Transformation Groups ©Springer Science+Business Media New York (2023)

LATTICE VERTEX ALGEBRAS AND LOOP GRASSMANNIANS

IVAN MIRKOVIĆ

Department
of Mathematics and Statistics
University
of Massachusetts at Amherst
Amherst MA 01003-4515, USA
mirkovic@math.umass.edu

To James (Jim) Humphreys

Abstract. For an integral symmetric matrix κ we construct a new "nonabelian homology localization" of the lattice vertex algebra $\boldsymbol{L}^\kappa$ on the corresponding loop Grassmannian space $\mathcal{G}^\kappa$. We attempt to motivate our construction by presenting related topics in the language of "interaction of particles over algebraic curves".

I would like to recall some of the influences of Jim Humphreys on me. Before coming to UMass I thought that Jim's field, the modular representation theory, was a jungle of random coincidences that one should never touch. Through Jim I learned of hidden beauties of this new world, including many of Jim's foundational results. Consequently, this became one of principal fields of my research. My student Dmitriy Rumynin was in reality a joint student with Jim. Jim taught him the classical modular representation and presented him with a problem that eventually lead to Dmitriy's thesis and a radical geometric solution resulted in a joint paper [BMR] in Annals of Mathematics.

Jim and I had offices at UMass next to each other and many occasions to talk. In time my first student Elena Galaktionova married his student Cornelius Pillen, they had a long happy marriage with two great kids. My long term study of physics of Quantum Field Theory that has colored much of my view on mathematics and the world, has started when Jim gifted me 1500 pages of books on this subject.

In my experience Jim was always taking care of somebody. For years he was organizing the representation theory seminar and used it to take care of many less known mathematicians, with huge emphasis on people rather than on fame. For me it was important to have a close mathematician that I could adore as a human being.

DOI: 10.1007/s00031

Received December 9, 2022. Accepted June 16, 2023.

Corresponding Author: Ivan Mirković, e-mail: mirkovic@math.umass.edu

0. Overview

The main result in this paper is a certain "homology localization" of lattice vertex algebras on "loop Grassmannian" spaces associated to a symmetric integral matrix (Theorem 3.2). Much of the paper recalls relevant topics in a slightly nonstandard ("quasi-physical") language of "interaction of particles on an algebraic curve over a commutative ring".

While one could describe our setting in various ways, our choice of interaction language is motivated by the construction of loop Grassmannians associated to symmetric matrices from [MYZ] (see Section 3 below) as well as by geometric forms of Class Field Theory recalled in Section 1. The motivating goal is to try to set the stage and a uniform language, in order to (hopefully) understand better something old rather than to directly compute something new.

0.1. Heuristics

One wish in the background in this paper is to reconstruct key objects of the geometric Langlands program over a curve C (local or global) by a single procedure $\mathbb{H}_*(C,\mathcal{C})$ (using a coefficient system $\mathcal{C}$). This will be variously described as *inner homology in Algebraic Geometry*, *free generation*, *distributions*, or *moduli of interactions of particles on a curve.*

0.2. Contents

Section 1 deals with the abelian case, i.e., the geometric Class Field Theory. In the above story, the coefficients $\mathcal{C}$ are just the integers $\mathbb{Z}$, or even $\mathbb{N}$. The construction $\mathbb{H}_*$ recovers the Hilbert scheme $\mathcal{H}_C$ of points of C, and the Jacobian $\mathrm{Bun}^c_C(\mathbb{G}_m)$, i.e., the moduli of compactly supported line bundles in the sense of 1.1.2.

In Section 2, we recall some local nonabelian constructions. Here the coefficient system is a symmetric integral matrix κ, i.e., the corresponding lattice ([M1], [M2], [MYZ]). We get *zastava* spaces Z^κ, *semiinfinite* spaces[1] S^κ and *loop Grassmannians* $\mathcal{G}^\kappa$. When κ is the Cartan matrix of a semisimple group G, these are the same-named traditional spaces $Z(G), S(G), \mathcal{G}(G)$ that arise for G. Spaces $Z^\kappa, S^\kappa, \mathcal{G}^\kappa$ are constructed from a simple interaction of particles given by the matrix κ. We prove some foundational facts on these objects.

In Section 3, our main Theorem 3.2 provides localization of a lattice vertex algebra $\boldsymbol{L}^\kappa$ (associated to the matrix κ) on the Grassmannian $\mathcal{G}^\kappa$. This is a version of the Beilinson–Drinfeld "abelian" localization of $\boldsymbol{L}^\kappa$ on the loop Grassmannian $\mathcal{G}(T)$ of a torus T ([BD, Prop. 3.10.8]). In our picture, torus T is $\mathbb{G}_m^I$ where I is the set of kinds of particles. The κ-Grassmannian $\mathcal{G}^\kappa$ contains the loop Grassmannian $\mathcal{G}(T)$, but it also remembers κ and provides more space for constructions.

0.3. The inner homology idea

In Geometric Representation Theory one restates problems in Algebraic Geometry. Then one uses some kind of a "topological" formalism in Algebraic Geometry, say, cohomology, perverse sheaves or D-modules. The above "inner homology" $\mathbb{H}_*$ should be an example of such topological formalisms.

[1] Not to be confused with the semi-infinite Grassmannians.

As we will see, the "free generation" intuition on $\mathbb{H}_*$ is inspiring in simple cases but may have a limited range of literal validity. The situation is analogous to (and inspired by) the Dold–Thom theorem in algebraic topology. It says that the homology groups of a based topological space X can be calculated as homotopy groups of the group completion $\widehat{S}^\infty X$ of the monoid $S^\infty X$, the infinite symmetric power of X. (An elementary proof is in [Ha], [BL], and for manifolds, a modern proof is in [Ba].) My present terminology of "free generation mechanisms" originates from replacing the precise constructions $S^\infty X$ and $\widehat{S}^\infty X$ by terminology of a commutative topological monoid and group "freely generated" by X.

The Dold–Thom theorem has also been used as a motivation for introducing Suslin's construction of the motivic complexes ([Fr, Sect. 1]). Suslin's construction led to Voevodsky's motivic homology of algebraic varieties with values in abelian groups ([Voe]). In contrast, I would like to have a "homology theory" $\mathbb{H}_*(X,\mathcal{C})$ which is *inner* to Algebraic Geometry. Ideally, $\mathbb{H}_*(X,\mathbb{Z})$ would be the part of motivic homology of X that is representable by an algebro-geometric object. For these reasons, I will heuristically describe $\mathbb{H}_*(X,\mathbb{Z})$ as an "abelian group object in Algebraic Geometry", which is "freely generated" by variety X.[2]

I find that the need for such an inner (nonabelian) (co)homology theory is suggested by the standard Geometric Langlands Program, because it is formulated via categories that are defined only because cohomology spaces for curves carry algebro-geometric structures.[3]

We will only consider candidates for $\mathbb{H}_*(C,\mathcal{C})$ for smooth curves C. Here the construction is known in a sense (see Section 1). Also, this $\mathbb{H}_*$ produces various kinds of "spaces of distributions' of additive, multiplicative, and projective nature. In 1.2.1, we comment on how $\mathbb{H}_*(C,\mathbb{Z})$ reflects the idea of free generation. Similarly, in 1.2.2, $\mathbb{H}_*(C,\mathbb{Z})$ is interpreted as a space of distributions.

0.4. Local spaces over a curve C

The *locality principle* in physics says that the experiments at causally separated points a,b in a spacetime are independent. When we denote by E_x the set of possible outcomes of experiment at x, we can write this as $E_{a,b}\cong E_a\times E_b$. On the quantum level, one has a vector space V_x of states of a system at x and locality becomes $V_{a,b}\cong V_a\otimes V_v$. We say that a single state $v\in V_{a,b}$ is compatible with the spacetime separation if it is a pure tensor $v=v_a\otimes v_b$. For simplicity, I say that such state v is *local* though the standard terminology is *unentangled.*[4]

In the case of complex curves, our framework is closely related to Vertex Algebras and therefore to very special Quantum Field Theories, the 2-dimensional chiral Conformal Field Theories on complex curves. In this case, the causal separation of points a,b on a curve reduces to $a\neq b$ (see [Kac]), which is what we use on algebraic curves.

[2] While not adequate in general, this description is still going to be precise in cases of interest in this paper.

[3] Also, this application suggests that $\mathbb{H}_*$ should not be homotopy invariant in contrast to Voevodsky's construction.

[4] The definition of unentangled states in QFT is algebraically the same as what we call local states—the pure tensors in a tensor product.

These ideas will motivate the definition of *C-local* versions of some standard notions over the Hilbert scheme of points $\mathcal{H}_{C\times I}$ of a curve C colored by I colors (1.1.1) Over $\mathcal{H}_{C\times I}$, we consider *local spaces* Y, *local vector bundles* V, and their *local projective spaces* $\mathbb{P}^{\rm loc}(V) \subseteq \mathbb{P}(V)$ (see 2.1.2). The last is the "local" or "unentangled" part of the projective bundle $\mathbb{P}(V)$. I will think of $\mathbb{P}^{\rm loc}(V)$ as recording all collisions of fibers $\mathbb{P}(V_{ai})$ at points $ai \in C\times I$, that are allowed by the "rules" supplied by the local vector bundle V — this will really mean that collisions are inside the projective bundle $\mathbb{P}(V)$.

Our zastava space Z^κ is constructed as such local projective space. The corresponding local vector bundle V^κ comes from the interaction κ.

0.5. Interactions and loop Grassmannians

In our setup, the interaction between I-many kinds of particles on a curve C is given by a symmetric integer matrix $(\kappa_{ij}),\ i,j \in I$. We view it also as a symmetric bilinear form on the based lattice $\mathbb{Z}[I]$. Its geometric form will be a local line bundle L^κ over $\mathcal{H}_{C\times I}$ (see 2.1.1).

We will induce the local line bundle L^κ to a local vector bundle $\mathrm{Ind}(\mathsf{L}^\kappa)$ over $\mathcal{H}_{C\times I}$ by an "exponentiation" process of passing from a finite subscheme $D \in \mathcal{H}_{C\times I}$ to its "Grassmannian" $\mathrm{Gr}(D) \subseteq \mathcal{H}_{C\times I}$, which is the moduli of all subschemes of D.[5] This produces the *zastava* space over the Hilbert scheme $\mathcal{H}_{C\times I}$, as the corresponding local projective space $Z^\kappa = \mathbb{P}^{\rm loc}[\mathrm{Ind}(\mathsf{L}^\kappa)^\vee]$.

We define the *loop Grassmannian* $\mathcal{G}^\kappa$ of the matrix κ in Subsection 2.3 as a certain repackaging of pieces of $Z(\kappa)$. An intermediate step in repackaging will be the semi-infinite space S^κ (see Subsection 2.3). This is a "positive half" of $\mathcal{G}^\kappa$ in analogy with a Borel in a reductive Lie group. The spaces $Z^\kappa, S^\kappa, \mathcal{G}^\kappa$ will turn out to be certain moduli of unentangled collisions of projective lines $\mathbb{P}^1$. In the case, when κ is the Cartan matrix of a semisimple simply laced group G, then $\mathcal{G}^\kappa$ is just the Beilinson–Drinfeld version $\mathcal{G}_{\mathcal{H}_C}(G) \to \mathcal{H}_C$ of the standard *loop Grassmannian* $\mathcal{G}(G) \stackrel{\rm def}{=} G((z))/G[[z]]$ of G, see Theorem 2.5.

Acknowledgments. I thank Andreas Hayash and Zhijie Dong for mathematical help. I thank Eric Sommers, Renata Slovinić and the Transformation Groups journal for patience and extensive effort towards getting this paper finished. I particularly thank Roman Bezrukavnikov for insisting on precision and completeness of arguments, which forced me to clarify ideas and correct errors. I thank the referee for prolonged work towards making the paper more readable. This paper has been partially supported by an NSF grant.

1. Particles on curves and Class Field Theory (CFT)

The space through which our particles wonder will be a smooth algebraic curve C (local or global). Unless otherwise stated the curve C is defined over a commutative ring $\Bbbk$, for instance, as in the work of Contou-Carrère [CC].

[5] We also use notation $\mathrm{Gr}(X)$ for the moduli $\mathcal{H}_X$ of finite subschemes of X.

1.1. Configuration spaces of particles

We will consider two models.

1.1.1. The Hilbert scheme moduli $\mathcal{H}_C$ of finite subschemes of C.

This is $\bigsqcup_{\mathbb{N}} \mathcal{H}_C^n$ for the categorical quotient $\mathcal{H}_C^n = C^n /\!\!/ S_n$, interpreted as a moduli of subschemes D of C which are flat and of length n, i.e., of effective divisors in C of degree n. The subspace of n-tuples of disjoint points $C_r^n \subseteq C^n$ gives an open $\mathcal{H}_{C,r}^n \stackrel{\text{def}}{=} C_r^n /\!\!/ S_n$ consisting of n element subsets $D = \{a_1, \dots, a_n\}$ of C. We view $\mathcal{H}_C^n$ as the (classical) configuration space of a system of n identical particles on C. Then the degeneration of finite subsets of C to finite subschemes models the collision of particles.

