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A fundamental lemma for the Hecke algebra: The Jacquet–Rallis case

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ABSTRACT

We formulate a relative fundamental lemma for a relative spherical Hecke algebra in the context of Jacquet–Rallis transfer. We then use results of Beuzart-Plessis on the Ichino–Ikeda conjecture to verify this fundamental lemma.

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1. Introduction

In this note, we formulate a fundamental lemma for certain modules of a spherical Hecke algebra in the context of Jacquet–Rallis transfer [Zha14b]. We then prove this comparison of relative orbital integrals, extending the result for the unit function due to [Yun11]. More precisely, we introduce an isomorphism between two modules for a spherical Hecke algebra, using work of Offen [Off04]. We then use results of Beuzart-Plessis on the local Ichino–Ikeda conjecture to verify the matching of orbital integrals.

We were led to this statement by considering another relative trace formula (see Remark 1.2). It serves as an example of the types of comparisons one ought to expect based on global comparisons of relative trace formulae, as well as a simple illustration of the method of reducing such fundamental lemma to a comparison of (relative) characters (for example, [Clo90,Hal95,Les19]).

1.1. Smooth transfer for Jacquet-Rallis

Let E/F be an unramified quadratic extension of p-adic fields with odd residue characteristic and let $n \geq 2$. We recall the context of the comparison of orbital integrals á la Jacquet and Rallis. Let $\mathrm{GL}_{n,E} = \mathrm{Res}_{E/F}\,\mathrm{GL}_n$ denote the restriction of scalars of the general linear group from E to F. Consider the symmetric space $\mathrm{S}_n = \mathrm{GL}_{n,E}\,/\,\mathrm{GL}_n$, on which GL_n acts via conjugation.

Now fix a *split* Hermitian space V_n of dimension n over E; for concreteness, we may take $V_n = E_n$ equipped with the Hermitian pairing $\langle \cdot, \cdot \rangle$ represented by the anti-diagonal matrix $J = (J_{ij})$ with $J_{ij} = (-1)^{i-1}\delta_{i,d+1-j}$. Let $\mathrm{U}_n := \mathrm{U}(V_n)$ denote the associated unitary group.

We recall the Jacquet–Rallis transfer between the spaces S_n and U_n. Fix a decomposition $V_n = V_{n-1} \oplus Ee_0$, where $\langle e_0, e_0 \rangle = 1$. Then we have an embedding $\mathrm{GL}_{n-1,E} \hookrightarrow \mathrm{GL}_{n,E}$ by

$$g \longmapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}.$$

We have a similar embedding $\mathrm{U}_{n-1} \hookrightarrow \mathrm{U}_n$, as well as $\mathrm{GL}_{n-1} \hookrightarrow \mathrm{GL}_n$.

Considering the conjugation action of $\mathrm{GL}_{n-1}(F)$ on $\mathrm{S}_n(F)$, an element $x \in \mathrm{S}_n(F)$ is called *relatively regular semi-simple* if its orbit is closed and its stabilizer in $\mathrm{GL}_{n-1}(F)$ is trivial. We denote the locus of such elements as S^{rrs}. Let $\eta = \eta_{E/F}$ denote the quadratic character of $F^\times$ associated to the extension E/F. We consider the orbital integrals

$$\mathrm{Orb}^{\mathrm{GL}_{n-1},\eta}(f', x) = \int\limits_{\mathrm{GL}_{n-1}(F)} f'(g^{-1}xg)\eta(g)dg,$$

where $f' \in C_c^\infty(\mathrm{S}_n(F))$ and $x \in \mathrm{S}_n^{rrs}(F)$ and dg is the Haar measure normalized to give $\mathrm{GL}_{n-1}(\mathcal{O}_F)$ volume 1.

Similarly, the subgroup $\mathrm{U}_{n-1}(F)$ acts via conjugation on $\mathrm{U}_n(F)$, and we say that an element $x \in \mathrm{U}_n(F)$ is relatively regular semi-simple if its orbit is closed and its stabilizer in $\mathrm{U}_{n-1}(F)$ is trivial. We similarly denote the locus of such elements as U_n^{rrs}, and define the Jacquet–Rallis orbital integrals to be

$$\mathrm{Orb}^{\mathrm{U}_{n-1}}(f, y) = \int\limits_{\mathrm{U}_{n-1}(F)} f(h^{-1}yh)g,$$

where $f \in C_c^\infty(\mathrm{U}_n(F))$ and $y \in \mathrm{U}_n^{rrs}(F)$.

We now recall the notions of matching orbits and smooth transfer of test functions. Two elements $x \in \mathrm{S}_n^{rrs}(F)$ and $y \in \mathrm{U}_n^{rrs}(F)$ are said to have *matching orbits* if they are conjugate in $\mathrm{GL}_n(E)$ by an element in the subgroup $\mathrm{GL}_{n-1}(E)$. For a general $x \in \mathrm{S}_n^{rrs}(F)$, there need not exist a matching $y \in \mathrm{U}_n(F)$.

For $x \in \mathrm{S}_n^{rrs}(F)$, we define

$$\Omega(x) = \eta\left(\det(x)^{-[\frac{n+1}{2}]} \det(e_0, xe_0, \dots, x^{n-1}e_0)\right).$$

With this definition, two functions $f \in C_c^\infty(\mathrm{U}_n(F))$ and $f' \in C_c^\infty(\mathrm{S}_n(F))$ are called *smooth transfers* of each other if

$$\Omega(x)\,\mathrm{Orb}^{\mathrm{GL}_{n-1},\eta}(f', x) = \mathrm{Orb}^{\mathrm{U}_{n-1}}(f, y)$$

for all matching $x \in \mathrm{S}_n^{rrs}(F)$ and $y \in \mathrm{U}_n^{rrs}(F)$.

1.2. The fundamental lemma

For both $\mathrm{S}_n(F)$ and $\mathrm{U}_n(F)$, there is a natural vector space of *spherical functions*. That is, on $\mathrm{S}_n(F)$ we consider the compactly-supported $K_n := \mathrm{GL}_n(\mathcal{O}_E)$-invariant functions

$$\mathcal{H}_{K_n}(\mathrm{S}_n(F)) := C_c^\infty(\mathrm{S}_n(F))^{K_n},$$

and on $\mathrm{U}_n(F)$ we consider the usual spherical Hecke algebra of $K_n^{un} := \mathrm{U}_n(\mathcal{O}_F)$-invariant function

$$\mathcal{H}_{K^{un}}(\mathrm{U}_n(F)) := C_c^\infty(\mathrm{U}_n(F))^{K_n^{un}}.$$

In [Off04], Offen studied $\mathcal{H}_{K_n}(\mathrm{S}_n)$ as a module for the spherical Hecke algebra of $\mathrm{GL}_n(E)$. His work implies that these two spaces are isomorphic as $\mathcal{H}_{K_n}(\mathrm{GL}_n(E))$-modules and we introduce an isomorphism of modules (see Section 3 for details), which we call the *relative Satake transform*,

$$\mathcal{RS} : \mathcal{H}_{K_n}(\mathrm{S}_n(F)) \longrightarrow \mathcal{H}_{K_n^{un}}(\mathrm{U}_n(F)).$$

Our main result is the following theorem.

Theorem 1.1. *The relative Satake transform $\mathcal{RS}$ realizes Jacquet–Rallis smooth transfer. That is, for every matching pair $x \in \mathrm{S}_n^{rrs}(F)$ and $y \in \mathrm{U}_n^{rrs}(F)$, and for every $\varphi \in \mathcal{H}_{K_n}(\mathrm{S}_n(F))$, we have the identity*

$$\Omega(x) \operatorname{Orb}^{\mathrm{GL}_{n-1},\eta}(\varphi, x) = \operatorname{Orb}^{\mathrm{U}_{n-1}}(\mathcal{RS}(\varphi), y). \tag{1}$$

Moreover, if there is no class $y \in \mathrm{U}_n(F)$ matching x, then $\operatorname{Orb}^{\mathrm{GL}_{n-1},\eta}(\varphi, x) = 0$.

