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NOVEL RESOLUTION ANALYSIS FOR THE RADON TRANSFORM IN $\mathbb{R}^2$ FOR FUNCTIONS WITH ROUGH EDGES*

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Abstract. Let f be a compactly supported, piecewise C^2 function in $\mathbb{R}^2$ with a jump across a sufficiently smooth, non-self-intersecting curve $\mathcal{S}$. Consider a family of modified functions f_ϵ^{m} so that f_ϵ^{m} has a jump across a curve $\mathcal{S}_\epsilon$. Each $\mathcal{S}_\epsilon$ is an $O(\epsilon)$-size perturbation of $\mathcal{S}$, which scales like $O(\epsilon^{-1/2})$ along $\mathcal{S}$. The functions f_ϵ^{m} are obtained by extending continuously the smooth components of f on either side of $\mathcal{S}$ all the way to $\mathcal{S}_\epsilon$ so that the location of the jump shifts from $\mathcal{S}$ to $\mathcal{S}_\epsilon$. By linearity of the Radon transform and its inversion formula, we can consider only the perturbation $f_\epsilon^{\text{p}} := f - f_\epsilon^{\text{m}}$. Let $f_\epsilon^{\text{p-rec}}$ be the reconstruction of f_ϵ^{p} from its discrete Radon transform data using a filtered backprojection inversion formula, where ϵ is the data sampling rate. A simple asymptotic (as $\epsilon \to 0$) formula to approximate $f_\epsilon^{\text{p-rec}}$ in any $O(\epsilon)$-size neighborhood of $\mathcal{S}$ was derived heuristically in an earlier paper of the author. Numerical experiments revealed that the formula is highly accurate even for nonsmooth (i.e., only Hölder continuous) $\mathcal{S}_\epsilon$. In this paper we provide a full proof of this result, which says that the magnitude of the error between $f_\epsilon^{\text{p-rec}}$ and its easily and explicitly computable approximation is $O(\epsilon^{1/2}\ln(1/\epsilon))$. The main assumption is that the level sets of the function $H_0(\cdot;\epsilon)$, which parametrizes the perturbation $\mathcal{S} \to \mathcal{S}_\epsilon$, are not too dense.

Key words. Radon inversion, resolution, rough edges, fractal boundary

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1. Introduction.

1.1. Local resolution analysis: Original and new. Let f be a compactly supported, piecewise C^2 function in $\mathbb{R}^2$, and $\mathcal{S}$ be some curve. We assume that f has a jump discontinuity across $\mathcal{S}$, and f is C^2 away from $\mathcal{S}$. Let f_ϵ^{rec} be a reconstruction from discrete tomographic data (i.e., discrete values of the appropriately averaged classical Radon transform of f; see (2.2) and (2.3)), where ϵ represents the data sampling rate. The reconstruction is computed by substituting interpolated data into a "continuous" filtered backprojection (FBP) inversion formula (see (2.4) and (2.5)). This is always assumed whenever we mention reconstruction in what follows. In many applications it is important to know the resolution of reconstruction from discrete data, including medical imaging, materials science, and nondestructive testing.

In [16] the author initiated analysis of resolution, called *local resolution analysis* (LRA), by focusing specifically on the behavior of f_ϵ^{rec} near $\mathcal{S}$. One of the main results of [16] is the computation of the limit

$$\text{DTB}(\check{x}; x_0, f) = \lim_{\epsilon \to 0} f_\epsilon^{\text{rec}}(x_0 + \epsilon \check{x}) \tag{1.1}$$

in a 2D setting under the assumptions that (a) $\mathcal{S}$ a sufficiently smooth curve with nonzero curvature, (b) f has a jump discontinuity across $\mathcal{S}$, (c) $x_0 \in \mathcal{S}$ is generic, and (d) $\check{x}$ is confined to a bounded set. The definition of a generic point in [16] is similar

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in spirit to the one used here (see Definition 2.6) but is more relaxed. It is important to emphasize that both the size of the neighborhood around x_0 and the data sampling rate go to zero simultaneously in (1.1). The limiting function $\mathrm{DTB}(\check{x}; x_0, f)$, which we call the discrete transition behavior (DTB), contains complete information about the resolution of reconstruction. The practical use of the DTB is based on the relation

$$f_\epsilon^{\mathrm{rec}}(x_0 + \epsilon\check{x}) = \mathrm{DTB}(\check{x}; x_0, f) + \text{error term}(\check{x}, x_0, \epsilon, f). \tag{1.2}$$

When $\epsilon > 0$ is sufficiently small, the error term is negligible, and $\mathrm{DTB}(\check{x}; x_0, f)$, which is given by a simple formula, is an accurate approximation to $f_\epsilon^{\mathrm{rec}}$. Hence whatever one wants to know about $f_\epsilon^{\mathrm{rec}}$ can be found by looking at an easily accessible $\mathrm{DTB}(\check{x}; x_0, f)$. Numerical experiments reported in [16] demonstrate that the error term in (1.2) is indeed quite small for realistic values of ϵ. LRA was extended to much more general settings in subsequent papers [17, 18, 19, 20].

Functions, which have been investigated in the LRA framework are, for the most part, nonsmooth across sufficiently smooth surfaces. On the other hand, in many applications discontinuities of f occur across nonsmooth (rough) surfaces. Examples include soil and rock imaging, where the surfaces of cracks and pores and boundaries between adjacent regions with different properties are highly irregular and frequently simulated by fractals [1, 8, 25, 31, 33, 34, 38].

It was proven in [21] that the original LRA based on (1.1) still works for functions with jumps across nonsmooth curves (i.e., Hölder continuous with some exponent $\gamma \in (0, 1]$). Our approach to nonsmooth boundaries is asymptotic. We begin by picking a compactly supported, piecewise C^2 function f, which has a jump across a sufficiently smooth (C^4), non-self-intersecting curve $\mathcal{S}$ (exactly as in the original LRA). Then we construct a family of functions f_ϵ^{m}, each with a jump across some curve $\mathcal{S}_\epsilon$. Here f_ϵ^{m} and $\mathcal{S}_\epsilon$ are modified versions of f and $\mathcal{S}$, respectively, and $\mathcal{S}_\epsilon$ is not necessarily smooth. Denote also the perturbation $f_\epsilon^{\mathrm{p}} := f - f_\epsilon^{\mathrm{m}}$. The superscripts "m" and "p" stand for "modified" and "perturbation," respectively.

Let $f_\epsilon^{*\text{-rec}}$ denote the reconstruction of f_ϵ^*, where $*$=p, m. Since $f_\epsilon^{\mathrm{m}} = f - f_\epsilon^{\mathrm{p}}$, the linearity of the Radon transform and FBP inversion formula imply $f_\epsilon^{\text{m-rec}} = f_\epsilon^{\mathrm{rec}} - f_\epsilon^{\text{p-rec}}$. By construction, f has only smooth boundaries, so we can concentrate on $f_\epsilon^{\text{p-rec}}$. We show that under certain assumptions on f_ϵ^{p}, the error term in (1.2) goes to zero as $\epsilon \to 0$ (with f and $f_\epsilon^{\mathrm{rec}}$ replaced by f_ϵ^{p} and $f_\epsilon^{\text{p-rec}}$, respectively). Nevertheless, numerical experiments in [21] demonstrate that this approach is not entirely satisfactory: a significant mismatch between the reconstruction $f_\epsilon^{\text{p-rec}}(x_0+\epsilon\check{x})$ and its approximation $\mathrm{DTB}(\check{x}; x_0, f_\epsilon^{\mathrm{p}})$ is observed. This means that the error term in (1.2) decays slowly as $\epsilon \to 0$ when $\mathcal{S}_\epsilon$ is rough.

To overcome this problem, a new LRA has been developed in [21]. It is based on allowing the DTB to depend on ϵ:

$$f_\epsilon^{*\text{-rec}}(x_0 + \epsilon\check{x}) = \mathrm{DTB}_{new}(\check{x}; x_0, \epsilon, f_\epsilon^*) + \text{error term}_1(\check{x}, x_0, \epsilon, f_\epsilon^*), \ * = \text{p, m}. \tag{1.3}$$

The idea is that since the new DTB is more flexible (due to its ϵ-dependence), the error term in (1.3) can be smaller than the one in (1.2). The new DTB proposed in [21] (see (2.14)) is given by the convolution of an explicitly computed and suitably scaled kernel with f_ϵ^*. Thus, analysis of resolution based on (1.3) is as simple as the one based on (1.2), and it can be used quite easily to investigate the partial volume effect (PVE), resolution, and many other properties of reconstruction in the case of rough boundaries. PVE arises due to limited resolution when the reconstructed value $f_\epsilon^{\text{m-rec}}(x)$ at some point x is not the true value $f_\epsilon^{\mathrm{m}}(x)$ but is an average of $f_\epsilon^{\mathrm{m}}(x')$ over all x' near

x [6]. PVE is most noticeable near the jumps of $f_\epsilon^{\rm m}$. It is especially detrimental when accurate identification of the boundaries between regions in a reconstructed image is required; see, e.g., subsection 1.2 below. See also section 3. Numerical experiments presented in [21] show an excellent match between $\text{DTB}_{new}(\check x; x_0, \epsilon, f_\epsilon^{\rm p})$ and the actual reconstruction $f_\epsilon^{\text{p-rec}}(x_0+\epsilon\check x)$, even when $\mathcal{S}_\epsilon$ is fractal.

To prove that the new DTB works well for nonsmooth $\mathcal{S}_\epsilon$, it is not sufficient to show that the error term in (1.3) goes to zero as $\epsilon\to 0$. We need to establish that the magnitude of the error is *independent* of how rough $\mathcal{S}_\epsilon$ is, i.e., independent of its Hölder exponent γ. This proved to be a difficult task. In [21] it was conjectured that the error term in (1.3) is $O(\epsilon^{1/2}\ln(1/\epsilon))$, and a partial result towards proving this conjecture was established. In this paper we provide a full proof of the conjecture.

Let $\mathcal{I}\ni s\to y(s)\in\mathcal{S}$, where $\mathcal{I}$ is an interval (or a union of intervals), be a $C^4(\mathcal{I})$, regular parametrization of $\mathcal{S}$. Let $\vec\theta(s)$ be a unit vector orthogonal to $\mathcal{S}$ at $y(s)$ (which depends on s continuously). The parametrization of $\mathcal{S}_\epsilon$ is given by $\mathcal{I}\ni s\to y(s)+\epsilon H_0(\epsilon^{-1/2}s;\epsilon)\vec\theta(s)$, where H_0 defines the perturbation. Then $f_\epsilon^{\rm m}$ is obtained by extending continuously the smooth components of f on either side of $\mathcal{S}$ all the way to $\mathcal{S}_\epsilon$ so that the location of the jump shifts from $\mathcal{S}$ to $\mathcal{S}_\epsilon$ (see Figure 3). The main assumption is that the function H_0 (more precisely, the family $H_0(\cdot;\epsilon)$) has level sets that are not too dense. This assumption does allow fairly nonsmooth H_0. For example, in [21] we construct a function on $\mathbb{R}$, whose level sets are not too dense as required, which is Hölder continuous with exponent γ for any prescribed $0<\gamma<1$, but which is not Hölder continuous with any exponent $\gamma'>\gamma$ on a dense subset of $\mathbb{R}$. Our construction ensures that the size of the perturbation is $O(\epsilon)$ in the direction normal to $\mathcal{S}$, and the perturbation scales like $\epsilon^{-1/2}$ in the direction tangent to $\mathcal{S}$.

The ideas behind the proof in this paper are quite different from those used in the original approach [16, 17, 18, 19, 20]. The latter proofs revolve around the smoothness of the singular support of f. The new proofs are based on cancellations occuring in certain exponential sums. The assumption about the level sets of H_0 is what allows for the cancellations to occur.

1.2. Practical application of our results. An important application of our results is in micro–computed tomography (CT) (i.e., CT capable of achieving micrometer resolution), which is a valuable tool for the imaging of rock samples extracted from wells. The reconstructed images are used to investigate properties of the samples. A collection of numerical methods that determine various rock properties using digital cores is collectively called digital rock physics (DRP) [7, 36]. Here the term "digital core" refers to a digital representation of the rock sample (rock core) obtained as a result of micro-CT scanning, reconstruction, and image analysis (segmentation and classification, feature extraction, etc.) [12]. DRP "is a rapidly advancing technology that relies on digital images of rocks to simulate multiphysics ... and predict properties of complex rocks (e.g., porosity, permeability, compressibility). ... For the energy industry, DRP aims to achieve more, cheaper, and faster results as compared to conventional laboratory measurements" [36]. Furthermore, "The simulation of various rock properties based on three-dimensional digital cores plays an increasingly important role in oil and gas exploration and development. *The accuracy of 3D digital core reconstruction is important for determining rock properties*" [38] (italic font is added here).

As stated above, boundaries between regions with different properties inside the rock are typically rough (see also [4]), i.e., they contain features across a wide range of scales, including scales below what is accessible with micro-CT. Clearly, the quality of micro-CT images (denoted here $f_\epsilon^{\text{m-rec}}$) stongly affects the accuracy with which various features of the rock (denoted here f_ϵ^{m}) are captured by its digital representation. This, in turn, strongly affects how accurate numerical simulations based on the digital rock are. The goal of DRP is to ensure that the physical properties of the rock computed using the digital core are as close as possible to the actual properties of the sample. Therefore, effects that degrade the resolution of micro-CT (e.g., due to finite data sampling) and how these effects manifest themselves in the presence of rough boundaries require careful investigation. For example, using (3.8) and the known scan parameters (i.e., the data sampling rate), one can determine how accurately image segmentation based on thresholding of $f_\epsilon^{\text{m-rec}}$ allows one to recover the actual boundaries in f_ϵ^{m}. Once fully understood and quantified, these effects can be accounted for at the step of image analysis, thereby leading to more accurate digital rock models and more accurate DRP results.

1.3. Related results. Organization of the paper. To situate the paper in a more general context, not much is known about how the Radon transform $\mathcal{R}$, its inverse $\mathcal{R}^{-1}$, and adjoint $\mathcal{R}^*$ act on distributions with complicated singularities. Since $\mathcal{R}$ and $\mathcal{R}^*$ are Fourier integral operators, their action on distributions and continuity in various L^p spaces have been studied in detail (see, e.g., [13, section 2.4], [11, 10, 23], and references therein). Nevertheless, specifically the case of functions with rough edges has never been explored to the best of our knowledge. A recent literature search reveals a small number of works which investigate the Radon transform acting on random fields [15, 35, 26]. The author did not find any publication on the Radon transform of characteristic functions of domains with rough boundaries. This appears to be the first paper that contains a result on the Radon transform of functions with rough edges.

An alternative way to study resolution of tomographic reconstruction is based on sampling theory. Applications of the classical sampling theory to Radon inversion are in papers such as [29, 32, 5], just to name a few. Analysis of sampling for distributions with semiclassical singularities is in [37, 27]. This line of work determines the sampling rate required to reliably recover features of a given size and describes aliasing artifacts if the sampling requirements are violated.

The paper is organized as follows. In section 2 we describe the setting of the problem, state the definition of a generic point, formulate all the assumptions (including assumptions about the perturbation H_0), and formulate the main result (Theorem 2.7). In section 3 we briefly discuss how DTB_{new} can be used in practice. Also, we demonstrate by numerical experiments that $\text{DTB}_{new}(\check{x}; x_0, \epsilon, f_\epsilon^{\text{m}})$ is indeed a good approximation to $f_\epsilon^{\text{m-rec}}(x_0 + \epsilon\check{x})$ even when $\mathcal{S}_\epsilon$ is fractal. The fact that $\text{DTB}_{new}(\check{x}; x_0, \epsilon, f_\epsilon^{\text{p}})$ is a good approximation to $f_\epsilon^{\text{p-rec}}(x_0 + \epsilon\check{x})$ is demonstrated numerically in [21]. The beginning of the proof is in section 4. We consider three cases:

(A) $x_0 \in \mathcal{S}$;

(B) $x_0 \notin \mathcal{S}$, and there is a line through x_0, which is tangent to $\mathcal{S}$; and

(C) $x_0 \notin \mathcal{S}$, and no line through x_0 is tangent to $\mathcal{S}$.

Sections 5–7 contain a nearly complete proof of the theorem in case (A). What is left is one additional assertion, which is proven in a later section. Likewise, section 8 and section 9 contain a nearly complete proof of the theorem in case (B). The final assertion of the theorem in cases (A) and (B) is proven in section 10. Originally, in case (C) the theorem is proven in [21] under the assumption that the curvature of $\mathcal{S}$ is nonzero at every point. Its proof without any restriction on the curvature is in Appendix D. Proofs of all lemmas and some auxiliary results are in Appendices A–C.

2. Preliminaries. Consider a function $f(x)$, $x \in \mathbb{R}^2$ in the plane, and let $\mathcal{S}$ be some curve.

Assumptions 2.1 (properties of the function f).
F1. $\mathcal{S}$ is a C^4 curve;
F2. f is compactly supported and $f \in C^2(\mathbb{R}^2 \setminus \mathcal{S})$; and
F3. for each $x_0 \in \mathcal{S}$ there exist a neighborhood $\mathcal{U} \ni x_0$, domains $D_\pm$, and functions $f_\pm \in C^2(\mathbb{R}^2)$ such that

$$
\begin{aligned}
&f(x) = \chi_{D_-}(x) f_-(x) + \chi_{D_+}(x) f_+(x),\ x \in \mathcal{U} \setminus \mathcal{S},\\
&D_- \cap D_+ = \varnothing,\ D_- \cup D_+ = \mathcal{U} \setminus \mathcal{S},
\end{aligned} \tag{2.1}
$$

where $\chi_{D_\pm}$ are the characteristic functions of $D_\pm$.

Assumptions 2.1 describe a typical function, which has a jump discontinuity across a smooth curve (see Figure 1).

The discrete tomographic data are given by

$$
\hat{f}_\epsilon(\alpha_k, p_j) := \frac{1}{\epsilon} \iint_{\mathbb{R}^2} w\left(\frac{p_j - \vec{\alpha}_k \cdot y}{\epsilon}\right) f(y) \mathrm{d}y,\ p_j = \bar{p} + j\Delta p,\ \alpha_k = \bar{\alpha} + k\Delta\alpha, \tag{2.2}
$$

where w is the detector aperture function, $\Delta p = \epsilon$, $\Delta\alpha = \kappa\epsilon$, and $\kappa > 0$, $\bar{p}$, $\bar{\alpha}$, are fixed. Here and below, $\vec{\alpha}$ and α in the same equation are always related by $\vec{\alpha} = (\cos\alpha, \sin\alpha)$. The same applies to $\vec{\theta} = (\cos\theta, \sin\theta)$ and θ. Sometimes we also use $\vec{\theta}^\perp = (-\sin\theta, \cos\theta)$.

If $w(p)$ satisfies certain conditions, in the limit $\epsilon \to 0^+$ the values $\hat{f}_{\epsilon=0^+}(\alpha_k, p_j)$, $k, j \in \mathbb{Z}$, represent the discretized (classical) Radon transform that integrates functions along lines [30]:

$$
\hat{f}(\alpha, p) := \iint_{\mathbb{R}^2} \delta(p - \alpha \cdot y) f(y) \mathrm{d}y. \tag{2.3}
$$

In reality, line integrals cannot be measured due to the finite resolution of the CT detector. In parallel beam geometry, which (2.2) describes, for each α_k, p_j the actual measurement represents a weighted average of $\hat{f}(\alpha_k, p)$ over a small neighborhood of p_j. The size of the neighborhood and the averaging kernel w (also known as the detector aperture function) depend on the detector. For our purposes, all we need to assume is that w is some sufficiently smooth, compactly supported, and normalized function.

Assumptions 2.2 (properties of the aperture function w).
AF1. w is even, and $w \in C_0^{\lceil\beta\rceil+1}(\mathbb{R})$ (i.e., w is compactly supported, and $w^{(\lceil\beta\rceil+1)} \in L^\infty(\mathbb{R})$) for some $\beta \geq 3$.
AF2. Normalization: $\int w(p)dp = 1$.

Here $\lceil\beta\rceil$ is the ceiling function, i.e., the integer n such that $n-1 < \beta \leq n$. The required value of β is stated below in Theorem 2.7. Later we also use the floor function $\lfloor\beta\rfloor$,

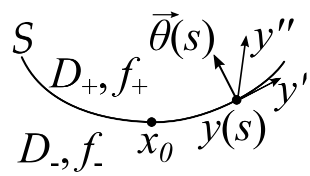

FIG. 1. *Illustration of a function f discontinuous across $\mathcal{S}$.*

which gives the integer n such that $n \le \beta < n+1$, and the fractional part function $\{\beta\} := \beta - \lfloor \beta \rfloor$.

Reconstruction from discrete data is achieved by the formula

$$f_\epsilon^{\text{rec}}(x) = -\frac{\Delta\alpha}{2\pi} \sum_{|\alpha_k| \le \pi/2} \frac{1}{\pi} \int \frac{\partial_p \sum_j \varphi\left(\frac{p-p_j}{\epsilon}\right) \hat{f}_\epsilon(\alpha_k, p_j)}{p - \alpha_k \cdot x} \mathrm{d}p, \tag{2.4}$$

where φ is an interpolation kernel, and the integral is understood in the principal value sense. Equation (2.4) is a discretized version of the "continuous" inversion formula [30]

$$f(x) = -\frac{1}{2\pi} \int_{|\alpha| \le \pi/2} \frac{1}{\pi} \int \frac{\partial_p \hat{f}(\alpha, p)}{p - \alpha \cdot x} \mathrm{d}p \, \mathrm{d}\alpha. \tag{2.5}$$

Equations (2.4) and (2.5) describe discrete and continuous FBP-type reconstruction, respectively [30].

Assumptions 2.3 (properties of the interpolation kernel φ).

IK1. φ is even, compactly supported, and its Fourier transform satisfies $\tilde{\varphi}(\lambda) = O(|\lambda|^{-(\beta+1)})$, $\lambda \to \infty$;

IK2. φ is exact up to order 1, i.e.,

$$\sum_{j \in \mathbb{Z}} j^m \varphi(u - j) \equiv u^m, \ m = 0, 1, \ u \in \mathbb{R}. \tag{2.6}$$

Here β is the same as in Assumption 2.2 (AF1). As is easily seen, Assumption 2.3 (IK2) implies $\int \varphi(p) dp = 1$. See section IV.D in [2], which shows that φ with the desired properties can be found for any $\beta > 0$ (i.e., for any regularity of φ).

Let $H_0(u; \epsilon)$, $u \in \mathbb{R}$, be a family of functions defined for all $\epsilon > 0$ sufficiently small. We use H_0 to parametrize perturbations of $\mathcal{S}$. Define the function

$$\chi(t, r) := \begin{cases} 1, & 0 < t \le r, \\ -1, & r \le t < 0, \\ 0 & \text{otherwise}. \end{cases} \tag{2.7}$$

Assumptions 2.4 (properties of the perturbation function H_0). There exist constants $c, \rho, L > 0$, which are independent of ϵ, such that for all $\epsilon > 0$ sufficiently small,

H1. $|H_0(u; \epsilon)| \le c$ for all $u \in \mathbb{R}$;

H2. the function $(t, u) \to \chi(t, H_0(u; \epsilon))$ is measurable in $\mathbb{R}^2$; and

H3. for any interval I of any length $L \ge L_0$ the set

$$U(I, t, \epsilon) := \{u \in I : \text{sgn}(t)(H_0(u; \epsilon) - t) \ge 0\} \tag{2.8}$$

is either empty or a union of no more than ρL intervals U_n, $\text{dist}(U_{n_1}, U_{n_2}) > 0$, $n_1 \ne n_2$, for almost all $t \ne 0$.