Notice that $\mathcal{H}_C$ is a commutative monoid for the operation of adding the divisors.

Example. If $C = \mathbb{A}^1$ we can encode a finite subset $\{a_1, \dots, a_n\}$ by a polynomial $P = \prod(X - a_i)$. Then $\mathcal{H}_{\mathbb{A}^1}^n$ is identified with all monic polynomials P of degree n. Such P gives a subscheme of $\mathbb{A}^1$, $D \stackrel{\text{def}}{=} \{P = 0\} = \mathrm{Spec}(\Bbbk[X]/P)$.

1.1.2. The moduli $A_C = \mathrm{Bun}_C^c(\mathbb{G}_m)$ of compactly supported line bundles.

By this we mean a line bundle trivialized at infinity in the following sense. Let C be open in a smooth projective curve $\overline{C}$. Consider the formal neighborhood $\widehat{\partial C}$ of the boundary $\partial C = \overline{C} - C$ and its punctured version $\widetilde{\partial C} = \widehat{\partial C} - \partial C$.

Then we define $\mathrm{Bun}_C^c(\mathbb{G}_m)$ as the moduli of pairs (L, s) of a line bundle L over C and an invertible section s on the part $\widetilde{\partial C}$ of C at infinity. This is a commutative group object for the operation of tensoring line bundles.

In particle terms, I will think of "vacuum" on C (i.e., absence of particles), as represented by $L = \mathcal{O}_C$ and $s = 1$. Then the general pair (L, s) in $\mathrm{Bun}_{\mathbb{G}_m}^c(C)$ will model a disturbance of the vacuum localized to the divisor $\mathrm{div}(s)$.

1.1.3. The Abel–Jacobi map $\mathcal{H}_C \to A_C$.

It sends a finite subscheme $D \in \mathcal{H}_C$ to $AJ_C(D) \stackrel{\text{def}}{=} (\mathcal{O}_C(-D), 1)$, where the subsheaf $\mathcal{O}_C(-D) \subseteq \mathcal{O}_C$ consists of functions that vanish on D and 1 is the standard trivialization at ∞ (actually valid outside D).

Heuristically, one can visualize $\mathrm{AJ}_C(D)$, i.e., the inclusion $\mathcal{O}_C(-D) \subseteq \mathcal{O}_C$, as the space C with a tiny hole at D. (As in General Relativity, where a particle mass "bends" the space around it.)

1.2. Particles and Class Field Theory

The Class Field Theory is an organizational principle of Number Theory. I will only consider it in the geometric case when the global field $\mathbb{Q}$ is simplified to the field $\mathbb{F}_q(X)$ of rational functions over a curve over a finite field. Then we will see that the above configuration spaces $\mathcal{H}_C$ and A_C can be viewed as the standard setting for Class Field Theory.

1.2.1. Contou-Carrère's Class Field Theory in Zariski topology.

I will restate the results of Contou-Carrere in a "free generation" formulation. I will only illustrate the approach in some cases though I expect the formulation to be eventually uniform for any smooth curve C (global or local) over any commutative ring $\Bbbk$.

Examples.

(0) If C is a smooth curve or a formal disc in such curve, then the Hilbert scheme $\mathcal{H}_C$ is the commutative monoid freely generated by C in indschemes.

Any map f of C into a commutative monoid M extends uniquely along the inclusion $C = \mathcal{H}_C^1 \subseteq \mathcal{H}_C$ to a homomorphism $F : \mathcal{H}_C \to M$ by using the norm map $F(D) = N_D f|_D$.

(1) The following is known to experts though there may be no reference in this formulation. If C is complete, then the stacky Jacobian $A_C = \mathrm{Bun}^c_{\mathbb{G}_m}(C) = \mathrm{Bun}_{\mathbb{G}_m}(C)$ is the commutative group object freely generated by C in commutative group stacks. Then the Abel–Jacobi map $AJ_C : \mathcal{H}_C \to A_C$ is the map defined by free generation properties in settings of monoids and groups.

(2) The work of Contou-Carrère [CC], [CC1] covers much more than I have stated. The simplest case is that for a formal disc d (viewed as an indscheme), the freely generated commutative group (in indschemes) is the loop Grassmannian $\mathcal{G}(\mathbb{G}_m) = \mathrm{Bun}^c_{\mathbb{G}_m}(d)$ of the multiplicative group [CC].

However, the corresponding statements in [CC] get more complicated for the punctured formal disc d^* and consequently also for the open curves.[6] A history of these ideas for global curves is found in [Se].

1.2.2. Distributional interpretations.

We consider versions of the following formalism. Consider a category of commutative monoids $\mathcal{A}$ (with a "dualizing object" $\underline{\omega}$) defined inside a category $\mathcal{B}$ of algebro-geometric objects that contains curves. We assume existence of inner homomorphism functors $\mathcal{H}om_{\mathcal{A}}$ and $\mathcal{M}ap_{\mathcal{B}}$ in $\mathcal{A}$ and $\mathcal{B}$. This defines a notion of $\underline{\omega}$-distributions on C as the double dual $Dis^{\underline{\omega}}(C) \stackrel{\text{def}}{=} \mathcal{H}om_{\mathcal{A}}[\mathcal{M}ap_{\mathcal{B}}(X,\underline{\omega}),\underline{\omega}] = \mathbb{D}^{\omega}\mathcal{M}ap_{\mathcal{B}}(X,\underline{\omega})$ for the "$\underline{\omega}$-Cartier duality" $\mathbb{D}^{\underline{\omega}}(A) = \mathcal{H}om_{\mathcal{A}}(A,\underline{\omega})$. Under some assumptions (that $(\mathbb{D}^{\underline{\omega}})^2$ is identity), $Dis^{\underline{\omega}}(C)$ is the universal object of $\mathcal{A}$ that C maps to, i.e., a freely generated object of $\mathcal{A}$.

For simplicity, let C be a smooth affine curve over a commutative ring $\Bbbk$ and let $\mathcal{B}$ be the category $IndSch_{\Bbbk}$ of indschemes over $\Bbbk$.

Examples.

(i) If $\mathcal{A}$ is the category $\mathcal{V}ec_{\Bbbk}$ of vector spaces over $\Bbbk$, and $\underline{\omega} = \mathbb{G}_a$ is the additive group, we get ordinary distributions $Dis^{\mathbb{G}_a}(C) = \mathcal{H}om_{\mathcal{V}ec}[\mathcal{M}ap_{IndSch_{\Bbbk}}(C,\mathbb{G}_a),\mathbb{G}_a] = \mathcal{O}(C)^{\vee}$.

(ii) When $Ind\mathcal{A}b_{\Bbbk}$ are the commutative affine group indschemes over $\Bbbk$, we get *multiplicative distributions* $Dis^{\mathbb{G}_m}_{\mathcal{V}ec_{\Bbbk}}(C) = \mathcal{H}om_{Ind\mathcal{A}b_{\Bbbk}}[\mathcal{M}ap_{IndSch_{\Bbbk}}(C,\mathbb{G}_m),\mathbb{G}_m] = \mathbb{D}^{\mathbb{G}_m}(\mathcal{O}^*(C))$ for the Cartier duality functor $\mathbb{D}^{\mathbb{G}_m}$.

(iii) Let $\mathcal{A}$ be the category $Ind\mathcal{M}_{\Bbbk}$ of commutative affine indscheme monoids and $\underline{\omega} = \overline{\mathbb{G}_m}$ be the monoid $\mathbb{A}^1$ with multiplication. We get the *monoidal multiplicative distributions* $Dis^{\overline{\mathbb{G}_m}}(C) = \mathcal{H}om_{Ind\mathcal{M}_{\Bbbk}}[\mathcal{M}ap_{IndSch_{\Bbbk}}(C,\overline{\mathbb{G}_m}),\overline{\mathbb{G}_m}] = \mathbb{D}^{\overline{\mathbb{G}_m}}(\mathcal{O}(C))$. Here, $\mathbb{D}^{\overline{\mathbb{G}_m}}$ is the Cartier duality functor in commutative monoids introduced by Lurie. Then $Dis^{\overline{\mathbb{G}_m}}(C)$ is the Hilbert scheme $\mathcal{H}_C$ of points of C.

[6] The reason seems to be that the punctured disc d^* is treated as a scheme while it is more natural to use some version of rigid geometry.

Remarks.

(0) We will also think of the projective space $\mathbb{P}[Dis^{\mathbb{G}_a}(C)]$ as "projective distributions" on C. This will be the setting for the construction of zastava spaces Z^κ in 2.2.3.

(1) For complete curves one needs a stacky version of distributions (using stacky $\mathcal{H}om$ and $\mathcal{M}ap$).

(2) The following conjectural $(\mathcal{A},\underline{\omega})$-version of algebro-geometric Poincaré duality for smooth curves C is known in many cases for $\mathcal{A},\underline{\omega}$ in our examples above. The traditional formulation is that

$$\mathrm{R}\Gamma_c(C,\underline{\omega})[1] \;\cong\; \mathbb{D}^{\underline{\omega}}[\mathrm{R}\Gamma(C,\underline{\omega})].$$

When the distributional formula on the right-hand side is known to be the free $\mathcal{A}$-object generated by C, the claim is that this free object is the compactly supported $\underline{\omega}$-cohomology on the left-hand side.[7]

(3) The idea that the inner homology $\mathbb{H}_*$ from 0.1 should literally be given by "free generation" works fairly well for curves, however, it does not seem adequate already for surfaces (Sam Raskin). This is analogous to the topological case where "free generation" is only heuristics for the symmetric power S^∞ (see 0.3).[8]

1.2.3. Class Field Theory in étale topology.

The Langlands program is a conjectural extension of the traditional Class Field Theory, which features Galois groups, hence *étale* topology. For a local or global field F, it describes the largest commutative quotient of the Galois group Γ_F of F in terms of the Galois cohomology of a certain canonical Galois module, the "Class Field Theory module" $\mathcal{C}_F$ for Γ_F (see [AT]).

When F contains a finite field $\mathbb{F}_q$, then F corresponds to the generic point η of a smooth curve C over $\Bbbk=\mathbb{F}_q$ (local or global). If the field F is local, the module $\mathcal{C}_F$ is given by invertible elements $\overline{F}^*$ of the separable closure. For a global field, $\mathcal{C}_F$ is the idele class group of $\overline{F}$.

We will describe the Class Field Theory module $\mathcal{C}_F$ for F, uniformly in the local and global case, in terms of the above Zariski Class Field Theory for the associated curve η.

Theorem. *The Class Field Theory module is the group of $\mathbb{F}_q$-points of $H^0(A_\eta)$:*

$$\mathcal{C}_F \;=\; [H^0(A_\eta)](\mathbb{F}_q).$$

Following the above description of the Class Field Module, one checks this separately in local and global cases.

[7] The versions of Poincaré duality in [Ma], [La], [HR] are rather in the spirit of the étale Class Field Theory in 1.2.3 below.

[8] A better formulation may involve factorization homology as in [Ba] or in Lurie's noncommutative Poincaré duality?

2. Interaction of particles, zastavas, and loop Grassmannians

The algebraic notion of a vertex algebra will only be used in stating the story. We will actually work with its geometric incarnations from [BD] (see Subsection 3.1 below).

In Subsection 2.1, we start with an algebro-geometric model of I-many kinds of particles on a smooth algebraic curve C. This is the Hilbert scheme of points $\mathcal{H}_{C\times I}$ as in Subsection 1.1, except that the curve C has been replaced by the colored curve $C{\times}I$. In this model, the interactions are given by a symmetric integral matrix $\kappa\in M_I(\mathbb{Z})$. This setting involves the notion of "locality" for objects over $\mathcal{H}_{C\times I}$ but with respect to the curve C. This includes local spaces over $\mathcal{H}_{C\times I}$, local vector bundles over $\mathcal{H}_{C\times I}$ and their local projective spaces (see 2.1.2).

These ideas lead to construction of several moduli of collisions of these particles. The basic one is the *zastava* space Z^κ introduced in Subsection 2.2. One repackages it into the *loop Grassmannian* $\mathcal{G}^\kappa$ in 2.4.5, and there is an intermediate space, the semi-infinite spaces S^κ (see Subsection 2.3). In the sequence $Z^\kappa, S^\kappa, \mathcal{G}^\kappa$, the next space is a union of copies of the preceding space. More on structure and relations between these spaces is summarized in Lemma in 2.4.5.

The spaces $\mathcal{G}^\kappa$ generalize loop Grassmannians of reductive groups (see Subsection 2.5).

