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Rigidity of homogeneous gradient soliton metrics and related equations

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ABSTRACT

We prove structure results for homogeneous spaces that support a non-constant solution to two general classes of equations involving the Hessian of a function and an invariant 2-tensor. We also consider trace-free versions of these systems. Our results generalize earlier rigidity results for gradient Ricci solitons and warped product Einstein metrics. In particular, our results apply to homogeneous gradient solitons of any invariant curvature flow and give a new structure result for homogeneous conformally Einstein metrics.

1. Introduction

Let G be a group acting by isometries on a Riemannian manifold (M, g) and f a real valued function on M. If f is invariant under G, then the Hessian of f is also invariant. In this paper we are interested in rigidity phenomena that occur when we conversely assume that the Hessian is invariant but the function is not. We focus on the case where G acts transitively so that any invariant function is constant. The prototypical example of a function which has invariant Hessian but is not invariant is a linear function on $\mathbb{R}^n$ whose Hessian, being zero, is invariant under the full isometry group. Another prominent example is the restriction of coordinate functions x^i in $\mathbb{R}^{n+1}$ to the sphere S^n, whose Hessian on the sphere satisfies $\mathrm{Hess} x^i = -x^i g$. In this case, while $\mathrm{Hess} x^i$ is not invariant under the full isometry group, its trace-free part is and it also

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satisfies an equation of the form $\mathrm{Hess}x^i = x^i q$ where $q = -g$ is invariant under the isometry group. Note that the coordinate functions in $\mathbb{R}^{n,1}$ restricted to hyperbolic space satisfy a similar equation $\mathrm{Hess}x^i = x^i g$.

More complicated examples come from gradient solitons to curvature flows. These satisfy $\mathrm{Hess}f = \lambda g - q$, where q is an expression involving the curvatures of the metric. Equations involving the Hessian of a function and the curvature also come up naturally in the study of warped products and conformal changes of metrics.

Motivated by these examples we consider the following general classes of equations involving q, a symmetric two tensor on a Riemannian manifold,

$$\mathrm{Hess}f = q, \tag{1.1}$$

$$\mathrm{Hess}w = wq, \tag{1.2}$$

where f, w are smooth functions. Given a Riemannian manifold (M, g) and a fixed tensor q we denote by $F(M, g, q)$ and $W(M, g, q)$ the space of all solutions to equation (1.1) and (1.2) respectively. We will often simply write $F(q)$ and $W(q)$.

When q is fixed, equations (1.1) and (1.2) are overdetermined in f or w respectively, as there is only one unknown function but $\frac{n(n+1)}{2}$ equations. Thus the solution spaces F and W are small except in exceptional circumstances. On the other hand, if q is invariant under G, a group of isometries, then G acts on F and W. Thus if G is a large group we have a large group acting on a small space and this also leads to rigidity. Roughly speaking, this is the approach we use to prove general structure theorems for any G-homogeneous Riemannian metric that supports non-constant solutions to (1.1) and (1.2) for a G-invariant q.

Our results build on previous work of the authors in two cases involving the Ricci curvature. Namely, functions in $F(\lambda g - \mathrm{Ric})$ corresponding to gradient Ricci solitons and functions in $W(\frac{1}{m}(\mathrm{Ric} - \lambda g))$, $m \in \mathbb{N}$ corresponding to warped product Einstein metrics [15,6]. These equations on homogeneous manifolds were studied by the authors in [20] and by the authors along with He in [14] respectively. The main idea of this paper is that a general structure extends to the more general equations, with some important variation.

In [20] the authors showed that a homogeneous gradient Ricci soliton is the product of an Einstein metric and a Euclidean space. We prove the following generalization of this result.

Theorem 1.1. *Let (M, g) be a G-homogeneous manifold and q a G-invariant symmetric two-tensor which is divergence free. If there is a non-constant function in $F(q)$, then (M, g) is a product metric $N \times \mathbb{R}^k$ and f is a function on the Euclidean factor.*

Remark 1.2. Note that $2\mathrm{div}\mathrm{Ric} = d\mathrm{scal}$, so on a homogeneous space the Ricci tensor is divergence free. By Proposition 3.7 the divergence free assumption on q can also be replaced with the assumption that $\mathrm{Ric}(\nabla f, \nabla f) \geq 0$ for $f \in F(q)$, which is also satisfied for homogeneous gradient Ricci solitons as $\mathrm{Ric}(\nabla f) = 0$.

Remark 1.3. Griffin applies Theorem 1.1 to study homogeneous gradient solitons for the four-dimensional Bach flow in [11].

On the other hand, Theorem 1.1 is not true if we do not assume q is divergence free, see Example 3.2. We prove a general structure theorem for $F(q)$ without the divergence free assumption (Theorem 3.6), whose precise statement we delay until section 3. The general rigidity we obtain involves spaces we call *one-dimensional extensions.*

Definition 1.4. A G-homogeneous space $(M = G/G_x, g)$ is called a one-dimensional extension if there is a closed subgroup, $H \subset G$ that contains G_x such that there is a surjective Lie group homomorphism from G to the additive real numbers whose kernel is H.

The algebraic condition of being a one-dimensional extension implies a geometric/topological product structure such that M is diffeomorphic to $\mathbb{R} \times (H/G_x)$ and $g = dr^2 + g_r$ where g_r is a one-parameter family of homogeneous metrics on H/G_x. Moreover, G acts as a semi-direct product $G = H \rtimes \mathbb{R}$ on g. Theorem 3.6 roughly says that if $F(q)$ contains a non-constant function then M is either a one-dimensional extension, a product of a one-dimensional extension with Euclidean space, or a space as in Theorem 1.1. In particular, Theorem 3.6 applies to any homogeneous gradient soliton for an invariant curvature flow. We are not aware of any examples of flows where examples of gradient solitons on one-dimensional extensions have arisen.

One-dimensional extensions play a larger role in the study of $W(q)$ as they occur even in the warped product Einstein case. In fact, in [17] Lafuente showed that a homogeneous space admits a one-dimensional extension which is the base of a warped product Einstein manifold if and only if it is an algebraic Ricci soliton. For general q, we obtain the following structure result.

Theorem 1.5. *Let (M, g) be a G-homogeneous manifold and q a G-invariant symmetric two-tensor. If $W(q)$ is non-trivial, then (M^n, g) is isometric to one of the following*

(1) *a space of constant curvature and* $\dim W = n + 1$,
(2) *the product of a homogeneous space and a space of constant curvature with W consisting of functions on the constant curvature factor and* $2 \leq \dim W \leq n$,
(3) *the quotient of the product of a homogeneous space and $\mathbb{R}$, $(H \times \mathbb{R})/\pi_1(M)$, with $W = \{w : \mathbb{R} \to \mathbb{R} \mid w'' = \tau w\}$ where $\tau < 0$ is constant, or*
(4) *a one-dimensional extension and* $\dim W = 1$.

If, in addition, q is Codazzi, then (M, g) is isometric to one of the cases (1)-(3).

Remark 1.6. A symmetric 2-tensor is Codazzi if its covariant derivative is symmetric, i.e. $(\nabla_X q)(Y, Z) = (\nabla_Y q)(X, Z)$, for all vectors X, Y, Z. In general, divergence free and Codazzi are different conditions. However, a Codazzi tensor is divergence free if and only if it has constant trace. Thus a Codazzi tensor that is invariant under a transitive group of isometries is divergence free. See section 6 for further discussion of examples in case (4) where q is divergence free.

We also consider the trace-free versions of these equations,

$$\mathring{\mathrm{Hess}} f = \mathring{q}, \tag{1.1a}$$

$$\mathring{\mathrm{Hess}} w = w\mathring{q}, \tag{1.2a}$$

where $\mathring{q}$ is the trace-free part of q, $\mathring{q} = q - \frac{\mathrm{tr} q}{\dim(M)} g$. We write $\mathring{F}(q)$ and $\mathring{W}(q)$ for the solution spaces to (1.1a) and (1.2a) respectively. Non-trivial functions in $\mathring{F}(-\mathrm{Ric})$ are called Ricci almost solitons in the literature, see for example [4]. Non-trivial functions in $\mathring{W}(\frac{1}{2-n}\mathrm{Ric})$ are called almost Einstein metrics in the literature. In this case, if the function is positive then the metric is conformal to an Einstein metric. See, for example, [7,8,10,18,16] and the references there-in.

The study of the solution spaces $\mathring{F}$ and $\mathring{W}$ can in the homogeneous case be reduced to the study of a corresponding F or W space. A space of functions $\mathring{F}$ (or $\mathring{W}$) is called *essential* if $\mathring{F}(q) \neq F(q')$ for all q' (or $\mathring{W}(q) \neq W(q')$ for all q'). We have the following rigidity result for essential spaces of solutions.

Theorem 1.7. *Let (M, g) be a G-homogeneous manifold and q a G-invariant symmetric two-tensor. If $\mathring{F}(q)$ is essential then (M, g) is a space of constant curvature. If $\mathring{W}(q)$ is essential, then (M, g) is locally conformally flat.*

Note that homogeneous locally conformally flat metrics are classified by Takagi in [21] (see also Theorem 2.6). Theorem 1.7 combined with structure results for F and W as well as Takagi's classification yields the following corollaries.

Corollary 1.8. *Let (M, g) be a G-homogeneous manifold and q a G-invariant symmetric two-tensor which is divergence free. If there is a non-constant function in $\mathring{F}(q)$, then (M, g) is either a space of constant curvature or is a product metric $N \times \mathbb{R}^k$ with f being a function on the Euclidean factor.*

Corollary 1.9. *If (M, g) is a G-homogeneous manifold and q is a G-invariant symmetric two-tensor such that $\mathring{W}$ is non-trivial, then (M, g) is isometric to either*

(1) $S^n(\kappa)/\Gamma$, $\mathbb{R}^n/\Gamma$, $H^n(-\kappa)$, $(S^k(\kappa)/\Gamma) \times H^{n-k}(-\kappa)$, $(\mathbb{R}^1/\Gamma) \times H^{n-1}(-\kappa)$, *or* $(S^{n-1}(\kappa) \times \mathbb{R}^1)/\Gamma$,
(2) *a direct product of a homogeneous space and a space of constant curvature with $\mathring{W}$ consisting of functions on the constant curvature factor,*
(3) *the quotient of the product of a homogeneous space and $\mathbb{R}$, $(H \times \mathbb{R})/\pi_1(M)$, with $W = \{w : \mathbb{R} \to \mathbb{R} \mid w'' = \tau w\}$ where $\tau < 0$ is constant, or*
(4) *a one-dimensional extension of a homogeneous space.*

Moreover, when (M, g) is not in case (1), $\mathring{W}(q) = W(q')$, where q' is a G-invariant tensor of the form $q' = q - \lambda g$ for some $\lambda \in \mathbb{R}$. If, in addition, q is Codazzi, then (M, g) is isometric to one of the cases (1)-(3).

In the case of Ricci almost solitons, Corollary 1.8 already follows from [4, Theorem 1.1]. For almost Einstein metrics, Corollary 1.9 generalizes Theorem 5.2 in [18] to the non-compact case. In dimension 4, homogeneous conformally Einstein spaces were classified in [5] where is it shown that if a space is not a symmetric space, then it is one of three families of one-dimensional extensions. In higher dimensions, Corollary 1.9 reduces the problem of classifying homogeneous almost Einstein spaces and thus conformally Einstein spaces to studying one-dimensional extensions. We discuss this case further in section 7, where we also discuss the application of Corollary 1.9 to more general "generalized m-quasi-Einstein metrics."

As a final application of the theorems above, we consider the case of a compact locally homogeneous manifold admitting non-trivial functions in F, $\mathring{F}$, W, or $\mathring{W}$ for a local isometry invariant q. First note that $F(q)$ can never be non-trivial because if $f \in F(q)$ then $\Delta f = \mathrm{tr} q$ and $\mathrm{tr} q$ is constant as q is a local isometry invariant tensor. A function on a compact manifold with constant Laplacian is constant, so f is constant. On the other hand, the sphere supports invariant tensors q such that $\mathring{F}$, W and $\mathring{W}$ are all non-trivial. In this case we get the following rigidity result. The proof follows from inspecting the possibilities for simply connected examples in Corollaries 1.8 and 1.9 to admit non-trivial $\mathring{F}$, W and $\mathring{W}$ that are invariant under co-compact actions of deck transformations.

Theorem 1.10. *Suppose that (M, g) is a compact locally homogeneous manifold and q a local isometry invariant symmetric two tensor.*

(1) *If $\mathring{F}(q)$ contains a non-constant function, then (M, g) is a spherical space form.*
(2) *If $\mathring{W}(q)$ is non-trivial, then either (M, g) is isometric to a direct product of a homogeneous space N and a spherical space form, isometric to $(N \times \mathbb{R})/\pi_1(M)$, or isometric to $(S^{n-1}(\kappa) \times \mathbb{R}^1)/\Gamma$.*

In particular any positive function in $\mathring{F}(q)$ or $\mathring{W}(q)$ must be constant.

