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Locally homogeneous non-gradient quasi-Einstein 3-manifolds

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Abstract: In this paper, we classify the compact locally homogeneous non-gradient m-quasi Einstein 3-manifolds. Along the way, we also prove that given a compact quotient of a Lie group of any dimension that is m-quasi Einstein, the potential vector field X must be left invariant and Killing. We also classify the nontrivial m-quasi Einstein metrics that are a compact quotient of the product of two Einstein metrics. We also show that S^1 is the only compact manifold of any dimension which admits a metric which is nontrivially m-quasi Einstein and Einstein.

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1 Introduction

Non-gradient m-quasi Einstein manifolds are of particular interest in the study of near-horizon geometries; see [10], [11], and [13]. In this paper, we study non-gradient m-quasi Einstein manifolds as a generalization of Einstein manifolds, gradient m-quasi Einstein manifolds, and Ricci solitons. In order to define the m-quasi Einstein equation, we must first give the definition of the m-Bakry Émery Ricci tensor:

Definition 1.1. Let X be a vector field on a Riemannian manifold (M^n, g). The m-Bakry Émery tensor is

$$\mathrm{Ric}_X^m := \mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g - \frac{1}{m} X^* \otimes X^*$$

where $\mathcal{L}_X g$ is the Lie derivative of g with respect to X, and $X^* : T_pM \to \mathbb{R}$, $Y \mapsto g(X, Y)$.

If $X = \nabla\phi$ where $\phi : M \to \mathbb{R}$ is a smooth function, the m-Bakry Émery Ricci tensor is

$$\mathrm{Ric}_\phi^m := \mathrm{Ric} + \mathrm{Hess}\,\phi - \frac{1}{m} d\phi \otimes d\phi,$$

and we call this the gradient m-Bakry Émery Ricci tensor. Note that when ϕ is a constant, the gradient m-Bakry Émery Ricci tensor is the Ricci tensor. If $m = \infty$, the m-Bakry Émery Ricci tensor becomes $\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g$.

The ∞-Bakry Émery Ricci curvature was first studied by Lichnerowicz in 1971 in [15], and Qian first studied the gradient m-Bakry Émery Ricci curvature with $m \neq \infty$ in [23]. Bakry and Émery further studied the Bakry Émery Ricci curvature in relation to diffusion processes in [2]. They also arise in the study of optimal transport, Ricci flow, and general relativity. In [17], Lott gives topological consequences and relations to the measured Gromov–Hausdorff limits to lower bounds on the Bakry Émery Ricci curvature. Wei–Wylie prove Bakry Émery Ricci curvature analogs of the comparison theorems and the volume comparison theorem in [25]. There have been many more papers written about the subject, too many to summarize here. Now, we are ready to define the m-quasi Einstein equation.

Definition 1.2. A manifold (M, g) satisfies the m-quasi Einstein equation if $\mathrm{Ric}_X^m = Ag$ for some constant A.

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Remark 1.3. Many authors consider only the gradient case and/or the manifolds with boundary case of the m-quasi Einstein equation. We will assume neither condition in this paper.

The case $m = \infty$ of the m-quasi Einstein equation corresponds to the Ricci soliton equation, $\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = Ag$. Ivey showed in [9] that compact Ricci solitons must be shrinking, i.e. A must be positive. Perelman showed in [19] that compact shrinking Ricci solitons must be gradient. Then Petersen–Wylie showed in [21] that any compact locally homogeneous gradient Ricci soliton is Einstein. Therefore, by Ivey, Perelman, and Petersen–Wylie, there are no non-Einstein non-trivial locally homogeneous compact Ricci solitons.

If (M, g) is m-quasi Einstein and if $X = \nabla\phi$, then we call the space gradient m-quasi Einstein. If $X = 0$, then we call the space trivial. Our first result is the following theorem and gives us a classification of manifolds which are Einstein and m-quasi Einstein.

Theorem 1.4. *Let M^n be a compact Einstein manifold. Then M is non-trivial m-quasi Einstein for $m \neq \infty$ if and only if M is S^1.*

Gradient m-quasi Einstein metrics with $m > 0$ where first systematically considered by Case–Shu–Wei in [4] and Kim–Kim in [12]. They show that gradient m-quasi Einstein metrics correspond to warped product Einstein metrics. In [4, Theorem 2.1], Case–Shu–Wei prove that a compact gradient m-quasi Einstein with constant curvature must be trivial if $m > 0$. Since locally homogeneous manifolds have constant scalar curvature, this shows that compact locally homogeneous manifolds which satisfy $\mathrm{Ric}_\phi^m = Ag$ with $m > 0$ must be trivial. The case $m < 0$ follows from [22, Theorem 1.9]. In [6, Theorem 1.3], He–Petersen–Wylie prove that if (M^3, g) has no boundary, satisfies $\mathrm{Ric}_\phi^m = Ag$ with $m > 1$, and has constant scalar curvature, then M^3 is a quotient of S^3, $S^2 \times \mathbb{R}$, $\mathbb{R}^3$, $H^2 \times \mathbb{R}$, or H^3 with the standard metric. In [7, Theorem 1.4], He–Petersen–Wylie show that if (M^n, g) is a non-compact Ricci soliton with $m > 0$ and $A < 0$, under certain conditions, M admits a non-trivial homogeneous gradient m-quasi Einstein ($\mathrm{Ric}_\phi^m = Ag$) one-dimensional extension. In [14, Theorem 1.1], Lafuente proves a converse to this result.

On the other hand, Chen–Liang–Zhu construct some examples of non-gradient m-quasi Einstein manifolds in [5]. In [13, Corollary 4.1,4.2], Kunduri–Lucietti study the non-gradient m-quasi Einstein metrics with $m = 2$ in the context of vacuum, homogeneous near-horizon geometries, which gives us motivation to study non-gradient m-quasi Einstein metrics.

Our main theorems give us a characterization of Lie groups which have a discrete group of isometries acting cocompactly and which satisfy $\mathrm{Ric}_X^m = Ag$.

Theorem 1.5. *Let G be a Lie group and let Γ be a discrete group of isometries which acts cocompactly on G. Let X be a vector field which is invariant under Γ. If (G, g, X) satisfies $\frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^* \otimes X^* = q$, where q and g are left invariant, then X is left invariant. If we also assume that $\mathrm{tr}(q \circ ad_X) = 0$, then X is a Killing vector field.*

Theorem 1.5 was proven by Chen–Liang–Zhu in [5, Theorem 1.1] in the case when G is a compact Lie group and $q = \mathrm{Ric}$. Our next theorem gives us a characterization of the product of Einstein manifolds of any dimension which satisfy the m-quasi Einstein equation.

Theorem 1.6. *Consider the compact quotient of $M \times N$ with the product metric, where M and N are simply connected complete Einstein manifolds. Then the only nontrivial solutions to $\mathrm{Ric}_X^m = Ag$ occur when either M is $\mathbb{R}$ or N is $\mathbb{R}$.*

We apply the results above to classify the m-quasi Einstein solutions for locally homogeneous 3-manifolds which admit compact quotient.

Theorem 1.7. *Let M^3 be a compact locally homogeneous Riemannian manifold with $\mathrm{Ric}_X^m = Ag$.*

(1) *If $m > 0$ and $A > 0$, then there exist m-quasi Einstein solutions if and only if M^3 is a compact quotient of $\mathrm{SU}(2)$.*
(2) *If $m > 0$ and $A = 0$, then there exist solutions if and only if M^3 is a compact quotient of $\mathrm{SU}(2)$ or $\mathbb{R}^3$, where the solution on $\mathbb{R}^3$ is $X = 0$.*

(3) *If $m > 0$ and $A < 0$, then there exist solutions if and only if M^3 is a compact quotient of* SU(2), Nil, *or* $H^2 \times \mathbb{R}$.

(4) *If $m < 0$ and $A > 0$, then there exist solutions if and only if M^3 is a compact quotient of* SU(2) *or* $S^2 \times \mathbb{R}$.

(5) *If $m < 0$ and $A = 0$, then there exist solutions if and only if M^3 is a compact quotient of* $\mathbb{R}^3$ *or* $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, *where the solution on* $\mathbb{R}^3$ *is trivial.*

(6) *If $m < 0$ and $A < 0$, then there are no m-quasi Einstein solutions on M^3.*

Remark 1.8. In a related paper, Buttsworth [3] studied the prescribed Ricci tensor problem on these spaces. This result when $m = 2$ was also proven by Kunduri–Lucietti in [13].

If M^n is a homogeneous Einstein manifold, where $\mathrm{Ric} = Ag$, then if $A > 0$, then M is compact by Myers' Theorem; if $A = 0$, then M is flat by Alekseevskii–Kimel'fel'd [1], and if $A < 0$, then M is not compact by Bochner's Theorem, which can be found in Section 5. If we compare this to Theorem 1.7, we see that this structure does not hold for m-quasi Einstein metrics. When $A = 0$, there exist solutions on (compact quotients of) SU(2), which are not flat. Similarly, in the case $A < 0$ there exist solutions on compact quotients of SU(2).

In [27, Lemma 4.4], we see that if M^n is a compact manifold with infinite fundamental group satisfying $\mathrm{Ric}_\phi^m = Ag$ where $A = 0$, with $m = 1 - n < 0$, then the universal cover has a warped product splitting. By Theorem 1.7, there exist solutions for the compact quotient of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ if M^n satisfies $\mathrm{Ric}_X^m = Ag$ when $m < 0$ and $A = 0$. This is interesting because $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ clearly does not split.

We organize the paper in the following way. In Section 3, we give a characterization, due to Singer, of locally homogeneous 3-manifolds. We then explain our approach for the rest of the paper to compute solutions to the m-quasi Einstein equation.