2.1. A geometric model of interaction of particles

We consider I kinds of particles on our curve C. We can think of particles of type $i\in I$ as living on the curve $C_i \overset{\text{def}}{=} C{\times}i\subseteq\ C{\times}I$.

If there are a_i particles of the kind $i\in I$, the total number of particles is $\alpha=\sum_{i\in I}a_i i\in\mathbb{N}[I]$ and the configuration space of these is $\mathcal{H}^\alpha_{C\times I}\overset{\text{def}}{=}\ \prod_{i\in I}\mathcal{H}^{a_i}_{C_i}$. An *$I$-colored divisor* $D\in\mathcal{H}_{C\times I}$ is therefore a family of component divisors $D_i\subseteq C$ for $i\in I$. The total space is $\mathcal{H}_{C\times I}\overset{\text{def}}{=}\ \bigsqcup_{\alpha\in\mathbb{N}[I]}\mathcal{H}^\alpha_{C\times I}$.

Interactions between these particles will be given by a matrix $\kappa\in M_I(\mathbb{Z})$ (which will be symmetric as in Newton's first law). Intuitively, κ_{ij} is the intensity of interaction of i and j particles at $(x,y)\in C_i{\times}C_j$ near the diagonal $x=y$, i.e., as they collide when y approaches x.

2.1.1. A geometrization of interaction.

For $i,j\in I$ we define the (i,j)-diagonal divisor $\Delta_{ij}\subseteq\mathcal{H}=\mathcal{H}_{C\times I}$. When $i\neq j$, the condition for $D\in\Delta_{ij}$ is that the component divisors D_i and D_j meet. Also, Δ_{ii} is a discriminant divisor, a subscheme $D\in\mathcal{H}_C$ lies in Δ_{ii} if D_i is not discrete, i.e., some point has multiplicity >1. We call the complement $\mathcal{H}^{\mathrm{r}}_{C\times I}$ of $\bigcup\Delta_{ij}\subseteq\mathcal{H}_{C\times I}$ the *regular* part of the configuration space.

Our geometric encoding of the interaction $\kappa\in M_I(\mathbb{Z})$ will be the line bundle on the configuration space $\mathcal{H}_{C\times I}$ denoted

$$\mathsf{L}=\mathsf{L}^\kappa\overset{\text{def}}{=}\mathcal{O}_{\mathcal{H}}(-\kappa\Delta),$$

where $\kappa\Delta$ is the divisor $\sum_{i\leq j}\kappa_{ij}\Delta_{ij}$ in $\mathcal{H}_{C\times I}$.

Remark. We can visualize L^κ as the space $\mathcal{H}$ with holes of depth κ_{ij} along the divisors Δ_{ij}.

2.1.2. Locality ([M1], [M2], [MYZ]).

A *locality structure* on a space Y over $\mathcal{H}_{C\times I}$ is a system of isomorphisms for fibers at disjoint $D', D'' \in \mathcal{H}_{C\times I}$,

$$Y_{D'\bigsqcup D''} \xleftarrow{\cong} Y_{D'}\times Y_{D''},$$

that satisfies the associative, commutative and unital properties.[9] Similarly, a local structure on a vector bundle V over $\mathcal{H}_{C\times I}$ is a system of isomorphisms $V_{D'\bigsqcup D''} \xleftarrow{\cong} V_{D'}\otimes V_{D''}$ that satisfies the same kind of properties. Our basic example of local line bundles are the interaction line bundles L^κ for symmetric matrices κ.

Remark. One can likewise define local spaces and local vector bundles over any local space $Y\to\mathcal{H}_{C\times I}$.

To a local vector bundle V over $\mathcal{H}_{C\times I}$ we will now associate its *local projective space* $\mathbb{P}^{\mathrm{loc}}(V)\subseteq\mathbb{P}(V)$. First, at a point $D=ai\in C\times I\subseteq\mathcal{H}_{C\times I}$ the locality condition is empty, so $\mathbb{P}^{\mathrm{loc}}(V)_{ai}$ is the whole fiber $\mathbb{P}(V)_{ai}=\mathbb{P}(V_{ai})$. The locality requirement now forces the fiber at a regular divisor $D\in\mathcal{H}^{\mathrm{r}}_{C\times I}$, i.e., a finite colored subset of C, to be

$$\mathbb{P}^{\mathrm{loc}}(V)_D \overset{\mathrm{def}}{=} \prod_{ai\in D}\mathbb{P}(V_{ai}).$$

The locality structure isomorphism $V_D\cong\bigotimes_{ai\in D}\ V_{ai}$ now gives the Segre embedding $\mathbb{P}^{\mathrm{loc}}(V)_D\hookrightarrow\mathbb{P}(V)_D$. So we have constructed a subspace $\mathbb{P}^{\mathrm{loc}}_{\mathrm{reg}}(V)\subseteq\mathbb{P}(V)|_{\mathcal{H}^{\mathrm{r}}_{C\times I}}$ over the open subspace $\mathcal{H}^{\mathrm{r}}_{C\times I}\subseteq\mathcal{H}_{C\times I}$ of regular divisors.

Finally, define the local projective space $\mathbb{P}^{\mathrm{loc}}(V)$ as the closure of $\mathbb{P}^{\mathrm{loc}}_{\mathrm{reg}}(V)$ in $\mathbb{P}(V)$. By its definition $\mathbb{P}^{\mathrm{loc}}_{\mathrm{reg}}(V)$ has a canonical structure of a local space over $\mathcal{H}^{\mathrm{r}}_{C\times I}$, so its closure $\mathbb{P}^{\mathrm{loc}}(V)$ is a local space over $\mathcal{H}_{C\times I}$.

Remark. Heuristically, the fiber $\mathbb{P}^{\mathrm{loc}}(V)_D$ at $D\in\mathcal{H}_{C\times I}$ consists of lines M in V_D which in some sense respect the geometry of the subscheme D of C. We call such lines "local" (see Subsection 0.4) or "unentangled".

Moreover, one can view a fiber $\mathbb{P}^{\mathrm{loc}}(V)_D$ at a general $D\in\mathcal{H}_{C\times I}$ as a limit of nearby fibers $\mathbb{P}^{\mathrm{loc}}(V)_E\cong\ \prod_{ai\in E}\mathbb{P}(V_{ai})$ for regular $E\in\mathcal{H}^{\mathrm{r}}_{C\times I}$. This degeneration of products of projective spaces models "*collision*" of "quantum particles" $\mathbb{P}(V_{ai})$.

2.1.3. A motivation: Lattice Vertex Algebras.

By a lattice we mean a pair $(K,\widetilde{\kappa})$ of a group $K\cong\mathbb{Z}^n$ and an integral symmetric bilinear form $\widetilde{\kappa}$ on K (a "quadratic form"). These parameterize the lattice vertex algebras $\mathbb{L}_{\widetilde{\kappa}}$ (see [Kac]). A symmetric matrix $\kappa\in M_I(\mathbb{Z})$ is the same as a quadratic form $\widetilde{\kappa}$ on a based lattice $\mathbb{Z}[I]$, so it defines a vertex algebra that we denote $\boldsymbol{L}^\kappa$.

However, we have noticed that such κ also defines a local line bundle L^κ on the configuration space $\mathcal{H}_{C\times I}$. This will lead in Subsections 2.2–2.4 to certain indschemes $Z^\kappa, S^\kappa, \mathcal{G}^\kappa$ associated to the matrix κ, and this will provide (see Theorem in Subsection 3.2) a geometric setting for the lattice vertex algebra $\boldsymbol{L}^\kappa$.

[9] This is just a version of the Beilinson–Drinfeld factorization structure from [BD].

2.2. Zastava space Z^κ

2.2.1. Induction of local vector bundles over $\mathcal{H}_{C\times I}$.

Our first step is to induce the local line bundle L^κ over $\mathcal{H}_{C\times I}$ to a local vector bundle $I^\kappa = \mathrm{Ind}(\mathsf{L}^\kappa)$ over $\mathcal{H}_{C\times I}$. This will use at each $D \in \mathcal{H}_{C\times I}$ the "Grassmannian" moduli $\mathrm{Gr}(D) = \mathcal{H}_D$ of all subschemes $D' \subseteq D$. This is a subscheme of $\mathcal{H}_{C\times I}$ of length $|\mathrm{Gr}(D)| = 2^{|D|} = \prod_{i\in I} 2^{|D_i|}$. Then the fiber of the vector bundle I at $D \in \mathcal{H}$ is the space $I_D = \Gamma[\mathrm{Gr}(D), \mathsf{L}]$ of sections of L on $\mathrm{Gr}(D)$. This is the space of states of all subschemes of D.

We can state this in terms of the subscheme $\mathcal{T} \subseteq \mathcal{H}_{C\times I}\times(C\times I)$ which is locally finite and flat over $\mathcal{H}_{C\times I}$. This is the tautological scheme with fibers $\mathcal{T}_D = D$ at $D \in \mathcal{H}_{C\times I}$. Then the relative Hilbert scheme $\mathrm{Gr}(\mathcal{T}/\mathcal{H}_{C\times I})$ has fibers $\mathrm{Gr}(\mathcal{T}/\mathcal{H}_{C\times I})_D = \mathrm{Gr}(D)$. In terms of sheaves, the induction is $I^\kappa = q_* p^* \mathsf{L}^\kappa$ for the correspondence $\mathcal{H}_{C\times I} \xleftarrow{p} Gr(\mathcal{T}/\mathcal{H}_{C\times I}) \xrightarrow{q} \mathcal{H}_{C\times I}$ given by $D' \hookleftarrow (D' \subseteq D) \mapsto D$. Now the fact that I is indeed a vector bundle follows from the next lemma.

Lemma. *For $p = (p_1 \leq \cdots \leq p_k)$, let $\mathrm{Gr}_p(\mathcal{T}/\mathcal{H}_{C\times I})$ be the scheme over $\mathcal{H}_{C\times I}$ such that the fiber at $D \in \mathcal{H}_{C\times I}$ consists of flags of subschemes $D_1 \subseteq \cdots \subseteq D_k \subseteq D$ with D_i of length p_i. This is a flat locally finite scheme over $\mathcal{H}_{C\times I}$, which is irreducible over each connected component of $\mathcal{H}_{C\times I}$.*

Proof. For simplicity, let I be a point and fix a component $\mathcal{H}^n_C = S^n C$ of the base. The claim holds for full flags $\mathrm{Gr}_{1,\ldots,n}$ since this space is just C^n. The claim for general p follows from the forgetful map $\mathrm{Gr}_{1,\ldots,n} \to \mathrm{Gr}_p$ since its fibers are products of smaller full flags. $\square$

2.2.2. Interaction κ defines a local vector bundle $V = V^\kappa$ over $\mathcal{H}_{C\times I}$.

Local vector bundles are closed under duality and tensoring. We will really use the dual $V^\kappa \overset{\text{def}}{=} (I^\kappa)^\vee$ and think of the fibers $V^\kappa_D = \Gamma[\mathrm{Gr}(D), \mathsf{L}]^\vee$ at $D \in \mathcal{H}_{C\times I}$ as "L-twisted additive distributions $Dis(\mathrm{Gr}(D), \mathsf{L})$" over $\mathrm{Gr}(D)$. Dualizing gives the Kodaira embedding of $\mathrm{Gr}(D)$ into the space $\mathbb{P}(V_D)$ of "L-twisted projective distributions over $\mathrm{Gr}(D)$".

So, at any $D \in \mathcal{H}_{C\times I}$, we have a vector space V_D and its projective space $\mathbb{P}(V)_D$, defined as some kind of spaces of "quantum states of a subscheme of D". (We often just say "quantum states of D".) By definition, $\mathbb{P}(V_D)$ contains the space $\mathrm{Gr}(D)$ of corresponding "classical states".

Example. Let D be a point ai in $C\times I$, i.e., a point a in C, colored by $i \in I$. Then $\mathrm{Gr}(ai) = \{\varnothing, ai\}$, so $V_{ai} = \mathsf{L}^*_\varnothing \oplus \mathsf{L}^*_{ai} \cong \Bbbk \oplus \mathsf{L}^*_{ai}$ and the corresponding fiber of the projective bundle is $\mathbb{P}(V)_{ai} \cong \mathbb{P}^1$.

2.2.3. The zastava space Z^κ of a symmetric matrix κ.

It is defined as the local projective space of the local vector bundle V^κ, so it is the space $Z^\kappa = Z^\kappa_{\mathcal{H}_{C\times I}} \overset{\text{def}}{=} \mathbb{P}^{\mathrm{loc}}(V) \subseteq \mathbb{P}(V^\kappa)$ over $\mathcal{H}_{C\times I}$.

Heuristically, we can say that Z^κ models the moduli of (L^κ-twisted) *local* quantum states of subschemes of $D \in \mathcal{H}_{C\times I}$. Moreover, these "quantum states" are just the collisions allowed by the interaction κ of quantum particles $\mathbb{P}(V^\kappa_{ai}) \cong \mathbb{P}^1$ that are indexed by $i \in I$ and positioned at points $a \in C$. Here, the "collision rules" κ are felt through the background $\mathbb{P}(V^\kappa)$, in which the collisions are allowed to happen.