The phrase “fundamental lemma” in our title refers to this comparison of orbital integrals between certain modules of Hecke algebras.

We prove this theorem by relating the relative Satake transform $\mathcal{RS}$ to a base-change homomorphism associated to the products of unitary and general linear groups arising in the unitary Gan-Gross-Prasad set up and a fundamental lemma with respect to this map; see Section 4 for more details. Utilizing results of Beuzart-Plessis on relative characters [BP20], we reduce this fundamental lemma to a certain identity between two such relative characters. This final identity may be readily checked, proving the theorem.

Remark 1.2. We were led to formulate this result by considerations in our proof of the fundamental lemma for unitary Friedberg-Jacquet periods [Les19]. In that work, we consider the symmetric space

$$\mathrm{X} = \mathrm{U}_{2n} \,/\, \mathrm{U}_n \times \mathrm{U}_n,$$

and study orbital integrals on the tangent space $T_{x_0}\mathrm{X}$ at a certain point.

It turns out that after a sequence of reductions, the unstable (or κ-)orbital integrals arising in that setting may be related to a comparison of orbital integrals analogous to the statement of the theorem above. In particular, the reduction to the spectral identity in Section 5 in part motivated the twisted Jacquet–Rallis comparison introduced in [Les19].

In general, one expects relative fundamental lemmas of the form (1) to hold when ever one expects a global comparison of relative trace formulae. The first such comparison in the literature is that of Jacquet [Jac05] in the study of unitary periods, which played a crucial role for the global applications of the Jacquet–Ye relative trace formula [FLO12].

In the literature on the Jacquet–Rallis trace formula comparison [Zha14b], the automorphic Tchebotarev density theorem of Ramakrishnan [Ram18] has allowed one to bypass the lack of a fundamental lemma as in Theorem 1.1. Roughly speaking, if E/F is a quadratic extension of number fields, that result states that one need only consider places of F which split in E–where the comparison of orbital integrals is trivial (see [Zha14b, Proposition 2.15])–to isolate cuspidal representations on the spectral side of the relative trace formula. This simplification is special to the base change functoriality on general linear groups and is not available for more general comparisons, so it is of interest to understand the relations between unramified relative orbital integrals even in this case. We wish to emphasize however that our main result relies on Ramakrishnan's result, which is used in the proof of Beuzart-Plessis's result [BP20]; a purely local proof of the comparison local relative characters would remove this dependence.

We now outline the contents. In Section 2, we fix notations and define the varieties of interest. Section 3 recalls the relevant work of Offen [Off04] and defines the isomorphism $\mathcal{RS}:\mathcal{H}_{K_n}(\mathrm{S}_n(F))\longrightarrow\mathcal{H}_{K^{un}}(\mathrm{U}_n(F))$. In Section 4, we show how to reduce Theorem 1.1 to Proposition 4.2, which we prove in Section 5 by way of a result of Beuzart-Plessis and an explicit calculation of local spherical characters.

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2. Preliminaries

Let F be a non-archimedean local field of characteristic zero with odd residue characteristic and let E/F be an unramified quadratic extension. Under this assumption, we may choose an element $\varpi\in F\subset E$ giving a uniformizer for both fields. Set $\mathcal{O}_F$ for the valuation ring of F and $\mathcal{O}_E$ for that of E. Set $q=\#(\mathcal{O}_F/\varpi\mathcal{O}_F)$, and note that $q^2=\#(\mathcal{O}_E/\varpi\mathcal{O}_E)$. Fix a non-trivial additive character $\psi:F\to\mathbb{C}^\times$ with conductor $\mathcal{O}_F$, and set $\psi_E=\psi\circ Tr_{E/F}$. Let $\eta=\eta_{E/F}$ denote the quadratic character associated to this extension by local class field theory. By abuse of notation, we also write $\eta:E^\times\to\mathbb{C}^\times$ for the *unique unramified* extension of η to $E^\times$.

For any F-group H, we set $C_c^\infty(H)$ to be the space of compactly-supported smooth test functions. Haar measures are always normalized so that the appropriate maximal compact subgroup has volume 1.

2.1. The symmetric space

Let $\mathrm{G}_n=\mathrm{Res}_{E/F}\,\mathrm{GL}_n$ and let $K_n=\mathrm{GL}_n(\mathcal{O}_E)\subset\mathrm{G}_n(F)$. Let dg denote the Haar measure on $\mathrm{G}_n(F)$ normalized so that $\mathrm{vol}_{dg}(K_n)=1$. We equip G_n with an involutive Galois action $\tau\in\mathrm{Gal}(E/F)$ induced by applying the Galois involution $x\mapsto\overline{x}$ to the matrix entries; we will also denote $\tau(g)$ by $\overline{g}$. Then $\mathrm{G}_n^\tau=\mathrm{GL}_n$ and the map $g\mapsto g\overline{g}^{-1}$ gives a morphism of F-varieties

$$\pi:\mathrm{G}_n\longrightarrow\mathrm{S}_n=\{g\in\mathrm{G}_n:g\overline{g}=I_n\}.$$

This induces an isomorphism $\mathrm{G}_n / \mathrm{G}_n^\tau \cong \mathrm{S}_n$ of F-varieties, and a bijection of F-points $\mathrm{G}_n(F)/ \mathrm{G}_n(F)^\tau \cong \mathrm{S}_n(F)$ by Hilbert's theorem 90. The group $\mathrm{G}_n(F)$ acts on $\mathrm{S}_n(F)$ by the τ-twisted action: for $g \in \mathrm{G}_n(F)$ and $x \in \mathrm{S}_n(F)$, we set

$$g \cdot x = gx\overline{g}^{-1}.$$

This induces an action of $\mathrm{G}_n(F)$ on the Schwartz space $C_c^\infty(\mathrm{S}_n(F))$ by $g\cdot f(x) = f(g^{-1}\cdot x)$.

Let

$$\mathcal{H}_{K_n}(\mathrm{S}_n(F)) = C_c^\infty(\mathrm{S}_n(F))^{K_n}$$

denote the subspace of K_n-fixed functions. We refer to this space as the (spherical) Hecke module of the symmetric variety $\mathrm{S}_n(F)$ as it is naturally equipped with the structure of a module of the spherical Hecke algebra $\mathcal{H}_{K_n}(\mathrm{G}_n(F))$. More precisely, this algebra acts on $\mathcal{H}_{K_n}(\mathrm{S}_n(F))$ via the convolution action: for $f \in \mathcal{H}_{K_n}(\mathrm{G}_n(F))$ and $\varphi \in \mathcal{H}_{K_n}(\mathrm{S}_n(F))$, we set

$$(f * \varphi)(x) = \int_{\mathrm{G}_n(F)} f(g)\varphi(g^{-1} \cdot x)dg.$$

The basic structure of this module was studied by Offen in his thesis work [Off04] (see also [Off08] for a correction of certain computations pertaining to the case at hand). We recall the relevant facts here.