In Section 2, we introduce theory which simplifies the m-quasi Einstein equation when M^n is a unimodular Lie group, and we compute the solutions in Section 4. In Section 5, we discuss using the Ric_X^m version of Myers' Theorem and the Splitting Theorem in order to study the case when $m > 0$, $A \geq 0$ as in Theorem 1.7.

In Section 6, we analyze the equation $\frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^* \otimes X^* = \lambda g$ in order to classify the m-quasi Einstein equations of the locally homogeneous 3-manifolds that admit compact quotient which are not Lie groups. We also classify the nontrivial m-quasi Einstein metrics that can be the product of two Einstein metrics in Section 6. Then, we finish our classification and we also show that there are no solutions to $\mathrm{Ric}_X^m = Ag$ on compact hyperbolic manifolds of any dimension. In Section 7, we give a table which summarizes our results.

2 Unimodular Lie groups

In [5, Theorem 1.1], Chen–Liang–Zhu proved that if M is a compact Lie group with a left-invariant metric g, and if X is a vector field on M such that $\mathrm{Ric}_X^m = Ag$ for $m \neq 0$, then X is a left-invariant. Furthermore, X is a Killing vector field by [5, Theorem 2.3].

Chen–Liang–Zhu prove [5, Theorem 1.1] by first proving that X is left-invariant, and then proving that X is Killing using properties of the Ricci tensor. We will consider $\frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^* \otimes X^* = q$ where q is a left-invariant tensor, which is more general than $\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^* \otimes X^* = Ag$. Rather than considering a compact Lie group G, we assume G admits a discrete group of isometries Γ which acts cocompactly on G.

Next, we give the definition for ad_X in order to state a linear algebra fact to prove that X is Killing given that X is a left-invariant vector field which satisfies $\mathrm{Ric}_X^m = Ag$.

Definition 2.1. If G is a Lie group and if $\mathfrak{g}$ is the Lie algebra of G, then we define $ad_X : \mathfrak{g} \to \mathfrak{g}$ by $ad_X(Y) = [X, Y]$, where X, Y are vector fields in $\mathfrak{g}$.

If G is a Lie group which admits a discrete subgroup Γ with compact quotient, then G must be unimodular. It is a linear algebra fact that if G is a unimodular Lie group, then there exists a basis $\{X_i\}_{i=1}^n$ of $\mathfrak{g}$, the Lie Algebra of G, such that $g(ad_X(X_i), X_i) = 0$ for all i. We will use these facts about Lie groups to prove our main lemmas, which are generalizations of Chen–Liang–Zhu's [5, Theorem 1.1] and [5, Theorem 2.3].

Lemma 2.2. *Let G be a connected Lie group and let Γ be a discrete group of isometries which acts cocompactly on G. Let X be a vector field which is invariant under Γ. If (G, g, X) satisfies $\frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^* \otimes X^* = q$, where q and g are left invariant, then X is a left-invariant vector field.*

Proof. Because G is a Lie group which admits a discrete subgroup with compact quotient, G is unimodular. Let $M = G/\Gamma$ and let $\pi : G \to M$. By our discussion above, we can choose a basis $\{X_i\}$ of the Lie algebra of G such that $g(ad_X(X_i), X_i) = 0$ for all i. Let $X = \sum_{k=1}^n f_k X_k$ where $f_k : G \to \mathbb{R}$. Using the technique from [5, Theorem 1.1], for all i we obtain

$$\begin{aligned}\frac{1}{2}\mathcal{L}_X g(X_i, X_i) - \frac{1}{m}X^* \otimes X^*(X_i, X_i) &= X_i f_i + \sum_{k=1}^n f_k g(\nabla_{X_i} X_k, X_i) - \frac{1}{m}f_i^2 \\ &= X_i f_i + \sum_{k=1}^n f_k g([X_i, X_k], X_i) - \frac{1}{m}f_i^2 = X_i f_i + g(-ad_X(X_i), X_i) - \frac{1}{m}f_i^2 = X_i f_i - \frac{1}{m}f_i^2.\end{aligned}$$

Since M is compact, there exists a maximum and a minimum of the function f_i on M. Let r be a point in M such that $f_i(r)$ is maximal and let s be a point in M such that $f_i(s)$ is minimal and let $q(\pi(X_i), \pi(X_i)) = \lambda_i$. Then

$$\lambda_i = X_i f_i(r) - \frac{1}{m}f_i^2(r) = -\frac{1}{m}f_i^2(r) \quad \text{and} \quad \lambda_i = X_i f_i(s) - \frac{1}{m}f_i^2(s) = -\frac{1}{m}f_i^2(s)$$

Thus $f_i^2(r) = f_i^2(s) = -m\lambda_i$. We now rule out the case $f_i(r) = -f_i(s)$ in order to show that f_i must be constant.

Let $c(t)$ be an integral curve of X_i. Then along $\pi \circ c(t)$ we have $f_i'(t) - \frac{1}{m}f_i^2(t) = \lambda_i$. Solving this equation (see Lemma 2.3), we obtain $f_i(t) = \sqrt{-\lambda_i m}$, $-\sqrt{-\lambda_i m}$, 0, or $-\sqrt{-\lambda_i m}\tanh(\frac{\sqrt{-\lambda_i m}}{m}(t + C))$.

Assume for the sake of contradiction that $f_i(t)$ is not constant, i.e. $f_i(t) = -\sqrt{-\lambda_i m}\tanh(\frac{\sqrt{-\lambda_i m}}{m}(t + C))$ where C is a constant. Let $\pi \circ c(t_i)$ be a sequence of points such that $t_i \to \infty$. Since M is compact, there exists a subsequence of $\{\pi \circ c(t_i)\}$ which converges to a point on M. The set $\overline{\{\pi \circ c(t) : t \in \mathbb{R}\}}$ is closed, hence f_i has a maximal point t_{max} on this set. The supremum of the tanh function is 1, thus the maximum of $f_i(t)$ on $\overline{\{\pi \circ c(t) : t \in \mathbb{R}\}}$ is $\sqrt{-\lambda_i m}$. Let $b(t)$ be an integral curve of X_i such that $b(0) = c(t_{max}) = \sqrt{-\lambda_i m}$ and consider the set $\{\pi \circ b(t) : t \in \mathbb{R}\}$. Along $b(t)$, $f_i(t)$ is either $\sqrt{-\lambda_i m}$ or $-\sqrt{-\lambda_i m}\tanh(\frac{\sqrt{-\lambda_i m}}{m}(t + C))$. Since the supremum of $f_i(t)$ on $\{\pi \circ b(t) : t \in \mathbb{R}\}$ is $\sqrt{-\lambda_i m}$ and tanh never achieves its maximum on its domain, $f_i(t)$ must be constantly $\sqrt{-\lambda_i m}$ on the set $\{\pi \circ b(t) : t \in \mathbb{R}\}$. Finally, since $\overline{\{\pi \circ b(t) : t \in \mathbb{R}\}} = \overline{\{\pi \circ c(t) : t \in \mathbb{R}\}}$, $f_i(t)$ is constant on $\overline{\{\pi \circ c(t) : t \in \mathbb{R}\}}$. Then, since $f_i(t)$ is constant along every integral curve and since G is connected, $f_i(t)$ is constant. □

Lemma 2.3. *Let $f'(t) - \frac{1}{m}f^2(t) = \lambda$, where $f : \mathbb{R} \to \mathbb{R}$ is defined for all t in $\mathbb{R}$ and λ and m are constants.*

(1) *If $\lambda = 0$, then $f(t) = 0$.*
(2) *If $\lambda m > 0$, then there are no solutions f.*
(3) *If $\lambda m < 0$, then $f(t) = \pm\sqrt{-\lambda m}$ or $\sqrt{-\lambda m}\tanh(\frac{\sqrt{-\lambda m}}{m}(t + C))$.*

Proof. Let $\lambda = 0$. Clearly $f(t) = 0$ is a solution. If $f(0)$ is not 0, then $f'(t) = f(t)^2/m$, hence $f(t) = (C - t/m)^{-1}$ for some real number C. However, at $t = mC$, t blows up, which is a contradiction since f exists for all time.

Let $\lambda m > 0$. Then $f'(t) = \frac{f(t)^2}{m} + \lambda$ is never zero. Integrating and rearranging, we obtain

$$\int \frac{f'(t)}{\frac{f^2(t)}{m} + \lambda}dt = \int 1dt \quad \Longrightarrow \quad \frac{m}{\lambda}\int \frac{f'(t)}{1 + (\frac{f(t)}{\sqrt{\lambda m}})^2}dt = t + C \quad \Longrightarrow \quad \sqrt{\frac{m}{\lambda}}\tan^{-1}\Big(\frac{f(t)}{\sqrt{\lambda m}}\Big) = t + C,$$

hence $f(t) = \sqrt{\lambda m}\tan\Big(\sqrt{\frac{\lambda}{m}}(t + C)\Big)$. Since the tan function does not exist everywhere, $f(t)$ also does not exist everywhere. Thus for $\lambda m > 0$ there are no solutions.

