



# Agol cycles of pseudo-Anosov 3-braids

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Received: 14 October 2022 / Accepted: 16 June 2023  
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## Abstract

An Agol cycle is a complete invariant of the conjugacy class of a pseudo-Anosov mapping class. We study necessary and sufficient conditions for equivalent Agol cycles of pseudo-Anosov 3-braids.

**Keywords** Agol cycle · Pseudo-Anosov mapping class

**Mathematics Subject Classification (2020)** 57M50

## 1 Introduction

Nielsen-Thurston's classification states that every surface homeomorphism is isotopic to either a periodic, reducible or pseudo-Anosov map [16]. If a map  $\phi$  is pseudo-Anosov, there are two transverse, singular measured foliations on the surface and a dilatation  $\lambda > 1$  such that  $\phi$  stretches along one foliation by  $\lambda$  and the other by  $1/\lambda$ . A dilatation is also called a stretch factor.

By definition, the dilatation is an invariant of the conjugacy class of a pseudo-Anosov map. The dilatation can be the first tool for the conjugacy problem since it is often easy to compute. However, the downside of the dilatation is that it is not always an effective conjugacy class invariant as demonstrated in Theorem 10:

**Theorem 10** *There are infinitely many integers  $x$ ,  $y$  and  $z$ , such that pseudo-Anosov 3-braids  $\beta = \sigma_1^x \sigma_2^{-1} \sigma_1^y \sigma_2^z$  and  $\beta' = \sigma_1^x \sigma_2^z \sigma_1^y \sigma_2^{-1}$  belong to distinct conjugacy classes but have the same dilatation*

$$\lambda = \frac{1}{2}(\gamma + \sqrt{\gamma^2 - 4})$$

where

$$\gamma = \gamma(x, y, z) = \operatorname{sgn}(xyz)(-2 - x - y + xz + yz + xyz). \quad (1)$$

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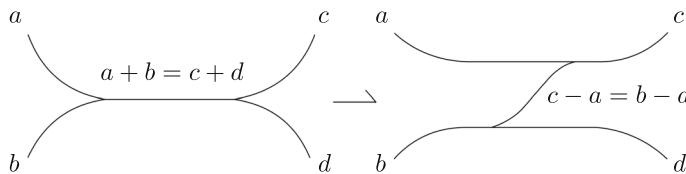
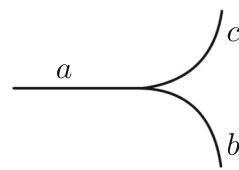
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**Fig. 1** A vertex of a train track where the weights satisfy  $a = b + c$



**Fig. 2** Maximal splitting when  $a < c$  (equivalently  $b > d$ ). The horizontal edge has the largest weight  $a + b = c + d$ . After the splitting, the largest weight will be either  $b$  or  $c$

Fortunately, there exists a stronger invariant. The Agol cycle [1] (see Definition 2) is another conjugacy class invariant of a pseudo-Anosov map on an orientable surface  $S_{g,n}$  with genus  $g$  and  $n$  punctures. More importantly, it is a complete conjugacy invariant.

An Agol cycle is a sequence of measured train tracks generated by maximal splitting.

A measured train track is a combinatorial object encoding the transverse measured foliation of a pseudo-Anosov map. (It is also called an invariant train track since the train track is invariant under action of the pseudo-Anosov map, and often we call it a train track without the adjective measured or invariant for simplicity.) It is a graph and each edge is labeled by a positive number called its weight (or measure). As depicted in Fig. 1, at each vertex three edges meet tangentially and the weights satisfy the *switch condition*  $a = b + c$ . For a detailed definition of a measured train track, see Sect. 2 of Agol's paper [1] or Chapter 15 of Farb and Margalit's book [8].

Maximal splitting is defined as follows:

**Definition 1** *Maximal splitting* ( $\rightarrow$ ) on a train track is an operation along all edges with the largest weight simultaneously as depicted in Fig. 2. Note that maximal splitting preserves the numbers of edges and vertices of the train track.

**Definition 2** [1] Given a pseudo-Anosov map  $\beta \in \mathcal{MCG}(S)$  with dilatation  $\lambda > 1$  and a measured train track  $(\tau_0, \mu_0)$  which is suited to the stable lamination for  $\beta$ , we can create an infinite maximal splitting sequence of train tracks:  $(\tau_0, \mu_0) \rightarrow (\tau_1, \mu_1) \rightarrow (\tau_2, \mu_2) \rightarrow \dots$ .

If  $\tau_q = \beta(\tau_p)$  and  $\mu_q = \lambda^{-1}\beta_*(\mu_p)$  for some  $0 \leq p < q$  we say that an *Agol cycle*

$$(\tau_p, \mu_p) \rightarrow \dots \rightarrow (\tau_{q-1}, \mu_{q-1}) \rightarrow (\tau_q, \mu_q) = (\beta(\tau_p), \lambda^{-1}\beta_*(\mu_p))$$

for  $\beta$  is formed. The *length of the Agol cycle* is  $q - p$ .