Assumption H2 is informally interpreted as saying that if $t > 0$ (resp., $t < 0$), then $H_0(u; \epsilon) \ge t$ (resp., $H_0(u; \epsilon) \le t$) for u in a measurable set for almost all t (and each $\epsilon > 0$ sufficiently small). Thus, the meaning of the argument t of χ in (2.7) is the value that defines a level set of H_0. The meaning of the argument r is the value of

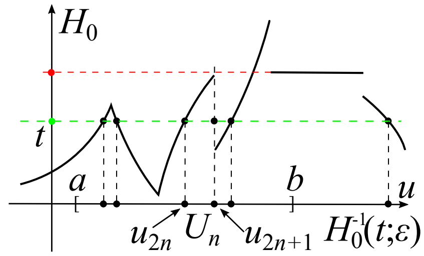

FIG. 2. *Illustration of the level sets of* H_0 *and Assumptions* 2.4 (H2, H3). *Here* $I=[a,b]$.

$H_0(u;\epsilon)$, which is being compared with t. In particular, $U(I,t,\epsilon)$ can be equivalently defined as follows:

$$\begin{aligned} U(I,t,\epsilon) &= \{u\in I : \mathrm{sgn}(t)=\chi(t,H_0(u;\epsilon))\} \\ &= \{u\in I : H_0(u;\epsilon)\geq t \text{ if } t>0, \text{ and } H_0(u;\epsilon)\leq t \text{ if } t<0\}. \end{aligned} \tag{2.9}$$

Assumption H3 further specifies that this set is just a union of intervals U_n, and the number of U_n for any interval $I=[a,b]$, $b-a\geq L_0$ is bounded by $\rho(b-a)$.

Our assumptions allow H_0 to be discontinuous. The endpoints of U_n are denoted u_{2n} and u_{2n+1}: $\overline{U}_n=[u_{2n},u_{2n+1}]$, where the bar denotes closure (see Figure 2). The distance between the intervals is positve, so $u_n<u_{n+1}$ for all n. The intervals U_n and the points u_n depend on t, I, and ϵ. If H_0 is continuous, then for each t and $\epsilon>0$ the collection of u_n's is simply the level set $\{u\in\mathbb{R} : H_0(u;\epsilon)=t\}$.

Assumptions 2.4 (H1, H2) imply that $\int_I\int_{\mathbb{R}}|\chi(t,H_0(u;\epsilon))|\mathrm{d}t\mathrm{d}u$ is well defined and bounded for any bounded interval I. By the Fubini theorem, Assumption 2.4 (H3), and (2.9),

$$\begin{aligned} \int_I\int_{\mathbb{R}} g(t,u)\chi(t,H_0(u;\epsilon))\mathrm{d}t\mathrm{d}u &= \int_{\mathbb{R}}\int_I g(t,u)\chi(t,H_0(u;\epsilon))\mathrm{d}u\mathrm{d}t \\ &= \int_{\mathbb{R}} \mathrm{sgn}(t)\sum_n\int_{U_n} g(t,u)\mathrm{d}u\mathrm{d}t \end{aligned} \tag{2.10}$$

for any sufficiently regular function g. Equation (2.10) is the main reason why we assume 2.4 (H2). In the proof of Theorem 2.7, we integrate over a domain bounded by the u-axis and the graph of $H_0(u;\epsilon)$. The first integral in (2.10) reflects the most common way to do that: the outer integral is with respect to u, while inside we integrate with respect to t between $t=0$ and $t=H_0(u;\epsilon)$ (due to the χ function). However, in the proof we need to change the order of integration and integrate with respect to t outside (as is done in the second and third integrals in (2.10)). Assumption 2.4 (H2) and the Fubini theorem allow us to change the order of integration.

Suppose $\mathcal{S}$ is parametrized by $\mathcal{I}\ni s\to y(s)\in\mathcal{S}$, where $\mathcal{I}$ is an interval (or a union of intervals), $y\in C^4(\mathcal{I})$, and $|y'(s)|\neq 0$ for any $s\in\mathcal{I}$. The normal direction to $\mathcal{S}$ at the point $y(s)$ is denoted $\vec{\theta}(s)$ (and the corresponding polar angle is $\theta(s)$); see Figure 1. Define

(2.11)
$$f_\epsilon^{\mathrm{p}}(x) := (f_+(x)-f_-(x))\chi(t,H_\epsilon(s)),\ x=y(s)+t\vec{\theta}(s),\ H_\epsilon(s):=\epsilon H_0(\epsilon^{-1/2}s;\epsilon).$$

Consider the modified function $f_\epsilon^{\mathrm{m}}:=f-f_\epsilon^{\mathrm{p}}$ and the curve $\mathcal{S}_\epsilon$ parametrized by

$$\mathcal{I}\ni s\to y(s)+H_\epsilon(s)\vec{\theta}(s)\in\mathcal{S}_\epsilon. \tag{2.12}$$

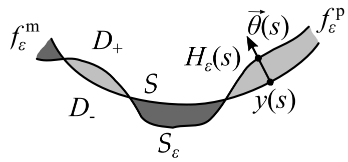

FIG. 3. *Illustration of the perturbations* $\mathcal{S}\to\mathcal{S}_\epsilon$ *and* f_ϵ^p. *The perturbation* f_ϵ^p *is supported in the shaded regions.*

By Assumption 2.4 (H1), $\mathcal{S}_\epsilon$ is an $O(\epsilon)$-size perturbation of $\mathcal{S}$. At the points where $H_\epsilon(s)>0$, a small region is removed from D_+ and added to D_- (see lighter shaded regions in Figure 3). At the points where $H_\epsilon(s)<0$, a small region is removed from D_- and added to D_+ (see darker shaded regions in Figure 3). Thus, $f_\epsilon^{\mathrm{m}}(x)$ is discontinuous across $\mathcal{S}_\epsilon$ instead of $\mathcal{S}$.

Let $\hat f_\epsilon^{\mathrm{m}}$ and $\hat f_\epsilon^{\mathrm{p}}$ denote the data for f_ϵ^{m} and f_ϵ^{p} defined by (2.2), with f replaced by f_ϵ^{m} and f_ϵ^{p}, respectively. Similarly, let $f_\epsilon^{\text{m-rec}}$ and $f_\epsilon^{\text{p-rec}}$ denote the reconstruction of f_ϵ^{m} and f_ϵ^{p} from the data $\hat f_\epsilon^{\mathrm{m}}$ and $\hat f_\epsilon^{\mathrm{p}}$, respectively. In [16, 18, 20], we obtained the DTB in the case of a sufficiently smooth $\mathcal{S}$. By linearity of the measurement process described by (2.2) and the inversion formula (2.4) (which correspond to the steps $f\to\hat f_\epsilon$ and $\hat f_\epsilon\to f_\epsilon^{\mathrm{rec}}$, respectively), we can ignore the original function f and consider the reconstruction of only the perturbation f_ϵ^{p}: $f_\epsilon^{\mathrm{p}}\to\hat f_\epsilon^{\mathrm{p}}\to f_\epsilon^{\text{p-rec}}$.

By (2.2) and (2.4),

$$f_\epsilon^{\text{p-rec}}(x)=-\frac{\Delta\alpha}{2\pi}\frac{1}{\epsilon^2}\sum_{|\alpha_k|\le\pi/2}\sum_j\mathcal{H}\varphi'\left(\frac{\vec\alpha_k\cdot x-p_j}{\epsilon}\right)\iint w\left(\frac{p_j-\vec\alpha_k\cdot y}{\epsilon}\right)f_\epsilon^{\mathrm{p}}(y)\mathrm{d}y, \tag{2.13}$$

where $\mathcal{H}$ denotes the Hilbert transform (see the integral with respect to p in (2.5)). Clearly, $f_\epsilon^{\text{p-rec}}$ can be written as the first integral in (2.10) with some g. In the proof of Theorem 2.7 below, we will change the order of integration using (2.10) in some very similar integrals (see (4.11) and (8.6)).

Following [21], replace the sums with respect to k and j with integrals to obtain the new DTB:

$$\begin{aligned}f_\epsilon^{\text{p-rec}}(x)&\approx\frac{1}{\epsilon^2}\iint K\left(\frac{x-y}{\epsilon}\right)f_\epsilon^{\mathrm{p}}(y)\mathrm{d}y=:\mathrm{DTB}_{\mathrm{new}}(\check x;x_0,\epsilon,f_\epsilon^{\mathrm{p}}),\\ K(z)&:=-\frac{1}{2\pi}\int_0^\pi(\mathcal{H}\varphi'*w)(\vec\alpha\cdot z)\mathrm{d}\alpha.\end{aligned} \tag{2.14}$$

As is easily seen, K is radial and compactly supported.

For a real number s, let $\langle s\rangle$ denote the distance from s to the nearest integer: $\langle s\rangle:=\min_{l\in\mathbb{Z}}|s-l|=\min(\{s\},1-\{s\})$. The following definition is in [24, p. 121] (after a slight modification in the spirit of [28, p. 172]).

DEFINITION 2.5. *Let* $\eta>0$. *The irrational number* s *is said to be of type* η *if for any* $\eta_1>\eta$, *there exists* $c(s,\eta_1)>0$ *such that*

$$m^{\eta_1}\langle ms\rangle\ge c(s,\eta_1)\ \textit{for any}\ m\in\mathbb{N}. \tag{2.15}$$

The irrational number s *is said to be of constant type or badly approximable if there exists* $c(s)>0$ *such that* $m\langle ms\rangle\ge c(s)$ *for any* $m\in\mathbb{N}$.

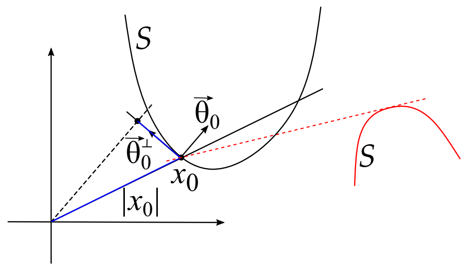

Fig. 4. *Illustration of the assumptions* P1–P4 *in Definition* 2.6. *The red segment of* $\mathcal{S}$ *and the red dashed line illustrate the situation prohibited by* P1. *The two blue line segments are involved in* P3 *and* P4.

See also [28], where the numbers which satisfy (2.15) are called $(\eta-1)$-order Roth numbers. It is known that $\eta \geq 1$ for any irrational s. The set of irrationals of each type $\eta \geq 1$ is of full measure in the Lebesgue sense, while the class of constant type is dense in the space of real numbers, but it is of null measure [28].

Definition 2.6. *A point* $x_0 \in \mathbb{R}^2$ *is said to be generic if the following assumptions are satisfied:*

P1. *No line, which is tangent to* $\mathcal{S}$ *at a point where the curvature of* $\mathcal{S}$ *is zero, passes through* x_0*;*

P2. *the line through the origin and* x_0 *is not tangent to* $\mathcal{S}$*;*

P3. $\kappa|x_0|$ *is irrational and of finite type; and*

P4. *if* $x_0 \in \mathcal{S}$*, then the number* $\kappa\vec{\theta}_0^{\perp} \cdot x_0$*, where* $\theta_0^{\perp}$ *is a unit vector tangent to* $\mathcal{S}$ *at* x_0*, is irrational and of finite type.*

In the rest of the paper we consider only generic points x_0.

Definition 2.6 is illustrated by Figure 4. The geometric meaning of assumptions P1 and P2 is clear. Since $\kappa = \Delta\alpha/\Delta p$, the units of κ are 1/length. Assumptions P3 and P4 imply that the two blue line segments in Figure 4 have irrational lengths when expressed in the units of $1/\kappa$ (thereby making the two quantities dimensionless).

The deeper meaning of assumption P4 is that the singularity of f at x_0 is in general position with respect to the data grid (α_k, p_j). Consider the function $p(\alpha) = \vec{\alpha} \cdot x_0$, whose graph is a curve in the Radon (or data) space. Let θ_0 be a value such that the line $\{x \in \mathbb{R}^2 : \vec{\theta}_0 \cdot x = p(\theta_0)\}$ is tangent to $\mathcal{S}$ at x_0. This is precisely the line in the data that "sees" the singularity of f at x_0. Clearly,

$$|\kappa\vec{\theta}_0^{\perp} \cdot x_0| = (\Delta\alpha/\Delta p)|\mathrm{d}p(\alpha = \theta_0)/\mathrm{d}\alpha| = \kappa|p'(\theta_0)|. \tag{2.16}$$

Assumption P4 says that $p(\alpha)$ has an irrational slope at $\alpha = \theta_0$ if the scales along the p- and α-axes are Δp and $\Delta\alpha$, respectively (see (2.2)). This assumption is quite natural; it and its analogues always appear even when boundaries are sufficiently smooth [16, 17, 18, 19, 20].

In the case of sufficiently smooth boundaries, we do not need to know the type of the irrational number $\kappa p'(\theta_0)$ since we establish only convergence to zero of the error term in (1.2). In this paper we establish the rate of convergence of the error term in (1.3). To quantify this rate we need more information about $\kappa p'(\theta_0)$, namely, its type (see also the second paragraph below).

Similarly, if θ_1 is a value such that $p(\theta_1) = 0$, then $\kappa|x_0| = (\Delta\alpha/\Delta p)|\mathrm{d}p(\alpha = \theta_1)/\mathrm{d}\alpha|$. Assumption P3 requires that the slope $p'(\theta_1)$ be irrational when the p- and α-axes are scaled the same way as before. This assumption is new. It is imposed only when $f_\epsilon^{\rm p}$ has a jump across a rough curve. Assumption P3 is needed because the wave front set of $f_\epsilon^{\rm p}$ above $x\in\mathcal{S}_\epsilon$ contains not only directions perpendicular to $\mathcal{S}_\epsilon$ (which would be the case if $\mathcal{S}_\epsilon$ were smooth) but many other directions as well. If $\mathcal{S}_\epsilon$ is nonsmooth almost everywhere (which is allowed by Assumptions 2.4), it may happen that $\mathcal{S}_\epsilon\cup(\mathbb{R}^2\setminus 0)\subset WF(f_\epsilon^{\rm p})$. Reconstruction of such a function is clearly more difficult than the one with sufficiently smooth boundaries, so an additional assumption makes sense.

More specifically, we need to know the type of $\kappa|x_0|$ (and the type of $\kappa\vec\theta_0^\perp\cdot x_0$ if $x_0\in\mathcal{S}$) in order to control the magnitude of certain exponential sums, which are central to the proof of Theorem 2.7 (see subsection 7.5 and section 9).

By P4, $x_0\neq 0$. Clearly, the set of generic x_0 is dense in the plane. Let η_0 denote the type of $\kappa|x_0|$ if $x_0\notin\mathcal{S}$, and the larger of the two types (in P3 and P4)—if $x_0\in\mathcal{S}$. Our main result is the following theorem.

Theorem 2.7. *Suppose*

1. *f satisfies Assumptions* 2.1*,*
2. *the detector aperture function w satisfies Assumptions* 2.2*,*
3. *the interpolation kernel φ satisfies Assumptions* 2.3*,*
4. *the perturbation H_0 satisfies Assumptions* 2.4*, and*
5. *point x_0 is generic.*

If $\beta>\eta_0+2$, one has

$$f_\epsilon^{\text{p-rec}}(x_0+\epsilon\check x)=\frac{1}{\epsilon^2}\iint K\left(\frac{(x_0+\epsilon\check x)-y}{\epsilon}\right)f_\epsilon^{\rm p}(y)dy+O(\epsilon^{1/2}\ln(1/\epsilon)),\ \epsilon\to 0, \tag{2.17}$$

where K is given by (2.14) *and the big-O term is uniform with respect to $\check x$ in any compact set.*

3. Discussion of the practical use of Theorem 2.7.

Even though Theorem 2.7 is about the reconstruction of $f_\epsilon^{\rm p}$, in practice one does not separate the object being scanned, $f_\epsilon^{\rm m}$, into its constituent parts. This is usually done only for theoretical analysis. Therefore, in this section we discuss how to relate (2.17) to the accuracy of the reconstruction of $f_\epsilon^{\rm m}$.

By construction (see (2.11) and the following sentence) we can represent $f_\epsilon^{\rm m}$ in the form $f_\epsilon^{\rm m}=f-f_\epsilon^{\rm p}$, where f is piecewise C^2 and has jumps across sufficiently smooth curves (cumulatively denoted $\mathcal{S}$), and $f_\epsilon^{\rm p}$ is a perturbation of the kind (2.11) supported between $\mathcal{S}$ and some nonsmooth curves (cumulatively denoted $\mathcal{S}_\epsilon$). Thus, $f_\epsilon^{\rm p}$ may have jumps across both $\mathcal{S}$ and $\mathcal{S}_\epsilon$.

As was mentioned before, the map $f\to\{\hat f_\epsilon(\alpha_j,p_k)\}_{j,k\in\mathbb{Z}}\to f_\epsilon^{\rm rec}$ is linear. Hence $f_\epsilon^{\text{m-rec}}=f_\epsilon^{\rm rec}-f_\epsilon^{\text{p-rec}}$. Pick any generic $x_0\in\mathcal{S}$ as in Theorem 2.7. Denote $K_\epsilon(x):=\epsilon^{-2}K(x/\epsilon)$. The error of the approximation $(f_\epsilon^{\text{p-rec}}-K_\epsilon*f_\epsilon^{\rm p})(x_0+\epsilon\check x)$ is estimated in Theorem 2.7.

Denote

$$f_0(x;x_0):=\begin{cases}\lim_{t\to 0^+}f(x_0+t\vec\theta_0), & \vec\theta_0\cdot(x-x_0)>0,\\ \lim_{t\to 0^-}f(x_0+t\vec\theta_0), & \vec\theta_0\cdot(x-x_0)<0,\end{cases} \tag{3.1}$$

where $\vec\theta_0$ is perpendicular to $\mathcal{S}$ at x_0 (see Figure 5). Thus, $f_0(x;x_0)$ takes only two values: one value in each of the half-planes $\pm\vec\theta_0\cdot(x-x_0)>0$ (assuming x_0 is fixed). It is proven in [20, Theorem 4.7, case $\kappa=0$] that the *original* DTB (1.1) satisfies

$$\text{DTB}(\check x;x_0,f)\equiv(K_\epsilon*f_0)(x_0+\epsilon\check x). \tag{3.2}$$

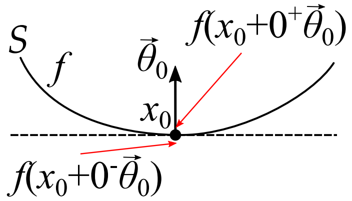

FIG. 5. *Construction of the function f_0. Red arrows indicate two limiting values of f on each side of x_0. The dashed line is $\vec{\theta}_0 \cdot (x - x_0) = 0$.*

The right-hand side of (3.2) is independent of ϵ because $f_0(x_0+\epsilon\check{x};x_0)$ is independent of $\epsilon > 0$. In the notation of [20], our case corresponds to theoretically exact reconstruction, i.e., $\check{f} \equiv f$. The statement of the theorem in [20] is not exactly the same as (3.2) but can be easily seen to be equivalent to it. Also, [20, Theorem 4.7, case $\kappa = 0$] asserts that the error $f_\epsilon^{\text{rec}}(x_0+\epsilon\check{x}) - \text{DTB}(\check{x};x_0,f) \to 0$ as $\epsilon \to 0$ when there are no global artifacts. The rate of convergence is not established in [20], but numerical experiments in [16] show that the error term is negligible for realistic values of ϵ.

Observing that $\mathcal{S}$ is a C^4 curve (so $\mathcal{S}$ is locally well approximated by a tangent line at x_0) and that$f$ is C^2 on each side of $\mathcal{S}$ (in the sense of (2.1)), it is obvious that $(K_\epsilon * (f - f_0))(x_0 + \epsilon\check{x}) = O(\epsilon)$. Therefore, by (3.2),

$$|\text{DTB}(\check{x};x_0,f) - \text{DTB}_{\text{new}}(\check{x};x_0,\epsilon,f)| = O(\epsilon). \tag{3.3}$$

This and (2.14) imply that the approximation $(f_\epsilon^{\text{rec}} - K_\epsilon * f)(x_0+\epsilon\check{x})$ is expected to be accurate. Our numerical experiments confirm that this is indeed the case.

We experiment with the same fractal phantom as in [21]. Here f is the characteristic function of the disc centered at $x_c = (0.1, 0.2)$ with radius $R = 0.3$, and $\mathcal{S}$ is its boundary. The perturbed boundary $\mathcal{S}_\epsilon$ is given by $r(\theta) = R + \epsilon H_0(\theta/\epsilon^{1/2})$ in polar coordinates with the origin at the center of the disc, where

$$H_0(s) = 5\sum_{n=c}^{\infty} r^{-\gamma n}\sin(r^n s),\ c = \lfloor \log_r(\pi)\rfloor,\ r = \sqrt{12},\ \gamma = 1/2. \tag{3.4}$$

The function H_0 is a real Weierstrass-type function (see [3]), which is continuous everywhere and differentiable nowhere and whose graph is a curve whose fractal dimension exceeds one [3]. It is well known that H_0 is bounded and Hölder continuous with exponent γ.

We set

$$\epsilon = \Delta p = 1.2/(N_p - 1),\ \Delta\alpha = \pi/N_\alpha,\ N_\alpha = N_p - 1,$$

and consider two values, $N_p = 501$ and $N_p = 1001$. Results of the experiments with $N_p = 501$ and $N_p = 1001$ are shown in Figures 6 and 7, respectively. In each of the experiments we consider two points $x_0 \in \mathcal{S}$:

$$x_0 = x_c - R\vec{\theta}_0,\ \theta_0 = 0.33\pi, 0.49\pi$$

and reconstruct $f_\epsilon^{\text{m-rec}}(x)$:

1. inside a region of interest (ROI), which is a square centered at x_0 with side length 100ϵ; and
2. along a segment of the line through x_0 and perpendicular to $\mathcal{S}$: $x = x_0 + \epsilon h\vec{\theta}_0$, $|h| \le 15$.

