T(x,y) = 0, \quad W_T(x,y) = 0.$$

If the tropical tangency component is just a point, $T = \{p\}$, then this amounts to solving the equations of the initial forms of Q, ℓ and W at p. If T is a segment of intersection involving two vertices of an edge E of $\mathrm{Trop}(C)$, then Q_T contains not only the terms of the edge $E^\vee$ dual to E, but also the terms corresponding to the vertices which form triangles with $E^\vee$ in the dual subdivision. If T is a segment containing one vertex V of $\mathrm{Trop}(C)$ and the vertex of Λ, then $\ell_T = \ell$ and Q_T

contains the terms corresponding to $E^\vee$ and the vertex of the triangle dual to V. If T consists of three segments joined at a vertex V, Q_T contains the terms of the triangle dual to V plus the terms of its neighbouring triangles, and $\ell_T = \ell$.

If T is more than just a point, the equations above are hard to solve. One uses a technical trick which is called a tropical modification or tropical refinement to produce solutions up to the order which appears as biggest valuation of the terms involved. For a discussion on how to treat the case of a segment T containing two vertices of $\mathrm{Trop}(C)$, see [25] Proposition 3.7 and 3.9. For the case of a segment containing the vertex of Λ, see Lemma 5.2, Case (3a) [6] and Proposition 3.12 in [25]. For the case of three segments joined at a vertex, see Proposition 3.12 in [25]. For general background on the technique of tropical modification, see e.g. [4, 14, 29]. Once we solved in such a way for x, y up to terms of higher valuation, it follows from the Henselian property (see also [8], Exercises 7.25, 7.26 and [14], Chapter 2) that these solutions can uniquely be completed to an element in K. For more details, see Section 2.3 in [25]. If the initials are in the residue field k, then the complete solutions are in K.

Example 3.2. We compute the initials of the four lifts of the bitangent class from Example 2.16. Consider first the upper vertex. There are two lifts over the algebraic closure which tropicalize to it. We denote the equations of those two lifts by $M_1 + N_1 x + y = 0$ resp. $M_2 + N_2 x + y = 0$. Let p_1 be the left tropical tangency component. In $\mathrm{Trop}(C)$, it is at the vertex dual to $A_{01}y + A_{12}xy^2 + A_{22}x^2y^2$. In the tropical bitangent line, it is in the horizontal ray, i.e. dual to $M_i + y$. We denote the initials of the A_{ij} by a_{ij} and analogously for the coefficients of the line equations. Then the initials (x_i, y_i) of the lift of the left tropical tangency component satisfy

$$\begin{aligned} &a_{01}y_{i1} + a_{12}x_{i1}y_{i1}^2 + a_{22}x_{i1}^2y_{i1}^2 = 0,\\ &m_i + y_{i1} = 0,\\ &2 \cdot a_{22} \cdot x_{i1} \cdot y_{i1}^2 + a_{12} \cdot y_{i1}^2 = 0, \end{aligned}$$

for $i = 1, 2$. Solving for m_i, x_{i1} and y_{i1} (for example by computing a Gröbner basis of the ideal defined by the three equations using a computer algebra system such as Singular [7] or Oscar [30]) we obtain, for $i = 1, 2$,

$$m_i = -\frac{4 \cdot a_{01} \cdot a_{22}}{a_{12}^2}, \quad x_{i1} = -\frac{a_{12}}{2 \cdot a_{22}}, \quad y_{i1} = \frac{4 \cdot a_{01} \cdot a_{22}}{a_{12}^2}.$$

The second tropical tangency component is the same for all liftable tropical bitangents in this bitangent class (i.e. for the tropical line with vertex the upper vertex of the red edge, and for the tropical line with vertex the lower vertex of the red edge in Figure 6). Therefore, we now first study the two lifts tropicalizing to the lower vertex and their tropical tangency component on the left. It is at the

vertex dual to $A_{01}y + A_{11}xy + A_{22}x^2y^2$. The three equations to solve are thus:

$$\begin{aligned} &a_{01}y_{i1} + a_{11}x_{i1}y_{i1} + a_{22}x_{i1}^2y_{i1}^2 = 0,\\ &m_i + y_{i1} = 0,\\ &2 \cdot a_{22} \cdot x_{i1} \cdot y_{i1}^2 + a_{11} \cdot y_{i1} = 0, \end{aligned}$$

for $i = 3, 4$, and the solutions are

$$m_i = -\frac{a_{11}^2}{4 \cdot a_{01} \cdot a_{22}}, \;\; x_{i1} = -\frac{2 \cdot a_{01}}{a_{11}}, \;\; y_{i1} = \frac{a_{11}^2}{4 \cdot a_{01} \cdot a_{22}}.$$

Let p_2 be the tangency point in the right tropical tangency component. This tropical tangency component appears not only for the two lifts of the upper vertex of the bitangent class, but also for the lower ones. That is, we can use the corresponding local lifting equations to solve for one tropical tangency component for each of the four lifts. The point p_2 is contained in a connected component of the intersection $\mathrm{Trop}(C)$ and a tropical bitangent which consists of a bounded edge of $\mathrm{Trop}(C)$. Dual to the vertices of this bounded edge are two triangles of the dual subdivision which meet along the edge joining $(2, 1)$ and $(3, 1)$. Therefore, we must take as local equation

$$A_{21}x^2y + A_{31}x^3y + A_{30}x^3 + A_{22}x^2y^2.$$

In the tropical line, the tropical tangency component p_2 is contained in the vertical ray, thus the local equation to consider is $m_i + n_i x$ for $i = 1, \ldots, 4$. As we saw earlier, m_i is determined by the left (upper or lower) tropical tangency component. In the local equations, we now solve for the ratio $\frac{m_i}{n_i}$, which is then sufficient to compute all coefficients of the lifts. It turns out that there are two solutions whose initials coincide. In order to be able to differentiate the two solutions whose initials coincide, we also list some contributions of higher valuation.

The solutions we obtain are

$$\begin{aligned} \frac{M_i}{N_i} &= \frac{A_{21}}{A_{31}} \pm \frac{2A_{22}}{A_{31}}\sqrt{-\frac{A_{30}A_{21}}{A_{22}A_{31}}} + \ldots\\ X_{i2} &= -\frac{A_{21}}{A_{31}} \mp \frac{2A_{22}}{A_{31}}\sqrt{-\frac{A_{30}A_{21}}{A_{22}A_{31}}} + \ldots\\ Y_{i2} &= \pm\sqrt{-\frac{A_{30}A_{21}}{A_{22}A_{31}}} + \ldots \end{aligned}$$

for $i = 1, \ldots, 4$. Here, we write the solutions not in terms of the initial a_{ij} but in terms of the whole coefficients A_{ij}, to keep track of the contributions of higher valuation.

Combining the results of the local lifting equations, we can now list the initials of the coefficient of the bitangent equation and the tangency point for all four lifts:

i	m_i	n_i	x_{i1}	y_{i1}	x_{i2}	y_{i2}
1	$-\frac{4\cdot a_{01}\cdot a_{22}}{a_{12}^2}$	$-\frac{4\cdot a_{01}\cdot a_{22}\cdot a_{31}}{a_{12}^2\cdot a_{21}}$	$-\frac{a_{12}}{2\cdot a_{22}}$	$\frac{4\cdot a_{01}\cdot a_{22}}{a_{12}^2}$	$-\frac{a_{21}}{a_{31}}$	$\sqrt{-\frac{a_{30}a_{21}}{a_{22}a_{31}}}$
2	$-\frac{4\cdot a_{01}\cdot a_{22}}{a_{12}^2}$	$-\frac{4\cdot a_{01}\cdot a_{22}\cdot a_{31}}{a_{12}^2\cdot a_{21}}$	$-\frac{a_{12}}{2\cdot a_{22}}$	$\frac{4\cdot a_{01}\cdot a_{22}}{a_{12}^2}$	$-\frac{a_{21}}{a_{31}}$	$-\sqrt{-\frac{a_{30}a_{21}}{a_{22}a_{31}}}$
3	$-\frac{a_{11}^2}{4\cdot a_{01}\cdot a_{22}}$	$-\frac{a_{11}^2\cdot a_{31}}{4\cdot a_{01}\cdot a_{22}\cdot a_{21}}$	$-\frac{2\cdot a_{01}}{a_{11}}$	$\frac{a_{11}^2}{4\cdot a_{01}\cdot a_{22}}$	$-\frac{a_{21}}{a_{31}}$	$\sqrt{-\frac{a_{30}a_{21}}{a_{22}a_{31}}}$
4	$-\frac{a_{11}^2}{4\cdot a_{01}\cdot a_{22}}$	$-\frac{a_{11}^2\cdot a_{31}}{4\cdot a_{01}\cdot a_{22}\cdot a_{21}}$	$-\frac{2\cdot a_{01}}{a_{11}}$	$\frac{a_{11}^2}{4\cdot a_{01}\cdot a_{22}}$	$-\frac{a_{21}}{a_{31}}$	$-\sqrt{-\frac{a_{30}a_{21}}{a_{22}a_{31}}}$

As in the example above, these local lifting equations can be solved for any case appearing in the combinatorial classification in Appendix A.1, and the solutions are always given by Laurent terms in the initials a_{ij} of the coefficients of Q, or by square roots thereof.

The consequence for lifting can be summed up as follows:

Theorem 3.3. *Let $C = V(Q)$ be a quartic curve defined over K with* $\mathrm{Trop}(C)$ *a generic tropicalized quartic and suppose Λ is a liftable tropical bitangent to C. Then whether or not Λ lifts over K is determined by* $\mathrm{Trop}(C)$ *and the equivalence classes of initials of the coefficients of Q in $k^\times/(k^\times)^2$.*

The proof follows along the same lines as in the complex and real case. we go through the classification in Appendix A.1 and solve the local lifting equations, no over the ground field K.

3.2. **Lifting conditions and twisted edges.** It follows from the classification of bitangent shapes in [6] and their lifting multiplicities that some bitangent shapes lift over any field (this holds for tropical bitangent lines with lifting multiplicity one, the rough argument being that a uniquely solvable system of linear equations over a field k also has its solution over k). Obstructions to lifting arise due to higher lifting multiplicities. In the following, we study tropical tangency components which produce a factor leading to such higher lifting multiplicities. Roughly, such factors arise whenever a tropical tangency component is locally fixed, i.e. we cannot vary the local part of the tropical bitangent without destroying the tangency. This is the case for segments of intersections, or for intersections involving the vertex of the tropical bitangent line.

In the case of a tropical tangency component on a segment of intersection, the potential obstruction for lifting can be phrased in terms of twisted edges or relatively twisted edges. By going through the cases in Appendix A.1, it turns out that the cases for lifting we study here cover all bitangent shapes except (C), for which lifting is more involved. For our purpose of understanding arithmetic multiplicities in Section 4, precise lifting conditions for (C) are not needed and we therefore do not consider these details here.

We start by stating the lifting solutions, which can be determined with the methods described in Section 3.1. In the following, we denote the vertex of the

dual subdivision corresponding to the term $A_{ij}x^iy^j$ in (the dehomogenized version of) Q by V_{ij}.

Lemma 3.4. *Assume the tropical line Λ intersects the tropicalized quartic $\mathrm{Trop}(C)$ in two connected components of which one is a segment of a horizontal edge dual to an edge connecting V_{1l} and V_{1l+1}. This edge forms triangles with two points V_{0i} and V_{j2}. Then there are two lifts for the coefficient M of a line $V(M+Nx+y)$ and for the tangency points (x,y) tropicalizing to this segment.*

If the vertex of Λ is not contained in the segment, we have

$$M = \frac{A_{1l}}{A_{1l+1}} \pm 2\sqrt{(-1)^{i+j}A_{0i}A_{2j}\left(\frac{A_{1l}}{A_{1l+1}}\right)^{i+j}}\frac{(-1)^lA_{1l+1}^{l-1}}{A_{1l}^l} + \dots,$$

$$y = -\frac{A_{1l}}{A_{1l+1}} \mp 2\sqrt{(-1)^{i+j}A_{0i}A_{2j}\left(\frac{A_{1l}}{A_{1l+1}}\right)^{i+j}}\frac{(-1)^lA_{1l+1}^{l-1}}{A_{1l}^l} + \dots,$$

$$x = \pm\sqrt{(-1)^{i-j}\frac{A_{0i}}{A_{2j}}\left(\frac{A_{1l}}{A_{1l+1}}\right)^{i-j}} + \dots,$$

where we assume that the lift of Λ has the equation $y+M+Nx$. Here, the dots stand for terms of higher valuation.

If the vertex of Λ is contained in the segment, we have

$$M = \frac{A_{1l}}{A_{1l+1}} \pm 2\sqrt{(-1)^{i+l+1}A_{0i}A_{1l+1}N\left(\frac{A_{1l}}{A_{1l+1}}\right)^{i+l}}\frac{(-1)^lA_{1l+1}^{l-1}}{A_{1l}^l} + \dots,$$

$$y = -\frac{A_{1l}}{A_{1l+1}} \mp 2\sqrt{(-1)^{i+l+1}A_{0i}A_{1l+1}N\left(\frac{A_{1l}}{A_{1l+1}}\right)^{i+l}}\frac{(-1)^lA_{1l+1}^{l-1}}{A_{1l}^l} + \dots,$$

$$x = \pm\sqrt{(-1)^{i-l+1}\frac{A_{0i}}{A_{1l+1}N}\left(\frac{A_{1l}}{A_{1l+1}}\right)^{i-l}} + \dots.$$

This follows from a computation using the local lifting equations, see also Proposition 5.2 in [6].

In the following, for a vertex r in the dual Newton subdivision of Q, we let a_r be the initial of the corresponding coefficient A_r of Q.

Definition 3.5 (Twisted edge)**.** Let C be a curve defined over K with smooth tropicalization $\mathrm{Trop}(C)$. Let e be a bounded edge in $\mathrm{Trop}(C)$ so that the dual subdivision corresponding to e is the segment joined by lattice points $q, q' \in \mathbb{Z}^2$ and the two triangles of the subdivision intersecting along this segment have vertices $r, r' \in \mathbb{Z}^2$. We say that the edge e is *twisted over k* if

$$\sqrt{(-1)^{\delta(e)}a_ra_{r'}(a_qa_{q'})^{\delta(e)}} \notin k,$$

FIGURE 9. The dual subdivision to a bounded edge e and its endpoints in a tropicalized curve, first in the case when $\delta(e) = 0$ on the left, and then when $\delta(e) = 1$ on the right.

where $\delta(e) = 0$ if $r = r' \mod 2$ and $\delta(e) = 1$ if $r \neq r' \mod 2$.

Example 3.6. To see the two possible cases for $\delta(e)$ consider the lattice polytopes in Figure 9. Notice that on the left hand side the coordinates of the vertices of the two triangles opposite the overlapping edge e have vector difference $(0, 0)$ modulo 2. Therefore, the points r and r' are equal modulo 2 and $\delta(e) = 0$. On the right hand side the vector difference of the coordinates of the two vertices has difference $(1, 1)$ modulo 2. Therefore we have $\delta(e) = 1$ in this case.

Example 3.7. The motivation for the terminology of twisting comes from considering amoebas of curves defined over the real numbers. Throughout this example, let $k = \mathbb{R}$ and $K = \mathbb{R}\{\{t\}\}$. Let C be a curve defined over K with smooth tropicalization $\mathrm{Trop}(C) = \Gamma$. Suppose that the defining polynomial Q of C has convergent coefficients for t sufficiently small. Then we can consider the family of real curves C_t defined by the family of polynomials Q_t for t sufficiently small. The tropical curve Γ is also obtained as the limit as s tends to infinity of amoebas of the family of curves $\widetilde{C}_s$ defined by the family of polynomials Q_s, where $s = t^{-1}$. The amoeba of a curve $\widetilde{C}_s$ is defined to be $\mathcal{A}_s := \mathrm{Log}_s(\widetilde{C}_s)$, where Log_s denotes the coordinatewise base s logarithm of absolute values, see [3, Section 2]. Moreover, for s sufficiently large, the real amoeba $\mathbb{R}\mathcal{A}_s := \mathrm{Log}_s(\mathbb{R}\widetilde{C}_s)$ either crosses a bounded edge e of Γ or it does not, see Figure 10. Following [3, Section 3], call the edge e *twisted* about the edge e if the amoeba crosses it.

The twisting of the edges of Γ are determined solely by the signs of the leading terms of the coefficients of the defining polynomial of C as described in [3, Remark 3.9]. Again let q, q' and r, r' denote the lattice points dual to the edge e as in Definition 3.5. Letting γ_q,$\gamma_{q'}$,γ_r and $\gamma_{r'}$ denote the signs of the leading terms of the corresponding coefficients, then

(1) if r and r' are distinct modulo 2 then e is twisted if and only if $\gamma_q\gamma_{q'}\gamma_r\gamma_{r'} > 0$;
(2) if r and r' are equal modulo 2 then e is twisted if and only if $\gamma_r\gamma_{r'} < 0$.

This is equivalent to our notion of twist in Definition 3.5 when $k = \mathbb{R}$. See Figure 9 for two examples of dual subdivisions exhibiting both cases in terms of r and r' above.

Using our local lifting solutions, obstructions to lifting can be phrased in terms of the twisting of an edge:

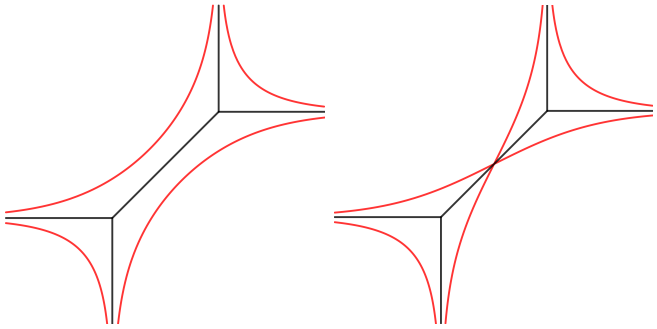

FIGURE 10. An illustration of the non-twisting and twisting of the real amoeba (in red) from Example 3.7.