The head  $(\tau_0, \mu_0) \rightarrow (\tau_1, \mu_1) \rightarrow \dots \rightarrow (\tau_{p-1}, \mu_{p-1})$  of the sequence is called the *pre-periodic part*. The *length of the pre-periodic part* is  $p$ .

**Theorem 1** [1, Theorem 3.5] *Let  $S$  be a closed oriented surface possibly with punctures. Every pseudo-Anosov map  $\beta : S \rightarrow S$  has an Agol cycle.*

Hodgson, Issa and Segerman defined the following equivalence relation.

**Definition 3** [9, Definition 7] Let  $\phi, \phi' : S \rightarrow S$  be pseudo-Anosov homeomorphisms. We say that two Agol cycles  $\{(\tau_i, \mu_i)\}_{i=p}^q$  and  $\{(\tau'_i, \mu'_i)\}_{i=p+k}^{q+k}$  are *combinatorially isomorphic* if there exists an orientation preserving homeomorphism  $h : S \rightarrow S$  (possibly permuting the punctures) and a positive constant  $s \in \mathbb{R}_{>0}$  such that

1.  $\phi' = h \circ \phi \circ h^{-1}$  and
2.  $h(\tau_i) = \tau_{i+k}$  and  $h_*(\mu_i) = s\mu'_{i+k}$  for all  $i \geq p$ .

In this paper, if the above condition (2) is satisfied (regardless of the truth of condition (1)), we say that the Agol cycles are *equivalent*.

The following theorem is proved in [9, Theorem 2]. We note that it is implicit in Agol's paper [1, Section 7].

**Theorem 2** *Pseudo-Anosov maps  $\phi$  and  $\phi'$  are conjugate in  $\mathcal{MCG}(S)$  if and only if their Agol cycles are combinatorially isomorphic.*

In this paper we only consider a 3-punctured disk, denoted  $D_3$ , as a surface. Let  $\mathcal{MCG}(D_3)$  be the mapping class group of  $D_3$ ; that is, the group of isotopy classes of orientation preserving homeomorphisms of  $D_3$  that fix the boundary  $\partial D_3$  point-wise. It is proved that  $\mathcal{MCG}(D_3)$  and the 3-stranded braid group  $B_3$  are isomorphic (see [2] and Chapter 9 of Farb and Margalit's book [8]).

Clearly  $D_3$  has boundary. On the other hand, Agol assumes surfaces to have no boundary. To align with Agol's setting, we consider mapping classes in the quotient group  $\mathcal{MCG}(D_3)/Z$  where  $Z = Z(\mathcal{MCG}(D_3))$  is the center of  $\mathcal{MCG}(D_3)$  and  $Z$  is generated by a positive Dehn twist along the boundary. Therefore,

$$\mathcal{MCG}(D_3)/Z \simeq B_3/Z(B_3)$$

where  $Z(B_3)$  is the center of  $B_3$  and generated by the full twist  $\Delta^2 = (\sigma_2\sigma_1)^3$ .

A known example of an Agol cycle for a pseudo-Anosov 4-braid can be found in Agol's paper [1]. Margalit's talk slides [11] contain an Agol cycle for the pseudo-Anosov 3-braid  $\sigma_1\sigma_2^{-1}$ . We also compute this 3-braid in detail in Example 5. In general, it is not easy to compute Agol cycles by hand. Therefore, necessary or sufficient conditions for combinatorially isomorphic Agol cycles will be helpful to solve the conjugacy problem.

In Theorem 6, we give three equivalent conditions for combinatorially isomorphic Agol cycles.

**Theorem 6** *Let  $\beta$  and  $\beta'$  be pseudo-Anosov 3-braids with the same dilatation. The following conditions are equivalent. Equivalence of (1) and (2) is due to Hodgson, Issa and Segerman (see Theorem 2) and implicit in Agol's paper [1].*

1.  $\beta$  and  $\beta'$  are conjugate in  $B_3/Z(B_3)$ . In other words,  $\beta' = \Delta^{2k} w \beta w^{-1}$  for some  $w \in B_3$  and  $k \in \mathbb{Z}$ .
2.  $\beta$  and  $\beta'$  have combinatorially isomorphic Agol cycles.
3. (Topological condition) There exist  $l$  and  $m \in \mathbb{N}$  such that  $\text{sgn}(\mathcal{T}_l) = \text{sgn}(\mathcal{T}'_m)$  and the triple-weight train track sequences  $\mathcal{T}_l \rightarrow \mathcal{T}_{l+1} \rightarrow \mathcal{T}_{l+2} \rightarrow \dots$  (for  $\beta$ ) and  $\mathcal{T}'_m \rightarrow \mathcal{T}'_{m+1} \rightarrow \mathcal{T}'_{m+2} \rightarrow \dots$  (for  $\beta'$ ) give the same periodic I-II-I'-II'-sequence (cf. Definition 12).
4. (Number theoretic condition) There exist  $l$  and  $m \in \mathbb{N}$  such that  $\text{sgn}(\mathcal{T}_l) = \text{sgn}(\mathcal{T}'_m)$  and the nested Farey interval sequences  $J_l \supset J_{l+1} \supset J_{l+2} \supset \dots$  for  $\beta$  and  $J'_m \supset J'_{m+1} \supset J'_{m+2} \supset \dots$  for  $\beta'$  give the same periodic **LR**-sequences (cf. Definition 16).
5. (Numerical condition) There exist  $l$  and  $m \in \mathbb{N}$  such that  $\text{sgn}(\mathcal{T}_l) = \text{sgn}(\mathcal{T}'_m)$  and the 4-ratios (cf. Definition 9) of  $\mathcal{T}_l$  and  $\mathcal{T}'_m$  are the same.