Let $\lambda m < 0$; then clearly $f(t) = \pm\sqrt{-\lambda m}$ is a solution to the equation. Assume that $f(0)$ is not $\pm\sqrt{-\lambda m}$. Then we integrate and rearrange as follows:

$$\int \frac{f'(t)}{\frac{f^2(t)}{m} + \lambda}dt = \int 1dt \quad \Longrightarrow \quad \frac{m}{2\sqrt{-\lambda m}}\ln\left|\frac{1 - \frac{f(t)}{\sqrt{-\lambda m}}}{1 + \frac{f(t)}{\sqrt{-\lambda m}}}\right| = t + C \quad \Longrightarrow \quad \left|\frac{1 - \frac{f(t)}{\sqrt{-\lambda m}}}{1 + \frac{f(t)}{\sqrt{-\lambda m}}}\right| = e^{2\frac{\sqrt{-\lambda m}}{m}(t+C)}.$$

If $\dfrac{1-\frac{f(t)}{\sqrt{-\lambda m}}}{1+\frac{f(t)}{\sqrt{-\lambda m}}} = e^{2\frac{\sqrt{-\lambda m}}{m}(t+C)}$, then $f(t) = \sqrt{-\lambda m}\left(\frac{1-e^{2\frac{\sqrt{-\lambda m}}{m}(t+C)}}{1+e^{2\frac{\sqrt{-\lambda m}}{m}(t+C)}}\right) = \sqrt{-\lambda m}\tanh\left(\frac{\sqrt{-\lambda m}}{m}(t+C)\right)$. If $\dfrac{1-\frac{f(t)}{\sqrt{-\lambda m}}}{1+\frac{f(t)}{\sqrt{-\lambda m}}} = -e^{2\frac{\sqrt{-\lambda m}}{m}(t+C)}$, then $f(t) = \sqrt{-\lambda m}\left(\frac{1+e^{2\frac{\sqrt{-\lambda m}}{m}(t+C)}}{1-e^{2\frac{\sqrt{-\lambda m}}{m}(t+C)}}\right)$ and $f(t)$ does not exist at $t=-C$, which is a contradiction. □

Lemma 2.4. *Let G be a unimodular Lie group with left-invariant metric g. If X is left-invariant,* $\operatorname{tr}(q\circ ad_X)=0$, *and $\frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^*\otimes X^* = q$ where q is left-invariant, then X is Killing.*

Proof. Let $\{X_i\}$ be an orthonormal basis relative to g and let $X = a_1X_1 + a_2X_2 + \cdots + a_nX_n$. Then, plugging in (X_i, X_j) into $q = \frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^*\otimes X^*$, we get

$$q(X_i,X_j) = \frac{1}{2}(g([X_i,X],X_j) + g([X_j,X],X_i)) - \frac{1}{m}g(X,X_i)g(X,X_j).$$

We denote the projection of X_i onto X by $\operatorname{proj}_X X_i$. Since $\operatorname{proj}_X X_i = \frac{g(X,X_i)X}{|X|^2}$ and $ad_X(X_i) = [X,X_i]$, we have

$$q(X_i,X_j) = \frac{1}{2}(g(ad_X(X_i),X_j) + g(ad_X(X_j),X_i)) - \frac{|X|^2}{m}g(\operatorname{proj}_X X_i, X_j).$$

Thus we have the following equation, where we view q, ad_X, and proj_X as matrices:

$$q = \frac{1}{2}(ad_X + ad_X^T) - \frac{|X|^2}{m}\operatorname{proj}_X.$$

We denote by “$\cdot$" the matrix multiplication. Multiplying both sides by the matrix ad_X we get

$$q\cdot ad_X = \frac{1}{2}(ad_X + ad_X^T)\cdot ad_X - \frac{|X|^2}{m}\operatorname{proj}_X\cdot ad_X = \frac{1}{2}(ad_X^2 + ad_X^T\cdot ad_X) - \frac{|X|^2}{m}\operatorname{proj}_X\cdot ad_X.$$

Taking the trace of both sides, we get

$$\operatorname{tr}(q\cdot ad_X) = \frac{1}{2}\operatorname{tr}\left(ad_X^2 + ad_X^T\cdot ad_X\right) - \frac{|X|^2}{m}\operatorname{tr}(\operatorname{proj}_X\cdot ad_X).$$

Since $\operatorname{tr}(q\cdot ad_X)=0$ and $\operatorname{tr}(A^2) = \operatorname{tr}((A^T)^2)$ for any $n\times n$ matrix A, we obtain

$$0 = \frac{1}{4}\operatorname{tr}\left((ad_X + ad_X^T)^2\right) - \frac{|X|^2}{m}\operatorname{tr}(\operatorname{proj}_X\cdot ad_X).$$

Now, plugging in X_i, one of the orthonormal basis vectors into $ad_X\cdot\operatorname{proj}_X$ and using that $\operatorname{tr}(AB)=\operatorname{tr}(BA)$ for any two matrices A and B, we get $ad_X\cdot\operatorname{proj}_X(X_i) = \frac{a_i}{|X|^2}[X,X] = 0$. Thus we have $0 = \frac{1}{4}\operatorname{tr}\left((ad_X + ad_X^T)^2\right)$.

Since $ad_X + ad_X^T$ is symmetric, we can diagonalize $ad_X + ad_X^T$, and we call the diagonalized matrix D. Then $\operatorname{tr}((ad_X + ad_X^T)^2) = \operatorname{tr}(D^2)$. Since the eigenvalues in D^2 are nonnegative and $\operatorname{tr}(D^2)$ is the sum of the eigenenvalues of D^2, we obtain $\frac{1}{2}(ad_X + ad_X^T) = 0$. Thus X is Killing. □

Next, we apply Lemma 2.2 to metrics which satisfy $\mathrm{Ric}_X^m = Ag$.

Theorem 2.5. *Let G be a Lie group and let Γ be a discrete group of isometries which acts cocompactly on G, where $\pi : G \to G/\Gamma$ is a covering map. If $(G/\Gamma, g, X)$ satisfies $\mathrm{Ric}_X^m = Ag$, then $\tilde{X} = \pi^*(X)$ is left invariant and Killing.*

Proof. Let $\tilde{g} = \pi^*(g)$ be the pullback metric of g. Since π is a local isometry, $\mathrm{Ric}_{\tilde{X}}^m = A\tilde{g}$. Since $A\tilde{g} - \mathrm{Ric}_{\tilde{g}}$ is left-invariant, by Lemmas 2.2 and 2.4, $\tilde{X}$ is left-invariant and Killing. □

We immediately get the following corollary, which we will use throughout Section 4.

Corollary 2.6. *If M^n is a unimodular Lie group and if $\mathrm{Ric}_X^m = Ag$ with X a left-invariant vector field and g a left-invariant metric, then X is a Killing field.*

Lemma 2.7. *Suppose that (M^n, g) is a Lie group which satisfies $\mathrm{Ric}_X^m = Ag$ where X is nonzero, left-invariant, and Killing. If $\{X_1, X_2, \ldots, X_n\}$ is an eigenbasis of the Ricci tensor of left invariant fields, then X is a multiple of one of the eigenbasis vectors (i.e. there exists an m with $1\le m\le n$ such that $X = a_mX_m$).*

Proof. Since X is left-invariant and Killing, we have $\mathrm{Ric}_X^m(X_i, X_j) = -\frac{1}{m}a_i a_j$ for $1 \le i, j \le n$ and $i \ne j$. Now $\mathrm{Ric}_X^m(X_i, X_j) = Ag(X_i, X_j) = 0$ for all sets of i, j if and only if at least $n - 1$ sets of a_k are 0. Thus, $X = a_m X_m$ for some m with $1 \le m \le n$. □

3 Preliminaries about locally homogeneous 3-manifolds

In this section, we will discuss locally homogeneous three-manifolds, which we will use to prove our main results. We first give definitions of locally homogeneous and homogeneous, which can be found in [8].

Definition 3.1. Let (M, g) be a Riemmanian manifold. Then (M, g) is locally homogeneous if for every pair of points $x, y \in M$, there exists neighborhoods U_x of x and V_y of y such that there is an isometry ψ mapping $(U_x, g|_{U_x})$ to $(V_y, g|_{V_y})$, with $\psi(x) = y$.

Definition 3.2. Let (M, g) be a Riemmanian manifold. Then (M, g) is homogeneous if for every pair of points $x, y \in M$, there exists an isometry ψ with $\psi(x) = y$.

According to Singer in [24], for every locally homogeneous geometry (M^3, g), the universal cover $(\widetilde{M^3}, \tilde{g})$ is homogeneous. If $(\widetilde{M}^3, \tilde{g})$ is a homogeneous, simply connected manifold that admits a compact quotient, then it is one of the following: $\mathbb{R}^3$, SU(2), $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, Nil, E(1, 1), E(2), H^3, $S^2 \times \mathbb{R}$, or $H^2 \times \mathbb{R}$; see [8, Table 1].

Since $\tilde{X}$ is a left-invariant solution to $\mathrm{Ric}_{\tilde{X}}^m = A\tilde{g}$ if and only if $d\pi(\tilde{X})$ is a solution to $\mathrm{Ric}_X^m = Ag$, where $\pi : \widetilde{M} \to M$ is the universal covering map, we study these nine geometries in order to classify m-quasi Einstein metrics on locally homogeneous three manifolds. Of the nine geometries, $\mathbb{R}^3$, SU(2), $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, Nil, E(1, 1), and E(2) are Lie groups. We can also use that H^2 is a Lie group to study $H^2 \times \mathbb{R}$. We will explicitly calculate the metrics on the Lie groups which satisfy $\mathrm{Ric}_X^m = Ag$ using the methods of Section 2. We will study the equation $\frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^* \otimes X^* = \lambda g$ in order to calculate the m-quasi Einstein metrics on $S^2 \times \mathbb{R}$ and H^3.

