Broadcast Packet Erasure Channels with Alternating Single-User Feedback

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***Abstract*—Delayed channel state information (CSI) feedback was shown to be very helpful in enlarging the capacity region of the two-user broadcast packet erasure channel (PEC), even with single-user feedback. However, feedback link itself requires additional resources and may also cause additional delay to data transmission. In this work, we aim to study how to optimally tradeoff the number of feedback bits and the reliable forward communication rate. In our model, one receiver does not provide its CSI while the other one can alternate between delayed CSI feedback and no feedback. This model includes the intermittent single-user feedback as a special case. Our achievability is an extension of previous opportunistic network coding such that the network coding gain can still be enjoyed even when the single-user feedback is not always available. Interestingly, when two users have the same link erasure probabilities, boundaries of the capacity regions are identified since they can be achieved by the proposed schemes. Our results also reveal that even when the single-user feedback is alternating, strictly positive capacity benefits can be attained over the no-feedback capacity.**

I. Introduction

Channel state information (CSI) at the transmitter is a key ingredient for interference management in broadcast channel (BC) [1][2]. For example, in multiple-input single-output (MISO) BC even the delayed CSI is proven to be very helpful in enlarging the degree-of-freedom (DoF) region [1][3][4]. Similar to studies on MISO BCs, delayed CSI at transmitter can also help to enlarge the capacity region of broadcast packet erasure channel (PEC)s. In a packet-based communication network, each hop can be modeled as a packet erasure channel [5], and thus, studying the broadcast PECs provides a good understanding of multi-session uni-casting in small wireless networks [2][6]. However, reliably gathering the CSI in future large-scale networks, especially those in higher frequency bands, would be challenging, and to make matters worse, most control channels are unprotected making them vulnerable to security attacks [7]. Recently, the capacity region of the two-user broadcast PEC with single-user delayed CSI was found [8]. Surprisingly, this capacity region matches that of the broadcast PEC with global delayed CSI of both users.

Typically, the CSI is estimated at the receiver and then fed back to the transmitter. However, maintaining a reliable and secure feedback link also requires wireless resource. Moreover, the CSI feedback itself can contribute to the delay in addition to that from the data transmission link [9]. For 5G low-latency communication, which has a stringent delay constraint, reducing the feedback delay by limiting the number of feedback bits is also important [10]. Target on these design issues, in [11] an intermittent feedback model is considered, where at each time instant, the CSI feedback is either available with unit delay or erased and omitted. When both receivers have intermittent feedback, capacity regions are partially characterized for special cases in [11].

In this work, unlike [11], we consider a single-user feedback setting, meaning that one user never feeds back its CSI to the other nodes in a two-user broadcast PEC. In practice, the CSIT can be heterogeneous (hybrid or even alternating) in a network, mainly because different links may have different compatibilities to know the time-varying channel states [12]. Moreover, for the user that does provide feedback (the feedback user), we remove the restriction in [11] that its feedback state must be independent and identically distributed (i.i.d.) over time enabling more complex feedback strategies. Following the nomenclature in [3], the feedback state of the two receivers is either in "DN" or "NN", where the former means only the first receiver provides delayed CSI feedback while the latter means none of the receivers feed back their CSI. Note that in [3], the DoF region for alternating between feedback states "DN" and "NN" is not considered and any feedback arrangement is symmetric with respect to the receivers. Moreover, the discrete nature on erasure CSI in our broadcast PEC also prohibits techniques in [3], which are tailored for continuous fading channels, to be directly applied here. Note that a single-flow interference network with asymmetric amount of CSI was studied in [13], while the setting was fundamentally different to us.

In this work, we extend the two-phase opportunistic network coding of [8] such that gains to both receivers can be still enjoyed even when the number of single-user feedback is limited. Note that our alternating single-user feedback model includes [8] as a special case when the feedback state is always "DN" during the whole coding block. For the network coding in [8], in Phase, I the transmitter broadcasts coded bits of the no-feedback user and records bits to be recycled, while in Phase II, the transmitter simultaneously sends fresh bits of the feedback user using Automatic Repeat-reQuest (ARQ) alongside the recycled bits. The network coding gain in Phase II is obtained since the ARQ retransmission of