The equivalence (1)  $\Leftrightarrow$  (2) is due to Agol. See [1, Section 7] and also Margalit's talk slides [11]. Our new conditions (3), (4) and (5) have different characteristics as follows:

An Agol cycle contains a large amount of information and each train track in the cycle can be very complicated with many twists. After untwisting (which is a homeomorphism operation), we will show that there are only four types of train tracks that appear in a cycle (Type I, II, I', II' defined in Fig. 6). In Condition (3), we focus on the topological types (Type I, II, I', II') and forget the numerical data of the weights of the train track edges.

In Condition (4), we estimate a particular algebraic number  $\alpha$  (the MP-ratio defined in Proposition 1) associated to the pseudo-Anosov braid in the Farey tessellation. Since the number  $\alpha$  is irrational, there exists an infinite sequence of nested intervals ( $J_l \supset J_{l+1} \supset J_{l+2} \supset \dots$ ) in the Farey tessellation that converges to  $\alpha$ . For each pair of adjacent intervals  $J_l \supset J_{l+1}$ , we may forget their exact location in the Farey tessellation, and instead focus on the relative location of the sub-interval  $J_{l+1}$  with respect to  $J_l$ . In the Farey tessellation,  $J_l$  splits into two sub-intervals. We record whether  $J_{l+1}$  is in the left-subinterval (**L**) or the right-subinterval (**R**) of  $J_l$ . This is how we generate an **LR**-sequence.

In Condition (5), we focus on just one train track in the infinite sequence. In contrast to (3), we forget the topological side and focus on the numerical data of the edge weights of train tracks. Here is a useful consequence of (5): If the algebraic numbers (MP-ratios)  $\alpha$  and  $\alpha'$  do not satisfy  $A + B\alpha + C\alpha' + D\alpha\alpha' = 0$  for any  $A, B, C, D \in \mathbb{Z}$ , then the 3-braids  $\beta$  and  $\beta'$  are not conjugate.

A concrete example is presented in Example 4.

**Definition 4** Two Agol cycles are *mirror-combinatorially isomorphic* if they are combinatorially isomorphic after taking mirror image.

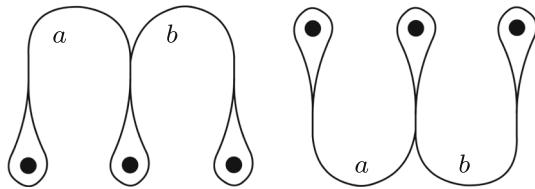
In Theorem 7, we study a more general statement regarding the direction of (5)  $\Rightarrow$  (2) in Theorem 6.

**Weak version of Theorem 7** *Let  $\beta$  and  $\beta'$  be pseudo-Anosov 3-braids with the same dilatation. Suppose that there exist  $l$  and  $m$  such that triple-weight train tracks  $T_l$  and  $T'_m$  have the same 4-ratio. Then the Agol cycles of  $\beta$  and  $\beta'$  are combinatorially isomorphic or mirror-combinatorially isomorphic.*

Examples for Theorem 7 are given in Example 6.

The outline of the paper is as follows: In Sect. 2, we study the stability property of a maximal splitting sequence which is summarized in Theorem 3. In Sect. 3, we focus on proving that there is a closed system of triple-weight train tracks in Theorem 4. The goal of Sect. 4 is to establish a connection between triple-weight train tracks and Farey sequences. In Sect. 5, we define nested intervals in the Farey tessellation and LR-sequences which will be needed to state our main result in Theorem 6. In Sect. 6, we state and prove our main result Theorem 6 which provides equivalent conditions for combinatorially isomorphic Agol cycles. In Sect. 7, we show that the same 4-ratio implies that the Agol cycles are combinatorially isomorphic or mirror combinatorially isomorphic. Lastly, we show in Sect. 8 that dilatation is a weaker invariant than Agol cycles in Theorem 10 by showing that dilatation is preserved under flype moves.

**Fig. 3** Train tracks  $M(a, b)$  and  $W(a, b)$



## 2 Stability of maximal splitting sequence

The goal of this section is to study the stability property of a maximal splitting sequence in Theorem 3. We show that a maximal splitting sequence arrives at a train track with only three distinct weights and any subsequent train track also has triple-weight type.