Note that in the statement of part (2) we allow N to be a point, so that the space could be isometric to a spherical space form.

The paper is organized as follows. In the next section we discuss preliminaries including the basic algebraic structure of the spaces F and W and the rigidity theorems for homogeneous spaces which we use to prove the structure theorems. In the next four sections we prove the results for F, $\mathring{F}$, W, and $\mathring{W}$. In the final section we discuss the application of the results to conformally Einstein and generalized m-Quasi Einstein metrics. We also include an appendix with a discussion of these spaces of functions on Kähler manifolds.

2. Preliminaries

In this section we discuss some basic properties about the spaces of functions $F(M, g, q)$, $\mathring{F}(M, g, q)$, $W(M, g, q)$, and $\mathring{W}(M, g, q)$ as well as some rigidity results for homogeneous spaces that will be the main tools in the proofs of our structure theorems.

2.1. Basic structure

First note that the spaces of functions F and $\mathring{F}$ are affine as $f_1, f_2 \in F$ (resp, $\mathring{F}$) implies $f_1 - f_2 \in V$ (resp, $\mathring{V}$), where

$$V = \{v \mid \mathrm{Hess} v = 0\}$$
$$\mathring{V} = \{v \mid \mathring{\mathrm{Hess}} v = 0\}.$$

Both V and $\mathring{V}$ are vector spaces of functions that contain the constant functions. Moreover, it is well known that if V or $\mathring{V}$ contain a non-constant function, then the metric must be special. If there is a non-constant function $v \in V$, then (M, g) must split as a product with a Euclidean factor, and v is a coordinate function in the Euclidean direction (see Proposition 3.4). If there is a non-constant function $v \in \mathring{V}$, then (M, g) must split as a warped product over a 1-dimensional base. This was first proven locally by Brinkmann [3] and later globally by Tashiro [22]. The complete study of the full space $\mathring{V}$ is due to Osgood-Stowe [19].

The spaces W and $\mathring{W}$ are vector spaces of functions. In fact, note that V and $\mathring{V}$ are special cases of W and $\mathring{W}$ where $q = 0$. Rigidity for metrics which admit linearly independent solutions in W was studied in [12] (also see Theorem 5.2 below). It gives a weaker warped product splitting than for V or $\mathring{V}$.

A tensor q is invariant under a subgroup, G, of isometries of (M, g), if $\gamma^* q = q$ for all $\gamma \in G$. If q is invariant under G, then G acts on the spaces F, $\mathring{F}$, W, and $\mathring{W}$ via $f \mapsto \gamma^* f$, $\gamma \in G$. Conversely, we also have that if F or W is invariant under the action of G then so is q.

Proposition 2.1. *If $F(M, g, q)$ or $W(M, g, q)$ are non-trivial and invariant under the action of $G \subset \mathrm{Isom}(M, g)$, then q is also invariant under G.*

Proof. We consider the case where W is invariant. The case for F is similar. Fix a non-trivial $w \in W$ and $\gamma \in G$. We have:

$$(w \circ \gamma) q = \mathrm{Hess}(w \circ \gamma) = \gamma^* \mathrm{Hess} w = \gamma^*(wq) = (w \circ \gamma)(\gamma^* q).$$

This shows that $\gamma^* q = q$ wherever $w \circ \gamma \neq 0$. Since this is a set of full measure unless $w \equiv 0$ (see [12, Proposition 1.1]) we conclude that q is γ invariant. □

2.2. Some rigidity results on homogeneous spaces

In this section we discuss some rigidity results for certain functions and vector fields on homogeneous spaces. We first recall the algebraic formulation of the rigidity we require from the introduction.

Definition 2.2. A G-homogeneous space $(M = G/G_x, g)$ is called a one-dimensional extension if there is a closed subgroup, $H \subset G$ that contains G_x such that there is a surjective Lie group homomorphism from G to the additive real numbers whose kernel is H.

This algebraic property has the following geometric consequences.

Proposition 2.3. *If a G-homogeneous space $(M = G/G_x, g)$ is a one-dimensional extension of H, then*

(1) *G acts on M as a semi-direct product group $G = H \rtimes \mathbb{R}$,*
(2) *M is diffeomorphic to $(H/G_x) \times \mathbb{R}$,*
(3) *$g = g_r + dr^2$ where g_r is a one-parameter family of homogeneous metrics on H/G_x.*

Proof. Let $\phi : G \to \mathbb{R}$ be a surjective Lie group homomorphism with kernel H. Since $G_x \subset H$ it follows that $M/H = (G/G_x)/H = G/H = \mathbb{R}$. Therefore, the action of H on M has cohomogeneity one. Let $r : M \to M/H$. By re-parametrizing the range, M/H, we can assume that r is a distance function. H acts transitively on the level sets of r, which gives the diffeomorphic splitting (2) as well as the metric of the form (3).

To see (1), let γ_t be a one-parameter family of isometries in G. It follows that $t \mapsto \phi(\gamma_t)$ is an additive group homomorphism from $\mathbb{R}$ to $\mathbb{R}$ and thus either trivial or an isomorphism. Since ϕ is assumed to be surjective, we can find a γ_t such that this map is an isomorphism. Let $\gamma \in G$. There is t such that $\phi(\gamma_{-t}) = \phi(\gamma)$, which implies that $\gamma_t \circ \gamma \in H$. This shows that G is a semi-direct product group $G = H \rtimes \mathbb{R}$. □

Now we are ready to prove the main Lemma which we use to show that spaces are one-dimensional extensions. It roughly says that when there is function which is "almost" invariant by a transitive group in the sense that it changes only by an additive or multiplicative constant, then we obtain a one-dimensional extension.

Lemma 2.4. *Let M be a G-homogeneous space, assume that either*

(1) *there is a non-constant function f such that for all $\gamma \in G$ there is $C_\gamma \in \mathbb{R}$ so that*

$$\gamma^* f = f + C_\gamma, \text{ or}$$

(2) *there is a non-constant function w such that for all $\gamma \in G$ there is $C_\gamma \in \mathbb{R}$ so that*

$$\gamma^* w = C_\gamma w.$$

In either case (M, g) becomes a one-dimensional extension of H, the subgroup of G that fixes the function f or w. Moreover, in case (1) $f = ar + b$ and in case (2) $w = be^{ar}$ for some $a, b \in \mathbb{R}$.

Proof. First consider case (1). The assumption $\gamma^* f = f + C_\gamma$, gives a homomorphism $\gamma \mapsto C_\gamma$ into the additive real numbers with kernel $H = \{\gamma \in G \mid \gamma^* f = f\}$. To see that $G_x \subset H$ note that if $\gamma(x) = x$, then $\gamma^* f(x) = f(x)$ implying that $C_\gamma = 0$. Observe that the image of $\gamma \mapsto C_\gamma$ is either trivial or $\mathbb{R}$ and in case it is trivial f is forced to be constant. Therefore, we have a one-dimensional extension of H and the diffeomorphic splitting $M = H/G_x \times \mathbb{R}$ with metric $g = g_r + dr^2$. As f is invariant under H we must have $f = f(r)$, $\nabla f = f'(r)\nabla r$. Since the group G preserves ∇f this implies that $f'(r)$ is constant, so $f = ar + b$ for constants $a, b \in \mathbb{R}$. This completes case (1).

Case (2) is similar. Since $\gamma^* w = C_\gamma w$, the action of G preserves both the zeros and the critical points of w. Since G is transitive and w is non-constant we must have that w has no zeros nor critical points so, by possibly switching to $-w$, we can assume that w is positive. The map $\gamma \mapsto C_\gamma$ is a group homomorphism into the multiplicative group of positive real numbers. But then $\ln(C_\gamma)$ gives a homomorphism into the additive reals whose kernel consists of the isometries that preserve w. We then obtain $M = H/G_x \times \mathbb{R}$ with metric $g = g_r + dr^2$ and $w = w(r)$.

To see that $w = be^{ar}$ consider that any isometry γ preserves the vector field $\frac{\nabla w}{w}$ as

$$d\gamma\left(\frac{\nabla w}{w}(\gamma^{-1}x)\right) = \frac{d\gamma(\nabla w(\gamma^{-1}x))}{w(\gamma^{-1}x)} = \frac{C_\gamma \nabla w(x)}{C_\gamma w(x)} = \frac{\nabla w}{w}(x).$$

So $|\nabla w|/w = |w'(r)|/w(r)$ is constant and so $w = be^{ar}$ for some $a, b \in \mathbb{R}$. □

Finally in this section we prove a fact about conformal fields on homogeneous spaces. Recall that a vector field X is a conformal field if $\overset{\circ}{L_X g} = 0$ which is equivalent to the 1-parameter family of (local) diffeomorphisms generated by X being conformal diffeomorphisms of g. We have the following rigidity for conformal fields on homogeneous spaces. This result was established and used in [4, Proof of Theorem 1.1], but the resulting formula there does not appear to be entirely correct.

Proposition 2.5. *Let (M, g) be a homogeneous space and X a conformal field, then either (M, g) is locally conformally flat, or X is a Killing field.*

Proof. All two-dimensional spaces are locally conformally flat, so there is nothing to prove in this case. In dimensions larger than 2 there is always a conformally invariant $(1, 3)$ tensor, C, on (M, g) such that $C = 0$ if and only if (M, g) is locally conformally flat. In dimension 3 it is the Cotton tensor, in higher dimensions the Weyl tensor.

The conformal invariance of C implies that $L_X C = 0$ as X is a conformal field. We claim that $D_X|C|^2 = -2\mathrm{tr}(L_X g)|C|^2$. To see this consider a point $p \in M$ where $V(p) \neq 0$ and select coordinates $x^1, \ldots, x^n$ such that $V = \partial_1$. The Lie derivative of any tensor can now be calculated by computing the directional derivatives of the components of the tensor in these coordinates. With this in mind it follows that the components of the metric tensor satisfy: $D_X g_{ij} = \mathrm{tr}(L_X g) g_{ij}$ and its inverse: $D_X g^{ij} = -\mathrm{tr}(L_X g) g^{ij}$, while $D_X C^l_{ijk} = 0$. We can now calculate

$$\begin{aligned} D_X|C|^2 &= D_X(g^{is} g^{jt} g^{ku} g_{lv} C^l_{ijk} C^v_{stu}) \\ &= (-3\mathrm{tr}(L_X g) + \mathrm{tr}(L_X g))(g^{is} g^{jt} g^{ku} g_{lv} C^l_{ijk} C^v_{stu}) \\ &= -2\mathrm{tr}(L_X g))(g^{is} g^{jt} g^{ku} g_{lv} C^l_{ijk} C^v_{stu}). \end{aligned}$$

Finally, the formula trivially holds on any open set where X vanishes. (In fact, a non-trivial conformal field cannot vanish on an open set as its zero set has components that are either points or totally umbilic hypersurfaces.) So the formula $D_X|C|^2 = -2\mathrm{tr}(L_X g)|C|^2$ must hold globally.

Since the space is homogeneous, $|C|^2$ is constant, so either $\mathrm{tr}(L_X g) = 0$ everywhere, and the field is Killing, or there is a point where $|C|^2 = 0$. However, again by homogeneity, if $C = 0$ at a point then $C = 0$ everywhere and then the space is locally conformally flat. □

Finally in this section we point out that locally conformally flat homogeneous spaces have a rigid classification due to Takagi.

Theorem 2.6. *[21, Theorem B] Let (M^n, g) be a homogeneous space which is locally conformally flat, then (M, g) is isometric to either $S^n(\kappa)/\Gamma$, $\mathbb{R}^n/\Gamma$, $H^n(-\kappa)$, $(S^k(\kappa)/\Gamma) \times H^{n-k}(-\kappa)$, $(\mathbb{R}^1/\Gamma) \times H^{n-1}(-\kappa)$, or $(S^{n-1}(\kappa) \times \mathbb{R}^1)/\Gamma$.*

3. *F*

Now we begin the study of the space of solutions to (1.1), $F(M, g, q)$. We start by offering two examples of spaces that typify situations where q is invariant under a group of isometries but not all the functions in $F(q)$ are.

Example 3.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = \frac{A}{2}|x|^2 + L(x) + C$ where A, C are constants and $L : \mathbb{R}^n \to \mathbb{R}$ is a linear function. Then $\mathrm{Hess} f = A g_0$ where g_0 denotes the Euclidean dot product. Clearly $\mathrm{Hess} f$ is invariant under the full isometry group, but f is not.

Example 3.2. Let $g = dr^2 + e^{2kr} g_0$, where g_0 is the Euclidean metric on $\mathbb{R}^{n-1}$. Then g is the Euclidean metric if $k = 0$ and is the hyperbolic space if $k \neq 0$. Consider $f = cr$ and $G = \{\phi \mid \phi(r, x) = (r + a, e^{-ka}\tau(x)),$ where $a \in \mathbb{R}$ and $\tau \in \mathrm{Isom}(\mathbb{R}^{n-1})\}$. In this case G is a group of isometries of g that acts transitively and $\mathrm{Hess} f = cke^{2kr} g_0$ which is invariant under the group G.