In (2.4) we use the Keys interpolation kernel [22, 2]

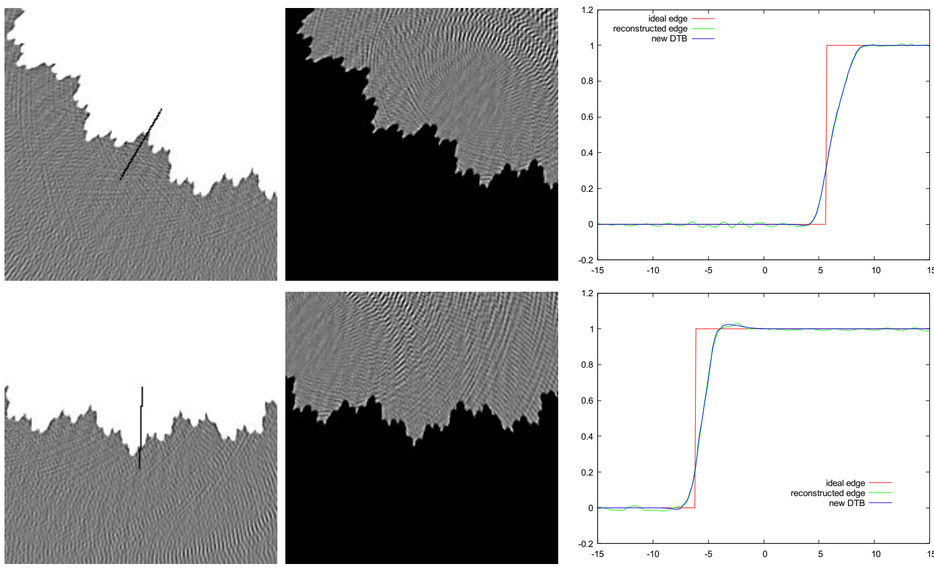

FIG. 6. *ROI in the fractal phantom of* [21]*,* $N_p = 501$*. Top row:* $\theta_0 = 0.33\pi$*, bottom row:* $\theta_0 = 0.49\pi$*. Left to right: density plot of the reconstructed ROI with the location of the profile shown,* $WL = 0$*,* $WW = 0.1$*; density plot of the reconstructed ROI,* $WL = 1$*,* $WW = 0.1$*; profiles of the ideal edge (red), reconstructed edge (green), and predicted edge (or* DTB_{new}*, blue). Values of* h *are on the horizontal axis.*

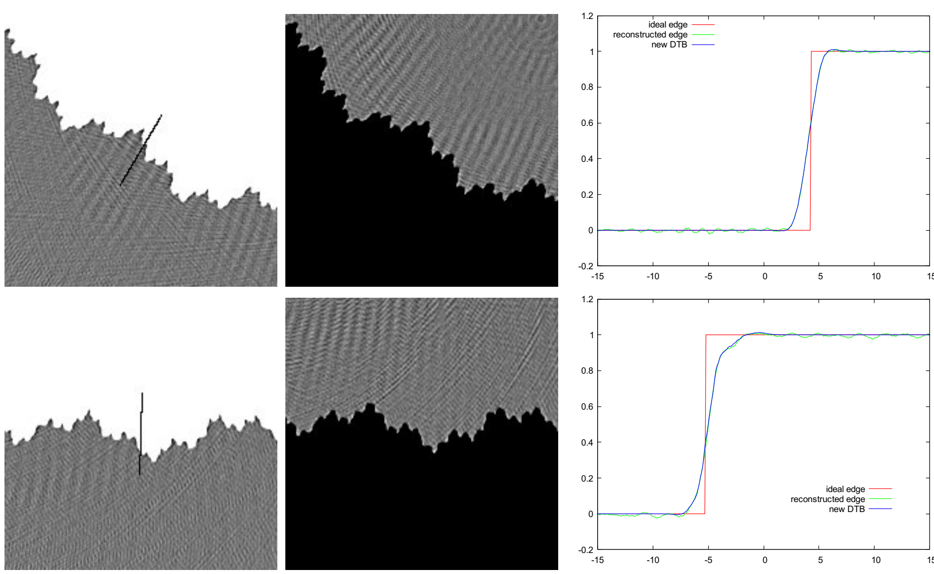

FIG. 7. *ROI in the fractal phantom of* [21]*,* $N_p = 1001$*. Top row:* $\theta_0 = 0.33\pi$*, bottom row:* $\theta_0 = 0.49\pi$*. Left to right: density plot of the reconstructed ROI with the location of the profile shown,* $WL = 0$*,* $WW = 0.1$*; density plot of the reconstructed ROI,* $WL = 1$*,* $WW = 0.1$*; profiles of the ideal edge (red), reconstructed edge (green), and predicted edge (or* DTB_{new}*, blue). Values of* h *are on the horizontal axis.*

$$\varphi(t) = 3B_3(t+2) - (B_2(t+2) + B_2(t+1)), \tag{3.5}$$

where B_n is the cardinal B-spline of degree n supported on $[0, n+1]$. Thus, $\text{supp}(\varphi) = [-2, 2]$. As the detector aperture function we take

$$w(p) = 1/\Delta p \text{ if } |p| \le \Delta p/2 \quad \text{and} \quad w(p) = 0 \text{ if } |p| > \Delta p/2. \tag{3.6}$$

Density plots of the reconstructed ROIs are shown in the left and center panels of Figure 6 and Figure 7. In the left panels, window level (WL) = 0 and window width (WW) = 0.1. In the center panels, WL = 0 and WW = 0.1. This means that only the values in the range $|f_\epsilon^{\text{m-rec}}(x) - \text{WL}| \le \text{WW}/2$ are shown accurately. Values outside the window are changed as follows. If $f_\epsilon^{\text{m-rec}}(x) < \text{WL} - \text{WW}/2$ at some pixel x, then the value is replaced by $\text{WL} - \text{WW}/2$. Similarly, if $f_\epsilon^{\text{m-rec}}(x) > \text{WL} + \text{WW}/2$, then the value is replaced by $\text{WL} + \text{WW}/2$. This is done in order to better see small artifacts that might be invisible if the full window with $\text{WW} \sim 1$ were used.

In all density plots, a lighter shade of grey stands for a smaller reconstructed value. Thus, in the left panels grey color stands for pixel values around 0 and white color for pixel values above 0.05. In the middle panels, black color stands for pixel values below 0.95 and grey color for pixel values around 1.

The reconstructed profiles of $f_\epsilon^{\text{m-rec}}(x_0 + \epsilon\check{x})$ (using (2.2) and (2.4) with f replaced by f_ϵ^{m}), along with the predicted reconstruction

$$\text{DTB}_{\text{new}}(\check{x}; x_0, \epsilon, f_\epsilon^{\text{m}}) = (K_\epsilon * f_\epsilon^{\text{m}})(x_0 + \epsilon\check{x}), \tag{3.7}$$

and the ideal edge $f_\epsilon^{\text{m}}(x_0 + \epsilon\check{x})$ are shown in the right panels. Here $\check{x} = h\vec{\theta}$, $|h| \le 15$. On the vertical axis are the values of the three functions and on the horizontal axis the values of h. The locations of the profiles are shown in the left panels. As expected, we see an excellent match between $f_\epsilon^{\text{m-rec}}$ and DTB_{new}.

In summary, our discussion and experiments show that for realistically small values $\epsilon > 0$, we have an accurate approximation

$$f_\epsilon^{\text{m-rec}}(x) \approx \text{DTB}_{new}(\check{x}; x_0, \epsilon, f_\epsilon^{\text{m}}) = (K_\epsilon * f_\epsilon^{\text{m}})(x), \ x = x_0 + \epsilon\check{x}, \ x_0 \in \mathcal{S}, \tag{3.8}$$

where K is a radial, compactly supported, and easily computable kernel (2.14). *Equation* (3.8) *provides a simple, easy to use relationship between the unknown object* f_ϵ^{m} *and the actual reconstruction from discrete data* $f_\epsilon^{\text{m-rec}}$ in a neighborhood of $\mathcal{S}_\epsilon$ (the jump of f_ϵ^{m}). Using (3.8), one can answer any question one might have about the reconstruction, e.g., what the resolution of the reconstruction is for a given class of objects f_ϵ^{m}, how accurately the location of the jump of f_ϵ^{m} can be determined by thresholding $f_\epsilon^{\text{m-rec}}$, and many others.

Theoretical estimation of the approximation accuracy $f_\epsilon^{\text{rec}} - K_\epsilon * f$ (i.e., when only smooth boundaries are present) as well as numerical analysis of resolution, segmentation accuracy, and other applied questions in the presence of rough boundaries are beyond the scope of the paper and will be the subject of future work.

Consider now the reconstruction of only f. Suppose $x_0 \in \mathcal{S}$ is not generic. Recall that assumption P3 in Definition 2.6 is not needed when boundaries are sufficiently smooth. Thus, we assume that assumption P4 is violated for the selected x_0. Numerical experiments in [17] with the Radon transform in $\mathbb{R}^3$ demonstrate that in this case it may happen that $f_\epsilon^{\text{rec}}(x_0 + \epsilon\check{x}) - \text{DTB}(\check{x}; x_0, f) \not\to 0$ as $\epsilon \to 0$. Hence it is reasonable to expect that the same phenomenon occurs in $\mathbb{R}^2$ as well. In this case, due to (3.3), we have that (3.8) and (2.17) may not hold either, i.e.,

$$f_\epsilon^{\text{rec}}(x) - \text{DTB}_{new}(\check{x}; x_0, \epsilon, f) = f_\epsilon^{\text{rec}}(x) - (K_\epsilon * f)(x) \not\to 0, \ x = x_0 + \epsilon\check{x}, \tag{3.9}$$

as $\epsilon \to 0$ (if x_0 is not generic). On the other hand, experiments in [17] showed that the generic behavior is sufficiently robust. In other words, even if $\kappa\vec{\theta}_0^\perp \cdot x_0$ is rational, but is not too close to an integer, the behavior of the reconstruction is numerically indistinguishable from the generic one. Numerical stability of $\mathrm{DTB}_{\mathrm{new}}$ is confirmed also by the experiments in the present paper. Even though the aperture function w in (3.6) does not satisfy the smoothness assumption 2.2 (AF1), the match between the reconstruction and prediction is excellent (when x_0 is generic). Comprehensive analysis of the numerical stability of $\mathrm{DTB}_{\mathrm{new}}$ requires a separate investigation.

4. Beginning of the proof of Theorem 2.7. From this point forward we consider only the reconstruction of f_ϵ^{p}.

In this section we (a) rewrite the inversion formula (2.13) as an exponential sum (4.5); (b) formulate two assertions from which Theorem 2.7 follows (see (4.6), (4.7)); (c) describe the phenomenon of cancellation, which informally explains why the exponential sum in (4.6) is small (establishing this fact is the most difficult part of the proof); and (d) identify three distinct settings of the theorem based on the location of x_0 relative to $\mathcal{S}$ that require separate consideration.

By the linearity of the map $f \to f_\epsilon^{\mathrm{rec}}$ given by (2.2) and (2.4), in what follows we can consider only one domain U and suppose the following.

Assumptions 4.1 (modified assumptions about f). In addition to Assumptions 2.1, f satisfies the following:

F1′. U is a sufficiently small open neighborhood of some $x \in \mathcal{S}$, i.e., $\mathcal{S} \cap U$ is sufficiently short;

F2′. $\mathrm{supp}(f) \subset U$;

F3′. $f \equiv 0$ in a neighborhood of the endpoints of $\mathcal{S} \cap U$; and

F4′. if $x_0 \in \mathcal{S}$ or there is a line through x_0 which is tangent to $\mathcal{S}$, then $\mathcal{S}$ has nonzero curvature at every point $x \in \mathcal{S}$.

Even though the reconstruction of f_ϵ^{p} is the main object of our analysis, we construct f_ϵ^{p} from f (more precisely, from $f_\pm$ in (2.1)). This is why Assumptions 4.1 are about f and not about f_ϵ^{p}.

Since $f \equiv 0$ outside U, in what follows we assume $\mathcal{S} = \mathcal{S} \cap U$. Also, we may assume without loss of generality that $\mathcal{S}$ is parametrized by $[-a, a] \ni s \to y(s) \in \mathcal{S}$ for some small $a > 0$. If $x_0 \in \mathcal{S}$, we assume $y(0) = x_0$.

Throughout the paper we use the following convention. If an inequality involves an unspecified constant c, this means that the inequality holds for some $c > 0$. If an inequality (or a string of inequalities) involves multiple unspecified constants c, then the values of c in different places can be different. If some additional information about the value of c is necessary (e.g., $c \gg 1$ or $c > 0$ small), then it is stated.

Following [21], consider the function (see (2.13))

$$\psi(q,t) := \sum_j (\mathcal{H}\varphi')(q-j)w(j-q-t). \tag{4.1}$$

Then

$$\begin{aligned} &\psi(q,t) = \psi(q+1,t),\ q,t \in \mathbb{R}; \quad \psi(q,t) = O(t^{-2}),\ t \to \infty,\ q \in \mathbb{R}; \\ &\int \psi(q,t)\,dt \equiv 0,\ q \in \mathbb{R}. \end{aligned} \tag{4.2}$$

The last property follows from Assumption 2.3 (IK2) (see (2.6)). By (4.2), we can represent ψ in terms of its Fourier series:

$$\psi(q,t)=\sum_m \tilde\psi_m(t)e(-mq),\ e(q):=\exp(2\pi iq),$$
(4.3)
$$\tilde\psi_m(t)=\int_0^1 \psi(q,t)e(mq)\,dq=\int_{\mathbb{R}}(\mathcal{H}\varphi')(q)w(-q-t)e(mq)\,dq.$$

Introduce the function $\rho(s):=(1+|s|^\beta)^{-1}$, $s\in\mathbb{R}$.

LEMMA 4.2. *One has*

$$|\tilde\psi_m(t)|,|\tilde\psi_m'(t)|\le c\rho(m)(1+t^2)^{-1}. \tag{4.4}$$

By the lemma, the Fourier series for ψ converges absolutely.

From (2.2), (2.13), (4.1), and (4.3), the reconstructed image becomes

$$f_\epsilon^{\text{p-rec}}(x)=-\frac{\Delta\alpha}{2\pi}\sum_m\sum_{|\alpha_k|\le\pi/2} e\left(-m\frac{\vec\alpha_k\cdot x-\bar p}{\epsilon}\right)A_m(\alpha_k,\epsilon),$$
(4.5)
$$A_m(\alpha,\epsilon):=\epsilon^{-2}\iint\tilde\psi_m\left(\frac{\vec\alpha\cdot(y-x)}{\epsilon}\right)f_\epsilon^{\text{p}}(y)\mathrm{d}y.$$

To prove Theorem 2.7, in (2.13) we should be able to replace the sum with respect to k by an integral with respect to α and ignore all $m\ne 0$ terms (that make up ψ). We will show that

$$\Delta\alpha\sum_{m\ne0}\left|\sum_{|\alpha_k|\le\pi/2} e\left(-m\frac{\vec\alpha_k\cdot x}{\epsilon}\right)A_m(\alpha_k,\epsilon)\right|=O(\epsilon^{1/2}\ln(1/\epsilon)), \tag{4.6}$$

$$\sum_{|\alpha_k|\le\pi/2}\int_{\alpha_k-\Delta\alpha/2}^{\alpha_k+\Delta\alpha/2}|A_0(\alpha,\epsilon)-A_0(\alpha_k,\epsilon)|\,\mathrm{d}\alpha=O(\epsilon^{1/2}\ln(1/\epsilon)), \tag{4.7}$$

where $x=x_0+\epsilon\check x$. The factor $e(m\bar p/\epsilon)$ is dropped because it is independent of k.

From here through the end of section 9 we will be assuming that Assumption 4.1 (F4′) applies. This corresponds to cases (A) and (B) in the introduction. Hence we can use θ to parametrize $\mathcal{S}$; i.e., $s\equiv\theta$, and $y(\theta)$ satisfies

$$\vec\theta\cdot y'(\theta)\equiv0,\ \vec\theta^\perp=-y'(\theta)/|y'(\theta)|,\ \mathcal{R}(\theta)\equiv\vec\theta\cdot y''(\theta)>0,\quad |\theta|\le a. \tag{4.8}$$

Here $\mathcal{R}(\theta)$ is the radius of curvature of $\mathcal{S}$ at $y(\theta)$. Thus, $\vec\theta$ points towards the center of curvature of $\mathcal{S}$. If $x_0\in\mathcal{S}$, then $\vec\theta_0:=\vec\theta(0)$ is the unit vector orthogonal to $\mathcal{S}$ at x_0 since $x_0=y(0)$. The case when Assumption 4.1 (F4′) does not apply (case (C) in the introduction) is considered in section 10. By Assumption 4.1 (F3′), $f(x)\equiv0$ in a neighborhood of $y(\pm a)$. The requirements on the smallness of a are formulated later as needed.

All the estimates below are uniform with respect to $\check x$, so the $\check x$-dependence of various quantities is frequently omitted from notation. Transform the expression for A_m (cf. (4.5)) by changing variables $y\to(\theta,t)$, where $y=y(\theta)+t\vec\theta$:

$$A_m(\alpha,\epsilon)=\frac{1}{\epsilon^2}\int_{-a}^{a}\int_0^{H_\epsilon(\theta)}\tilde\psi_m\left(\frac{\vec\alpha\cdot(y(\theta)-x_0)}{\epsilon}+h(\theta,\alpha)\right)F(\theta,t)\mathrm{d}t\mathrm{d}\theta,$$
(4.9)
$$F(\theta,t):=\Delta f(y(\theta)+t\vec\theta)(\mathcal{R}(\theta)-t),\ h(\theta,\alpha):=-\alpha\cdot\check x+\hat t\cos(\theta-\alpha),$$

where $\mathcal{R}(\theta)-t=\det(\mathrm{d}y/\mathrm{d}(\theta,t))>0$. Recall that $\mathcal{R}(\theta)$ is the radius of curvature of $\mathcal{S}$ at $y(\theta)$. The dependence of h on $\hat t$ and $\check x$ is irrelevant and omitted from notation.

Consider the function

$$R_1(\theta,\alpha) := \vec{\alpha}\cdot(y(\theta)-x_0). \tag{4.10}$$

Change variables $\theta=\epsilon^{1/2}\tilde{\theta}$ and $t=\epsilon\hat{t}$ in (4.9) and then use (2.10):

$$\begin{aligned}\epsilon^{1/2}A_m(\alpha,\epsilon) &= \int_{-a\epsilon^{-1/2}}^{a\epsilon^{-1/2}}\int_{\mathbb{R}}\tilde{\psi}_m\left(\epsilon^{-1}R_1(\theta,\alpha)+h(\theta,\alpha)\right)F(\theta,\epsilon\hat{t})\chi(\hat{t},H_0(\tilde{\theta};\epsilon))\mathrm{d}\hat{t}\mathrm{d}\tilde{\theta}\\ &= \int_{\mathbb{R}}\operatorname{sgn}(\hat{t})\sum_n\int_{U_n}\tilde{\psi}_m\left(\epsilon^{-1}R_1(\theta,\alpha)+h(\theta,\alpha)\right)F(\theta,\epsilon\hat{t})\mathrm{d}\tilde{\theta}\mathrm{d}\hat{t},\end{aligned} \tag{4.11}$$

where θ is a function of $\tilde{\theta}$: $\theta=\epsilon^{1/2}\tilde{\theta}$, and χ is defined in (2.7). Recall that $U_n=[u_{2n},u_{2n+1}]$ (see Assumption 2.4 (H3), where these intervals are introduced). Here and in what follows, the interval I used in the construction of U_n's is always $I=[-a\epsilon^{-1/2},a\epsilon^{-1/2}]$. The intervals U_n can be closed, open, and half-closed. Since what kind they are is irrelevant, with some abuse of notation we write them as if they are closed. The dependence of U_n and u_n on $\hat{t}$ and ϵ is omitted from notation for simplicity.

In view of (4.5), (4.6), and (4.11) we need to estimate the quantity

$$\begin{aligned}W_m(\hat{t}) &:= \epsilon^{1/2}\sum_{\Delta\alpha|k|\le\pi/2}e\left(-mq_k\right)\left[g(\tilde{\alpha}_k;\cdot)e\left(-m\vec{\alpha}_k\cdot\check{x}\right)\right],\ q_k:=\frac{\vec{\alpha}_k\cdot x_0}{\epsilon},\\ g(\tilde{\alpha};m,\epsilon,\hat{t},\check{x}) &:= \sum_n\int_{U_n}\tilde{\psi}_m\left(\epsilon^{-1}R_1(\theta,\alpha)+h(\theta,\alpha)\right)F(\theta,\epsilon\hat{t})\mathrm{d}\tilde{\theta}.\end{aligned} \tag{4.12}$$

For convenience, we express g as a function of $\tilde{\alpha}$ rather than α. The sum in (4.6) is bounded by $\int\sum_{m\ne0}|W_m(\hat{t})|\mathrm{d}\hat{t}$. By Assumption 2.4 (H1), H_0 is bounded, so the integral with respect to $\hat{t}$ is over a bounded set.

Throughout the paper we frequently use rescaled variables $\tilde{\theta},\hat{\theta}$ and $\tilde{\alpha},\hat{\alpha}$:

$$\tilde{\alpha}=\epsilon^{-1/2}\alpha,\ \hat{\alpha}=\alpha/\Delta\alpha,\ \tilde{\theta}=\epsilon^{-1/2}\theta,\ \hat{\theta}=\theta/\Delta\alpha. \tag{4.13}$$

Whenever an original variable (e.g., α) is used together with its rescaled counterpart (e.g., $\hat{\alpha}$ or $\tilde{\alpha}$) in the same equation or sentence, they are always assumed to be related according to (4.13). The same applies to θ and its rescaled versions. The only exception is Appendix D, where the relationship between θ and $\tilde{\theta}$ is slightly different from the one in (4.13).

We distinguish two cases: $x_0\in\mathcal{S}$ and $x_0\notin\mathcal{S}$. In the former case, $x_0=y(0)$. The proof of (4.6) is much more difficult than the proof of (4.7), so we discuss the intuition behind the former. Due to the remark in the paragraph following (4.12), we just have to show that $\sum_{m\ne0}|W_m(\hat{t})|$ satisfies the same estimate as in (4.6). Summation with respect to m does not bring any complications, so we consider the sum with respect to k for a fixed $m\ne0$. The factor $g(\tilde{\alpha}_k;\cdot)$ is bounded and goes to zero as $\tilde{\alpha}\to\infty$. In addition, the factor $e(-mq_k)$ oscillates rapidly with k. If the product $[g(\tilde{\alpha}_k;\cdot)e(-m\vec{\alpha}_k\cdot\check{x})]$ changes slowly with k (in an appropriate sense), the rapidly oscillating exponentials nearly cancel each other out, thereby making $|W_m(\hat{t})|$ small.

An additional phenomenon is that the cancellation does not occur near the points $\alpha_{j,m}$, where the derivative of the phase is an integer: $m\vec{\alpha}^{\perp}(\alpha_{j,m})\cdot x_0=j\in\mathbb{Z}$. The reason is that near these points $e(-mq_k)$ does not oscillate rapidly with k. Hence the contributions into the sum in (4.12) coming from small neighborhoods of $\alpha_{j,m}$ should be investigated separately. If cancellations do not occur, these contributions are small only if $|g(\tilde{\alpha}_{j,m};\cdot)|$ are all small. If $\alpha_{j,m}=0$ for some j and m, (4.6) may fail because $|g(\tilde{\alpha}_{j,m}=0;\cdot)|$ is not small. Fortunately, this does not happen for a generic x_0 due to the assumption P4 in Definition 2.6 (see the paragraph following (7.2)).

The above argument applies when $x_0\in\mathcal{S}$ (case (A)). If $x_0\notin\mathcal{S}$, and there is a line through x_0 which is tangent to $\mathcal{S}$ (case (B)), then the argument is somewhat similar. One of the differences between the cases is that the function g is now expressed in terms of $\hat{\alpha}$ rather than $\tilde{\alpha}$. Also, estimates for $g(\hat{\alpha};\cdot)$ and its derivative are different from those for $g(\tilde{\alpha};\cdot)$.

Recall that (4.6), (4.7) imply Theorem 2.7. Estimates for $g(\tilde{\alpha}_k;\cdot)$ and its derivative in case (A) are obtained in section 6, and the sum with respect to k is estimated in section 7. This completes the proof of (4.6) in case (A). Likewise, estimates for $g(\hat{\alpha}_k;\cdot)$ and its derivative in case (B) are obtained in section 8, and the sum with respect to k is estimated in section 9. This completes the proof of (4.6) in case (B). The proof of (4.7) in cases (A) and (B) is in section 10. This completes the proof of Theorem 2.7 in these two cases.

Finally, suppose $x_0\notin\mathcal{S}$ and no line through x_0 is tangent to $\mathcal{S}$ (case (C)). The statement of Theorem 2.7 in this case is formulated as Lemma 8.1. Its proof is in Appendix D. The proof is not based on (4.6), (4.7) and does not use the cancellation property. Instead we prove that that $f_\epsilon^{\text{p-rec}}(x)\to 0$ sufficiently fast for all x in a small (but finite size) neighborhood of x_0 directly from (4.5). The proof is based on the smallness of $A_m(\alpha,\epsilon)$. Compared with the case (C) in [21], here we do not require that $\mathcal{S}$ have nonzero curvature. Interestingly, even though the proofs of all three cases are different (see also Remark 9.3), level sets of H_0 appear in all of them in an essential way.