Throughout this paper, we will use the following computations by Milnor:

Lemma 3.3 ([18, pages 305 and 307]). *Let G be a 3-dimensional unimodular Lie group with left invariant metric. If L is self-adjoint, then there exists an orthonormal basis $\{X_1, X_2, X_3\}$ consisting of eigenvectors $LX_i = \lambda_i^* X_i$. We obtain the following:*

$$[X_2, X_3] = \lambda_1^* X_1, \quad [X_3, X_1] = \lambda_2^* X_2, \quad [X_1, X_2] = \lambda_3^* X_3.$$

The following table gives the signs of λ_i^* for SU(2), $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, E(2), E(1, 1), Nil, *and* $\mathbb{R}^3$.

Lie group	λ_1^*	λ_2^*	λ_3^*
Nil	$\lambda_1^* > 0$	$\lambda_2^* = 0$	$\lambda_3^* = 0$
$\widetilde{\mathrm{SL}_2(\mathbb{R})}$	$\lambda_1^* > 0$	$\lambda_2^* > 0$	$\lambda_3^* < 0$
E(1, 1)	$\lambda_1^* > 0$	$\lambda_2^* < 0$	$\lambda_3^* = 0$
E(2)	$\lambda_1^* > 0$	$\lambda_2^* > 0$	$\lambda_3^* = 0$
$\mathbb{R}^3$	$\lambda_1^* = 0$	$\lambda_2^* = 0$	$\lambda_3^* = 0$
SU(2)	$\lambda_1^* > 0$	$\lambda_2^* > 0$	$\lambda_3^* > 0$

Table 1

From now on, let $\lambda_i = |\lambda_i^*|$. Because X is Killing for unimodular Lie groups with $\mathrm{Ric}_X^m = Ag$, it will be useful to calculate $\mathcal{L}_X g$.

Proposition 3.4. *Let $X = a_1 X_1 + a_2 X_2 + a_3 X_3$ be a left-invariant vector field on a 3-dimensional unimodular Lie group with left invariant metric. With the notation as in Lemma 3.3, we have $\mathcal{L}_X g(X_i, X_i) = 0$ for all i and*

$$\mathcal{L}_X g(X_1, X_2) = -a_3\lambda_2^* + a_3\lambda_1^*, \quad \mathcal{L}_X g(X_1, X_3) = -a_2\lambda_1^* + a_2\lambda_3^*, \quad \mathcal{L}_X g(X_2, X_3) = -a_1\lambda_3^* + a_1\lambda_2^*.$$

Proof. We have the following computation for $\mathcal{L}_X g$:

$$\begin{aligned}\mathcal{L}_X g(X_i, X_j) &= g(\nabla_{X_i}(a_1X_1 + a_2X_2 + a_3X_3), X_j) + g(\nabla_{X_j}(a_1X_1 + a_2X_2 + a_3X_3), X_i)\\ &= \sum_k a_k g(\nabla_{X_i}X_k, X_j) + a_k g(\nabla_{X_j}X_k, X_i)\\ &= \sum_k g(\nabla_{X_k}X_i + [X_i, X_k], X_j) + g(\nabla_{X_k}X_j + [X_j, X_k], X_i)\\ &= \sum_k a_k g([X_i, X_k], X_j) + a_k g([X_j, X_k], X_i) + DX_k g(X_i, X_j)\\ &= \sum_k a_k g([X_i, X_k], X_j) + a_k g([X_j, X_k], X_i).\end{aligned}$$

Using Lemma 3.3 we obtain the assertion. □

We recall the definition of the Ricci quadratic form $r(x)$ as introduced by Milnor in [18], and the signatures of the Ricci forms of Nil, E(1, 1), $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, E(2), $\mathbb{R}^3$, and SU(2) when the metric is left invariant.

Definition 3.5. The Ricci quadratic form $r(X)$ takes vectors $X \in TM$ to $\mathbb{R}$ and is defined as $g(r(X), Y) = \mathrm{Ric}(X, Y)$ for all $Y \in TM$.

The collection of signs of $r(e_i)$, namely $\{\mathrm{sign}(r(e_i))\}_{i=1}^n$, is called the signature of the quadratic form r, where $\{e_i\}_{i=1}^n$ is any orthonormal basis for the tangent space.

Lie group	$r(e_1)$	$r(e_2)$	$r(e_3)$	Reference
Nil	$r(e_1) > 0$	$r(e_2) < 0$	$r(e_3) < 0$	[18, Corollary 4.6]
E(1, 1), $\widetilde{\mathrm{SL}_2(\mathbb{R})}$	$r(e_1) > 0$	$r(e_2) < 0$	$r(e_3) < 0$	
	$r(e_1) = 0$	$r(e_2) = 0$	$r(e_3) < 0$	[18, Corollary 4.7]
E(2)	$r(e_1) > 0$	$r(e_2) < 0$	$r(e_3) < 0$	[18, Corollary 4.8]
$\mathbb{R}^3$	$r(e_1) = 0 <$	$r(e_2) = 0$	$r(e_3) < 0$	
SU(2)	$r(e_1) > 0$	$r(e_2) > 0$	$r(e_3) > 0$	
	$r(e_1) > 0$	$r(e_2) = 0$	$r(e_3) = 0$	
	$r(e_1) > 0$	$r(e_2) < 0$	$r(e_3) < 0$	[18, Corollary 4.5]

Table 2

4 *m*-quasi Einstein solutions for Nil, $\widetilde{\mathrm{SL}_2\,\mathbb{R}}$, E(1, 1), E(2) and $H^2 \times \mathbb{R}$

In this section we compute solutions to the m-quasi Einstein equation for the Lie groups Nil, $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, E(1, 1), and E(2). We also compute solutions to $H^2 \times \mathbb{R}$, using the Lie group structure of H^2. We use Tables 1 and 2 and the next remark to find examples of X which give us $\mathrm{Ric}_X^m = Ag$ for $m > 0$ and $A < 0$ for the space Nil.

Remark 4.1. By [18, Corollary 4.5], for any left invariant metric on Nil, the principal Ricci curvatures satisfy $|r(e_1)| = |r(e_2)| = |r(e_3)| = |\rho|$.

Proposition 4.2. *Consider* Nil *with* $\mathrm{Ric}_X^m = Ag$. *If g is a left-invariant metric and if X is a left-invariant vector field, then there exist examples of X such that* $\mathrm{Ric}_X^m = Ag$ *if and only if* $A < 0$ *and* $m > 0$.

Proof. Let $\{X_1, X_2, X_3\}$ be an orthonormal basis with $\mathrm{Ric}(X_1, X_1) = \rho$, $\mathrm{Ric}(X_2, X_2) = -\rho$, and $\mathrm{Ric}(X_3, X_3) = -\rho$ as in Table 2 and Remark 4.1. Let $X = a_1X_1 + a_2X_2 + a_3X_3$ where a_1, a_2, and a_3 are all constants. By Corollary 2.6, X is a Killing field, so we set $\mathcal{L}_X g(X_i, X_j) = 0$ for all $i, j = 1, 2, 3$ as follows:

$$\mathcal{L}_X g(X_1, X_2) = a_3\lambda_1 = 0, \quad \mathcal{L}_X g(X_1, X_3) = -a_2\lambda_1 = 0$$

where $\mathcal{L}_X g(X_i, X_j)$ is zero for every other combination of i, j, by definition of Nil. Thus, $a_2 = a_3 = 0$. We compute Ric_X^m as follows:

$$\mathrm{Ric}_X^m(X_1, X_1) = \rho - \tfrac{1}{m}a_1^2, \quad \mathrm{Ric}_X^m(X_2, X_2) = -\rho - \tfrac{1}{m}a_2^2 = -\rho, \quad \mathrm{Ric}_X^m(X_3, X_3) = -\rho - \tfrac{1}{m}a_3^2 = -\rho.$$

Thus, $\mathrm{Ric}_X^m = Ag$ if and only if $X = \pm\sqrt{2m\rho}X_1$. In this case, $m > 0$ and $A = -\rho < 0$. □

Now we find examples of X which satisfy $\mathrm{Ric}_X^m = Ag$ for the spaces $\mathrm{E}(1,1)$ and $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

Proposition 4.3. *Consider* $\widetilde{\mathrm{SL}_2(\mathbb{R})}$. *If g is a left-invariant metric and if X is a left-invariant vector field, then there exist examples of* $\mathrm{Ric}_X^m = Ag$ *if and only if $m < 0$ and $A = 0$.*

Proof. Let g be a left-invariant metric and let X be a left-invariant vector field, where $X = a_1X_1 + a_2X_2 + a_3X_3$ with $\{X_1, X_2, X_3\}$ an orthonormal basis. By Corollary 2.6, X must be a Killing field if $\mathrm{Ric}_X^m = Ag$, so we set $\mathcal{L}_X g(X_i, X_j) = 0$ for all $i, j = 1, 2, 3$ as follows:

$$\mathcal{L}_X g(X_1, X_2) = a_3(\lambda_1 - \lambda_2) = 0, \quad \mathcal{L}_X g(X_1, X_3) = a_2(-\lambda_1 - \lambda_3) = 0, \quad \mathcal{L}_X g(X_2, X_3) = a_1(\lambda_2 + \lambda_3) = 0$$

where all other $\mathcal{L}_X g(X_i, X_j) = 0$ by properties of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$. By the above, we must have $a_1 = a_2 = 0$ and either $a_3 = 0$ or $\lambda_1 = \lambda_2$.