**Definition 5** We define two types of measured train tracks called Types M and W as in Fig. 3. Each type has exactly six edges and four vertices. With the weights  $a, b > 0$ , the left measured train track is denoted by  $M(a, b)$  and the right by  $W(a, b)$ . The weights are considered up to scaling.

The next Proposition 1 follows from the fact that the projective measured foliation space for a 4-punctured sphere is homeomorphic to  $\mathbb{RP}^1 = S^1$ , see Figure 15.6 in [8].

**Proposition 1** *For every pseudo-Anosov 3-braid  $\beta$ , there exists a unique irrational number  $\alpha \in (0, 1)$  such that exactly one of*

$$M_\alpha := M(1, \alpha), \quad M'_\alpha := M(\alpha, 1), \quad W_\alpha := W(\alpha, 1), \quad W'_\alpha := W(1, \alpha)$$

*yields a train track of  $\beta$ . Since  $\alpha$  describes the width-height ratio of the Markov partition rectangle, we call  $\alpha$  the MP-ratio for  $\beta$ .*

In the proof of Lemma 9, we show how to compute a transition matrix and determine the type (Type M or W) for certain pseudo-Anosov 3-braids. It is straightforward to generalize the construction to general pseudo-Anosov 3-braids. With a transition matrix in hand, the dilatation  $\lambda$  is its largest eigenvalue. The eigenvector for  $\lambda$  is  $\begin{bmatrix} \alpha \\ 1 \end{bmatrix}$  or  $\begin{bmatrix} 1 \\ \alpha \end{bmatrix}$  where  $0 < \alpha < 1$  is the MP-ratio. Since the dilatation  $\lambda$  is irrational, the MP-ratio is also an irrational number.

Note that  $M_\alpha$  and  $W_\alpha$  are related to each other by a  $180^\circ$ -rotation, and so are  $M'_\alpha$  and  $W'_\alpha$ . We further note that  $M_\alpha$  and  $M'_\alpha$  are related by a mirror reflection across a vertical line, and so are  $W_\alpha$  and  $W'_\alpha$ .

Based on Proposition 1, we present the Triple-Weight Train Track Theorem:

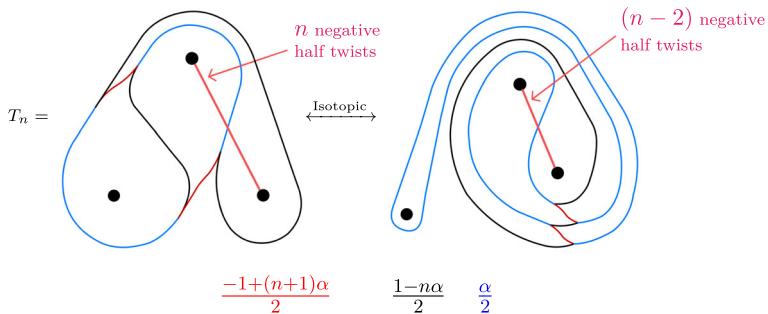
**Theorem 3** *Let  $\alpha \in (0, 1)$  be an irrational number and  $n = \lfloor \frac{1}{\alpha} \rfloor$ . Thus,  $\frac{1}{n+1} < \alpha < \frac{1}{n}$ . Consider the maximal splitting sequence starting from  $W_\alpha$ .*

$$W_\alpha = \tau_0 \rightarrow \tau_1 \rightarrow \tau_2 \rightarrow \dots$$

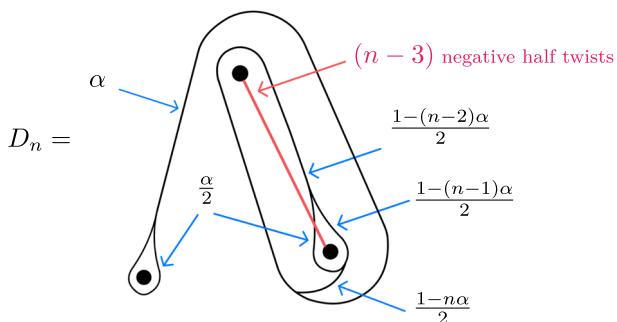
*The train track  $\tau_{n+4}$  in the sequence is the train track  $T_n$  shown in Fig. 4. We call it a triple-weight train track as it has only three different weights. This is the first triple-weight train track in the sequence and all the train tracks after  $\tau_{n+4}$  have triple-weight type.*

*If  $\tau_0 = W'_\alpha$ ,  $M'_\alpha$ , or  $M_\alpha$ , then the same result holds with the reflection of the figure about a vertical axis, the reflection about a horizontal axis, or a  $180^\circ$ -rotation, respectively.*

An immediate consequence is:



**Fig. 4** The train track  $T_n$ . The weight of a red edge is  $\frac{-1+(n+1)\alpha}{2}$  and of a black edge is  $\frac{1-n\alpha}{2}$ . The blue edges have the largest weight which is  $\frac{\alpha}{2} = \frac{-1+(n+1)\alpha}{2} + \frac{1-n\alpha}{2}$ . (Color figure online)