For $m \in \mathbb{Z}_{\geq 1}$, let

$$\Lambda_m^+ = \{\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m : \lambda_i \geq \lambda_{i+1} \geq 0 \text{ for all } i = 1, \dots m-1\}$$

be the set of partitions of length m. For any $\lambda \in \Lambda_m^+$, set the dual partition $\lambda^* = (-\lambda_m, \dots, -\lambda_1)$. For any $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{Z}^m$, let

$$\varpi^\mu = \begin{pmatrix} & & \varpi^{\mu_1} \\ & \iddots & \\ \varpi^{\mu_m} & & \end{pmatrix}.$$

Finally, for any $\lambda \in \Lambda_m^+$, define

$$d_\lambda = \begin{pmatrix} & \varpi^\lambda \\ \varpi^{\lambda^*} & \end{pmatrix},$$

if $n = 2m$ is even; if $n = 2m + 1$ is odd, then set

$$d_\lambda = \begin{pmatrix} & & \varpi^\lambda \\ & 1 & \\ \varpi^{\lambda^*} & & \end{pmatrix}.$$

These elements give representatives for the K_n-orbits on $\mathrm{S}_n(F)$.

Lemma 2.1. *[Off04, Proposition 3.1] The K_n-orbits of $\mathrm{S}_n(F)$ are given by the disjoint union*

$$\mathrm{S}_n(F) = \bigsqcup_{\lambda \in \Lambda_m^+} K \cdot d_\lambda.$$

This decomposition is an analogue of the Cartan decomposition for the space $\mathrm{S}_n(F)$, and immediately implies that the functions $\{\mathbf{1}_\lambda := \mathbf{1}_{K \cdot d_\lambda} : \lambda \in \Lambda_m^+\}$ form a basis for $\mathcal{H}_{K_n}(\mathrm{S}_n(F))$.

2.2. The unitary side

Now suppose that $\langle \cdot, \cdot \rangle_n$ is a split Hermitian form on E^n. Up to isometry, we may choose this form to be represented by the anti-diagonal matrix $J = (J_{ij})$ with $J_{ij} = (-1)^{i-1}\delta_{i,d+1-j}$, and we make this choice to align with that of [Off04]. In particular, the lattice $\mathcal{O}_E^n$ is self-dual with respect to $\langle \cdot, \cdot \rangle_n$. Let $\mathrm{U}_n \subset \mathrm{G}_n$ be the associated (quasi-split) unitary group. The self-dual lattice $\mathcal{O}_E^n$ induces a hyperspecial maximal compact group $K_n^{un} := \mathrm{U}_n(\mathcal{O}_F) \subset \mathrm{U}_n(F)$.

With these choices, we have the spherical Hecke algebra of K_n^{un}-biinvariant functions $\mathcal{H}_{K_n^{un}}(\mathrm{U}_n(F))$ with unit $\mathbf{1}_{K_n^{un}}$.

2.2.1. Symmetric polynomials

We recall the notation for symmetric polynomials used in [Off04]. Set $W \cong S_n$ to be the Weyl group of type A_{n-1}. For $m = \lfloor n/2 \rfloor$, we let $z = (z_1, \dots, z_m) \in \mathbb{C}^m$.

Let $\mathbb{C}[X_1^{\pm 1}, \cdots X_n^{\pm 1}]^W$ denote the algebra of W-invariant polynomials in $X_i^{\pm 1}$. For any $P \in \mathbb{C}[X_1^{\pm 1}, \cdots X_n^{\pm 1}]^W$, we will abuse notation and write $P(q^{-z})$ for the holomorphic function $z \in \mathbb{C}^m \mapsto P(q^{-z}) := P(q^{-z_1}, \dots, q^{-z_m})$, where $q = \#(\mathcal{O}_F/\varpi\mathcal{O}_F)$. We denote the resulting space of functions as $\mathbb{C}[q^{\pm z_1}, \cdots q^{\pm z_m}]^W$.

Define the map $\nu : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ by

$$\nu(z_1, \dots, z_m) = \begin{cases} (-z_1, \dots - z_m, z_m, \dots, z_1) & : n = 2m \text{ is even,} \\ (-z_1, \dots - z_m, 1, z_m, \dots, z_1) & : n = 2m+1 \text{ is odd.} \end{cases}$$

Following the notational convention of [Off04], we define the $\mathbb{C}$-algebra $\mathbb{C}[q^{-2z}, q^{2z}]^W$ is the algebra of holomorphic functions on $\mathbb{C}^m$ given as

$$z \in \mathbb{C}^m \longmapsto P(q^{2\nu(z)}),$$

for all $P \in \mathbb{C}[X_1^{\pm 1}, \cdots X_n^{\pm 1}]^W$. That is,

$$\mathbb{C}[q^{-2z}, q^{2z}]^W = \left\{ P(q^{-2z_1}, \dots, q^{-2z_m}, q^{2z_m}, \dots, q^{2z_1}) \, : \, P \in \mathbb{C}[X_1^{\pm 1}, \cdots X_n^{\pm 1}]^W \right\},$$

if $n = 2m$ is even and

$$\mathbb{C}[q^{-2z}, q^{2z}]^W = \left\{P(q^{-2z_1}, \dots, q^{-2z_m}, 1, q^{2z_m}, \dots, q^{2z_1}) \,:\, P \in \mathbb{C}[X_1^{\pm 1}, \cdots X_n^{\pm 1}]^W\right\},$$

when $n = 2m + 1$ is odd. The resulting algebra is isomorphic to the space of $S_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m$-invariant polynomials in m-variables, where the $\mathbb{Z}/2\mathbb{Z}$-factors act by inversion on the appropriate variables [Off04, Section 2.1, Lemma 4.8].

3. The relative Satake homomorphism

In [Off04], Offen completely describes the $\mathcal{H}_{K_n}(\mathrm{G}_n(F))$-module $\mathcal{H}_{K_n}(\mathrm{S}_n(F))$. This is accomplished by constructing the family of *relative spherical functions* for $\mathrm{S}_n(F)$

$$\Omega_z : \mathrm{S}_n(F) \longrightarrow \mathbb{C},$$

where $z \in \mathbb{C}^m$. These are normalized eigenfunctions for the convolution action of $\mathcal{H}_{K_n}(\mathrm{G}_n(F))$; we only recall the relevant facts about these, referring the reader to [Off04] for additional details.

For any value $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{C}^n$, let Φ_ν be the function on G such that

$$\Phi_\nu(g) = \prod_{i=1}^{n} |a_i|^{\nu_i - \frac{1}{2}(n+1-2i)},$$

where $g = nak$ is the Iwasawa decomposition of g and $a = \mathrm{diag}(a_1, \dots, a_n)$. Recall that for a function $f \in \mathcal{H}_{K_n}(\mathrm{G}_n(F))$, the Satake transform of f is given by

$$\mathrm{Sat}(f)(\nu) := \int_{\mathrm{G}_n} f(g)\Phi_\nu(g)dg.$$

This gives an isomorphism of $\mathbb{C}$-algebras

$$\mathcal{H}_{K_n}(\mathrm{G}_n(F)) \cong \mathbb{C}[q^{\pm 2\nu_1}, \dots, q^{\pm 2\nu_n}]^W$$

satisfying the following: if $\chi_\nu : \mathrm{T}_n(F) \longrightarrow \mathbb{C}^\times$ is the unramified character

$$\chi_\nu(\mathrm{diag}(t_1, \dots, t_n)) = \prod_{i=1}^{n} |t_i|^{\nu_i}$$

of the diagonal torus T_n and $I(\nu)$ is the associated spherical representation, then

$$\mathrm{Sat}(\phi)(\nu) = Trace(\phi \mid I(\nu)) \tag{2}$$

for any $\phi \in \mathcal{H}_{K_n}(\mathrm{G}_n(F))$. Note here that the Satake parameters of $I(\nu)$ are $(q^{-\nu_1}, \dots, q^{-\nu_n})$.