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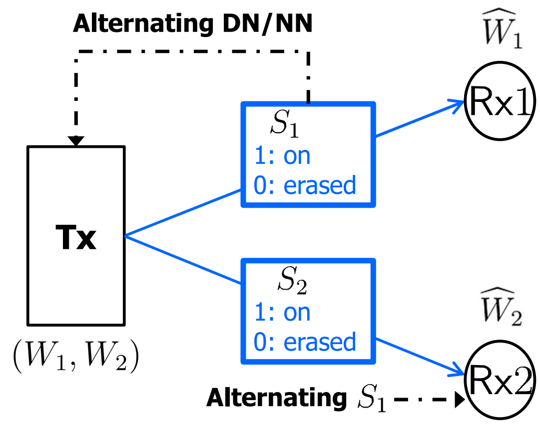

Fig. 1. Broadcast PEC with alternating single-user feedback, where private message W_i is targeted for receiver Rx$i, i = 1, 2$ and the feedback state is alternating between "DN" and "NN".

the feedback user helps aligning its interference at the other one. To meet the constraint on the number of feedback bits, we alternate the feedback erasure probabilities such that the feedback state is in "DN" between Phase I and Phase II. Next, we design a new ARQ control signal when feedback is not always available, which can still align the interference. After optimizing our scheme, the boundary of the rate region match that of the outer bound. Thus, the boundaries of the capacity regions are identified when the two users have the same link erasure probabilities. By modifying the outer bound under i.i.d. feedback state over time, the capacity boundary of the intermittent single-user feedback can also be identified. Finally when there is no feedback at all, the capacity is achieved by simple routing, or time-division multiple access between the two users [14]. Our results also reveal that even when the single-user feedback is alternating, strictly positive capacity benefits can still be obtained over no-feedback capacity [14].

II. Problem Formulation and Review of [8]

We consider the two-user broadcast PEC in which one transmitter wishes to communicate two independent messages W_1 and W_2 to Rx1 and Rx2, respectively, over n channel uses. Here, we assume the messages are independently distributed (from each other and channel parameters), and that each W_i is an nR_i-dimensional vector in a finite field $\mathbb{F}_q$ and uniformly distributed, for $i = 1, 2$. As [6][8], the unit of our rate R_i is packets per time slot and can be converted to the traditional unit bits per time slot by multiplying a factor of $\log_2(q)$. The two messages are mapped to the channel input $X[t] \in \mathbb{F}_q$ and the corresponding received signals at Rx1 and Rx2 are

$$Y_1[t] = S_1[t]X[t], Y_2[t] = S_2[t]X[t] \tag{1}$$

respectively, where $\{S_i[t]\}$ denotes the Bernoulli $(1 - \delta_i)$ process that governs the erasure at Rxi, and it is independent and identically distributed (i.i.d.) over time. When $S_i[t] = 1$, Rxi receives $X[t]$ noiselessly; and when $S_i[t] = 0$, it receives an erasure.

In our single-user feedback framework, we assume Rx2 will not share its erasure states $S_2[t]$ with the other nodes and only Rx1 will feed back its CSI. For the CSI feedback model of $S_1[t]$, we first present the alternating feedback model and argue that the classical intermittent model studied in [11] can be viewed as a special case.

Alternating single-user feedback: As depicted in Fig. 1, there are two possible feedback states from Rx1 and Rx2 at a certain time index: "DN" where Rx1 feeds back its CSI such that transmitter and Rx2 will know S_1 with an unit delay, and "NN" where no receivers provides CSI feedback. Over the total of n channel uses, the fraction for feedback state "DN" is limited to be $0 \leq \lambda_{DN} \leq 1$, while that for feedback state "NN" is $\lambda_{NN} = 1 - \lambda_{DN}$. Note that λ_{DN} is proportional to the number of feedback bits. We define a new binary feedback state $S_{DN}[t]$ such that when $S_{DN}[t] = 1$ the feedback is in "DN"; when $S_{DN}[t] = 0$ the feedback is in "NN". The encoding function $f_t(.)$ at time index t is constrained as

$$X[t] = f_t(W_1, W_2, S_{DN}^{t-1}, \{S_{DN}S_1\}^{t-1}) \ , \tag{2}$$

where

$$S_{DN}^{t-1} = (S_{DN}[1], \ldots, S_{DN}[t-1]) \ , \tag{3}$$

$$\{S_{DN}S_1\}^{t-1} = (S_{DN}[1]S_1[1], \ldots, S_{DN}[t-1]S_1[t-1]). \tag{4}$$