Our results come from considering the cases when the dimension of V is one and larger than one separately. When the dimension is one we have an almost trivial action of a transitive group of isometries while, when the dimension is larger than one, we have a rigidity result for the metric. Example 3.2 is in the case where V is one dimensional and Example 3.1 is in the case where V is higher dimensional.

Let us now be more precise. First in the case where $\dim(V) = 1$, we can apply Lemma 2.4.

Proposition 3.3. *Let (M, g) be a G-homogeneous manifold and let q be a G-invariant symmetric two tensor. If $\dim(V) = 1$ and $f \in F(q)$ is non-constant, then (M, g) is a one-dimensional extension and $f = kr$.*

Proof. Recall that $\gamma^* f = f \circ \gamma^{-1}$. Since q is invariant under γ we have $\gamma^* f \in F$. Therefore, $\gamma^* f - f \in V$ and this is a real number since V consists only of constants. This shows that $\gamma^* f = f + C_\gamma$ for a constant C, so we can apply Lemma 2.4. □

The rigidity statement for complete spaces which have non-constant functions in V is the following.

Proposition 3.4. *Suppose (M, g) is a complete Riemannian manifold and suppose that $\dim(V) = k + 1$ for some $k \geq 1$, then M splits isometrically as $\mathbb{R}^k \times N$ for some space N and $\mathrm{Isom}(M) = \mathrm{Isom}(\mathbb{R}^k) \times \mathrm{Isom}(N)$. Moreover, $\dim(V(N)) = 1$ and $V(M)$ consists of the space of affine functions $\mathbb{R}^k \to \mathbb{R}$.*

Proof. The metric splitting follows from the fact that all elements in V have parallel gradient. Moreover, $\mathbb{R}^k$ must be the Euclidean de Rham factor as otherwise $\dim V > k + 1$. This shows that the isometry group splits. Finally if $\dim(V(N)) > 1$, then also $\dim V > k + 1$. □

The previous two propositions show that if $f \in F(q)$ is a non-constant function and q is invariant under a transitive group of isometries, then the metric is either a one-dimensional extension or splits as a product. In the case of a product splitting, we do not assume that the tensor q necessarily splits, however a further application of Lemma 2.4 allows us to determine the function f when the metric splits.

Proposition 3.5. *Let $M = B \times F$ be a direct product and let $G = G_1 \times G_2$ where G_1, G_2 are transitive groups of isometries on B and F respectively. Suppose that there is a function f on $B \times F$ such that*

$$(\gamma^* f)(x, y) - f(x, y) = \phi_\gamma(y)$$

for all $\gamma \in G$, where ϕ is a function of F that depends on γ. Either

(1) *$f = \psi(y)$, or*
(2) *B is a one-dimensional extension, $g_B = dr^2 + g_r$, and $f = ar + \psi(y)$*

where ψ is a function of F.

Proof. Fix a point $y_0 \in F$, and let $f_0 : B \times \{y_0\} \to \mathbb{R}$ be defined as $f_0(x) = f(x, y_0)$. Let $\gamma_1 \in G_1$, by assumption we have

$$((\gamma_1 \times \mathrm{id})^* f)(x, y_0) - f(x, y_0) = \phi_1(y_0),$$
$$((\gamma_1)^* f_0)(x) - f_0(x) = \phi_1(y_0).$$

So, applying Lemma 2.4 we get that either f_0 is constant in x or $B \times \{y_0\}$ is a one-dimensional extension and $f_0 = a(0)r + b(0)$.

If $f_0(x) = d$ for a constant d, then let $\gamma_2 \in G_2$ and consider

$$((\mathrm{id} \times \gamma_2)^* f)(x, y_0) - f(x, y_0) = \phi_2(y_0),$$
$$f(x, \gamma_2(y_0)) - d = \phi_2(y_0).$$

Since G_2 acts transitively, this implies that f is constant in the x direction everywhere.

On the other hand, if f_0 is non-constant and $B \times \{y_0\}$ is a one-dimensional extension, then $B \times \{y\}$ is a one-dimensional extension for all y since M is assumed to be a product metric. Applying Lemma 2.4 to each $f_y(x) = f(x, y)$ we obtain that $f(x, y) = a(y)r + b(y)$ where a, b could a priori be functions of y. But then a must be constant as

$$(\mathrm{id} \times \gamma_2)^*(f)(x, y_0) - f(x, y_0) = ((\gamma_2^* a)(y_0) - a(y_0))r + (\gamma_2^* b)(y_0) - b(y_0).$$

Since the right hand side is assumed to only be a function of y it follows that $(\gamma_2^* a)(y_0) = a(y_0)$ for all $\gamma_2 \in G_2$ and a is constant. □

This gives us the following theorem.

Theorem 3.6. *Let (M, g) be a G-homogeneous manifold and let q be a G-invariant symmetric two tensor. Suppose that $f \in F(q)$ is a non-constant function then either*

(1) *(M, g) is isometric to a product, $N \times \mathbb{R}^k$ where f is constant on N,*
(2) *(M, g) is a one-dimensional extension, $g = dr^2 + g_r$, and $f(x, y) = ar + b$, or*
(3) *(M, g) is isometric to a product, $N \times \mathbb{R}^k$ where N is a one-dimensional extension and $f(x, y) = ar(x) + v(y)$ where v is a function on $\mathbb{R}^n$ and r is a distance function on N.*

Proof. We have already seen that the theorem is true when $\dim(V) = 1$. So suppose $\dim(V) > 1$ and note that the metric splits as a direct product, $N \times \mathbb{R}^k$. Moreover, $G = G_1 \times G_2$ because unit tangent vectors to the $\mathbb{R}^k$ factor are characterized as gradients to functions in V. We also have that $\gamma^* f - f$ is a function of the $\mathbb{R}^k$ factor for any γ. So we may apply Proposition 3.5 to obtain the result. □

The natural question coming from Theorem 3.6 is what conditions imply that a one-dimensional extension is a product, the next proposition gives two such conditions.

Proposition 3.7. *Let (M, g) be a one-dimensional extension. The following properties hold:*

(1) Δr *is constant,*
(2) $\mathrm{Ric}(\nabla r, \nabla r) \leq 0$,
(3) *If* $\mathrm{Ric}(\nabla r, \nabla r) = 0$*, then* $g = g_0 + dr^2$ *is a product,*
(4) *If* $\mathrm{div}(\nabla\nabla r) = 0$*, then* $g = g_0 + dr^2$ *is a product.*

Proof. The transitive group G preserves ∇r and $\nabla\nabla r$ is invariant by G so $\Delta r = \mathrm{tr}(\nabla\nabla r)$ is constant.

To see (2) and (3) consider the Bochner formula applied to r:

$$\frac{1}{2}\Delta|\nabla r|^2 = \mathrm{Ric}(\nabla r, \nabla r) + |\mathrm{Hess} r|^2 + g(\nabla\Delta r, \nabla r).$$

Since $|\nabla r|$ and Δr are constant, we obtain

$$\mathrm{Ric}(\nabla r, \nabla r) = -|\mathrm{Hess} r|^2.$$

So if $\mathrm{Ric}(\nabla r, \nabla r) = 0$ then $|\mathrm{Hess} r|^2 = 0$, which implies that M splits isometrically as $N \times \mathbb{R}$.

Finally, for (4) note that

$$\mathrm{div}(\nabla\nabla r) = \nabla\Delta r + \mathrm{Ric}(\nabla r).$$

So, as Δr is constant, the condition $\mathrm{div}(\nabla\nabla r) = 0$ implies that $\mathrm{Ric}(\nabla r) = 0$ and we have a product splitting. □

This allows us to prove Theorem 1.1

Proof of Theorem 1.1. We have that $\mathrm{Hess} f = q$ for a tensor q that is invariant under a transitive group of isometries. Assume that f is non-constant, then since q is invariant under isometries Theorem 3.6 implies that either M is a one-dimensional extension or M splits as a product metric $M = N \times \mathbb{R}^k$, $g = g_1 + g_2$. Assume also that this splitting is maximal in the sense that M does not split off more than k Euclidean factors.

If $\mathrm{div}(q) = 0$ then $\mathrm{div}(\nabla\nabla r) = 0$, so by Proposition 3.7 the one-dimensional extension in the splitting is itself a product $\mathbb{R} \times N$, where r is the coordinate in the $\mathbb{R}$ direction. But, this contradicts the maximality of the splitting.

Therefore, we have $M = N \times \mathbb{R}^k$, by Theorem 3.6 we also have a splitting of the function f of the form $f = f_1 \times f_2$ where f_1 is a function on N and f_2 is a function on $\mathbb{R}^k$. In particular, $q = \mathrm{Hess} f = \mathrm{Hess}(f_1) + \mathrm{Hess}(f_2)$ so q splits as $q_1 + q_2$ where q_1 is a tensor on N and q_2 is a tensor on $\mathbb{R}^k$. In particular, $\mathrm{div} q = \mathrm{div}(q_1) + \mathrm{div}(q_2)$, so $\mathrm{div}(q_1) = 0$. If f_1 is non-constant then, by Theorem 3.6, (N, g_1) is a one-dimensional extension with $\mathrm{Hess} f_1 = q_1$ and $\mathrm{div} q_1 = 0$. So we also obtain that the one-dimensional extension in this case is a product, again contradicting the maximality of the splitting. Therefore, for the maximal splitting, we must have that f_1 is constant on the N factor. □

4. Traceless *F*

Now we consider spaces of functions $\mathring{F}(M, g, q)$ of solutions to (1.1a). Given our established results about the corresponding space $F(M, g, q)$, we consider the question of when $\mathring{F}(M, g, q) \neq F(M, g, q)$. There is a trivial way to produce such examples by adding a factor of g to q. Namely, if $f \in F(M, g, q - \phi g)$ for $\phi \in C^\infty(M)$, $\phi \neq 0$, then $f \notin F(M, g, q)$, but $f \in \mathring{F}(M, g, q)$. This motivates the following definition.

Definition 4.1. Let (M, g) be a Riemannian manifold and q a symmetric two-tensor, then $\mathring{F}(M, g, q)$ is inessential if $\mathring{F}(M, g, q) = F(M, g, q')$ for some quadratic form q'. $\mathring{F}(M, g, q)$ is essential if it is not inessential.

The next proposition shows that essential spaces are easily characterized in terms of the spaces $\mathring{V}$ and V. It also shows that the property of $\mathring{F}$ being essential is a property of the space (M, g) but not the choice of q.

Proposition 4.2. *Let (M, g) be a Riemannian manifold and q a symmetric two-tensor, then the following are equivalent:*

(1) *$\mathring{F}(M, g, q)$ is essential,*
(2) *$\mathring{F}(M, g, q) \neq F(M, g, q - \phi g)$ for all $\phi \in C^\infty(M)$,*
(3) *The map $\Delta : \mathring{F}(M, g, q) \to C^\infty(M)$ is non-constant, and*
(4) *$\mathring{V} \neq V$.*

Moreover, if $\mathring{F}(M, g, q) = F(M, g, q')$ is inessential and q is invariant under $G \subset \mathrm{Isom}(M, g)$ then q' is also invariant under G.

Proof. (1) ⇒ (2) is obvious. To see (2) ⇒ (1) consider that if (1) is not true then $\mathring{F}(M, g, q) = F(M, g, q')$. So $\mathrm{Hess} f = q'$ and

$$\mathring{q} = \mathring{\mathrm{Hess}} f = q' - \frac{\mathrm{tr}(q')}{n} g.$$

So $q' = q + \frac{\mathrm{tr}(q') - tr(q)}{n} g$ which would contradict (2).

(1) and (3) are equivalent because if two quadratic forms have the same trace-free part, then they are the same if and only if they have the same trace.

To see that (3) and (4) are equivalent note that $w \in \mathring{V}$ is an element of V if and only if $\Delta w = 0$. If $f, f' \in \mathring{F}(M, g, q)$ then $f - f' \in \mathring{V}$, so Δ being non-constant on $\mathring{F}(M, g, q)$ is equivalent to there being a function in $\mathring{V}$ with non-zero Laplacian.

The final statement follows from Proposition 2.1. □

The next example shows that for simply connected spaces of constant curvature, $\mathring{F}$ is essential.

Example 4.3. Let (M^n, g) be a simply connected space of constant curvature. Then $\dim(\mathring{V}) = n + 2$ and $\mathring{F}(M, g, q)$ is essential. If (M^n, g) is Euclidean space then V is the $n+1$ dimensional space of affine functions and $\mathring{V}$ is spanned by V along with the function $|x|^2$. If M^n is a sphere or hyperbolic space then V just contains constant functions. For the sphere $\mathring{V}$ also contains the restriction of the coordinate functions in $\mathbb{R}^{n+1}$ while for hyperbolic space $\mathring{V}$ contains the restriction of the coordinate functions in $\mathbb{R}^{1,n}$. See [12] for more details.