Proposition 3.8. *Let C be a curve defined over K such that* $\mathrm{Trop}(C)$ *is a smooth tropicalized curve. Let Λ be a tropical line and suppose a bounded edge e of* $\mathrm{Trop}(C)$ *is a tropical tangency component of $\Lambda \cap \mathrm{Trop}(C)$ strictly contained in a ray of Λ. If $L = V(\ell)$ is a lift of Λ which is tangent to C at a point tropicalizing to e then the local equation ℓ_e is defined over K if and only if e is not twisted over k.*

Proof. The statement follows by applying Lemma 3.4. □

We now want to consider the situation when $\mathrm{Trop}(C)$ and Λ have a tropical tangency component T which is a segment strictly contained in an edge e of $\mathrm{Trop}(C)$, i.e. the vertex of Λ is contained in the segment of intersection, see Figure 11, This happens e.g. for the shape (D) we discuss in the proof of Theorem 4.23. Let s, s' be the lattice points dual to the edge of Λ containing T and q, q' be the lattice points dual to e in C. Moreover we choose the labels such that s and q correspond to the region of $\mathbb{R}^2 \setminus C$ resp. $\mathbb{R}^2 \setminus \Lambda$ on the same side of e, and s' and q' correspond to the region on the other side of e. As before, for a vertex r in the dual Newton subdivision of Q, we let a_r be the initial of the corresponding coefficient A_r of Q. Similarly, for a vertex s of the triangle dual to Λ, we let b_s denote the coefficient of the defining equation of a lift L of Λ. By Lemma 3.4 (where the coefficient M of the line can be expressed as $\frac{b_s}{b_{s'}}$ in the present notation, as the y-coefficient there was 1), in order for a lift L of Λ to have a tangency point P with C which tropicalizes to T we must have

$$\frac{b_s}{b_{s'}} = \frac{a_q}{a_{q'}}. \tag{1}$$

Definition 3.9 (Relatively twisted edge)**.** Let $\mathrm{Trop}(C)$ be smooth, and let Λ be a tropical line which has a tropical tangency component T which is a segment strictly contained in an edge e of $\mathrm{Trop}(C)$. Assume e is dual to the edge qq' and Λ meets the vertex of e which is dual to the triangle spanned by qq' and r. Let $a_q, a_{q'}, a_r$ denote the corresponding initials. We use the analogous indices for the initials of the coefficients of the polynomial defining L, i.e. b_s and $b_{s'}$ are the

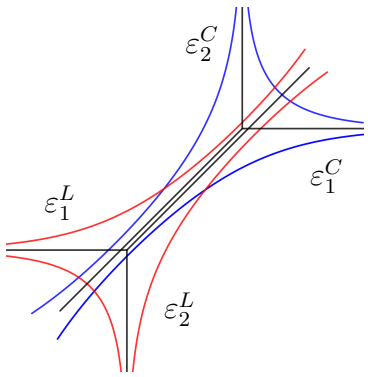

FIGURE 11. The picture shows the real amoebas of the two curves, the one of the line in red and (locally) of the quartic in blue. The segment in the overlap of Γ and Λ is relatively twisted if the two branches of the two amoebas which are on the same side of the tropicalized curve Γ are the image under the coordinatewise logarithm map of pieces of the real algebraic curves which live in the same orthants of $\mathbb{R}^2$.

initials of the monomials of the endpoints of the edge in the dual subdivision to the ray of Λ which contains e, and $b_{r'}$ is the initial of the remaining coefficient of the defining polynomial of L.

We say that the edge e is relatively twisted over k with respect to C and L if

$$\frac{b_s}{b_{s'}} = \frac{a_q}{a_{q'}} \qquad \text{and} \qquad \sqrt{(-1)^{\delta(e,L)+1} a_r (a_q a_{q'})^{\delta(e,L)} b_{r'} a_q b_s} \in k,$$

where $\delta(e, L) = 0$ if the edge in the dual subdivision with endpoints a_r, a_q is parallel modulo 2 to the edge in the dual subdivision with endpoints $b_{r'}, b_s$, and otherwise $\delta(e, L) = 1$.

Notice that the first condition $\frac{b_s}{b_{s'}} = \frac{a_q}{a_{q'}}$ implies that the second condition is equivalent to

$$\sqrt{(-1)^{\delta(e,L)+1} a_r (a_q a_{q'})^{\delta(e,L)} b_{r'} a_{q'} b_{s'}} \in k.$$

Also notice that by the balancing condition for tropical curves, the term $\delta(e, L)$ in the above definition can be equivalently be computed by comparing the directions mod 2 of the edges with endpoints r, q' and r', s'.

Example 3.10. Once again the above definition of relative twists is motivated by the geometric picture over the real numbers, as was explained in the case of twisted edges in Example 3.7. When considering real tropical intersections of curves, Le Texier defines the notion of relative twists [23]. As in Example 3.7, consider the real amoebas of both the family of lines and the family of curves, as drawn in Figure 11. The coordinatewise logarithm map is the composition of

the absolute value map and the logarithm map $\log_t \mathbb{R}_{>0} \to \mathbb{R}$. Therefore, each branch of the amoeba of a real curve can be labelled by the orthant of $(\mathbb{R}^\times)^2$ from which it came. In Figure 11, the branches of the amoebas of the family C along the edge e are labelled $\varepsilon_1^C, \varepsilon_2^C \in \{+,-\}^2$, and the branches of the amoebas of the family L along the edge e are labelled $\varepsilon_1^L, \varepsilon_2^L \in \{+,-\}^2$. The condition that

$$\frac{b_s}{b_{s'}} = \frac{a_q}{a_{q'}}$$

amounts to the equality of the sets

$$\{\varepsilon_1^C, \varepsilon_2^C\} = \{\varepsilon_1^L, \varepsilon_2^L\}.$$

In the real case, the edge e being relatively twisted with respect to C and L amounts to $\varepsilon_1^C = \varepsilon_1^L$ and $\varepsilon_2^C = \varepsilon_2^L$, where ε_i^C and ε_i^L are signs of the branches the amoebas of C and L respectively, which are on opposite sides of the bounded edge e. See Figure 11 for the labelling and [23, Proposition 4.13] for a proof of this statement.

In order for C and L to have a real tangency point tropicalizing to the segment, the edge e must be relatively twisted, see [23, Theorem 1.4]. Otherwise there would be two real points in the intersection. Notice that this condition is opposite to the case when the tropical tangency component is an entire edge. In that case, in order to have a real tangency we required the edge to be non-twisted.

Proposition 3.11. *Let C be a curve defined over K such that $\mathrm{Trop}(C)$ is a smooth tropicalized curve. Let Λ be a tropical line and suppose the segment T is a tropical tangency component of $\Lambda \cap \mathrm{Trop}(C)$ strictly contained in an edge of Λ and of $\mathrm{Trop}(C)$. Then a lift $L = V(\ell)$ of Λ which is tangent to C at a point tropicalizing to T has local equation ℓ_T defined over K if and only if the edge T is relatively twisted over k.*

Proof. The statement once again follows from Lemma 3.4. □

A condition similar to twisting can be given also for another type of tangency, as follows.

Proposition 3.12. *Let C be a curve defined over K such that $\mathrm{Trop}(C)$ is a smooth tropicalized curve. Let Λ be a tropical line and suppose p is an isolated point of $\Lambda \cap \mathrm{Trop}(C)$ which is a vertex p of Λ with intersection multiplicity 2. Then a lift L of Λ tangent to C at a point tropicalizing to p is defined over K if and only if $b_{r'}, b_{s'} \in k$ and*

$$\sqrt{-a_r a_s b_{r'} b_{s'}} \in k,$$

where $r, s \in \mathbb{Z}^2$ are the lattice endpoints of the edge dual to the edge E of $\mathrm{Trop}(C)$ and $r', s' \in \mathbb{Z}^2$ are lattice endpoints dual to the unique edge of Λ which has intersection multiplicity 2 with E.

Proof. Let Q_p and ℓ_p denote the local equations for the curve C and a lift of the tropical line Λ at p, respectively. Then Q_p is the binomial $A_r x^{r_1} y^{r_2} + A_s x^{s_1} y^{s_2}$ and $\ell_p = \ell$ is a trinomial. We denote the coefficients of the linear form ℓ by $B_{q'}, B_{r'}, B_{s'}$.

If Λ and C have intersection multiplicity 2 at the vertex p and $r', s' \in \mathbb{Z}^2$ are lattice endpoints dual to the unique edge of Λ which has intersection multiplicity 2 with E, then the direction of E is determined. For example, if $r' = (0,0)$ and $s' = (1,0)$ then the direction of E must be $(2,1)$ and hence $Q_p = A_r x^{r_1} y^{r_2} + A_s x^{r_1+1} y^{r_2-2}$ and set $\ell = B_{q'} y + B_{s'} x + B_{r'}$. Solving this system of two equations in the torus reduces to the degree two equation in y given by

$$A_r y^2 - \frac{A_s B_t}{B_{s'}} y - \frac{A_s B_{r'}}{B_{s'}} = 0.$$

In order to have a tangency tropicalizing to p we require the discriminant of this equation to be equal to zero, so that $A_r^2 B_{q'}^2 + 4 A_r A_s B_{r'} B_{s'} = 0$. Passing to the initials this implies that $b_{q'} = \pm\sqrt{\frac{-4 a_r a_s b_{r'} b_{s'}}{a_r^2}}$. Thus we have $b_{q'} \in k$ if and only if $\sqrt{-a_r a_s b_{r'} b_{s'}} \in k$ and the statement is proven. The other two cases for r' and s' are solved analogously. □

By going through the cases in Appendix A.1, one can see that Propositions 3.8, 3.11 and 3.12 cover all necessary lifting conditions of generic bitangent shapes except for shape (C). Shape (C) is more involved (see also Proposition 6.4 in [6] for the case of real lifting), and we leave it out here.

3.3. **Comparing lifting for different fields.** In this subsection, we sum up the results about lifting of tropical bitangents, and use our study of lifting to compare lifting over different fields.

Proposition 3.13. *Let $C = V(Q)$ be a quartic defined over K with generic tropicalization* $\mathrm{Trop}(C)$ *and Λ a tropical bitangent, and assume that the residue characteristic of K is not 2 or 3. Then the initials of the coefficients of the defining equation ℓ of a lift $L = V(\ell)$ of Λ, viewed in the algebraic closure of k, are Laurent terms in the initials a_{ij} of the coefficients of Q or square roots thereof.*

If Λ is not contained in a bitangent class of shape (II) in the classification in [6]*, the coefficients of these Laurent terms are of the form $\pm 2^m$ for some $m \in \mathbb{Z}$.*

For some of the liftable tropical bitangents of a class of shape (II), the coefficients involve $\sqrt{2}$ and $\sqrt{3}$.

Proof. This is a generalization of Theorem 1.2 in [6]. As the proof of this theorem, the result follows from a case-by-case analysis involving the classification of shapes of bitangent classes from [6], see also Appendix A.1. Unlike in Theorem 1.2 [6], which focused on the case of lifting over $\mathbb{R}$, we have to pay attention to the coefficients of the solutions of our local lifting equations in order to decide liftability over K. □

If we check all 4 local lifting computations for a tropical bitangent class, we observe that the obstruction for lifting to K is the same for all 4, i.e. the radicands which appear are all equal up to square (see e.g. Example 3.2). Thus we can conclude:

Theorem 3.14. *Assume the residue characteristic of K is not 2 or 3. Given a generic tropicalization $\mathrm{Trop}(C)$ of a quartic $C = V(Q)$ defined over K, and a bitangent class S of $\mathrm{Trop}(C)$, then either all four lifts of S are defined over K or none of them are.*

As noted above, the obstructions for lifting are equal not only for the coefficients of the equations of the bitangent lines, but also for the tangency points. Taking the special behaviour of shape (II) into account (see Proposition 3.13), we can deduce:

Corollary 3.15. *Assume $\sqrt{2}, \sqrt{3}$ exist in $k^\times$. Given a generic tropicalization $\mathrm{Trop}(C)$ of a quartic $C = V(Q)$ defined over K, and a bitangent class S of $\mathrm{Trop}(C)$ such that the equations of all four lifts are defined over K. Then the tangency points are also all defined over K.*

To compare lifting over different fields K_1 and K_2 with residue fields k_1 and k_2 respectively, we first suppose there exists an isomorphism $\phi : k_1^\times/(k_1^\times)^2 \to k_2^\times/(k_2^\times)^2$. For an element $a \in k^\times$, let $\overline{a}$ denote its class in $k^\times/(k^\times)^2$. As a main example, consider the case when k is the real numbers or any finite field. Then

$$k^\times/(k^\times)^2 \cong \mathbb{Z}/2\mathbb{Z},$$

and an isomorphism $k^\times/(k^\times)^2 \to \mathbb{Z}/2\mathbb{Z}$ is given by the Legendre symbol. For a finite field k we let $\left(\frac{a}{k}\right)$ denote the Legendre symbol of $a \in k^\times$. Concretely, we have

$$\overline{a} = \left(\frac{a}{k}\right) = \begin{cases} +1 \text{ if } x^2 = a \text{ has a solution in } k \\ -1 \text{ if } x^2 = a \text{ has no solution in } k \end{cases}$$

If we take $k = \mathbb{R}$ then the Legendre symbol is simply remembering whether $a \in k^\times$ is positive or negative. For two fields k_1, k_2 with isomorphisms $k_i^\times/(k_i^\times)^2 \to \mathbb{Z}/2\mathbb{Z}$, there is a unique isomorphism $\phi : k_1^\times/(k_1^\times)^2 \to k_2^\times/(k_2^\times)^2$.

Using Theorem 3.3 we can relate the lifts over different fields when $k_1^\times/(k_1^\times)^2$ and $k_2^\times/(k_2^\times)^2$ are isomorphic.

Theorem 3.16. *Let K_1 and K_2 be fields with residue fields k_1 and k_2, respectively. Suppose there exists an isomorphism of groups $\phi : k_1^\times/(k_1^\times)^2 \to k_2^\times/(k_2^\times)^2$ such that $\phi(\overline{-1}) = \overline{-1}$.*

For $i = 1, 2$, let $C_i = V(Q_i)$ be a quartic curve defined over K_i such that $\mathrm{Trop}(C_i)$ is generic. Let $Q_1 = \sum A_{ij} x^i y^j z^{4-i-j}$ and $Q_2 = \sum B_{ij} x^i y^j z^{4-i-j}$. We assume $val(B_{ij}) = val(A_{ij})$, in particular $\mathrm{Trop}(C_1) = \mathrm{Trop}(C_2)$, and $\phi(\overline{a_{ij}}) = \overline{b_{ij}}$ for all i, j, where a_{ij}, b_{ij} are the initials of A_{ij} and B_{ij}, respectively.

Then a tropical bitangent Λ to the tropicalized quartic $\mathrm{Trop}(C_1)$ *lifts to a bitangent of C_1 over K_1 if and only if Λ lifts to a bitangent of C_2 defined over K_2.*

This follows from Theorem 3.3.

The following corollary deals with the special case of comparing lifting over a finite field k with the real numbers. Here we can make use of the Legendre symbols. This is particularly useful, because lifting over the real numbers can be checked computationally using the POLYMAKE-extension of Geiger and Panizzut [10, 11, 12].

Corollary 3.17. *Let the residue field k be finite and of characteristic p with $p \equiv 3 \mod 4$ and order p^{2l+1}.*

Let $C = V(Q)$ be a quartic curve defined over K with $Q = \sum A_{ij}x^iy^j$. Then a tropical bitangent Λ to the tropicalized curve $\mathrm{Trop}(C)$ *lifts to a bitangent over K if and only if L lifts to a bitangent of $C' = V(Q')$ defined over $\mathbb{R}\{\{t\}\}$, where $Q' = \sum B_{ij}x^iy^j$ and $B_{ij} \in \mathbb{R}\{\{t\}\}$ are such that $val(B_{ij}) = val(A_{ij})$ and*

$$\left(\frac{a_{ij}}{k}\right) = \left(\frac{b_{ij}}{\mathbb{R}}\right).$$

This follows from Theorem 3.16 by inserting $K_2 = \mathbb{R}\{\{t\}\}$.

4. The Grothendieck-Witt ring and $\mathbb{A}^1$-enumerative invariants

Inspired by and building upon a broader program using $\mathbb{A}^1$-homotopy theory to introduce arithmetic refinements to enumerative geometry via quadratic forms [13, 18, 19, 26], Larson and Vogt considered an arithmetic count of bitangents for smooth plane quartics with values in the Grothendieck-Witt ring of the ground field [22]. One of our main goals is to demonstrate, using the arithmetic count of bitangents as an illustrative test case, how tropical methods can be useful for computing such arithmetic counts.

4.1. **The Grothendieck-Witt ring.** We now recall the definition and basic properties of the Grothendieck-Witt rings in which these arithmetic counts take their values. See [21] for a beautiful and comprehensive expository treatment, which includes the proofs that we omit. In this section, K denotes a field of characteristic not equal to two.

A *quadratic space* is a finite-dimensional K-vector space V equipped with a symmetric bilinear form $q\colon V \times V \to K$. Two quadratic spaces (V, q) and (V', q') are isomorphic if there is an isomorphism of k-vector spaces $\phi\colon V \to V'$ such that $q(v, w) = q'(\phi(v), \phi(w))$ for all v, w in V.

Definition 4.1. For $a \in K^\times$, we write $\langle a\rangle$ for the 1-dimensional quadratic space (k, q) with $q(x, y) = axy$. The *hyperbolic plane* is $\mathbb{H} = \langle 1\rangle \oplus \langle -1\rangle$.

Lemma 4.2. *For any $a \in K^\times$, the quadratic space $\langle a\rangle \oplus \langle -a\rangle$ is isomorphic to $\mathbb{H}$, as is the quadratic space K^2 with quadratic form $\big((x_1,x_2),(y_1,y_2)\big) \mapsto ax_1y_2 + ax_2y_1$.*

For a proof, see e.g. Corollary 16 in [21].

Isomorphism classes of quadratic spaces naturally form a semiring with addition and multiplication given by orthogonal direct sum and tensor product, respectively. A theorem of Witt shows that this addition is cancellative, i.e., if V, W, and W' are quadratic spaces such that $V \oplus W$ is isomorphic to $V \oplus W'$ then W is isomorphic to W'. As a consequence, this semiring injects into its associated ring of formal differences.

Definition 4.3. The *Grothendieck-Witt ring* $\mathrm{GW}(K)$ is the ring of formal differences of isomorphism classes of quadratic spaces over K.

In other words, elements of $\mathrm{GW}(K)$ are formal differences $V - W$, where V and W are isomorphism classes of quadratic spaces.

Definition 4.4. The degree map $\deg\colon \mathrm{GW}(k) \to \mathbb{Z}$ takes a quadratic space V to its dimension as a K-vector space.

For instance, $\deg\langle a\rangle = 1$ and $\deg \mathbb{H} = 2$. Note that deg is a map of rings. All of the $\mathrm{GW}(K)$-valued arithmetic counts that we consider specialize to the classical integer valued enumerative invariants when composed with the degree map.

Lemma 4.5. *As an additive group, $\mathrm{GW}(K)$ is generated by $\{\langle a\rangle : a \in K^\times\}$, with relations generated by*

(1) $\langle a\rangle = \langle ab^2\rangle$ *for all a, b in $K^\times$, and*
(2) $\langle a\rangle + \langle b\rangle = \langle a+b\rangle + \langle ab(a+b)\rangle$, *for all a, b in $K^\times$ such that $a + b \neq 0$.*

When no confusion seems possible, we write $\langle a\rangle$, $\mathbb{H}$, and so on, not only for a given quadratic space, but also for its class in $\mathrm{GW}(K)$.

We will frequently consider how Grothendieck-Witt rings behave with respect to finite field extensions. Suppose K'/K is a finite extension. Then any finite dimensional K'-vector space V is also a finite-dimensional as a K-vector space, and we write V_K to denote V, viewed as a K-vector space. If (V,q) is a quadratic space over K', then $(V_K, \mathrm{Tr}_{K'|K} \circ q)$ is a quadratic space over K. One writes

$$\mathrm{Tr}_{K'|K}\colon \mathrm{GW}(K') \to \mathrm{GW}(K)$$

for the induced map of Grothendieck-Witt rings. Note that $\mathrm{Tr}_{K'|K}$ multiplies degrees by a factor of $[K' : K]$, since $\dim_K V_K = [K' : K] \dim_{K'} V$.