For the CSI at the receivers, following [3][11], each receiver knows its own CSI across the entire transmission block but only Rx2 knows the additional CSI $\{S_{DN}S_1\}$ via the feedback channel. Each receiver Rxi uses its own decoding function

$$\widehat{W}_1 \triangleq g_1 \ (Y_1^n, S_1^n, S_{DN}^n) \ , \tag{5}$$

$$\widehat{W}_2 \triangleq g_2 \ (Y_2^n, S_2^n, \{S_{DN}S_1\}^n) \ , \tag{6}$$

to get the estimate $\widehat{W}_i$ of W_i, respectively for $i = 1, 2$. An error occurs whenever $\widehat{W}_i \neq W_i$ and the average probability of error is given by $\mathbb{E}[P(\widehat{W}_i \neq W_i, i = 1, 2)]$, where the expectation is taken over random and uniform transmitted messages. We say that a rate pair (R_1, R_2) is achievable if there exists an (R_1, R_2)-code and a block decoder at each receiver, such that the average probability of error goes to zero as the block length n goes to infinity. The capacity region, $\mathcal{C}$, is the closure of the set of all achievable rate pairs.

Intermittent single-user feedback: With intermittent feedback, the feedback channel from Rx1 is a binary erasure channel with erasure probability δ_{F1}. In other words, the successful delivery of CSI $S_1[t]$ is governed by a Bernoulli $(1-\delta_{F1})$ process, $\{S_{F1}[t]\}$. More precisely, only if $S_{F1}[t] = 1$ Rx1's CSI $S_1[t]$ is known at other nodes. The feedback process $\{S_{F1}[t]\}$ is i.i.d. over time and independent of the forward delivery process. Compared with alternating single-user feedback, at time index t, if feedback is not erased CSI $S_1[t-1]$ of Rx1 is shared with the other nodes, which corresponds to $S_{DN}[t-1] = S_{F1}[t-1] = 1$ in (4). Also as $n \to \infty$, the successful probability of $1 - \delta_{F1}$ will equal to λ_{DN}. Thus, intermittent feedback can be treated as a special case of alternating feedback when the feedback process is limited to be i.i.d. over time. The encoding function then comes from replacing $S_{DN}[t]s$ in (2) with $S_{F1}[t]$s, while the decoding functions (5) (6) are modified similarly.

Here, we briefly review [8] which characterizes the capacity region for $\lambda_{DN} = 1$, or $S_{DN}[t] = 1, \forall t$. The capacity

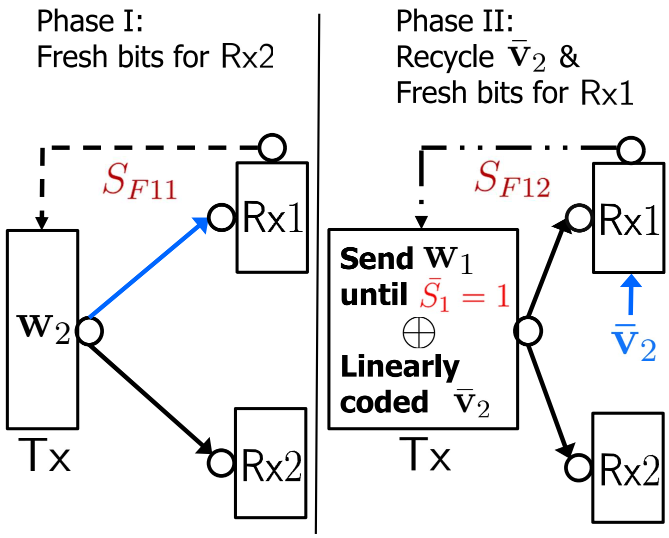

Fig. 2. Proposed achievability for alternating single-user feedback. Compared with [8], the feedback state S_{DN} is now governed by two processes S_{F11} and S_{F12} respectively in Phase I and II, and new control for retransmission is $\bar{S}_1$ in (10).

achieving scheme is a two-phase opportunistic network coding scheme as follows. In Phase I, the transmitter broadcasts coded bits of W_2 for user 2 and records those coded bits received at Rx1 as $\bar{\mathbf{v}}_2$. While in Phase II, the transmitter will simultaneously send fresh bits of user 1 using ARQ and recycled bits $\bar{\mathbf{v}}_2$. Specially, each bits in W_1 is retransmitted as standard ARQ while $\bar{\mathbf{v}}_2$ is re-encoded as a fountain code, and the superposition of them via XORing is sent. Since $\bar{\mathbf{v}}_2$ is already known at Rx1 in Phase I, it will not interfere the decoding of W_1 in Phase II. Also the ARQ retransmission of W_1 helps to align its interference at Rx2. Details please refer to [8].