Recall the map $\nu : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ in Section 2.2.1; this gives rise to a surjective map of $\mathbb{C}$-algebra

$$\begin{aligned}
&\mathbb{C}[q^{\pm 2\nu_1}, \dots, q^{\pm 2\nu_n}]^W \xrightarrow{\nu} \mathbb{C}[q^{-2z}, q^{2z}]^W \\
&P(q^{-2\nu_1}, \dots, q^{-2\nu_n}) \longmapsto \begin{cases} P(q^{-2z_1}, \dots, q^{-2z_m}, q^{2z_m}, \dots, q^{2z_1}) & : n = 2m, \\ P(q^{-2z_1}, \dots, q^{-2z_m}, 1, q^{2z_m}, \dots, q^{2z_1}) & : n = 2m+1, \end{cases}
\end{aligned}$$

for all $P \in \mathbb{C}[X_1^{\pm 1}, \cdots X_n^{\pm 1}]^W$. Following the notation of [Off04, Section 4], set $\widetilde{f}(z) = \mathrm{Sat}(f)(\nu(z))$ for $z \in \mathbb{C}^m$. The composition

$$\mathcal{H}_{K_n}(\mathrm{G}_n(F)) \xrightarrow{Sat} \mathbb{C}[q^{\pm 2\nu_1}, \dots, q^{\pm 2\nu_n}]^W \xrightarrow{\nu} \mathbb{C}[q^{-2z}, q^{2z}]^W$$

is thus a surjection.

Proposition 3.1. *[Off04, Lemma 4.2, Theorem 1.2] For $z = (z_1, \dots, z_m) \in \mathbb{C}^m$, there exists a unique function $\Omega_z \in \mathcal{H}_{K_n}(\mathrm{S}_n(F))$, called the normalized relative spherical function, satisfying $\Omega_z(d_0) = 1$ and for any $f \in \mathcal{H}_{K_n}(\mathrm{G}_n(F))$,*

$$f * \Omega_z(x) = \widetilde{f}(z)\Omega_z(x).$$

Indeed, for any $\lambda \in \Lambda_m^+$, we have the evaluation

$$\Omega_z(d_\lambda) = q^{-\langle \lambda, \rho \rangle} \frac{V_\lambda}{V_0} P_z(\lambda),$$

where $P_z(\lambda)$ is a Hall–Littlewood-type polynomial (see [Off08] for more details) and V_λ is an explicit rational function in q.

We endow $\mathrm{S}_n(F)$ with a quotient measure dx normalized so that $\mathrm{vol}(\mathrm{S}_n(F) \cap K_n) = 1$.

Theorem 3.2. *[Off04, Proposition 4.10] Define the* spherical Fourier transform *by*

$$\mathcal{SF}(\varphi)(z) = \int_{\mathrm{S}_n(F)} \varphi(x)\Omega_z(x)dx,$$

where $\varphi \in \mathcal{H}_{K_n}(\mathrm{S}_n(F))$. The transform $\mathcal{SF}$ induces an isomorphism of $\mathcal{H}_{K_n}(\mathrm{G}_n(F))$-modules

$$\mathcal{H}_{K_n}(\mathrm{S}_n(F)) \xrightarrow{\sim} \mathbb{C}[q^{-2z}, q^{2z}]^W,$$

where $\mathcal{H}_{K_n}(\mathrm{G}_n(F))$ acts on the left-hand side through $\nu \circ \mathrm{Sat}$.

In particular, $\mathcal{H}_{K_n}(\mathrm{S}_n(F))$ is cyclic as a $\mathcal{H}_{K_n}(\mathrm{G}_n(F))$-module: for every $f \in \mathcal{H}_{K_n}(\mathrm{S}_n(F))$, there exists a $\phi \in \mathcal{H}_{K_n}(\mathrm{G}_n(F))$ such that

$$f(x) = \phi * \mathbf{1}_0(x).$$

This follows from the simple computation that $\mathcal{SF}(\mathbf{1}_0) = 1$ along with the action of $\mathcal{H}_{K_n}(\mathrm{G}_n(F))$ on $\mathbb{C}[q^{-2z}, q^{2z}]^W$. This motivates the following lemma, which will prove useful later.

Lemma 3.3. *For each $\phi \in \mathcal{H}_{K_n}(\mathrm{G}_n(F))$, we define $\pi_!(\phi) \in \mathcal{H}_{K_n}(\mathrm{S}_n(F))$ by*

$$\pi_!(\phi)(h\overline{h}^{-1}) = \int\limits_{\mathrm{GL}_n(F)} \phi(hg)\eta^n(hg)dg, \tag{3}$$

where dg is normalized so that $\mathrm{vol}(K_n) = 1$*. Then*

$$\pi_!(\phi) = \phi * \mathbf{1}_0.$$

In particular, for each $f \in \mathcal{H}_{K_n}(\mathrm{S}_n(F))$, there exists $\phi \in \mathcal{H}_{K_n}(\mathrm{G}_n(F))$ such that $\pi_!(\phi) = f$.

Proof. First we prove the claim for $\phi = \mathbf{1}_0$. In this special case, it is immediate that $\mathbf{1}_{K_n} * \mathbf{1}_0 = \mathbf{1}_0$. On the other hand,

$$\pi_!(\mathbf{1}_{K_n})(g\overline{g}^{-1}) = \int\limits_{\mathrm{GL}_n(F)} \mathbf{1}_0(gh)\eta^n(hg)dh.$$

The right-hand side is only non-zero if there exists $h \in \mathrm{GL}_n(F)$ such that $hg \in K_n$, in which case $\eta^n(hg) = 1$. This implies that the left-hand side is non-zero only if $g\overline{g}^{-1} \in S \cap K = K \cdot d_0$ [Off04, Lemma 3.3]. Thus, $\pi_!(\mathbf{1}_{K_n}) = c\mathbf{1}_0$ for some constant $c \in \mathbb{C}$. We check that

$$c = \pi_!(\mathbf{1}_{K_n})(1) = \int\limits_{\mathrm{GL}_n(F)} \mathbf{1}_{K_n}(h)dh = \mathrm{vol}(K_n) = 1.$$

For a general $\phi \in \mathcal{H}_{K_n}(\mathrm{G}_n(F))$, then for any other $\phi_1 \in \mathcal{H}_{K_n}(\mathrm{G}_n(F))$,

$$\begin{aligned}\phi * \pi_!(\phi_1)(g\overline{g}^{-1}) &= \int\limits_{\mathrm{GL}_n(E)} \phi(h^{-1})\pi_!(\phi_1)(hg\overline{g}^{-1}\overline{h}^{-1})dh \\ &= \int\limits_{\mathrm{GL}_n(E)} \int\limits_{\mathrm{GL}_n(F)} \phi_1(h^{-1})\phi(hgy)\eta^n(hgy)dgdy\end{aligned}$$

$$= \int_{\mathrm{GL}_n(F)} (\phi * (\phi_1 \cdot \eta^n))(gy)\eta^n(hg)dy = \pi_!(\phi * (\phi_1 \cdot \eta^n))(g\overline{g}^{-1}),$$

where $(\phi_1 \cdot \eta^n)(g) = \phi_1(g)\eta^n(g)$ and $\phi * \phi_1 \in \mathcal{H}_{K_n}(\mathrm{G}_n(F))$ denotes the convolution product. Now set $\phi_1 = \mathbf{1}_{K_n}$ and note that $\mathbf{1}_{K_n} = \mathbf{1}_{K_n} \cdot \eta^n$ since η is unramified. Using the $\mathcal{H}_{K_n}(\mathrm{GL}_n(E))$-module structure of $\mathcal{H}_{K_n}(\mathrm{S}_n(F))$ and our calculation that $\pi_!(\mathbf{1}_{K_n}) = \mathbf{1}_0$, we find that

$$\pi_!(\phi) = \pi_!(\phi * \mathbf{1}_{K_n}) = \phi * \pi_!(\mathbf{1}_{K_n}) = \phi * (\mathbf{1}_{K_n} * \mathbf{1}_0) = (\phi * \mathbf{1}_{K_{n,E}}) * \mathbf{1}_0 = \phi * \mathbf{1}_0,$$

proving the lemma. □

The surjective morphism $\nu \circ Sat : f \mapsto \widetilde{f}$ arises in another context: it realizes the (stable) base change morphism from $\mathrm{GL}_n(E)$ to the unitary group U_n.