Example 4.6. Let K'/K be a field extension of degree 2, and let $a \in (K')^\times$. We now explain how to compute $\mathrm{Tr}_{K'|K}\langle a\rangle \in \mathrm{GW}(K)$. Suppose $K' = K(\beta)$, and $b = \beta^2$. Use $\{1, \beta\}$ as a K-basis for K', and write

$$a = r + s\beta.$$

Then the bilinear form on $\mathrm{Tr}_{K'|K}\langle a\rangle$ may be expressed by the symmetric matrix

$$\begin{pmatrix} 2r & 2bs \\ 2bs & 2br \end{pmatrix}.$$

If $r = 0$, then $\mathrm{Tr}_{K'|K}\langle a\rangle \cong \mathbb{H}$, by Lemma 4.2. If $r \neq 0$, then we can diagonalize the symmetric matrix to get $\mathrm{Tr}_{K'|K}\langle a\rangle \cong \langle 2r\rangle \oplus \langle 2br(r^2 - bs^2)\rangle$.

A case of particular interest is when $K = \mathbb{R}$ and $b = -1$. Then $r^2 - bs^2$ is positive and hence is a square, so the above quadratic space is isomorphic to $\langle 2r\rangle \oplus \langle -2r\rangle \cong \mathbb{H}$. Thus, for any $a \in \mathbb{C}^\times$, we have $\mathrm{Tr}_{\mathbb{C}|\mathbb{R}}\langle a\rangle \cong \mathbb{H}$.

4.2. **Grothendieck-Witt rings over valued fields.** We now return to the case where K is a Henselian valued field with residue field k. We assume the characteristic of k is not 2 and fix a section of the valuation $\sigma\colon \mathrm{val}(K^\times) \to K^\times$. The initial of an element $a \in K^\times$, denoted $\mathrm{in}(a)$, is the image of $\sigma(-\mathrm{val}(a)) \cdot a$ in $k^\times$. We also make the simplifying assumption that the value group of K is 2-divisible, cf. Remark 1.6.

Theorem 4.7. *Let K be a Henselian valued field of residue characteristic not equal to 2, with 2-divisible value group, and with a section of the valuation.*

There is an isomorphism of Grothendieck-Witt rings

$$g : \mathrm{GW}(K) \xrightarrow{\sim} \mathrm{GW}(k),\ \langle A\rangle \mapsto \langle \mathrm{in}(A)\rangle.$$

Moreover, this isomorphism is independent of the section σ.

Proof. We first show that g is well-defined, i.e., that $\langle A\rangle \mapsto \langle \mathrm{in}(A)\rangle$ respects the relations (1) and (2) from Lemma 4.5. Let $a = \mathrm{in}(A)$ and $b = \mathrm{in}(B)$. For (1), we have $\langle \mathrm{in}(AB^2)\rangle = \langle ab^2\rangle$. For (2), we consider a few subcases. First, suppose $\mathrm{val}(A) \neq \mathrm{val}(B)$. Without loss of generality we may assume $\mathrm{val}(A) < \mathrm{val}(B)$. Then $\mathrm{in}(\langle A\rangle + \langle B\rangle) = \langle a\rangle + \langle b\rangle$, while $\mathrm{in}\langle A+B\rangle = \langle a\rangle$ and $\mathrm{in}\langle AB(A+B)\rangle = \langle a^2b\rangle$. Using 4.5(1), we see that $g(\langle A + B\rangle + \langle AB(A + B)\rangle) = \langle a\rangle + \langle a^2b\rangle = \langle a\rangle + \langle b\rangle$, as required. If $\mathrm{val}(A) = \mathrm{val}(B)$ but $a + b \neq 0$, then (2) is clear. It remains to consider the case where $\mathrm{val}(A) = \mathrm{val}(B)$ and $a + b = 0$. Let $c = \mathrm{in}(A + B)$. Then $g(\langle A + B\rangle) = \langle c\rangle$ and $g(\langle AB(A + B)\rangle) = \langle abc\rangle = \langle -a^2c\rangle$, which is equal to $\langle -c\rangle$ in $\mathrm{GW}(k)$, again by 4.5(1). Finally, $g(\langle A\rangle + \langle B\rangle) = \langle a\rangle + \langle -a\rangle = \langle c\rangle + \langle -c\rangle$, by Lemma 4.2. This proves that g is well-defined.

It is evident that g respects addition and multiplication, so it is a ring map. We now construct its inverse. For each $a \in k^\times$, choose some $A \in R$ with residue a. We claim that $\langle a\rangle \mapsto \langle A\rangle$ gives a well-defined map $\mathrm{GW}(k) \to \mathrm{GW}(K)$. Indeed, if A and A' are two lifts of a, then A/A' has valuation 0 and initial 1. Since the characteristic of k is not 2 and K is Henselian, it follows that A/A' is a square and hence $\langle A\rangle \cong \langle A'\rangle$. Moreover, since σ is a section and $\mathrm{val}(K^\times)$ is 2-divisible, t^v has a square root for every $v \in \mathrm{val}(K^\times)$. Thus $\mathrm{GW}(K)$ is generated by quadratic spaces $\langle A\rangle$ such that $\mathrm{val}(A) = 0$, and hence the map so defined surjects onto $\mathrm{GW}(K)$ and gives an inverse to g, as required.

Finally, again using the fact that $\mathrm{val}(K^\times)$ is 2-divisible, the image of the section σ is contained in the multiplicative subgroup of squares. Thus, if we choose a different section, the resulting initials of any $a \in K^\times$ will differ by a square and hence the isomorphism $\mathrm{GW}(K) \to \mathrm{GW}(k)$ is independent of this choice. □

4.3. **The Qtype of a bitangent to a quartic.** A line L in $\mathbb{P}^2_k$ is a closed point of $(\mathbb{P}^2_k)^\vee$. Let K_L denote the field of definition of L. So K_L/K is a finite extension and L corresponds to a Galois orbit of geometric lines defined over the finite extension K_L/K. Suppose L is a bitangent to our plane quartic C. We assume furthermore that the intersection $L \cap C$ is disjoint from the line at infinity L_∞ given in homogeneous coordinates by $z = 0$. The GW-*multiplicity of the bitangent* L with respect to the fixed reference line L_∞ is an element of degree $[K_L : K]$ in $\mathrm{GW}(K)$ that is defined as follows.

Let $\mathbb{A}^2 = \mathbb{P}^2 \setminus L_\infty$. Let $Q(x, y)$ be a (non-homogeneous) quartic polynomial that vanishes on the affine plane quartic $C \cap \mathbb{A}^2$, and let ∂_L denote the derivation with respect to a linear form (defined over K_L) that vanishes on L. Note that both $Q(x, y)$ and ∂_L are defined only up to nonzero scalars (in $K^\times$ and $K_L^\times$, respectively).

Definition 4.8. Let P_1, $P_2 \in C(\overline{K})$ be the points where L is tangent to C. Then the Qtype of L with respect to L_∞ is

$$\mathrm{Qtype}_{L_\infty}(L) := \langle \partial_L Q(P_1) \cdot \partial_L Q(P_2) \rangle \in \mathrm{GW}(K_L), \tag{2}$$

and the GW-*multiplicity of* L is

$$\mathrm{mult}_{\mathrm{GW}}(L) := \mathrm{Tr}_{K_L|K}(\mathrm{Qtype}_{L_\infty}(L)) \in \mathrm{GW}(K).$$

To see that $\mathrm{Qtype}_{L_\infty}(L)$ is well-defined, first suppose P_1 and P_2 are rational over K_L. Note that rescaling Q or ∂_L by a nonzero factor a multiplies $\partial_L Q(P_1) \cdot \partial_L Q(P_2)$ by a^2. Then since $\langle a^2 b \rangle = \langle b \rangle$, the $\mathrm{Qtype}_{L_\infty}$ of L is well-defined.

Otherwise, if P_1 and P_2 are not rational over K_L then they are rational over some quadratic extension K_1/K_L. In this case $\partial_L Q(P_1)$ is conjugate to $\partial_L Q(P_2)$, and so $\partial_L Q(P_1) \cdot \partial_L Q(P_2)$ is in K_L. Once again, rescaling Q or L changes $\partial_L Q(P_1) \cdot \partial_L Q(P_2)$ by a square, and so $\mathrm{Qtype}_{L_\infty}(L)$ is well-defined in $\mathrm{GW}(K_L)$.

Example 4.9. When $K = \mathbb{R}$, the $\mathrm{Qtype}_{L_\infty}$ has a natural topological interpretation, as explained by Larson and Vogt, see page 3 in [22]. Suppose a bitangent is defined over $\mathbb{R}$. If each of the bitangency points of L is defined over $\mathbb{R}$, then near these two points in the affine plane $\mathbb{A}^2(\mathbb{R})$, the real locus of C is either on the same side of the bitangent line, in which case the $\mathrm{Qtype}_{L_\infty}$ is $\langle 1 \rangle$, or on different sides, in which case the $\mathrm{Qtype}_{L_\infty}$ is $\langle -1 \rangle$. See Figure 12. When these points are not defined over $\mathbb{R}$, the $\mathrm{Qtype}_{L_\infty}$ is $\langle 1 \rangle$. If the bitangent line is not defined over $\mathbb{R}$ then the $\mathrm{Qtype}_{L_\infty}$ of L is $\langle 1 \rangle \in \mathrm{GW}(\mathbb{C})$ and $\mathrm{mult}_{\mathrm{GW}}(L) = \langle 1 \rangle + \langle -1 \rangle = \mathbb{H}$ by Example 4.6.

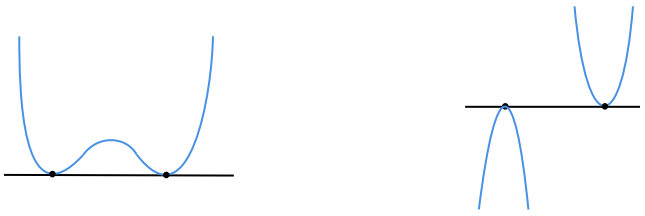

FIGURE 12. Geometric characterization of $\mathrm{Qtype}_{L_\infty}$ for the reals.

Example 4.10. Suppose $K = \mathbb{R}$ (or $\mathbb{R}\{\{t\}\}$). If L is a bitangent line that is not rational over K, (i.e., if L is the point of $(\mathbb{P}^2_{\mathbb{R}})^\vee$ corresponding to a pair of complex conjugate lines) then, by Example 4.6, we have $\mathrm{mult}_{\mathrm{GW}}(L) = \mathbb{H}$.

4.4. **The GW-multiplicity of a tropical bitangent class.** In the following, we study GW-multiplicities for the four liftable members of a tropical bitangent class.

We start, in the following Lemma, with picking a vector for the computation of $\partial_L Q(P)$ for a bitangent line L. It turns out that our choice is particularly useful for the tropical degeneration.

Lemma 4.11. *Let $L = V(y + M + Nx)$ be a bitangent to a quartic curve defined by Q, and let P be a point of tangency. Then $\partial_L Q(P) = (\frac{\partial}{\partial y} + \frac{1}{N} \cdot \frac{\partial}{\partial x})Q(P)$.*

Proof. Up to rescaling, the direction vector of the line L is $(-1, N)$. To compute ∂_L, we can take any vector which is not parallel to the direction vector and multiply it with the gradient. If $(\frac{1}{N}, 1)$ was parallel to $(-1, N)$, then $-\frac{1}{N} \cdot (-1, N) = (\frac{1}{N}, -1) = (\frac{1}{N}, 1)$ and thus $-1 = 1$, which is not the case as we are not in characteristic 2. □

Remark 4.12. In the following theorem, we make use of our choice of gradient and express GW-multiplicity in terms of derivatives of initial forms, which can be determined using the tropicalization of the quartic.

Using the classification in Appendix A.1, one can show that any such derivative of an initial form that can appear equals, after inserting initials of the tangency point, a Laurent monomial in the initials of the coefficients of the quartic times possibly a square root thereof.

Theorem 4.13. *Assume the bitangent L is given by the equation $M + NX + Y$ with $M, N \in K$, and let m, n denote their initials. If the tropicalization p of a tangency point P is contained in the interior of the horizontal ray of* $\mathrm{Trop}(L)$*, then*

$$\mathrm{in}(\partial_L Q(P)) = 2\frac{\partial}{\partial y}\,\mathrm{in}_p(Q)(\mathrm{in}(P)).$$

If it is contained in the interior of the vertical ray of $\mathrm{Trop}(L)$*, then*

$$\mathrm{in}(\partial_L Q(P)) = 2\frac{1}{n}\frac{\partial}{\partial x}\,\mathrm{in}_p(Q)(\mathrm{in}(P)).$$

If it is contained in the interior of the diagonal ray or at the vertex, then

$$\mathrm{in}(\partial_L Q(P)) = (\frac{\partial}{\partial y} + \frac{1}{n}\cdot\frac{\partial}{\partial x})\,\mathrm{in}_p(Q)(\mathrm{in}(P)).$$

In particular, the GW*-multiplicity* $\langle \partial_L Q(P_1)\cdot\partial_L Q(P_2)\rangle$ *can be expressed in terms of derivatives of initial forms.*

Proof. Without restriction, we can assume $p=(0,0)$ and Q only contains terms whose coefficients have valuation 0 or higher. The terms of valuation 0 are then precisely the terms contributing to $\mathrm{in}_p(Q)$.

By Lemma 4.11, $\partial_L Q(P) = (\frac{\partial}{\partial y} + \frac{1}{N}\cdot\frac{\partial}{\partial x})Q(P)$. The initial equals

$$\mathrm{in}(\partial_L Q(P)) = \mathrm{in}\Big(\Big(\frac{\partial}{\partial y} + \frac{1}{N}\cdot\frac{\partial}{\partial x}\Big)Q(P)\Big) = \mathrm{in}\Big(\frac{\partial}{\partial y}Q(P) + \frac{1}{N}\cdot\frac{\partial}{\partial x}Q(P)\Big). \tag{3}$$

If the two summands above are of the same valuation, then cancellation could happen, which would be an obstruction to expressing $\mathrm{in}(\partial_L Q(P))$ in terms of derivatives of initial forms.

Assume cancellation happens, then

$$\mathrm{in}\Big(\frac{\partial}{\partial y}Q(P)\Big) = -\frac{1}{n}\,\mathrm{in}\Big(\frac{\partial}{\partial x}Q(P)\Big).$$

But the Wronskian $W = N\cdot\frac{\partial}{\partial y}Q(P) - \frac{\partial}{\partial x}Q(P) = 0$ vanishes, and thus also its initial vanishes, so $n\cdot\mathrm{in}(\frac{\partial}{\partial y}Q(P)) = \mathrm{in}(\frac{\partial}{\partial x}Q(P))$. We substitute $n\cdot\mathrm{in}(\frac{\partial}{\partial y}Q(P))$ for $\mathrm{in}(\frac{\partial}{\partial x}Q(P))$ in the above equation, obtaining $\mathrm{in}(\frac{\partial}{\partial y}Q(P)) = -\mathrm{in}(\frac{\partial}{\partial y}Q(P))$ from which we deduce $1=-1$ which is a contradiction as we are not in characteristic 2. Thus no cancellation happens.

There are three cases to distinguish:

- Assume p is on the horizontal ray. Then the vertex of $\mathrm{Trop}(L)$ is at $(a,0)$ for some $a>0$. The valuation of N is a, the valuation of M is zero.

  Consider the Wronskian $W = N\cdot\frac{\partial}{\partial y}Q(P) - \frac{\partial}{\partial x}Q(P) = 0$. Consequently, $\mathrm{in}(N\cdot\frac{\partial}{\partial y}Q(P) - \frac{\partial}{\partial x}Q(P)) = 0$. But the summand $N\cdot\frac{\partial}{\partial y}Q(P)$ has terms of valuation a or higher, and the summand $\frac{\partial}{\partial x}Q(P)$ has terms of valuation 0 or higher. Cancellation can thus only appear if $\frac{\partial}{\partial x}Q(P)$ vanishes up to valuation a.

  Thus both summands in the above equation (3) for $\mathrm{in}(\partial_L Q(P))$ have expected valuation 0.

  From the computations for GW-multiplicities we perform in Appendix A.2 and A.3 (see Remark 4.12), we conclude that $\mathrm{in}(\frac{\partial}{\partial y}Q(P)) =$

$\frac{\partial}{\partial y}(\mathrm{in}_p(Q)(\mathrm{in}(P)) \neq 0$ and thus this summand is of the expected valuation. The equality $\mathrm{in}(\frac{\partial}{\partial y}Q(P)) = \frac{\partial}{\partial y}(\mathrm{in}_p(Q)(\mathrm{in}(P))$ holds since for a polynomial $F(x,y)$ and a point $P = (P_x, P_y)$ that we insert, we have $\mathrm{in}(F(P_x,P_y)) = \mathrm{in}_{(p_x,p_y)}(F)(\mathrm{in}(P))$, where (p_x,p_y) contains the valuations, if no cancellation happens in the initial form when inserting $\mathrm{in}(P)$. Furthermore we have $\mathrm{in}_p(\frac{\partial}{\partial y}Q) = \frac{\partial}{\partial y}(\mathrm{in}_p(Q))$. This is true since the p-weight of a term of Q equals the $p+p_y$-weight of a term of $\frac{\partial}{\partial y}Q$.

Thus $\mathrm{val}(N \cdot \frac{\partial}{\partial y}Q(P)) = a$ and since there is cancellation in the Wronskian, also $\mathrm{val}(\frac{\partial}{\partial x}Q(P)) = a$. Furthermore, since there is cancellation in W, $n\,\mathrm{in}(\frac{\partial}{\partial y}Q(P)) = \mathrm{in}(\frac{\partial}{\partial x}Q(P))$.

Inserting this into the above equation (3), we obtain

$$\begin{aligned}
\mathrm{in}(\partial_L Q(P)) &= \mathrm{in}\left(\frac{\partial}{\partial y}Q(P)\right) + \mathrm{in}\left(\frac{1}{N}\cdot\frac{\partial}{\partial x}Q(P)\right)\\
&= \mathrm{in}\left(\frac{\partial}{\partial y}Q(P)\right) + \frac{1}{n}\,\mathrm{in}\left(\frac{\partial}{\partial x}Q(P)\right)\\
&= \mathrm{in}\left(\frac{\partial}{\partial y}Q(P)\right) + \frac{1}{n}\cdot n\,\mathrm{in}\left(\frac{\partial}{\partial y}Q(P)\right)\\
&= \mathrm{in}\left(\frac{\partial}{\partial y}Q(P)\right) + \mathrm{in}\left(\frac{\partial}{\partial y}Q(P)\right)\\
&= 2\,\mathrm{in}\left(\frac{\partial}{\partial y}Q(P)\right) = 2\frac{\partial}{\partial y}(\mathrm{in}_p(Q))(\mathrm{in}(P)).
\end{aligned}$$

- Assume p is on the vertical ray. Then the vertex of $\mathrm{Trop}(L)$ is at $(0,a)$ for some $a > 0$. The valuation of N and M is $-a$.

  Consider the Wronskian $W = N \cdot \frac{\partial}{\partial y}Q(P) - \frac{\partial}{\partial x}Q(P) = 0$. Consequently, $\mathrm{in}(N \cdot \frac{\partial}{\partial y}Q(P) - \frac{\partial}{\partial x}Q(P)) = 0$. But the summand $N \cdot \frac{\partial}{\partial y}Q(P)$ has terms of valuation $-a$ or higher, and the summand $\frac{\partial}{\partial x}Q(P)$ has terms of valuation 0 or higher. Cancellation can thus only appear if $\frac{\partial}{\partial y}Q(P)$ vanishes up to valuation a.