5. Proof of (4.6) in case (A): Preparation. In sections 5–7, $x_0=y(0)$, so the function R_1 in (5.1) becomes

$$R_1(\theta,\alpha)=\vec{\alpha}\cdot(y(\theta)-y(0))\,. \tag{5.1}$$

The convexity of $\mathcal{S}$ and our convention imply that the nonzero vector $(y(\theta)-y(0))/\theta$ rotates counterclockwise as θ increases from $-a$ to a. Thus, for each $\theta\in[-a,a]$ there is $\alpha=\mathcal{A}_1(\theta)\in(-\pi/2,\pi/2)$ such that $\vec{\alpha}(\mathcal{A}_1(\theta))\cdot(y(\theta)-y(0))\equiv 0$, and the function $\mathcal{A}_1$ is injective. By continuity, $\mathcal{A}_1(0):=0$. Define

$$\Omega:=\text{ran}\mathcal{A}_1. \tag{5.2}$$

Clearly, $\Omega\subset(-a,a)$, and the inverse of $\mathcal{A}_1(\theta)$ is sufficiently smooth and well defined on Ω. In what follows, we need rescaled versions of R_1 and $\mathcal{A}_1$:

$$R(\tilde{\theta},\tilde{\alpha}):=R_1(\theta,\alpha)/\epsilon,\ \mathcal{A}(\tilde{\theta}):=\mathcal{A}_1(\theta)/\epsilon^{1/2}. \tag{5.3}$$

For simplicity, the dependence of R and $\mathcal{A}$ on ϵ is omitted from notation.

Definition 5.1. *We say $f(x)\asymp g(x)$ for $x\in U\subset\mathbb{R}^n$ if there exist $c_{1,2}>0$ such that*

$$c_1\le f(x)/g(x)\le c_2 \text{ if } g(x)\ne 0 \text{ and } f(x)=0 \text{ if } g(x)=0 \tag{5.4}$$

for any $x\in U$.

Lemma 5.2. *One has*

$$\mathcal{A}(\tilde{\theta})\asymp\tilde{\theta},\ \mathcal{A}'(\tilde{\theta})\asymp 1,\ |\theta|\le a,|\alpha|\le\pi/2;\ \max_{|\tilde{\theta}|\le a\epsilon^{-1/2}}|\mathcal{A}(\tilde{\theta})/\tilde{\theta}|<1, \tag{5.5}$$

and

$$
\begin{aligned}
&R(\tilde\theta,\tilde\alpha)\asymp\tilde\theta(\mathcal{A}(\tilde\theta)-\tilde\alpha),\ \partial_{\tilde\theta}R(\tilde\theta,\tilde\alpha)\asymp\tilde\theta-\tilde\alpha,\ \partial_{\tilde\alpha}R(\tilde\theta,\tilde\alpha)=O(\tilde\theta),\\
&\partial^2_{\tilde\alpha}R(\tilde\theta,\tilde\alpha),\,\partial^2_{\tilde\theta}R(\tilde\theta,\tilde\alpha),\,\partial_{\tilde\theta}\partial_{\tilde\alpha}R(\tilde\theta,\tilde\alpha)=O(1),\quad |\theta|\le a,|\alpha|\le\pi/2.
\end{aligned}\tag{5.6}
$$

Then, from (4.11),

$$
\epsilon^{1/2}A_m(\alpha,\epsilon)=\int_{\mathbb{R}}\operatorname{sgn}(\hat t)\sum_n\int_{U_n}\tilde\psi_m\left(R(\tilde\theta,\tilde\alpha)+h(\theta,\alpha)\right)F(\theta,\epsilon\hat t)\mathrm{d}\tilde\theta\mathrm{d}\hat t.\tag{5.7}
$$

Recall that θ is a function of $\tilde\theta$ in the arguments of h and F. Clearly,

$$
h,F=O(1),\ \partial_{\tilde\theta}h,\partial_{\tilde\alpha}h,\partial_{\tilde\theta}F=O(\epsilon^{1/2}),\quad |\theta|\le a,|\alpha|\le\pi/2,\tag{5.8}
$$

uniformly with respect to all variables. Here α is a function of $\tilde\alpha$.

Fix some small $\delta>0$ and define three sets

$$
\Xi_1:=[-\delta,\delta],\ \Xi_2:=\{\tilde\theta:|\mathcal{A}(\tilde\theta)-\tilde\alpha|\le\delta\},\ \Xi_3:=[-a\epsilon^{-1/2},a\epsilon^{-1/2}]\setminus(\Xi_1\cup\Xi_2)\tag{5.9}
$$

and the associated functions:

$$
g_l(\tilde\alpha;m,\epsilon,\hat t,\check x):=\sum_n\int_{U_n\cap\Xi_l}\tilde\psi_m\left(R(\tilde\theta,\tilde\alpha)+h(\theta,\alpha)\right)F(\theta,\epsilon\hat t)\mathrm{d}\tilde\theta,\ l=1,2,3.\tag{5.10}
$$

If $\alpha\notin\Omega$, we assume $\Xi_2=\varnothing$ and $g_2(\tilde\alpha;m,\epsilon,\hat t,\check x)=0$. To simplify notations, the arguments $m,\epsilon,\hat t$, and $\check x$ of g_l are omitted in what follows, and we write $g_l(\tilde\alpha)$. In view of (4.12), $g=g_1+g_2+g_3$.

6. Proof of (4.6) in case (A): Estimates for $g_{1,2,3}$.

Using Lemma 5.2 and (5.8) and estimating the model integral $\int(1+\tilde\theta^2((\tilde\theta/2)-\tilde\alpha)^2)^{-1}\mathrm{d}\tilde\theta$ over various domains, it is straightforward to conclude that

$$
g_{1,2}(\tilde\alpha)=\rho(m)O(|\tilde\alpha|^{-1}),\quad g_3(\tilde\alpha)=\rho(m)O(|\tilde\alpha|^{-2}),\quad \tilde\alpha\to\infty,\ |\alpha|\le\pi/2.\tag{6.1}
$$

Similarly, estimating the model integral $\int|\tilde\theta|(1+\tilde\theta^2((\tilde\theta/2)-\tilde\alpha)^2)^{-1}\mathrm{d}\tilde\theta$ implies

$$
\partial_{\tilde\alpha}g_{1,3}(\tilde\alpha)=\rho(m)O(|\tilde\alpha|^{-1}),\quad \tilde\alpha\to\infty,\ |\alpha|\le\pi/2.\tag{6.2}
$$

Thus, it remains to estimate $\partial_{\tilde\alpha}g_2$. We assume $\alpha\in\Omega$ because $g_2(\tilde\alpha)=0$ if $\alpha\notin\Omega$. Change variables $\tilde\theta\to r=R(\tilde\theta,\tilde\alpha)$ in (5.10) so that $\tilde\theta=\Theta(r,\tilde\alpha)$:

$$
\begin{aligned}
&g_2(\tilde\alpha)=\sum_n\int_{R_n\cap[r_{\rm mn},r_{\rm mx}]}\tilde\psi_m(r+h(\theta,\alpha))\frac{F(\theta,\epsilon\hat t)}{|(\partial_{\tilde\theta}R)(\Theta(r,\tilde\alpha),\tilde\alpha)|}\mathrm{d}r,\\
&R_n:=R(U_n,\tilde\alpha),\ r_{\rm mn}:=R(\mathcal{A}^{-1}(\tilde\alpha-\delta),\tilde\alpha),\ r_{\rm mx}:=R(\mathcal{A}^{-1}(\tilde\alpha+\delta),\tilde\alpha),
\end{aligned}\tag{6.3}
$$

where $\theta=\epsilon^{1/2}\Theta(r,\tilde\alpha)$. If R is decreasing in $\tilde\theta$ and $r_{\rm mn}>r_{\rm mx}$, the domain in (6.3) is understood as $R_n\cap[r_{\rm mx},r_{\rm mn}]$. Denote also $r_n:=R(u_n,\tilde\alpha)$ and $v_n:=\mathcal{A}(u_n)$. Clearly, r_n's are the endpoints of R_n's: $R_n=[r_{2n},r_{2n+1}]$ or $R_n=[r_{2n+1},r_{2n}]$, depending on whether $R(\tilde\theta,\tilde\alpha)$ is increasing or decreasing as a function of $\tilde\theta$.

Lemma 6.1. *For $\alpha\in\Omega$, $|\tilde\alpha|\ge c$, one has*

$$
\begin{aligned}
&r_n\asymp\tilde\alpha(\mathcal{A}(u_n)-\tilde\alpha),\ \partial_{\tilde\alpha}r_n\asymp-\tilde\alpha\quad\text{if}\quad |v_n-\tilde\alpha|\le\delta;\\
&r_{mn}\asymp-\tilde\alpha,\ r_{mx}\asymp\tilde\alpha,\ \partial_{\tilde\alpha}r_{mn},\partial_{\tilde\alpha}r_{mx}=O(1),\\
&\partial_{\tilde\theta}R(\tilde\theta,\tilde\alpha)\asymp\tilde\alpha\quad\text{if}\quad |\mathcal{A}(\tilde\theta)-\tilde\alpha|\le\delta;
\end{aligned}\tag{6.4}
$$

and

$$\partial_{\tilde\alpha}\Theta(r,\tilde\alpha) = -\left.\frac{\partial_{\tilde\alpha}R(\tilde\theta,\tilde\alpha)}{\partial_{\tilde\theta}R(\tilde\theta,\tilde\alpha)}\right|_{\tilde\theta=\Theta(r,\tilde\alpha)} = O(1) \quad \textit{if} \quad r\in[r_{mn},r_{mx}]. \tag{6.5}$$

In particular, $\partial_{\tilde\theta}R(\tilde\theta,\tilde\alpha)\neq 0$ if $|\mathcal{A}(\tilde\theta)-\tilde\alpha|\le\delta$, $\alpha\in\Omega$, and $|\tilde\alpha|\ge c$, so the change of variables in (6.3) is justified. Differentiating (6.3) and using (5.8) and Lemma 5.2, Lemma 6.1 gives

$$\begin{aligned}|\partial_{\tilde\alpha}g_2(\tilde\alpha)| &\le c\frac{\rho(m)}{|\tilde\alpha|}\left[\sum_{n:|v_n-\tilde\alpha|\le\delta}\frac{|\tilde\alpha|}{1+\tilde\alpha^2(v_n-\tilde\alpha)^2}+\frac{1}{|\tilde\alpha|^2}+\epsilon^{1/2}+\frac{1}{|\tilde\alpha|}\right]\\ &\le c\rho(m)\left[\sum_{n:|v_n-\tilde\alpha|\le\delta}\frac{1}{1+\tilde\alpha^2(v_n-\tilde\alpha)^2}+\frac{1}{\tilde\alpha^2}\right].\end{aligned} \tag{6.6}$$

Here we have used that $\epsilon^{1/2}\tilde\alpha=O(1)$. To summarize, we have

$$\begin{aligned}&|g(\tilde\alpha)|\le c\rho(m)(1+|\tilde\alpha|)^{-1},\ |\alpha|\le\pi/2;\\ &|\partial_{\tilde\alpha}g(\tilde\alpha)|\le c\rho(m)\left[\sum_{n:|v_n-\tilde\alpha|\le\delta}\frac{1}{1+\tilde\alpha^2(v_n-\tilde\alpha)^2}+\frac{1}{1+\tilde\alpha^2}\right],\ \alpha\in\Omega;\\ &|\partial_{\tilde\alpha}g(\tilde\alpha)|\le c\rho(m)(1+|\tilde\alpha|)^{-1},\ \alpha\in[-\pi/2,\pi/2]\setminus\Omega,\end{aligned} \tag{6.7}$$

where g is given by (4.12).

7. End of proof of (4.6) in case (A): Summation with respect to *k*.

7.1. Preliminary results. The goal of this section is to estimate the sum in (4.12). This is done by breaking up the interval $[-\pi/2,\pi/2]$ into a union of smaller intervals and estimating individually the sums over each of these intervals. The sums over the smaller intervals are estimated using the partial integration identity (7.7) and the Kusmin–Landau inequality (7.9). Some of these intervals require special consideration (e.g., those that contain $\alpha_{j,m}$ mentioned in the third paragraph following (4.13)).

Denote

$$\phi(\alpha)=m\kappa r_x\sin(\alpha-\alpha_x),\ \vartheta(\hat\alpha)=-m\vec\alpha\cdot x_0/\epsilon,\ \mu=-\kappa r_x\sin\alpha_x, \tag{7.1}$$

where $x_0=r_x\vec\alpha(\alpha_x)$. Recall that (4.13) is always assumed. Clearly, $\vartheta'(\hat\alpha)\equiv\phi(\alpha)$, $\phi(0)=m\mu$, and $\mu=-\kappa\theta_0^\perp\cdot x_0$. Without loss of generality we may assume $m\ge1$. We begin by estimating the top sum in (4.12). There are four cases to consider: $m\ge1$ or $m\le-1$ combined with $\alpha_k\in[0,\pi/2]$ or $\alpha_k\in[-\pi/2,0]$. We will consider only one case: $m\ge1$ and $\alpha_k\in[0,\pi/2]$; the other three cases are completely analogous.

Let $\alpha_*>0$ be the smallest angle such that $\phi'(\alpha_*)=0$; i.e., $\alpha_*=\alpha_x+(\pi/2)$ (mod π). Assumption P2 in Definition 2.6 implies $\alpha_*\neq0$. Otherwise, $\vec\theta_0\cdot x_0=0$, $\vec\theta_0\cdot y'(0)=0$, and $x_0=y(0)$ imply that the line through the origin and x_0 is tangent to $\mathcal{S}$. By assumption P4 in Definition 2.6, $\alpha_*\neq\pi/2$. If not, $\kappa\theta_0^\perp\cdot x_0=0$ is rational.

Let $\alpha_{s,m}$ satisfy

$$\phi(\alpha_{s,m})=s,\ |s|\le m\kappa r_x,\ s\in(1/2)\mathbb{Z},\ \alpha_{s,m}\in[0,\pi/2]. \tag{7.2}$$

See Figure 8, where $\alpha_{s,m}$ are shown as thick dots for integer values of s (and without the subscript m). If $\alpha_*\in(0,\pi/2)$, then for some s there may be two solutions:

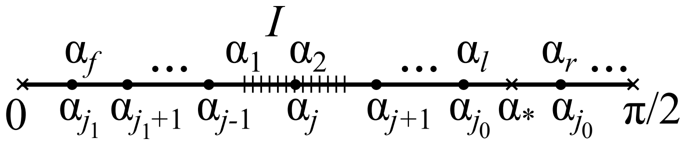

FIG. 8. *Illustration of the interval* $[0,\pi/2]$ *with various angles used in the derivation of the estimates.*

$\alpha_{s,m}\in(0,\alpha_*)$ and $\alpha_{s,m}\in(\alpha_*,\pi/2]$. If $\alpha_*\notin(0,\pi/2)$, there is at most one solution for each s. Note that the solution(s) may exist only for some of the indicated s. By assumption P3 in Definition 2.6, κr_x is irrational, so $\alpha_{s,m}\neq\alpha_*$ for any s,m. By assumption P4, $\phi(0)$ is irrational, so $\alpha_{s,m}\neq 0$ for any s,m.

Consider an interval $I\subset[0,\pi/2]$ and its rescaled versions:

$$\hat I:=(1/\Delta\alpha)I,\ \tilde I:=\epsilon^{-1/2}I. \tag{7.3}$$

In view of (4.6) and (4.12), consider the expression

$$W_m(I):=\epsilon^{1/2}\left|\sum_{\tilde\alpha_k\in\tilde I}e\left(-mq_k\right)\left[g(\tilde\alpha_k)e\left(-m\vec\alpha_k\cdot\check x\right)\right]\right|,\ q_k:=\frac{\vec\alpha_k\cdot x_0}{\epsilon}. \tag{7.4}$$

The dependence of $W_m(I)$ on ϵ and $\hat t$ is omitted from notations. The goal is to estimate the sum $\sum_{m\neq0}W_m([0,\pi/2])$. Obviously, $W_m([0,\pi/2])\le\sum_j W_m(I_j)$ if $\cup_j I_j=[0,\pi/2]$. From (6.7),

$$W_m([0,\pi/2])\le O(\epsilon^{1/2})\rho(m)\sum_{k=0}^{1/\epsilon}(1+\epsilon^{1/2}k)^{-1}=\rho(m)O(\ln(1/\epsilon)) \tag{7.5}$$

because there are $O(1/\epsilon)$ values $\alpha_k\in[0,\pi/2]$. Therefore,

$$\sum_{|m|\ge c\epsilon^{-1/2}}W_m([0,\pi/2])\le O(\ln(1/\epsilon))\sum_{m\ge c\epsilon^{-1/2}}\rho(m)=O(\epsilon\ln(1/\epsilon)) \tag{7.6}$$

if $\rho(m)=O(|m|^{-3})$ and $c>0$. Thus, in what follows, we will assume $\epsilon^{1/2}|m|\le c$ for some small $c>0$.

Next we investigate the individual $W_m(I)$ for smaller intervals I. For this we need a partial integration identity [14, p. 89] (written in a slightly different form):

$$\sum_{k=K_1}^{K_2}G(k)\Phi(k)=G(K_2)\sum_{k=K_1}^{K_2}\Phi(k)-\int_{K_1}^{K_2}G'(\tau)\sum_{k=K_1}^{\tau}\Phi(k)\mathrm{d}\tau, \tag{7.7}$$

where $G(\tau)$ is continuously differentiable on the interval $[K_1,K_2]$. Here and throughout the paper, $\sum_{k=c_1}^{c_2}$, where $c_{1,2}$ are not necessarily integers, denotes the sum over $k\in[c_1,c_2]$. Hence

$$\left|\sum_{k=K_1}^{K_2}G(k)\Phi(k)\right|\le\left(|G(K_2)|+\int_{K_1}^{K_2}|G'(\tau)|\mathrm{d}\tau\right)\max_{k'\in[K_1,K_2]}\left|\sum_{k=K_1}^{k'}\Phi(k)\right|. \tag{7.8}$$

The following result is also needed [9, p. 7].

THEOREM 7.1 (Kusmin–Landau inequality). *If $\vartheta(\tau)$ is continuously differentiable, $\vartheta'(\tau)$ is monotonic, and $\langle\vartheta'(\tau)\rangle \geq \lambda > 0$ on an interval $\hat{I}$, then there exists $c > 0$ (independent of ϑ and $\hat{I}$) such that*

$$\left|\sum_{n\in\hat{I}} e(\vartheta(n))\right| \leq c/\lambda. \tag{7.9}$$

In view of (7.4), (7.7), define

$$G(\hat{\alpha}) := g(\tilde{\alpha})e\left(-m\vec{\alpha}\cdot\check{x}\right), \Phi(\hat{\alpha}) := e(\vartheta(\hat{\alpha})), \tag{7.10}$$

where $\vartheta(\hat{\alpha})$ is defined in (7.1).

LEMMA 7.2. *One has, for any $b, L \geq 0$, $L = O(1)$, $[b, b+L] \subset \epsilon^{-1/2}\Omega$,*

$$\int_b^{b+L} \sum_{n:|v_n-\tilde{\alpha}|\leq\delta} \frac{1}{1+\tilde{\alpha}^2(v_n-\tilde{\alpha})^2}\, d\tilde{\alpha} = O((1+b)^{-1}). \tag{7.11}$$

COROLLARY 7.3. *One has, for any $b, L \geq 0$, $L = O(1)$, $b + L \leq \epsilon^{-1/2}\pi/2$,*

$$\int_b^{b+L} |\partial_{\tilde{\alpha}} g(\tilde{\alpha})|\, d\tilde{\alpha} = \rho(m)O((1+b)^{-1}). \tag{7.12}$$

The assertion is obvious because (a) the second term in brackets in (6.7) is $O(\tilde{\alpha}^{-2})$ and (b) by (6.7), $\partial_{\tilde{\alpha}} g(\tilde{\alpha})$ satisfies the same estimate as in (7.11) if $\tilde{\alpha} \notin \epsilon^{-1/2}\Omega$ (with $\rho(m)$ accounted for).

Pick any interval $I \subset [0, \pi/2]$ such that $|\tilde{I}| = O(1)$. From (7.10),

$$\int_{\hat{I}} |G'(\hat{\alpha})|\, d\hat{\alpha} \leq c\int_{\tilde{I}} \left(|\partial_{\tilde{\alpha}} g(\tilde{\alpha})| + \epsilon^{1/2}|m||g(\tilde{\alpha})|\right) d\tilde{\alpha}. \tag{7.13}$$

Also, by (6.7) and (7.12),

$$\begin{aligned} &|G(\hat{\alpha}_0)| = |g(\tilde{\alpha}_0)| = \rho(m)O((1+\tilde{\alpha}_0)^{-1}),\\ &\int_{\tilde{I}} |g(\tilde{\alpha})|\, d\tilde{\alpha}, \int_{\tilde{I}} |\partial_{\tilde{\alpha}} g(\tilde{\alpha})|\, d\tilde{\alpha} = \rho(m)O((1+\tilde{\alpha}_0)^{-1}) \text{ for any } \alpha_0 \in I. \end{aligned} \tag{7.14}$$

Hence

$$\max_{\hat{\alpha}\in\hat{I}} |G(\hat{\alpha})| + \int_{\hat{I}} |G'(\hat{\alpha})|\, d\hat{\alpha} = \rho(m)O((1+\tilde{\alpha}_0)^{-1}) \text{ for any } \alpha_0 \in I, \tag{7.15}$$

where we have used that $\epsilon^{1/2}m \leq c$.

We suppose that $\alpha = \alpha_*$ is a local maximum of $\phi(\alpha)$. The case when α_* is a local minimum is analogous.

Let $[0, \alpha_f]$ be the shortest interval such that $\phi(\alpha_f)$ is an integer, i.e., $\phi(\alpha_f) \in \mathbb{Z}$ and $\phi(\alpha) \notin \mathbb{Z}$ for any $\alpha \in [0, \alpha_f)$. Similarly, let $[\alpha_l, \alpha_r]$ be the shortest interval centered at α_* such that $\phi(\alpha_l) = \phi(\alpha_r) \in \mathbb{Z}$ and $\phi(\alpha) \notin \mathbb{Z}$ for any $\alpha \in (\alpha_l, \alpha_r)$. Clearly, $\phi(\alpha_l) = \phi(\alpha_r) = \lfloor\phi(\alpha_*)\rfloor$. Set

$$I^{(0)} := [0, \pi/2] \setminus ([0, \alpha_f] \cup [\alpha_l, \alpha_r]). \tag{7.16}$$

Thus, $W_m([0, \pi/2]) \leq W_m(I^{(0)}) + W_m([0, \alpha_f]) + W_m([\alpha_l, \alpha_r])$.

7.2. Towards estimation of $W_m(I^{(0)})$. In this subsection we assume $\alpha_f < \alpha_l$, i.e., $\lceil\phi(0)\rceil \le \lfloor\phi(\alpha_*)\rfloor$, because otherwise $I^{(0)} = \varnothing$, and $W_m(I^{(0)}) = 0$.

Pick any interval $I := [\alpha_1, \alpha_2] \subset I^{(0)}$ (if one exists) such that $2\phi(\alpha_{1,2}) \in \mathbb{Z}$, $|\phi(\alpha_2) - \phi(\alpha_1)| = 1/2$, and $2\phi(\alpha) \notin \mathbb{Z}$ for any $\alpha \in (\alpha_1, \alpha_2)$. By construction, $\phi(\alpha)$ is monotone on I. Let j be the integer value in the pair $\phi(\alpha_1), \phi(\alpha_2)$, i.e., $\alpha_1 = \alpha_{j,m}$ or $\alpha_2 = \alpha_{j,m}$ (see (7.2)). The other value in the pair is $j - 1/2$ or $j + 1/2$.