Proposition 3.4. *There is a commutative diagram*

$$\begin{tikzcd} \mathcal{H}_{K_n}(\mathrm{G}_n(F)) \arrow[r, "\mathrm{BC}_n"] \arrow[d, "\mathrm{Sat}"] & \mathcal{H}_{K_n^{un}}(\mathrm{U}_n(F)) \arrow[d, "\mathrm{Sat}"] \\ \mathbb{C}[q^{\pm 2\nu_1}, \ldots, q^{\pm 2\nu_n}]^W \arrow[r, "\nu"] & \mathbb{C}[q^{-2z}, q^{2z}]^W, \end{tikzcd} \tag{4}$$

where the top horizontal arrow is the base change homomorphism.

Proof. With our choice of Hermitian form, a maximal torus $M_n \subset \mathrm{U}_n$ is given by elements of the form

$$t = \begin{cases} \mathrm{diag}(t_1, \ldots, t_m, t_m^{-1}, \ldots, t_1^{-1}) & : \text{ if } n = 2m \text{ is even,} \\ \mathrm{diag}(t_1, \ldots, t_m, s, t_m^{-1}, \ldots, t_1^{-1}) & : \text{ if } n = 2m+1 \text{ is odd,} \end{cases}$$

where $s, t_i \in E^\times$ where $\mathrm{Nm}_{E/F}(s) = 1$. The Satake isomorphism for $\mathrm{U}_n(F)$ gives an isomorphism [Mí11, Theorem 2.4]

$$\mathrm{Sat} : \mathcal{H}_{K_n^{un}}(\mathrm{U}_n(F)) \xrightarrow{\sim} \mathcal{H}_{K_n^{un} \cap M_n(F)}(M_n(F))^{W_n^{un}}$$

where $W_n^{un} \cong S_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m$ is the Weyl group of $\mathrm{U}_n(F)$. Since $\mathcal{H}_{K_n^{un} \cap M_n(F)}(M_n(F)) \cong \mathbb{C}[X_1^{\pm 1}, \ldots, X_m^{\pm 1}]$ and

$$\mathbb{C}[q^{-2z}, q^{2z}]^W \cong \mathbb{C}[X_1^{\pm 1}, \ldots, X_m^{\pm 1}]^{S_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m}$$

[Off04, Section 2.1, Lemma 4.8], we see that

$$\mathrm{Sat} : \mathcal{H}_{K_n^{un}}(\mathrm{U}_n(F)) \xrightarrow{\sim} \mathbb{C}[q^{-2z}, q^{2z}]^W.$$

Now the base change map $\mathrm{BC}_n : \mathcal{H}_{K_n}(\mathrm{G}_n(F)) \longrightarrow \mathcal{H}_{K_n^{un}}(\mathrm{U}_n(F))$ may be computed as follows; see [MÍ1, Section 4.1.3]. For $z \in \mathbb{C}^m$, let $\chi_z : M_n(F) \longrightarrow \mathbb{C}^\times$ be the unramified character

$$\chi_z(t) = \prod_{i=1}^{m} |t_i|^{z_i}.$$

Let $I(z)$ denote the associated spherical representation of $\mathrm{U}_n(F)$ with Satake parameters $(q^{-z_1}, \dots, q^{-z_m})$. Let $\mathrm{BC}_n(I(z))$ denote the (stable) base change of $I(z)$ to $\mathrm{G}_n(F)$. Then for any $\phi \in \mathcal{H}_{K_n}(\mathrm{G}_n(F))$, we have $\mathrm{BC}_n(\phi) \in \mathcal{H}_{K_n^{un}}(\mathrm{U}_n(F))$ is the unique spherical function such that

$$Trace(\mathrm{BC}_n(\phi) \mid I(z)) = Trace(\phi \mid \mathrm{BC}_n(I(z))).$$

By [MÍ1, Theorem 4.1], the Satake parameters of $\mathrm{BC}_n(I(z))$ are

$$\begin{cases} \mathrm{diag}(q^{-z_1}, \dots, q^{-z_m}, q^{z_m}, \dots, q^{z_1}) & : \text{ if } n \text{ is even,} \\ \mathrm{diag}(q^{-z_1}, \dots, q^{-z_m}, 1, q^{z_m}, \dots, q^{z_1}) & : \text{ if } n \text{ is odd.} \end{cases}$$

In particular, recalling (2) we conclude that

$$Trace(\phi \mid \mathrm{BC}_n(I(z))) = \mathrm{Sat}(\phi)(\nu(z)),$$

and the proposition follows. □

Combining Theorem 3.2 with Proposition 3.4, we define an isomorphism of $\mathcal{H}_{K_n}(\mathrm{G}_n(F))$-modules

$$\mathcal{RS} := \mathrm{Sat}^{-1} \circ \mathcal{SF} : \mathcal{H}_{K_n}(\mathrm{S}_n(F)) \longrightarrow \mathcal{H}_{K_n^{un}}(\mathrm{U}_n(F)),$$

which we refer to as the **relative Satake isomorphism** for $\mathrm{S}_n(F)$.

Corollary 3.5. *Suppose that $\phi \in \mathcal{H}_{K_n}(\mathrm{G}_n(F))$ and set $f = \phi * \mathbf{1}_0 \in \mathcal{H}_{K_n}(\mathrm{S}_n(F))$. Then*

$$BC_n(\phi) = \mathcal{RS}(f).$$

Proof. Lemma 3.3 states that $\pi_!(\phi) = f$. Theorem 3.2 tells us that $\mathcal{SF}(f)(z) = \widetilde{\phi}(z)$. But the commutativity of the diagram in Proposition 3.4 now implies that

$$\mathcal{SF}(f)(z) = \widetilde{\phi}(z) = \mathrm{Sat} \circ \mathrm{BC}_n(\phi).$$

Applying the inverse Satake transform to both sides proves the claim. □

3.1. Statement of main result

Recalling the orbital integrals and notion of matching orbits from the introduction, we now restate the main theorem.

Theorem 3.6. *The isomorphism $\mathcal{RS} : \mathcal{H}_{K_n}(\mathrm{S}_n(F)) \longrightarrow \mathcal{H}_{K_{ur}}(\mathrm{U}_n(F))$ realizes Jacquet–Rallis smooth transfer. That is, for every regular semi-simple $x \in \mathrm{S}_n(F)$ matching $y \in \mathrm{U}_n(F)$, and for every $f \in \mathcal{H}_{K_n}(\mathrm{S}_n(F))$, we have the identity*

$$\Omega(x)\,\mathrm{Orb}^{\mathrm{GL}_{n-1},\eta}(f, x) = \mathrm{Orb}^{\mathrm{U}_{n-1}}(\mathcal{RS}(f), y).$$

Moreover, if there is no class $y \in \mathrm{U}_n(F)$ matching x, then $\mathrm{Orb}^{\mathrm{GL}_{n-1},\eta}(f, x) = 0$.

As outlined in the introduction, we relate this relative Satake transform to a base change homomorphism associated to larger groups and a fundamental lemma with respect to this map in the next section. We then use work of Beuzart-Plessis [BP20] to reduce this fundamental lemma to a spectral identity. This final identity is checked in Section 5.