Example.

(a) At a point $D = ai \in C{\times}I \subseteq \mathcal{H}_{C\times I}$, there are no locality conditions and the space Z_{ai} of local states of a quantum particle at ai is the whole fiber $\mathbb{P}(V)_{ai} = \mathbb{P}(V_{ai}) \cong \mathbb{P}^1$ (independently of the color $i \in I$).

(b) For two particles, the fiber of Z^κ at a regular $D = ai+bj \in \mathcal{H}_{C\times I}$ (meaning that $a \neq b$), is $\mathbb{P}^1{\times}\mathbb{P}^1$, which is given in $\mathbb{P}(V_{ai+bj}) = \mathbb{P}^3$ by the locality equation $xy = uv$. Then the collision as $b \to a$ degenerates this smooth quadric into one of the form $z_1^2 + \cdots + z_k^2 = 0$ in $\mathbb{P}^3$ with $1 \leq k \leq 4$. The simplest degeneration (the case $k = 3$), is the one point compactification of the tangent bundle of $\mathbb{P}^1$. It appears when the quadratic form κ corresponds to the group $G = \mathrm{PGL}(2)$ in the sense of Theorem in Subsection 2.5 below.

2.2.4. K-triples.

For a sheaf L on a space A over S, we denote the corresponding sheaf on S by $L_{A/S} \stackrel{\text{def}}{=} (A \to S)_* L$.

The K-triples (A, L, B) over S will consist of a very ample line bundle L over A and an embedding $B \hookrightarrow \mathbb{P}[L^\vee_{A/S}]$ such that

- the Kodaira embedding of A is into B;
- the restriction $\mathcal{O}_B(1)_{B/S} \to L_{A/S}$ is an isomorphism,

where $\mathcal{O}_B(1)$ is the restriction of $\mathcal{O}_{\mathbb{P}[\Gamma(A,L)^\vee]}(1)$ to B.

Lemma (The "weak flatness" [MYZ, Cor. 2.4.3]**).**

(a) $(\mathrm{Gr}(\mathcal{T}/\mathcal{H}_{C\times I}), \mathrm{pr}_2^* \mathsf{L}^\kappa, Z^\kappa_{\mathcal{H}_C})$ *is a K-triple over* $\mathcal{H}_{C\times I}$*. In other words, at* $D \in \mathcal{H}_{C\times I}$*, one has* $\Gamma(Z^\kappa_D, \mathcal{O}_{\mathbb{P}(V^\kappa)}(1)) \xrightarrow{\cong} \Gamma(\mathrm{Gr}(D), \mathsf{L}^\kappa)$.

(b) *For the torus* $T = \mathbb{G}_m^I$*, the fixed point subscheme* $(Z^\kappa_{\mathcal{H}_{C\times I}})^T$ *is* $\mathrm{Gr}(\mathcal{T}/\mathcal{H}_{C\times I})$.

So we will denote by $\mathcal{O}_{Z^\kappa}(1)$ the restriction of $\mathcal{O}_{\mathbb{P}(V^\kappa)}(1)$ for the embedding $Z^\kappa_{\mathcal{H}_C} \subseteq \mathbb{P}(V^\kappa)$.

2.3. Semi-infinite spaces $S^\kappa_{\mathcal{H}_C}$ of matrices κ

Let $\kappa \in M_I(\mathbb{Z})$ be a symmetric matrix.

2.3.1. The defect action.

Let $\mathcal{T} \to \mathcal{H}_{C\times I}$ denote the tautological scheme, i.e., $\mathcal{T}_D = D$ for $D \in \mathcal{H}_{C\times I}$. Then the relative Hilbert scheme $\mathrm{Gr}(\mathcal{T}/\mathcal{H}_{C\times I})$ with fibers $\mathrm{Gr}(\mathcal{T}/\mathcal{H}_{C\times I})_D = \mathrm{Gr}(D)$ lies in Z^κ. First, $\mathrm{Gr}(\mathcal{T}/\mathcal{H}_{C\times I}) \hookrightarrow \mathbb{P}(V^\kappa)$ is the Kodaira embedding as $V_D = \Gamma[\mathrm{Gr}(D), \mathsf{L}^\kappa]^*$. Then $\mathrm{Gr}(\mathcal{T}/\mathcal{H}_{C\times I})$ actually lies in $\mathbb{P}^{\mathrm{loc}}(V^\kappa) = Z^\kappa$ because this is true over the regular part $\mathcal{H}^{\mathrm{r}}_{C\times I}$ and $\mathrm{Gr}(\mathcal{T}/\mathcal{H}_{C\times I})$ is irreducible over each connected component of $\mathcal{H}_{C\times I}$ (Lemma in 2.2.1).

Lemma.

(a) *There is a unique action of* $\mathcal{H}_{C\times I}$ *on the space* Z^κ *over* $\mathcal{H}_{C\times I}$ *that preserves* $\mathrm{Gr}(\mathcal{T}/\mathcal{H}_{C\times I})$ *and such that for* $E, D \in \mathcal{H}_{C\times I}$*, the action* $\mathrm{Gr}(\mathcal{T}/\mathcal{H}_{C\times I})_D \xrightarrow{E} \mathrm{Gr}(\mathcal{T}/\mathcal{H}_{C\times I})_{D+E}$*, i.e.,* $\mathrm{Gr}(D) \to \mathrm{Gr}(D+E)$*, is just the inclusion. This is called the* defect action.[10] *Then the maps* $Z^\kappa_D \xrightarrow{E} Z^\kappa_{D+E}$ *are closed inclusions.*

[10] Because it takes the generic part of Z^κ_D to the boundary of the generic part of Z^κ_{D+E}.

(b) *The zastava space* $\big(Z^\kappa, \mathcal{O}_{Z^\kappa}(1)\big)$ *over* $\mathcal{H}_{C\times I}$ *has a "growth structure", i.e., a consistent system of closed embedding* $\gamma_{D',D}: Z_{D'} \hookrightarrow Z_D$ *for* $D' \subseteq D \in \mathcal{H}_{C\times I}$*, compatible with the line bundle* $\mathcal{O}_{Z^\kappa}(1)$.

Proof. (a) For an affine test scheme S, let E, D be two S-points of $\mathcal{H}_{C\times I}$ so that $F = D + E$ contains D. The inclusion $\mathrm{Gr}(D) \subseteq \mathrm{Gr}(F)$ of affine schemes yields a surjection $\Gamma[\mathrm{Gr}(F), \mathsf{L}^\kappa] \to \Gamma[\mathrm{Gr}(D), \mathsf{L}^\kappa]$. Dually we have $V_D \subseteq V_F$ hence $\mathbb{P}(V_D) \subseteq \mathbb{P}(V_F)$. If F is regular, the first inclusion is the tensor product of inclusions $V_{ai\cap D} \subseteq V_{ai}$ over points $ai \in F$. Then the F-product of inclusions $\mathbb{P}(V_{ai\cap D}) \subseteq \mathbb{P}(V_{ai})$ gives $\mathbb{P}^{\mathrm{loc}}(V_D) \subseteq \mathbb{P}^{\mathrm{loc}}(V_F)$. By the closure definition of $\mathbb{P}^{\mathrm{loc}}(V)$, this implies the same for all $F \in \mathcal{H}_{C\times I}$.

(b) This is a restatement of (a) since $\gamma_{D',D}$ is given by the defect action of $E = D - D'$. The identification of line bundles $\mathcal{O}_{Z^\kappa}(1)|_{Z^\kappa_{D'}} \xrightarrow{\cong} \gamma^*_{D,D'}\mathcal{O}_{Z^\kappa}(1)|_{Z^\kappa D}$ is also by the construction of the defect action in (a). ☐

2.3.2. Semi-infinite spaces S^κ.

Due to Lemma in 2.3.1, we can define the space $S^\kappa = S^\kappa_{\mathcal{H}_C}$ over $\mathcal{H}_C$ with fibers at the divisor $\mathsf{D} \in \mathcal{H}_C$:

$$S^\kappa_{\mathsf{D}} \stackrel{\mathrm{def}}{=} \varinjlim_{\mathcal{H}_{C\times I}\ni F\subseteq \widehat{D}\times I} Z_F = \varinjlim_{\alpha\in\mathbb{N}[I]}\ Z^\kappa_{\alpha\mathsf{D}},$$

given by the union over all colored divisors $F \in \mathcal{H}_{C\times I}$ that (once we forget the colors) lie in the formal neighborhood $\widehat{D}$ of the divisor D. Space $Z^\kappa_{\mathcal{H}_C}$ carries the line bundle $\mathcal{O}_S(1)$ that on pieces $Z^\kappa_F \subseteq S^\kappa_{\mathsf{D}}$ equals $\mathcal{O}_{Z^\kappa}(1)|_{Z^\kappa_F}$ (by part (b) of the same Lemma).

2.4. Loop Grassmannians $\mathcal{G}^\kappa_{\mathcal{H}_C}$ of matrices κ

As in passing from Z^κ to S^κ by the defect action (in 2.3.2), one glues $\mathcal{G}^\kappa_a$ from Z^κ_a (in 2.4.5), using the "shift" action on S^κ_a of the monoid $\mathbb{N}[I] = \pi_0(\mathcal{H}_C)$.

However, such shift action and formulation of $\mathcal{G}^\kappa$ are only valid in the presence of a local coordinate z. A global formulation uses the shift action of a submonoid $\mathcal{L}^\kappa_{\mathcal{H}_C}(T)^\dagger$ of the central extension $\mathcal{L}^\kappa_{\mathcal{H}_C}(T)$ of the loop group $\mathcal{L}_{\mathcal{H}_C}(T)$ over $\mathcal{H}_C$ (the notation is in 2.4.1–2.4.3). Again, one has $\pi_0(\mathcal{L}^\kappa_{\mathcal{H}_C}(T)^\dagger) = \mathbb{N}[I]$.

While the *defect* action on Z^κ becomes trivial already on the level of the semiinfinite space S^κ (see 2.3.2), on the loop Grassmannian $\mathcal{G}^\kappa$ the *shift* action will shift the semiinfinite orbits by z^λ for $\lambda \in X_*(T)$ (see 2.4.5).

2.4.1. Formal discs.

Here we introduce notation for loop groups defined over the Hilbert scheme $\mathcal{H}_C$. For a subscheme $Y \subseteq X$, we denote by $\widehat{Y}$ the formal neighborhood of Y in X and by $\widetilde{Y}$ the punctured formal neighborhood $\widehat{Y} - Y$.

In particular, for a finite subscheme D of a curve C, we have the "formal disc" $\widehat{D} \subseteq C$ and the "formal punctured disc" $\widetilde{D} \subseteq C$. These give for affine schemes X, the mapping spaces $\mathcal{L}_D X = \mathrm{Map}(\widetilde{D}, X)$ and $\mathcal{L}^+_D X = \mathrm{Map}(\widehat{D}, X)$ of "loops" and "discs" in X. Each of these spaces is defined as a functor from $\Bbbk$-algebras A to sets. For instance, when D is a point $a \in C$ with a local parameter z, then $(\mathcal{L}^+_a X)(A) = X\big(A[[z]]\big) \subseteq (\mathcal{L}_a X)(A) = X\big(A((z))\big)$.

An example is given by the lifts of functions on $\widetilde{D}\subseteq\widehat{D}$ to indschemes $\mathcal{K}_{\mathcal{H}_C}\overset{\text{def}}{=}\mathcal{L}_{\mathcal{H}_C}\mathbb{A}^1\supseteq\mathcal{O}_{\mathcal{H}_C}\overset{\text{def}}{=}\mathcal{L}^+_{\mathcal{H}_C}\mathbb{A}^1$ over $\mathcal{H}_C$ with fibers $\mathcal{K}_D=\mathcal{L}_D\mathbb{A}^1$ and $\mathcal{O}_D=\mathcal{L}^+_D\mathbb{A}^1$. Actually, both $\mathcal{K}_{\mathcal{H}_C}\supseteq\mathcal{O}_{\mathcal{H}_C}$ have structures of *Tate vector bundles* over $\mathcal{H}_C$.[11]

2.4.2. Central extensions of loop groups over the Hilbert scheme of a curve.

We will use 2.4.1 for the curve $C\times I$. The Tate vector bundle structure on $\mathcal{K}_{\mathcal{H}_{C\times I}}$ produces for the group indscheme bundle $\mathrm{GL}(\mathcal{K}_{\mathcal{H}_{C\times I}})$ of $\Bbbk$-linear operators on $\mathcal{K}_{\mathcal{H}_{C\times I}}$, the *Tate central extension* $GL^\flat(\mathcal{K}_{\mathcal{H}_{C\times I}})$ by $\mathbb{G}_m$. These are given by the determinant line bundle on the space of lattices in the Tate vector bundle $\mathcal{K}_{\mathcal{H}_{C\times I}}$ over $\mathcal{H}_{C\times I}$ (see [BBE]). This central extension restricts to Tate's extension $0\to\mathbb{G}_m\to\mathcal{L}^\flat_{\mathcal{H}_{C\times I}}\mathbb{G}_m\to\mathcal{L}_{\mathcal{H}_{C\times I}}\mathbb{G}_m\to 0$ of the subgroup $\mathcal{L}_{\mathcal{H}_{C\times I}}\mathbb{G}_m$ of $\mathrm{GL}(\mathcal{K}_{\mathcal{H}_{C\times I}})$.