  Thus both summands in the above equation for $\mathrm{in}(\partial_L Q(P))$ (3) have expected valuation 0.

  From the computations for GW-multiplicities we perform in Appendix A.2 and A.3 (see Remark 4.12), we conclude that $\mathrm{in}(\frac{\partial}{\partial x}Q(P)) = \frac{\partial}{\partial x}(\mathrm{in}_p(Q)(\mathrm{in}(P)) \neq 0$ and thus this summand is of the expected valuation. Thus $\mathrm{val}(\frac{\partial}{\partial x}Q(P)) = 0$ and since there is cancellation in the Wronskian, also $\mathrm{val}(N \cdot \frac{\partial}{\partial y}Q(P)) = 0$. Furthermore, since there is cancellation in W, $n\,\mathrm{in}(\frac{\partial}{\partial y}Q(P)) = \mathrm{in}(\frac{\partial}{\partial x}Q(P))$.

Inserting this into the above equation for $\mathrm{in}(\partial_L Q(P))$, we obtain

$$\begin{aligned}\mathrm{in}(\partial_L Q(P)) &= \mathrm{in}\left(\frac{\partial}{\partial y}Q(P)\right) + \mathrm{in}\left(\frac{1}{N}\cdot\frac{\partial}{\partial x}Q(P)\right)\\ &= \mathrm{in}\left(\frac{\partial}{\partial y}Q(P)\right) + \frac{1}{n}\,\mathrm{in}\left(\frac{\partial}{\partial x}Q(P)\right)\\ &= \frac{1}{n}\,\mathrm{in}\left(\frac{\partial}{\partial x}Q(P)\right) + \frac{1}{n}\cdot n\,\mathrm{in}\left(\frac{\partial}{\partial y}Q(P)\right)\\ &= 2\frac{1}{n}\,\mathrm{in}\left(\frac{\partial}{\partial x}Q(P)\right) = 2\frac{1}{n}\frac{\partial}{\partial x}(\mathrm{in}_p(Q))(\mathrm{in}(P)).\end{aligned}$$

- Assume p is on the vertical ray or the vertex. Then the vertex of $\mathrm{Trop}(L)$ is at $(-a,-a)$ for some $a \geq 0$. The valuation of M is a, the valuation of N is zero. Then both summands in equation (3) are of valuation zero and we can directly express

$$\begin{aligned}\mathrm{in}(\partial_L Q(P)) &= \mathrm{in}\left(\frac{\partial}{\partial y}Q(P)\right) + \mathrm{in}\left(\frac{1}{N}\cdot\frac{\partial}{\partial x}Q(P)\right)\\ &= \mathrm{in}\left(\frac{\partial}{\partial y}Q(P)\right) + \frac{1}{n}\,\mathrm{in}\left(\frac{\partial}{\partial x}Q(P)\right).\\ &= \frac{\partial}{\partial y}(\mathrm{in}_p(Q))(\mathrm{in}(P)) + \frac{1}{n}\frac{\partial}{\partial x}(\mathrm{in}_p(Q))(\mathrm{in}(P)).\end{aligned}$$

For the statement in particular, note that by Theorem 4.7, we have $\langle \partial_L Q(P_1)\cdot \partial_L Q(P_2)\rangle = \langle \mathrm{in}(\partial_L Q(P_1)\cdot \partial_L Q(P_2))\rangle$. The initial of the product $\mathrm{in}(\partial_L Q(P_1)\cdot \partial_L Q(P_2))$ equals the product of initials $\mathrm{in}(\partial_L Q(P_1))\cdot \mathrm{in}(\partial_L Q(P_2))$. The factors can be expressed in terms of derivatives of initial forms as described above. □

Remark 4.14. By Example 4.10, a tropical bitangent with lifting multiplicity 2 which does not lift to K contributes $\mathbb{H}$ to the arithmetic count of bitangents: Such a lift is defined over a field extension of degree 2. The element whose class in $\mathrm{GW}(K)$ we take equals a Laurent term in the initials of Q times a square root of such a term by Remark 4.12 (see also Appendix A.2 and A.3). Thus, in the notation of Example 4.10, $r = 0$ and we get a contribution of $\mathbb{H}$.

In the following example, we consider tropical bitangents of lifting multiplicity four whose lifts live in a field extension.

Example 4.15. There are tropical bitangent classes for which we need to add two roots to the ground field to lift to 4 bitangent lines. These are precisely the zero-dimensional classes (A), (B) and (C) in the classification in [6]. Assume these two roots are α and β and $K(\alpha,\beta)/K$ is of degree 4.

Let $a \in K(\alpha,\beta)^\times$. We now compute $\mathrm{Tr}_{K(\alpha,\beta)|K}\langle a\rangle \in \mathrm{GW}(K)$. A basis for $K(\alpha,\beta)$ is given by $1, \alpha, \beta, \alpha\cdot\beta$. We write a in this basis:

$$a = c_1\cdot 1 + c_2\cdot\alpha + c_3\cdot\beta + c_4\cdot\alpha\cdot\beta.$$

We study the bilinear map

$$\tilde{q} : K(\alpha,\beta)\times K(\alpha,\beta) \to K(\alpha,\beta) \to K,$$

where the first arrow is the map $(x,y)\mapsto axy$ and the second is the trace. Let us first compute the trace for an arbitrary element $b = b_1\cdot 1 + b_2\cdot\alpha + b_3\cdot\beta + b_4\cdot\alpha\cdot\beta$. The matrix we obtain is

$$\begin{pmatrix} b_1 & b_2\alpha^2 & b_3\beta^2 & b_4\alpha^2\beta^2 \\ b_2 & b_1 & b_4\beta^2 & b_3\beta^2 \\ b_3 & b_4\alpha^2 & b_1 & b_2\alpha^2 \\ b_4 & b_3 & b_2 & b_1 \end{pmatrix}.$$

The trace is thus $4b_1$. Now we can insert basis vectors in the above bilinear map to obtain the symmetric matrix

$$\begin{pmatrix} 4c_1 & 4c_2\alpha^2 & 4c_3\beta^2 & 4c_4\alpha^2\beta^2 \\ 4c_2\alpha^2 & 4c_1\alpha^2 & 4c_4\alpha^2\beta^2 & 4c_3\alpha^2\beta^2 \\ 4c_3\beta^2 & 4c_4\alpha^2\beta^2 & 4c_1\beta^2 & 4c_2\alpha^2\beta^2 \\ 4c_4\alpha^2\beta^2 & 4c_3\alpha^2\beta^2 & 4c_2\alpha^2\beta^2 & 4c_1\alpha^2\beta^2 \end{pmatrix}.$$

As we can see in our computations for lifts of tropical bitangents and their $\mathrm{Qtype}_{L_\infty}$ (see Remark 4.12 and Appendix A.1), the bilinear map we obtain for a lift of a tropical bitangent class of type (A), (B) or (C) is of the form $c_2\alpha$ or $c_4\alpha\beta$. Thus, we can insert $c_1 = c_3 = c_4 = 0$ or $c_1 = c_2 = c_3 = 0$ in the above matrix. In both cases, we obtain a decomposition into two hyperbolic planes.

This computation shows that the GW-multiplicity of a tropical bitangent class whose 4 lifts live in a field extension of degree 4 which we obtain by adjoining two roots is $2\mathbb{H}$.

Remark 4.16 (Computation of GW-multiplicities). Theorem 4.13 together with the theory of tropical bitangents allows to compute GW-multiplicities of quartics as follows:

- Compute the tropicalization $\mathrm{Trop}(C)$, e.g. using the software system Polymake [10] or the libraby tropical.lib in the computer algebra system Singular [15, 7].
- Using the Polymake-extension on tropical bitangents of quartics by Geiger-Panizzut [11], compute all its tropical bitangent classes.
- For each liftable tropical bitangent, use Theorem 4.13 to compute its $\mathrm{Qtype}_{L_\infty}$.
- For each tropical bitangent class which does not lift, deduce from Remark 4.14 and 4.15 that the GW-multiplicity of the four bitangents equals $2\mathbb{H}$.

In the following, we prepare statements which will be used for the further study of the $\mathrm{Qtype}_{L_\infty}$ of tropical bitangent lines which lift over K. Our goal is to provide tools for the proof of Theorem 4.23, stating that many bitangent shapes yield a contribution of $2\mathbb{H}$. Lemma 4.17 is an observation that simplifies computations in general. Lemmas 4.18, 4.19, 4.20 and 4.22 will be needed to pair up contributions to arithmetic multiplicities for tropical bitangents in the same bitangent class for various bitangent shapes in Theorem 4.23, spelled out in more detail in the Appendix A.2, see Theorem A.2. Going through Appendix A.2, one can see that these four Lemmas are not sufficient to cover all cases. They cover a large subset of the cases however, and each of them appears several times (i.e. for many bitangent shapes), which is why we include these statements in the main part of the paper. The remaining cases appear more individually, so we discuss them only in the Appendix after going through the case-by-case analysis of bitangent shapes.

Lemma 4.17. *Assume the initial form of Q at tropical tangency component p_1 equals a monomial m times a form q, i.e. $\mathrm{in}_{p_1}(Q) = m \cdot q$. Let ∂_L be as above. Let P_1 be a lift of p_1. Then $\partial_L(\mathrm{in}_{p_1}(Q))(\mathrm{in}(P_1)) = m \cdot \partial_L(q)(\mathrm{in}(P_1))$.*

Proof. This holds true since the second summand we obtain from the derivative of the product disappears since $q(\mathrm{in}(P_1)) = 0$. The monomial cannot vanish by inserting nonzero values. □

Lemma 4.18. *Let Q be a quartic polynomial over K and assume $\mathrm{Trop}(V(Q))$ is smooth and generic. Assume there is a tropical tangency component in the interior of an untwisted horizonal bounded edge E of $\mathrm{Trop}(V(Q))$. We denote the liftable tangency point by p. Then the lifts of the tropical bitangents at p come in pairs L_1, L_2 such that the tangency points $P \in L_1 \cap V(Q)$ and $\widetilde{P} \in L_2 \cap V(Q)$ which tropicalize to p satisfy*

$$\mathrm{in}(\partial_{L_1}(Q(P)) = -\mathrm{in}(\partial_{L_2}(Q(\widetilde{P})).$$

Proof. By Theorem 4.13, $\mathrm{in}(\partial_{L_1}(Q(P)) = 2\frac{\partial}{\partial y}\mathrm{in}_p(Q)(\mathrm{in}(P))$, and anlogously for L_2, as the tropical tangency component p is on the horizontal ray. The initial form is $\mathrm{in}_p(Q) = a_{1k}xy^k + a_{1k+1}xy^{k+1}$. By Lemma 4.17, after inserting the initials of the solutions (x_0, y_0) of the tangency point we have

$$\mathrm{in}(\partial_{L_i}(Q(P_i)) = 2xy^k\frac{\partial}{\partial y}(a_{1k} + a_{1k+1}y)(x_0, y_0) = 2a_{1k+1}x_0y_0^k.$$

Plugging in the solutions for the tangency points P and $\widetilde{P}$ from Lemma 3.4 we obtain the result. □

The following is the analogue of Lemma 4.18 where the untwisted horizontal bounded edge E is replaced by a diagonal bounded edge. As we broke symmetry by declaring $\{z = 0\}$ to be our infinite line, we cannot expect a similar statement. It is interesting to note that while in Lemma 4.18, the contributions for the two

different lifts were negatives of each other (equal up to sign), here we obtain the same contribution.

Lemma 4.19. *Let Q be a quartic polynomial over K and assume* $\mathrm{Trop}(V(Q))$ *is smooth and generic.*

Assume there is a tropical tangency component in the interior of an untwisted diagonal bounded edge E of $\mathrm{Trop}(V(Q))$*. We denote the liftable tangency point in it by p. As in Lemma 4.18, the lifts of the tropical bitangents at p come in pairs L_1, L_2 but now the tangency points $P \in L_1 \cap V(Q)$ and $\widetilde{P} \in L_2 \cap V(Q)$ which tropicalize to p satisfy*

$$\mathrm{in}(\partial_{L_1}(Q(P)) = \mathrm{in}(\partial_{L_2}(Q(\widetilde{P})).$$

Proof. Using Theorem 4.13 we have to derive with $\frac{\partial}{\partial y} + \frac{1}{n} \cdot \frac{\partial}{\partial x}$, and by Lemma 4.17, after inserting the initials of the solutions (x_0, y_0) of the tangency point we have

$$\partial_L(\mathrm{in}_p(Q))(x_0, y_0) = x^m y^{2-m} \frac{\partial}{\partial y} + \frac{1}{n} \cdot \frac{\partial}{\partial x}(a_{m,3-m}y + a_{m+1,2-m}x) = 2a_{m,3-m}x_0^m y_0^{2-m},$$

as $n = \frac{a_{m+1,2-m}}{a_{m,3-m}}$ by Lemma 3.4. Also, $\frac{y_0}{x_0} = -n$, so we obtain

$$2a_{m,3-m}\Big(- \frac{a_{m+1,2-m}}{a_{m,3-m}}\Big)^{2-m} \cdot x_0^{2m-2}.$$

Up to squares, this equals

$$(-1)^m \cdot 2 \cdot a_{m,3-m}^{m-1} a_{m+1,2-m}^m,$$

no matter whether we insert the positive or the negative root which is the solution for x_0. □

Lemma 4.20. *Assume the horizontal or vertical ray of a tropical bitangent line can move along a bounded edge maintaining tangency, and the tropicalization of the tangency points p_1 and p_2 of the liftable members L_1, L_2 of the corresponding bitangent class are the two end vertices. Then*

$$\mathrm{in}(\partial_{L_1}(Q(P_1)) = -\,\mathrm{in}(\partial_{L_2}(Q(P_2)) \text{ up to squares.}$$

Proof. By symmetry, we can without restriction assume that the ray is horizontal. The only direction vectors of dual edges that fit into a the Newton polygon of a quartic, i.e. the triangle with vertices $(0,0)$, $(0,4)$ and $(4,0)$, and intersect with multiplicity 2 a horizontal edge are $\binom{2}{1}$, $\binom{2}{-1}$ and $\binom{2}{3}$. Because of smoothness, the adjacent vertices correspond to triangles as depicted in Figure 13.

We perform the computation for the left picture in Figure 13, for the others, it is completely analogous. Here, $\mathrm{in}_{p_1}(Q) = x^i y^j \cdot (a + bxy + cx^2y)$ and $\mathrm{in}_{p_2}(Q) = x^i y^j \cdot (a + dx + cx^2y)$. As the horizontal ray cannot meet any other point of the tropicalized quartic, we can conclude that $i = 0$. Using local lifting equations, we

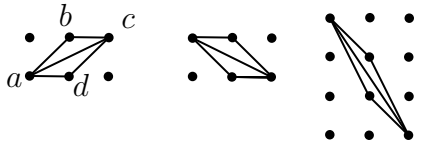

FIGURE 13. Possible duals of bounded edges intersecting a horizontal ray with multiplicity 2.

obtain $\mathrm{in}(P_1) = (-\frac{b}{2c}, \frac{4ac}{b^2})$ and $\mathrm{in}(P_2) = (-\frac{2a}{d}, \frac{d^2}{4ac})$. Using Theorem 4.13 we have to derive with $\frac{\partial}{\partial y}$, and by Lemma 4.17, we obtain

$$\mathrm{in}(\partial_{L_1}(Q(P_1)) = \Big(\frac{4ac}{b^2}\Big)^j \cdot \Big(- b\frac{b}{2c} + c\frac{b^2}{4c^2}\Big) = \Big(\frac{4ac}{b^2}\Big)^j \cdot \Big(- \frac{b^2}{4c}\Big),$$
$$- \mathrm{in}(\partial_{L_2}(Q(P_2)) = \Big(\frac{d^2}{4ac}\Big)^j \cdot \Big(c\frac{4a^2}{d^2}\Big).$$

Up to squares, this equals $\pm a^j c^{j+1}$. □

Remark 4.21. An analogous statement for intersections with the diagonal ray does not hold. (See e.g. Appendix A.3, shape (Ea), where one tropical tangency component contributes a_{21} and the other $-a_{20}a_{31}a_{30}$).

Lemma 4.22. *Assume a liftable tropical bitangent meets a tropicalized quartic at a point p in the interior of an edge of direction $(1,-1)$ dual to $a_{00} + a_{11}xy$ with its vertex. Assume the second tropical tangency component is on the diagonal ray. The lifts come in pairs L_1, L_2 with tangency points P_1 and P_2 tropicalizing to p, and we have*

$$\mathrm{in}(\partial_{L_1}(Q(P_1)) = - \mathrm{in}(\partial_{L_2}(Q(P_2)).$$

Proof. By a local lifting computation, we obtain for the initials of the two lifts P_1 and P_2 the coordinates $(x_0, y_0) = \Big(\mp \frac{1}{2n}\sqrt{-\frac{4a_{00}n}{a_{11}}}, \mp\frac{1}{2}\sqrt{-\frac{4a_{00}n}{a_{11}}}\Big)$, where n as usual denotes the coefficient of x in the normalized equation defining the bitangent lift. Its initial is imposed by the second tropical tangency component which is on the diagonal ray.

Using Theorem 4.13 we have to derive with $\frac{\partial}{\partial y} + \frac{1}{n}\frac{\partial}{\partial x}$, and by Lemma 4.17, we obtain

$$a_{11} \cdot x_0 + \frac{1}{n} a_{11} y_0 = 2a_{11}x_0.$$

Inserting the two solutions for x_0, which are negative of each other, the statement follows. □

Theorem 4.23. *A bitangent class S of a generic tropicalized quartic* $\mathrm{Trop}(C)$ *contributes either $2\mathbb{H}$ to the $\mathbb{A}^1$-enumerative count of bitangents to C, or a sum of four monomials in the initials of the coefficients of the defining polynomial of C.*

In particular, the total $\mathbb{A}^1$-enumerative count of bitangents to C is determined by its tropicalization $\mathrm{Trop}(C)$ *and the square classes of the initials of its coefficients.*

The exceptional cases (which do not give $2\mathbb{H}$) and their GW-multiplicities are listed in the Appendix A.2, details are spelled out in Theorem A.2.

Proof. The strategy of the proof is to use Lemmas 4.18, 4.20 and 4.22 to argue that the four lifts come in pairs contributing $\langle \pm a \rangle$ for some a, which sums up to $2\mathbb{H}$. This works if the lifts exist in the field, else we use Examples 4.6 and 4.15 to show that nevertheless, we obtain $2\mathbb{H}$ as total contribution. The proof is a case-by-case analysis relying on the classification of bitangent shapes in Appendix A.1. Here, we show one case and refer to the detailed version, Theorem A.2 in Appendix A.2, for the remaining ones.

We focus on a bitangent shape of type (D) as in Figure 6 of [6]. The bitangent shape is a bounded segment which partly overlaps with an edge of the tropicalized quartic. The liftable points are the two end vertices of the segment, each such point has lifting multiplicity 2. Thus there are two tropical bitangent lines in the shape which each have two algebraic lifts.