III. Main results

Our first main result is for alternating single-user feedback as follows, where the first constraint in the outer bound $\mathcal{C}^{ALT}_{out}$ matches that for the inner bound $\mathcal{C}^{ALT}_{in}$.

Theorem 3.1: The outer bound on the capacity region for the two-user broadcast PEC with alternating single-user feedback is

$$\mathcal{C}^{ALT}_{out} = \left\{ (R_1, R_2) \;\middle|\; \begin{array}{l} \dfrac{R_1}{1-\delta_1\delta_2} + \dfrac{R_2}{1-\delta_2} \leq 1 \\ \dfrac{R_1}{1-\delta_1} + \dfrac{R_2}{1-\delta_1\delta_2} \leq 1 \end{array} \right\} \tag{7}$$

while the inner bound is

$$\mathcal{C}^{ALT}_{in} = \left\{ (R_1, R_2) \;\middle|\; \begin{array}{l} \dfrac{R_1}{1-\delta_1\delta_2} + \dfrac{R_2}{1-\delta_2} \leq 1 \\ \dfrac{R_1}{1-\delta_1} + \dfrac{R_2}{1-\delta_2\delta'_{11}} \leq 1 \end{array} \right\} \tag{8}$$

where $(1-\delta'_{11}) = (1-\delta_1)(1-\delta_{F11})$ with

$$\delta_{F11} = 1 - \frac{\lambda_{DN}(1-\delta_2)(\delta_1-\delta_1\delta_2)}{(1-\delta_2)(\delta_1-\delta_1\delta_2)+(1-\lambda_{DN})\delta_2(1-\delta_1)(1-\delta_1\delta_2))} \tag{9}$$

and λ_{DN} being the fraction of feedback state "DN".

Sketch of proof : The capacity region for $\lambda_{DN} = 1$ is proved in [8], and this region naturally serves as outer-bounds for any $\lambda_{DN} < 1$ as $\mathcal{C}^{ALT}_{out}$ in (7). Our main contribution is the inner-bound for $\lambda_{DN} < 1$. We generalize the two-phase opportunistic network coding [8] reviewed in Section II with two main ingredients, as shown in Fig. 2. First, since we can not always let $S_{DN}[t] = 1$ as [8], the feedback probabilities of $S_{DN}[t] = 1$ of Phase I and Phase II are selected to be different. More specially, the feedback process $\{S_{F11}[t]\}$ in Phase I is a Bernoulli$(1-\delta_{F11})$ process while $\{S_{F12}[t]\}$ in Phase II is a Bernoulli$(1-\delta_{F12})$ process. When $S_{DN}[t] = S_{F11}[t] = 1$ in Phase I or $S_{DN}[t] = S_{F12}[t] = 1$ in Phase 2, the feedback state is "DN". The second ingredient is that now the retransmission for W_1 in Phase II can not always be controlled by previous $S_1[t-1]$ as [8], we propose a new ARQ control from $S_{F12}[t-1]$ and $S_{F12}[t-1]S_1[t-1]$ as

$$\bar{S}_1[t-1] = \left\{ \begin{array}{l} 1, \text{ if } S_1[t-1] = S_{F12}[t-1] = 1 \\ 0, \text{ otherwise} \end{array} \right\}. \tag{10}$$

Only when $\bar{S}_1 = 1$ the transmitter is very sure that a bit in W_1 is delivered and proceeds the next fresh message bit for Rx1. Unlike the scheme reviewed in Sec II, now $\bar{\mathbf{v}}_2$ in Phase I of Fig 2 is only parts of the coded bits received at Rx1 since the transmitter needs $S_{F11}[t] = 1$ for recording and recycling them in Phase II. For fixed λ_{DN}, if one wish to have more recycled bits $\bar{\mathbf{v}}_2$ the feedback probability of $S_{F11}[t]$ must be larger, which makes feedback probability $S_{F12}[t]$ in Phase II smaller and the total ARQ retransmission time for W_1 will be longer from (10). This tradeoff does not exists in [8] since when $\lambda_{DN} = 1$ one can simply choose $\delta_{F11} = \delta_{F12} = 0$. Network coding gain is *optimized* over feedback parameters δ_{F11} and δ_{F12} under $\lambda_{DN} < 1$, we have the partially-matched inner bound. Detailed rate analysis please refer to Section IV. ■