On the other hand $\mathring{F}$ is inessential for product spaces.

Proposition 4.4. *If $(M, g) = (M_1^{n_1} \times M_2^{n_2}, g_1 + g_2)$, then $\mathring{V} = V$, so $\mathring{F}$ is inessential.*

Proof. Consider $f(x_1, x_2) \in \mathring{V}$. Then $\mathrm{Hess} f(X, U) = 0$ for $X \in TM_1$ and $U \in TM_2$ so by [20, Lemma 2.1] $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$. Thus

$$\mathrm{Hess}_g(f) = \mathrm{Hess}_{g_1} f_1 + \mathrm{Hess}_{g_2} f_2 = \frac{\Delta_{g_1} f_1 + \Delta_{g_2} f_2}{n} g.$$

If we restrict this equation to M_1 and M_2 this tells us that $\mathring{\mathrm{Hess}}_{g_1} f_1 = 0$ and $\mathring{\mathrm{Hess}}_{g_2} f_2 = 0$. Thus

$$\frac{\Delta_{g_1} f_1 + \Delta_{g_2} f_2}{n} = \frac{\Delta_{g_1} f_1}{n_1} = \frac{\Delta_{g_2} f_2}{n_2},$$

which shows that $\Delta_{g_1} f_1 = 0$ and $\Delta_{g_2} f_2 = 0$. Consequently, $\mathring{V} = V$. □

This gives us the following characterization of essential $\mathring{F}$ in the homogeneous case.

Theorem 4.5. *Suppose that (M, g) is a homogeneous Riemannian manifold. If $\mathring{F}(M, g, q)$ is essential, then (M, g) is a space of constant curvature.*

Proof. Suppose that $\mathring{F}$ is essential. Let $w \in \mathring{V} \neq V$, then ∇w is a conformal field which is not Killing. By Proposition 2.5, (M, g) is locally conformally flat. By Takagi, the universal cover of M is either a space of constant curvature or a product of spaces of constant curvature. Note that if $\pi : \widetilde{M} \to M$ is the universal cover of M, $w \in \mathring{V}(M)$ implies $(w \circ \pi) \in \mathring{V}(\widetilde{M})$ and $v \in V(M)$ implies $(v \circ \pi) \in V(\widetilde{M})$. Therefore, if $\mathring{V}(M) \neq V(M)$ then $\mathring{V}(\widetilde{M}) \neq V(\widetilde{M})$, so M essential implies that $\widetilde{M}$ is. Then by Proposition 4.4, the universal cover does not split as a product and so must be a space of constant curvature. □

Theorem 4.6. *Let (M, g) be a G-homogeneous Riemannian manifold and q be a G-invariant symmetric two-tensor. If $f \in \mathring{F}$ is a non-constant function then either*

(1) *(M, g) is a space of constant curvature,*
(2) *(M, g) is isometric to a product, $N \times \mathbb{R}^k$ where f is constant on N,*
(3) *(M, g) is a one-dimensional extension, $g = dr^2 + g_r$, and $f(x, y) = ar + b$, or*
(4) *(M, g) is isometric to a product, $N \times \mathbb{R}^k$ where N is a one-dimensional extension and $f(x, y) = ar(x) + v(y)$, where v is a function on $\mathbb{R}^n$ and r is a distance function on N.*

Proof. If $\mathring{F}$ is essential, then by Theorem 4.5 (M, g) is a space of constant curvature. If F is inessential, then $\mathring{F}(q) = F(q')$ where q' is also invariant by G, then Theorem 3.6 implies the result. □

This allows us to prove Corollary 1.8

Proof of Corollary 1.8. By Theorem 4.5 either (M, g) is constant curvature or $\mathring{F}(q) = F(q')$ and by (2) of Proposition 4.2 $q' = q - \phi g$ for a function ϕ. But then since q and q are both invariant by the transitive group G we must have ϕ constant. In particular, $\operatorname{div}(q') = \operatorname{div}(q)$, so q' is also divergence free and the Corollary follows from applying Theorem 1.1 to $F(q')$. □

5. *W*

Now we consider the space $W(M, g, q)$ of solutions to equation (1.2). When this is a one-dimensional space we have the following statement.

Theorem 5.1. *Let (M, g) be a G-homogeneous manifold and let q be a non-zero G-invariant two-tensor. If $\dim(W) = 1$, then (M, g) is a one-dimensional extension and $W = \{be^{ar} \mid b \in \mathbb{R}\}$ for some constant $a \in \mathbb{R}$.*

Proof. Let G be a transitive group of isometries and w be a non-constant function in W. Since W is one-dimensional and G acts on W, for $\gamma \in G$, we have $w \circ \gamma = C_\gamma w$ for some constant C_γ. The theorem now follows from Lemma 2.4. □

When $\dim(W) > 1$ we have the following result of He-Petersen-Wylie.

Theorem 5.2. *[12, Theorem A and B and Proposition 6.5] Suppose (M, g) is a complete Riemannian manifold such that $\dim(W) = k + 1$, $k \geq 1$. If $k > 1$ or M is simply connected, then M is isometric to a warped product $B \times_u F$ where F is a space of constant curvature. Moreover,*

$$W = \{w(x, y) = u(x)v(y) \mid v \in W(F, -\tau g_F)\}.$$

If $k = 1$, then M is isometric to $(B \times_u \mathbb{R})/\pi_1(M)$, where $u > 0$ and $\pi_1(M)$ acts by translations on $\mathbb{R}$.

Before applying these theorems, we need some basic results about warped products which are homogeneous.

By a warped product, $M = B \times_u F$ we mean a metric of the form $g_M = g_B + u^2 g_F$ where $u : B \to \mathbb{R}$. In general, it is possible to obtain a smooth metric g_M even in case u vanishes on the boundary of B. However, in this paper we will be able to conclude that $u > 0$ and M is diffeomorphic to $B \times F$. Let γ be a map of $B \times_u F$, we will say that γ respects the warped product splitting if $\gamma = \gamma_1 \times \gamma_2$ with $\gamma_1 : B \to B$ and $\gamma_2 : F \to F$. A group of isometries is said to respect the splitting if all its elements do. We have the following simple result about the isometries of a warped product that respect the splitting.

Proposition 5.3. *[13, Lemma 5.1] Suppose $M = B \times_u F$ with $u > 0$, then a map γ which respects the splitting is an isometry of g_M if and only if (1) $\gamma_1 \in \mathrm{Isom}(g_B)$, (2) there is a $C \in \mathbb{R}^+$ such that $\gamma_1^*(u) = Cu$, and (3) γ_2 is a C-homothety of g_F.*

Let $\mathrm{Isom}(B)_u$ be the isometries of g_B that preserve u. Proposition 5.3 implies that $\mathrm{Isom}(B)_u \times \mathrm{Isom}(F)$ is a group of isometries that respects the splitting. Recall also that a complete Riemannian manifold admits a C-homothety with $C \neq 1$ if and only if it is a Euclidean space. Therefore, if F is not a Euclidean space, then any subgroup of isometries that preserves the splitting is a subgroup of $\mathrm{Isom}(B)_u \times \mathrm{Isom}(F)$. In general, a warped product can have isometries that do not respect the splitting, so we will have to justify this assumption when we apply the Proposition below.

Combining Proposition 5.3 with Lemma 2.4 gives us the following characterization of when a warped product admits a transitive group of isometries which preserves the splitting.

Lemma 5.4. *Let $M = B \times_u F$ with $u > 0$ be a warped product manifold which admits a transitive group of isometries, G, that respects the splitting. Then either*

(1) *$M = B \times F$ and u is constant, or*
(2) *M is a one-dimensional extension such that*

$$g_M = dr^2 + g_r + u^2 g_{\mathbb{R}^k} \quad \textit{and} \quad u = be^{ar}.$$

Proof. Since G splits we have the projection $\pi : G \to \mathrm{Isom}(B)$ given by $\pi(\gamma) = \gamma_1$. Since G acts transitively on M, the image $\pi(G)$ acts transitively on B. By Proposition 5.3, for all $\gamma_1 \in \pi(G)$ there is a C such that $\gamma_1^*(u) = Cu$, so by Lemma 2.4 case (2) either u is constant or B is a one-dimensional extension, $g_B = dr^2 + g_r$ and $u = be^{ar}$. □

Theorem 5.5. *Let (M, g) be a G-homogeneous manifold and let q be a G-invariant two-tensor. If W is non-trivial, then (M^n, g) is isometric to one of the following*

(1) *a space of constant curvature with $\dim W = n + 1$,*
(2) *the product of a homogeneous space and a space of constant curvature with W consisting of functions on the constant curvature factor with $2 \leq \dim W \leq n$,*

(3) *the quotient of the product of a homogeneous space and* $\mathbb{R}$, $(H \times \mathbb{R})/\pi_1(M)$, *with* $W = \{w : \mathbb{R} \to \mathbb{R} \mid w'' = \tau w\}$ *where* $\tau < 0$ *is constant, or*
(4) *a one-dimensional extension with* $\dim W = 1$.

Proof. If $\dim(W) = 1$, then we obtain a one-dimensional extension by Theorem 5.1. Assume M is not a space of constant curvature. Then, if $\dim(W) > 2$ or if M is simply connected and $\dim(W) = 2$, then from Theorem 5.2 we obtain the warped product splitting $M = B \times_u F$ and we have that all w are of the form $w(x, y) = u(x)v(y)$. First we want to show that $u > 0$. To see this suppose that $u(x_0) = 0$ for some x_0, then $w(x_0, y) = u(x_0)v(y) = 0$, so there is a singular point where all functions in w vanish. But since G acts on W and is transitive this would imply that all functions in W are zero, a contradiction.

Next we observe that G respects the splitting $M = B \times_u F$. In fact, the tangent distributions to the leaves $\{b\} \times F$ are given by $\mathcal{F} = \{\nabla w \mid w \in W_p\}$ where $W_p = \{w \in W \mid w(p) = 0\}$. Since G preserves W it must also preserve $\mathcal{F}$ as well as the orthogonal distribution.

In case M is not simply connected and $\dim(W) = 2$ we reach the same conclusion for the universal cover of M. Here $W = \{w : \mathbb{R} \to \mathbb{R} \mid w'' = \tau w\}$ becomes a space of functions on $\mathbb{R}$ that is invariant under a cyclic group of translations. Since our quadratic form is invariant under a homogeneous group the function τ must be constant.

We can now apply Lemma 5.4 to see that either M is a one-dimensional extension, a direct product, or the universal cover is a direct product with $\mathbb{R}$. Once M or its universal cover is a direct product we have that u is constant, so $w = u(x)v(y)$ shows that all the functions in W are only on the constant curvature factor, F. □

Now we consider what more we can say in the case that q is assumed to be divergence free or Codazzi. Note that in cases (1)-(3) of Theorem 5.5 q is either a constant multiple of the metric or, on the products, a constant sum of the metrics on the factors. In particular, q is both divergence free and Codazzi. We show that the Codazzi property in fact characterizes these examples, while there are many more examples which are divergence free. First we establish some properties of the metrics in case (4) of the previous theorem.

Proposition 5.6. *Let* $w = e^{ar}$, $a > 0$, *where* $r : M \to \mathbb{R}$ *is a distance function. If* $q = \frac{1}{w}\operatorname{Hess} w$, *then*

$$q = a^2 dr^2 + a \operatorname{Hess} r, \tag{5.1}$$

$$(\nabla_X Q)(\nabla w) = a^2 w Q(X) - w Q^2(X) \tag{5.2}$$

where Q *dual* $(1,1)$ *tensor to* q. *If* q *is divergence free we further have:*

$$\operatorname{tr} q^2 = \operatorname{tr} q, \tag{5.3}$$

$$|\operatorname{Hess} r|^2 = a\Delta r. \tag{5.4}$$

In particular, if q *is invariant under a transitive group of isometries, then so is* $\operatorname{Hess} r$.

Proof. (5.1) follows directly from $q = \frac{1}{w}\operatorname{Hess} w$ as $w = e^{ar}$. To prove (5.2) note that we have that $wQ(X) = \nabla_X \nabla w$ so we obtain

$$\begin{aligned}(\nabla_X Q)(\nabla w) &= \nabla_X Q(\nabla w) - Q(\nabla_X \nabla w) \\ &= a^2 \nabla_X \nabla w - w Q^2(X) \\ &= a^2 w Q(X) - w Q^2(X),\end{aligned}$$

where the formula $Q(\nabla w) = a^2 \nabla w$ follows from (5.1).