4. Reduction to stable base change

We first introduce the groups and orbital integrals present in the Jacquet–Rallis comparison; this is discussed in [Zha14b]. Recall our decomposition $E^n = E^{n-1} \oplus Ee_0$, where $\langle e_0, e_0\rangle = 1$. This gave an embedding $\mathrm{GL}_{n-1} \hookrightarrow \mathrm{GL}_n$ by

$$g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}.$$

Now consider the group $\mathrm{G}_{n-1} \times \mathrm{G}_n$ as well as the subgroups $H_1' = \mathrm{G}_{n-1}$ embedded diagonally and $H_2' = \mathrm{G}_{n-1}^{\tau} \times \mathrm{G}_n^{\tau} = \mathrm{GL}_{n-1} \times \mathrm{GL}_n$. We say that an element of $\mathrm{G}_{n-1}(F) \times \mathrm{G}_n(F)$ is *regular semi-simple* if its $(H_1'(F), H_2'(F))$-orbit is closed and the stabilizer subgroup is trivial.

We have a similar embedding $\mathrm{U}_{n-1} \hookrightarrow \mathrm{U}_n$. Consider the group $\mathrm{U}_{n-1} \times \mathrm{U}_n$, and diagonal subgroup $H = \mathrm{U}_{n-1}$. We say that an element of $\mathrm{U}_{n-1}(F) \times \mathrm{U}_n(F)$ is *regular semi-simple* if its $(H(F), H(F))$-orbit is closed and the stabilizer subgroup is trivial.

Note that as F-varieties

$$H_1' \backslash \mathrm{G}_{n-1} \times \mathrm{G}_n \cong \mathrm{G}_n,$$

where the right H_2'-action intertwines with the action

$$\gamma \cdot (g_1, g_2) = g_1^{-1} \gamma g_2.$$

Further quotienting out the free action of $\mathrm{GL}_n = \{I_{n-1}\} \times \mathrm{GL}_n$, we obtain a GL_{n-1}-equivariant isomorphism map

$$\pi : H_1' \backslash \mathrm{G}_{n-1} \times \mathrm{G}_n / \{I_{n-1}\} \times \mathrm{GL}_n \longrightarrow \mathrm{S}_n.$$

Moreover, there is a corresponding matching of regular semi-simple orbits: for $(\gamma_{n-1}, \gamma_n) \in \mathrm{G}_{n-1}(F) \times \mathrm{G}_n(F)$ set

$$x = \pi(g_{n-1}^{-1} g_n) = (g_{n-1}^{-1} g_n)(\overline{g_{n-1}^{-1} g_n})^{-1} \in S_n(F).$$

Similarly, there is a U_{n-1}-equivariant isomorphism

$$\begin{aligned} H \backslash \mathrm{U}_{n-1} \times \mathrm{U}_n &\xrightarrow{\sim} \mathrm{U}_n \\ (h_{n-1}, h_n) &\longmapsto y = h_{n-1}^{-1} h_n, \end{aligned}$$

taking regular semi-simple elements to regular semi-simple elements.

We say that regular semi-simple elements

$$(g_{n-1}, g_n) \in \mathrm{G}_{n-1}(F) \times \mathrm{G}_n(F) \text{ and } (h_{n-1}, h_n) \in \mathrm{U}_{n-1}(F) \times \mathrm{U}_n(F)$$

are matching if y and x are in the sense of Section 1.1.

4.1. Orbital integrals redux

We recall the relative orbital integrals with occur in the Jacquet–Rallis transfer: for $f' \in C_c^\infty(\mathrm{G}_{n-1}(F) \times \mathrm{G}_n(F))$ and $(g_{n-1}, g_n) \in (\mathrm{G}_{n-1}(F) \times \mathrm{G}_n(F))^{rrs}$, we set

$$\mathrm{Orb}^{JR,\eta}(f', (g_{n-1}, g_n)) = \int\limits_{\mathrm{GL}_{n-1}(E)} \int\limits_{\mathrm{GL}_{n-1}(F) \times \mathrm{GL}_n(F)} f'(h_1 (g_{n-1}, g_n) h_2^{-1}) \eta(h_2) dh_2 dh_1,$$

where for $h_2 = (h, h') \in \mathrm{GL}_{n-1}(F) \times \mathrm{GL}_n(F)$,

$$\eta(h_2) = \eta(\det(h))^n \eta(\det(h'))^{n-1}.$$

We define the linear map $m' : C_c^\infty(\mathrm{GL}_{n-1}(E) \times \mathrm{GL}_n(E)) \longrightarrow C_c^\infty(\mathrm{GL}_n(E))$ given by

$$m'(\phi')(g) = \int\limits_{\mathrm{GL}_{n-1}(E)} \phi'((h', h'g)) dh'.$$

We also need the transfer factor. For all $(g_{n-1}, g_n) \in (\mathrm{G}_{n-1}(F) \times \mathrm{G}_n(F))^{rrs}$, set

$$\Omega(g_{n-1}, g_n) = \begin{cases} \Omega(x) & : \ n \text{ is even}, \\ \eta(g_{n-1}^{-1} g_n) \Omega(x) & : \ n \text{ is odd}, \end{cases}$$

where $x = \pi(g_{n-1}^{-1}g_n)$. Setting $\overline{f'} = \pi_!(m'(f')) : S_n \to \mathbb{C}$ with $\pi_!$ as in (3), we have the reduction

$$\Omega(g_{n-1}, g_n)\,\mathrm{Orb}^{JR,\eta}(f', (g_{n-1}, g_n)) = \Omega(x)\,\mathrm{Orb}^{\mathrm{GL}_{n-1},\eta}(\overline{f'}, x), \tag{5}$$

where the orbital integral on the right is as in the statement of Theorem 3.6.

The unitary orbital integrals in this case are of the form

$$\mathrm{Orb}^{JR}(f, (\gamma_{n-1}, \gamma_n)) = \int\limits_{\mathrm{U}_{n-1}(F)} \int\limits_{\mathrm{U}_{n-1}(F)} f(h(\gamma_{n-1}, \gamma_n)h')dhdh',$$

for $f \in C_c^\infty(\mathrm{U}_{n-1}(F) \times \mathrm{U}_n(F))$ and $(h_{n-1}, h_n) \in (\mathrm{U}_{n-1}(F) \times \mathrm{U}_n(F))^{rrs}$. We similarly define the linear map $m : C_c^\infty(\mathrm{U}_{n-1}(F) \times \mathrm{U}_n(F)) \longrightarrow C_c^\infty(\mathrm{U}_n(F))$ given by

$$m(\phi)(y) = \int\limits_{\mathrm{U}_{n-1}(F)} \phi((h, hy))dh.$$

If we set $\overline{f} = m(f)$ and $y = h_{n-1}^{-1}h_n$, then we similarly have

$$\mathrm{Orb}^{JR}(f, (h_{n-1}, h_n)) = \mathrm{Orb}^{\mathrm{U}_{n-1}}(\overline{f}, y) \tag{6}$$

With this notation, two functions $f \in C_c^\infty(\mathrm{U}_{n-1}(F)\times\mathrm{U}_n(F))$ and $f' \in C_c^\infty(\mathrm{G}_{n-1}(F)\times \mathrm{G}_n(F))$ are said to be **smooth transfers** of each other if we have

$$\Omega(g_{n-1}, g_n)\,\mathrm{Orb}^{JR,\eta}(f', (g_{n-1}, g_n)) = \mathrm{Orb}^{JR}(f, (h_{n-1}, h_n))$$

for all $(g_{n-1}, g_n) \in (\mathrm{G}_{n-1}(F) \times \mathrm{G}_n(F))^{rrs}$ and $(h_{n-1}, h_n) \in (\mathrm{U}_{n-1}(F) \times \mathrm{U}_n(F))^{rrs}$ with matching orbits.

4.2. Reduction

We need to know that spherical functions all have nice lifts to the group case.