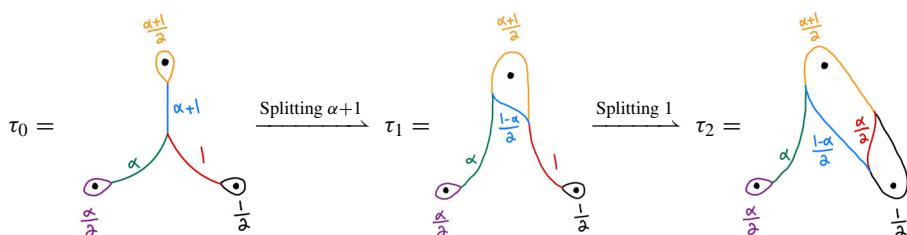
**Fig. 5** Train track  $D_n$

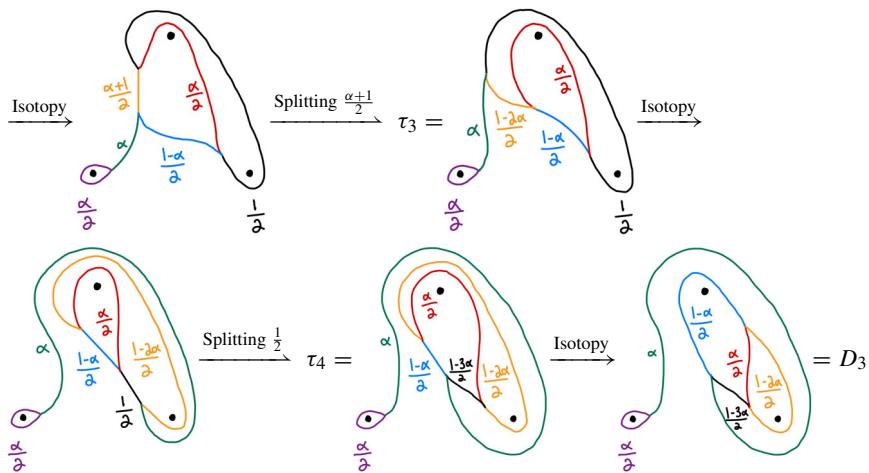
**Corollary 1** The length  $l$  of the pre-periodic part must satisfy  $l \geq n + 4$ .

Here is a lemma for Theorem 3.

**Lemma 1** Suppose that  $\tau_0 = W(\alpha, 1)$ . If  $n = \lfloor \frac{1}{\alpha} \rfloor \geq 3$ , then  $\tau_{n+1} = D_n$  as in Fig. 5.

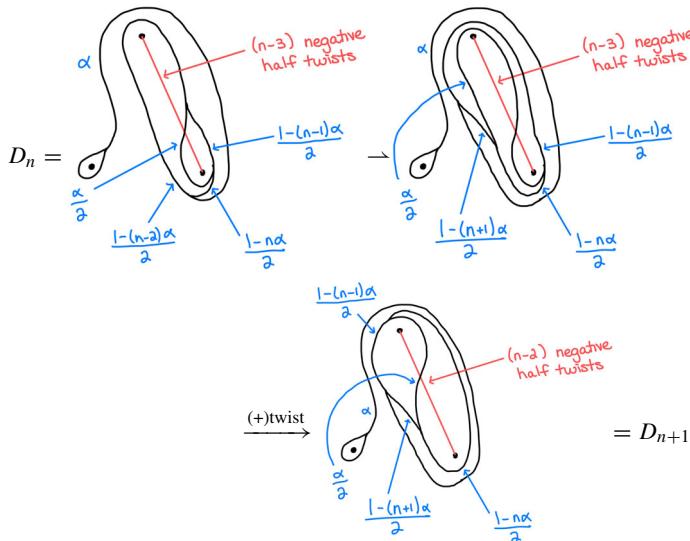
**Proof of Lemma 1** This is a proof by induction. We begin with the base case of  $n = 3$  and detail the splittings below with the labeled edges.





**Inductive Hypothesis:** Suppose that if  $\alpha < 1/n$  and  $n \geq 3$ , then after  $(n+1)$  maximal splittings, the train track  $\tau_0$  becomes  $D_n$ .

**Inductive Step:** Assume that  $\alpha < \frac{1}{n+1}$ . Since  $\frac{1}{n+1} < \frac{1}{n}$ ,  $\alpha < 1/n$  and we can apply the inductive hypothesis. Thus, after  $(n+1)$  maximal splittings,  $\tau_0$  becomes  $D_n$ . We perform another maximal splitting to the edge  $\frac{1-(n-2)\alpha}{2}$  to obtain  $D_{n+1}$ .