Tracing (5.1) also gives us

$$\mathrm{tr}q = a^2 + a\Delta r$$

and

$$\mathrm{tr}q^2 = |q|^2 = a^4 + a^2 |\mathrm{Hess}|^2 .$$

Thus (5.4) follows from (5.3). To see (5.3), consider the trace of (5.2)

$$\mathrm{div}q\,(\nabla w) = w\left(a^2 \mathrm{tr}q - \mathrm{tr}q^2\right),$$

which implies (5.3). □

We now show the characterization in the Codazzi case.

Theorem 5.7. *With (M, g) and q as in Theorem 5.5, q is Codazzi if and only if (M, g) is isometric to one of the cases (1)-(3).*

Proof. The fact that q is Codazzi, (5.2) and $\frac{\nabla w}{w} = a\nabla r$, implies that

$$a(\nabla_{\nabla r} q)(X, X) = a^2 q(X, X) - q^2(X, X). \tag{5.5}$$

At a point p, let X be an eigenvector for q perpendicular to ∇r, with eigenvalue λ. Let β be the geodesic at p in the ∇r direction and let ϕ_t be a smooth curve of isometries in G such that $\phi_t(p) = \beta(t)$. Define $X_t = d\phi_t(X)$. Then X_t is a vector field along β with $|X_t| = 1$. Since ϕ_t preserves ∇r and q is invariant under ϕ_t we also have that $X_t \perp \nabla r$ and X_t an eigenvector of q with eigenvalue λ for all t. Using X_t we can then calculate,

$$\begin{aligned}(\nabla_{\nabla r} q)(X, X) &= D_{\nabla r}(\lambda) - 2q(\nabla_{\nabla r} X, X)\\ &= D_{\nabla r}(\lambda) - 2\lambda D_{\nabla r}|X_t|^2\\ &= 0.\end{aligned}$$

Plugging this back into (5.5) gives that either $\lambda = 0$ or $\lambda = a^2$ so q has only two possible eigenvalues. By invariance of q, the multiplicity of the eigenvalues is constant, so the corresponding eigenspace decomposition gives us a pair of orthogonal distributions on M. Moreover, since q is Codazzi, these eigendistributions are integrable (see Chapter 16 of [2]). Consequently, we can write the one dimensional extension, M, as

$$\begin{aligned}M &= \mathbb{R} \times N_1 \times N_2\\ g &= dr^2 + (g^1)_r + (g^2)_r\end{aligned}$$

where the tangent space to N_1 corresponds to the eigenvectors for q with eigenvalue a^2 and the tangent space to N_2 corresponds to null vectors for q. But then (5.1) implies that

$$\mathrm{Hess}r = a(g^1)_r$$

which implies that we have a warped product splitting

$$g = dr^2 + e^{2ar}(g^1)_0 + g_0^2.$$

Since the group G preserves the eigenspaces of q it preserves the warped product splitting in the sense of Lemma 5.4 and then the Lemma implies that $(g^1)_0$ is a flat metric on Euclidean space. Then we have that $dr^2 + e^{2ar}(g^1)_0$ is a hyperbolic metric.

Putting this all together we have three cases, if the only eigenvalue if q is a^2 then M is hyperbolic space which is contained in (1) of Theorem 5.5, if the only eigenvalue of q is 0 then we have a direct product as in case (3) of Theorem 5.5, finally if both eigenvalues occur we have a product of a homogeneous space and hyperbolic space as in case (2). □

Now we consider the divergence free case. The only case we need to consider is evidently when $\dim W = 1$ and is spanned by $w = e^{ar}$, $a > 0$, where $r : M \to \mathbb{R}$ is a distance function. An interesting special case occurs when

$$\operatorname{Hess} w = \frac{w}{m}\left(\operatorname{Ric} - \lambda g\right).$$

This is the so-called quasi-Einstein or warped product Einstein equation as it is the equation on B that makes a warped product $B \times_w F$ an Einstein metric when F is an appropriately chosen Einstein metric. Interestingly there are many such examples that are 1-dimensional extensions of algebraic solitons (see [14], [17]). The quasi-Einstein equation is studied in more detail in section 7.

With these examples in mind we cannot expect the same rigid behavior in the divergence free case. In fact, we will produce examples of one-dimensional extensions $G = H \rtimes \mathbb{R}$ such that H is not an algebraic soliton and $\frac{1}{w}\operatorname{Hess} w$ is divergence free, where $w = e^{ar}$.

Before discussing the examples, we identify some situations where we do obtain products and warped products.

Corollary 5.8. *Let $w = e^{ar}$, $a > 0$, where $r : M \to \mathbb{R}$ is a distance function on a homogeneous space (M, g). If $q = \frac{1}{w}\operatorname{Hess} w$ is invariant under a transitive group of isometries and divergence free, then $\Delta r \in [0, (n-1)\, a]$. When $\Delta r = 0$, the metric splits as a product $g = dr^2 + g_0$, and when $\Delta r = (n-1)\, a \neq 0$ the metric is isometric to $H^n\left(-a^2\right)$. Moreover, these are the only possibilities for g to be a warped product of the type $dr^2 + \rho^2(r)\, g_N$, where $\rho : \mathbb{R} \to (0, \infty)$.*

Proof. From the last formula in the previous proposition and Cauchy-Schwarz we have

$$\frac{(\Delta r)^2}{n-1} \leq |\operatorname{Hess} r|^2 = a\Delta r.$$

This establishes the range of possible values for Δr. When $\Delta r = 0$, the Hessian vanishes and we obtain a product metric. While when Δr is maximal we must have that $\operatorname{Hess} r = a g_r$, where $g = dr^2 + g_r$. This shows that $L_{\nabla r} g_r = 2a g_r$ and consequently that $g_r = e^{2ar} g_0$. This shows that

$$q = a^2 dr^2 + a \operatorname{Hess} r = a^2 g.$$

When $a \neq 0$ this shows that ∇e^{ar} is a conformal field that is not a Killing field. As the metric is homogeneous we conclude that it must be locally conformally flat. Theorem 2.6 then shows that the space is isometric to $H^n\left(-a^2\right)$.

Finally if we assume that $g = dr^2 + \rho^2(r)\, g_N$, then $\operatorname{Hess} r = \frac{\rho'}{\rho} g_r$. In particular, the Cauchy-Schwarz inequality $\frac{(\Delta r)^2}{n-1} \leq |\operatorname{Hess} r|^2$ must be an equality. This forces us to be in one of the two previous situations. □

The goal for the remainder of the section is to construct examples indicating that there is little hope for classifying the general situation where q is divergence free. To that end it is convenient to use the following condition.

Proposition 5.9. *Assume that G is a transitive group of isometries on M and that $r : M \to \mathbb{R}$ is a smooth distance function whose Hessian is invariant under G. If the hypersurface $N = \{x \in M \mid r(x) = 0\}$, has divergence free second fundamental form at one point, then it is possible to find $a \in \mathbb{R}$ such that $q = \frac{1}{w}\operatorname{Hess} w$ is divergence free and G invariant, where $w = e^{ar}$.*

Proof. First note that as G acts transitively we only need to check that an invariant tensor is divergence free at a specific point p.

When $w = e^{ar}$ we have that $q = \frac{1}{w}\operatorname{Hess} w = a^2 dr^2 + a \operatorname{Hess} r$. Thus q is also invariant under G. The divergence is:

$$\operatorname{div} q = \operatorname{div}\operatorname{Hess} r + a\Delta r dr.$$

By invariance, it follows that $\operatorname{div}\operatorname{Hess} r(\nabla r)$ is constant. Thus we can choose a so that $\operatorname{div} q(\nabla r) = 0$. This shows that we obtain $\operatorname{div} q = 0$ when $\operatorname{div}\operatorname{Hess} r(X) = 0$ for $X \perp \nabla r$. As $\operatorname{Hess} r$ is the second fundamental form for the level sets for r we need to check that $\operatorname{div}\operatorname{Hess} r(X) = \operatorname{div}_N \mathrm{II}(X)$. This follows provided $(\nabla_{\nabla r}\operatorname{Hess} r)(\nabla r, X) = 0$ and that calculating this divergence intrinsically on N is the same as calculating it with the connection on M. We will check this for the type changed $(1,1)$-tensor $S(X) = \nabla_X \nabla r$. For the intrinsic part use an orthonormal frame E_i for N:

$$\begin{aligned}
\left(\nabla^M_{E_i}\mathrm{II}\right)(E_i, X) &= g\left(\left(\nabla^M_{E_i} S\right)(X), E_i\right) \\
&= g(\nabla_{E_i}(S(X)) - S(\nabla_{E_i} X), E_i) \\
&= g(\nabla^N_{E_i}(S(X)) + g(\nabla_{E_i}(S(X)), \nabla r)\nabla r - S(\nabla^N_{E_i} X) - g(\nabla_{E_i} X, \nabla r) S(\nabla r), E_i) \\
&= g\left(\left(\nabla^N_{E_i} S\right)(X), E_i\right),
\end{aligned}$$

since $\nabla r \perp E_i$ and $S(\nabla r) = 0$. Finally, we also have

$$\begin{aligned}
(\nabla_{\nabla r}\operatorname{Hess} r)(\nabla r, X) &= g\left((\nabla_{\nabla r} S)(\nabla r), X\right) \\
&= g\left(\nabla_{\nabla r}(S(\nabla r)), X\right) - g\left(S(\nabla_{\nabla r}\nabla r), X\right) \\
&= 0. \quad \square
\end{aligned}$$

The general set-up for constructing a 1-dimensional extension is a Lie group H with a derivation D on the Lie algebra $\mathfrak{h}$. This gives us a Lie algebra $\mathfrak{g} = \mathfrak{h} \rtimes \mathbb{R}$ and corresponding Lie group G. The metric is left invariant and preserves orthogonality in the semi-direct splitting $T_e G = \mathfrak{g} = \mathfrak{h} \rtimes \mathbb{R}$. Thus it is determined by a left invariant metric on H. Finally, as in [14], the tensor T that corresponds to the second fundamental form for H is proportional to the symmetric part of the derivation.

Specifically, fix an n-dimensional Lie group H and a left invariant basis X_i for its Lie algebra $\mathfrak{h}$. The structure constants are given by

$$[X_i, X_j] = c^k_{ij} X_k.$$

The Lie group is said to be unimodular if $\operatorname{tr}(\operatorname{ad}_X) = 0$ for all X. This is equivalent to $c^j_{ij} = 0$ for all i. We fix a derivation D, but in what follows the derivation property is not used, only that it is a linear operator on the Lie algebra.

Our calculations will be with respect to a general left invariant metric $g_{ij} = g(X_i, X_j)$. The corresponding connection is given by

$$\begin{aligned}
2g(\nabla_{X_i} X_j, X_k) &= g([X_i, X_j], X_k) - g([X_i, X_k], X_j) - g([X_j, X_k], X_i) \\
&= g_{kl} c^l_{ij} - g_{jl} c^l_{ik} - g_{il} c^l_{jk}.
\end{aligned}$$

The symmetric part of D is given by

$$S = \frac{1}{2}D + \frac{1}{2}D^*,$$
$$S^i_j = \frac{1}{2}D^i_j + \frac{1}{2}g^{il}\left(D^t\right)^k_l g_{kj} = \frac{1}{2}D^i_j + \frac{1}{2}g^{il}D^l_k g_{kj}.$$

This can be type changed to two symmetric bilinear forms: S^{ij} and S_{ij}. Note that

$$S^k_i g_{kj} = S_{ij} = g\left(S\left(X_i\right), X_j\right) = S_{ji} = S^k_j g_{ki}$$

and similarly

$$S^i_k g^{kj} = S^{ij} = S^j_k g^{ki}.$$

Proposition 5.10. *With these assumptions and notation it follows that:*

$$2g\left(\text{div}S, X\right) = \text{tr}\left(D \circ \text{ad}_X\right) + g\left(D, \text{ad}_X\right) - 2\text{tr}\left(\text{ad}_{S(X)}\right).$$