In [BBE], one considers the case $I=\mathrm{pt}$ and $D=a$, a point of $C\subseteq\mathcal{H}_C$, i.e., the extension $0\to\mathbb{G}_m\to\mathcal{L}^\flat_a\mathbb{G}_m\to\mathcal{L}_a\mathbb{G}_m\to 0$, and proves that the corresponding commutator map $\mathcal{L}_a\mathbb{G}_m\times\mathcal{L}_a\mathbb{G}_m\to\mathbb{G}_m$ is Contou-Carrère's tame symbol [CC].

The projection $C\times I\to C$ comes with a map $\Pi:\mathcal{H}_C\to\mathcal{H}_{C\times I},\ D\mapsto D\times I$. The Π-pullback $\Pi^*\mathcal{L}_{\mathcal{H}_{C\times I}}(\mathbb{G}_m)$ is the analogous loop group indscheme $\mathcal{L}_{\mathcal{H}_C}T$ over $\mathcal{H}_C$. In [BD, 3.10.13], the Tate central extension $\mathcal{L}^\flat_{\mathcal{H}_{C\times I}}\mathbb{G}_m$ was combined with a symmetric bilinear form κ over $\mathbb{Z}[I]$ to give a central $\mathbb{G}_m$-extension $\mathcal{L}^\kappa_{\mathcal{H}_C}T$ of $\mathcal{L}_{\mathcal{H}_C}T$ (unique up to an isomorphism), with fibers $\mathcal{L}^\kappa_{\mathsf{D}}T$ at $\mathsf{D}\in\mathcal{H}_C$.[12] In particular, one gets the commutator map $\mathcal{L}_{\mathcal{H}_C}T\times_{\mathcal{H}_C}\mathcal{L}_{\mathcal{H}_C}T\to\mathbb{G}_m$.

2.4.3. Positive submonoids of loop groups.

The loop Grassmannian of a group G at a point $a\in C$ is defined as

$$\mathcal{G}_a(G)\overset{\text{def}}{=}\mathrm{Bun}^c_G(\widehat{a}),$$

i.e., as the moduli of G-torsors over $\widehat{a}$ trivialized at its infinity $\widetilde{a}\overset{\text{def}}{=}\widehat{a}-a$. (A local coordinate z at a identifies this with $G((z))/G[[z]]$.) The Abel–Jacobi map $AJ_{\widehat{a}}:\mathcal{H}_{\widehat{a}}\to\mathcal{G}_a(\mathbb{G}_m)$ embeds the Hilbert scheme monoid of finite subschemes of the formal neighborhood $\widehat{a}$ into the loop Grassmannian at a (see 1.1.3). The restriction of an object X over $\mathcal{G}_a(\mathbb{G}_m)$ to $\mathcal{G}_a(\mathbb{G}_m)^\dagger\overset{\text{def}}{=}\mathcal{H}_C$ will be called the "positive part" $X^\dagger$ of X.

We will extend this formally from Grassmannians $\mathcal{G}_a(\mathbb{G}_m)=\mathcal{L}_a(\mathbb{G}_m)/\mathcal{L}^+_a(\mathbb{G}_m)$ to their Beilinson–Drinfeld deformations $\mathcal{G}_{\mathcal{H}_C}(\mathbb{G}_m)\overset{\text{def}}{=}\mathcal{L}_{\mathcal{H}_C}(\mathbb{G}_m)/\mathcal{L}^+_{\mathcal{H}_C}(\mathbb{G}_m)$ over $\mathcal{H}_C$.

First, we modify the tautological subbundle $\mathcal{T}$ of $\mathcal{H}_{C\times I}\times(C\times I)$ to the pullback $\boldsymbol{\mathcal{T}}$ of $\mathcal{T}$ by the map $\mathcal{H}_C\hookrightarrow\mathcal{H}_{C\times I}$, $D\mapsto D\times I$. So, for $D\in\mathcal{H}_C$ we have $(\boldsymbol{\mathcal{T}}/\mathcal{H}_C)_D=\mathcal{T}_{D\times I}=D\times I$. Now let $\widehat{\boldsymbol{\mathcal{T}}}$ (a shorthand for $\widehat{\boldsymbol{\mathcal{T}}/\mathcal{H}_C}$) be the relative formal neighborhood of the $\mathcal{H}_C$-subspace $\boldsymbol{\mathcal{T}}/\mathcal{H}_C$ of $\mathcal{H}_C\times(C\times I)$, i.e., $(\widehat{\boldsymbol{\mathcal{T}}})_D=\widehat{\boldsymbol{\mathcal{T}}_D}=\widehat{D}\times I$.

11 "Tate modules" and more generally "Tate vector bundles" are algebro-geometric versions of the notion of Tate vector spaces, see [Dr] or [BBE, 2.11].

12 Formally, this argument in [BD] was written at a point $a\in C$ rather than over the Hilbert scheme $\mathcal{H}_C$ but the argument is the same.

Again, we define the *positive part* $\mathcal{G}_{\mathcal{H}_C}(T)^\dagger \subseteq \mathcal{G}_{\mathcal{H}_C}(T)$ of the Beilinson–Drinfeld loop Grassmannian of a torus, as the relative ("vertical") Hilbert scheme of points $\mathcal{H}(\widehat{\mathcal{T}}/\mathcal{H}_C) = \mathrm{Gr}(\widehat{\mathcal{T}}/\mathcal{H}_C)$. The fiber at $D \in \mathcal{H}_C$ is $\mathrm{Gr}(\widehat{\mathcal{T}}/\mathcal{H}_C)_D = \mathcal{H}_{\widehat{D}\times I}$, hence the space is $\{(D,F) \in \mathcal{H}_C \times \mathcal{H}_{C\times I};\ F \subseteq \widehat{D}\times I\}$, a nonsymmetric colored Hilbert scheme version of $\widehat{\Delta}_C$.

It is embedded by the relative Abel–Jacobi map $AJ_{\mathcal{H}_C}\colon \mathcal{H}(\widehat{\mathcal{T}}/\mathcal{H}_C) \hookrightarrow \mathcal{G}_{\mathcal{H}_C}(T)$ with the usual formula $AJ_{\mathcal{H}_C}(D,F) \overset{\text{def}}{=} (D, \mathcal{O}_{\widehat{D}}(-D), 1)$ (see 1.1.3).

Now we can restrict $\mathcal{L}^\kappa_{\mathcal{H}_C}(T) \twoheadrightarrow \mathcal{L}_{\mathcal{H}_C}(T) \twoheadrightarrow \mathcal{G}_{\mathcal{H}_C}(T)$ to monoids $\mathcal{L}^\kappa_{\mathcal{H}_C}(T)^\dagger \twoheadrightarrow \mathcal{L}_{\mathcal{H}_C}(T)^\dagger \twoheadrightarrow \mathcal{G}_{\mathcal{H}_C}(T)^\dagger$ by fibered products such as

$$\mathcal{L}^\kappa_{\mathcal{H}_C}(T)^\dagger \overset{\text{def}}{=} \mathcal{L}^\kappa_{\mathcal{H}_C}(T) \times_{\mathcal{G}_{\mathcal{H}_C}(T)} \mathcal{G}_{\mathcal{H}_C}(T)^\dagger.$$

By the argument in [BBE], the $\mathbb{G}_m$-extension $\mathcal{L}^\kappa_{\mathcal{H}_C}(T)$ splits canonically over the disc subgroup $\mathcal{L}^+_{\mathcal{H}_C}(T) \subseteq \mathcal{L}_{\mathcal{H}_C}(T)$. Then the quotient $\mathcal{L}^\kappa_{\mathcal{H}_C}(T)/\mathcal{L}^+_{\mathcal{H}_C}(T)$ is an $\mathcal{L}^\kappa_{\mathcal{H}_C}(T)$-equivariant $\mathbb{G}_m$-torsor over the loop Grassmannian $\mathcal{G}_{\mathcal{H}_C}(T) = \mathcal{L}_{\mathcal{H}_C}(T)/\mathcal{L}^+_{\mathcal{H}_C}(T)$. The corresponding line bundle $\underline{\mathsf{L}}^\kappa$ over $\mathcal{G}_{\mathcal{H}_C}(T)$ is $\mathcal{L}^\kappa_{\mathcal{H}_C}(T)$-equivariant.

On the subspace $\mathcal{G}_{\mathcal{H}_C}(T)^\dagger$, we have another line bundle, the pullback $\mathbf{pr}_2^*\mathsf{L}^\kappa$ for the natural map

$$\mathcal{G}_{\mathcal{H}_C}(T)^\dagger = \mathrm{Gr}(\widehat{\mathcal{T}}/\mathcal{H}_C) \xrightarrow{\mathbf{pr}_2} \mathcal{H}_{C\times I}.$$

Lemma. *The line bundle $\mathbf{pr}_2^*\mathsf{L}^\kappa$ over $\mathcal{G}_{\mathcal{H}_C}(T)^\dagger$ naturally extends to the $\mathcal{L}^\kappa_{\mathcal{H}_C}(T)$-equivariant line bundle $\underline{\mathsf{L}}^\kappa$ over $\mathcal{G}_{\mathcal{H}_C}(T)$.*

Proof. The apparent difference between $\underline{\mathsf{L}}^\kappa$ and $\mathbf{pr}_2^*\mathsf{L}^\kappa$ is that the first has a factorization property along the base $\mathcal{H}_C$ of $\mathcal{G}_{\mathcal{H}_C}(T)$ and the second has a locality structure on the fibers $\mathcal{G}^\kappa_D(T)^\dagger$ (for $D \in \mathcal{H}_C$) of $\mathcal{G}^\kappa_{\mathcal{H}_C}(T)^\dagger \to \mathcal{H}_C$.

In [BD, Prop. 3.10.7], one finds an equivalence of categories of factorization line bundles λ on $\mathcal{G}_{\mathcal{H}_C}(T)$ (which is there called "$X_*(T)$-divisors" and denoted $\mathrm{Div}\big(X, X_*(T)\big)$) and the "*theta data*" $(\kappa(\lambda), \lambda', c)$ where $\kappa(\lambda)$ is a symmetric bilinear form on $\mathbb{Z}[I]$ and (λ', c) amounts to a line bundle μ on $C\times I$. Here, $\kappa(\lambda)_{ij}$ is the order of vanishing of the factorization isomorphism for λ along the diagonal in C^2 and μ is the restriction of λ to $C \subseteq \mathcal{H}_C$. In our case, $\lambda = \underline{\mathsf{L}}^\kappa$, the restriction μ is the trivial line bundle on $C\times I$, and $\kappa(\lambda)$ is κ.

In [MYZ, Rem. 2.1.3], it was noticed that when one restricts such λ to $\mathcal{H}_{\widehat{a}\times I} = \mathcal{G}_a(T)^\dagger$ that lies in the fiber $\mathcal{G}_a(T)$ at a point $a \in C$, one gets our local line bundle $\mathsf{L}^{\kappa(\lambda)}$ on $\mathcal{H}_{\widehat{a}\times I}$ (from 2.1.1).[13] This proves that $\underline{\mathsf{L}}^\kappa|_{\mathcal{G}_{\mathcal{H}_C}(T)^\dagger}$ is $\mathbf{pr}_2^*\mathsf{L}^\kappa$. $\square$

2.4.4. The shift action on $S^\kappa_{\mathcal{H}_C}$.

By Lemma in 2.4.3, the action of the extended loop group $\mathcal{L}^\kappa_{\mathcal{H}_C}(T)$ on the line bundle $\underline{\mathsf{L}}^\kappa$ over $\mathcal{G}_{\mathcal{H}_C}(T)$ restricts to an action of the submonoid $\mathcal{L}_{\mathcal{H}_C}(T)^\dagger$ on the line bundle $\mathbf{pr}_2^*\mathsf{L}^\kappa$ over $\mathcal{G}_{\mathcal{H}_C}(T)^\dagger = \mathrm{Gr}(\widehat{\mathcal{T}}/\mathcal{H}_C)$.