The discrete nature of the channel distributions as well as the asymmetricity of the CSI in this work pose new challenges compared to symmetric [15] and the continuous [3] settings. In particular, [8] revealed a surprising result that even with no CSI feedback from one user, the capacity may not degrade in the discrete setting as longs as the other user shares its CSI, which is in sharp contrast to the continuous setting [16]. Further, [3] assumes the alternating CSIT is symmetric for the two users meaning that if for some period of time the network is in "DN", then for an equivalent portion of times it will be in "ND". To have (8), our asymmetric setting prohibits the transmitter to reuse any of the $S_2[t]s$ and we solve this challenge using opportunistic network coding. Finally, if ones follow the steps in [3], the same set of outer-bounds as (7) would be obtained.

Corollary 3.1: When $\delta_1 \geq \delta_2$, the outer bound on the capacity region for the two-user broadcast PEC with intermittent single-user feedback is

$$\mathcal{C}^{IF}_{out} = \left\{ (R_1, R_2) \;\middle|\; \begin{array}{l} \dfrac{R_1}{1-\delta_2\delta'_1} + \dfrac{R_2}{1-\delta_2} \leq 1 \\ \dfrac{R_1}{1-\delta_1} + \dfrac{R_2}{1-\delta_2\delta'_1} \leq 1 \end{array} \right\} \tag{11}$$

while the inner bound is given by

$$\mathcal{C}_{in}^{IF} = \left\{ (R_1, R_2) \left| \begin{array}{l} \frac{R_1}{1-\delta_2\delta_1'} + \frac{R_2}{1-\delta_2} \leq 1 \\ \frac{R_1}{1-\delta_1} + \frac{R_2}{\frac{1-\delta_2\delta_1'}{1+\frac{\delta_2(1-\delta_1')(1-\delta_2\delta_1')(\delta_1'-\delta_1)}{\delta_1'(1-\delta_1)(1-\delta_2)^2}}} \leq 1 \end{array} \right. \right\} \tag{12}$$

where $(1-\delta_1') = (1-\delta_1)(1-\delta_{F1})$.

Proof sketch: Our achievability is a special case of that for (8) by restricting $\delta_{F11} = \delta_{F12} = \delta_{F1}$, reflecting that in the intermittent feedback $S_{DN}[t] = S_{F1}[t]$ for both phases and the feedback process $\{S_{F1}[t]\}$ must be i.i.d. over time. In other words, now (10) is changed to

$$\bar{S}_1[t-1] = \left\{ \begin{array}{l} 1, \text{ if } S_1[t-1] = S_{F1}[t-1] = 1 \\ 0, \text{ otherwise} \end{array} \right\}. \tag{13}$$

The outer bound (11) is from channel enhancement and the results of [15], [11] where the feedback links are available from both receivers. More precisely, to apply [15], [11], we enhance the channel described in Section II by assuming that on top of Rx1, Rx2 also provides intermittent feedback to the other nodes. Furthermore, we choose this virtual feedback link to be fully correlated with the original one from Rx1 and governed by the same Bernoulli$(1-\delta_{F1})$. The capacity of this enhanced channel provides outer-bound $\mathcal{C}_{out}^{IF}$ on the capacity of the channel we study here, with details in Appendix B. ■

In fact, if we follow the steps in [3], for intermittent feedback looser outer-bounds than (11) will be obtained.

From (13), we adopt a conservative scheme for Rx1 which only stops retransmission at time t only when both the feedback state is "DN" and the link to Rx1 is on. Though the retransmission time for Rx1 is long, Rx2 can still benefits from the interference alignment to get useful equations for decoding. This is why our scheme is optimal for intermittent feedback with large enough rate for user 2, as in the matched bounds in $\mathcal{C}_{out}^{IF}$ and $\mathcal{C}_{in}^{IF}$. Also, the channel condition in Corollary 3.1, i.e., $\delta_1 \geq \delta_2$, favors user 2. Unlike the alternating feedback model, one cannot allocate feedback probabilities in two phases of Fig 2 and C_{in}^{IF} is smaller than C_{in}^{ALT}.