Proof. The goal is to calculate $\text{div}S = g^{ij}\left(\nabla_{X_i}S\right)\left(X_j\right)$. Since it is easier to calculate the corresponding 1-form we calculate instead:

$$\begin{aligned}
2g^{ij}g\left(\left(\nabla_{X_i}S\right)\left(X_j\right), X_k\right) &= 2g^{ij}g\left(\nabla_{X_i}S\left(X_j\right), X_k\right) - 2g^{ij}g\left(\nabla_{X_i}X_j, S\left(X_k\right)\right) \\
&= 2g^{ij}S^\alpha_j g\left(\nabla_{X_i}X_\alpha, X_k\right) - 2g^{ij}S^\alpha_k g\left(\nabla_{X_i}X_j, X_\alpha\right) \\
&= g^{ij}S^\alpha_j\left(g_{k\beta}c^\beta_{i\alpha} - g_{i\beta}c^\beta_{\alpha k} - g_{\alpha\beta}c^\beta_{ik}\right) - g^{ij}S^\alpha_k\left(g_{\alpha\beta}c^\beta_{ij} - g_{i\beta}c^\beta_{j\alpha} - g_{j\beta}c^\beta_{i\alpha}\right) \\
&= S^{i\alpha}g_{k\beta}c^\beta_{i\alpha} - S^\alpha_j c^j_{\alpha k} - S^{i\alpha}g_{\alpha\beta}c^\beta_{ik} - g^{ij}S^\alpha_k g_{\alpha\beta}c^\beta_{ij} + S^\alpha_k c^j_{j\alpha} + S^\alpha_k c^i_{i\alpha} \\
&= S^{i\alpha}c^\beta_{i\alpha}g_{k\beta} - S^\alpha_j c^j_{\alpha k} - S^i_\beta c^\beta_{ik} - g^{ij}c^\beta_{ij}S^\alpha_k g_{\alpha\beta} + S^\alpha_k c^j_{j\alpha} + S^\alpha_k c^i_{i\alpha} \\
&= 0 - 2S^\alpha_j c^j_{\alpha k} - 0 + 2S^\alpha_k c^i_{i\alpha} \\
&= 2\text{tr}\left(S \circ \text{ad}_{X_k}\right) - 2\text{tr}\left(\text{ad}_{S(X_k)}\right) \\
&= \text{tr}\left(D \circ \text{ad}_{X_k}\right) + \text{tr}\left(D^* \circ \text{ad}_{X_k}\right) - 2\text{tr}\left(\text{ad}_{S(X_k)}\right) \\
&= \text{tr}\left(D \circ \text{ad}_{X_k}\right) + g\left(D, \text{ad}_{X_k}\right) - 2\text{tr}\left(\text{ad}_{S(X_k)}\right).
\end{aligned}$$

In other words:

$$2g\left(\text{div}S, X\right) = \text{tr}\left(D \circ \text{ad}_X\right) + g\left(D, \text{ad}_X\right) - 2\text{tr}\left(\text{ad}_{S(X)}\right). \quad \square$$

With a view toward concrete examples note that: $\text{tr}\left(D \circ \text{ad}_X\right)$ does not depend on the metric; while $\text{tr}\left(\text{ad}_{S(X)}\right) = 0$ when the Lie group is unimodular. Keep in mind that $g\left(D, \text{ad}_X\right)$ is not linear in g_{ij}, in the given frame it looks like

$$g^{ij}g\left(D\left(X_i\right), \text{ad}_X\left(X_j\right)\right) = g^{ij}g_{\alpha\beta}D^\alpha_i\left(\text{ad}_X\right)^\beta_j.$$

Example 5.11. The simplest examples are on the 3-dimensional Heisenberg group. This algebra has the single relation: $[X, Y] = Z$. In this basis the adjoint actions have the matrices

$$\text{ad}_X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \text{ad}_Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \text{ad}_Z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We use any derivation of the form:

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

The composition of this derivation with any of the adjoint actions clearly vanishes. So for any metric we get the three equations:

$$\begin{aligned} g(D, \mathrm{ad}_X) &= \lambda_1 g_{31} g^{21} + \lambda_2 g_{32} g^{22} + \lambda_3 g_{33} g^{23} = 0, \\ g(D, \mathrm{ad}_Y) &= -\lambda_1 g_{31} g^{11} - \lambda_2 g_{32} g^{12} - \lambda_3 g_{33} g^{31} = 0, \\ g(D, \mathrm{ad}_Z) &= 0. \end{aligned}$$

These equations are clearly satisfied for any metric of the form

$$\begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & g_{33} \end{bmatrix}$$

as the inverse satisfies $g^{23} = g^{13} = 0$.

Example 5.12. Consider the three dimensional simple (and unimodular) Lie algebra with relations:

$$[X_1, X_2] = X_3, [X_2, X_3] = X_1, [X_3, X_1] = X_2.$$

In this basis we have

$$\mathrm{ad}_{X_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \mathrm{ad}_{X_2} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \mathrm{ad}_{X_3} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We will use $D = \mathrm{ad}_{X_1}$. This derivation is skew-symmetric with respect to the standard biinvariant metric that makes the basis elements have equal length and be orthogonal. To calculate $\mathrm{tr}(D \circ \mathrm{ad}_X)$ we note that:

$$\begin{aligned} \mathrm{tr}(D \circ \mathrm{ad}_{X_1}) &= -2, \\ \mathrm{tr}(D \circ \mathrm{ad}_{X_2}) &= 0, \\ \mathrm{tr}(D \circ \mathrm{ad}_{X_3}) &= 0. \end{aligned}$$

Next we find $g(D, \mathrm{ad}_X)$ for a general metric:

$$\begin{aligned} g(D, \mathrm{ad}_{X_1}) &= g_{33} g^{22} + g_{22} g^{33} - 2 g_{23} g^{23}, \\ g(D, \mathrm{ad}_{X_2}) &= -g_{33} g^{21} + g_{31} g^{23} - g_{21} g^{33} + g_{23} g^{31}, \\ g(D, \mathrm{ad}_{X_3}) &= g_{32} g^{21} - g_{13} g^{22} - g_{22} g^{31} + g_{21} g^{32}. \end{aligned}$$

We then restrict attention a metric of the form

$$\begin{bmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & g_{23} \\ 0 & g_{23} & g_{33} \end{bmatrix}$$

The inverse is

$$\begin{bmatrix} \frac{1}{g_{11}} & 0 & 0 \\ 0 & \frac{g_{33}}{g_{22}g_{33}-g_{23}^2} & -\frac{g_{23}}{g_{22}g_{33}-g_{23}^2} \\ 0 & -\frac{g_{23}}{g_{22}g_{33}-g_{23}^2} & \frac{g_{22}}{g_{22}g_{33}-g_{23}^2} \end{bmatrix}$$

and

$$\begin{aligned} g\left(D, \mathrm{ad}_{X_1}\right) &= 2\frac{g_{22}g_{33}}{g_{22}g_{33}-g_{23}^2} - 2\frac{g_{23}^2}{g_{22}g_{33}-g_{23}^2} = 2, \\ g\left(D, \mathrm{ad}_{X_2}\right) &= -g_{33}\cdot 0 + 0\cdot g^{23} - 0\cdot g^{33} + g_{23}\cdot 0 = 0, \\ g\left(D, \mathrm{ad}_{X_3}\right) &= g_{32}\cdot 0 - 0\cdot g^{22} - g_{22}\cdot 0 + 0\cdot g^{32} = 0. \end{aligned}$$

This results in a 4-dimensional family of metrics with the property that $\mathrm{div}S = 0$. This family includes the Berger spheres.

6. Traceless W

Now we consider the vector space of functions $\mathring{W}(M, g, q)$ satisfying (1.2a). We have the following definition.

Definition 6.1. Let (M, g) be a Riemannian manifold and q a quadratic form. The space of functions $\mathring{W}(M, g, q)$ is essential if $\mathring{W}(M, g, q) \neq W(M, g, q')$ for all quadratic forms q'.

For simply connected spaces of constant curvature, $\mathring{W}(M, g, 0)$ is essential since it is $(n + 2)$-dimensional and $W(M, g, q)$ has maximal dimension $n + 1$ [12, Proposition 1.1]. We have the following result for essential/inessential $\mathring{W}$. The proof is exactly analogous to the proposition in the F case, so we omit it.

Proposition 6.2. *Let (M, g) be a Riemannian manifold and q a quadratic form. $\mathring{W}(M, g, q)$ is essential if and only if $\mathring{W}(M, g, q) \neq W(M, g, q - \phi g)$ for all $\phi \in C^\infty(M)$. Moreover, if $\mathring{W}(M, g, q) = W(M, g, q')$ is inessential and q is invariant under $G \subset \mathrm{Isom}(M, g)$ then q' is also invariant under G.*

This gives the following characterization of essential $\mathring{W}$.

Lemma 6.3. *Let (M, g) be a G-homogeneous manifold and q be a G-invariant tensor. If $\mathring{W}(M, g, q)$ is essential, then (M, g) is locally conformally flat.*

Proof. If $\dim(\mathring{W}) = 1$, take $w \in \mathring{W}$, then clearly $\mathring{W} = W(M, g, q - \frac{\Delta w}{n} g)$, so $\mathring{W}$ is inessential. Therefore, $\dim(\mathring{W}) > 1$. Let w_1, w_2 be linearly independent functions in $\mathring{W}$ and define $V = w_1\nabla w_2 - w_2\nabla w_1$ and note that

$$L_V g = w_1 \mathrm{Hess} w_2 - w_2 \mathrm{Hess} w_1 = \frac{w_1\Delta w_2 - w_2\Delta w_1}{n} g.$$

So V is a conformal field. If V is Killing for all $w_1, w_2 \in \mathring{W}$, then

$$w_1 \mathrm{Hess} w_2 = w_2 \mathrm{Hess} w_1. \tag{6.1}$$

Let $p \in M$ and define $\mathring{W}_p = \{w \in \mathring{W} \mid w(p) = 0\}$. If $\mathring{W}_p = \mathring{W}$ at some point p, then all functions in $\mathring{W}$ vanish at p. Since $\mathring{W}$ is invariant under the transitive group of isometries, this would imply that $\mathring{W}$ is trivial. Therefore, $\mathring{W}_p \neq \mathring{W}$ $\forall p$. In fact, if we define q' by the formula

$$q'_p = \frac{\mathrm{Hess}_p w}{w(p)} \quad \text{where} \quad w \in W \setminus \mathring{W}_p, \tag{6.2}$$

then q' is well defined on all of M by (6.1). We then have that $\mathring{W}(q) = W(q')$ which contradicts that W is essential.

Therefore, if $\mathring{W}$ is essential, then (M, g) must support a non-Killing conformal field and by Proposition 2.5 the space is locally conformally flat. □

In the $\mathring{F}$ case, we showed that any product space was inessential, the following proposition shows that this is not the case for $\mathring{W}$.

Proposition 6.4. *Suppose that $(M^n, g) = (M_1^k \times M_2^{n-k}, g_1 + g_2)$ is a product manifold and $q = c_1 g_1 + c_2 g_2$ where $c_i \in \mathbb{R}$ and $c_1 \neq c_2$, then*

$$\mathring{W}(M, g, q) = W(M_1, g_1, \lambda g_1) \oplus W(M_2, g_2, -\lambda g_2),$$

where $\lambda = c_1 - c_2$.

Proof. If $w \in \mathring{W}$, then $\mathrm{Hess} w(X, U) = 0$ for $X \in TM_1$, $U \in TM_2$, so by [20, Proposition 2.1], $w = w_1 + w_2$ where w_i is a function on M_i. This shows that for $X, Y \in TM_1$

$$\mathring{\mathrm{Hess}} w(X, Y) = \mathrm{Hess}_{g_1} w_1(X, Y) - \frac{\Delta_{g_1} w_1 + \Delta_{g_2} w_2}{n} g_1(X, Y).$$

However, as

$$\mathring{q} = c_1 g_1 + c_2 g_2 - \frac{k c_1 + (n-k) c_2}{n} (g_1 + g_2) = \frac{n-k}{n} \lambda g_1 - \frac{k}{n} \lambda g_2$$

we also have

$$\mathring{\mathrm{Hess}} w(X, Y) = w \mathring{q}(X, Y) = \frac{n-k}{n} \lambda w g_1 (X, Y).$$

Setting these equations for $\mathring{\mathrm{Hess}} w$ equal shows that $\mathrm{Hess}_{g_1} w_1$ is conformal to g_1. Thus

$$\left(\frac{n-k}{n} \lambda w + \frac{\Delta_{g_1} w_1 + \Delta_{g_2} w_2}{n} \right) g_1 = \mathrm{Hess}_{g_1} w_1 = \frac{\Delta_{g_1} w_1}{k} g_1.$$

This implies that there is a constant α such that

$$\frac{n-k}{nk} \Delta_{g_1} w_1 - \frac{n-k}{n} \lambda w_1 = \frac{n-k}{n} \lambda w_2 + \frac{1}{n} \Delta_{g_2} w_2 = \alpha.$$

α is constant as the left side of the equation depends on M_1 only and the right depends on M_2 only.