[13] The reason in [MYZ] is that the "horizontal" version of the Abel–Jacobi map that is used to measure $\kappa(\lambda)$ differs from the "vertical" Abel–Jacobi map that measures the corresponding matrix for L^κ only in a nonessential way — both are pullbacks of an Abel–Jacobi map into the *rational loop Grassmannian* $\mathcal{G}_{\mathrm{rat}}(T)$.

Proposition.

(a) *Space* $\mathcal{G}_{\mathcal{H}_C}(T)^\dagger = \mathrm{Gr}(\widehat{\boldsymbol{\mathcal{T}}}/\mathcal{H}_C)$ *has a Kodaira embedding into* $S^\kappa_{\mathcal{H}_C}$ *by the line bundle* $\mathbf{pr}_2^*\mathsf{L}^\kappa$.

(b) *The action of the monoid* $\mathcal{L}_{\mathcal{H}_C}(T)^\dagger$ *on* $\mathcal{G}_{\mathcal{H}_C}(T)^\dagger$ *extends canonically to the space* $S^\kappa_{\mathcal{H}_C}$. *We will call this the* shift *action of* $\mathcal{L}_{\mathcal{H}_C}(T)^\dagger$.

(c) *This lifts canonically to an action of its central extension* $\mathcal{L}^\kappa_{\mathcal{H}_C}(T)^\dagger$ *on the line bundle* $\mathcal{O}_{S^\kappa_{\mathcal{H}_C}}(1)$.

Proof. (a) At a fixed $D \in \mathcal{H}_C$ this embedding $\mathrm{Gr}(\widehat{D}\times I) \hookrightarrow S^\kappa_D = \varinjlim_\alpha Z^\kappa_{\alpha D\times I}$ is the union of the Kodaira embeddings from the definition of zastava $\mathrm{Gr}(F) \hookrightarrow Z^\kappa_F$ (over $F \in \mathcal{H}_{\widehat{D}\times I}$) that uses the line bundle L^κ restricted to $\mathrm{Gr}(F) \subseteq \mathcal{H}_{C\times I}$.

(b) Since $\mathcal{L}^\kappa_{\mathcal{H}_C}(T)^\dagger$ acts on the line bundle $\mathbf{pr}^*\mathsf{L}^\kappa$ over $\mathcal{G}_{\mathcal{H}_C}(T)^\dagger$, it also acts on the projective space $\mathbb{P}[(\mathbf{pr}_2^*\mathsf{L}^\kappa)_{\mathcal{G}_{\mathcal{H}_C}(T)^\dagger/\mathcal{H}_C}{}^\vee]$ over $\mathcal{H}_C$ and on its line bundle $\mathcal{O}(1)$. We will see that it preserves $S^\kappa_{\mathcal{H}_C}$ and therefore also the restriction of the line bundle line $\mathcal{O}(1)$ to $S^\kappa_{\mathcal{H}_C}$.

First, the action of $\mathcal{L}_{\mathcal{H}_C}(T)^\dagger$ on the base $\mathcal{G}_{\mathcal{H}_C}(T)^\dagger$ factors through the quotient $\mathcal{L}_{\mathcal{H}_C}(T)^\dagger \twoheadrightarrow \mathcal{G}_{\mathcal{H}_C}(T)^\dagger$. If $f \in \mathcal{L}_D(T)^\dagger$ has image $E \in \mathcal{G}_D(T)^\dagger = \mathcal{H}_{\widehat{D}\times I}$, then f acts on $X \in \mathcal{H}_{\widehat{D}\times I}$ by addition $X \mapsto X+E$.

This gives for $F \in \mathcal{H}_{\widehat{D}\times I}$ an isomorphisms of $\mathrm{Gr}(F)$ with $\mathrm{Gr}(E+F/E)$, the moduli of all $A \in \mathcal{H}_{\widehat{D}\times I}$ such that $E \subseteq A \subseteq E+F$. Now, any lift $\phi \in \mathcal{L}^\kappa_D(T)$ of f gives an isomorphism of line bundles $\mathsf{L}^\kappa|_{\mathrm{Gr}(F)}$ and $\mathsf{L}^\kappa|_{\mathrm{Gr}(F,E)}$ (above the isomorphism of spaces given by E). The corresponding isomorphism of projective spaces of global sections is independent of the choice of ϕ, so we get

$$\mathbb{P}[\Gamma(\mathrm{Gr}(F),\mathsf{L}^\kappa)^\vee] \underset{\cong}{\xrightarrow{\phi}} \mathbb{P}[\Gamma(\mathrm{Gr}(F+E,E),\mathsf{L}^\kappa)^\vee] \subseteq \mathbb{P}[\Gamma(\mathrm{Gr}(F+E),\mathsf{L}^\kappa)^\vee].$$

This restricts to an inclusion $\iota_f; Z^\kappa_F \hookrightarrow Z^\kappa_{F+E}$ (as in the construction of the defect action).

Passing to the inductive limit over all $F \in \mathcal{H}_{\widehat{D}\times I}$, this yields $\iota_f; S^\kappa_D \hookrightarrow S^\kappa_D$.

Claim (c) is essentially the action of the lift ϕ in the argument of (b). $\square$

Remark. The shift action is "vertical" in the sense that it preserves fibers of $S^\kappa_{\mathcal{H}_C} \to \mathcal{H}_C$ and the defect action is "horizontal", i.e., nontrivial on the base $\mathcal{H}_{C\times I}$.

2.4.5. Loop Grassmannian $\mathcal{G}^\kappa_{\mathcal{H}_C}$ *of matrices* κ.

The space $\mathcal{G}^\kappa_{\mathcal{H}_C}$ lies over $\mathcal{H}_C$ and carries a line bundle $\mathcal{O}_{\mathcal{G}^\kappa}(1)$. These are defined as an associated bundle for the shift action of $\mathcal{L}_{\mathcal{H}_C}(T)^\dagger$ from Proposition in 2.4.4, which amounts to a certain inductive system (which is just a union):[14]

[14] For a submonoid M of a group A we have the action category A_M with objects $\mathrm{Ob}(A_M) = A$ and there is a unique morphism $a \xrightarrow{m} a+m$ for $a \in A,\ m \in M$. A_M is directed iff $M - M = A$ (for any $a,b \in A$, we need $m,n \in M$ with $a+m = b+n$, i.e., $b-a = m = n \in M-M$).

Any M-space X defines a functor $\widetilde{X}$ on A_M so that for $a \in A$, we have $\widetilde{X}_a \overset{\mathrm{def}}{=} X$, and for $m \in M$, the morphism $\widetilde{X}_a \xrightarrow{m} \widetilde{X}_{a+m}$ is given by $X \xrightarrow{m} X$. Then by the "associated bundle" $A\times_M X$ we mean $\varinjlim_{A_M} \widetilde{X}$. For instance, $A\times_M M = A$. If for each $m \in M$ the action $m : X \to X$ is a closed embedding, then $A\times_M X$ is a union of copies of X.

$$\big(\mathcal{G}^\kappa_{\mathcal{H}_C},\mathcal{O}_{\mathcal{G}^\kappa}(1)\big)\overset{\text{def}}{=}\mathcal{L}^\kappa_{\mathcal{H}_C}(T)\times_{\mathcal{L}^\kappa_{\mathcal{H}_C}(T)^\dagger}\ \big(S^\kappa,\mathcal{O}_{S^\kappa}(1)\big)=\varinjlim_{\mathcal{L}^\kappa_{\mathcal{H}_C}}\big(S^\kappa_{\mathcal{H}_C},\mathcal{O}_{S^\kappa}(1)\big).$$

Remarks.

(0) The definition of the space itself does not involve the central extension: $\mathcal{G}^\kappa \overset{\text{def}}{=} \mathcal{L}_{\mathcal{H}_C}(T)\times_{\mathcal{L}_{\mathcal{H}_C}(T)^\dagger}S^\kappa$.

(1) Once one chooses a local coordinate z at a point a, then the restriction of the shift action to the fiber $S_{\kappa,a}$ can be viewed as an action of $\mathbb{N}[I]$, where $\alpha\in\mathbb{N}[I]$ acts by $z^\alpha\in\mathcal{L}_a(T)$, which is the composition $\widetilde{a}\xrightarrow{z}\mathbb{G}_m\xrightarrow{\lambda}T$. So, in a presence of a coordinate, the definition can be stated as $\mathcal{G}^\kappa=\ \mathbb{Z}[I]\times_{\mathbb{N}[I]}S^\kappa$. This was the point of view in [MYZ].

(2) S^κ and $\mathcal{G}^\kappa$ are just repackaged versions of the zastava space Z^κ. Say, $\mathcal{G}^\kappa$ is covered by copies of S^κ, which is also a union of copies of Z_D's for $D\in\mathcal{H}_{C\times I}$. (Actually, this procedure can be reversed.)

(3) For any lattice L that contains $\mathbb{Z}[I]$, we can define the corresponding *loop Grassmannian* as $\mathcal{G}(L,\kappa)\overset{\text{def}}{=}\mathcal{L}_{\mathcal{H}_C}(T_L)\times_{\mathcal{L}_{\mathcal{H}_C}(T)}\mathcal{G}^\kappa$ for the torus $T_L=L\otimes_{\mathbb{Z}[I]}T$.

Lemma.

(a) *We have embeddings* (i) $\subseteq$ (ii) $\subseteq$ (iii) *of K-triples* (*see* 2.2.4) *for*
  (i) $\big(\mathrm{Gr}(\mathcal{T}/\mathcal{H}_C),\mathrm{pr}_2^*\mathsf{L}^\kappa,Z^\kappa_{\mathcal{H}_C}\big)$*;*
  (ii) $\big(\mathrm{Gr}(\widehat{\boldsymbol{\mathcal{T}}}/\mathcal{H}_C),\mathbf{pr}_2^*\mathsf{L}^\kappa,S^\kappa_{\mathcal{H}_C}\big)$*;*
  (iii) $\big(\mathcal{G}_{\mathcal{H}_C}(T),\underline{\mathsf{L}}^\kappa,\mathcal{G}^\kappa_{\mathcal{H}_C}\big)$.[15]

(b) $S^\kappa,\mathcal{G}^\kappa$ *descend to the de Rham space* $\mathcal{H}_{C_{\mathrm{dR}}}$ *of* $\mathcal{H}_C$. (*Similarly, one can define the Ran space versions* $S^\kappa_{\mathrm{Ran}},\mathcal{G}^\kappa_{\mathrm{Ran}}$ *over the Ran space* Ran_C *of the curve.*)

(c) *The compatible locality and growth structures on zastava* $\big(Z^\kappa,\mathcal{O}_{Z^\kappa}(1)\big)$ *induce the same on* $\big(S^\kappa,\mathcal{O}_{S^\kappa}(1)\big)$ *and then also on* $\big(\mathcal{G}^\kappa,\mathcal{O}_{Z^\kappa}(1)\big)$.

Also, these locality structures on $S^\kappa_{\mathcal{H}_C},\mathcal{G}^\kappa_{\mathcal{H}_C}$ *become the* factorization structures *from* [BD] *on the corresponding Ran space versions* $S^\kappa_{\mathrm{Ran}},\mathcal{G}^\kappa_{\mathrm{Ran}}$.

Proof. (a) The K-triple (i) is established in the lemma 2.2.4, its Kodaira embedding is given on fibers by $\mathrm{Gr}(D)\hookrightarrow Z^\kappa_D$.

The remaining claims follow because one defines triples (ii) form (i), and (iii) from(ii), by inductive limits (which are just unions) by definitions of S^κ in 2.3.2, and $\mathcal{G}^\kappa$ in 2.4.5.

---

[15] We picture these embeddings starting with the vertical inclusions of spaces $A\subseteq B$; then we list the compatible line bundles on these spaces:

$$\begin{array}{ccccc} \mathrm{Gr}(\mathcal{T}/\mathcal{H}_C) & \xrightarrow{\subseteq} & \mathrm{Gr}(\widehat{\boldsymbol{\mathcal{T}}}/\mathcal{H}_C) & \xrightarrow{\subseteq} & \mathcal{G}_{\mathcal{H}_C}(\mathbb{G}_m)\\ \hookrightarrow\downarrow & & \hookrightarrow\downarrow & & \hookrightarrow\downarrow\\ Z^\kappa & \xrightarrow{\subseteq} & S^\kappa_{\mathcal{H}_C} & \xrightarrow{\subseteq} & \mathcal{G}^\kappa_{\mathcal{H}_C}(\mathbb{G}_m)\end{array}\quad\text{and}\quad\begin{array}{ccc} \mathrm{pr}_2^*\mathsf{L}^\kappa & \mathbf{pr}_2^*\mathsf{L}^\kappa & \underline{\mathsf{L}}^\kappa\\ & & \\ \mathcal{O}_{Z^\kappa}(1) & \mathcal{O}_{S^\kappa}(1) & \mathcal{O}_{Z^\kappa}(1)\end{array}.$$

(b) For S^κ, this is by definition since $S^\kappa_D = \bigcup_{\alpha\in\mathbb{N}[I]} Z_{\alpha D}$ (see 2.3.2) depends only on the formal neighborhood of D. Then the same holds for $\mathcal{G}^\kappa$.