Lemma 4.1. *For any $f \in \mathcal{H}_{K_n}(\mathrm{GL}_n(E))$, there is a spherical function*

$$\phi \in \mathcal{H}_{K_{n-1}\times K_n}(\mathrm{G}_{n-1}(F) \times \mathrm{G}_n(F))$$

such that $m'(\phi) = f$. The analogous statement holds in the unitary case as well.

Proof. Set $\phi = \mathbf{1}_{K_{n-1}} \otimes f$. It is simple to see that this satisfies $m'(\phi) = f$. □

Now consider the tensor product of the base change maps

$$\mathrm{BC}_{n-1} \otimes \mathrm{BC}_n : \mathcal{H}_{K_{n-1}\times K_n}(\mathrm{G}_{n-1}(F) \times \mathrm{G}_n(F)) \longrightarrow \mathcal{H}_{K_{n-1}^{un}\times K_n^{un}}(\mathrm{U}_{n-1}(F) \times \mathrm{U}_n(F)).$$

Proposition 4.2. *The base change map* $\mathrm{BC} := \mathrm{BC}_{n-1} \otimes \mathrm{BC}_n$ *realizes Jacquet–Rallis smooth transfer. That is, for every* $\phi \in \mathcal{H}_{K_{n-1}\times K_n}(\mathrm{G}_{n-1}(F) \times \mathrm{G}_n(F))$ *we have the identity*

$$\Omega(g_{n-1}, g_n) \operatorname{Orb}^{JR,\eta}(\phi, (g_{n-1}, g_n)) = \operatorname{Orb}^{JR}(\mathrm{BC}(\phi), (h_{n-1}, h_n))$$

for all $(g_{n-1}, g_n) \in (\mathrm{G}_{n-1}(F) \times \mathrm{G}_n(F))^{rrs}$ *and* $(h_{n-1}, h_n) \in (\mathrm{U}_{n-1}(F) \times \mathrm{U}_n(F))^{rrs}$ *with matching orbits. Moreover,* $\operatorname{Orb}^{JR,\eta}(\phi, (g_{n-1}, g_n)) = 0$ *if there exists no matching* (h_{n-1}, h_n).

We postpone the proof of this proposition to the next section as it requires introducing more notation. We now show that this suffices to prove Theorem 3.6.

Proposition 4.3. *Proposition 4.2 implies Theorem 3.6.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathcal{H}_{K_{n-1}\times K_n}(\mathrm{G}_{n-1}(F) \times \mathrm{G}_n(F)) \arrow[rrrr, "\mathrm{BC}_{n-1}\otimes\mathrm{BC}_n"] \arrow[drr, "m'"] & & & & \mathcal{H}_{K^{un}_{n-1}\times K^{un}_n}(\mathrm{U}_{n-1}(F) \times \mathrm{U}_n(F)) \arrow[dd, "m"] \\
& & \mathcal{H}_{K_n}(\mathrm{G}_n(F)) \arrow[dll, "-*\mathbf{1}_0"'] \arrow[drr, "\mathrm{BC}_n"] & & \\
\mathcal{H}_{K_n}(\mathrm{S}_n(F)) \arrow[rr, "\mathcal{SF}"] & & \mathbb{C}[q^{-2z}, q^{2z}]^W & & \mathcal{H}_{K^{un}}(\mathrm{U}_n(F)), \arrow[ll, "Sat"']
\end{tikzcd}$$

where the morphisms m and m' were introduced in the previous section. Beginning in the lower-left corner, we will show that the assumption that the top horizontal arrow affects smooth transfer implies that the bottom horizontal arrow does.

Let $f \in \mathcal{H}_{K_n}(\mathrm{S}_n(F))$. By Lemma 3.3, there is a $\phi \in \mathcal{H}_{K_n}(\mathrm{G}_n(F))$ such that $\pi_!(\phi) = f$ and $\mathcal{SF}(f)(z) = \widetilde{\phi}(z)$. By Lemma 4.1, the function $\overline{\phi} = \mathbf{1}_{K_{n-1}} \otimes \phi \in \mathcal{H}_{K_{n-1}\times K_n}(\mathrm{G}_{n-1}(F) \times \mathrm{G}_n(F))$ satisfies $m'(\overline{\phi}) = \phi$. Similarly, if we consider $\mathrm{BC}_n(\phi) \in \mathcal{H}_{K^{un}}(U_n)$, we have the lift $\overline{\mathrm{BC}_n(\phi)} = \mathbf{1}_{K^{un}_{n-1}} \otimes \mathrm{BC}(\phi) \in \mathcal{H}_{K^{un}_{n-1}\times K^{un}_n}(\mathrm{U}_{n-1}(F))$.

Note that $\mathrm{BC}(\overline{\phi}) = \overline{\mathrm{BC}(\phi)}$, since we are multiplying both sides by the unit of the appropriate Hecke algebra. In particular, Proposition 4.2 implies that $\overline{\phi}$ and $\overline{\mathrm{BC}(\Phi)}$ are Jacquet–Rallis transfers of one another. By the relations of orbital integrals and transfer factors in (5) and (6),

$$\begin{aligned}
\Omega(x) \operatorname{Orb}^{\mathrm{GL}_{n-1},\eta}(f, x) &= \Omega(\delta_{n-1}, \delta_n) \operatorname{Orb}^{JR,\eta}(\overline{\phi}, (\delta_{n-1}\delta_n)) \\
&= \operatorname{Orb}^{JR}(\overline{\mathrm{BC}(\phi)}, (\gamma_{n-1}, \gamma_n)) \\
&= \operatorname{Orb}^{\mathrm{U}_{n-1}}(\mathrm{BC}(\phi), y).
\end{aligned}$$

By Corollary 3.5, we know $\mathrm{BC}(\phi) = \mathcal{RS}(f) \in \mathcal{H}_{K^{un}}(\mathrm{U}_n(F))$. Noting that the appropriate vanishing is similarly implied, we are done. $\square$

5. Proof of the theorem

In this final section, we prove Proposition 4.2. To simplify things, we will adopt new notation for this section alone. Set $G = \mathrm{U}_{n-1}(F) \times \mathrm{U}_n(F)$ and $H = \mathrm{U}_{n-1}(F)$ embedded diagonally. We also set $G' = \mathrm{G}_{n-1}(F) \times \mathrm{G}_n(F)$, $H_1' = \mathrm{GL}_{n-1}$ embedded diagonally, and $H_2' = \mathrm{GL}_{n-1}(F) \times \mathrm{GL}_n(F)$. Recall our fixed additive character ψ_E. Let $N_n(E) \subset \mathrm{GL}_n(E)$ denote the maximal unipotent subgroup of upper triangular matrices, and denote by ψ_E the generic character of $N_n(E)$ formed in the standard fashion.

5.1. Local spherical characters

We recall the definitions of the local spherical characters in this setting (see [BP20, Section 3.2] for more detail). Let π be a tempered representation for G and define the distribution $J_\pi : C_c^\infty(G) \to \mathbb{C}$ to be

$$J_\pi(f) = \int_H Trace(\pi(h)\pi(f))dh$$

for $f \in C_c^\infty(G)$. This is absolutely convergent, and defines a tempered distribution on G called the spherical character of π.