Here, we show some numerical examples for Theorem 3.1 and Corollary 3.1 in Fig. 3 and 4, under $\delta_1 = \delta_2 = 0.5$. In Fig. 3, we compare the inner bound (8) in Theorem 3.1 with (12) in Corollary 3.1 under $\lambda_{DN} = 0.8$. Both (8) and (12) are "strictly" larger than the capacity region with no feedback at all [14]. The sum rate of alternating single-user feedback is 0.5800 bpcu, which is higher than the sum rate 0.5552 of the intermittent feedback. Furthermore in Fig. 4, we show that the outer and inner bounds in Theorem 3.1 partially match when $\lambda_{DN} = 0.9$, which means $\mathcal{C}_{in}^{ALT}$ partially matches $\mathcal{C}_{out}^{ALT}$. The sum rates of the outer and inner bounds are 0.6000 and 0.5900, respectively, and the gap is small. Since the outer bound $\mathcal{C}_{out}^{ALT}$ comes from the case where $\lambda_{DN} = 1$, Fig. 4 also shows that the sum rate drop is only 1.7% by saving 10% feedback from the proposed $\mathcal{C}_{in}^{ALT}$.

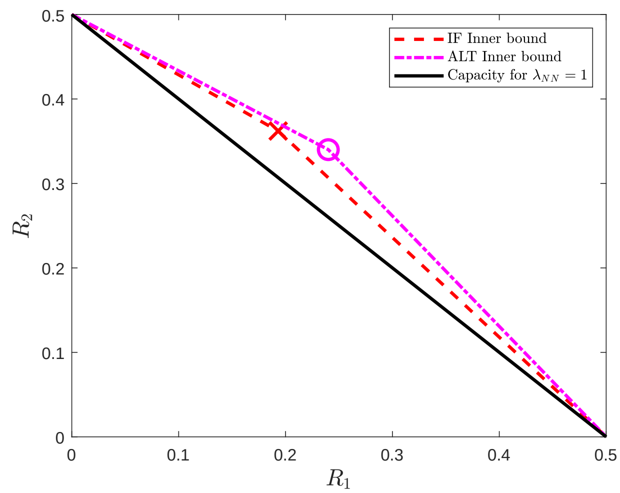

Fig. 3. Comparison between the achievable rate regions of alternating and intermittent single-user feedback under $\lambda_{DN} = 0.8, \delta_1 = \delta_2 = 0.5$.

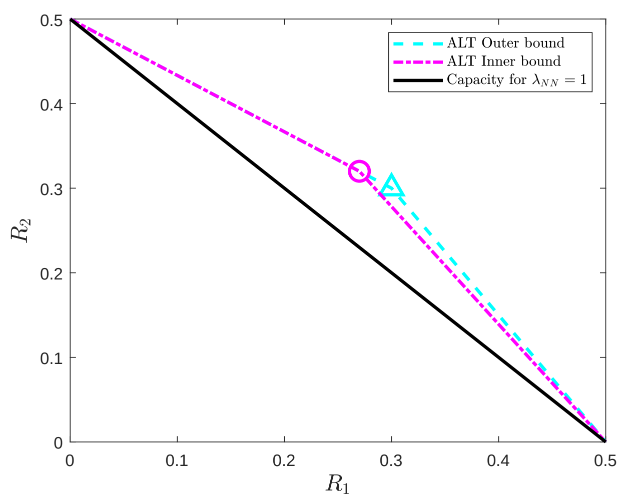

Fig. 4. Comparison between the outer and inner bounds for alternating single-user feedback in Theorem 3.1 under $\lambda_{DN} = 0.9, \delta_1 = \delta_2 = 0.5$.

IV. Proofs of Inner bounds for alternating single-user feedback

We will prove a more general form of $\mathcal{C}_{in}^{ALT}$ in (8) as

$$\begin{array}{l} \frac{R_1}{1-\delta_2\delta_{12}'} + \frac{R_2}{1-\delta_2} \leq 1 \\ \frac{R_1}{1-\delta_1} + \frac{R_2}{\frac{1-\delta_2\delta_{11}'}{1+\frac{\delta_2(1-\delta_{11}')(1-\delta_2\delta_{12}')(\delta_{12}'-\delta_1)}{\delta_{12}'(1-\delta_1)(1-\delta_2)^2}}} \leq 1 \end{array} \tag{14}$$

where $1-\delta_{11}' = (1-\delta_1)(1-\delta_{F11})$ and $1-\delta_{12}' = (1-\delta_1)(1-\delta_{F12})$. After optimizing δ_{F11} and δ_{F12} this inner bound becomes (8) in Theorem 3.1. Now we show the details for the generalization of the two-phase opportunistic network coding [8] reviewed in Section II. In Fig. 2, we alternate the feedback erasure probability to be different between Phase I and Phase II as δ_{F11} and δ_{F12}, respectively. Also in Phase II, we propose a new retransmission control trigger (10) for the next fresh message bit for Rx1. In the following proof, without loss of generality, we assume the channel input is binary with $q = 2$. We also view the two messages W_1 and W_2 as bit vectors $\mathbf{w}_1 \in \mathbb{F}_2^{1\times R_1 n}$ and $\mathbf{w}_2 \in \mathbb{F}_2^{1\times R_2 n}$ respectively, where n is the length of the total two phases.