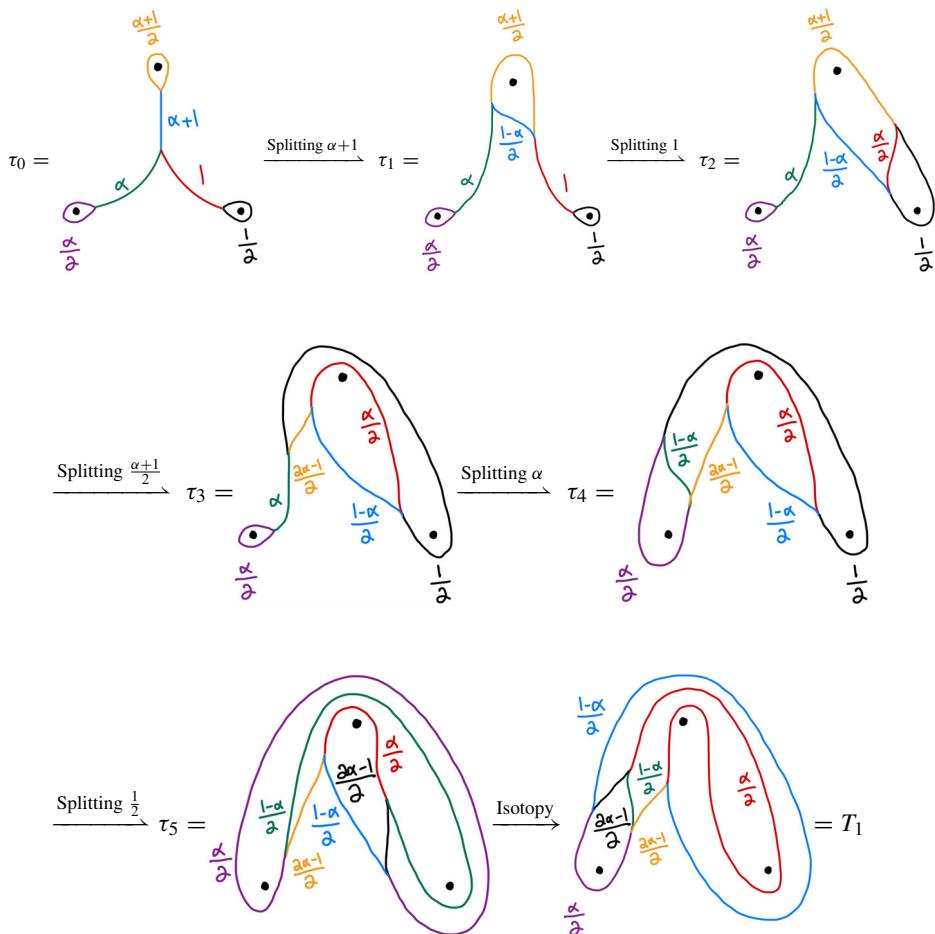
This completes the proof of Lemma 1.  $\square$

Now we are ready to prove the Triple-Weight Train Track Theorem.

**Proof of Theorem 3** We prove that the train track  $\tau_{n+4}$  is the train track in Fig. 4. Once this is done, it is easy to see that  $\tau_i$  has triple-weight type for all  $i \geq n+3$  since  $\tau_{n+3}$  has six edges and the maximal splitting preserves the numbers of edges and vertices.

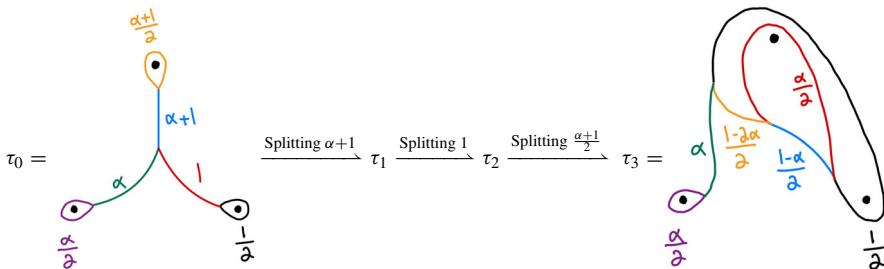
We first provide the explicit details for  $n = 1$  and  $n = 2$  and then apply Lemma 1 for  $n \geq 3$ .

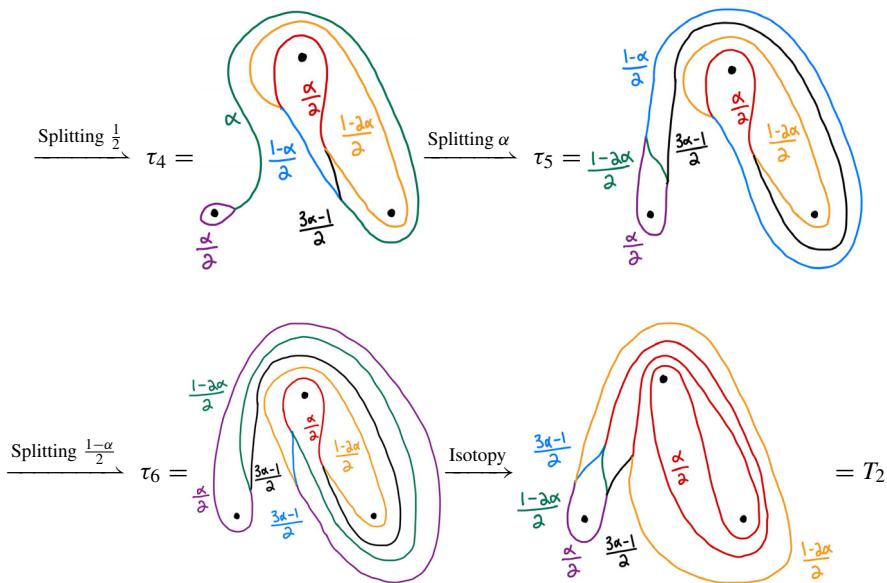
( $n = 1$ ): Note that  $\frac{1}{2} < \alpha < 1$ .