By Table 2, the signature for the Ricci form is $(+, -, -)$ or $(0, 0, -)$. If the Ricci form is $(+, -, -)$, let $|\mathrm{Ric}(X_i, X_i)| = \rho_i$. Then, plugging in (X_i, X_j) where $i, j = 1, 2, 3$ into $\mathrm{Ric}_X^m = Ag$, we obtain

$$\mathrm{Ric}_X^m(X_1, X_1) = \rho_1 - \tfrac{1}{m}a_1^2 = \rho_1, \quad \mathrm{Ric}_X^m(X_2, X_2) = -\rho_2 - \tfrac{1}{m}a_2^2 = -\rho_2, \quad \mathrm{Ric}_X^m(X_3, X_3) = -\rho_3 - \tfrac{1}{m}a_3^2.$$

In this case, we cannot have $\mathrm{Ric}_X^m = Ag$ since $\mathrm{Ric}_X^m(X_1, X_1) > 0$ and $\mathrm{Ric}_X^m(X_2, X_2) < 0$.

If the signature of the Ricci form is $(0, 0, -)$, then we obtain

$$\mathrm{Ric}_X^m(X_1, X_1) = -\tfrac{1}{m}a_1^2 = 0, \quad \mathrm{Ric}_X^m(X_2, X_2) = -\tfrac{1}{m}a_2^2 = 0, \quad \mathrm{Ric}_X^m(X_3, X_3) = -\rho_3 - \tfrac{1}{m}a_3^2.$$

Then, $\mathrm{Ric}_X^m = Ag$ if and only if $a_3 = \sqrt{-m\rho_3}$, $A = 0$, and $m < 0$. □

Proposition 4.4. *Consider* $\mathrm{E}(1,1)$. *If g is a left-invariant metric and if X is a left-invariant vector field, then there are no solutions to* $\mathrm{Ric}_X^m = Ag$.

Proof. Let g be a left-invariant metric and let X be a left-invariant vector field, where $X = a_1X_1 + a_2X_2 + a_3X_3$ with $\{X_1, X_2, X_3\}$ an orthonormal basis. By Corollary 2.6, X must be a Killing field if $\mathrm{Ric}_X^m = Ag$, so we set $\mathcal{L}_X g(X_i, X_j) = 0$ for all $i, j = 1, 2, 3$ as follows:

$$\mathcal{L}_X g(X_1, X_2) = a_3(\lambda_2 + \lambda_1) = 0, \quad \mathcal{L}_X g(X_1, X_3) = -a_1\lambda_2 = 0, \quad \mathcal{L}_X g(X_2, X_3) = -a_2\lambda_1 = 0,$$

where all other $\mathcal{L}_X g(X_i, X_j) = 0$ by properties of $\mathrm{E}(1,1)$. By the three equations above, $a_1 = a_2 = a_3 = 0$. By Table 2, the signature for the Ricci form is $(+, -, -)$ or $(0, 0, -)$. If the signature is $(+, -, -)$, let $|\mathrm{Ric}(X_i, X_i)| = \rho_i$. Then, plugging in all pairs (X_i, X_j) with $i, j = 1, 2, 3$, we obtain

$$\mathrm{Ric}_X^m(X_1, X_1) = \rho_1 - \tfrac{1}{m}a_1^2 = \rho_1, \quad \mathrm{Ric}_X^m(X_2, X_2) = -\rho_2 - \tfrac{1}{m}a_2^2 = -\rho_2, \quad \mathrm{Ric}_X^m(X_3, X_3) = -\rho_3 - \tfrac{1}{m}a_3^2 = -\rho_3.$$

Ric_X^m cannot equal Ag since $\mathrm{Ric}_X^m(X_1, X_1) > 0$ and $\mathrm{Ric}_X^m(X_2, X_2) < 0$.

If the signature is $(0, 0, -)$, then we get the following set of equations:

$$\mathrm{Ric}_X^m(X_1, X_1) = -\tfrac{1}{m}a_1^2 = 0, \quad \mathrm{Ric}_X^m(X_2, X_2) = -\tfrac{1}{m}a_2^2 = 0, \quad \mathrm{Ric}_X^m(X_3, X_3) = -\rho_3 - \tfrac{1}{m}a_3^2.$$

In this case, we cannot have $\mathrm{Ric}_X^m = Ag$ since $\mathrm{Ric}_X^m(X_1, X_1) = \mathrm{Ric}_X^m(X_2, X_2) = 0$ and $\mathrm{Ric}_X^m(X_3, X_3) < 0$. □

Finally, we find that there are no examples of X on $\mathrm{E}(2)$ which give us $\mathrm{Ric}_X^m = Ag$.

Proposition 4.5. *Consider* $\mathrm{E}(2)$. *If g is a left-invariant metric and if X is a left-invariant vector field, then there are no solutions to* $\mathrm{Ric}_X^m = Ag$.

Proof. Let g be a left-invariant metric and let X be a left-invariant vector field, where $X = a_1X_1 + a_2X_2 + a_3X_3$ with $\{X_1, X_2, X_3\}$ an orthonormal basis. By Corollary 2.6, X must be a Killing field if $\mathrm{Ric}_X^m = Ag$, so we set $\mathcal{L}_X g(X_i, X_j) = 0$ for all $i, j = 1, 2, 3$ as follows:

$$\mathcal{L}_X g(X_1, X_2) = a_3(\lambda_1 - \lambda_2) = 0, \quad \mathcal{L}_X g(X_1, X_3) = -a_2\lambda_1 = 0, \quad \mathcal{L}_X g(X_2, X_3) = a_1\lambda_2 = 0.$$

All other $\mathcal{L}_X g(X_i, X_j) = 0$ by properties of E(2). By the three equations above, $a_1 = a_2 = 0$ and either $\lambda_1 = \lambda_2$ or $a_3 = 0$. By Table 2, the signature for the Ricci form is $(+, -, -)$. Letting $|\mathrm{Ric}(X_i, X_i)| = \rho_i$, we plug in all pairs (X_i, X_j) with $i, j = 1, 2, 3$ as follows:

$$\mathrm{Ric}_X^m(X_1, X_1) = \rho_1 - \tfrac{1}{m}a_1^2 = \rho_1, \quad \mathrm{Ric}_X^m(X_2, X_2) = -\rho_2 - \tfrac{1}{m}a_2^2, \quad \mathrm{Ric}_X^m(X_3, X_3) = -\rho_3 - \tfrac{1}{m}a_3^2.$$

Ric_X^m cannot equal Ag since $\mathrm{Ric}_X^m(X_1, X_1) > 0$ and $\mathrm{Ric}_X^m(X_2, X_2) < 0$. □

Proposition 4.6. *Consider* $\mathbb{R}^3$. *If g is a left-invariant metric and if X is a left-invariant vector field, then the only solutions of* $\mathrm{Ric}_X^m = Ag$ *occur when $m \neq 0$, $A = 0$, and $X = 0$.*

Proof. Let g be a left-invariant metric and let X be a left-invariant vector field, where $X = a_1X_1 + a_2X_2 + a_3X_3$ with $\{X_1, X_2, X_3\}$ an orthonormal basis of left-invariant vector fields. By Corollary 2.6, X must be a Killing field if $\mathrm{Ric}_X^m = Ag$. By [18, page 307], $\mathcal{L}_X g(X_i, X_j) = 0$ for all $i, j = 1, 2, 3$ and $\mathrm{Ric}(X_i, X_j) = 0$ for all $i, j = 1, 2, 3$, so we have the following sets of equations for $\mathrm{Ric}_X^m(X_i, X_j)$:

$$\mathrm{Ric}_X^m(X_1, X_1) = -\tfrac{1}{m}a_1^2, \quad \mathrm{Ric}_X^m(X_2, X_2) = -\tfrac{1}{m}a_2^2, \quad \mathrm{Ric}_X^m(X_3, X_3) = -\tfrac{1}{m}a_3^2.$$

Setting $\mathrm{Ric}_X^m = Ag$, solutions exist only when $m \neq 0$, $A = 0$, and $X = 0$. □

Remark 4.7. Since $\mathbb{R}^3$ is Ricci flat, Proposition 4.6 also follows from Proposition 6.7.

Proposition 4.8. *If g is a left-invariant metric on $H^2 \times \mathbb{R}$ and if X is a left-invariant vector field, then there exist solutions to* $\mathrm{Ric}_X^m = Ag$ *if and only if $A < 0$ and $m > 0$.*

Proof. Let $\{X_1, X_2, \frac{\partial}{\partial r}\}$ be an orthonormal basis where $\{X_1, X_2\}$ are in TH^2 and $\frac{\partial}{\partial r}$ is in $T\mathbb{R}$. Let $X = a_1X_1 + a_2X_2 + a_3\frac{\partial}{\partial r}$. We compute the Lie derivatives as follows:

$$\begin{aligned}
&\mathcal{L}_X g(X_1, X_1) = 2g(\nabla_{X_1}X, X_1) = 2g(-a_2X_2, X_1) = 0\\
&\mathcal{L}_X g(X_2, X_2) = 2g(\nabla_{X_2}X, X_2) = 2g(-a_1X_2 + a_2X_1, X_2) = -2a_1\\
&\mathcal{L}_X g(\tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial r}) = 0\\
&\mathcal{L}_X g(X_1, X_2) = g(\nabla_{X_1}X, X_1) + g(\nabla_{X_1}X, X_1) = g(-a_1X_2 + a_2X_1, X_1) = a_2\\
&\mathcal{L}_X g(X_2, \tfrac{\partial}{\partial r}) = g(\nabla_{X_2}X, \tfrac{\partial}{\partial r}) + g(\nabla_{\frac{\partial}{\partial r}}X, X_2) = 0.
\end{aligned}$$

By Corollary 2.6, X must be a Killing field, so we set $\mathcal{L}_X g = 0$ to get that $a_1 = a_2 = 0$. We have $\mathrm{Ric}(X_1, X_1) = \mathrm{Ric}(X_2, X_2) = -\rho g$ where $\rho > 0$, and $\mathrm{Ric}(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = 0$, so we can compute Ric_X^m as follows:

$$\mathrm{Ric}_X^m(X_1, X_1) = -\rho, \quad \mathrm{Ric}_X^m(X_2, X_2) = -\rho, \quad \mathrm{Ric}_X^m(\tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial r}) = -\tfrac{1}{m}a_3^2.$$

Thus, $\mathrm{Ric}_X^m = Ag$ if and only if $X = \pm\sqrt{\rho m}\frac{\partial}{\partial r}$, where $A = -\rho < 0$ and $m > 0$. □

We show that we can find examples of X such that $\mathrm{Ric}_X^m = 0$ on SU(2) with left-invariant metric.