The $\mathbb{S}_3/\mathbb{S}_2$-orbit of this shape consists of three cases (Da), (Db) and (Dc). In (Da), the bounded segment is horizontal, in (Db) it is diagonal and in (Dc) vertical. Figure 14 shows the dual motifs and local pictures of the tropicalized quartic and the tropical tangency components. The pictures contain the two liftable tropical bitangent lines in blue. In each case, the bitangent class is the segment connecting the two vertices of the two blue tropical lines in each picture. Observe that the segment partially overlaps with an edge of the tropicalized quartic, as mentioned before.

For (Da) and (Dc), each of the two liftable tropical bitangents has a tropical tangency component which is an overlap of a horizontal resp. vertical edge. Each liftable tropical bitangent has lifting multiplicity 2, and the two lifts are given by the two choices we obtain for the tangency point on the overlap. The second tangency point is the same for both lifts, for each of the two liftable bitangents. By Lemma 4.18, we can pair up the two lifts for each such tropical bitangent such that their Qtypes are negative of each other. Altogether, we obtain $2\mathbb{H}$.

For (Db) this argument does not work, but a computation shows that the contribution of the other tropical tangency component , which is not in the interior of the diagonal edge, equals a_{21} for one resp. $-a_{21}$ for the second, up to squares: for the upper tropical tangency component p_1, we have $\mathrm{in}_{p_1}(Q) = a_{02}y^2 + a_{12}xy^2 + a_{21}x^2y$. Solving for the initials of the coordinates of the tangency point and the coefficient M of the line equation $y + Mx + N$, we obtain $m = \mathrm{in}(M) = -4\frac{a_{02}a_{21}}{a_{12}^2}$, $x = -2\frac{a_{02}}{a_{12}}$ and $y = 4\frac{a_{02}a_{21}}{a_{12}^2}$. Using Lemma 4.17 and Theorem 4.13, we obtain

$$2\frac{\partial}{\partial y}\,\mathrm{in}_{p_1}(Q(\mathrm{in}(P_1)) = 8\frac{a_{02}a_{21}}{a_{12}^2}\cdot\Big(a_{02} + a_{12}\cdot\Big(-2\frac{a_{02}}{a_{12}}\Big)\Big) = -8\frac{a_{02}^2}{a_{12}^2}\cdot a_{21}.$$

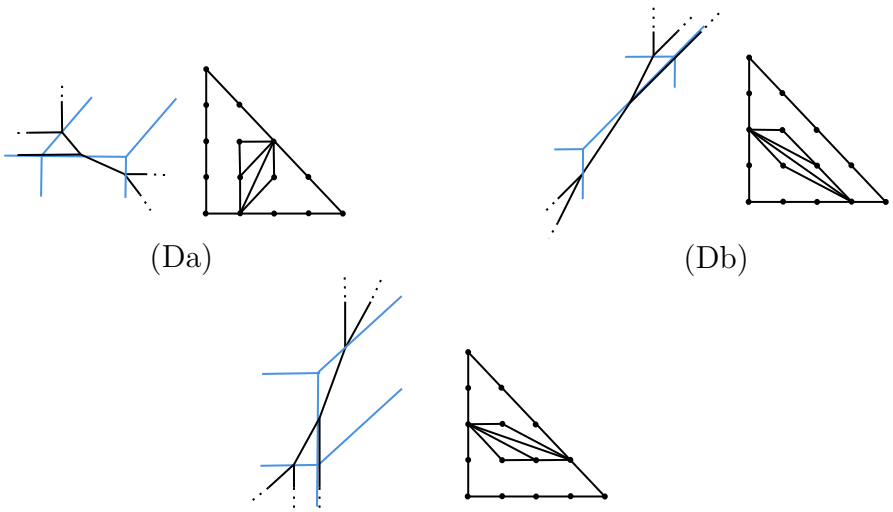

FIGURE 14. Liftable tropical bitangents and motifs for shapes (Da), (Db) and (Dc).

For the lower tropical tangency component p_2, solving for the initials of the coordinates of the tangency point and the ratio $\frac{M}{N}$ of the coefficients of the line equation, we obtain $\frac{m}{n} = \operatorname{in}\frac{M}{N} = -\frac{a_{11}^2}{4a_{02}a_{30}}$, $x = \frac{a_{11}^2}{4a_{02}a_{30}}$ and $y = -\frac{a_{11}^3}{8a_{02}^2a_{30}}$. We have $\operatorname{in}_{p_2}(Q) = a_{02}y^2 + a_{11}xy + a_{30}x^3$ and

$$\begin{aligned}
&2\frac{1}{n}\frac{\partial}{\partial x}\operatorname{in}_{p_2}(Q(\operatorname{in}(P_2)) = 2\frac{1}{m}\cdot\frac{m}{n}\frac{\partial}{\partial x}\operatorname{in}_{p_2}(Q)\\
&= 2\frac{1}{m}\cdot\frac{m}{n}\cdot\Big(a_{11}\cdot\Big(-\frac{a_{11}^3}{8a_{02}^2a_{30}}\Big) + 3a_{30}\cdot\Big(\frac{a_{11}^2}{4a_{02}a_{30}}\Big)^2\Big)\\
&= 2\frac{1}{m}\cdot\frac{m}{n}\cdot\Big(\frac{a_{11}^4}{16a_{02}^2a_{30}}\Big)\\
&= 2\Big(-\frac{a_{12}^2}{4a_{02}a_{21}}\Big)\Big(-\frac{a_{11}^2}{4a_{02}a_{30}}\Big)\Big(\frac{a_{11}^4}{16a_{02}^2a_{30}}\Big)\\
&= 2\frac{1}{16^2}\frac{a_{12}^2a_{11}^6}{a_{21}^2a_{02}^4a_{30}^2}a_{21}.
\end{aligned}$$

Using Lemma 4.19, we can now pair up a lift of one with a lift of the other tropical bitangent and conclude that the total contribution is again $2\mathbb{H}$. □

4.5. **Comparing the** $\mathrm{Qtype}_{L_\infty}$ **for different fields.** Building on our comparison of lifting for different fields in Section 3.3, we can also discuss the comparison of the $\mathrm{Qtype}_{L_\infty}$ for different fields, as a consequence of Theorem 4.23. Let K_1

and K_2 be fields with residue fields k_1 and k_2, respectively. Suppose there exists an isomorphism of groups $\phi : k_1^\times/(k_1^\times)^2 \to k_2^\times/(k_2^\times)^2$ such that $\phi(\overline{-1}) = \overline{-1}$ and $\phi(\overline{2}) = \overline{2}$. Then ϕ induces an isomorphism $\mathrm{GW}(k_1) \sim \mathrm{GW}(k_2)$ by sending $\langle a\rangle$ to $\langle\phi(a)\rangle$, as the Grothendieck-Witt ring is generated by elements of the form $\langle a\rangle$ for $a \in k_i^\times/(k_i^\times)^2$. As $\mathrm{GW}(K_i) \sim \mathrm{GW}(k_i)$ by Theorem 4.7, we also obtain $\mathrm{GW}(K_1) \sim \mathrm{GW}(K_2)$.

Theorem 4.24. *Let K_1 and K_2 be fields with residue fields k_1 and k_2, respectively. Suppose there exists an isomorphism of groups $\phi : k_1^\times/(k_1^\times)^2 \to k_2^\times/(k_2^\times)^2$ such that $\phi(\overline{-1}) = \overline{-1}$ and $\phi(\overline{2}) = \overline{2}$.*

For $i = 1, 2$, let $C_i = V(Q_i)$ be a quartic curve defined over K_i such that $\mathrm{Trop}(C_i)$ is generic. Let $Q_1 = \sum A_{ij}x^iy^jz^{4-i-j}$ and $Q_2 = \sum B_{ij}x^iy^jz^{4-i-j}$. We assume $val(B_{ij}) = val(A_{ij})$, in particular $\mathrm{Trop}(C_1) = \mathrm{Trop}(C_2)$, and $\phi(\overline{a_{ij}}) = \overline{b_{ij}}$ for all i, j, where a_{ij}, b_{ij} are the initials of A_{ij} and B_{ij}, respectively.

Suppose all lifts of Λ are defined over K_1 and K_2 then

$$\sum_{L \text{ lift of } \Lambda} \phi(QType(L)) = \sum_{L' \text{ lift of } \Lambda} QType(L')$$

where the lifts L are over K_1 and the lifts L' are over K_2.

Proof. The formulas for the $\mathrm{Qtype}_{L_\infty}$ of tropical bitangents lifting over k are expressed in terms of a Laurent monomials in the $\{a_{ij}\}$ or the $\{b_{ij}\}$ with coefficients of -1, 2, or squares in the residue fields. Moreover, the Qtypes are defined modulo squares so the statement follows from the assumptions on the $\{a_{ij}\}$ and $\{b_{ij}\}$. □

We present the special case comparing QTypes over finite fields and the real numbers. Let K be a field with finite residue field k of size p^{2l+1} for a prime p satisfying $p \equiv 7 \mod 8$. Let $\phi : k^\times/(k^\times)^2 \to \mathbb{R}^\times/(\mathbb{R}^\times)^2$ denote the unique isomorphism between these two groups. As before, ϕ induces an isomorphism of the Grothendieck-Witt rings.

Corollary 4.25. *Let $C = V(Q)$ be a quartic curve defined over K with $Q = \sum A_{ij}x^iy^j$, and $C' = V(Q')$ be defined over $\mathbb{R}\{\{t\}\}$, where $Q' = \sum B_{ij}x^iy^j$ and $B_{ij} \in \mathbb{R}\{\{t\}\}$. Assume that $val(B_{ij}) = val(A_{ij})$ and*

$$\left(\frac{a_{ij}}{k}\right) = \left(\frac{b_{ij}}{\mathbb{R}}\right).$$

Suppose all lifts of Λ are defined over K and $\mathbb{R}\{\{t\}\}$, then

$$\sum_{L \text{ lift of } \Lambda} \phi(QType(L)) = \sum_{L' \text{ lift of } \Lambda} QType(L')$$

where the lifts L are over K_1, the lifts L' are over $\mathbb{R}\{\{t\}\}$.

Proof. The statement follows immediately from Theorem 4.24, since in a field k of characteristic p with $p \equiv 7 \mod 8$ and with order p^{2l+1} we have $\left(\frac{-1}{k}\right) = \left(\frac{-1}{\mathbb{R}}\right)$ and $\left(\frac{2}{k}\right) = \left(\frac{2}{\mathbb{R}}\right)$. □

5. The real GW-multiplicity in tropical geometry

Let $K = \mathbb{R}\{\{t\}\}$ the field of Puiseux series with real coefficients. By the Tarski principle (see e.g. Chapter 1 in [16]), i.e. elimination of quantifiers in the first order theory of real closed fields, $\mathbb{R}\{\{t\}\}$ is equivalent to the reals, so we can count real bitangents by lifting tropical bitangents to $\mathbb{R}\{\{t\}\}$. This principle has been applied to other problems in tropical geometry, see e.g. [1].

Recall from Example 4.9, that the $\mathrm{Qtype}_{L_\infty}$ of a real bitangent is $\langle 1\rangle$ or $\langle -1\rangle$, and the two cases can be characterized geometrically (see [22]): Assume first that the tangency points are real. In the affine chart of $\mathbb{P}^2$ we obtain by taking out the line at infinity, the quartic $V(Q)$ can pass on the same side of the bitangent line for the two tangency points, or on opposite sides (see Figure 12). If the tangency points are not real, the $\mathrm{Qtype}_{L_\infty}$ is $\langle 1\rangle$.

Larson and Vogt obtained interesting results for counts of real bitangents according to the sign of their $\mathrm{Qtype}_{L_\infty}$: If $V(Q)$ does not meet the line at infinity, then the number of real bitangents with $\mathrm{Qtype}_{L_\infty}$ equal to $\langle 1\rangle$ minus the number of real bitangents with $\mathrm{Qtype}_{L_\infty}$ equal to $\langle -1\rangle$ equals 4 [22]. Based on a randomized search, they formulate the following conjecture:

Conjecture 5.1 ([22, Conjecture 2])**.** *Let $V(Q)$ be a real quartic such that for each bitangent L, $L \cap V(Q) \cap L_\infty = \emptyset$, where L_∞ denotes the line at infinity. Then the number of real bitangents with $\mathrm{Qtype}_{L_\infty}$ equal to $\langle 1\rangle$ minus the number of real bitangents with $\mathrm{Qtype}_{L_\infty}$ equal to $\langle -1\rangle$ is in*

$$\{0, 2, 4, 6, 8\}.$$

Larson and Vogt prove that the numbers are nonnegative.

In the following, we give a partial proof for the conjecture, for the cases of quartics whose tropicalization is smooth and generic. In particular, the quartics we consider all intersect the infinite line in four points. It is interesting to observe that we do not obtain all the possible numbers with this restriction:

Theorem 5.2. *Let $C = V(Q)$ be a real quartic whose tropicalization is smooth and generic. Then the number of real bitangents with $\mathrm{Qtype}_{L_\infty}$ equal to $\langle 1\rangle$ minus the number of real bitangents with $\mathrm{Qtype}_{L_\infty}$ equal to $\langle -1\rangle$ is in*

$$\{0, 2, 4\}.$$

Our approach to prove this theorem is via GW-multiplicities of bitangent classes of the tropicalization. It builds on our classification of generic bitangent shapes and their dual motifs in Appendix A.1, on our study of GW-multiplicities of tropical bitangents in Theorem 4.23, A.2 and Appendix A.3. In the following lemma, we use the letters for the bitangent shapes which are introduced in the classification in Appendix A.1.

Lemma 5.3. *Let $V(Q)$ be a real quartic whose tropicalization is smooth and generic. Let S be a liftable tropical bitangent class of $\mathrm{Trop}(V(Q))$. Let s denote*

the number of lifts of S with $\mathrm{Qtype}_{L_\infty}$ *equal to* $\langle 1\rangle$ *minus the number of lifts of S with* $\mathrm{Qtype}_{L_\infty}$ *equal to* $\langle -1\rangle$.

If S is of shape (Nb), (Ob), (Oc), (Pb), (Qb), (Rb), (Rc), (Sb), (Ub), (Vb), (YaI), (Yb), (CCb) or (IIc), then $s = 2$. If S is of shape (BBb), then $s = 4$.

Otherwise $s = 0$.

Proof. This follows by going through the table in Appendix A.3 and only paying attention to the signs. □

Proof of Theorem 5.2. By checking the dual motifs of a tropical bitangent class of one of the shapes with non-zero contribution in Lemma 5.3 (and their symmetric cases w.r.t. $x-y$-symmetry), we can see that the only ones which are not mutually exclusive are (Qb), (Rb), (Ub), (Vb), resp. (YbII), (BBb), (CCb). For each of those, within a symmetry class, it is always the same edges which contain the tropicalization of the tangency points, thus, it is only the lengths of the edges of the tropical curve which decide which of (Qb), (Rb), (Ub) or (Vb) (resp. which of (YbII), (BBb) or (CCb)) shows up, in particular, they do not show up together either. The only possibility is to combine a shape from (Ub) or (Vb) with another such shape after applying $x - y$-symmetry. Also, the three possibilities for (Nb) do not exclude each other on the level of Newton subdivisions, but the vertex dual to triangle can only align with one of the possible three edges, and so it is precisely one of the types of (Nb) that can occur together. We can see that we can combine at most two shapes that contribute 2, all other shapes with nonzero contribution appear exclusively. As the maximal nonzero contribution from such a shape is 4, we obtain the desired result. □

One does not need to rely on our results on GW-multiplicities of tropical bitangent classes from Theorem 4.23, A.2 and Appendix A.3 to show this result. An alternative approach is to use Viro's patchworking method, and read off the signs of the GW-multiplicities using Figure 12. We illustrate this approach in an example:

Example 5.4. In Viro's combinatorial patchworking [35, 14], we start from a regular subdivision dual to a tropical plane curve C given by a tropical polynomial F. We associate signs to each term of the polynomial, resp. to each region of $\mathbb{R}^2 \setminus C$. These signs represent the signs of the coefficients of a polynomial f over the real Puiseux series tropicalizing to F. Consider the tropical plane curve C inside an orthant (the tropicalization of affine space is $(\mathbb{R} \cup \{-\infty\})^2$) and glue four such orthants as usual, coming with the reflected versions of C. For the reflected copies of C, we add signs to the regions by inserting signs for the x and y-coordinates into the corresponding term of f as given by the respective orthant. Then, we take only those edges of the four copies of C that disconnect a positive and a negative region. Viro's patchworking theorem states that the object we obtain in this way (viewed in the real projective plane which we obtain from the four glued orthants

by compactifying and identifying boundaries accordingly) is homeomorphic to a real plane curve defined by f after inserting a small value for the parameter t.

Figure 15 shows the dual Newton subdivision of a tropicalized quartic, and the four copies of the tropicalized quartic. We pick the signs of all coefficients except a_{03} to be positive, and $s_{03} < 0$. The signs in the four copies are disctributed accordingly. With thick black lines, the patchworked quartic is depicted. By Viro's Theorem, real quartics close to the tropical limit are homeomorphic to this picture. We consider the bitangent class of shape (E) which is also depicted in Figure 6. It has two liftable members. Their patchworked versions are drawn with thick red lines and denoted Λ_1 and Λ_2 in Figure 15. We can see altogether 4 tangency points, $p_1, \ldots, p_4$, shown in green in Figure 15. The two lifts of the tropical bitangent line for Λ_1 are represented by the same thick red line, but they are tangent at different points: one lift, call it L_{11}, is tangent at p_1 and p_3, the other, L_{12}, at p_1 and p_4. Analogously, one lift, L_{21}, of the tropical bitangent for Λ_2 is tangent at p_2 and p_3, the other, L_{22}, at p_2 and p_4. If we move along the thick red line for Λ_1, starting at the top right corner, we meet the first tangency point, p_1, on the right, and the second, p_3, also on the right. Thus, the $\mathrm{Qtype}_{L_\infty}$ of L_{11} is $\langle 1\rangle$, see Figure 12. For L_{12}, we follow the same path, but now we meet p_1 on the right and p_4 on the left, so the $\mathrm{Qtype}_{L_\infty}$ of L_{12} is $\langle -1\rangle$. Analogously, the $\mathrm{Qtype}_{L_\infty}$ for L_{21} is $\langle -1\rangle$ and for L_{22} $\langle 1\rangle$. With this patchworked picture, we thus confirm the result from Lemma 5.3 that the overall contribution of a tropical bitangent class of shape (E) is zero.

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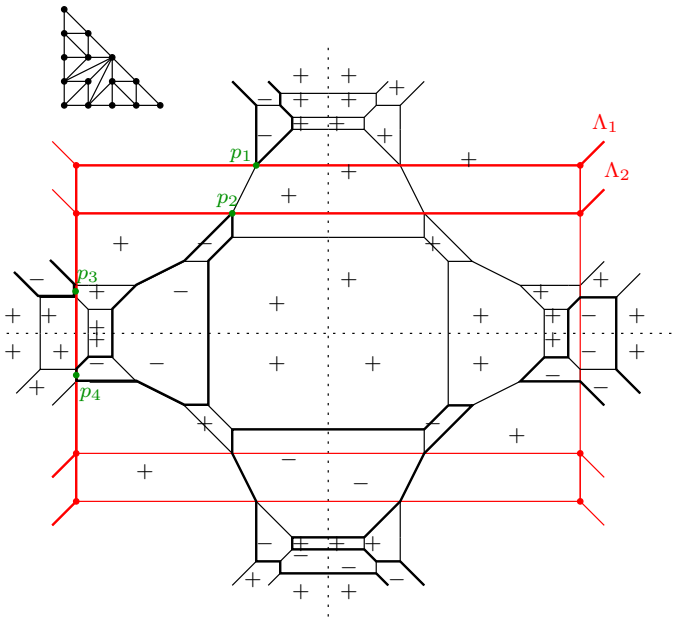

FIGURE 15. A patchworked quartic and the lifts of a tropical bitangent class.