Generically, one has

$$|\cos(\alpha_1 - \alpha_x)| \asymp |\cos(\alpha_2 - \alpha_x)|. \tag{7.17}$$

The only exceptions are the two cases when $[\alpha_1, \alpha_2]$ is close to α_*: $\alpha_2 = \alpha_l$ and $\alpha_1 = \alpha_r$. In these cases one of the expressions in (7.17) can be arbitrarily close to zero (as $m \to \infty$), while the other can stay away from zero. If $\alpha_2 = \alpha_l$, then $\phi(\alpha_2) = j$ and $\phi(\alpha_1) = j - 1/2$. If $\alpha_1 = \alpha_r$, then $\phi(\alpha_1) = j$ and $\phi(\alpha_2) = j - 1/2$. Therefore, when $m \gg 1$,

$$\begin{aligned} &|\cos(\alpha_1 - \alpha_x)| > |\cos(\alpha_{j,m} - \alpha_x)| \text{ if } \alpha_{j,m} = \alpha_l, \\ &|\cos(\alpha_2 - \alpha_x)| > |\cos(\alpha_{j,m} - \alpha_x)| \text{ if } \alpha_{j,m} = \alpha_r. \end{aligned} \tag{7.18}$$

Clearly, $\alpha_2 - \alpha_1 = O(m^{-1/2})$. Away from a neighborhood of α_*, this difference is actually $O(1/m)$. To cover all the cases, we use a more conservative estimate.

Subdivide $\tilde{I}$ into N subintervals of length $\asymp 1$ (see Figure 8):

$$\begin{aligned} &\tilde{I} = \cup_{n=0}^{N-1} [\tilde{\alpha}_{j,m} - (n+1)L, \tilde{\alpha}_{j,m} - nL],\ \tilde{\alpha}_{j,m} - NL = \tilde{\alpha}_1 \text{ if } \alpha_2 = \alpha_{j,m}, \\ &\tilde{I} = \cup_{n=0}^{N-1} [\tilde{\alpha}_{j,m} + nL, \tilde{\alpha}_{j,m} + (n+1)L],\ \tilde{\alpha}_{j,m} + NL = \tilde{\alpha}_2 \text{ if } \alpha_1 = \alpha_{j,m}. \end{aligned} \tag{7.19}$$

Clearly, $\tilde{\alpha}_2 - \tilde{\alpha}_1 \ge c/(\epsilon^{-1/2} m)$, so the requirement $\epsilon^{-1/2} m \le c$ implies $\tilde{\alpha}_2 - \tilde{\alpha}_1 \ge c$. Hence we can choose $N = 1$ and $L = \tilde{\alpha}_2 - \tilde{\alpha}_1$ if $\tilde{\alpha}_2 - \tilde{\alpha}_1 < 1$, and $N = \lfloor\tilde{\alpha}_2 - \tilde{\alpha}_1\rfloor$ and $L = (\tilde{\alpha}_2 - \tilde{\alpha}_1)/N$ if $\tilde{\alpha}_2 - \tilde{\alpha}_1 \ge 1$.

One has

$$\begin{aligned} &|\sin\alpha - \sin\alpha_1| \ge (\alpha - \alpha_1)\min(|\cos\alpha_1|, |\cos\alpha_2|),\ \alpha \in [\alpha_1, \alpha_2], \\ &|\sin\alpha_2 - \sin\alpha| \ge (\alpha_2 - \alpha)\min(|\cos\alpha_1|, |\cos\alpha_2|),\ \alpha \in [\alpha_1, \alpha_2], \end{aligned} \tag{7.20}$$

for any $\alpha_1 < \alpha_2$ such that $\sin\alpha \ne 0$ on the interval (α_1, α_2). The statement is immediate in view of the mean value theorem and the monotonicity of $\cos\alpha$ on the interval $[\alpha_1, \alpha_2]$.

By construction, j is the integer closest to $\vartheta'(\hat{\alpha})$ if $\hat{\alpha} \in \hat{I}$:

$$\langle\vartheta'(\hat{\alpha})\rangle = |\vartheta'(\hat{\alpha}) - j| = |\vartheta'(\hat{\alpha}) - \vartheta'(\hat{\alpha}_{j,m})| \text{ if } \hat{\alpha} \in \hat{I}. \tag{7.21}$$

From (7.17), (7.18), (7.20), and (7.21),

$$\begin{aligned} &\langle\vartheta'(\hat{\alpha})\rangle \ge c(\epsilon^{1/2} nL) m\kappa r_x |\cos(\alpha_{j,m} - \alpha_x)| = c\epsilon^{1/2} n((m\kappa r_x)^2 - j^2)^{1/2} \\ &\text{if } \tilde{\alpha} \in [\tilde{\alpha}_{j,m} - (n+1)L, \tilde{\alpha}_{j,m} - nL] \text{ or } \tilde{\alpha} \in [\tilde{\alpha}_{j,m} + nL, \tilde{\alpha}_{j,m} + (n+1)L]. \end{aligned} \tag{7.22}$$

This follows from the top inequality in (7.20) if $\alpha_1 = \alpha_{j,m}$ and from the bottom one if $\alpha_2 = \alpha_{j,m}$. Also,

$$\tilde{\alpha} \ge c\tilde{\alpha}_{j,m} \text{ if } \alpha \in I. \tag{7.23}$$

Indeed, notice that $\tilde{\alpha} \ge \tilde{\alpha}_{j,m}$ if $\alpha_1 = \alpha_{j,m}$. If $\alpha_2 = \alpha_{j,m}$, then it suffices to assume that $m \gg 1$ is large enough. From $\alpha_* \ne 0$, $\alpha_* \notin [0, c']$ for some $c' > 0$. By construction,

$|\phi(\alpha_1)-\phi(0)|>0.5$, $|\phi(\alpha_2)-\phi(0)|>1$, and $|\phi(\alpha_2)-\phi(\alpha_1)|=1/2$. If $m\gg 1$, (7.23) is obvious if $\alpha_2>c'$, and it follows from the mean value theorem if $[\alpha_1,\alpha_2]\subset[0,c']$.

The partial integration identity (7.7) and the Kusmin–Landau inequality (7.9) imply

$$\begin{aligned}W_m(I)&\le c\epsilon^{1/2}\rho(m)\left(\frac{\epsilon^{-1/2}}{\tilde\alpha_{j,m}}+\frac{\epsilon^{-1/2}}{\tilde\alpha_{j,m}((m\kappa r_x)^2-j^2)^{1/2}}\sum_{n=1}^{N}\frac{1}{n}\right)\\&\le c\frac{\rho(m)}{\tilde\alpha_{j,m}}\left(1+\frac{\ln(1/(\epsilon m))}{((m\kappa r_x)^2-j^2)^{1/2}}\right).\end{aligned}\tag{7.24}$$

The first term in parentheses on the first line in (7.24) bounds the contribution from the subinterval $[\tilde\alpha_{j,m}-L,\tilde\alpha_{j,m}]$ or $[\tilde\alpha_{j,m},\tilde\alpha_{j,m}+L]$ (depending on the case), which is adjacent to $\tilde\alpha_{j,m}$. Since $\phi(\alpha_{j,m})=j$, we cannot use the Kusmin–Landau inequality, so $W_m(\cdot)$ for this subinterval is estimated directly from (7.4) using the top line in (6.7) and (7.23).

Clearly, $I^{(0)}$ can be represented as a union of intervals $I=[\alpha_1,\alpha_2]$ of the kind considered in this subsection. Summing the estimates in (7.24) for all $I\subset I^{(0)}$ to obtain a bound for $W_m(I^{(0)})$ is done in subsection 7.5 below. Therefore, it is left to consider $W_m([\alpha_l,\alpha_r])$ and $W_m([0,\alpha_f])$.

7.3. Estimation of $W_m([\alpha_l,\alpha_r])$. Suppose first that $\alpha_*\in(0,\pi/2)$. Since α_* is a local maximum, $\phi(\alpha)$ is increasing on $[\alpha_l,\alpha_*]$ and decreasing on $[\alpha_*,\alpha_r]$. When $m\gg 1$ is sufficiently large, we have $0<\alpha_l<\alpha_*<\alpha_r\le\pi/2$.

Suppose $\alpha_l>0$ (see Appendix C if this does not hold). Then $\tilde\alpha\asymp\epsilon^{-1/2}$ if $\tilde\alpha\in[\tilde\alpha_l,\tilde\alpha_r]$. Split $[\tilde\alpha_l,\tilde\alpha_*]$ into $N=\lfloor\tilde\alpha_*-\tilde\alpha_l\rfloor$ subintervals of length $L\asymp 1$:

$$[\tilde\alpha_l,\tilde\alpha_*]=\cup_{n=0}^{N-1}[\tilde\alpha_l+nL,\tilde\alpha_l+(n+1)L],\ \tilde\alpha_l+NL=\tilde\alpha_*.\tag{7.25}$$

Since $\tilde\alpha_*-\tilde\alpha_l\asymp[\{m\kappa r_x\}/(\epsilon m)]^{1/2}$, we have

$$c[\langle m\kappa r_x\rangle/(\epsilon m)]^{1/2}\le\tilde\alpha_*-\tilde\alpha_l\le c/(\epsilon m)^{1/2}.\tag{7.26}$$

Applying (7.15) to each of the subintervals in (7.25) gives the estimate $\rho(m)O((1+\tilde\alpha_*)^{-1})=\rho(m)O(\epsilon^{1/2})$. Also,

$$\begin{aligned}\langle\vartheta'(\hat\alpha)\rangle&\ge\min\left(\frac{nL}{\tilde\alpha_*-\tilde\alpha_l}\{\phi(\alpha_*)\},1-\{\phi(\alpha_*)\}\right)\\&\ge\min\left(cn(\epsilon m\langle m\kappa r_x\rangle)^{1/2},\langle m\kappa r_x\rangle\right),\ \tilde\alpha\in[\tilde\alpha_l+nL,\tilde\alpha_l+(n+1)L].\end{aligned}\tag{7.27}$$

Completely analogous estimates hold for $[\tilde\alpha_*,\tilde\alpha_r]$ if $\alpha_r\le\pi/2$ (see also Appendix C). Therefore,

$$\begin{aligned}W_m([\alpha_l,\alpha_r])&\le c\epsilon^{1/2}\rho(m)\left(1+\frac{\epsilon^{1/2}}{(\epsilon m)^{1/2}\langle m\kappa r_x\rangle}+\frac{1}{(m\langle m\kappa r_x\rangle)^{1/2}}\sum_{n=1}^{N-1}\frac{1}{n}\right)\\&\le c\epsilon^{1/2}\rho(m)\left(\frac{1}{m^{1/2}\langle m\kappa r_x\rangle}+\frac{\ln(1/(\epsilon m))}{(m\langle m\kappa r_x\rangle)^{1/2}}\right),\ \tilde\alpha_*-\tilde\alpha_l\ge 1.\end{aligned}\tag{7.28}$$

The first term in parentheses on the first line in (7.28) corresponds to the subinterval $[\tilde\alpha_l,\tilde\alpha_l+L]$, since $\phi(\tilde\alpha_l)\in\mathbb{Z}$ and its contribution is estimated directly from (7.4) using the top line in (6.7).

If $\tilde\alpha_*-\tilde\alpha_l<1$, we can estimate $W_m([\alpha_l,\alpha_r])$ directly from (7.4). There are $O(\epsilon^{-1/2})$ terms in the sum; each of them is $O(\tilde\alpha_*^{-1})=O(\epsilon^{1/2})$, so

$$W_m([\alpha_l,\alpha_r])\le c\epsilon^{1/2}\rho(m),\ \tilde\alpha_*-\tilde\alpha_l<1.\tag{7.29}$$

7.4. Estimation of $W_m([0,\alpha_f])$. Since α_* is a local maximum, $\phi(\alpha)$ is increasing on $[0,\alpha_*]$. If there is no $\alpha \in (0,\alpha_*)$ such that $\phi(\alpha)\in\mathbb{Z}$, then $\alpha_l<0$, and this case is addressed in Appendix C. Therefore, in this subsection we assume that $\phi(\alpha_{j_1,m})=j_1$, where $j_1:=\lceil\phi(0)\rceil$ for some $\alpha_{j_1,m}\in(0,\alpha_*)$. Clearly, $\alpha_f=\alpha_{j_1,m}$.

Split $[0,\tilde\alpha_{j_1,m}]$ into $N=\lfloor\tilde\alpha_{j_1,m}\rfloor$ intervals of length $L\asymp 1$:

$$[0,\tilde\alpha_{j_1,m}]=\cup_{n=0}^{N-1}[nL,(n+1)L],\ NL=\tilde\alpha_{j_1,m} \text{ if } \tilde\alpha_{j_1,m}\geq 2. \tag{7.30}$$

Since $\tilde\alpha_{j_1,m}\asymp(1-\{m\mu\})/(\epsilon^{1/2}m)$, we have

$$c\langle m\mu\rangle/(\epsilon^{1/2}m)\leq\tilde\alpha_{j_1,m}\leq c/(\epsilon^{1/2}m). \tag{7.31}$$

Applying (7.15) to the interval $\hat I_n=(\kappa\epsilon^{1/2})^{-1}[nL,(n+1)L]$ gives

$$\max_{\hat\alpha\in\hat I_n}|G(\hat\alpha)|+\int_{\hat I_n}|G'(\hat\alpha)|\mathrm{d}\hat\alpha=\rho(m)O((1+n)^{-1}). \tag{7.32}$$

By the bottom line in (7.20),

$$j_1-\phi(\alpha)\geq\epsilon^{1/2}(N-(n+1))L(m\kappa r_x)\min(|\cos(-\alpha_x)|,|\cos(\alpha_{j_1,m}-\alpha_x)|) \tag{7.33}$$

if $\tilde\alpha\in[nL,(n+1)L]$; therefore

$$\langle\vartheta'(\hat\alpha)\rangle\geq\min\left(\{m\mu\},c\epsilon^{1/2}m(N-(n+1))\right),\ \hat\alpha\in\hat I_n, \tag{7.34}$$

because $|\cos(-\alpha_x)|,|\cos(\alpha_{j_1,m}-\alpha_x)|\asymp 1$. Combining the inequalities, using (7.31), and simplifying give

$$\begin{aligned} W_m([0,\alpha_{j_1,m}])&\leq c\epsilon^{1/2}\rho(m)\bigg(\frac{\epsilon^{-1/2}}{\tilde\alpha_{j_1,m}}+\frac{1}{\{m\mu\}}\sum_{n=0}^{N-2}\frac{1}{1+n}\\ &\quad+\frac{1}{\epsilon^{1/2}m}\sum_{n=0}^{N-2}\frac{1}{(1+n)(N-(n+1))}\bigg)\\ &\leq c\epsilon^{1/2}\rho(m)\frac{m+\ln\left(1/(\epsilon^{1/2}m)\right)}{\langle m\mu\rangle},\ \tilde\alpha_{j_1,m}\geq 2. \end{aligned} \tag{7.35}$$

The first term in parentheses on the first line in (7.35) bounds the contribution from the last subinterval $\tilde I_{N-1}=[\tilde\alpha_{j_1,m}-L,\tilde\alpha_{j_1,m}]$. Since $\phi(\alpha_{j_1,m})=j_1$, we cannot use the Kusmin–Landau inequality, so $W_m(I_{N-1})$ is estimated directly from (7.4) using the top line in (6.7). From these two equations we get also

$$W_m([0,\alpha_{j_1,m}])\leq c\rho(m),\ \tilde\alpha_{j_1,m}<2. \tag{7.36}$$

7.5. Combining all the estimates. We begin by summing (7.24) over all the intervals $I=[\alpha_1,\alpha_2]\subset I^{(0)}$ in order to finish estimating $W_m(I^{(0)})$. The analysis in subsection 7.2 shows that $W_m(I)$ admits the same bound (7.24) regardless of whether $\alpha_1=\alpha_{j,m}$ or $\alpha_2=\alpha_{j,m}$. Hence we need to sum the right-hand side of (7.24) over all integers $j\in\phi(I^{(0)})$. Recall that $\alpha_{j,m}$ denote the angles such that $\phi(\alpha_{j,m})=j$ (see (7.2)). We need to distinguish two cases: $0<\alpha_{j,m}\leq\min(\alpha_l,\pi/2)$ and $\alpha_r\leq\alpha_{j,m}<\pi/2$. The latter case may occur only if $\alpha_*<\pi/2$.

First, suppose $0<\alpha_{j,m}\leq\min(\alpha_l,\pi/2)$. Denote $j_1:=\lceil\phi(0)\rceil$, $j_0:=\lfloor\phi(\alpha_*)\rfloor$. Then

$$\alpha_{j,m}\geq[\langle m\mu\rangle+(j-j_1)]/(m\kappa r_x),\ j_1\leq j\leq j_0, \tag{7.37}$$

and

$$
\begin{aligned}
W_m([\alpha_f,\alpha_l]) \le c\epsilon^{1/2}\rho(m)\Bigg[&\frac{m}{\langle m\mu\rangle}\left(1+\frac{\ln(1/(\epsilon m))}{m}\right)+\left(1+\frac{\ln(1/(\epsilon m))}{m^{1/2}\langle m\kappa r_x\rangle^{1/2}}\right)\\
&+\sum_{j=j_1+1}^{j_0-1}\frac{m}{j-j_1}\left(1+\frac{\ln(1/(\epsilon m))}{m^{1/2}(m\kappa r_x-j)^{1/2}}\right)\Bigg].
\end{aligned}
\tag{7.38}
$$

The first term in brackets corresponds to $j=j_1$, the second term to $j=j_0$, and the sum to all $j_1<j<j_0$. If $j_0-j_1=1$, the sum is assumed to be zero.

If $\alpha_l>\pi/2$, $W_m([\alpha_f,\pi/2])$ is assumed instead of $W_m([\alpha_f,\alpha_l])$. Clearly, $m\kappa r_x-j\ge 1$ if $j_1<j<j_0$. Simplifying and keeping only the dominant terms, we have

$$
W_m([\alpha_f,\alpha_l]) \le c\epsilon^{1/2}\rho(m)\left(\frac{m+\ln(1/(\epsilon m))}{\langle m\mu\rangle}+\frac{\ln(1/(\epsilon m))}{m^{1/2}\langle m\kappa r_x\rangle^{1/2}}\right).
\tag{7.39}
$$

If $\alpha_r<\pi/2$ and $\alpha_r<\alpha_{j,m}\le\pi/2$, then $\tilde\alpha_{j,m}\asymp\epsilon^{-1/2}$, so (7.24) gives

$$
W_m([\alpha_r,\pi/2]) \le c\epsilon^{1/2}\rho(m)\left(m+\frac{\ln(1/(\epsilon m))}{m^{1/2}}\left[\langle m\kappa r_x\rangle^{-1/2}+m^{1/2}\right]\right).
\tag{7.40}
$$

Here we used that there are $O(m)$ distinct values of j. Clearly, the estimate in (7.39) dominates the one in (7.40). This implies that $W_m(I^{(0)})$ satisfies the estimate in (7.39).

We have

$$
\rho(m)=O(m^{-\beta}),\ m\to\infty,\ \ \langle m\mu\rangle, \langle m|x_0|\rangle\ge c_\eta m^{-\eta},\ m\in\mathbb{N},
\tag{7.41}
$$

for any $\eta>\eta_0$ and some $c_\eta>0$ (see conditions P3, P4 in Definition 2.6). By summing each of the estimates (7.28) (contribution of $[\alpha_l,\alpha_r]$), (7.35) (contribution of $[0,\alpha_f]$), and (7.39) (contribution of $I^{(0)}$) with respect to m from 1 to ∞, we see that the dominating term is $\epsilon^{1/2}\sum_{m=1}^{\infty}\rho(m)m/\langle m\mu\rangle$ (cf. (7.35) and (7.39)). The series converges if $\beta>\eta_0+2$.

Next consider special cases. Comparing (7.29) and (7.36), we see that the latter grows faster as $m\to\infty$. From (7.31) and (7.36),

$$
\sum_{m\ge1,\tilde\alpha_{j_1,m}<2} W_m([0,\alpha_{j_1,m}]) \le c \sum_{\substack{m\ge1\\ \langle m\mu\rangle/(\epsilon^{1/2}m)\le c}} \rho(m) = O(\epsilon^{(\beta-1)/(2(\eta+1))}) = O(\epsilon^{1/2})
\tag{7.42}
$$

if $\beta\ge\eta_0+2$.

The contribution of the exceptional cases that take place for finitely many m (see Appendix C) is of order $O(\epsilon^{1/2}\ln(1/\epsilon))$. Hence we proved that

$$
\sum_{1\le|m|\le O(\epsilon^{-1/2})} W_m([-\pi/2,\pi/2]) = O(\epsilon^{1/2}\ln(1/\epsilon)) \text{ if } \beta>\eta_0+2.
\tag{7.43}
$$

Given that $\hat t$ is confined to a bounded interval and (7.43) is uniform with respect to $\hat t$ (see the paragraph following (4.12)), we prove (4.6) in the case $x_0\in\mathcal{S}$.

8. Proof of (4.6) in case (B): Preparation.

8.1. Preliminaries. Now we consider the case $x_0 \notin \mathcal{S}$. On a high level, our approach is similar to the one in case (A). In section 8 we show after several simplifications that the analogue of $g(\tilde{\alpha})$ decays sufficiently fast with its derivative (see Lemma 8.6). In section 9 we break up the sum with respect to α_k in (4.6) into smaller sums and use the same methods as in section 7 to estimate the latter. Adding all the estimates and summing with respect to $m \neq 0$ proves (4.6).

The following result is proven in [21].

LEMMA 8.1. *Pick $x_0 \notin \mathcal{S}$ such that no line through x_0, which intersects $\mathcal{S}$, is tangent to $\mathcal{S}$. This includes the endpoints of $\mathcal{S}$, in which case the one-sided tangents to $\mathcal{S}$ are considered. Under the assumptions of Theorem* 2.7, *one has*

$$f_\epsilon^{\text{p-rec}}(x) = O(\epsilon^{1/2}\ln(1/\epsilon)),\ \epsilon \to 0, \tag{8.1}$$

uniformly with respect to x in a sufficiently small neighborhood of x_0.

To make this paper self-contained, a slightly simplified proof of the lemma is in Appendix D. By the above lemma and a partition of unity, only a small neighborhood of a point of tangency can be considered. Therefore, in this section also we can assume that $\mathcal{S}$ is as short ($a > 0$ as small) as we like.

In this section we use R_1 in its original form (see (4.10)),

$$R_1(\theta,\alpha) = \vec{\alpha}\cdot(y(\theta)-x_0)\,, \tag{8.2}$$

because $x_0 \neq y(0)$. Let $\alpha = \mathcal{A}_1(\theta)$: $[-a,a] \to [-\pi/2,\pi/2]$ be the function such that $\vec{\alpha}(\mathcal{A}_1(\theta))\cdot(y(\theta)-x_0) \equiv 0$. Suppose, for example, that x_0 is on the side of $\mathcal{S}$ for which $\mathcal{A}_1(\theta) \geq 0$. The case when $\mathcal{A}_1(\theta) \leq 0$ is analogous. In contrast with section 5, $\mathcal{A}_1$ is now quadratic near $\theta = 0$. In what follows we need rescaled versions of the functions R_1 and A_1:

$$R(\tilde{\theta},\hat{\alpha}) := R_1(\theta,\alpha)/\epsilon,\ \mathcal{A}(\tilde{\theta}) := \mathcal{A}_1(\theta)/\Delta\alpha. \tag{8.3}$$

As usual, the dependence of R and $\mathcal{A}$ on ϵ is omitted from notation for simplicity.