(c) For spaces S^κ, this is evident: $S_{D'\sqcup D''} = \varinjlim Z_{\alpha(D'\sqcup D'')} = \varinjlim Z_{\alpha D'}\times Z_{\alpha D''} = S_{D'}\times S_{D''}$. Also, if $D\subseteq E$, then $S^\kappa_D = \varinjlim Z_{\alpha D} \subseteq \varinjlim Z_{\alpha E} = S_E$.

For $\mathcal{G}^\kappa_D = \mathcal{L}_D(T)\times_{\mathcal{L}^+_D(T)} S^\kappa_D$, all three factors are manifestly local in $D = D'\sqcup D''$. The growth structure under $D\subseteq E$ is known for S^κ, we will check that the first functor $\mathcal{L}(T)$ is functorial in $D\subseteq E$.

Notice that each factor in $\mathcal{G}^\kappa_D$ is invariant under passing from D to a multiple nD. For $D\subseteq E$, intersection $E\cap\widehat{D}$ lies in some nD, therefore, $\widehat{E}-E\subseteq \widehat{D}-\widehat{nD} = \widehat{D}-D$ giving an injective restriction map $\mathcal{L}_D T\to\mathcal{L}_E T$. This implies the same for the positive part $\mathcal{L}_D(T)^\dagger$, and the resulting map $\mathcal{G}^\kappa_D\to\mathcal{G}^\kappa_E$ is injective (because $\mathcal{L}_D(T)\bigcap_{\mathcal{L}_E(T)}\mathcal{L}_E(T)^\dagger = \mathcal{L}_D(T)^\dagger$).

The above proof of claims for inductive limits $S^\kappa, \mathcal{G}^\kappa$ also carries over to the line bundles $\mathcal{O}_{S^\kappa}(1), \mathcal{O}_{\mathcal{G}^\kappa}(1)$. □

2.5. Spaces $\mathcal{G}^\kappa$ generalize loop Grassmannians of reductive groups

Theorem ([MYZ]). *Let I be the set of simple roots of a semisimple group G (simply connected and simply laced) and let κ be the Cartan matrix $\kappa_{ij} = \langle\alpha_i, \check{\alpha}_j\rangle$. Then the loop Grassmannian $\mathcal{G}^\kappa_{\mathcal{H}_C}$ is just the Beilinson–Drinfeld version $\mathcal{G}_{\mathcal{H}_C}(G)\to\mathcal{H}_C$ of the loop Grassmannian $\mathcal{G}(G) = G((z))/G[[z]]$ of G. (The same reconstruction holds for zastavas and semi-infinite spaces.)*

Proof. The proof in [MYZ] only covers the version at a point $a\in C$: $\mathcal{G}^\kappa_a\cong\mathcal{G}_a(G)$. However, with the above definitions it works as well for the Beilinson–Drinfeld version $\mathcal{G}^\kappa_{\mathcal{H}_C}$. □

Remarks.

(0) The paper [MYZ] with Yaping Yang and Gufang Zhao also contains a quantum version and an arithmetic version of constructions $Z^\kappa, S^\kappa, \mathcal{G}^\kappa$. It also explains the relation to representations of quivers and provides a conjectural extension of this theorem to all reductive groups.

(1) This reconstruction motivates a search for "*Noncommutative Poincaré duality in Zariski topology*" that should extend the multiplicative Poincaré duality conjectured in 1.2.2. First, recall that the usual loop Grassmannians $\mathcal{G}_a(G)$ of a group G is the compactly supported cohomology $\mathrm{Bun}^c_G(\widehat{a})$ of a formal disc $\widehat{a}$ with coefficients in G (see the first paragraph of 2.4.3). On the other hand, one can view the above construction of $\mathcal{G}^\kappa_a$ as a kind of a "noncommutative" homology of $\widehat{a}$ with coefficients in the interaction κ, because we reconstruct $\mathcal{G}(G)$ from pieces (finite subschemes) of the disc $\widehat{a}$. The Poincaré duality claim is the coincidence of the two constructions.

3. Lattice vertex algebras and Grassmannians $\mathcal{G}^\kappa$

Consider the lattice vertex algebra $\boldsymbol{L}^\kappa$ that corresponds to the lattice defined by a symmetric integral matrix κ. In Theorem in Subsection 3.2 we localize $\boldsymbol{L}^\kappa$ over the corresponding space $\mathcal{G}^\kappa$ from 2.4.5. This result is a byproduct of the "collision" reconstruction of loop Grassmannians of reductive groups in Theorem in

Subsection 2.5 above, which itself was motivated by developments in the geometric Langlands program.

3.1. Factorization algebras

We will be interested in two geometric incarnations of the notion of vertex algebras. On any smooth curve C, Beilinson and Drinfeld have introduced equivalent notions of a *chiral algebra* A and a *factorization algebra* A^ℓ. The equivalence is by tensoring with a canonical bundle: $A = A^\ell \otimes \omega_C$, this exchanges the natural structures of a left D-module over the curve C on A^ℓ and a right D-module structure on A. The chiral algebras are of "Lie nature" and more directly related to representations of affine Lie algebras. The factorization algebras are of "commutative nature".

Taking the global sections then provides an equivalence of chiral algebras on $\mathbb{A}^1$ equivariant under translations and vertex algebras. For versions of this equivalence see [BD, 0.15].

3.1.1. Factorization algebras [BD].

By a factorization algebra on a smooth curve C we will mean a quasicoherent sheaf B over $\mathcal{H}_C = \bigsqcup_{\mathbb{N}} S^n C$ together with the following two compatible structures:

- the *locality* (=factorization) structure c, which is a natural system of isomorphisms $c_{D',D''} : B_{D'} \otimes B_{D''} \xrightarrow{\cong} B_{D' \sqcup D''}$ that involve fibers of B at disjoint divisors $D', D'' \in \mathcal{H}_C$ (see Subsection 0.4),
- the *growth* structure γ, which is a natural system of maps $\gamma_{D',D''} : B_{D'} \to B_{D''}$ of fibers associated to inclusions $D' \subseteq D''$ in $\mathcal{H}_C$

such that[16]

(1) c is commutative and associative, while γ is functorial in the inclusion poset $\mathcal{H}_C$,
(2) restriction of B to the point $\mathcal{H}_C^0$ is $\Bbbk$,
(3) B has no nonzero local sections supported at the discriminant divisor in $\mathcal{H}_C$.

Remarks.

(a) In [BD], factorization algebras are given several equivalent formulations using slightly different settings. The standard notion uses the Ran space of the curve C (see [BD, 3.4.4]), we will use the one in terms of effective Cartesian divisors on C (see [BD, 3.4.6]). There, B is described as a functor $B_{Z,D}$ on Z-families D of effective Cartier divisors in C. This agrees with our definition since moduli of such divisors are represented by the Hilbert scheme $\mathcal{H}_C$.

(b) Condition (3) ensures that the restriction functor from factorization algebras to $\mathcal{O}_X$-modules is faithful.

---

[16] *Added in proof.* Actually, the definition of factorization algebras in 3.1.1 and of factorization monoids in 3.1.2 also requires a further condition that the growth structure γ stabilizes under multiples, i.e., $\gamma_{D,2D}$ is an isomorphism. We will call the above weaker notions "locality algebras" and "locality monoids".

As a consequence, in the main Theorem in Subsection 3.2 below, the space Z^κ (and its subspace) is only a locality monoid and provides a locality algebra, while the claims for $Z^\kappa, \mathcal{G}^\kappa$ (and their subspaces) are correct as stated.

3.1.2. Factorization monoids [BD].

A *factorization* (or *chiral*) monoid over a curve C is an indscheme $\mathcal{G}$ over $\mathcal{H}_C$ together with the following two compatible structures:

- the local space structure c (from 2.1.2), i.e., a natural system of isomorphisms $c_{D',D''}:\ \mathcal{G}_{D'}\times\mathcal{G}_{D''} \xrightarrow{\cong} \mathcal{G}_{D'\sqcup D''}$,
- the *growth* structure γ which is a natural[17] system of closed embeddings $\gamma_{D',D''}: \mathcal{G}_{D'} \to \mathcal{G}_{D''}$ of fibers associated to inclusions $D' \subseteq D''$ in $\mathcal{H}_C$.

One also requires that $\mathcal{G}_\varnothing$ is a point and that the restriction to the regular part $\mathcal{G}|_{\mathcal{H}_C^{\mathrm{r}}}$ is dense.

A $\mathbb{G}_m$-*extension* of a factorization monoid $\mathcal{G}$ (see [BD]) is a local line bundle over the local space $\mathcal{G}$ which is compatible with the growth structure.

3.1.3. Factorization algebras from $\mathbb{G}_m$-extensions of factorization monoids.

Proposition. *Let λ be a $\mathbb{G}_m$-extension of a factorization monoid $\pi: \mathcal{G} \to \mathcal{H}_C$. Suppose that $\mathcal{G}$ has a directed filtration by subschemes $\mathcal{G}_i$ projective over $\mathcal{H}_C$ such that each $(\mathcal{G}_i \to \mathcal{H}_C)_*\lambda$ is finite and flat over $\mathcal{H}_C$. Then*

$$A^\ell(\mathcal{G},\lambda) \overset{\mathrm{def}}{=} [\pi_*(\lambda)]^\vee \overset{\mathrm{def}}{=} \mathcal{H}om[\pi_*(\lambda), \mathcal{O}_{\mathcal{H}_C}]$$

has a canonical structure of a factorization algebra.

Proof. Since $(\mathcal{G}_i \to \mathcal{H}_C)_*\lambda$ is finite and flat over a smooth affine scheme $\mathcal{H}_C$ (over a ring $\Bbbk$), it is locally free. Since $\mathcal{G} \xrightarrow{\pi} \mathcal{H}_C$ is an ind-projective scheme, $\pi_*\lambda$ is a pro-coherent sheaf and its dual $A^\ell(\mathcal{G},\lambda)$ is an ind-coherent sheaf. Moreover, since the dual of each $(\mathcal{G}_i \to \mathcal{H}_C)_*\lambda$ is locally free, the ind-coherent sheaf $A^\ell(\mathcal{G},\lambda)$ is quasicoherent.

Moreover, even directly by the definition of $A^\ell(\mathcal{G},\lambda)$, as $\mathcal{H}om[\pi_*(\lambda),\mathcal{O}_{\mathcal{H}_C}]$ it has no sections supported on a divisor. Let us consider fibers $A^\ell(\mathcal{G},\lambda)_D \overset{\mathrm{def}}{=} \Gamma(\mathcal{G}_D,\lambda)^\vee$.

One has $\Gamma(\mathcal{G}_{D'\sqcup D''},\lambda) = \Gamma(\mathcal{G}_{D'}\times\Gamma_{D''},\lambda\boxtimes\lambda) = \Gamma(\mathcal{G}_{D'},\lambda)\otimes\Gamma(\Gamma_{D''},\lambda)$. Similarly, $D \subseteq E$ gives $\mathcal{G}_D \overset{i}{\subseteq} \mathcal{G}_E$, and then $\Gamma(\mathcal{G}_D,\lambda|_{\mathcal{G}_D}) \cong \Gamma(\mathcal{G}_D,\lambda|_{\mathcal{G}_E}) \twoheadleftarrow \Gamma(\mathcal{G}_E,\lambda|_{\mathcal{G}_E})$, hence $\Gamma(\mathcal{G}_D,\lambda)^\vee \hookrightarrow \Gamma(\mathcal{G}_E,\lambda)^\vee$. Also $\mathcal{G}_\varnothing = \{\varnothing\}$ and $A^\ell(\mathcal{G},\lambda) = \Gamma(\mathcal{G}_\varnothing,\lambda)^\vee = (\lambda_\varnothing)^\vee = \Bbbk^\vee = \Bbbk$. So, we have satisfied all conditions for a factorizations algebra from 3.1.1. □

Remarks.

(1) This proposition is a simplification of the more interesting point of view of Beilinson and Drinfeld [BD]. They ask whether the derived version of the above construction $RA^\ell(\mathcal{G},\lambda) \overset{\mathrm{def}}{=} \pi_!(\lambda\otimes\pi^!\mathcal{O}_{\mathcal{H}_C})$ is a DG-factorization algebra under suitable conditions (that $\mathcal{G}$ has a directed filtration by projective subschemes $\mathcal{G}_i$).

Following a parallel of sheaf theories in topology and algebraic geometry, one can call the proposed $RA^\ell(\mathcal{G},\lambda)$ the "algebro-geometric homology of $\mathcal{G}$ with coefficients in λ".

This "homology" localization is covariant in the space, hence different from standard localizations (for instance that of Beilinson–Bernstein [BB]) which use global sections, i.e., cohomology.