Proposition 4.9. *Consider* SU(2). *If g is a left-invariant metric and if X is a left-invariant vector field, then there exist solutions to* $\mathrm{Ric}_X^m = Ag$ *if and only if either $m > 0$ with A any real number or $m < 0$ with $A > 0$.*

Proof. Let $X = a_1X_1 + a_2X_2 + a_3X_3$. By Lemma 2.7, at least two a_i's must be zero. By Corollary 2.6, X is a Killing field, so we compute $\mathcal{L}_X g$ using Proposition 3.4 as follows:

$$\mathcal{L}_X g(X_1, X_2) = a_3(\lambda_1 - \lambda_2), \quad \mathcal{L}_X g(X_2, X_3) = a_1(\lambda_2 - \lambda_3), \quad \mathcal{L}_X g(X_1, X_3) = a_2(\lambda_3 - \lambda_1).$$

By Table 2, the siganture of the Ricci form is either $(+,+,+)$, $(+,0,0)$, or $(+,-,-)$. Let $|\operatorname{Ric}(X_i,X_i)) = \rho_i$ for $i = 1,2,3$. If the signature is $(+,+,+)$, then we have the following computations for Ric_X^m:

$$\operatorname{Ric}_X^m(X_1,X_1) = \rho_1 - \tfrac{1}{m}a_1^2, \quad \operatorname{Ric}_X^m(X_2,X_2) = \rho_2 - \tfrac{1}{m}a_2^2, \quad \operatorname{Ric}_X^m(X_3,X_3) = \rho_3 - \tfrac{1}{m}a_3^2$$

Setting $\operatorname{Ric}_X^m = Ag$, if all three a_i's are zero, then $X = 0$ and $\operatorname{Ric}_X^m = \rho g$ where $\rho = \rho_1 = \rho_2 = \rho_3$. If $a_1 = a_2 = 0$ and $a_3 \neq 0$, and $\rho = \rho_1 = \rho_2$, then $X = \pm\sqrt{m(\rho_3 - \rho)}X_3$. Similarly, if $a_1 = a_3 = 0$, and $\rho = \rho_1 = \rho_3$, then $X = \pm\sqrt{m(\rho_2 - \rho)}X_2$. If $a_2 = a_3 = 0$, and $\rho = \rho_2 = \rho_3$, then $X = \pm\sqrt{m(\rho_1 - \rho)}X_1$. In these cases, $\operatorname{Ric}_X^m = \rho g$, where $\rho > 0$, and m can be positive or negative, depending on the sign of $\rho_3 - \rho$, $\rho_2 - \rho$, and $\rho_1 - \rho$, respectively.

If the signature is $(+,0,0)$, then

$$\operatorname{Ric}_X^m(X_1,X_1) = \rho_1 - \tfrac{1}{m}a_1^2, \quad \operatorname{Ric}_X^m(X_2,X_2) = -\tfrac{1}{m}a_2^2, \quad \operatorname{Ric}_X^m(X_3,X_3) = -\tfrac{1}{m}a_3^2.$$

The solutions to the above equations are $X = \pm\sqrt{\rho m}X_1$ and $\operatorname{Ric}_X^m = 0$. In this case, m must be positive.

If the signature is $(+,-,-)$, then

$$\operatorname{Ric}_X^m(X_1,X_1) = \rho_1 - \tfrac{1}{m}a_1^2, \quad \operatorname{Ric}_X^m(X_2,X_2) = -\rho_2 - \tfrac{1}{m}a_2^2, \quad \operatorname{Ric}_X^m(X_3,X_3) = -\rho_3 - \tfrac{1}{m}a_3^2.$$

Setting $\operatorname{Ric}_X^m = Ag$, the solutions are $X = \pm\sqrt{m(\rho + \rho_1)}X_1$, where $\rho = \rho_2 = \rho_3$. In this case, $\operatorname{Ric}_X^m = -\rho g$ and m must be positive. □

5 Relations to the Splitting Theorem, Myers' Theorem and Bochner's Theorem

By Khuri–Woolgar–Wylie [11, Theorem 2], the Splitting Theorem holds for Ric_X^m if $m > 0$. We also recall that if (M,g) is a noncompact homogenous space, then it contains a line. Using the Ric_X^m version of the Splitting Theorem and the fact about noncompact homogeneous spaces, we show that of the 9 geometries which are 3-dimensional and homogeneous, the ones which do not split do not have solutions if $m > 0$ and $A \geq 0$.

Proposition 5.1. *H^3, $\widetilde{\mathrm{SL}_2\,\mathbb{R}}$, Nil,E(2), $H^2 \times \mathbb{R}$, and E(1, 1) do not admit metrics such that $\operatorname{Ric}_X^m = Ag$ for $m > 0$ and $A \geq 0$.*

Proof. H^3, $\widetilde{\mathrm{SL}_2\,\mathbb{R}}$, Nil,E(2), and E(1, 1) all admit lines and do not split as $N \times \mathbb{R}$. Thus, the proposition follows by the Bakry Émery Ricci version of the Splitting Theorem by Khuri–Woolgar–Wylie.

In the case of $H^2 \times \mathbb{R}$, by the Splitting Theorem, $\operatorname{Ric}_X^m \geq 0$ with $m > 0$ if and only if $\operatorname{Ric}_X^m \geq 0$ with $m > 0$ on H^2. Now H^2 admits lines and does not split as $N \times \mathbb{R}$, so the proposition follows. □

In [23, Theorem 5], Qian proves that Myers' Theorem holds for the gradient m-Bakry Émery Ricci curvature when $m > 0$. Limoncu showed in [16, Theorem 1.2] that Myers' Theorem holds for the non-gradient m-Bakry Émery Ricci curvature when $m > 0$. In [10] Khuri–Woolgar use Limoncu's version of Myers' Theorem to study Near Horizon Geometries. Using this version of Myers' Theorem, we see that since $S^2 \times \mathbb{R}$ and $\mathbb{R}^3$ are both noncompact, $S^2 \times \mathbb{R}$ and $\mathbb{R}^3$ do not admit metrics such that $\operatorname{Ric}_X^m = Ag$ for $m > 0$ and $A > 0$. In fact, since SU(2) is the only compact simply connected three-dimensional geometry, it is the only one that can admit a metric such that $\operatorname{Ric}_X^m = Ag$ for $m > 0$ and $A > 0$.

We discuss the case $m < 0$, $A < 0$ of the m-quasi Einstein metric. Bochner proved that if (M,g) is compact, oriented and if Ric < 0, then there are no nontrivial Killing fields; see [20, Theorem 36]. This leads to

Proposition 5.2. *If M^n is a compact locally homogeneous Riemannian manifold, and if M^n is a compact quotient of a Lie group G, then there are no solutions to $\operatorname{Ric}_X^m = Ag$ if $m < 0$ and $A < 0$.*

Proof. By Lemma 2.5, $\widetilde{X}$ is Killing on G. Then, $\operatorname{Ric} = A\widetilde{g} + \frac{1}{m}\widetilde{X}^* \otimes \widetilde{X}^*$ which is negative, giving us a contradiction by Bochner's Theorem. □

Corollary 5.3. *If M^3 is a compact locally homogeneous Riemannian manifold which satisfies $\operatorname{Ric}_X^m = Ag$ with $m < 0$ and $A < 0$, then M^3 cannot be a compact quotient of $\mathbb{R}^3$, SU(2), $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, Nil, E(1, 1), $H^2 \times \mathbb{R}$, or E(2).*

6 The m-quasi Einstein equation on geodesics

Our next definition and proposition deal with analyzing the equation $\frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^* \otimes X^* = Ag$, which we will use to find m-quasi Einstein solutions to $S^2 \times \mathbb{R}$ and H^3. We will also prove theorems for more general spaces using this analysis.

Definition 6.1. Let $\gamma(t)$ be a unit speed geodesic. We define $\varphi_\gamma(t)$ as $g(X_{\gamma(t)}, \dot{\gamma}(t))$. Note that $\varphi_\gamma(t)$ is well defined for all t such that $\gamma(t)$ is defined. If it is clear which $\gamma(t)$ we are using to define $\varphi_\gamma(t)$, then we write $\varphi(t)$ rather than $\varphi_\gamma(t)$.

Proposition 6.2. *Let (M, g) be a complete Riemannian manifold and let $\gamma : (-\infty, \infty) \to M$ be a unit speed geodesic. Suppose the equation*

$$\frac{1}{2}\mathcal{L}_X g(\dot{\gamma}, \dot{\gamma}) - \frac{1}{m}g(X, \dot{\gamma})g(X, \dot{\gamma}) = \lambda g(\dot{\gamma}, \dot{\gamma})$$

is satisfied at every point on γ.