Now suppose that $\Pi = \Pi_{n-1} \boxtimes \Pi_n$ is a generic unitary representation of G'. Let $\mathcal{W}(\Pi_{n-1}, \overline{\psi}_E)$ and $\mathcal{W}(\Pi_n, \psi_E)$ denote the Whittaker models of Π_{n-1} and Π_n respectively and set $\mathcal{W}(\Pi) = \mathcal{W}(\Pi_{n-1}, \overline{\psi}_E) \otimes \mathcal{W}(\Pi_n, \psi_E)$. By strong multiplicity one, we may identify Π with $\mathcal{W}(\Pi)$ unambiguously. For $k \in \{n-1, n\}$ and for ψ_E representing either ψ_k or $\overline{\psi}_E$ as our prior choices dictate, we define the *Flicker-Rallis period* $\beta_k : \mathcal{W}(\Pi_k, \psi_k) \to \mathbb{C}$ by

$$\beta_k(W_k) = \int_{N_{k-1}(F)\backslash \mathrm{GL}_{k-1}(F)} W_k(\epsilon_k(\tau)g_{k-1})\eta(\det(g_{k-1}))^{k-1}dg_{k-1},$$

for $W_k \in \mathcal{W}(\Pi_n, \psi_k)$ and where $\epsilon_{k-1}(\varepsilon) = \mathrm{diag}(\varepsilon^{k-1}, \varepsilon^{k-2}, \dots, \varepsilon, 1)$ for a fixed non-zero traceless element of E ε. Since E/F is assumed to be unramified, we may assume that ε is a unit and that $E = F(\varepsilon)$.

In addition, we recall the scalar products $\theta_k : \mathcal{W}(\Pi_k, \psi_k) \times \mathcal{W}(\Pi_k, \psi_k) \to \mathbb{C}$ defined by

$$\theta_k(W_k, W_k') = \int_{N_{k-1}(E)\backslash \mathrm{GL}_{k-1}(E)} W_k(g_{k-1})\overline{W_k'(g_{k-1})}dg_{k-1}$$

for all $W_k, W_k' \in \mathcal{W}(\Pi_n, \psi_k)$. Both β_k and θ_k are absolutely convergent as integrals (see [JS81, Proposition 1.3]). Set $\beta = \beta_{n-1} \otimes \beta_n$ and $\theta = \theta_{n-1} \otimes \theta_n$.

Finally, for $s \in \mathbb{C}$, we define the local Rankin-Selberg period $\lambda(s, \cdot) : \mathcal{W}(\Pi) \to \mathbb{C}$ by

$$\lambda(s, W_{n-1} \otimes W_n) = \int\limits_{N_{n-1}(E)\backslash \mathrm{GL}_{n-1}(E)} W_{n-1}(g_{n-1}) W_n(g_{n-1}) |\det(g_{n-1})|_E^s dg_{n-1},$$

for all $W_{n-1} \otimes W_n \in \mathcal{W}(\Pi)$. When Π is tempered, this integral is absolutely convergent whenever $Re(s) > -\frac{1}{2}$. In this setting, we denote $\lambda = \lambda(0, \cdot)$.

For a tempered representation Π of G', we may now state the definition of the local spherical character $I_\Pi : C_c^\infty(G') \to \mathbb{C}$. Let $f \in C_c^\infty(G')$ and fix a compact open subgroup $K_f \subset G'$ so that f is K_f-biinvariant. Let $\mathcal{B}_\Pi$ be a basis of Π^{K_f}, orthonormal with respect to the scalar product θ. Then we define

$$I_\Pi(f) = \sum_{W \in \mathcal{B}_\Pi} \lambda\left(\Pi(f) W\right) \overline{\beta(W)}.$$

This sum is independent of K_f and the basis $\mathcal{B}_\Pi$, and defines a tempered distribution on G'

We note that our definitions differ by a constant from those in [BP20] due to our choice in measures. With this in mind, we define

$$\lambda_G = \prod_{i=1}^{n} L(i, \eta^i),$$

where $L(s, \eta)$ is the local L-factor associated to the character η.

5.2. A result of Beuzart-Plessis

We now recall the crucial local result. Let $\mathrm{Temp}(G)$ denote the tempered dual of G.

Proposition 5.1. *[BP20, Corollary 4.5.1] Let $f \in C_c^\infty(G)$ and $f' \in C_c^\infty(G')$. Then f and f' are smooth transfers of each other if and only if we have*

$$I_{\mathrm{BC}(\pi)}(f') = \kappa(\pi) \lambda_G^{-1} J_\pi(f) \tag{7}$$

for all $\pi = \pi_{n-1} \boxtimes \pi_n \in \mathrm{Temp}(G)$ and where

$$\kappa(\pi) = |\varepsilon|_E^{(d_{n-1}+d_n)/2} \left(\frac{\epsilon(\frac{1}{2}, \eta, \psi)}{\eta(-2\varepsilon)} \right)^{(n-1)n/2} \omega_{\mathrm{BC}(\pi_{n-1})}(\varepsilon).$$

Here, $\omega_{\mathrm{BC}(\pi_{n-1})}$ denotes the central character of $\mathrm{BC}(\pi_{n-1})$ and $d_k = \binom{k}{3}$.

The necessity part is one of the main results of [Zha14a] and establishes a conjecture of Wei Zhang. The proof combines global results of [Zha14a], the principle of analytic continuation on the tempered dual $\mathrm{Temp}(G)$, and a local trace formula. Sufficiency is a simple result of the theory of this local trace formula.

Proof of Proposition 4.2. Let $f' \in \mathcal{H}_{K_{n-1}\times K_n}(\mathrm{G}_{n-1}(F)\times \mathrm{G}_n(F))$ and $f = \mathrm{BC}(f') \in \mathcal{H}_{K_{n-1}^{un}\times K_n^{un}}(\mathrm{U}_{n-1}(F)\times \mathrm{U}_n(F))$. By Proposition 5.1, we need to establish the spectral identity (7) for all $\pi = \pi_{n-1}\boxtimes\pi_n \in \mathrm{Temp}(G)$. Since our functions are spherical, both sides vanish away from the spherical component of the tempered dual $\mathrm{Temp}^{sph}(G)$, so that the identity need only be verified on this component. Note that $\kappa(\pi)=1$ for any such π.

Let $\pi = \pi_{n-1}\boxtimes\pi_n$ be an unramified tempered representation with Satake parameter $z=(z_{n-1},z_n)$, and set $\Pi = \mathrm{BC}(\pi)$. Then Π is also unramified, and we let W_0 denote the normalized spherical Whittaker function in $\mathcal{W}(\Pi)$. Then

$$I_\Pi(f') = \frac{\lambda(\Pi(f')W_0)\overline{\beta(W_0)}}{\theta(W_0,W_0)} = \widetilde{f'}(z)\frac{\lambda(W_0)\overline{\beta(W_0)}}{\theta(W_0,W_0)},$$

where we have used the fact that Π is the base change of π to compute the trace of f as $\widetilde{f'}(z)$. On the other hand, setting ϕ_0 for the normalized spherical vector of π we have

$$J_\pi(f) = \int_H \langle \pi(h)\pi(f)\phi_0,\phi_0\rangle dh = Sat(f)(z)\int_H \langle\pi(h)\phi_0,\phi_0\rangle dh.$$

With our choices of measure, we compute

$$\frac{\lambda(W_0)\overline{\beta(W_0)}}{\theta(W_0,W_0)} = \frac{L(\frac{1}{2},\mathrm{BC}(\pi))}{L(1,\pi,\mathrm{Ad})},$$

where

$$L(s,\pi,\mathrm{Ad}) = L(s,\mathrm{BC}(\pi_{n-1}),As^{(-1)^{n-1}})L(s,\mathrm{BC}(\pi_n),As^{(-1)^n})$$

is the product of local Asai L-factors, as in [BP20]. On the other hand, [Har12, Theorem 2.12] shows that

$$\int_H \langle\pi(h)\phi_0,\phi_0\rangle dh = \lambda_G\frac{L(\frac{1}{2},\mathrm{BC}(\pi))}{L(1,\pi,\mathrm{Ad})}.$$

The proposition now reduces to the equality

$$\widetilde{f'}(z) = Sat(f)(z),$$

which follows by Proposition 3.4 and the definition of the base change homomorphism. □

Data availability

No data was used for the research described in the article.

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