[17] Here, "natural" means that the definition should be interpreted in terms of families of objects.

(2) The case of the [BD] question when $\mathcal{G}$ is affine is established in [BD] after 3.10.16.1 (for the proof one is referred to the material in the proof of their Proposition 3.8.10). In particular, they provide a construction of lattice vertex algebras when $\mathcal{G}$ is $\mathcal{G}_{\mathcal{H}_C}(T)$, the loop Grassmannian of the torus $T = \mathbb{G}_m^I$ and the line bundle is $\underline{\mathsf{L}}^\kappa$ from 2.4.3.

3.2. Homology localizations of lattice vertex algebras on Grassmannians $\mathcal{G}^\kappa$

The following is the main result of this paper.

Theorem.

(a) *The spaces Z^κ and $S^\kappa_{\mathcal{H}_C}, \mathcal{G}^\kappa_{\mathcal{H}_C}$ associated to a symmetric matrix κ are factorization monoids (over $\mathcal{H}_{C\times I}$ and $\mathcal{H}_C$) with canonical $\mathbb{G}_m$-extensions $\mathcal{O}_{Z^\kappa}(1)$ and $\mathcal{O}_{S^\kappa_{\mathcal{H}_C}}(1)$, $\mathcal{O}_{\mathcal{G}^\kappa_{\mathcal{H}_C}}(1)$. These structures are compatible with the inclusions $Z^\kappa \subseteq S^\kappa_{\mathcal{H}_C} \subseteq \mathcal{G}^\kappa_{\mathcal{H}_C}$ and we get inclusions of the corresponding factorization algebras*

$$A^\ell\big(Z^\kappa, \mathcal{O}_{Z^\kappa}(1)\big) \subseteq A^\ell\big(S^\kappa, \mathcal{O}_{S^\kappa_{\mathcal{H}_C}}(1)\big) \subseteq A^\ell\big(\mathcal{G}^\kappa, \mathcal{O}_{\mathcal{G}^\kappa_{\mathcal{H}_C}}(1)\big).$$

(b) *The same holds for subobjects*

$$\big(\mathrm{Gr}(\mathcal{T}/\mathcal{H}_{C\times I}), \mathrm{pr}_2^* \mathsf{L}^\kappa\big) \ \ \textit{and} \ \ \big(\mathrm{Gr}(\widehat{\boldsymbol{\mathcal{T}}}/\mathcal{H}_C), \mathbf{pr}_2^* \mathsf{L}^\kappa\big), \big(\mathcal{G}_{\mathcal{H}_C}(T), \underline{\mathsf{L}}^\kappa\big).$$

The maps of factorization algebras corresponding to these inclusions of subobjects, are isomorphisms; say, $A^\ell(\mathcal{G}_{\mathcal{H}_C}(T), \underline{\mathsf{L}}^\kappa) \xrightarrow{\cong} A^\ell\big(\mathcal{G}^\kappa, \mathcal{O}_{\mathcal{G}^\kappa_{\mathcal{H}_C}}(1)\big)$.

(c) *The vertex algebra $A^\ell\big(\mathcal{G}_{\mathcal{H}_C}(T), \underline{\mathsf{L}}^\kappa\big)$ is the lattice vertex algebra $\boldsymbol{L}^\kappa$ from* 2.1.3.

Proof. (0) *The basic fact.* The three spaces in (a) are related by closed inclusions $Z^\kappa \subseteq S^\kappa \subseteq \mathcal{G}^\kappa$ in which the next space is union of copies of the preceding space. Moreover, this relation also holds for subspaces $\mathrm{Gr}(\mathcal{T}/\mathcal{H}_{C\times I}), \mathrm{Gr}(\widehat{\boldsymbol{\mathcal{T}}}/\mathcal{H}_C), \mathcal{G}_{\mathcal{H}_C}(T)$ and for the indicated line bundles on these six spaces (see Lemma (a) in 2.4.5).

(1) *Factorization monoid structures on six objects.* The compatible locality and growth structures on zastava $Z^\kappa, S^\kappa, \mathcal{G}^\kappa$ and their line bundles are in Lemma (c) in 2.4.5.

Z^κ is defined as the closure of the regular part and the restriction of $\mathcal{T}$ to each connected component of $\mathcal{H}_{C\times I}$ is irreducible. This implies that the regular parts are dense in all six spaces in (a) and (b) since the latter ones are unions of copies of preceding ones.

For the empty divisor $\varnothing$ by definitions one has $\{\varnothing\} = Z^\kappa_\varnothing = S^\kappa_\varnothing = \mathcal{G}^\kappa_\varnothing$.

(2) *Factorization algebras.* According to Proposition in 3.1.3, we now only need the filtrations required there.

For Z^κ, we can take the trivial filtration (all terms are the projective schemes Z^κ itself). This provides needed filtrations on Z^κ and $\mathcal{G}^\kappa$ since these spaces are unions of copies of Z^κ. For instance, S^κ has an $\mathbb{N}[I]$-filtration by projective schemes $F_\alpha S^\kappa_{\mathcal{H}_C} \overset{\text{def}}{=} Z_{\alpha\mathcal{H}_C}, \ \alpha \in \mathbb{N}[I]$, i.e., at $D \in \mathcal{H}_C$ we have $F_\alpha S^\kappa_D \overset{\text{def}}{=} Z_{\alpha D}, \ \alpha \in \mathbb{N}[I]$ (see 2.3.2).

One can get filtrations for subobjects in (b) by intersecting them with the filtrations from (a), but we actually get some obvious filtrations by finite flat schemes over configuration spaces.

(c) is the result of Beilinson and Drinfeld, see [BD, Prop. 3.10.8]. $\square$

Remarks.

(1) For special choices of the quadratic form κ one hopes that $\mathcal{G}^\kappa$ can be thought of as "loop Grassmannian" spaces of the corresponding affine Lie algebras, or more generally as "loop Grassmannian" spaces of (generalized) Kac–Moody Lie algebras.

(2) The question of localization of vertex algebras is a part of the program of constructing vertex algebra moduli, which is developed in the book of Ben-Zvi and Frenkel [BZFr].

(3) One may hope to extend the above localization to other vertex algebras that are built from lattice vertex algebras (say, by orbifolding or screening operators).

3.2.1. A "Verma type" vertex algebra $\mathbf{M}^\kappa$ *of a symmetric matrix* κ.

As we saw in the proof of Theorem in 3.1.2, we now have two localizations of the lattice vertex algebra $\boldsymbol{L}^\kappa$ on factorization monoids $\mathcal{G}(T) \subseteq \mathcal{G}^\kappa$. The interest in $\mathcal{G}^\kappa$ is that its geometry reflects the quadratic form κ. Example (2) below indicates how this can be used.

Lemma. *Let $\mathcal{G}' \subseteq \mathcal{G}$ be a factorization submonoid. The formal neighborhood $\widehat{\mathcal{G}'}$ is also a factorization submonoid. Also, if λ is a $\mathbb{G}_m$-extension of $\mathcal{G}$, then its restriction $\lambda' = \lambda|_{\mathcal{G}'}$ is also a $\mathbb{G}_m$-extension. Moreover, a filtration on $\mathcal{G}$ from Proposition in* 3.1.3 *induces such filtration on $\mathcal{G}'$.*

Example.

(0) In particular, a $\mathbb{G}_m$-extension λ of a factorization monoid $\mathcal{G}$ restricted to the formal neighborhood of the unit in $\mathcal{G}$ gives a factorization algebra $\mathcal{M}_\mathcal{G}$.

Then the corresponding vertex algebra $M_\mathcal{G}$ is the global sections of the D-module on $\mathcal{G}$ given by the twist of the delta distribution D-module at the unit in $\mathcal{G}$ by the line bundle λ on $\mathcal{G}$.

(1) When $\mathcal{G}$ is the loop Grassmannian $\mathcal{G}(G)$ of a semisimple simply laced group G, then the corresponding vertex algebra $\mathbb{M}^\kappa$ is the Verma type vertex algebra $V_1(\mathfrak{g})$ of $\mathfrak{g}$ of level one.

(2) The lattice algebra $\boldsymbol{L}^\kappa$ is a quotient of $\mathbf{M}^\kappa$ because $\mathcal{G}^\kappa$ is connected. Also, the same construction defines a deformation $\mathbf{M}^{k\kappa}$ for $k \in \Bbbk$.

In this generality, the vertex algebras $\mathbf{M}^\kappa$ may be new. The rational vertex algebra $\boldsymbol{L}^\kappa$ should in some sense be the "integrable" part of the nonrational vertex algebra $\mathbf{M}^\kappa$.

(3) In our setting of $\mathcal{G}^\kappa$, we can define another vertex algebra using the formal neighborhood $\widehat{S}^\kappa$ of $S^\kappa \subseteq \mathcal{G}^\kappa$.

References

[AT] E. Artin, J. Tate, *Class Field Theory*, Addison-Wesley, Redwood City, Ca., 1990.

[Ba] L. Bandklyder, *A proof of the Dold–Thom theorem via factorization homology*, `arXiv:math/1703.09170` (2017).

[BB] A. Beilinson, J. Bernstein, *Localisation de* $\mathfrak{g}$*-modules* C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 1, 15–18.

[BBE] A. Beilinson, S. Bloch, H. Esnault, *ε-factors for Gauss–Manin determinants*, `arXiv:math/0111277v3` (2003).

[BD] A. Beilinson, V. Drinfeld, *Chiral Algebras*, Colloquium Publications, Vol. 51, American Mathematical Society, Providence, RI, 2004.

[BL] T. Barnett-Lamb, *The Dold–Thom theorem*, available at `https://people.brandeis.edu/~tbl/dold-thom.pdf`.

[BMR] R. Bezrukavnikov, I. Mirković, D. Rumynin, *Localization of modules for a semisimple Lie algebra in prime characteristic*, Ann. of Math. **167** (2008), no. 3, 945–991.

[BZFr] E. Frenkel, D. Ben-Zvi, *Vertex Algebras and Algebraic Curves*, 2nd Edn., American Mathematical Society, Providence, RI, 2004.

[CC] C. E. Contou-Carrère, *Corps de classes local geometrique relatif*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 9, 481–484.

[CC1] C. E. Contou-Carrère, *Jacobienne locale d'une courbe formelle relative*, Rend. Sem. Mat. Univ. Padova **130** (2013), 1–106.

[Dr] V. Drinfeld, *Infinite-dimensional vector bundles in algebraic geometry*, `arXiv:math/0309155v4` (2004).

[FM] M. Finkelberg, I. Mirković, *Semiinfinite flags.* I. *Case of the global curve* $\mathbb{P}^1$, `arxiv.org/abs/alg-geom/9707010v2` (1998).

[FK] I. B. Frenkel, V. G. Kac, *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. **62** (1980), 23–66.

[Fr] E. M. Friedlander, *Motivic complexes of Suslin and Voevodsky*, Astérisque **245** (1997), Séminaire Bourbaki, exp. no. 833, 355–378.

[Ha] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

[HR] J. Hilburn, S. Raskin, *Tate's thesis in the de Rham setting*, `arXiv:2107.11325` (2021).

[Kac] V. Kac, *Vertex Algebras for Beginners*, 2nd Edn., American Mathematical Society, Providence, RI, 1998.

[La] G. Laumon, *Transformation de Fourier generalisee*, `arXiv:alg-geom/9603004` (1996).

[Ma] B. Mazur, *Notes on étale cohomology of number fields*, Ann. Sci. École Norm. Sup. **6** (1973), no. 4, 521–552.

[M1] I. Mirković, *Loop Grassmannians in the framework of local spaces over a curve*, in: *Recent Advances in Representation Theory, Quantum Groups, Algebraic Geometry, and Related Topics*, Contemp. Math. **623**, American Mathematical Society, Providence, RI, 2014, pp. 215–226.

[M2] I. Mirković, *Some extensions of the notion of loop Grassmannians*, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan., the Mardešić issue **532** (2017), 53–74.

[MV] I. Mirković, K. Vilonen, K., *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. **166** (2007), no. 1, 95–143.

[MYZ] I. Mirković, Y. Yang, G. Zhao, *Loop Grassmannians of quivers and affine quantum groups*, accepted for the conference volume for Alexander Beilinson and Victor Ginzburg.

[Se] J.-P. Serre, *Groupes Algébriques et Corps de Classes*, Hermann, Paris, 1959.

[TZ] J. Tao, Y. Zhao, *Extensions by K_2 and factorization line bundles*, `arXiv:1901.08760` (2019).

[Voe] V. Voevodsky, *The $\mathbb{A}^1$-homotopy theory*, Proc. ICM, Vol. 1, 1998, pp. 579–604.

[Zhu] X. Zhu, *Affine Demazure modules and T-fixed point subschemes in the affine Grassmannian*, Adv. Math. **221** (2009), no. 2, 570–600.