(1) *If $\lambda = 0$ for $m \neq 0$ at every point along γ, then $\varphi(t) = 0$.*

(2) *If $\lambda m > 0$ at every point along γ, then there are no complete solutions to $\frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^* \otimes X^* = \lambda g$.*

(3) *If $\lambda m < 0$ along a geodesic, then*

$$\varphi(t) = \sqrt{-\lambda m}\tanh\Big(\frac{\sqrt{-\lambda m}}{m}(t + C)\Big) \quad \textit{or} \quad \varphi(t) = \pm\sqrt{-\lambda m}.$$

Proof. We have the following set of equations:

$$\frac{d}{dt}(\varphi(t)) = \frac{1}{2}\mathcal{L}_X g(\dot{\gamma}, \dot{\gamma}) = \frac{1}{m}(X^* \otimes X^*)(\dot{\gamma}, \dot{\gamma})) + \lambda g(\dot{\gamma}, \dot{\gamma}) = \frac{1}{m}g(X, \dot{\gamma})^2 + \lambda = \frac{1}{m}\varphi^2(t) + \lambda.$$

The proposition follows from Lemma 2.3. □

Remark 6.3. If M^n is a compact manifold, then we can prove Proposition 6.2(2) using the Divergence Theorem. Taking the trace of both sides of $\frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^* \otimes X^* = \lambda g$, we get $\operatorname{div}(X) - \frac{1}{m}|X|^2 = \lambda n$. Integrating both sides over M, we get

$$\int_M |X|^2 = -\int_M \lambda mn = -\lambda mn \operatorname{vol}(M).$$

Either $X = 0$ and $\lambda = 0$ or the left hand side is positive which implies that λm must be negative.

Now we provide an example of a manifold which satisfies $\mathrm{Ric}_X^m = \lambda g$ with $\lambda m < 0$.

Example 6.4. Let $M = S^1$ with the usual metric, with basis vector $\{\frac{\partial}{\partial\theta}\}$. Let $X = \sqrt{-\lambda m}\frac{\partial}{\partial\theta}$ with $\lambda m < 0$. Since X is Killing and S^1 is Ricci flat, we get $\mathrm{Ric}_X^m = \lambda g$.

Next, we give a global analysis of $\frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^* \otimes X^* = \lambda g$ when $\lambda m < 0$. In order to do this, we first state a definition of critical point originally defined by Grove–Shiohama; see also [20].

Definition 6.5 ([20]). Fix $p \in M$. A point q is a critical point of the distance function to p (or is a critical point to p) if for every vector $V \in T_qM$, there is a minimal geodesic γ with $\gamma(0) = p$, $\gamma(d(p, q)) = q$ such that $g(\dot{\gamma}(d(p, q)), V) \leq 0$.

Lemma 6.6. [20, Corollary 43] *Suppose that there are no critical points of the distance function to p in the annulus $\{q : a \leq d(p, q) \leq b\}$. Then $B(p, a)$ is homeomorphic to $B(p, b)$, and $B(p, b)$ deformation retracts onto $B(p, a)$. Moreover, if there are no critical points of p in M, then M is diffeomorphic to $\mathbb{R}^n$.*

Using techniques similar to those of Wylie in the proof of [26, Proposition 1], we look for spaces which admit $\frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^* \otimes X^* = \lambda g$ with $\lambda m < 0$ everywhere. We find that the only possibility is S^1 if the space is compact.

Proposition 6.7. *If M is a compact manifold which satisfies $\frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^* \otimes X^* = \lambda g$ with $X \neq 0$ and $\lambda m < 0$ along every geodesic, then $M = S^1$.*

Proof. Since M is compact, the function $f(p) = |X(p)|^2$ achieves a maximum and a minimum value. At a minimum, $0 = D_X f = D_X g(X, X) = 2\mathcal{L}_X g(X, X)$. Then

$$\frac{1}{2}\mathcal{L}_X g(X, X) - \frac{1}{m}(X^* \otimes X^*)(X, X) = \lambda g(X, X) \quad \Longrightarrow \quad -\frac{1}{m}|X|^4 = \lambda |X|^2.$$

Thus either $|X|^2 = -\lambda m$ or $|X|^2 = 0$ at a minimum point. By a similar argument, $|X|^2 = -\lambda m$ or $|X|^2 = 0$ at a maximum point. Hence either $|X|^2 = -\lambda m$ for every point on M, or there exists a point $p \in M$ where $X(p) = 0$.

If $|X|^2 = -\lambda m$ for every point in M, then taking the trace of $\frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^* \otimes X^* = \lambda g$, we get $\operatorname{div}(X) - \frac{|X|^2}{m} = \lambda n$. Plugging in $|X|^2 = -\lambda m$, we obtain $\operatorname{div}(X) = \lambda(n-1)$. Taking the integral of both sides over M and using the Divergence Theorem, we get that $\lambda(n-1)\operatorname{vol}(M) = 0$. If $\lambda = 0$ then $X = 0$ by Proposition 6.2(1), so n is 1. Since M is compact, this means that $M = S^1$.

In the case where there exists a point $p \in M$ such that $X(p) = 0$, we prove that there are no critical points to p in M and we use Lemma 6.6 to show that M must be $\mathbb{R}^n$.

By Definition 6.5, we want to show that there exists a vector V such that every geodesic γ with $\gamma(0) = p$, $\gamma(d(p, q)) = q$ satisfies $g(\dot\gamma(d(p, q)), V) > 0$. Consider the case when $m < 0$. Let $\gamma(t)$ be a geodesic with $\gamma(0) = p$ and let $V = X$. If $\varphi(t) = g(X_{\gamma(t)}, \dot\gamma(t))$, then since $X(p) = 0$, $\varphi(0)$ must be 0, so $\varphi(t)$ cannot be constantly nonzero. Then by Proposition 6.2,

$$\varphi(t) = \sqrt{-\lambda m}\tanh\Big(\frac{\sqrt{-\lambda m}}{m}t\Big).$$

If $\varphi(t) = \sqrt{-\lambda m}\tanh\big(\frac{\sqrt{-\lambda m}}{m}t\big)$, then $\varphi(t) > 0$ when $t > 0$, so by Lemma 6.6 we have $M = \mathbb{R}^n$. This is a contradiction because M is compact.

If $m > 0$, then we again let $\gamma(t)$ be a geodesic with $\gamma(0) = p$. We let $V = -X$ so that the differential equation we have to solve is $-\frac{d}{dt}\varphi(t) = \frac{1}{m}\varphi^2(t) + \lambda$. The solutions are

$$\varphi(t) = \sqrt{-\lambda m}\tanh\Big(\frac{-\sqrt{-\lambda m}}{m}t\Big) \text{ or } \varphi(t) = \pm\sqrt{-\lambda m}.$$

$\varphi(t)$ cannot be $\pm\sqrt{-\lambda m}$ as in the case $m < 0$. If $\varphi(t) = \sqrt{-\lambda m}\tanh\big(\frac{-\sqrt{-\lambda m}}{m}t\big)$, then $\varphi(t)$ is positive for $t > 0$, giving us a contradiction by Lemma 6.6. □

Proposition 6.8. *On H^3, $\operatorname{Ric} = -\rho g$ where $\rho > 0$. Moreover, $\operatorname{Ric}_X^m = Ag$ if and only if $A + \rho = 0$ and $X = 0$.*

Proof. If $(A + \rho)m > 0$, then by Proposition 6.2, there are no solutions. If $(A + \rho)m < 0$, then by Proposition 6.7 there are no solutions. If $A + \rho = 0$, then by Proposition 6.2 we have $X = 0$. □

Corollary 6.9. *There are no solutions to $\operatorname{Ric}_X^m = Ag$ with $A > 0$ on a compact hyperbolic manifold.*

Next, we give an example of a space (M, g) which is non-Euclidean, m-quasi Einstein and Einstein, and X is not trivial.

Example 6.10. Consider H^2 with the metric $g = dr^2 + e^{2r}dx^2$ and let $X = -m\frac{\partial}{\partial r}$. Then we have the following:

$$\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}, \quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = -e^{2r}\frac{\partial}{\partial r}, \quad \nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} = 0.$$

Then, we have the following computations for the Ricci curvature:

$$\operatorname{Ric}(\tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial x}) = 0, \quad \operatorname{Ric}(\tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial r}) = -1, \quad \operatorname{Ric}(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}) = -e^{2r},$$

so we see that our metric satisfies $\operatorname{Ric} = -1g$. We have the following computations for Ric_X^m:

$$\operatorname{Ric}_X^m(\tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial x}) = 0, \quad \operatorname{Ric}_X^m(\tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial r}) = -1 - \tfrac{1}{m}(-m)^2 = -1 - m, \quad \operatorname{Ric}_X^m(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}) = e^{2r}(-1 - m),$$

so we see that $\operatorname{Ric}_X^m = (-1 - m)g$.

We are now ready to solve the m-quasi Einstein equation for $S^j \times \mathbb{R}$ when $j \geq 2$.

Proposition 6.11. *Consider $S^j \times \mathbb{R}$ with the product metric and $j \geq 2$, endowing S^j with a constant curvature metric of Ricci curvature ρ, and $\mathbb{R}$ with the flat metric. Then there exists a nontrivial m-quasi Einstein metric, $\mathrm{Ric}_X^m = Ag$ if and only if $A = \rho$ and $m < 0$.*

Proof. Let $\{X_1, X_2, \frac{\partial}{\partial r}\}$ be an orthonormal basis where $\{X_1, X_2\}$ is in TS^2 and $\{\frac{\partial}{\partial r}\}$ is in $T\mathbb{R}$.