LEMMA 8.2. *One has*

$$\mathcal{A}(\tilde{\theta}) \asymp \tilde{\theta}^2,\ \partial_{\tilde{\theta}}\mathcal{A}(\tilde{\theta}) \asymp \tilde{\theta},\ |\theta| \leq a, \tag{8.4}$$

and

$$\begin{aligned} &R(\tilde{\theta},\hat{\alpha}) \asymp \mathcal{A}(\tilde{\theta}) - \hat{\alpha},\ \partial_{\tilde{\theta}}R(\tilde{\theta},\hat{\alpha}) \asymp \tilde{\theta} - \kappa\epsilon^{1/2}\hat{\alpha},\ \partial_{\hat{\alpha}}R(\tilde{\theta},\hat{\alpha}) = O(1),\\ &\partial_{\tilde{\theta}}^2 R(\tilde{\theta},\hat{\alpha}) = O(1),\ \partial_{\tilde{\theta}}\partial_{\hat{\alpha}}R(\tilde{\theta},\hat{\alpha}) = O(\epsilon^{1/2}),\ |\theta| \leq a, |\alpha| \leq \pi/2. \end{aligned} \tag{8.5}$$

The proofs of this and all other lemmas in this section are in Appendix B. From (4.9),

$$\begin{aligned} \epsilon^{1/2}A_m(\alpha,\epsilon) &= \int_{-a\epsilon^{-1/2}}^{a\epsilon^{-1/2}}\int_{\mathbb{R}}\tilde{\psi}_m\left(R(\tilde{\theta},\hat{\alpha})+h(\theta,\alpha)\right)F(\theta,\epsilon\hat{t})\chi(\hat{t},H_0(\tilde{\theta};\epsilon))\mathrm{d}\hat{t}\mathrm{d}\tilde{\theta}\\ &= \int_{\mathbb{R}}\operatorname{sgn}(\hat{t})\sum_n\int_{U_n}\tilde{\psi}_m\left(R(\tilde{\theta},\hat{\alpha})+h(\theta,\alpha)\right)F(\theta,\epsilon\hat{t})\mathrm{d}\tilde{\theta}\mathrm{d}\hat{t}. \end{aligned} \tag{8.6}$$

Clearly,

$$h, F = O(1),\ \partial_{\tilde{\theta}}h, \partial_{\tilde{\theta}}F = O(\epsilon^{1/2}),\ \partial_{\hat{\alpha}}h = O(\epsilon),\quad |\theta| \leq a, |\alpha| \leq \pi/2, \tag{8.7}$$

uniformly with respect to all variables. By (8.6),

$$
\begin{aligned}
A_m(\alpha,\epsilon) &= \epsilon^{-1/2}\int_{\mathbb{R}} \operatorname{sgn}(\hat t) g(\hat\alpha; m,\epsilon,\hat t,\check x)\mathrm{d}\hat t,\\
g(\hat\alpha; m,\epsilon,\hat t,\check x) &:= \sum_n \int_{U_n} \tilde\psi_m\left(R(\tilde\theta,\hat\alpha)+h(\theta,\alpha)\right) F(\theta,\epsilon\hat t)\mathrm{d}\tilde\theta.
\end{aligned} \tag{8.8}
$$

As usual, the arguments $m,\epsilon,\hat t$, and $\check x$ of g are omitted in what follows, and we write $g(\hat\alpha)$.

Our goal is to estimate the sum in (4.6). To do that we first simplify the sum by reducing the range of indices k and simplifying the expression for A_m. Note that we no longer assume $m\ge 1$.

8.2. Simplification of the sum (4.6). From (8.4)–(8.7) and (4.4), it is easy to obtain $g(\hat\alpha)=\rho(m)O(|\hat\alpha|^{-3/2})$, $\hat\alpha\to-\infty$. This implies

$$
\Delta\alpha \sum_m \sum_{\hat\alpha_k\le\hat\alpha_*} |A_m(\alpha_k,\epsilon)| = O(\epsilon^{1/2}) \tag{8.9}
$$

for any fixed $\hat\alpha_*>0$. The meaning of α_* here is different from that in section 7. Similarly to (5.2), introduce the set $\Omega=\operatorname{ran}\mathcal{A}_1$ (with $\hat\Omega=(1/\Delta\alpha)\Omega$ according to our usual convention). We will show that the sum over $\alpha_k\in[0,\pi/2]\setminus\Omega$ makes only a negligible contribution to $f_\epsilon^{\text{p-rec}}$. By (8.9), the contribution of negative α_k and any finite number of $\alpha_k>0$ can be ignored. Introduce the variable τ:

$$
\hat\alpha=\mathcal{A}_{\text{mx}}+\tau,\ \mathcal{A}_{\text{mx}}:=\max(\mathcal{A}(-a\epsilon^{-1/2}),\mathcal{A}(a\epsilon^{-1/2})). \tag{8.10}
$$

Clearly, $\hat\Omega=\operatorname{ran}\mathcal{A}=[0,\mathcal{A}_{\text{mx}}]$.

LEMMA 8.3. *One has*

$$
\Delta\alpha \sum_m \sum_{\hat\alpha_k>\mathcal{A}_{\text{mx}}} |A_m(\alpha_k,\epsilon)| = O(\epsilon\ln(1/\epsilon)). \tag{8.11}
$$

LEMMA 8.4. *One has*

$$
g_1(\hat\alpha):=\sum_n \int_{\substack{\tilde\theta\in U_n\\ |R(\tilde\theta,\hat\alpha)|\ge\delta\hat\alpha^{1/2}}} \tilde\psi_m\left(R(\tilde\theta,\hat\alpha)+h(\theta,\alpha)\right)F(\theta,\epsilon\hat t)d\tilde\theta=\rho(m)O(\hat\alpha^{-1}) \tag{8.12}
$$

as $\hat\alpha\to+\infty$ *for any* $\delta>0$ *as small as we like.*

Equation (8.12) implies

$$
\Delta\alpha \sum_m \sum_{\hat\alpha_k\in[\hat\alpha_*,\mathcal{A}_{\text{mx}}]} |A_m^{(1)}(\alpha_k,\epsilon)| = O(\epsilon^{1/2}\ln(1/\epsilon)), \tag{8.13}
$$

where $A_m^{(1)}$ is obtained by the top line in (8.8) with g replaced by g_1.

The only remaining contribution to $f_\epsilon^{\text{p-rec}}$ comes from

$$
g_2(\hat\alpha):=\sum_n \int_{\substack{\tilde\theta\in U_n\\ |R(\tilde\theta,\hat\alpha)|<\delta\hat\alpha^{1/2}}} \tilde\psi_m\left(R(\tilde\theta,\hat\alpha)+h(\theta,\alpha)\right)F(\theta,\epsilon\hat t)d\tilde\theta,\ \hat\alpha\in\hat\Omega. \tag{8.14}
$$

A simple calculation shows that $g_2(\hat\alpha)=O(\hat\alpha^{-1/2})$, $\hat\alpha\to\infty$. By (8.4) and (8.5),

$$
\begin{aligned}
&\tilde\theta-\mathcal{A}^{-1}(\hat\alpha)=O(\hat\alpha^{1/2}/\mathcal{A}'(\mathcal{A}^{-1}(\hat\alpha)))=O(1),\quad \theta-\Theta(\hat\alpha)=O(\epsilon^{1/2}),\\
&\Theta(\hat\alpha):=\epsilon^{1/2}\mathcal{A}^{-1}(\hat\alpha),\ \hat\alpha\to\infty,\ \hat\alpha\in\hat\Omega,\ |R(\tilde\theta,\hat\alpha)|<\delta\hat\alpha^{1/2}.
\end{aligned} \tag{8.15}
$$

Despite the fact that $\mathcal{A}^{-1}$ is two-valued, (8.15) holds regardless of whether $\theta > 0$ and $\text{ran}\mathcal{A}^{-1} \subset [0,\infty)$ or $\theta < 0$ and $\text{ran}\mathcal{A}^{-1} \subset (-\infty,0]$. Hence by (4.4) we can replace $F(\theta,\epsilon\hat{t})$ and $h(\theta,\alpha)$ with $F(\Theta(\hat{\alpha}),\epsilon\hat{t})$ and $h(\Theta(\hat{\alpha}),\alpha)$, respectively, in (8.14):

$$\begin{aligned} &g_2(\hat{\alpha}) = g_2^+(\hat{\alpha}) + g_2^-(\hat{\alpha}) + \rho(m)O((\epsilon/\hat{\alpha})^{1/2}),\ \hat{\alpha}\in\hat{\Omega},\\ &g_2^\pm(\hat{\alpha}) = F(\Theta(\hat{\alpha}),\epsilon\hat{t})\sum_n \int_{\substack{\pm\tilde{\theta}>0,\tilde{\theta}\in U_n,\\ |R(\tilde{\theta},\hat{\alpha})|<\delta\hat{\alpha}^{1/2}}} \tilde{\psi}_m\left(R(\tilde{\theta},\hat{\alpha}) + h(\Theta(\hat{\alpha}),\alpha)\right) d\tilde{\theta}. \end{aligned} \tag{8.16}$$

The superscript $'+'$ is taken if $\text{ran}\mathcal{A}^{-1} = [0,\infty)$ and $'-'$ if $\text{ran}\mathcal{A}^{-1} = (-\infty,0]$. The same convention is assumed in what follows. In particular, the domain of integration in (8.16) is a subset of $(0,\infty)$ when g_2^+ is computed and a subset of $(-\infty,0)$ when g_2^- is computed. Omitting the big-O term in (8.16) leads to

$$\Delta\alpha \sum_m \sum_{\hat{\alpha}_k\in[\hat{\alpha}_*,\mathcal{A}_{\text{mx}}]} |A_m^{(1)}(\alpha_k,\epsilon) - (A_m^{(2+)}(\alpha_k,\epsilon) + A_m^{(2-)}(\alpha_k,\epsilon))| = O(\epsilon^{1/2}), \tag{8.17}$$

where $A_m^{(2\pm)}$ are obtained by the top line in (8.8) with g replaced by $g_2^\pm$, respectively. Due to this simplification, we can consider

$$g_3^\pm(\hat{\alpha}) := F\sum_n \int_{\substack{\pm\tilde{\theta}>0,\tilde{\theta}\in U_n,\\ |R(\tilde{\theta},\hat{\alpha})|<\delta\hat{\alpha}^{1/2}}} \tilde{\psi}_m\left(R(\tilde{\theta},\hat{\alpha}) + h\right) d\tilde{\theta}, \tag{8.18}$$

where $F = F(\Theta(\hat{\alpha}),\epsilon\hat{t})$, $h = h(\Theta(\hat{\alpha}),\alpha)$ are uniformly bounded and independent of $\tilde{\theta}$. As was done before, set $r_n(\hat{\alpha}) := R(u_n,\hat{\alpha})$. By shifting the index of u_n if necessary, we may suppose that u_0 is the smallest nonnegative u_n. This means that $u_n \geq 0$ if $n \geq 0$ and $u_n < 0$ if $n < 0$. In this case, $u_n \asymp n$ if $|u_n| \geq c$ by Assumption 2.4 (H3). Changing variables $\tilde{\theta} \to r = R(\tilde{\theta},\hat{\alpha})$ gives

$$g_3^\pm(\hat{\alpha}) := F\sum_{\pm n>0} \int_{\substack{r\in R_n\\ |r|\leq\delta\hat{\alpha}^{1/2}}} \tilde{\psi}_m(r+h)\frac{dr}{|(\partial_{\tilde{\theta}}R)(\tilde{\theta},\hat{\alpha})|}, \tag{8.19}$$

where $\tilde{\theta}$ is a function of r and $\hat{\alpha}$. As usual, $+n > 0$ in g_4^+ and $-n > 0$ – in g_4^-. The change of variables is justified because $(\partial_{\tilde{\theta}}R)(\tilde{\theta},\hat{\alpha}) \neq 0$ on the integration domain. Indeed, $\tilde{\theta}$ is bounded away from zero on the domain, and the following result holds.

LEMMA 8.5. *One has*

$$(\partial_{\tilde{\theta}}R)(\tilde{\theta},\hat{\alpha}) \asymp \tilde{\theta} \quad \text{if} \quad |R(\tilde{\theta},\hat{\alpha})| < \delta\hat{\alpha}^{1/2}, \hat{\alpha}\in[\hat{\alpha}_*,\mathcal{A}_{\text{mx}}]. \tag{8.20}$$

Further simplification is achieved by replacing $\tilde{\theta}$ with $\mathcal{A}^{-1}(\hat{\alpha})$ in the argument of $\partial_{\tilde{\theta}}R$. From (8.5) and (8.15),

$$\begin{aligned} &g_3^\pm(\hat{\alpha}) = g_4^\pm(\hat{\alpha}) + \rho(m)O(\hat{\alpha}^{-1}),\\ &g_4^\pm(\hat{\alpha}) := \frac{F}{|(\partial_{\tilde{\theta}}R)(\mathcal{A}^{-1}(\hat{\alpha}),\hat{\alpha})|}\sum_{\pm n>0}\int_{\substack{r\in R_n\\ |r|\leq\delta\hat{\alpha}^{1/2}}} \tilde{\psi}_m(r+h)\mathrm{d}r,\\ &F = F(\Theta(\hat{\alpha}),\epsilon\hat{t}),\ h = h(\Theta(\hat{\alpha}),\epsilon\hat{\alpha}). \end{aligned} \tag{8.21}$$

Neglecting the big-O term in (8.21) leads to a term of magnitude $O(\epsilon^{1/2}\ln(1/\epsilon))$ in $f_\epsilon^{\text{p-rec}}$.

Here is a summary of what we obtained so far:

$$\begin{aligned}&\Delta\alpha\sum_m\sum_{\hat\alpha_k\notin[\hat\alpha_*,\mathcal{A}_{\rm mx}]}|A_m(\alpha_k,\epsilon)|=O(\epsilon^{1/2}),\\&\Delta\alpha\sum_m\sum_{\hat\alpha_k\in[\hat\alpha_*,\mathcal{A}_{\rm mx}]}|A_m(\alpha_k,\epsilon)-(A_m^+(\alpha_k,\epsilon)+A_m^-(\alpha_k,\epsilon))|=O(\epsilon^{1/2}\ln(1/\epsilon)),\end{aligned}\tag{8.22}$$

where (cf. (8.8))

$$A_m^\pm(\alpha,\epsilon):=\epsilon^{-1/2}\int_{\mathbb{R}}g_4^\pm(\hat\alpha;m,\epsilon,\hat t,\check x)\mathrm{d}\hat t.\tag{8.23}$$

The first line in (8.22) follows from (8.9) and (8.11). The second line follows from (8.13), (8.17), and the comment following (8.21).

Finally, we need the following result.

LEMMA 8.6. *One has*

$$|g_4^\pm(\hat\alpha)|\le c\rho(m)\left(\hat\alpha^{-1/2}\left(1+r_{min}(\hat\alpha)\right)^{-1}+\hat\alpha^{-1}\right),\tag{8.24}$$

$$|\partial_{\hat\alpha}g_4^\pm(\hat\alpha)|\le c\rho(m)\left(\frac{1}{\hat\alpha^{1/2}(1+r_{min}^2(\hat\alpha))}+\frac{\epsilon^{1/2}}{\hat\alpha}\right),\tag{8.25}$$

where $\hat\alpha\in[\hat\alpha_*,\mathcal{A}_{\rm mx}]$, *and*

$$r_{min}(\hat\alpha):=\min_{\pm n>0}|\mathcal{A}(u_n)-\hat\alpha|.\tag{8.26}$$

To clarify, in the estimate for g_4^+, the minimum in (8.26) is over $n>0$ and in the estimate for g_4^- over $n<0$.

9. End of proof of (4.6) in case (B): Summation with respect to *k*. Denote $v_n:=\mathcal{A}(u_n)$ and consider the intervals

$$\begin{aligned}&V_n:=[v_{n-0.5},v_{n+0.5}],n>0,\ V_n:=[v_{n+0.5},v_{n-0.5}],n<0,\\&v_{n+0.5}:=(v_n+v_{n+1})/2,\ n\in\mathbb{Z}.\end{aligned}\tag{9.1}$$

Since the function $\mathcal{A}(\tilde\theta)$ in section 8 and section 9 is different from the one in sections 5–7, the v_n's in (9.1) are different from the v_n's in sections 5–7.

Clearly, the estimates in Lemma 8.6 increase if extra points are (formally) added to the list of u_n's. If $u_n\not\asymp n$ (i.e., there are too few u_n's), we can always add more points to the u_n's and enumerate them so that the enlarged collection satisfies $u_n\asymp n$. This property is assumed in what follows.

LEMMA 9.1. *If* $V_n\subset[\hat\alpha_*,\mathcal{A}_{\rm mx}]$, *one has*

$$v_n\asymp n^2,\ |V_n|=|v_{n+0.5}-v_{n-0.5}|\asymp|n|;\ \hat\alpha\asymp v_n,\ r_{min}(\hat\alpha)=|v_n-\hat\alpha|\ \textit{if}\ \hat\alpha\in V_n.\tag{9.2}$$

The proof of the lemma is immediate using Lemma 8.2, (9.1), and that $u_n\asymp n$.

LEMMA 9.2. *For all* V_n *such that* $V_n\subset[\hat\alpha_*,\mathcal{A}_{\rm mx}]$, *one has, as* $n\to\infty$,

$$|g_4^\pm(v_{n+1/2})|=\rho(m)O(n^{-2})\tag{9.3}$$

and

$$\int_{V_n}|g_4^\pm(\hat\alpha)|d\hat\alpha=\rho(m)O(|n|^{-1}\ln|n|),\ \int_{V_n}|\partial_{\hat\alpha}g_4^\pm(\hat\alpha)|d\hat\alpha=\rho(m)O(|n|^{-1}).\tag{9.4}$$

Arguing similarly to (7.5), (7.6), equation (9.4) implies

$$\epsilon^{1/2}\sum_{\hat\alpha_k\in[\hat\alpha_*,\mathcal{A}_{\rm mx}]}|A_m^{\pm}(\alpha_k,\epsilon)|\le\rho(m)\sum_{n=1}^{O(\epsilon^{-1/2})}n^{-1}\ln n=\rho(m)O(\ln^2(1/\epsilon)). \tag{9.5}$$

Here we used Lemma 9.1 and the following two arguments. (i) Since $\hat\alpha_*>0$ can be taken as large as we want, we can select $\hat\alpha_*=\hat\alpha_*(\hat t,\epsilon)$ so that (a) $c_1\le\hat\alpha_*\le c_2$ for some fixed $c_{1,2}>0$ and all $\hat t$ and $\epsilon>0$ and (b) $\hat\alpha_*=v_{n+0.5}$ for some n. (ii) If v_{n_l} is the last of the v_n's in the interval $[\hat\alpha_*,\mathcal{A}_{\rm mx}]$, the sum over $\hat\alpha_k\in[v_{n_l},\mathcal{A}_{\rm mx}]$ can be estimated directly. By (9.2), $\mathcal{A}_{\rm mx}-\hat\alpha_*=O(\mathcal{A}_{\rm mx}^{1/2})$, so the number of the additional $\hat\alpha_k$'s is $O(\mathcal{A}_{\rm mx}^{1/2})$. By (8.24), $g_4^{\pm}(\hat\alpha)=\rho(m)O(\mathcal{A}_{\rm mx}^{-1/2})$, $\hat\alpha_k\in[v_{n_l},\mathcal{A}_{\rm mx}]$. Therefore the contribution of $[v_{n_l},\mathcal{A}_{\rm mx}]$ is $O(\rho(m))$, which is absorbed by the right-hand side of (9.5).

From (9.5),

$$\Delta\alpha\sum_{|m|\ge\ln(1/\epsilon)}\sum_{\hat\alpha_k\in[\hat\alpha_*,\mathcal{A}_{\rm mx}]}|A_m^{\pm}(\alpha_k,\epsilon)|=O(\epsilon^{1/2}) \tag{9.6}$$

if $\rho(m)=O(|m|^{-3})$. Thus, in what follows, we will assume $|m|\le\ln(1/\epsilon)$.

Let $\alpha_m^{(0)}$ be the smallest positive root of the equation $|\phi(\alpha)-\phi(0)|=\langle\phi(0)\rangle/2$. Recall that the function ϕ depends on m. By (7.41) and the restriction on m (recall that $\phi(0)=m\mu$),

$$\langle m\mu\rangle\ge c(\ln(1/\epsilon))^{-\eta},\quad \hat\alpha_m^{(0)}\ge c(\epsilon\ln^{\eta}(1/\epsilon))^{-1}, \tag{9.7}$$

for any $\eta>\eta_0$. Find n such that $\hat\alpha_m^{(0)}\in V_n$ and set $\hat\alpha_m:=v_{n+0.5}$. By construction and (9.2),

$$0<\hat\alpha_m-\hat\alpha_m^{(0)}=O(v_n^{1/2})=O((\hat\alpha_m^{(0)})^{1/2})=O(\epsilon^{-1/2}); \tag{9.8}$$

therefore $\alpha_m-\alpha_m^{(0)}=O(\epsilon^{1/2})$.

In this section we use only two intervals:

$$\hat I_1:=[\hat\alpha_*,\hat\alpha_m],\ \hat I_2:=[\hat\alpha_m,\mathcal{A}_{\rm mx}],\ m\ne0. \tag{9.9}$$

Since $\hat\alpha_*=O(1)$, due to (9.7) we can assume $\hat\alpha_m\ge\hat\alpha_*$. The derivations for $g_4^{\pm}$ are the same, so we drop the superscript. Similarly to (7.4), define

$$W_m(I):=\epsilon^{1/2}\left|\sum_{\hat\alpha_k\in\hat I}e\left(-mq_k\right)\left[g_4(\hat\alpha_k)e\left(-m\vec\alpha_k\cdot\check x\right)\right]\right|. \tag{9.10}$$

The dependence of $W_m(I)$ on ϵ and $\hat t$ is omitted from notations.

Following the method in section 7, we use the partial integration identity (7.8) and the Kusmin–Landau inequality (7.9). The definitions of Φ and ϑ are the same as before, and the definition of G (cf. (7.10)) is modified slightly:

$$G(\hat\alpha):=g_4(\hat\alpha)e\left(-m\vec\alpha\cdot\check x\right). \tag{9.11}$$

We begin by applying (7.8) to the first interval, so we select $[K_1,K_2]$ as the interval such that $k\in[K_1,K_2]$ is equivalent to $\hat\alpha_k\in\hat I_1$. By (9.7) and the choice of $\alpha_m^{(0)}$, α_m,

$$\hat\alpha_m\ge c\langle m\mu\rangle/(\epsilon|m|),\quad \langle\vartheta'(\hat\alpha)\rangle\ge c\langle m\mu\rangle,\ \hat\alpha\in\hat I_1,\ m\ne0. \tag{9.12}$$

From (9.11),

$$\int_{\hat I_1} |G'(\hat\alpha)|\mathrm{d}\hat\alpha \le c\int_{\hat I_1} \left(|\partial_{\hat\alpha} g_4(\hat\alpha)| + \epsilon|m||g_4(\hat\alpha)|\right)\mathrm{d}\hat\alpha. \tag{9.13}$$

Our construction ensures that $\hat I_1 = \cup V_n$, where the union is taken over all $V_n \subset \hat I_1$. There are $N = O((\hat\alpha_m^{(0)})^{1/2})$ intervals V_n such that $V_n \subset \hat I_1$. By (9.3), (9.4),

$$\begin{aligned} &|G(K_2)| \le c|g_4(v_{N+0.5})| \le c\rho(m)N^{-2},\\ &\int_{\hat I_1} |g_4(\hat\alpha)|\mathrm{d}\hat\alpha \le c\rho(m)\sum_{n=1}^{N} O(n^{-1}\ln n) = \rho(m)O(\ln^2 N),\\ &\int_{\hat I_1} |\partial_{\hat\alpha} g_4(\hat\alpha)|\mathrm{d}\hat\alpha \le c\rho(m)\sum_{n=1}^{N} n^{-1} = \rho(m)O(\ln N). \end{aligned} \tag{9.14}$$

Hence

$$|G(K_2)| + \int_{\hat I_1} |G'(\hat\alpha)|\mathrm{d}\hat\alpha = \rho(m)O(\ln N). \tag{9.15}$$

Here we have used that $|m| \le \ln(1/\epsilon)$ implies $\epsilon|m|\ln N < 1$ for $\epsilon > 0$ sufficiently small. For the same reason we can assume $N \ge 1$. By (9.12), (9.15), (7.8), and the Kusmin–Landau inequality,

$$W_m(I_1) \le c\frac{\epsilon^{1/2}\rho(m)}{\langle m\mu\rangle}\ln\left(\frac{\langle m\mu\rangle}{\epsilon|m|}\right),\ m \ne 0. \tag{9.16}$$

The sum over the remaining $V_n \subset \hat I_2$, $N \le n \le O(\epsilon^{-1/2})$, can be estimated easily without utilizing exponential sums. By construction, the left endpoint of $\hat I_2$ coincides with $v_{n+0.5}$ for some n. As was established following (9.5), the contribution of $\hat\alpha_k$ beyond the last $V_n \subset \hat I_2$ is $O(\rho(m))$. Therefore Lemma 9.2 implies

$$\begin{aligned} \sum_{\hat\alpha_k \in \hat I_2} |g_4(\hat\alpha_k)| &\le c\int_{\hat I_2} |g_4(\hat\alpha)|\mathrm{d}\hat\alpha \le c\sum_{V_n \subset \hat I_2}\int_{V_n} |g_4(\hat\alpha)|\mathrm{d}\hat\alpha + O(\rho(m))\\ &\le c\rho(m)\sum_{n=N}^{O(\epsilon^{-1/2})} n^{-1}\ln n \le c\rho(m)(\ln^2(\epsilon^{-1/2}) - \ln^2(N))\\ &= \rho(m)O\left(\ln(1/\epsilon)\ln(|m|/\langle m\mu\rangle)\right). \end{aligned} \tag{9.17}$$

With some abuse of notation, in the two integrals on the top line above, we integrate the upper bound for g_4 obtained in (8.24) (with $\hat\alpha$ replaced by v_n by Lemma 9.1). This bound has better monotonicity properties (i.e., it can be made monotone within V_n on each side of v_n). Otherwise, we would not be able to estimate the sum in terms of an integral. We also used that $\hat\alpha_{k+1} - \hat\alpha_k = 1$.