Let Phases I and II have lengths n_1 and n_2, respectively, where $n_1 + n_2 = n$. The full encoding process comes as follows:

Phase I : Using $\mathbf{g}_2[t_1] \in \mathbb{F}_2^{R_2 n\times 1}$ with each entry generated from i.i.d. $\mathrm{Ber}(1/2)$, the transmitter sends coded bits $X[t_1] = (\mathbf{g}_2[t_1])^\intercal \mathbf{w}_2$, $0 \le t_1 \le n_1$, aimed for Rx2. After Phase I, the transmitter knows $\bar{\mathbf{v}}_2$, which is formed by bits $(\mathbf{g}_2[t_1])^\intercal \mathbf{w}_2$ received at Rx1 in Phase I where *both*

$$S_1[t_1] = S_{F11}[t_1] = 1, \tag{15}$$

and would have a length of

$$n_1(1-\delta_1)(1-\delta_{F11})$$

for large enough n_1. Note in [8], $S_{F11}[t_1] = 1, \forall t_1$.

Phase II : In this phase, all (fresh) bits of message $\mathbf{w}_1$ for user 1 are repeated according to the state feedback as in the standard ARQ, and after XORing a random linear combination of $\bar{\mathbf{v}}_2$, the resulting superposition is sent. Similar to (15), now the ARQ is controlled by the new $\bar{S}_1$ in (10) and the details are given in Algorithm 1. As described in Algorithm 1, at time index t_2, the i-th bit $w_{1,i}$ in $\mathbf{w}_1$ is repeated until $\bar{S}_1 = 1$ (lines 3,4,5), and after XORing a random linear combination $(\mathbf{g}_2[t_2])^\intercal \bar{\mathbf{v}}_2$, the resulting superposition is sent (lines 8,9).

As for decoding, we first focus on Rx2, which first decodes the recycled sequence $\bar{\mathbf{v}}_2$ by opportunistically obtaining pure linear equations of $\bar{\mathbf{v}}_2$ according to ARQ control $\bar{S}_1$. To be more specific, consider the i-th bit $w_{1,i}$ of the recycled sequence $\mathbf{w}_1$. Suppose it is repeated L_i times until $\bar{S}_1 = 1$, where the transmitter is sure its mixture with $\bar{\mathbf{v}}_2$ is successfully arrived at Rx1 in Phase II. Within this span, Rx2 gets

$$K_i \triangleq \sum_{\ell=1}^{L_i} S_{2,i}[\ell] \tag{16}$$

linear equations mixing $w_{1,i}$ and $\bar{\mathbf{v}}_2$, where $S_{2,i}[\ell]$ is the erasure state at Rx2 for the ℓ-th transmission of $w_{1,i}$. During the retransmissions of $w_{1,i}$ controlled by $\bar{S}_1$, if there are two time slots where $S_{2,i}[\ell]$ are both 1, the XOR of the corresponding received bits at Rx2 yields a pure equation of $\bar{\mathbf{v}}_2$, since the interfering bits cancel themselves. In this case, we get $(K_i - 1)^+$ pure equations, where $(x)^+ \equiv \max\{x, 0\}$. Rx2 then uses decoded $\bar{\mathbf{v}}_2$ together with other linear equations arrived during Phase I to decode $\mathbf{w}_2$. For Rx1, the side information $\bar{\mathbf{v}}_2$ is known at Rx1 during Phase I, and then the decoding of private $\mathbf{w}_1$ at Rx1 is straightforward with large enough n_2 after removing the interference from $\bar{\mathbf{v}}_2$.