By assumption $\lambda \neq 0$, so

$$\begin{aligned}
\mathrm{Hess}_{g_1} w_1 &= \frac{\Delta_{g_1} w_1}{k} g_1 = \left(\lambda w_1 + \alpha \frac{n}{n-k} \right) g_1 = \lambda \left(w_1 + \frac{\alpha}{\lambda} \frac{n}{n-k} \right) g_1, \\
\mathrm{Hess}_{g_2} w_2 &= \frac{\Delta_{g_2} w_2}{n-k} g_2 = \left(-\lambda w_2 + \alpha \frac{n}{n-k} \right) g_2 = -\lambda \left(w_2 - \frac{\alpha}{\lambda} \frac{n}{n-k} \right) g_2.
\end{aligned}$$

Taking

$$w_1' = w_1 + \frac{\alpha}{\lambda}\frac{n}{n-k} \qquad w_2' = w_2 - \frac{\alpha}{\lambda}\frac{n}{n-k}$$

we then have $w = w_1 + w_2 = w_1' + w_2'$ where

$$\mathrm{Hess}_{g_1} w_1' = \lambda w_1',$$
$$\mathrm{Hess}_{g_2} w_2' = -\lambda w_2'. \quad \square$$

This gives us the following partial converse to Lemma 6.3

Proposition 6.5. *Let (M, g) be a simply connected homogeneous locally conformally flat manifold. Then there is a unique $\mathrm{Isom}(M, g)$-invariant trace-free quadratic form q such that $\mathring{W}(q)$ is an essential $(n+2)$-dimensional space of functions.*

Proof. For a simply connected space form, since the isometry group acts isotropically, the only $\mathrm{Isom}(M, g)$-invariant trace-free quadratic form is the zero tensor and we have already seen that this is an essential $(n+2)$-dimensional space. For the product cases in Theorem 2.6, since q is assumed to be $\mathrm{Isom}(M, g)$-invariant and the isometry groups split in these cases and act isotropically on each factor, $q = c_1 g_B + c_2 g_F$. For $\mathbb{R}^1$, $\dim(W(\lambda g)) = 2$ for all λ. For $S^k(\kappa)$, $\dim(W(-\kappa g)) = k+1$, $\dim W(0) = 1$, and $\dim(W(\lambda g)) = 0$ for $\lambda \neq 0, -\kappa$. For $H^k(-\kappa)$, $\dim(W(\kappa g)) = k+1$, $\dim W(0) = 1$, and $\dim(W(\lambda g)) = 0$ for $\lambda \neq 0, -\kappa$.

Proposition 6.4 shows that if $c_1 - c_2 = -\kappa$ then $\mathring{W}$ has dimension $n+2$ and is essential. In addition

$$\mathring{q} = c_1 g_B + c_2 g_F - \frac{kc_1 + (n-k)\,c_2}{n}(g_B + g_F) = -\frac{n-k}{n}\kappa g_B + \frac{k}{n}\kappa g_F. \quad \square$$

This now gives us the structure theorem for $\mathring{W}$.

Theorem 6.6. *Let (M, g) be a G-homogeneous manifold and let q be a G-invariant tensor such that there is a non-constant function in $\mathring{W}(M, g, q)$, then (M, g) is isometric to either*

(1) *a locally conformally flat space,*
(2) *a direct product of a homogeneous space and a space of constant curvature with $\mathring{W}$ consisting of functions on the constant curvature factor,*
(3) *$(N \times \mathbb{R})/\pi_1(M)$ with $W = \{w : \mathbb{R} \to \mathbb{R} \mid w'' = \tau w\}$ where $\tau < 0$ is constant, or*
(4) *a one-dimensional extension of a homogeneous space.*

Moreover, when (M, g) is not conformally flat, $\mathring{W}(q) = W(q')$, where q' is a G-invariant tensor of the form $q' = q - \lambda g$ for some $\lambda \in \mathbb{R}$. If, in addition, q is Codazzi, then (M, g) is isometric to one of the cases (1)-(3).

Proof. If $\mathring{W}(q)$ is essential, then (M, g) is locally conformally flat. If $\mathring{W}(q)$ is inessential, then $\mathring{W}(q) = W(q')$ and $q' = q - \phi g$. Since q and q' are both invariant under the transitive group G, ϕ is constant. Then applying Theorem 5.5 to $W(q')$ gives the result. $\square$

Now we prove Theorem 1.10 from the introduction.

Theorem 6.7. *Suppose that (M, g) is a compact locally homogeneous manifold and q a local isometry invariant symmetric two tensor.*

(1) *If $\mathring{F}(q)$ contains a non-constant function, then (M,g) is a spherical space form.*
(2) *If $\mathring{W}(q)$ is non-trivial, then (M,g) is a direct product of a homogeneous space N and a spherical space form, isometric to $(N\times\mathbb{R})/\pi_1(M)$, or isometric to $(S^{n-1}(\kappa)\times\mathbb{R}^1)/\Gamma$.*

In particular any positive function in $\mathring{F}(q)$ or $\mathring{W}(q)$ must be constant.

Proof. Let (M,g) be locally homogeneous and $f\in\mathring{F}(q)$, let $(\widetilde{M},\widetilde{g})$ be the universal cover with covering metric and let $\widetilde{q}=\pi^*q$ be the pullback of q to the universal cover. Then the pullback function $\widetilde{f}=\pi^*f$ is in $\mathring{F}(\widetilde{M},\widetilde{g},\widetilde{q})$. Then, since q is invariant under local isometries of (M,g), it is invariant under the isometry group of $(\widetilde{M},\widetilde{g})$. We can then apply Theorem 4.6 to conclude that $(\widetilde{M},\widetilde{g})$ is a sphere as $\widetilde{f}$ is a bounded function and none of the other possibilities given by Theorem 4.6 admit a function in $\mathring{F}$ which is bounded.

The cases of W and $\mathring{W}$ are similar in that we can apply Theorems 5.5 and 6.6 respectively to the universal covers and the only possibilities for a having a bounded function are the ones given.

We note that none of these spaces in the conclusion of the theorem admit a positive function because all of the solutions on the sphere have zeroes. □

7. Quasi-Einstein and conformally Einstein metrics

In this section, we apply the structure theorems from the previous sections for W and $\mathring{W}$ to the tensor $q=\frac{1}{m}\mathrm{Ric}$ for a constant m. The corresponding equations

$$\mathrm{Hess}\,w=\frac{w}{m}(\mathrm{Ric}-\lambda g)$$
$$\mathring{\mathrm{Hess}}\,w=\frac{w}{m}\mathring{\mathrm{Ric}}$$

are often called the m-quasi Einstein and generalized m-quasi-Einstein equations respectively. In the literature, the equations are often considered in the case where w is a positive function and then the equations can be re-written in terms of f when $w=e^f$, but our results do not require w to be positive. When $m>1$ is an integer, solutions to the m-quasi Einstein equation correspond to warped product Einstein metrics. Theorem 7.1 is a generalization of structure results obtained by the authors with He in [14]. When $m>1$, Lafuente in [17] further showed that if M is a homogeneous m-quasi Einstein metric that is a one-dimensional extension of a homogeneous space, N, then N is an algebraic Ricci soliton.

Thus, directly combining this work with Theorem 6.6, we obtain a result for homogeneous generalized m-quasi Einstein metrics when $m>1$.

Theorem 7.1. *Suppose (M,g) is homogeneous Riemannian manifold that admits a non-constant function w which solves the generalized m-quasi Einstein equation for some $m>1$. Either M is a locally conformally flat space, a product of an Einstein metric and a space of constant curvature, the quotient of the product of a homogeneous space and $\mathbb{R}$, $(H\times\mathbb{R})/\pi_1(M)$, or a one-dimensional extension of an algebraic Ricci soliton metric.*

The construction in [14] also shows that any algebraic Ricci soliton metric can be extended to a warped product Einstein metric and that the derivation used to extend the soliton is a multiple of the soliton derivation.

When $m<0$ the m-quasi Einstein equation does not seem to have been studied in depth. In fact, we will see below that the question of which spaces have one-dimensional extensions that are quasi-Einstein is more complicated in this case. As a simple example of the difference between the $m>0$ and $m<0$ cases consider the m-quasi Einstein structures on S^n and H^n.

Example 7.2. Consider $S^n(\kappa)$ or $H^n(-\kappa)$, the spaces of constant curvature $\pm\kappa$. Clearly $\mathrm{Ric} = \pm\kappa(n-1)g$, but there are non-constant functions satisfying $\mathrm{Hess}w = \mp\kappa w g$. So

$$\mathrm{Ric} - \frac{m}{w}\mathrm{Hess}w = \pm\kappa(n+m-1)g.$$

In particular, when $m < -(n-1)$, then $H^n(-\kappa)$ has $\lambda > 0$ and $S^n(-\kappa)$ has $\lambda < 0$. Note that hyperbolic space is a one-dimensional extension of Euclidean space, so it is possible to have $\lambda > 0$ for a one-dimensional extension, at least when $m < -(n-1)$.

Of special interest is the case $m = 2-n$, $n \geq 3$, where the equation

$$\mathring{\mathrm{Hess}}w = \frac{w}{2-n}\mathring{\mathrm{Ric}}$$

is the almost Einstein equation. If there is a positive solution to this equation we call the space conformally Einstein. Theorem 6.6 shows that the only interesting homogeneous almost Einstein metrics are conformally Einstein.

In dimension 4, homogeneous conformally Einstein spaces are classified in [5] by studying the Bach tensor of homogeneous 4-manifolds. In the classification, any non-symmetric space example is homothetic to one of three families of one-dimensional extensions of 3-dimensional Lie algebras. One of the examples (case (ii) of [5, Theorem 1.1]) is a one-dimensional extension of the Ricci soliton on the 3-dimensional Heisenberg group, the other two families are extensions of the abelian Lie algebra and the extension derivations are not soliton derivations. In particular, these non-soliton families have $\lambda = 0$. Another difference when $m < 0$ is that not all algebraic solitons can be extended to m-quasi Einstein metrics when $m < 0$ as, for example, the solvable 3-dimensional soliton cannot be extended to a conformally Einstein metric.

Inspired by these examples we give two constructions of m-quasi Einstein metrics for any dimension n and parameter m. First we consider when we can extend an algebraic Ricci soliton to an m-quasi Einstein metric for general m.

Proposition 7.3. *Let (H^{n-1}, h) be an algebraic Ricci soliton metric*

$$\mathrm{Ric} = \lambda I + D.$$

There is a non-Einstein homogeneous m-quasi Einstein metric with Lie algebra $\mathbb{R}\xi \ltimes \mathfrak{h}$, where $\mathrm{ad}_\xi = \alpha D$ for some constant α, if and only if $\mathrm{tr}D > m\lambda$.

Remark 7.4. For an algebraic Ricci soliton, $\mathrm{tr}D > 0$ and $\lambda < 0$, so the condition is trivially satisfied when $m > 0$. Also note that tracing the soliton equation gives $\mathrm{tr}(D) = \mathrm{scal} - (n-1)\lambda$, so the condition is equivalent to $\mathrm{scal} > (n+m-1)\lambda$. For the conformal Einstein case, $m = 2-n$ the condition is $\mathrm{scal} > \lambda$.

Remark 7.5. For the soliton on the three-dimensional Heisenberg group $\mathrm{scal} = \lambda/3$ while for the soliton on the three-dimensional Lie group Sol $\mathrm{scal} = \lambda$. In particular, the three dimensional Heisenberg group can be extended to a conformally Einstein metric, but Sol can only be extended to a m-quasi Einstein metric when $m > -2$.

Proof. By [14, Lemma 2.9] the Ricci tensor of such a one-dimensional extension is

$$\begin{aligned}
\mathrm{Ric}\,(\xi,\xi) &= -\alpha^2\mathrm{tr}(S^2), \\
\mathrm{Ric}\,(X,\xi) &= -\alpha\mathrm{div}(S), \\
\mathrm{Ric}\,(X,X) &= \mathrm{Ric}^H(X,X) - \left(\alpha^2\mathrm{tr}S\right)h\,(S(X),X) - \alpha^2 h([S,A](X),X),
\end{aligned} \tag{7.1}$$

where $S = \frac{D+D^t}{2}$ and $A = \frac{D-D^t}{2}$. For an algebraic Ricci soliton, D is symmetric so $S = D$, $A = 0$, $\text{div}(D) = \text{div}(\text{Ric}) = 0$, and $\text{tr}(D^2) = -\lambda \text{tr}(D)$, so we have

$$\begin{aligned} \text{Ric}(\xi, \xi) &= \lambda \alpha^2 \text{tr} D, \\ \text{Ric}(X, \xi) &= 0, \\ \text{Ric}(X, X) &= \lambda g + \left(1 - \alpha^2 \text{tr} D\right) h(D(X), X). \end{aligned}$$

When we write $w = e^{ar}$, then $\text{Hess}w = wa^2 dr \otimes dr - wa\alpha h(S(\cdot), \cdot)$ (see the proof of [14, Theorem 3.3]) and

$$\begin{aligned} \left(\text{Ric} - \frac{m}{w}\text{Hess}w\right)(\xi, \xi) &= \lambda \alpha^2 \text{tr} D - ma^2, \\ \left(\text{Ric} - \frac{m}{w}\text{Hess}w\right)(X, X) &= \lambda h(X, X) + (1 - \alpha^2 \text{tr} D + ma\alpha) h(S(X), X). \end{aligned}$$

So, if we want to obtain $\text{Ric} - \frac{m}{w}\text{Hess}w = \lambda g$, then we have to solve the equations

$$\begin{aligned} \lambda &= \lambda \alpha^2 \text{tr} D - ma^2, \\ 1 &= \alpha^2 \text{tr} D - ma\alpha \end{aligned}$$

for the unknown constants α and a. Multiplying the second equation by λ and subtracting the two equations gives that either $a = 0$ or $a = \alpha\lambda$. The $a = 0$ case is the Einstein case, so we take $a = \alpha\lambda$. Plugging this back into the system gives

$$1 = \alpha^2(\text{tr} D - m\lambda)$$

so there exists such an α if and only if $\text{tr} D > m\lambda$. □

Proposition 7.6. *Let $\mathfrak{h}$ be an abelian Lie algebra and D a normal derivation of $\mathfrak{h}$ such that*

$$\text{tr}(S^2) = -\frac{\text{tr}(S)^2}{m},$$

where $S = \frac{D+D^t}{2}$, then there is a homogeneous m-quasi Einstein metric with Lie algebra $\mathbb{R}\xi \ltimes \mathfrak{h}$ where $\text{ad}_\xi = D$ and $\lambda = 0$.