Thus

$$W_m(I_2) = \rho(m)\ln(|m|/\langle m\mu\rangle)O(\epsilon^{1/2}\ln(1/\epsilon)),\ m \ne 0. \tag{9.18}$$

Comparing (9.16) and (9.18) with (7.39), we see that the case $x_0 \notin \mathcal{S}$ gives no additional restrictions on $\rho(m)$. Similarly to the end of section 7, we use here that the estimates (9.16) and (9.18) are uniform with respect to $\hat{t}$ and $\hat{t}$ is confined to a bounded interval.

Remark 9.3. Note that the order of operations in the proof of Lemma 8.1 (see Appendix D) is as follows:

$$\sum_{\alpha_k}(\cdot) \to \int (\cdot)\mathrm{d}\hat{t} \to \sum_m \int (\cdot)\mathrm{d}\tilde{\theta}. \tag{9.19}$$

In sections 4–9 the order is different:

$$\int (\cdot)\mathrm{d}\tilde{\theta} \to \sum_{\alpha_k}(\cdot) \to \sum_m \int (\cdot)\mathrm{d}\hat{t}. \tag{9.20}$$

10. Proof of (4.7) in cases (A) and (B). Here we prove (4.7). Begin with the case $x_0 = y(0)$, which is case (A). By (4.11), (4.12),

$$\begin{aligned} J_\epsilon &:= \sum_{|\alpha_k|\le\pi/2} \int_{\alpha_k-\Delta\alpha/2}^{\alpha_k+\Delta\alpha/2} |A_0(\alpha,\epsilon) - A_0(\alpha_k,\epsilon)|\,\mathrm{d}\alpha \le O(\epsilon^{1/2}) \int J_\epsilon(\hat{t})\mathrm{d}\hat{t},\\ J_\epsilon(\hat{t}) &:= \sum_{|\alpha_k|\le\pi/2} \max_{|\tilde{\alpha}-\tilde{\alpha}_k|\le\Delta\tilde{\alpha}/2} |\partial_{\tilde{\alpha}} g(\tilde{\alpha})|\,\Delta\tilde{\alpha},\ \Delta\tilde{\alpha} := \kappa\epsilon^{1/2}. \end{aligned} \tag{10.1}$$

As is seen from (6.7), the only term that requires careful estimation is given by

$$\begin{aligned} J_\epsilon^{(1)}(\hat{t}) &:= \Delta\tilde{\alpha} \sum_{\alpha_k\in\Omega} \max_{|\tilde{\alpha}-\tilde{\alpha}_k|\le\Delta\tilde{\alpha}/2} \sum_{n:|v_n-\tilde{\alpha}|\le\delta} g_1(\tilde{\alpha}, v_n),\\ g_1(\tilde{\alpha}, v_n) &:= \left[1 + v_n^2(v_n - \tilde{\alpha})^2\right]^{-1}. \end{aligned} \tag{10.2}$$

Here we used that $1 + \tilde{\alpha}^2(v_n - \tilde{\alpha})^2 \asymp 1 + v_n^2(v_n - \tilde{\alpha})^2$ if $|v_n - \tilde{\alpha}| \le \delta$. Replacing the inner sum with a larger sum over $n : |v_n - \tilde{\alpha}_k| \le 2\delta$ (using that $\delta + (\Delta\alpha/2) \le 2\delta$), we can write

$$J_\epsilon^{(1)}(\hat{t}) \le \Delta\tilde{\alpha} \sum_{|n|\le O(\epsilon^{-1/2})} \sum_{k:|v_n-\tilde{\alpha}_k|\le 2\delta} \max_{|\tilde{\alpha}-\tilde{\alpha}_k|\le\Delta\tilde{\alpha}/2} g_1(\tilde{\alpha}, v_n). \tag{10.3}$$

Since $\partial_{\tilde{\alpha}} g_1(\tilde{\alpha}, v_n) < 0$, $\tilde{\alpha} > v_n$,

$$\begin{aligned} \sum_{k:v_n\le\tilde{\alpha}_k\le v_n+2\delta} \max_{|\tilde{\alpha}-\tilde{\alpha}_k|\le\Delta\tilde{\alpha}/2} g_1(\tilde{\alpha}, v_n) &\le \sum_{0\le k\Delta\tilde{\alpha}\le 2\delta} g_1(v_n + k\Delta\tilde{\alpha}, v_n)\\ &\le 1 + \frac{1}{\Delta\tilde{\alpha}} \int_{v_n}^{\infty} g_1(\tilde{\alpha}, v_n)\mathrm{d}\tilde{\alpha} \le c\left(1 + \frac{1}{\Delta\tilde{\alpha}(1+|v_n|)}\right). \end{aligned} \tag{10.4}$$

The same argument applies to the left of v_n, so by (9.2)

$$J_\epsilon^{(1)}(\hat{t}) \le \sum_{|n|\le O(\epsilon^{-1/2})} \left(O(\epsilon^{1/2}) + (1+n^2)^{-1}\right) = O(1). \tag{10.5}$$

Consider now the remaining terms in (6.7). Define similarly to (10.2):

$$\begin{aligned} J_\epsilon^{(2)}(\hat{t}) &:= \Delta\tilde{\alpha} \sum_{\alpha_k\in\Omega} \max_{|\tilde{\alpha}-\tilde{\alpha}_k|\le\Delta\tilde{\alpha}/2} (1+\tilde{\alpha}^2)^{-1},\\ J_\epsilon^{(3)}(\hat{t}) &:= \Delta\tilde{\alpha} \sum_{\alpha_k\in[-\pi/2,\pi/2]\setminus\Omega} \max_{|\tilde{\alpha}-\tilde{\alpha}_k|\le\Delta\tilde{\alpha}/2} (1+|\tilde{\alpha}|)^{-1}. \end{aligned} \tag{10.6}$$

Arguing analogously to (10.4), (10.5), we get $J_\epsilon^{(l)}(\hat{t})=O(1)$, $l=2,3$. When estimating $J_\epsilon^{(3)}$ we use that the summation is over α_k, which satisfy $c\le|\alpha_k|\le\pi/2$. Combining with (10.5) and substituting into (10.2) lead to $J_\epsilon=O(\epsilon^{1/2})$.

Suppose now $x_0\notin\mathcal{S}$, which is case (B). It is obvious that the continuous analogue of (8.22) works for $m=0$:

$$\begin{aligned}&\int_{\hat\alpha\notin[\hat\alpha_*,\mathcal{A}_{\rm mx}]}|A_0(\alpha,\epsilon)|\mathrm{d}\alpha=O(\epsilon^{1/2}),\\&\int_{\hat\alpha\in[\hat\alpha_*,\mathcal{A}_{\rm mx}]}|A_0(\alpha,\epsilon)-(A_0^+(\alpha,\epsilon)+A_0^-(\alpha,\epsilon))|\mathrm{d}\alpha=O(\epsilon^{1/2}\ln(1/\epsilon)).\end{aligned}\tag{10.7}$$

Hence it remains to estimate

$$J_\epsilon^\pm:=\sum_{\hat\alpha_k\in[\hat\alpha_*,\mathcal{A}_{\rm mx}]}\int_{\alpha_k-\Delta\alpha/2}^{\alpha_k+\Delta\alpha/2}|A_0^\pm(\alpha,\epsilon)-A_0^\pm(\alpha_k,\epsilon)|\mathrm{d}\alpha.\tag{10.8}$$

By (8.23),

(10.9)

$$J_\epsilon^\pm\le c\epsilon^{1/2}\int J_\epsilon^\pm(\hat t)\mathrm{d}\hat t,\quad J_\epsilon^\pm(\hat t):=\sum_{V_n\subset[\hat\alpha_*,\mathcal{A}_{\rm mx}]}\sum_{\hat\alpha_k\in V_n}\max_{|\hat\alpha-\hat\alpha_k|\le1/2}\left|\partial_{\hat\alpha}g_4^\pm(\hat\alpha)\right|+O(1).$$

Here $O(1)$ is the contribution of $\hat\alpha_k$ beyond the last $V_n\subset[\hat\alpha_*,\mathcal{A}_{\rm mx}]$ (see the argument following (9.5)). Clearly, we can assume $\hat\alpha_*>1/2$. By (8.25), (8.26), and (9.2), it is easy to see that

$$\sum_{\hat\alpha_k\in V_n}\max_{|\hat\alpha-\hat\alpha_k|\le1/2}\left|\partial_{\hat\alpha}g_4^\pm(\hat\alpha)\right|\le c\sum_{\hat\alpha_k\in V_n}\left(\frac{1}{v_n^{1/2}(1+(v_n-\hat\alpha_k)^2)}+\frac{\epsilon^{1/2}}{v_n}\right).\tag{10.10}$$

Arguing similarly to (10.4), we conclude that the left-hand side of (10.10) is bounded by the same estimate as the integral of $|\partial_{\hat\alpha}g_4^\pm(\hat\alpha)|$ in (9.4). Combining (10.8)–(10.10) gives the desired result:

$$J_\epsilon^\pm\le c\epsilon^{1/2}\sum_{n=1}^{O(\epsilon^{-1/2})}n^{-1}=O(\epsilon^{1/2}\ln(1/\epsilon)).\tag{10.11}$$

Appendix A. Proofs of lemmas in sections 4–7.

A.1. Proof of Lemma 4.2. By Assumption 2.2 (AF1), $\tilde w(\lambda)=O(|\lambda|^{-(\lceil\beta\rceil+1)})$, $\lambda\to\infty$, where $\tilde w(\lambda)$ is the Fourier transform of w. If t is restricted to any compact set, the estimate for $\tilde\psi_m$ holds because $\widetilde{(\mathcal{H}\varphi')}(\lambda),\tilde w(\lambda)=O(\rho(\lambda))$ (cf. Assumption 2.3 (IK1)) implies

(A.1)

$$\left|\int|\mu|\tilde\varphi(\mu)\tilde w(\mu-\lambda)e^{i(\mu-\lambda)t}\mathrm{d}\mu\right|\le\int|\mu\tilde\varphi(\mu)\tilde w(\mu-\lambda)|\mathrm{d}\mu=O(\rho(\lambda)),\lambda=2\pi m\to\infty.$$

If $|t|\ge c$ for some $c\gg1$ sufficiently large, integrate by parts $\lceil\beta\rceil$ times and use that

$$\max_q|(\partial/\partial q)^{\lceil\beta\rceil}((\mathcal{H}\varphi')(q)w(-q-t))|=O(t^{-2}),\ t\to\infty.\tag{A.2}$$

The argument works because $(\mathcal{H}\varphi')(q)$ is smooth in a neighborhood of any q such that $w(-q-t)\ne0$.

The estimate for $\tilde\psi_m'$ follows by differentiating (4.3) and applying the above argument with w replaced by w'. The argument still works because $w'\in C_0^{\lceil\beta\rceil}(\mathbb{R})$ (by Assumption 2.2 (AF1)).

A.2. Proof of Lemma 5.2. To prove (5.5) we write

$$\vec{\alpha}(\mathcal{A}_1(\theta))\cdot\frac{y(\theta)-y(0)}{\theta}\equiv\vec{\alpha}(\mathcal{A}_1(\theta))\cdot\big(y'(0)+y''(0)(\theta/2)+O(\theta^2)\big)\equiv 0. \tag{A.3}$$

Differentiating with respect to θ and setting $\theta=0$ gives

$$\vec{\theta}_0^{\perp}\cdot y'(0)\mathcal{A}_1'(0)+\vec{\theta}_0\cdot y''(0)(1/2)=0. \tag{A.4}$$

Hence $\mathcal{A}_1'(0)=1/2$ (because $\vec{\alpha}^{\perp}\cdot y'(\alpha)+\vec{\alpha}\cdot y''(\alpha)\equiv 0$), and the desired properties of $\mathcal{A}$ follow by rescaling $\theta\to\tilde{\theta}$ and $\mathcal{A}_1\to\mathcal{A}$.

By the choice of coordinates,

$$\vec{\alpha}\cdot(y(\theta)-y(0))\,/(\theta(\mathcal{A}(\theta)-\alpha)) \tag{A.5}$$

is a sufficiently smooth positive function of $(\alpha,\theta)\in[-\pi/2,\pi/2]\times[-a,a]$. The positivity follows from the following statements: (i) $\mathcal{S}$ is convex; (ii) by construction, $\vec{\alpha}^{\perp}(0)\cdot y'(0)<0$ and $\vec{\alpha}(0)\cdot y''(0)>0$; and (iii) $\vec{\alpha}\cdot(y(\theta)-y(0))$ has first order zero at $\theta=0$ if $\alpha\neq 0$ and at $\mathcal{A}_1(\theta)=\alpha$ if $\alpha,\theta\neq 0$. Rescaling $\theta\to\tilde{\theta}$ and $\alpha\to\tilde{\alpha}$ proves the first property in (5.6).

The rest of (5.6) follows by differentiating $\vec{\alpha}\cdot(y(\theta)-y(0))$, using that $\vec{\alpha}\cdot y'(\alpha)\equiv 0$, and rescaling.

A.3. Proof of Lemma 6.1. Suppose $|\tilde{\alpha}|\geq c$. Assuming $|\mathcal{A}_1(\theta)-\alpha|\leq\delta\epsilon^{1/2}$, where $\delta>0$ is sufficiently small, the properties (see (5.5))

$$\mathcal{A}_1(\theta)\asymp\theta,\quad \max_{|\theta|\leq a}|\mathcal{A}_1(\theta)/\theta|<1, \tag{A.6}$$

imply $\theta\asymp\alpha$, $\theta/\alpha\geq c'$ for some $c'>1$, and $\partial_{\tilde{\theta}}R(\tilde{\theta},\tilde{\alpha})\asymp\tilde{\alpha}$ (see (5.6)). Also, differentiating R_1 in (5.1) we get $\partial_\alpha R_1(\theta,\alpha)\asymp-\theta$ and, hence, $\partial_{\tilde{\alpha}}R(\tilde{\theta},\tilde{\alpha})\asymp-\tilde{\alpha}$. The properties of r_n now follow immediately by rescaling and setting $\tilde{\theta}=u_n$. The magnitudes of r_{mn} and r_{mx} follow as well because $\mathcal{A}(\tilde{\theta})-\tilde{\alpha}=\pm\delta$ for the corresponding $\tilde{\theta}$.

Denote $B(\tilde{\alpha}):=\mathcal{A}^{-1}(\tilde{\alpha})$. To prove the statement about $\partial_{\tilde{\alpha}}r_{\text{mn}},\partial_{\tilde{\alpha}}r_{\text{mx}}$ we need to show that $\partial_{\tilde{\alpha}}R(B(\tilde{\alpha}\pm\delta),\tilde{\alpha})=O(1)$. Using that (i) $\partial_{\tilde{\alpha}}\partial_{\tilde{\theta}}R,\partial_{\tilde{\alpha}}^2R$, and B' are all $O(1)$ (see (5.6); $B'=O(1)$ follows from $A'=O(1)$) and (ii) $R(B(\tilde{\alpha}\pm\delta),\tilde{\alpha}\pm\delta)\equiv 0$, it is easy to see that the desired assertion holds.

To prove (6.5) we differentiate $R(\Theta(r,\tilde{\alpha}),\tilde{\alpha})\equiv 0$ and use that $\partial_{\tilde{\theta}}R(\tilde{\theta},\tilde{\alpha})\asymp\tilde{\alpha}$ and $\partial_{\tilde{\alpha}}R(\tilde{\theta},\tilde{\alpha})\asymp-\tilde{\alpha}$.

A.4. Proof of Lemma 7.2. Any v_n such that $|v_n-\tilde{\alpha}|\leq\delta$ for some $\tilde{\alpha}\in[b,b+L]$ satisfies $b-\delta\leq v_n\leq b+L+\delta$. By Lemma 5.2, u_n and v_n satisfy qualitatively the same assumptions (see Assumption 2.4 (H3)). Since $L=O(1)$, there are finitely many such v_n. Also, $1+\tilde{\alpha}^2(v_n-\tilde{\alpha})^2\asymp 1+b^2(v_n-\tilde{\alpha})^2$. Let A denote the expression on the left side of (7.11). Then

$$A\leq\sum_{b-\delta\leq v_n\leq b+L+\delta}\int_{\mathbb{R}}\frac{\mathrm{d}\tilde{\alpha}}{1+b^2(v_n-\tilde{\alpha})^2}=O(1/b),\ b\to\infty. \tag{A.7}$$

Appendix B. Proofs of lemmas in section 8 and section 9.

B.1. Proof of Lemma 8.2. Using that $\vec{\theta}_0\cdot y'(0)=0$ and $\vec{\theta}_0\cdot(y(0)-x_0)=0$, it is easy to see that the right-hand side of the identity

$$\sin(\mathcal{A}_1(\theta))=\vec{\theta}_0\cdot(y(\theta)-x_0)\,/|y(\theta)-x_0| \tag{B.1}$$

and its first derivative are zero at $\theta = 0$. Also, its second derivative at $\theta = 0$ equals $\vec{\theta}_0 \cdot y''(0)/|y(0) - x_0|$. By our choice of coordinates, this expression is positive. The properties in (8.4) now follow from the properties of $\sin^{-1}(t)$ and by rescaling.

To prove the first property in (8.5), consider the function $R_1(\theta,\alpha)/(\mathcal{A}_1(\theta) - \alpha)$ and note that $\vec{\alpha}(\mathcal{A}_1(\theta))$ is the only unit vector with $|\alpha| < \pi/2$ orthogonal to $y(\theta) - x_0$. Recall that a is sufficiently short, so $\mathcal{A}_1([-a,a]) \subset [-\pi/2,\pi/2]$. This function is clearly sufficiently smooth on $[-a,a] \times [-\pi/2,\pi/2]$. The ratio is positive because (i) $\mathcal{S}$ is convex, (ii) $\vec{\theta}_0 \cdot y''(0) > 0$ (cf. the proof of Lemma 5.2) and $\mathcal{A}_1(\theta) \geq 0$ by the assumption about x_0 (see the paragraph following (8.2)), and (iii) $R_1(\theta,\alpha)$ has a root of first order at $\alpha = \mathcal{A}_1(\theta)$, $\theta \neq 0$. Rescaling $\theta \to \tilde{\theta}$ and $\alpha \to \hat{\alpha}$, we finish the proof.

Differentiating R_1 and using that $\vec{\alpha} \cdot y'(\alpha) \equiv 0$, we find

$$\partial_\theta R_1(\theta,\alpha) \asymp \theta - \alpha; \quad \partial_\theta^2 R_1(\theta,\alpha), \partial_\theta\partial_\alpha R_1(\theta,\alpha), \partial_\alpha R_1(\theta,\alpha) = O(1). \tag{B.2}$$

Rescaling the variables proves the rest of (8.5).

B.2. Proof of Lemma 8.3. By (8.5),

$$|g(\mathcal{A}_{\rm mx} + \tau)| \leq c\rho(m) \int_{-a\epsilon^{-1/2}}^{a\epsilon^{-1/2}} \frac{1}{1 + (\tau + (\mathcal{A}_{\rm mx} - \mathcal{A}(\tilde{\theta})))^2} \mathrm{d}\tilde{\theta}, \ \tau > 0. \tag{B.3}$$

By construction, $\mathcal{A}_{\rm mx} - \mathcal{A}(\tilde{\theta}) \geq 0$. By (8.4), we can change variables $r = \mathcal{A}(\tilde{\theta})$ (separately on $(-a\epsilon^{-1/2}, 0]$ and $[0, a\epsilon^{-1/2})$). Then $\mathrm{d}\tilde{\theta}/\mathrm{d}r \asymp \pm r^{-1/2}$ and

$$|g(\mathcal{A}_{\rm mx} + \tau)| \leq c\rho(m) \int_0^{\mathcal{A}_{\rm mx}} \frac{1}{(1 + \tau + \mathcal{A}_{\rm mx} - r)^2 r^{1/2}} \mathrm{d}r, \ \tau > 0. \tag{B.4}$$

Here we used that $1 + x^2 \asymp (1+x)^2$, $x \geq 0$. Since $\mathcal{A}_{\rm mx} \asymp 1/\epsilon$, we obtain

$$\int_0^{\mathcal{A}_{\rm mx}/2} (\cdot)\mathrm{d}r \leq c\mathcal{A}_{\rm mx}^{-2} \int_0^{\mathcal{A}_{\rm mx}/2} \frac{\mathrm{d}r}{r^{1/2}} = O(\epsilon^{3/2}), \ \tau > 0, \tag{B.5}$$

and

$$\int_{\mathcal{A}_{\rm mx}/2}^{\mathcal{A}_{\rm mx}} (\cdot)\mathrm{d}r \leq c\mathcal{A}_{\rm mx}^{-1/2} \int_0^{\mathcal{A}_{\rm mx}/2} \frac{\mathrm{d}r}{(1 + \tau + r)^2} \leq c\frac{\epsilon^{1/2}}{1+\tau}, \ \tau > 0. \tag{B.6}$$

Consequently, the sum in (8.11) is bounded by

$$c\epsilon^{1/2} \sum_{0 \leq k \leq O(1/\epsilon)} \left[\epsilon^{3/2} + \frac{\epsilon^{1/2}}{1+k}\right] = O(\epsilon \ln(1/\epsilon)). \tag{B.7}$$

B.3. Proof of Lemma 8.4. By Lemma 8.2, there exists $\delta' > 0$ such that for any $0 < \alpha \leq \pi/2$, one has

$$\{|\theta| \leq a : |R(\tilde{\theta},\hat{\alpha})| \geq \delta\hat{\alpha}^{1/2}\} \subset \{|\theta| \leq a : |\mathcal{A}(\tilde{\theta}) - \hat{\alpha}| \geq \delta'\hat{\alpha}^{1/2}\}. \tag{B.8}$$

This implies

$$|g_1(\hat{\alpha})| \leq c\rho(m) \int_{\substack{|\tilde{\theta}| \leq a\epsilon^{-1/2} \\ |\mathcal{A}(\tilde{\theta}) - \hat{\alpha}| \geq \delta'\hat{\alpha}^{1/2}}} (1 + (\mathcal{A}(\tilde{\theta}) - \hat{\alpha})^2)^{-1} \mathrm{d}\tilde{\theta}. \tag{B.9}$$

Again by Lemma 8.2, on the sets $\tilde{\theta}>0$ and $\tilde{\theta}<0$ we can change variables $\tilde{\theta}\to r=\mathcal{A}(\tilde{\theta})$, where $d\tilde{\theta}/dr \asymp \pm|r|^{-1/2}$, to obtain

$$|g_1(\hat{\alpha})| \le c\rho(m)\int_{\substack{r>0\\ |r-\hat{\alpha}|\ge\delta'\hat{\alpha}^{1/2}}} (1+(r-\hat{\alpha})^2)^{-1}r^{-1/2}\mathrm{d}r. \tag{B.10}$$

By an easy calculation,

$$\int_{\hat{\alpha}+\delta'\hat{\alpha}^{1/2}}^{\infty}\frac{\mathrm{d}r}{(r-\hat{\alpha})^2r^{1/2}},\ \int_0^{\hat{\alpha}-\delta'\hat{\alpha}^{1/2}}\frac{\mathrm{d}r}{(\hat{\alpha}-r)^2r^{1/2}}=O(\hat{\alpha}^{-1}), \tag{B.11}$$

and the lemma is proven.