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Appendix A. Classification, computations and table

A.1. **Classification of generic bitangent shapes.** For the classification of bitangent shapes up to $\mathbb{S}_3$-symmetry, see Figure 6 in [6]. There are 41 shapes up to $\mathbb{S}_3$-symmetry, labeled with capital letters such as (A), double capital letters such as (BB) or dashed capital letters such as (T′). For bitangent shapes, we follow the color coding of Figure 6 in [6]: The black cells of each bitangent class miss the tropicalized quartic, whereas the red ones lie on it. The unfilled dots are vertices of the tropicalized quartic.

In our study, we fix the line $\{z = 0\}$ and do not have $\mathbb{S}_3$-symmetry for that reason, only $\mathbb{S}_2$-symmetry for exchanging the variables x and y. We label shapes in an $\mathbb{S}_3/\mathbb{S}_2$-orbit by adding lower case letters such as (Da). Sometimes, the shapes allow different cases of dual motifs in the dual Newton subdivision. These different cases play a role for the computation of Qtypes. We label such different cases of dual motifs by adding Roman numbers such as (TaI).

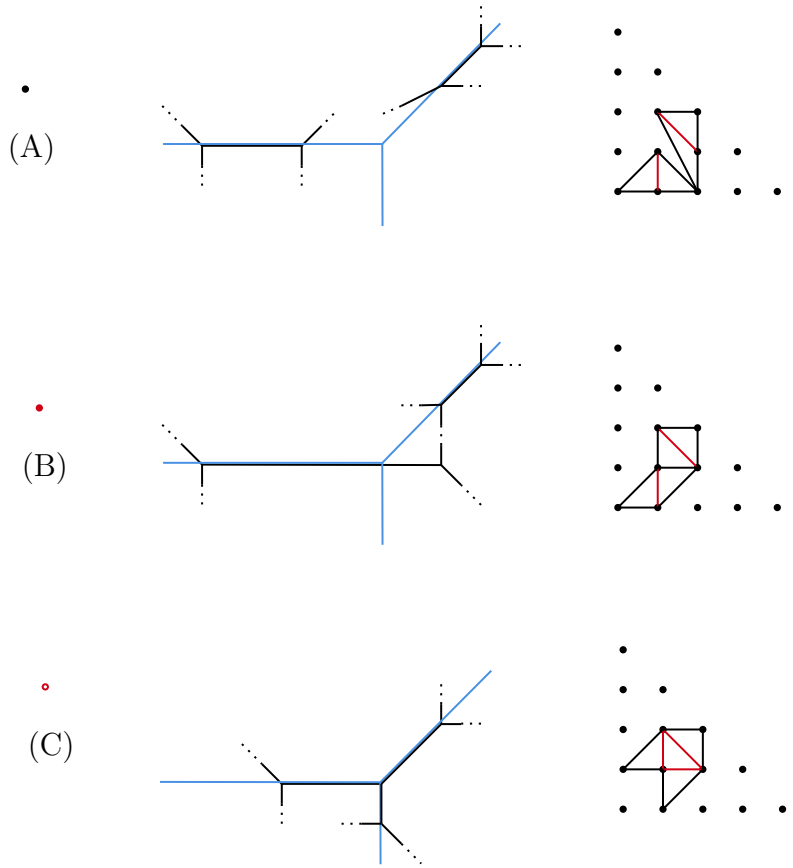

FIGURE 16. Bitangent shapes, local pictures and motifs of type (A), (B) and (C).

We consider only generic bitangent shapes, i.e. bitangent shapes which appear in the open cones of the subdivided secondary fan, where the subdivision is chosen such that the bitangent shapes are constant in each cell (see Remark 2.17). This subdivision has been computed in [11].

Lemma A.1. *Let S be a tropical bitangent class of a generic tropicalized quartic. Then, from the classification of possible shapes for S in* [6]*, Figure 6, S can have the shapes (A), (B), (C), (D), (E), (F), (G), (H), (N), (O), (P), (T), (S), (R), (Q), (U), (V), (W), (Y), (BB), (CC), (EE), or (II).*

Proof. This follows since the existence of a tropical bitangent class of one of the remaining shapes ((H′), (I), (J), (K), (L′), (L), (M), (T′), (T″), (U′), (Q′), (X), (Z), (AA), (DD), (FF), (GG), (HH)) requires some equalities on the lengths of the edges of the tropicalized quartic to hold, hence they appear only in lower-dimenional cones of our subdivision of the secondary fan. □

Figure 16 shows the three zero-dimensional bitangent shapes (A), (B) and (C). Next to the shape, we sketch the tropical bitangent in blue and parts of the tropicalized quartic which carry the tropicalization of the tangency points. Next, we draw an exemplary dual motif. The red edges are dual to the edges that overlap with the tropicalized quartic. The red edges could also be shifted for case (A) and (B). In case (A) and (B), the vertices on the vertical and diagonal boundary edge of the Newton triangle could also be shifted. In case (C), all vertices on the boundary could be shifted. We refrain from drawing the $\mathbb{S}_3/\mathbb{S}_2$-orbits of these bitangent shapes, as the arguments for the computation for the Qtypes do not change. Note that each such shape has a $\mathbb{S}_3/\mathbb{S}_2$-orbit of size two, as the tropicalization of the tangency points can be on the horizontal and vertical edge, or on the diagonal and horizontal edge.

Figure 14 shows bitangent shape (Da). Shape (Db) and (Dc) are diagonal resp. vertical. Figure 14 depicts the liftable tropical bitangents together with the local parts of the tropicalized quartic carrying the tropicalization of the tangency points, and the dual motifs.

Bitangent shape (E) is a segment which does not meet the tropicalized quartic, i.e. in Figure 6 [6] it is drawn completely in black. The two end points of the segment are the liftable tropical bitangents. Figure 17 shows the behaviour of the tropicalization of the tangency points (i.e. local parts of the tropicalized quartic) together with the two liftable tropical bitangents in blue, and the partial dual motifs for the shapes (Ea), (Eb) and (Ec) which we obtain as the $\mathbb{S}_3/\mathbb{S}_2$-orbit of (E). In the dual pictures, the red edge could also be shifted (upwards for (Ea) and (Eb), antidiagonally for (Ec)), as could the vertex which forms the left resp. upper triangle with the red edge.

Bitangent shape (F) is a segment which touches the tropicalized quartic, i.e. in the color coding we draw one vertex red. Again, the two end points of the segment are the liftable tropical bitangents. Figure 18 shows the behaviour of the tropicalization of the tangency points together with the two liftable tropical bitangents in blue, and the partial dual motifs for the shapes (Fa), (Fb) and (Fc) which are the $\mathbb{S}_3/\mathbb{S}_2$-orbit of (F). In the dual pictures, the red edge could also be shifted (upwards for (Fa) and (Fb), antidiagonally for (Fc)), as could the vertex which forms the left resp. upper triangle with the red edge.

Bitangent shape (G) is a segment which is completely contained in the tropicalized quartic. One tropical tangency is an overlap of an edge with the ray spanned by the segment. The two liftable members are the two end points of the segment. As we deform one liftable tropical bitangent to the other, the second tropical tangency component wanders over an edge which can have three possible slopes. We differentiate those three cases by using Roman letters (I), (II) and (III). Figure 19 combines the three possibilities (I), (II), (III) for the three $\mathbb{S}_3/\mathbb{S}_2$-orbits (Ga), (Gb) and (Gc) in one picture each, both for the local pictures of the tropicalization of the tangency points and for the partial dual motifs.

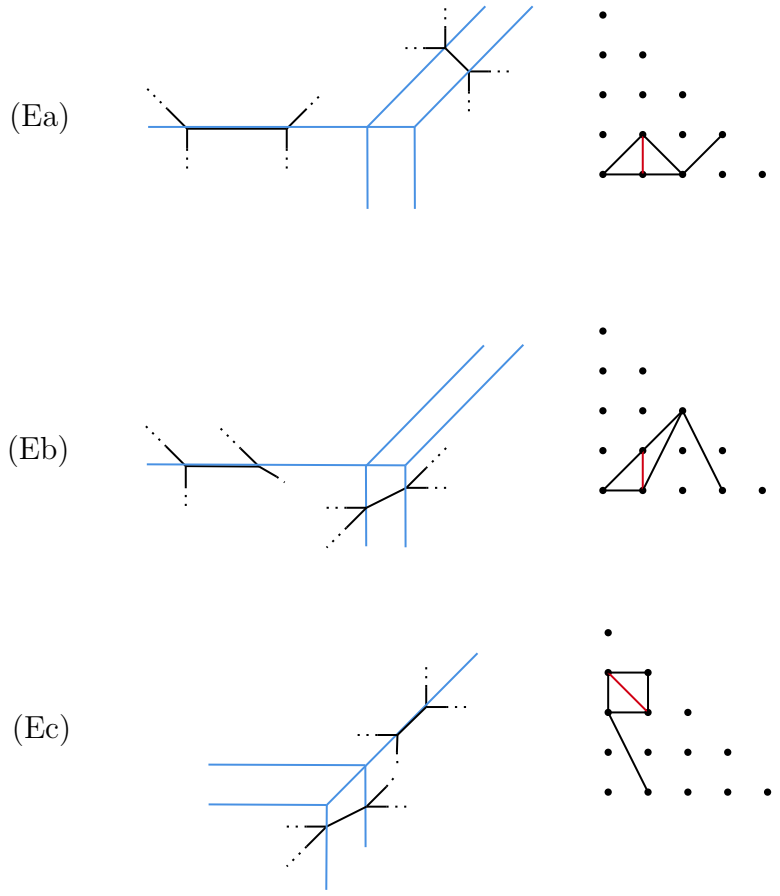

FIGURE 17. Local pictures of tropicalizations of tangency points and partial motifs for type (Ea), (Eb) and (Ec).

Bitangent shape (H) is a ray whose vertex corresponds to the only liftable tropical bitangent in the class. Figure 20 shows the local pictures of the tropicalization of the tangency points and the partial dual motifs for the two cases (Ha) and (Hb). The red edge could be shifted upwards resp. antidiagonally, as could the vertex of its adjacent right resp. upper triangle.

Bitangent shape (N) looks like a reversed tropical line with two rays cut off. The end points of the two cut rays are the two liftable members. As for shape (G) (see Figure 19), one tropical tangency is an overlap with an edge (although the two overlapping edges differ for the two liftable tropical bitangents, different from the situation in (G)), and one tropical tangency component wanders over an edge — which can have three different possible slopes (I), (II) and (III) — as we deform one liftable tropical tangency component to the other. Figure 21 shows local pictures of the tropicalization of the tangency points and partial dual motifs,

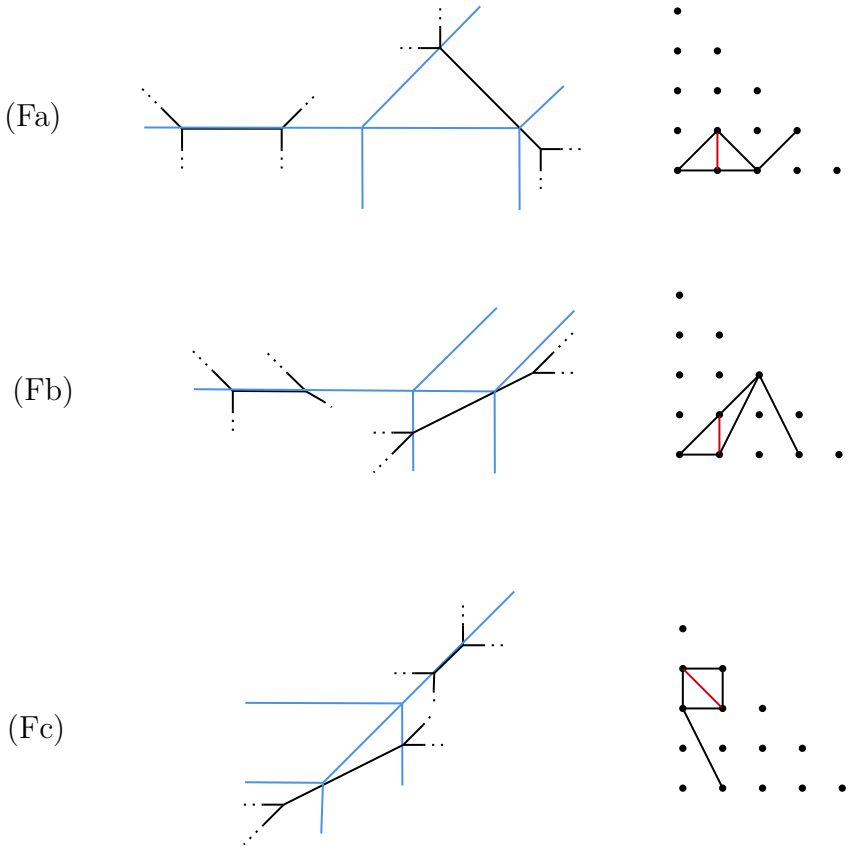

FIGURE 18. Local pictures of tropicalization of the tangency points and partial dual motifs for type (Fa), (Fb) and (Fc).

and combines the three options (I), (II) and (III) in one picture each for the two $\mathbb{S}_3/\mathbb{S}_2$-orbits (Na) and (Nb).

Shape (O) has two liftable tropical bitangents, the local pictures and dual motifs for the three symmetric cases (Oa), (Ob) and (Oc) are depicted in 22.

Shape (P) has two liftable tropical bitangents. Figure 23 shows local pictures of the tropical tangency components and dual motifs for the two symmetric cases (Pa) and (Pb).

Shape (Q) has two liftable tropical bitangents. Figure 24 shows the local pictures of the tropical tangency components and the dual motifs for the three symmetric cases (Qa), (Qb) and (Qc).

Figure 25 shows the local tropical tangency components and dual motifs for the three symmetric cases (Ra), (Rb) and (Rc) that we have for shape (R). Each has two liftable tropical bitangents.

In shape (S), there are two liftable tropical bitangents, shown in Figure 26 together with the dual motifs for the two symmetric cases (Sa) and (Sb).

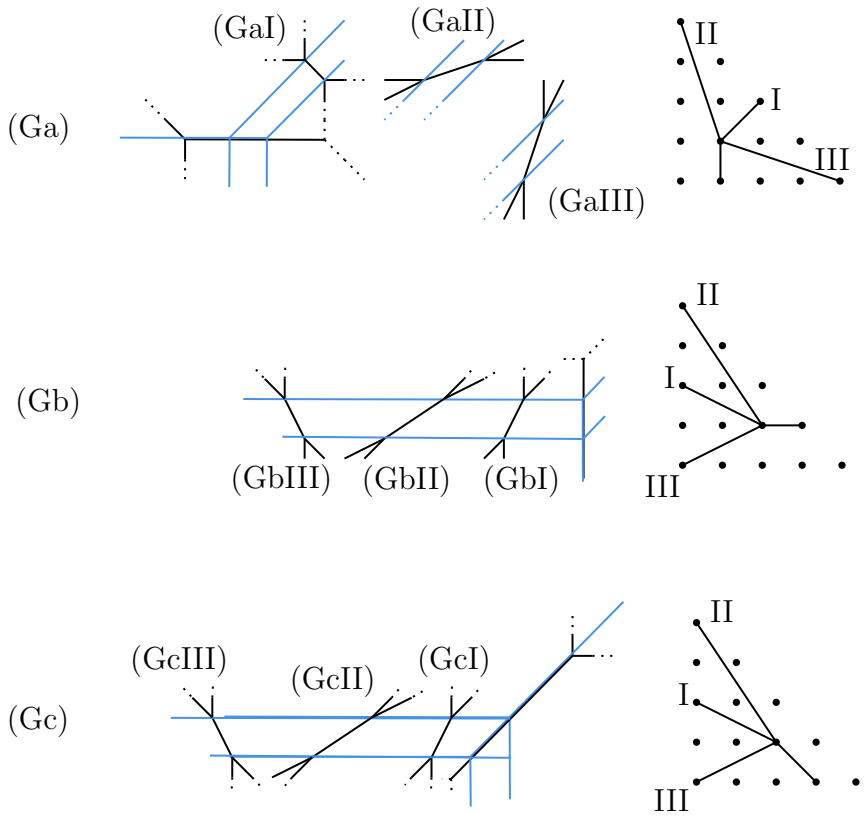

FIGURE 19. Local pictures of tropical tangency components and partial dual motifs for types (GaI), (GaII), (GaIII), (GbI), (GbII), (GbIII), (GcI), (GcII) and (GcIII).

Shape (T) is an unbounded 2-dimensional cell show two vertices each lift with multiplicity two. As for shape (G) and (N), there are three possibilities for the slope of the edge along which one of the tropical tangency component wanders, which we denote by (I), (II) and (III). Figure 27 combines these possibilities into one picture for (Ta) and one for (Tb), which are the two orbits we have to take into account.

Similar to shape (G), (N) and (T), shape (U) comes with three different possibilities for slopes for the wandering tropical tangency component, see Figure 28.

The behaviour of the tropicalization of the tangency points for shape (V) is the same as for shape (N), it is only the possible deformations that differ, but this has no effect on Qtypes, see Figure 29.

We now consider the bitangent shapes (W), (Y), (BB), (CC) and (EE) which each have four liftable members. For some of these shapes, tangencies can arise due to edges for which different slopes are possible. We label these different slopes

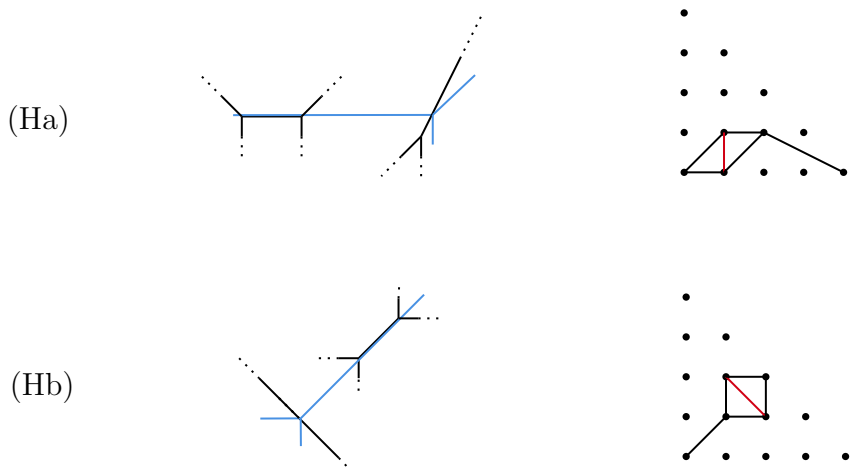

FIGURE 20. Local pictures for the tropical tangency components and partial dual motifs for shapes (Ha), (Hb).