This shows  $\tau_{1+4} = T_1$ .

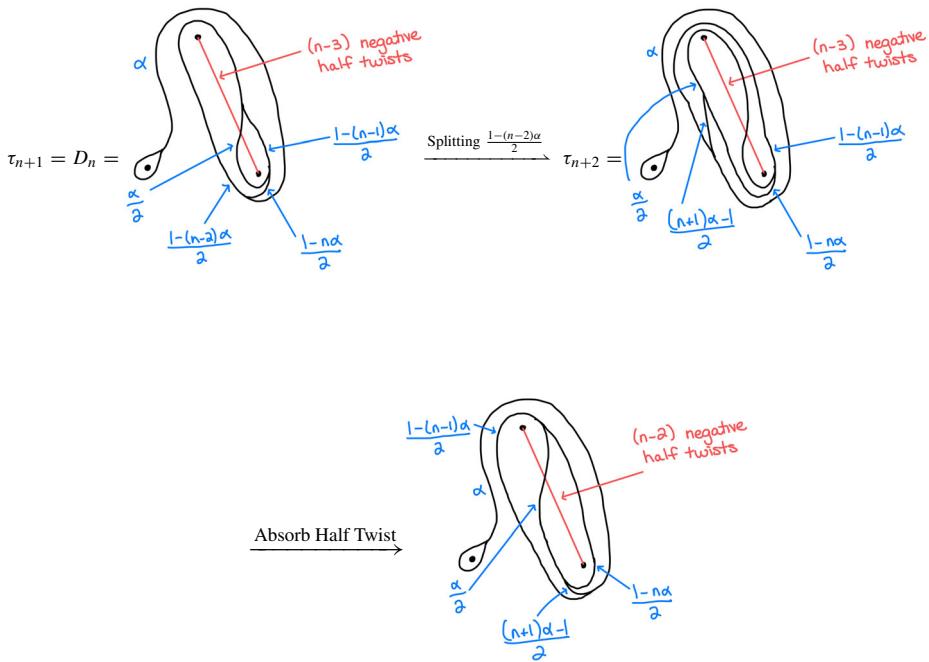
( $n = 2$ ): Note that  $\frac{1}{3} < \alpha < \frac{1}{2}$ . Thus the initial part of the sequence  $\tau_0 \rightarrow \tau_1 \rightarrow \tau_2$  is exactly the same as the  $n = 1$  case. The difference between  $n = 1$  and  $n = 2$  cases occur at  $\tau_3$ .