B.4. Proof of Lemma 8.5. By (8.5), we need to establish that $|\epsilon^{1/2}\hat{\alpha}/\tilde{\theta}|\le c$ for some sufficiently small $c>0$. Given that $\tilde{\theta}$, $\hat{\alpha}$, satisfy the conditions in (8.20), (8.4) and (8.5) imply

$$\left|\frac{\epsilon^{1/2}\hat{\alpha}}{\tilde{\theta}}\right| \le \frac{\epsilon^{1/2}\hat{\alpha}}{\min_{|R(\tilde{\theta},\hat{\alpha})|<\delta\hat{\alpha}^{1/2}}|\tilde{\theta}|}\le c_1\frac{\epsilon^{1/2}\hat{\alpha}}{(\hat{\alpha}-c_2\delta\hat{\alpha}^{1/2})^{1/2}} \tag{B.12}$$

for some $c_{1,2}>0$. Squaring both sides gives

$$\frac{\epsilon\hat{\alpha}^2}{\tilde{\theta}^2}\le c_1^2\frac{\epsilon\hat{\alpha}^2}{\hat{\alpha}-c_2\delta\hat{\alpha}^{1/2}}=c_1^2\frac{\alpha}{1-c_2(\delta/\hat{\alpha}^{1/2})}. \tag{B.13}$$

Given that $\alpha\in\Omega$, Ω can be made as small as we like (by selecting $a>0$ small), $\hat{\alpha}\ge\hat{\alpha}_*$, and $\hat{\alpha}_*$ can be made as large as we like, (8.20) is proven.

B.5. Proof of Lemma 8.6. We will consider only g_4^+ (i.e., $n>0$), since estimating g_4^- is completely analogous. For simplicity, the superscript $+$ is omitted.

By Lemma 8.2 and Lemma 8.5, $\partial_{\tilde{\theta}}R\asymp\tilde{\theta}$ and $\mathcal{A}^{-1}(\hat{\alpha})\asymp\hat{\alpha}^{1/2}$, so the coefficient in front of the integral in (8.21) is $O(\hat{\alpha}^{-1/2})$. Suppose first that one of the intervals $R_{n_0}=[r_{2n_0},r_{2n_0+1}]$ contains zero. Then $r_{2n_0}\le 0\le r_{2n_0+1}$ and

$$J:=\int_{r_{2n_0}}^{r_{2n_0+1}}\tilde{\psi}_m(r+h)\mathrm{d}r=-\left(\int_{-\infty}^{r_{2n_0}}+\int_{r_{2n_0+1}}^{\infty}\right)\tilde{\psi}_m(r+h)\mathrm{d}r. \tag{B.14}$$

Therefore, by (4.4),

$$J\le\frac{c\rho(m)}{1+\min(|r_{2n_0}+h|,|r_{2n_0+1}+h|)}. \tag{B.15}$$

If either $r_{2n_0}<-\delta\hat{\alpha}^{1/2}$ or $r_{2n_0+1}>\delta\hat{\alpha}^{1/2}$, then the corresponding limit is replaced by either $-\delta\hat{\alpha}^{1/2}$ or $\delta\hat{\alpha}^{1/2}$, as needed. If both r_{2n_0} and r_{2n_0+1} exceed the limits, then $J=O(\hat{\alpha}^{-1/2})$. By Lemma 8.2, $r_n\asymp v_n-\hat{\alpha}$, $v_n=\mathcal{A}(u_n)$, and (8.24) is proven.

Recall that $u_n>0$ if $n>0$. Then $u_n\asymp n$ and, by Lemma 8.2, $v_n\asymp n^2$. This implies that, on average, the distance between consecutive v_n increases as $n\to\infty$. In turn, this means that $v_n-\hat{\alpha}$ stays bounded for a progressively smaller fraction of $\hat{\alpha}$ as $\hat{\alpha}\to\infty$. Since the term h is uniformly bounded, it can be omitted from (B.15) to better reflect the essence of the estimate.

Contribution of all remaining intervals located on one side of zero $[r_{2n},r_{2n+1}]\subset(r_{2n_0+1},\delta\hat{\alpha}^{1/2}]$, $n>n_0$, and $[r_{2n},r_{2n+1}]\subset[-\delta\hat{\alpha}^{1/2},r_{2n_0})$, $0\le n<n_0$, can be estimated in a similar fashion:

$$\sum_{n>n_0}\int_{r_{2n}}^{r_{2n+1}}|\tilde{\psi}_m(r+h)|\mathrm{d}r\le\int_{r_{2n_0+1}}^{\infty}|\tilde{\psi}_m(r+h)|\mathrm{d}r\le\frac{c\rho(m)}{1+r_{2n_0+1}}, \tag{B.16}$$

and the same way for the other set of intervals. This proves (8.24).

To prove (8.25), we first collect some useful results, which follow from Lemma 8.2:

$$\partial_{\hat\alpha}\mathcal{A}^{-1}(\hat\alpha)=O(\hat\alpha^{-1/2}),\ \partial_{\hat\alpha}r_n=O(1). \tag{B.17}$$

Differentiating $g_4(\hat\alpha)$ in (8.21) and using (8.5), (8.7), and (B.17) gives

$$\begin{aligned}&\partial_{\hat\alpha}g_4^{\pm}(\hat\alpha)\\&\le c\rho(m)\left(\left(\frac{\epsilon^{1/2}}{\hat\alpha}+\frac{1}{\hat\alpha^{3/2}}\right)\frac{1}{1+r_{\min}(\hat\alpha)}+\frac{1}{\hat\alpha^{1/2}}\frac{1}{1+r_{\min}^2(\hat\alpha)}+\frac{\epsilon^{1/2}}{\hat\alpha}+\frac{\epsilon}{\hat\alpha^{1/2}}\right),\end{aligned} \tag{B.18}$$

and the desired result follows by keeping only the dominant terms. Here we have used that $\epsilon\hat\alpha=O(1)$, and there are finitely many n such that $[r_{2n},r_{2n+1}]$ intersects the set $|r|\le\delta\hat\alpha^{1/2}$. The last claim is proven by finding all $u_n\asymp n^2$ that satisfy $\hat\alpha-\delta'\hat\alpha^{1/2}\le\mathcal{A}(u_n)\le\hat\alpha+\delta'\hat\alpha^{1/2}$ for some $\delta'>0$.

B.6. Proof of Lemma 9.2. We consider only g_4^+; the proof for g_4^- is analogous. By Lemma 9.1, $\hat\alpha\asymp v_n$ if $\hat\alpha\in V_n$. Also, at the right endpoint of V_n, by (8.24)

$$|g_4^+(v_{n+1/2})|\le c\rho(m)\left(v_n^{-1/2}(1+(v_{n+1}-v_n))^{-1}+v_n^{-1}\right)=\rho(m)O(n^{-2}). \tag{B.19}$$

By Lemma 8.6, with $\hat\alpha=v_n+\tau$, $\hat\alpha\in V_n$,

$$|g_4^+(\hat\alpha)|\le c\rho(m)\left(v_n^{-1/2}(1+\tau)^{-1}+v_n^{-1}\right), \tag{B.20}$$

$$|\partial_{\hat\alpha}g_4^+(\hat\alpha)|\le c\rho(m)\left(v_n^{-1/2}(1+\tau^2)^{-1}+\epsilon^{1/2}v_n^{-1}\right) \tag{B.21}$$

if $v_{n-1/2}\ge\hat\alpha_*$. Then

$$\begin{aligned}&\int_{V_n}|g_4^+(\hat\alpha)|\mathrm{d}\hat\alpha\le c\rho(m)\left(v_n^{-1/2}\int_0^{O(n)}\frac{\mathrm{d}\tau}{1+\tau}+v_n^{-1}O(n)\right)=\rho(m)O(n^{-1}\ln n),\\&\int_{V_n}|\partial_{\hat\alpha}g_4^+(\hat\alpha)|\mathrm{d}\hat\alpha\le c\rho(m)\left(v_n^{-1/2}O(1)+\epsilon^{1/2}v_n^{-1}O(n)\right)=\rho(m)O(n^{-1})\end{aligned} \tag{B.22}$$

because $n=O(\epsilon^{-1/2})$, and the lemma is proven.

Appendix C. Analysis of exceptional cases. Consider now possible violations of the inequalities $0<\alpha_l<\alpha_*<\alpha_r\le\pi/2$ (see the beginning of subsection 7.2 and subsection 7.3). A violation may happen only for finitely many m. If $\alpha_l<0$ (i.e., the interval $(\phi(0),\phi(\alpha_*))$ does not contain an integer), we consider the interval $[0,\alpha_*]$ instead of $[\alpha_l,\alpha_*]$. The analogues of (7.25)–(7.27) become

$$\begin{aligned}&N=O(\tilde\alpha_*)=O(\epsilon^{-1/2});\ [0,\tilde\alpha_*]=\cup_{n=0}^{N-1}[nL,(n+1)L],\ \tilde\alpha_*=NL;\\&\langle\vartheta'(\hat\alpha)\rangle\ge\min\left(\langle m\kappa r_x\rangle,\langle m\mu\rangle\right),\ \alpha\in[0,\alpha_*].\end{aligned} \tag{C.1}$$

The analogue of (7.28) is (cf. (7.15))

$$W_m([0,\alpha_*])\le c\frac{\epsilon^{1/2}\rho(m)}{\min\left(\langle m\kappa r_x\rangle,\langle m\mu\rangle\right)}\sum_{n=0}^{N-1}\frac{1}{1+(N-(n+1))L}=O(\epsilon^{1/2}\ln(1/\epsilon)). \tag{C.2}$$

Here we have used that the number of different values of m for which $\alpha_l<0$ and the above estimate applies is finite.

If $\alpha_r > \pi/2$ (which implies $\langle\phi(\pi/2)\rangle > 0$), we consider the interval $[\alpha_*, \pi/2]$ instead of $[\alpha_*, \alpha_r]$. The analogue of (C.1) becomes

$$\begin{aligned}&N = O(\epsilon^{-1/2}),\\ &[\tilde\alpha_*, \epsilon^{-1/2}\pi/2] = \cup_{n=0}^{N-1}[\tilde\alpha_* + nL, \tilde\alpha_* + (n+1)L],\ \tilde\alpha_* + NL = \epsilon^{-1/2}\pi/2,\\ &\langle\vartheta'(\hat\alpha)\rangle \ge \min\left(\langle m\kappa r_x\rangle, \langle\phi(\pi/2)\rangle\right),\ \alpha \in [\alpha_*, \pi/2].\end{aligned} \tag{C.3}$$

The analogue of (C.2) is

$$W_m([\alpha_*, \pi/2]) \le c\frac{\epsilon^{1/2}\rho(m)}{\min\left(\langle m\kappa r_x\rangle, \langle\phi(\pi/2)\rangle\right)} = O(\epsilon^{1/2}). \tag{C.4}$$

Here we used that $\tilde\alpha = O(\epsilon^{-1/2})$ if $\alpha \ge \alpha_*$.

This completes the analysis of the case $\alpha_* < \pi/2$. If $\alpha_* > \pi/2$, then $\alpha_l > \pi/2$ when $m \gg 1$ is sufficiently large. Hence the only relevant case is when $\alpha_l < \pi/2$. As before, this happens for finitely many m. If $\alpha_l > 0$, the interval we should consider is $[\alpha_l, \pi/2]$. In this case, the estimates in (7.28), (7.29) still hold (some of the terms in the sum are not necessary). Given that m is bounded, the estimates imply $W_m([\alpha_l, \pi/2]) = O(\epsilon^{1/2}\ln(1/\epsilon))$. If $\alpha_l < 0$, the relevant interval is $[0, \pi/2]$. Arguing similarly to (C.1), (C.2), it is obvious that $W_m([0, \pi/2]) = O(\epsilon^{1/2}\ln(1/\epsilon))$.

Appendix D. Proof of Lemma 8.1. Pick any x sufficiently close to x_0. All the estimates below are uniform with respect to x in a small (but fixed) size neighborhood, so the x-dependence of various quantities is omitted from notation. Since $\mathcal{S}$ is not necessarily convex, the parametrization is not by the normal direction but is some regular $y(s) \in C^4$ (cf. Assumption 2.1 (F1)).

Let Ω be the set of all $\alpha \in [-\pi/2, \pi/2]$ such that the lines $\{y \in \mathbb{R}^2 : (y - x)\cdot\vec\alpha = 0\}$ intersect $\mathcal{S}$. Let $s = \mathcal{B}(\alpha)$, $\alpha \in \Omega$, be determined by solving $(y(s) - x)\cdot\vec\alpha = 0$. By using a partition of unity, if necessary, we can assume that (a) $\mathcal{S}$ is short, (b) the solution is unique for each $\alpha \in \Omega$, and (c) Ω is an interval. By assumption, the intersection is transverse for any $\alpha \in \Omega$ (up to the endpoints). Hence $|\mathcal{B}'(\alpha)| = |y(\mathcal{B}(\alpha)) - x|/|\vec\alpha\cdot y'(\mathcal{B}(\alpha))|$ and

$$0 < \min_{\alpha\in\Omega}|\vec\alpha\cdot y'(\mathcal{B}(\alpha))|,\ 0 < \min_{\alpha\in\Omega}|\mathcal{B}'(\alpha)| \le \max_{\alpha\in\Omega}|\mathcal{B}'(\alpha)| < \infty. \tag{D.1}$$

Transform the expression for A_m (cf. (4.5)) similarly to (4.9):

$$A_m(\alpha,\epsilon) = \frac{1}{\epsilon}\int_{-a}^{a}\int_0^{\epsilon^{-1}H_\epsilon(s)}\tilde\psi_m\left(\frac{\vec\alpha\cdot(y(s)-x)}{\epsilon} + h(\hat t, s, \alpha)\right)F(s, \epsilon\hat t)\mathrm{d}\hat t\mathrm{d}s, \tag{D.2}$$

where h and F are defined similarly to (4.9):

$$F(s,t) := \Delta f(y(\theta) + t\vec\theta)(\mathcal{R}(\theta) - t),\ h(\hat t, s, \alpha) := \hat t\cos(\theta - \alpha),\ \theta = \theta(s). \tag{D.3}$$

Note that in this section we assume $m \in \mathbb{Z}$, i.e., the case $m = 0$ is included. Since $\check x$ is absorbed by x, the term $\alpha\cdot\check x$ is no longer a part of h. Clearly,

$$A_m(\alpha,\epsilon) = \rho(m)O(\epsilon),\ \alpha \in [-\pi/2, \pi/2]\setminus\Omega,\ m \in \mathbb{Z}. \tag{D.4}$$

Next, consider the case $\alpha \in \Omega$. Setting $\tilde s = (s - \mathcal{B}(\alpha))/\epsilon^{1/2}$, (D.2) becomes

$$\begin{aligned}A_m(\alpha,\epsilon) = \epsilon^{-1/2}\int\int_0^{\epsilon^{-1}H_\epsilon(s)}\tilde\psi_m\left(\frac{\vec\alpha\cdot(y(s) - y(\mathcal{B}(\alpha)))}{\epsilon} + h(\hat t, s, \alpha)\right)\\ \times F(s, \epsilon\hat t)\mathrm{d}\hat t\mathrm{d}\tilde s,\quad s = \mathcal{B}(\alpha) + \epsilon^{1/2}\tilde s,\ \alpha\in\Omega,\ m\in\mathbb{Z}.\end{aligned} \tag{D.5}$$

Due to (4.4), we can integrate with respect to $\tilde{s}$ over any fixed neighborhood of 0:

$$
\begin{aligned}
&A_m(\alpha,\epsilon)\\
&=\epsilon^{-1/2}\int_{-\delta}^{\delta}\int_0^{\epsilon^{-1}H_\epsilon(s)}\tilde{\psi}_m\left(\frac{\vec{\alpha}\cdot y'(\mathcal{B}(\alpha))}{\epsilon^{1/2}}\tilde{s}+O(\tilde{s}^2)+h(\hat{t},\mathcal{B}(\alpha),\alpha)+O(\epsilon^{1/2})\right)\\
&\quad\times\left(F(\mathcal{B}(\alpha),0)+O(\epsilon^{1/2})\right)\mathrm{d}\hat{t}\mathrm{d}\tilde{s}+\rho(m)O(\epsilon^{1/2}),\qquad \alpha\in\Omega,\ m\in\mathbb{Z},
\end{aligned}\tag{D.6}
$$

for some $\delta>0$ sufficiently small. Using (4.4) it is easy to see that the terms $O(\epsilon^{1/2})$ and $O(\tilde{s}^2)$ can be omitted from the argument of $\tilde{\psi}_m$ without changing the error term:

$$
\begin{aligned}
&A_m(\alpha,\epsilon)\\
&=\frac{F(\mathcal{B}(\alpha),0)}{\epsilon^{1/2}}\int_{-\delta}^{\delta}\int_0^{H_0(\epsilon^{-1/2}\mathcal{B}(\alpha)+\tilde{s};\epsilon)}\tilde{\psi}_m\left(\frac{\vec{\alpha}\cdot y'(\mathcal{B}(\alpha))}{\epsilon^{1/2}}\tilde{s}+h(\hat{t},\mathcal{B}(\alpha),\alpha)\right)\mathrm{d}\hat{t}\mathrm{d}\tilde{s}\\
&\quad+\rho(m)O(\epsilon^{1/2}),\ \alpha\in\Omega,\ m\in\mathbb{Z}.
\end{aligned}\tag{D.7}
$$

By (D.1), $\vec{\alpha}\cdot y'(\mathcal{B}(\alpha))$ is bounded away from zero on Ω. By the last equation in (4.2), $\int\tilde{\psi}_m(\hat{t})\mathrm{d}\hat{t}=0$, $m\in\mathbb{Z}$, so we can replace the lower limit of the inner integral in (D.7) with any value independent of $\tilde{s}$. Again, we use here that the contribution to the integral with respect to $\tilde{s}$ of the domain outside $(-\delta,\delta)$ is of the same magnitude as the error term in (D.7). We choose the lower limit to be $H_0(\epsilon^{-1/2}\mathcal{B}(\alpha);\epsilon)$. Hence

$$
\begin{aligned}
&|A_m(\alpha,\epsilon)|\le\rho(m)\left[O(\epsilon^{-1/2})\int_{-\delta}^{\delta}\frac{\left|H_0(v+\tilde{s};\epsilon)-H_0(v;\epsilon)\right|}{1+(\tilde{s}^2/\epsilon)}\mathrm{d}\tilde{s}+O(\epsilon^{1/2})\right],\\
&\qquad v=\epsilon^{-1/2}\mathcal{B}(\alpha),\ \alpha\in\Omega,\ m\in\mathbb{Z}.
\end{aligned}\tag{D.8}
$$

Neglecting the $O(\epsilon^{1/2})$ term in (D.8) (the last term inside the brackets) leads to a term of magnitude $O(\epsilon^{1/2})$ in $f_\epsilon^{\text{p-rec}}$. Accounting for (D.4) in a similar fashion, (4.5) and (D.8) imply

$$
\begin{aligned}
f_\epsilon^{\text{p-rec}}(x)=&O(\epsilon^{1/2})\int_{-\delta}^{\delta}\frac{g(\tilde{s},\epsilon)}{1+(\tilde{s}^2/\epsilon)}\mathrm{d}\tilde{s}+O(\epsilon^{1/2}),\\
g(\tilde{s},\epsilon):=&\sum_{\alpha_k\in\Omega}\left|H_0(v_k+\tilde{s};\epsilon)-H_0(v_k;\epsilon)\right|,\ v_k:=\epsilon^{-1/2}\mathcal{B}(\alpha_k).
\end{aligned}\tag{D.9}
$$

Define similarly to (2.7):

$$
\chi_{t_1,t_2}(t):=\begin{cases}1, & t_1\le t\le t_2 \text{ or } t_2\le t\le t_1,\\ 0 & \text{otherwise.}\end{cases}\tag{D.10}
$$

Clearly,

$$g(\tilde{s},\epsilon)=\sum_{\alpha_k\in\Omega}\int \chi_{H_0(v_k;\epsilon),H_0(v_k+\tilde{s};\epsilon)}(\hat{t})\mathrm{d}\hat{t}=\int N(\hat{t},\tilde{s},\epsilon)\mathrm{d}\hat{t}, \tag{D.11}$$

where $N(\hat{t},\tilde{s},\epsilon)$ is the number of $\alpha_k\in\Omega$ such that either $H_0(v_k;\epsilon)\le\hat{t}\le H_0(v_k+\tilde{s};\epsilon)$ or $H_0(v_k+\tilde{s};\epsilon)\le\hat{t}\le H_0(v_k;\epsilon)$. By Assumption 2.4 (H3), N is finite for almost all $\hat{t}$. Our argument implies that the values of the index k counted by the function $N(\hat{t},\tilde{s},\epsilon)$ are such that the closed interval with the endpoints v_k and $v_k+\tilde{s}$ contains at least one $u_n\in H_0^{-1}(\hat{t};\epsilon)$. By Assumption 2.4 (H3), the number of $u_n\in H_0^{-1}(\hat{t};\epsilon)$ on any interval of length $O(\epsilon^{-1/2})$ is $O(\epsilon^{-1/2})$ uniformly in $\hat{t}$ for almost all $\hat{t}$. The summation in (D.11) is over $\alpha_k\in\Omega$, so we look only for u_n such that $\mathrm{dist}(u_n,\epsilon^{-1/2}\mathcal{B}(\Omega))\le|\tilde{s}|\le\delta$.

Fix any n. By (D.1) and the definition of s_k in (D.9), there are no more than $1+O(\epsilon^{-1/2}|\tilde{s}|)$ values of k such that $|v_k-u_n|\le|\tilde{s}|$. Hence $N(\hat{t},\tilde{s},\epsilon)=O(\epsilon^{-1/2})(1+\epsilon^{-1/2}|\tilde{s}|)$. Using that the range of H_0 is bounded (cf. Assumption 2.4 (H1)), the integral with respect to $\hat{t}$ in (D.11) is over a compact set, so

$$g(\tilde{s},\epsilon)=O(\epsilon^{-1/2})(1+\epsilon^{-1/2}|\tilde{s}|). \tag{D.12}$$

Substituting (D.12) into (D.9), we finish the proof:

$$f_\epsilon^{\text{p-rec}}(x)=O(1)\int_{-\delta}^{\delta}\frac{1+\epsilon^{-1/2}|\tilde{s}|}{1+(\tilde{s}^2/\epsilon)}\mathrm{d}\tilde{s}+O(\epsilon^{1/2})=O(\epsilon^{1/2}\ln(1/\epsilon)). \tag{D.13}$$

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