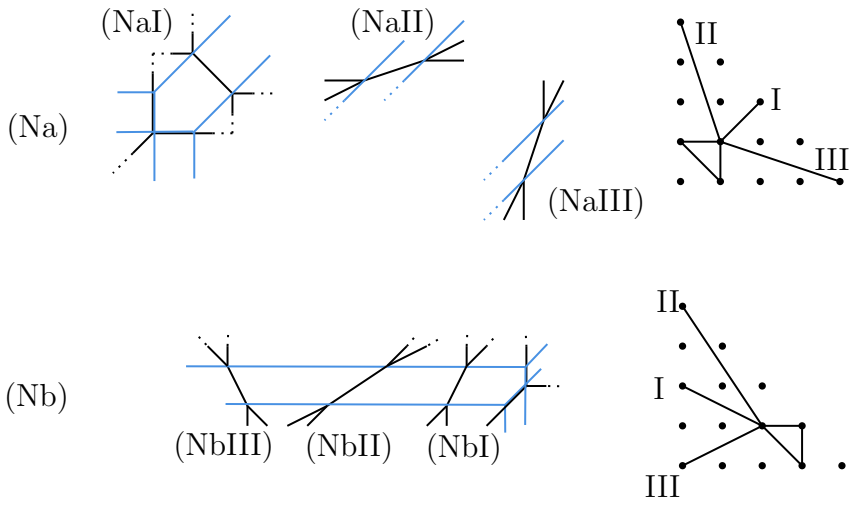

FIGURE 21. Local pictures of tropical tangency components and partial dual motifs for types (NaI), (NaII), (NaIII), (NbI), (NbII), (NbIII).

again using Roman numbers (I), (II), (III). Bitangent shape (W) is a parallelogram whose four vertices are the liftable members. The other shapes can be viewed as deformations of the parallelogram, where edges of the tropicalized quartic "cut off" pieces. Figure 30 shows the local pieces of the tropicalized quartic together with the four liftable tropical bitangents in blue for the cases (WaI) and (WbI). Next to it, the partial dual motifs are depicted. For (WaII), (WaIII), (WbIII) and

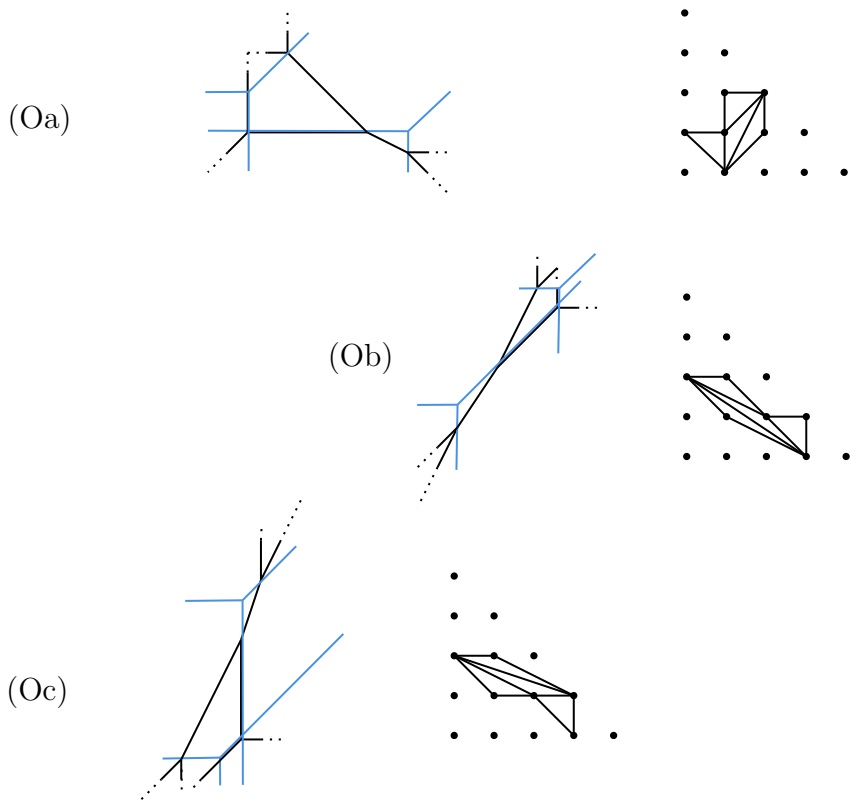

FIGURE 22. Local pictures for tropical tangency components and dual motifs for types (Oa), (Ob) and (Oc).

(WcII), the partial dual motifs are depicted in Figure 31. As the local pictures of the tropical tangency components and liftable tropical bitangents are similar to the ones shown in Figure 30, we do not include those.

Concerning shape (Y), the dual motif for (YaI) equals the one for (WbII), (YbI) equals (WcII), (YcI) equals (WaII), (YaII) and (YaIII) equals (WaIII), and (YbII) equals (WbIII). The shape (Y) only differs from (W) by "cutting off" a vertex of the parallelogram — one liftable tropical bitangent now has its vertex on an edge of the tropicalized quartic. Pictures of the partial dual motifs can thus be found in Figure 31. Figure 32 depicts the 4 liftable tropical bitangents in each case together with a local picture of the tropicalized quartic.

Shape (BB) differs from shape (W) by cutting off two vertices of the parallelogram. The dual motif for (BBa) equals the one for (WaIII), the one for (BBb) equals (WbIII). Again, we refer to Figure 31 for these partial dual motifs. The 4 liftable tropical bitangents together with local pictures of the tropicalized quartic are found in Figure 33.

In shape (CC), we also cut off two vertices, but we cut one of those off even further. The dual motif for (CCa) equals (WaIII), the one for (CCb) (WbIII), see

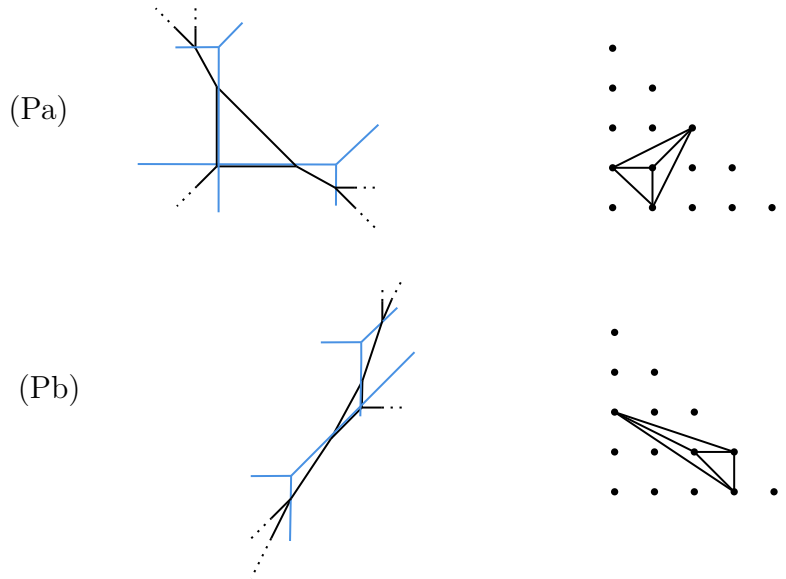

FIGURE 23. Local pictures of the tropical tangency components and dual motifs for shapes (Pa) and (Pb).

Figure 31. The corresponding local pictures of the tangencies are found in Figure 34. (CCaI) and (CCaII) differ by which of the two vertices is cut off further. For (CCb), this is symmetric.

In shape (EE), we only cut off one vertex, but we cut this one off even further again. The dual motif for (EEaI) and (EEaII) equals the one for (WaIII) (those two cases only differ by the choice which vertex of the parallelogram we cut off), the dual motif for (EEaIII) equals the one for (WaII), (EEbI) equals (WbIII), (EEbII) equals (WbII) and (EEcI) equals (WcII), see Figure 31. The corresponding local pictures of the tangencies appear in Figure 35.

Finally, Figure 36 shows the dual motifs for the bitangent shapes (IIa), (IIb) and (IIc). For (IIa), there is also a local picture of the tropicalized quartic with the liftable tropical bitangent lines. The two left ones have lifting multiplicity one, the right one lifting multiplicity two. The local pictures for (IIb) and (IIc) are similar. For the two left tropical bitangents, the two tangency points tropicalize to the same point, namely a vertex of the tropicalized quartic, where the tropical line and the tropicalized quartic meet with intersection multiplicity four.

A.2. **Details on Theorem 4.23 and its proof including the case-by-case analysis of bitangent shapes.** The following Theorem spells out the details of the case-by-case analysis of bitangent shapes for Theorem 4.23:

Theorem A.2. *If a tropical bitangent class is of shape (A), (B), (C), (D), (E), (F), (G), (H), (Na), (Oa), (Pa), (Qa), (Qc), (Ra), (Sa), (IIa), (IIb), (T), (Ua),*

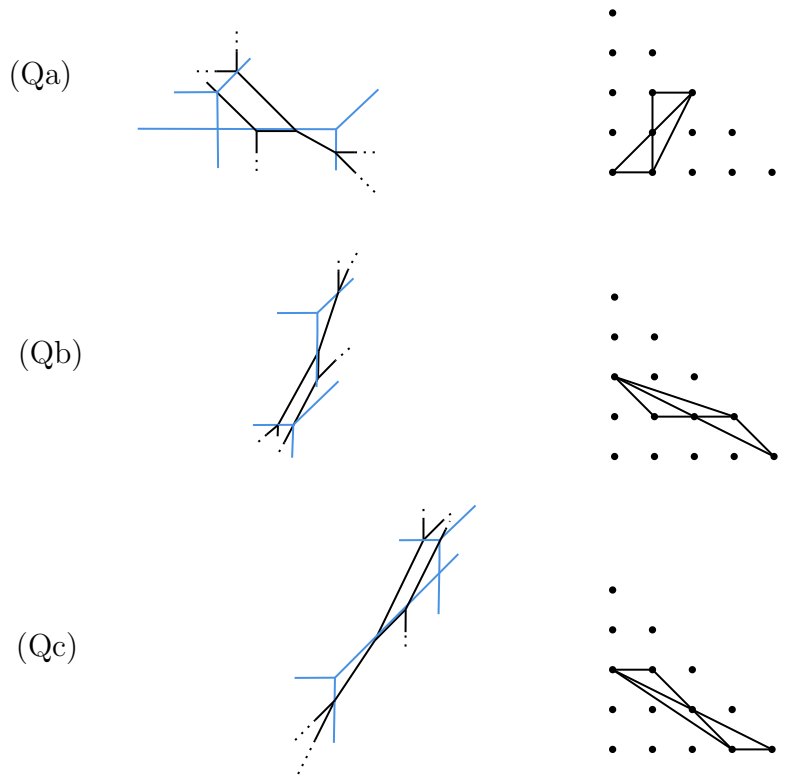

FIGURE 24. Local pictures of tropical tangency components and dual motifs for shapes (Qa), (Qb) and (Qc).

(Uc), (Va), (W), (YcI), (YaII), (CCaII), or (EE), then the total contribution of its $\mathrm{Qtype}_{L_\infty}$ *is* $2\mathbb{H}$.

Proof. The strategy of the proof is to use Lemmas 4.18, 4.20 and 4.22 to argue that the four lifts come in pairs contribution $\langle \pm a \rangle$ for some a, which sums up to $2\mathbb{H}$. This works if the lifts exist in the field, else we use Remark 4.14 and Example 4.15 to show that nevertheless, we obtain $2\mathbb{H}$ as total contribution.

If it is of shape (A) or (B), there are two tropical tangency components in the interior of edges, of which at least one has to be vertical or horizontal. Using Lemma 4.18, we can pair up the lifts in such a way that in total we obtain $2\mathbb{H}$. The argument is analogous for shape (C), as locally the tropical tangency components are in the interior of edges here, too. Shape (D) was already discussed in detail in the proof of Theorem 4.23.

For (E),(G), one of the tropical tangency components has to be on the horizontal or vertical ray, and we can use Lemma 4.18 resp. Lemma 4.20, depending on which tropical tangency component it is, to pair up.

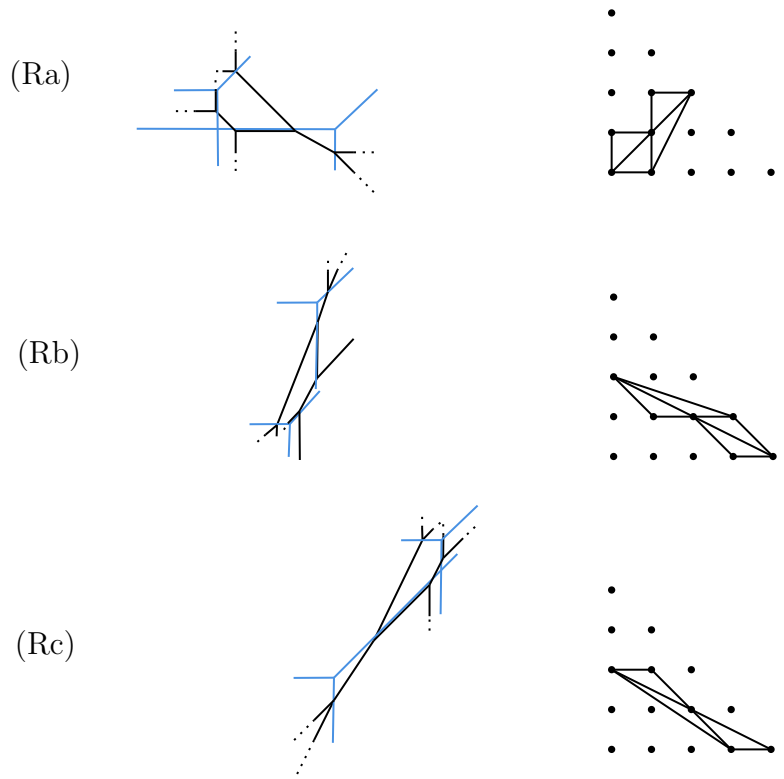

FIGURE 25. Local tangencies and dual motifs for shapes (Ra), (Rb) and (Rc).

For (Fa) and (Fb), we have a tropical tangency component in the interior of a horizontal or vertical edge, and we can use Lemma 4.18 to pair up. For (Fc), we have two liftable tropical bitangents. Both have a segment of intersection with $\mathrm{Trop}(Q)$ on the diagonal ray, so by Lemma 4.19, we get the same contribution. The second tropical tangency component differs: for the first, it is, as in (Ec), an intersection with the horizontal ray, for the second, it is an intersection with the vertex. A computation shows that the contributions are negative of each other, so we can pair up and obtain $2\mathbb{H}$.

For (Ha), one tropical tangency component is in the interior of a horizontal edge. For (Hb), we can pair up using Lemma 4.22 and Lemma 4.19.

For (Na), (Oa), (Pa), (Ra), (Sa) we can pair up using the tropical tangency components on the horizontal resp. vertical ray. For (Qa), for one liftable point we can use the tropical tangency component on the horizontal edge to pair up, for the other, the tropical tangency component at the vertex of the tropical line as in Lemma 4.22. For (Qc), a computation shows that we obtain $\langle 1\rangle + \langle 1\rangle + \langle -1\rangle + \langle -1\rangle = 2\mathbb{H}$. For (IIa) and (IIb), we can use a computation to show that

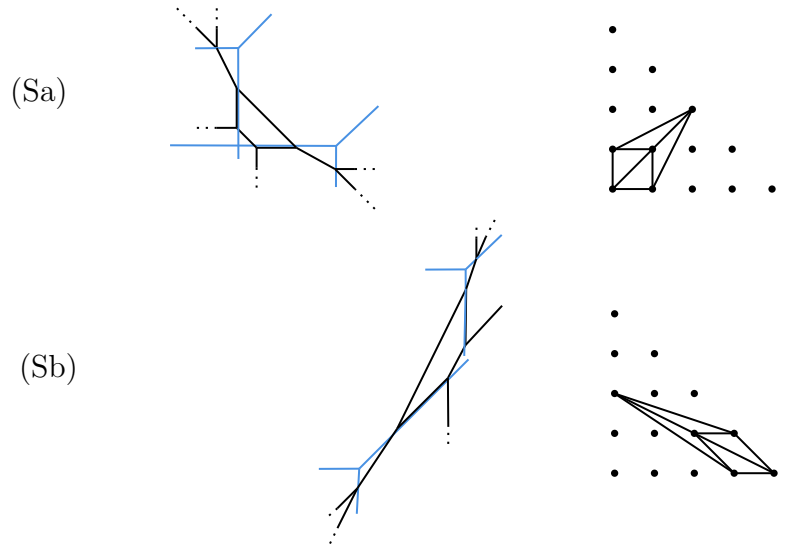

FIGURE 26. Local tangencies of liftable tropical bitangents and dual motifs for shapes (Sa) and (Sb).

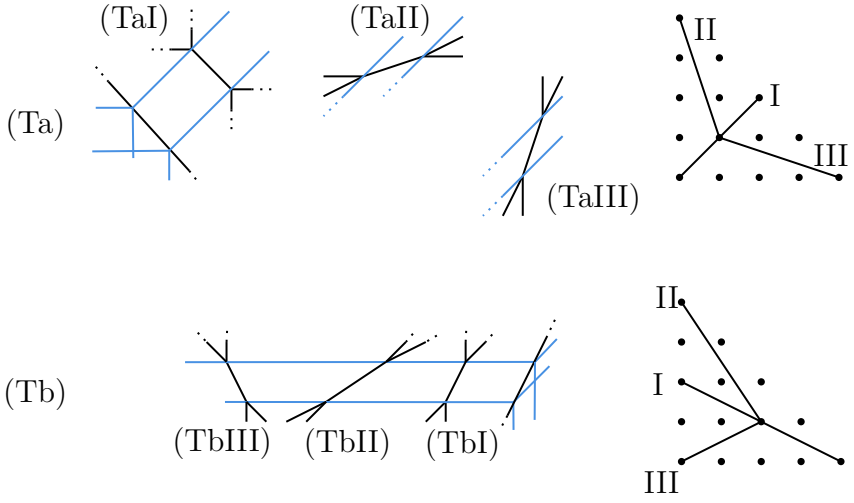

FIGURE 27. Local pictures of tropical tangency components and partial dual motifs for shapes (TaI0, (TaII), (TaIII), (TbI), (TbII) and (TbIII).

we obtain $2\mathbb{H}$. For (Ta), Lemma 4.22 shows that we can pair up. For (Tb), we can use Lemma 4.20 to pair up. For (Ua), we can use Lemma 4.22 and Lemma 4.18 to pair up.