First, consider the case $A - \rho = 0$. Let γ_{S^2} be a great circle on S^2 since the geodesics on S^2 are the great circles. We apply Proposition 6.2 (1). This says that X restricted to S^2 must be 0. Letting $\gamma_{\mathbb{R}}$ be a unit speed geodesic in $\mathbb{R}$, we have

$$\frac{1}{2}\mathcal{L}_X g(\dot\gamma_{\mathbb{R}}, \dot\gamma_{\mathbb{R}}) - \frac{1}{m}X^* \otimes X^*(\dot\gamma_{\mathbb{R}}, \dot\gamma_{\mathbb{R}}) = A = \rho.$$

If $A - \rho = 0$ and $m < 0$, then by Proposition 6.2(3), $\varphi_{\gamma_{\mathbb{R}}}(t)$ is either

$$\sqrt{-\rho m} \qquad \text{or} \qquad \sqrt{-\rho m}\tanh\Big(\frac{\sqrt{-\rho m}}{m}(t + C)\Big)$$

which implies

$$X = \sqrt{-\rho m}\frac{\partial}{\partial r} \qquad \text{or} \qquad \sqrt{-\rho m}\tanh\Big(\frac{\sqrt{-\rho m}}{m}(t + C)\Big)\frac{\partial}{\partial r}.$$

If $A - \rho = 0$ and $m > 0$, then by Proposition 6.2(2) there are no solutions.

If $(A - \rho)m > 0$, then applying Proposition 6.2(2) to γ_{S^2} in a similar fashion, we get that there are no solutions.

Consider the case $(A - \rho)m < 0$. Since S^2 has dimension greater than 1, we can choose γ_{S^2} perpendicular to X at 0 so that $\varphi_{\gamma_{S^2}}(0) = 0$ and we apply Proposition 6.2(3) to $\gamma_{S^2} \in S^2$. Then $\varphi_{S^2}(t)$ is either

$$\pm\sqrt{-(A - \rho)m} \qquad \text{or} \qquad \sqrt{-(A - \rho)m}\tanh\Big(\frac{\sqrt{(A - \rho)m}}{m}(t + C)\Big).$$

$\varphi_{S^2}(t)$ cannot be $\sqrt{-(A - \rho)m}\tanh(\frac{\sqrt{(A-\rho)m}}{m}(t+C))$ since γ_{S^2} must be periodic and $\varphi_{S^2}(t)$ cannot be $\sqrt{-(A - \rho)m}$ since $\varphi_{\gamma_{S^2}}(0) = 0$. This is a contradiction, so there are no solutions in this case as well. □

Now we generalize Proposition 6.11 to compact quotients of manifolds of the form $M \times N$ where M and N are Einstein manifolds. We prove this in a different way from Proposition 6.11 because we cannot use the argument that $\varphi(t)$ must be periodic on S^j.

Lemma 6.12. *Consider a compact quotient of $M \times N$ with the product metric where M is an Einstein manifold. If there is a nontrivial m-quasi Einstein solution on such a space, then either $X|_M = 0$ or M is one-dimensional.*

Proof. Without loss of generality, we can assume that M and N are simply connected because if either space is not simply connected, then we can lift it to the universal cover. Let $\pi : M \times N \to (M \times N)/\Gamma$ be the universal covering map and let $\mathrm{Ric}_M = \rho_M g_M$. Let $\gamma_M(t)$ be a unit speed geodesic in M. Then we have

$$\frac{1}{2}\mathcal{L}_X g(\dot\gamma_M, \dot\gamma_M) - \frac{1}{m}X^* \otimes X^*(\dot\gamma_M, \dot\gamma_M) = A - \rho_M.$$

We aim to show that either $A - \rho_M = 0$ or $M = \mathbb{R}$. If M is not $\mathbb{R}$ then M is not one-dimensional, so we can choose γ_M to be perpendicular to X at 0. In this case, $\varphi_{\gamma_M}(0)$ is zero, so $\varphi_{\gamma_M}(t)$ cannot be constantly nonzero. If $(A - \rho_M)m > 0$, then by Proposition 6.2(2), there are no complete solutions. If $(A - \rho_M)m < 0$, then by Proposition 6.2(3) $\varphi_{\gamma_M}(t)$ is

$$\sqrt{-(A - \rho_M)m}\tanh\Big(\frac{\sqrt{(A - \rho_M)m}}{m}(t + C)\Big).$$

To show that $\varphi_{\gamma_M}(t)$ cannot be $\sqrt{-(A - \rho_M)m}\tanh\Big(\frac{\sqrt{(A-\rho_M)m}}{m}(t + C)\Big)$, we will use an argument similar to the proof of Lemma 2.2.

Consider the set $\overline{\{\pi \circ \gamma_M(t) : t \in \mathbb{R}\}}$. Since this set is closed, $\varphi_{\gamma_M}(t)$ has a maximal point t_{max} on this set. Because the supremum of the tanh function is 1, we know that the maximum of $\varphi_{\gamma_M}(t)$ on $\overline{\{\pi \circ \gamma_M(t) : t \in \mathbb{R}\}}$ is $\sqrt{-(A - \rho_M)m}$.

Let $\beta(t)$ be a geodesic of X with $\beta(0) = \gamma_M(t_{max}) = \sqrt{-(A-\rho_M)m}$ and consider the set $\{\pi \circ \beta(t) : t \in \mathbb{R}\}$. Along $\beta(t)$, $\varphi_\beta(t)$ is either $\sqrt{-(A-\rho_M)m}$ or $-\sqrt{-(A-\rho_M)m}\tanh(\frac{\sqrt{-(A-\rho_M)m}}{m}(t+C))$. Since the supremum of $\varphi_\beta(t)$ on $\{\beta(t) : t \in \mathbb{R}\}$ is $\sqrt{-(A-\rho_M)m}$ and the tanh function never achieves its maximum on its domain, $\varphi_\beta(t)$ must be constantly $\sqrt{-(A-\rho_M)m}$ on the set $\{\pi\circ\beta(t) : t \in \mathbb{R}\}$. Since $\overline{\{\pi \circ \beta(t) : t \in \mathbb{R}\}} = \overline{\{\pi \circ \gamma_M(t) : t \in \mathbb{R}\}}$, $\varphi_{\gamma_M}(t)$ is constant on $\overline{\{\pi \circ \gamma_M(t) : t \in \mathbb{R}\}}$. Thus, $\varphi_{\gamma_M}(t)$ is constant. Since $\varphi_{\gamma_M}(0) = 0$, $\varphi_{\gamma_M}(t)$ cannot be $\pm\sqrt{-(A-\rho_M)m}$, and so we have arrived at a contradiction.

Thus either $M = \mathbb{R}$ or $A - \rho_M = 0$. If $A - \rho_M = 0$, then $\varphi_{\gamma_M} = 0$ by Proposition 6.2(1), which implies that $X|_M = 0$. □

Now we can prove Theorem 1.6.

Proof of Theorem 1.6. Let $\pi : M \times N \to (M \times N)/\Gamma$ be the universal covering map and let $\mathrm{Ric}_M = \rho_M g_M$ and $\mathrm{Ric}_N = \rho_N g_N$. Let $\gamma_M(t)$ be a unit speed geodesic in M and let $\gamma_N(t)$ be a unit speed geodesic in N. By Lemma 6.12, M is either one-dimensional or $X|_M = 0$ and $A - \rho_M = 0$. By symmetry, either $A - \rho_N = 0$ and $X|_N$ is zero, or $N = \mathbb{R}$.

Suppose without loss of generality that $N = \mathbb{R}$. Then

$$\frac{1}{2}\mathcal{L}_X g(\dot\gamma_N, \dot\gamma_N) - \frac{1}{m}X^*(\dot\gamma_N)X^*(\dot\gamma_N) = Ag.$$

By Proposition 6.2, $A = 0$, hence $X = 0$. If $Am > 0$, then there are no solutions, and if $Am < 0$, then

$$X = \sqrt{-\lambda m}\tanh\Big(\frac{\sqrt{-\lambda m}}{m}(t+C)\Big)\frac{\partial}{\partial r} \quad \text{or} \quad X = \pm\sqrt{-\lambda m}\frac{\partial}{\partial r}.$$

If we consider the set $\overline{\{\pi \circ \gamma_N(t) : t \in \mathbb{R}\}}$ and use the same argument as above, we see that

$$X = \sqrt{-\lambda m}\tanh\Big(\frac{\sqrt{-\lambda m}}{m}(t+C)\Big)\frac{\partial}{\partial r}$$

is not a solution. Thus, the only solutions are $X = 0$ when $A = \rho_M = \rho_N \neq 0$, and $X = \pm\sqrt{-Am}\frac{\partial}{\partial r}$ when either $N = \mathbb{R}$ or $M = \mathbb{R}$. □

7 Summary

In the following table, we summarize the solutions of locally homogeneous compact three-manifolds M^3 which have quasi-Einstein metrics. In the first column, named “Manifold", we have the manifolds which act cocompactly on M^3. In the other columns we consider the different signs of m and A in our m-quasi Einstein equation $\mathrm{Ric}_X^m = Ag$. If there are no solutions to the compact quotient of “Manifold", we write *None*. If the only solutions are when $X = 0$, then we say *Trivial*, and if there are nontrivial solutions, then we say *Exists*.

Manifold	$m>0$ $A>0$	$m>0$ $A=0$	$m>0$ $A<0$	$m<0$ $A>0$	$m<0$ $A=0$	$m<0$ $A<0$
$\mathbb{R}^3$	None	Trivial	None	None	Trivial	None
SU(2)	Exists	Exists	Exists	Exists	None	None
$\widetilde{\mathrm{SL}_2(\mathbb{R})}$	None	None	None	None	Exists	None
Nil	None	None	Exists	None	None	None
E(1, 1)	None	None	None	None	None	None
E(2)	None	None	None	None	None	None
$H^2 \times \mathbb{R}$	None	None	Exists	None	None	None
$S^2 \times \mathbb{R}$	None	None	None	Exists	None	None
H^3	None	None	Trivial	None	None	Trivial

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