The rate analysis is modified from that in [8] and given in Appendix A. The main modification is that for Rx2, the expected number of pure equations obtained for decoding $\bar{\mathbf{v}}_2$ in Phase II is

$$(R_1 n)\mathbb{E}[(K_i - 1)^+] = R_1 n \left(\frac{\delta'_{12} - \delta_2}{1-\delta'_{12}} + \frac{(1-\delta'_{12})\delta_2}{1-\delta'_{12}\delta_2} \right). \tag{17}$$

Algorithm 1 Retransmission via $\bar{S}_1$ in Phase II

1: **Initial** Set time index $t_2 = n_1$ the previous time index of Phase II, and $S_{F12}[n_1] = 0$.
2: **for** $i = 1$ to $R_1 n$ **do**
3:   **while** $\bar{S}_1[t-1] \neq 1$ ($S_1[t-1] \neq 1$ or $S_{F12}[t-1] \neq 1$) **do**
4:     **I: Pre-Encoder (PENC) 1 for $\mathbf{w}_1$**
5:       Output the i-th bit $w_{1,i}$ of the input $\mathbf{w}_1$ at t_2
6:     **II: Pre-Encoder (PENC) 2 for $\bar{\mathbf{v}}_2$**
7:       Output a new linear combination $(\mathbf{g}_2[t_2])^\intercal \bar{\mathbf{v}}_2$ of the input $\bar{\mathbf{v}}_2$ at t_2
8:     **III: Superposition**
9:       Send the XOR of outputs of PENC 1 and 2 at t_2
10:     Increase time index t_2 by 1
11:   **end while**
12: **end for**

To see it, from $\bar{S}_1$ in (10) and K_i in (16), the following two events are the same

$$\{L_i = \ell\} \equiv \{\bar{S}_{1,i}[1] = \ldots = \bar{S}_{1,i}[\ell-1] = 0 \text{ and } \bar{S}_{1,i}[\ell] = 1\},$$

where $\bar{S}_{1,i}[\ell]$ at Rx1 is defined similarly as $S_{2,i}[\ell]$. Furthermore, given this event, $\{S_{2,i}[1], ..., S_{2,i}[l]\}$ are independent Bernoulli random variables, and the first $(l-1)$ are Bernoulli with parameter

$$\Pr\{S_2 = 1 | \bar{S}_1 = 0\} = 1 - \delta_2$$

due to the independence of random erasure and feedback states while the last one is Bernoulli with parameter $\Pr\{S_2 = 1 | \bar{S}_1 = 1\} = 1 - \delta_2$. Now Geometric distributed L_i has a success probability $(1-\delta_1)(1-\delta_{F12}) = 1 - \delta'_{12}$ then

$$\mathbb{E}[L_i] = 1/(1-\delta'_{12}), \tag{18}$$

and we have

$$\mathbb{E}[K_i] = \mathbb{E}[L_i - 1](1-\delta_2) + 1 - \delta_2 = \frac{1-\delta_2}{1-\delta'_{12}},$$

$$\mathbb{E}[(K_i - 1)^+] = \mathbb{E}[K_i - 1] + \mathbb{E}\left[(\delta_2)^{L_i}\right] \tag{19}$$

$$= \frac{\delta'_{12} - \delta_2}{1-\delta'_{12}} + \frac{(1-\delta'_{12})\delta_2}{1-\delta'_{12}\delta_2}. \tag{20}$$

where (19) comes from when $K_i = 0$ we need to add back 1 to $K_i - 1$. Then, (17) is valid from (20).

To maximize the network coding gain, we focus on the boundary (14) of $\mathcal{C}_{in}^{ALT}$

$$\frac{R_1}{1-\delta_2\delta'_{12}} + \frac{R_2}{1-\delta_2} \le 1. \tag{21}$$

The slope is optimized when $1 - \delta_2\delta'_{12}$ is maximized, which corresponds to $\delta'_{12} = \delta_1$ or $\delta_{F12} = 0$. After optimization, we have the first inequality in (8). In the meantime, since we let $\delta_{F12} = 0$, the fraction of state "DN"

$$\lambda_{DN} = \left(n_1(1-\delta_{F11}) + n_2(1-\delta_{F12})\right)/n \tag{22}$$

becomes

$$\frac{(1-\delta_2)(\delta_1 - \delta_1\delta_2) + \delta_2(1-\delta_1)(1-\delta_1\delta_2)}{(1-\delta_2)(\delta_1 - \delta_1\delta_2)/(1-\delta_{F11}) + \delta_2(1-\delta_1)(1-\delta_1\delta_2)}.$$

For any $0 \le \lambda_{DN} \le 1$, one can find a corresponding $0 \le \delta_{F11} \le 1$ as (9).

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