For (Uc), a computation shows that the lifts of the right tropical tangency components yield a contribution of +2 for the lower liftable tropical bitangent,

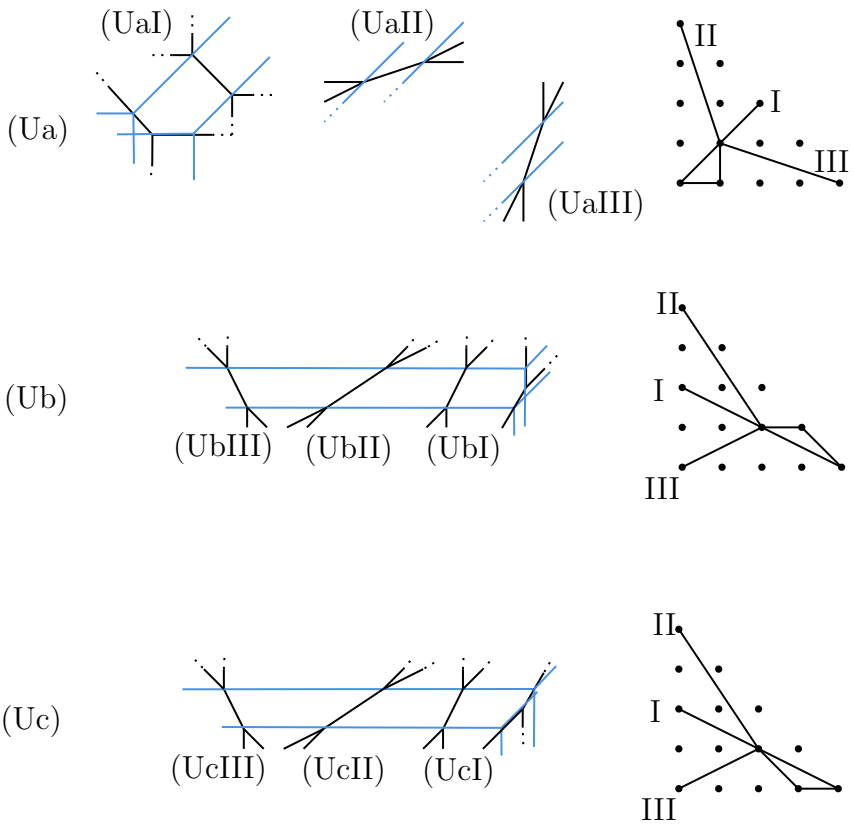

FIGURE 28. Local pictures of tropical tangency components and partial dual motifs for (UaI), (UaII), (UaIII), (UbI), (UbII), (UbIII), (UcI0, (UcII) and (UcIII).

and -2 for the upper liftable tropical bitangent. Combining with Lemma 4.20, we can pair up.

For (Va), we can use the tropical tangency components on the horizontal and vertical edge to pair up, using Lemma 4.18. For (W), one of the tropical tangency component has to be on a horizontal or vertical ray, and thus we can use Lemma 4.20 to pair up. For (EEaII), (EEbI), (EEbII), (EEcI, and (EEcII) we can use Lemma 4.20 to pair up. For (YcI), (YaII), (CCaII), (EEaI) and (EEaIII), we can pair up two liftable tropical bitangents using Lemma 4.20. For the remaining two, a computation shows that we can again lift up, similar to (Fc).

□

Following this classification we can prove the version of this Theorem stated in the introduction:

Proof of Theorem 1.7. The compact bitangent shapes are (A), (B), (C), (D), (E), (F), (G) and (W). These all yield $2\mathbb{H}$ as $\mathrm{Qtype}_{L_\infty}$ by Theorem A.2. □

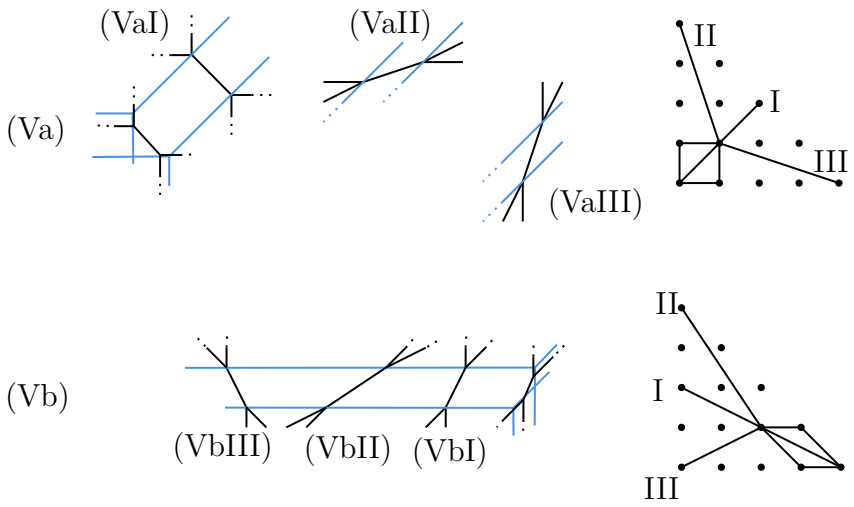

FIGURE 29. Local pictures of tropical tangency components and partial dual motifs for shapes (VaI0, (VaII), (VaIII), (VbI), (VbII) and (VbIII).

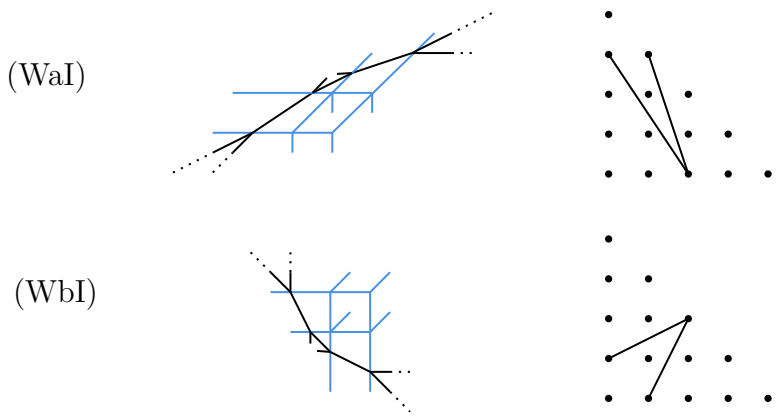

FIGURE 30. Local pictures of tropical tangency components, liftable tropical bitangents and partial dual motifs for shapes (WaI) and (WbI).

A.3. **Exceptional GW-multiplicities.** Here we discuss the GW-multiplicities of the special bitangent shapes which do not yield $2\mathbb{H}$ by Theorem 4.23 resp. A.2.

The shapes (Nb), (Ob), (Oc), (Pb), (Qb), (Rb), (Rc), (Sb), (Ub), (Vb), (IIc), (YbI), (YbII) and (CCb) all have similar behaviour of the tropical tangency components as described in the classification in Appendix A.1. The GW-multiplicity of a bitangent class of one of these shapes equals

$$\langle 1\rangle + \langle 1\rangle + \mathbb{H}.$$

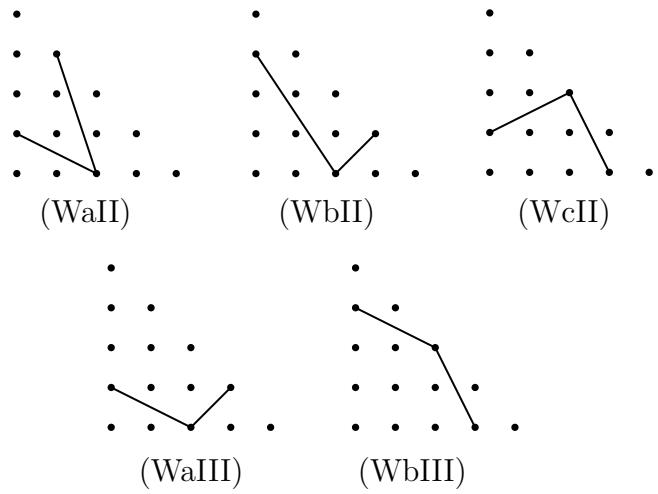

FIGURE 31. Partial dual motifs for the shapes (WaII), (WbII), (WcII), (WaIII), (WBIII).

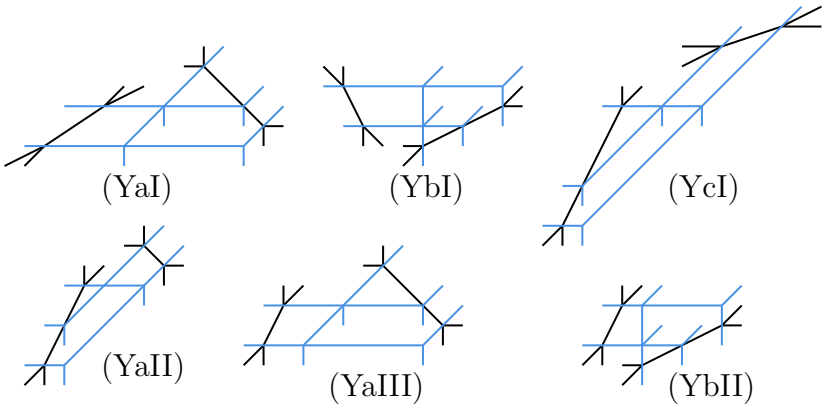

FIGURE 32. Local pictures of liftable tropical bitangent lines for (YaI), (YbI), (YcI), (YaII), (YaIII), (YbII).

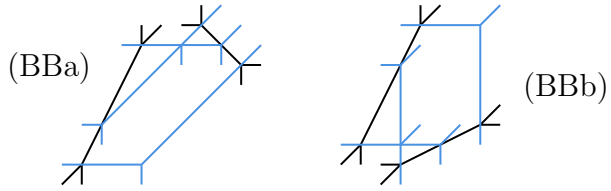

FIGURE 33. Local pictures of liftable tropical bitangent lines for (BBa) and (BBb).

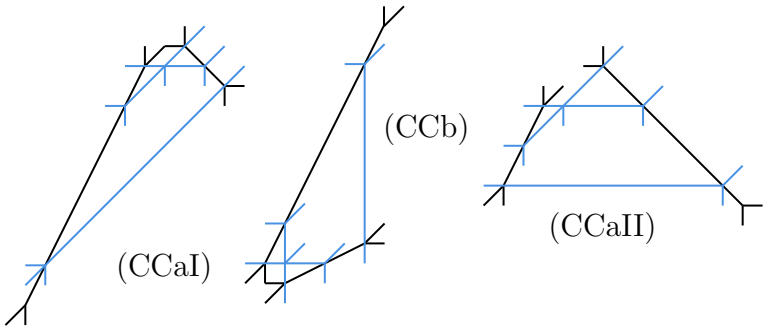

FIGURE 34. Local pictures of liftable tropical bitangent lines for (CCaI), (CCaII), (CCb).

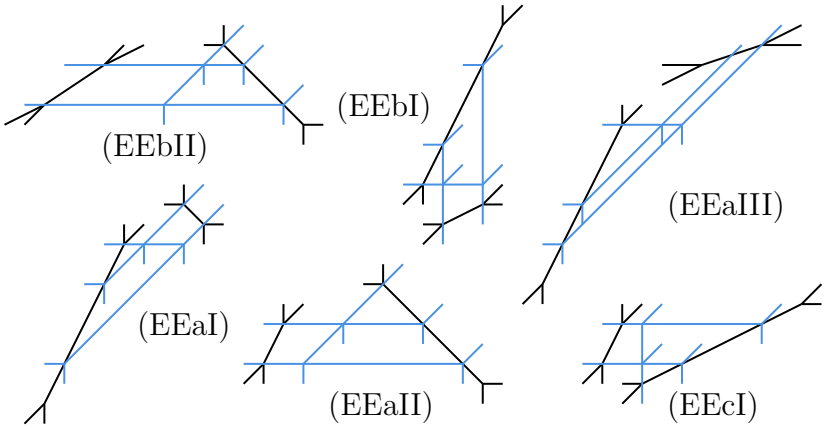

FIGURE 35. Local pictures of liftable tropical bitangent lines for (EEaI), (EEaII), (EEaIII), (EEbI), (EEbII), (EEcI).

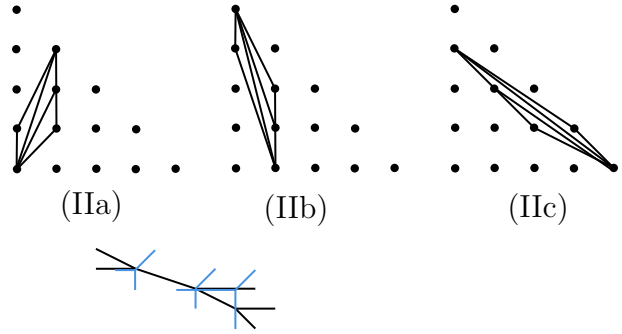

FIGURE 36. Dual Motifs for (IIa), (IIb) and (IIc). For (IIa), a local picture involving the three liftable tropical bitangent lines and their tropical tangency components.

The following table shows the GW-multiplicities of the remaining exceptional cases.

bitangent				
(YaI)	$\langle a_{20}a_{31}a_{30}a_{03}\rangle$	$\langle 2\rangle$	$\mathbb{H}$	
(YaIII)	$\langle -a_{01}a_{20}a_{31}a_{30}\rangle$	$\langle 2\rangle$	$\mathbb{H}$	
(BBa)	$\langle -a_{01}a_{20}a_{31}a_{30}\rangle$	$\langle 2\rangle$	$\mathbb{H}$	
(BBb)	$\langle 1\rangle$	$\langle 1\rangle$	$\langle 1\rangle$	$\langle 1\rangle$
(CCaI)	$\langle -2a_{20}a_{31}a_{30}a_{01}\rangle$	$\langle 2\rangle$	$\mathbb{H}$	

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BITANGENTS TO PLANE QUARTICS VIA TROPICAL GEOMETRY — ANSWERS TO REFEREES

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We thank the referees for the thorough reading of our paper and the many useful comments. In the following, we explain the changes we made.

Review 1:

(1) We have tried to go through these cases and make it more explicit by adding references or comments about the missing proofs. For 3.3, we added a comment, for 3.4 a comment and a reference, for 3.16 and 3.17 a comment. We hope that this is clearer now.
(2) We have included the Figure.
(3) Yes, we meant liftable bitangents. We added this with a reference to the definition of liftable now.
(4) Done.
(5) We reformulated it and hope it is clear now.
(6) Yes, we added it.
(7) Yes, we added Newton.
(8) Done.
(9) Changed.
(10) Yes, that is essentially right. We have added a short explanation about it.
(11) By constant coefficient we were referring to M in the equation $M + Nx + y$ of the line. We have removed the word constant now, hoping that it is clearer. There is no difference between h.o.t. and three dots. We have changed to three dots now throughout.
(12) Done.
(13) No, the edge can also be bounded. Hopefully this will also be clearer after we inserted the example of shape (D).
(14) That is corrected, we also added an explanation what belongs to what in the caption of the Figure.
(15) Yes, that is correct, we added this.
(16) We wanted to say that in one case we need an edge to be twisted in order for the tangency to lift, and in the other non-twisted. We reformulated this now, hoping that it is clearer.
(17) This happens for many shapes, we now mentioned (D) as an example in the paragraph before Def 3.9 where it comes up for the first time.
(18) Very true, thanks for catching this and sorry about it. We have changed it here and the other places it appears.
(19) Done.
(20) Inserted "isomorphisms" in front of the maps $k_i^\times/(k_i^\times)^2 \to \mathbb{Z}/2\mathbb{Z}$.
(21) We added a reference.
(22) Done.
(23) Done.
(24) This phrase was a left over referring to the Corollary 4.25. We have deleted it.

(25) Yes, thank you. It is changed.
(26) Done.
(27) Done.
(28) Yes, that is what we meant. We added it.
(29) Larson and Vogt had examples, we believe, they say they base their conjecture on a randomized search of over 10000 quartics. We have no interpretation of the fact that we do not get 6 or 8 for tropically smooth quartics. This is one of the intriguing questions that is still open.
(30) Done. Thanks for catching this.
(31) This is a very interesting question indeed. We are thinking about possible generalizations, but for anything beyond quartics certainly other techniques will have to be developed. That is to say, our techniques should form a subset of the techniques used in other situations. The thing is that the combinatorics of the tropical curves have more possibilities, also for local parts carrying a tangency. What could be done is to use our techniques for a subset of tropical curves with restricted combinatorial features — roughly, if every polygon that shows up in the Newton subdivision can also show up for a quartic, our techniques should be fine. Admittedly, this answer is kind of vague, but we would need more time to come up with a better answer.

Review 2:

Concerning the global view:

(1) We added a discussion at the beginning of Sec 3.2 and at the end of Sec 3.2. We hope that this makes the question of the tangency components and their lifting clearer.
(2) We included a discussion between Remark 4.16 and Lemma 4.17 which hopefully makes the more global perspective clearer. Because of the exceptional cases that do not give $2\mathbb{H}$ however, we do not see any other route except going through the classification to give all arguments why we "often" get $2\mathbb{H}$. It is also not obvious how to characterize the exceptional cases. We have thought about this, but did not reach a satisfying conclusion. Therefore, we still think that providing a test case in the main body and going through all cases in the appendix is the best solution for the presentation, even though it is admittedly difficult to go through this.

Other comments:

(1) Done.
(2) Done.
(3) Admittedly, that was strange. We changed it.
(4) Done.
(5) We have added an explanation.
(6) Done.
(7) Done.
(8) Done.
(9) Done.
(10) That is correct, we mean the liftable tangency point within the tangency component. We have added this now, also to 4.18 and 4.19.
(11) Done.

(12) This is not the general case for lifting bitangents, it is only a case that appears multiple times, whenever there is an overlap. We hope that the overall exposition makes this clearer now.
(13) Done.
(14) We have explained what b_s is and also added explanation to the choice of the labelings for the sides. We have also added an explanation concerning the application of Lemma 3.4: what is behind this is the normalization for the coefficients of the line we picked in 3.4. There, the y-coefficient was 1, and so we can express the coefficient M for which we solve also as a quotient $\frac{b_s}{b_{s'}}$ as in the notation here. Lemma 3.4 then shows that the corresponding quotient for Q must be the same. We hope that this is clearer now with our reformulation.
(15) Done.
(16) Done.
(17) We added the missing explanation that the sign pairs ε_i^L ε_i^C with the same subscripts are meant to label branches of the respective amoebas on opposites sides of the edge e. We also referenced the figure again, where this labelling is used.
(18) Done.
(19) Yes, thanks for catching this. Changed.
(20) Done.
(21) Done.
(22) Done.
(23) Indeed, there were initial forms missing in the statement, and we hope this is were the confusion arose mostly. We have added it now, and also expressed the statement in terms of an equation. We have also added an explanation in the proof. Since $Q(P_1) = 0$ as P_1 is a point in the quartic, also the initial must vanish. If the initial decomposes as a monomial times a form, the form must vanish as the monomial cannot. This decomposition is not necessarily unique, but of course there is a biggest monomial we can divide out from each term of the initial of Q, the lcm of the terms. We mostly use the lemma for this. We hope that this is clearer now with the changes.
(24) Done.
(25) Done.
(26) Changed. Thanks for the suggestion.
(27) We have inserted an explanation about this. It is correct, Lemma 4.18 is, first of all, only about the contribution of one tangency point. But for the two lifts of one of the liftable bitangents, the second tangency point is the same and thus also gives the same contribution to the Qtype. For that reason, we obtain $\mathbb{H}$ for each of the liftable bitangents.
(28) We agree that a proof of the statement in 1.7 was missing. We are not aware of any other way to prove it but to use the classification. If we use the classification, it is just 2 lines. We have added those two lines at the end of appendix A2. Going through this, we have noted that the description with the vertex does actually not increase the list, so we have shortened 1.7 and now only mention the compact condition there.
(29) That is right. These sentences actually belong to the next proof, but there they already appear in different formulation. Therefore, we removed them.
(30) Indeed, this was a note to ourselves which we accidentally overlooked. Sorry about that, and thanks for catching it. We have now introduced a short paragraph about Viro's patchworking, as requested by the note.
(31) Done.

Additional change:

(1) We added a reference for the new preprint by Kummer and McKean in which they prove Conjecture 5.1 by Larson and Vogt about the signed real count.