Remark 7.7. Taking $n = 4$, $m = -2$, we obtain the condition that $2\text{tr}(S^2) = \text{tr}(S)^2$. The examples in [5] have these properties.

Proof. We again use the equations (7.1). Since $\mathfrak{h}$ is abelian, $\text{Ric}^H = 0$, and $\text{div}(S) = 0$ for any D, it follows that

$$\begin{aligned} \left(\text{Ric} - \frac{m}{w}\text{Hess}w\right)(\xi, \xi) &= -\text{tr}(S^2) - ma^2, \\ \left(\text{Ric} - \frac{m}{w}\text{Hess}w\right)(X, \xi) &= 0, \\ \left(\text{Ric}(X, X) - \frac{m}{w}\text{Hess}w\right)(X, X) &= -(\text{tr} S - ma) h(S(X), X). \end{aligned}$$

When $a = \frac{\text{tr} S}{m}$ the condition $\text{tr}(S^2) = -\frac{\text{tr}(S)^2}{m}$ shows that both equations vanish. □

Conversely, we have the following necessary conditions for any m-quasi Einstein metric and, if the derivation is normal, the following partial converse.

Proposition 7.8. *Suppose that there is a homogeneous m-quasi Einstein metric with Lie algebra $\mathbb{R}\xi \ltimes \mathfrak{h}$ where $\mathrm{ad}_\xi = D$ and $w = e^{ar}$. It follows that $\mathrm{div}(S) = 0$ and $\mathrm{tr}(S^2) = -a\mathrm{tr}(S)$. Moreover, if D is normal, then either (H^{n-1}, h) is a Ricci soliton or (H, h) is a flat space and $\mathrm{tr}(S^2) = -\frac{(\mathrm{tr}(S))^2}{m}$.*

Proof. Consider again the equations (7.1). First note that $\left(\mathrm{Ric} - \frac{m}{w}\mathrm{Hess}w\right)(X, \xi) = 0$ implies that $\mathrm{div}(S) = 0$ is necessary.

We also have $q = \frac{\mathrm{Hess}w}{w}$, for q with $\mathrm{div}q = 0$. In terms of r, this gives

$$\mathrm{div}\mathrm{Hess}r = -a\Delta r dr.$$

By the Bochner identity,

$$\mathrm{div}\mathrm{Hess}r = \nabla\Delta r + \mathrm{Ric}(\xi) = \mathrm{Ric}(\xi).$$

So using the equation $\mathrm{Ric}(\xi, \xi) = -\mathrm{tr}(S^2)$ from (7.1) we have

$$-\mathrm{tr}(S^2) = \mathrm{Ric}(\xi, \xi) = \mathrm{div}\mathrm{Hess}r(\xi, \xi) = -a\Delta r = a\mathrm{tr}(S).$$

Now, if D is normal we obtain $[A, S] = 0$ so the equations become

$$\left(\mathrm{Ric} - \frac{m}{w}\mathrm{Hess}w\right)(\xi, \xi) = -\mathrm{tr}(S^2) - ma^2,$$
$$\left(\mathrm{Ric}\,(X, X) - \frac{m}{w}\mathrm{Hess}w\right)(X, X) = \mathrm{Ric}^H(X, X) - (\mathrm{tr}S - ma)\, h\,(S(X), X)\,.$$

Let $\beta = \mathrm{tr}S - ma$. When $\beta \neq 0$ we have $\mathrm{Ric}^H = \lambda + \beta S$, so H is a Ricci soliton.

Otherwise, for $\beta = 0$ it follows that $\mathrm{Ric}^H = \lambda g$, $\mathrm{tr}S = am$, and $\lambda = -\mathrm{tr}(S^2) - ma^2$. But then the equation $\mathrm{tr}(S^2) = -a\mathrm{tr}(S)$ implies that $\lambda = 0$ and consequently H is flat since homogeneous Ricci flat metrics are flat [1]. □

We finish with a final characterization of spaces that are conformally Einstein that comes from a different approach.

Lemma 7.9. *Assume (M^n, g) has a one-dimensional space of solutions to the conformal Einstein equation:*

$$\overset{\circ}{\mathrm{Hess}}w = \frac{w}{2-n}\overset{\circ}{\mathrm{Ric}},$$

i.e., $\tilde{g} = w^{-2}g$ is an Einstein metric. If G is a transitive group of isometries and $H \subset G$ is the co-dimension one normal subgroup that fixes w, then H acts isometrically on the conformally changed Einstein metric $\tilde{g}$ and G acts conformally. Moreover, either

(1) *w is constant and g is Einstein,*
(2) *$w = e^{ar}$ and (M, g) is isometric to $H^n\left(-a^2\right)$, or*
(3) *$w = e^{ar}$ and all conformal fields from the action of G have constant divergence with respect to $\tilde{g}$.*

Proof. Note that G clearly acts conformally with respect to $\tilde{g}$. If G acts isometrically, then w is forced to be constant and vice versa. Thus we can assume that $w = e^{ar}$, $a > 0$. Since H fixes w it follows that it acts isometrically on $\tilde{g}$. This shows that the Riemannian submersion $r : (M, g) \to \mathbb{R}$ can be altered to a Riemannian submersion $\frac{1}{aw} : (M, \tilde{g}) \to (0, \infty)$. We let $\mathfrak{h} \subset \mathfrak{g}$ denote the Lie algebras of vector fields on M that correspond to $H \subset G$. On $(M, \tilde{g})$ all of the fields in $\mathfrak{g}$ are conformal and the fields in $\mathfrak{h}$ are Killing. Consider $Z \in \mathfrak{g} - \mathfrak{h}$ so that $L_Z(\tilde{g}) = \frac{2}{n}(\mathrm{div}_{\tilde{g}} Z)\,\tilde{g}$. A well-known formula by Yano shows that if $u = \frac{\mathrm{div}_{\tilde{g}} Z}{n}$, then

$$L_Z \tilde{\mathrm{Ric}} = -(n-2)\,\mathrm{Hess}_{\tilde{g}}\, u - \Delta u \tilde{g}.$$

As $\tilde{\mathrm{Ric}}$ is Einstein this implies that $u \in \mathring{V}(M, \tilde{g})$. If some nonzero u is constant, then all fields in $\mathfrak{g}$ have constant divergence with respect to $\tilde{g}$ as in case (3). Otherwise, we have a non-constant $u \in \mathring{V}(M, \tilde{g})$. This gives a local warped product structure for $\tilde{g}$. We claim that it is global by showing that $u = u(r)$. Since $\mathfrak{h} \subset \mathfrak{g}$ is an ideal we have that $[X, Z] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$. Thus

$$0 = L_{[X,Z]}\tilde{g} = L_X L_Z \tilde{g} - L_Z L_X \tilde{g} = L_X(2u\tilde{g}) = 2(D_X u)\,\tilde{g}.$$

This shows that u is invariant under H and hence that $u = u(r)$. Thus

$$\tilde{g} = w^{-2} g = dt^2 + \varphi^2(t)\, g_N,$$
$$g = w^2\left(dt^2 + \rho^2 g_N\right) = dr^2 + \rho^2 g_N.$$

When the metric is inessential we can use Corollary 5.8 to conclude that we are in case (2). In case it is essential we can instead use Takagi's classification (see Theorem 2.6) to see that only hyperbolic space can admit solutions of the from $w = e^{ar}, a \neq 0$, to the conformal Einstein equation. □

Appendix A. Kähler manifolds

In this appendix we include a discussion of some of the spaces of functions discussed above on Kähler manifolds. No isometric symmetry is assumed in this section, but we will assume that the tensor q is Hermitian.

Recall that a Kähler manifold is a complex manifold, M, equipped with a Riemannian metric, g, such that the complex structure J is skew-adjoint and parallel with respect to g. A symmetric 2-tensor, q, is called *Hermitian* if $q(Jv, w) = -q(v, Jw)$. If q is Hermitian, then $\chi(v, w) = q(Jv, w)$ defines a 2-form. Note that the Ricci tensor and metric of a Kähler manifold are Hermitian with closed 2-form. Thus, for Kähler gradient Ricci solitons, quasi Einstein metrics, and conformally Einstein metrics the tensor q is Hermitian and the corresponding 2-form is closed.

In fact, the problem of when a Kähler manifold admits a non-trivial function with Hermitian Hessian has been investigated extensively by Derdzinski and Maschler where they obtain interesting results for Kähler conformally Einstein manfiolds [7–9]. Note that functions with Hermitian Hessian are also called Killing potentials because Derdzinski and Maschler show that a function has Hermitian Hessian if and only if J applied to the gradient is a Killing field. Case, Shu, and Wei also obtain a rigidity result for Kähler quasi-Einstein metrics which says that they must be a quotient of a product of a surface and an Einstein metric [6, Theorem 1.3]. In this appendix we verify that this result holds in general for functions in a solution space of the form $W(q)$ when q is Hermitian and the corresponding 2-form is closed.

Proposition A.1. *Let (M, g) be a simply connected Kähler manifold and q a Hermitian symmetric two-tensor such that the corresponding 2-form is closed. If $W(q)$ is non-trivial, then (M, g) is an isometric product $N_1^2 \times N_2^{n-2}$ and W consists of functions on the N_1 factor only.*

Proof. Let w be a non-constant function such that Hess$w = wq$. The proof proceeds as in [6] as the only properties used in the proof come from the general properties of q. We include an outline of the proof for completeness.

Let $\chi(v, w) = q(Jv, w)$, $\omega = g(Jv, w)$ and $\phi = \text{Hess}w(Jv, w) = \frac{1}{2}L_{\nabla w}\omega$. Then ϕ is closed as the Lie derivative of a closed form ω. By assumption $\frac{\phi}{w}$ is also closed as it is equal to χ. Therefore, $dw \wedge \phi = 0$.

Then

$$\begin{aligned}(dw \wedge \phi)(X, Y, Z) &= (D_X w)\phi(Y, Z) + (D_Y w)\phi(Z, X) + (D_Z w)\phi(X, Y) \\ &= (D_X w)g(\nabla_{JY}\nabla w, Z) + (D_Y w)g(\nabla_{JZ}\nabla w, X) + (D_Z w)g(\nabla_{JX}\nabla w, Y)\end{aligned}$$

Taking $X, Y \perp \nabla w$, $Z = \nabla w$ then gives

$$0 = |\nabla w|^2 g(\nabla_{JX}\nabla w, Y) = -|\nabla w|^2 g(\nabla_X \nabla w, JY).$$

So that $\nabla_X \nabla w \perp JY$ whenever $\nabla w \neq 0$. On the other hand, taking $X = \nabla w$, $Y = J\nabla w$ and $Z \perp \nabla w$ we also obtain

$$0 = |\nabla w|^2 \phi(JX, Z) = -|\nabla w|^2 g(\nabla_{\nabla w}\nabla w, Z).$$

Which implies that $\nabla_{\nabla w}\nabla w$ is parallel to ∇w when $\nabla w \neq 0$.

Putting this together shows that $\nabla.\nabla w \in \text{span}\{\nabla w, J\nabla w\}$. The fact that Hess$w$ is Hermitian also implies that $\nabla.J\nabla w = J(\nabla.\nabla w)$. So we also have that $\nabla.J\nabla w \in \text{span}\{\nabla w, J\nabla w\}$.

This implies that $\text{span}\{\nabla w, J\nabla w\}$ is a parallel distribution on the set where $\nabla w \neq 0$ and thus gives an isometric splitting on this set. Since $\nabla.\nabla w = wq$ and q is assumed to be smooth, we also have that this distribution is locally uniformly continuous, so that the isometric splitting extends to the closure of $\{\nabla w \neq 0\}$. However, since w is a Killing potential, by remark 5.4 in [7], $\nabla w \neq 0$ almost everywhere, so we have the isometric splitting on all of M. □

In contrast to this result for $W(q)$ note that there are many interesting examples of Kähler Ricci solitons and Kähler conformally Einstein spaces, so no such strong rigidity is possible for the spaces of function $F(q)$, $\mathring{F}(q)$ or $\mathring{W}(q)$ when q is assumed to be Hermitian. See [7–9] for further results on Kähler Killing potentials.

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