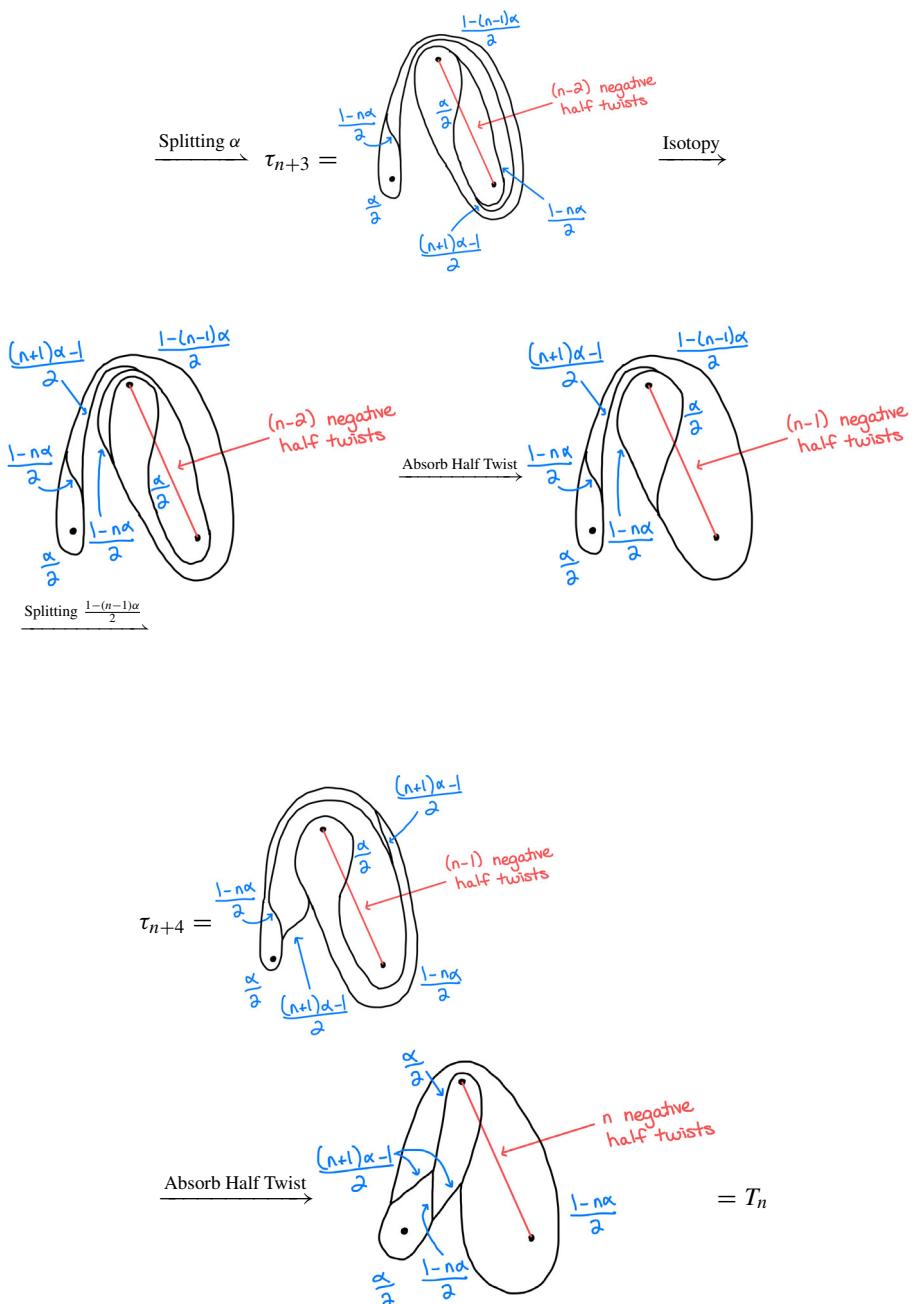




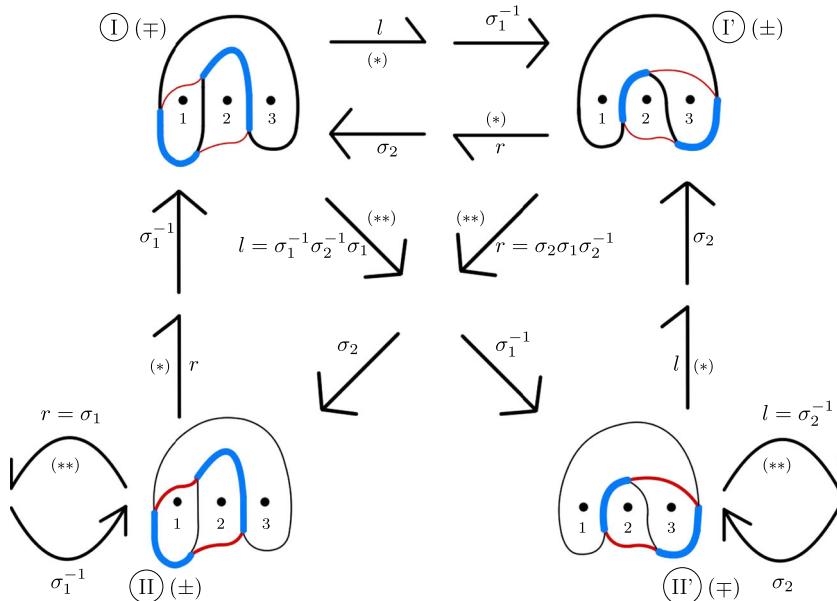
This shows  $\tau_{2+4} = T_2$ .

Assume  $n \geq 3$  and  $\frac{1}{n+1} < \alpha < \frac{1}{n}$ . By Lemma 1, after  $(n+1)$  maximal splittings, we arrive at train track  $\tau_{n+1} = D_n$ . We now apply three more maximal splittings and obtain  $\tau_{n+1+3} = T_n$ .





This completes the proof of Theorem 3. □



**Fig. 6** (Theorem 4) The I, I', II, and II' triple-weight train tracks. The symbol  $(\pm)$  represents the sign (Definition 9) of the train track explained in Corollary 4. The arrow  $\xrightarrow{(*)}$  (resp.  $\xrightarrow{(**)}$ ) means the maximal splitting is Type 1 (resp. Type 2) as defined in Proposition 2. Arrows  $\xrightarrow{l}$  (resp.  $\xrightarrow{r}$ ) means the  $l$ -splitting (resp.  $r$ -splitting) as defined in Fig. 9

### 3 System of triple-weight train tracks

The goal of this section is to study a closed system of triple-weight train tracks and prove Theorem 4.

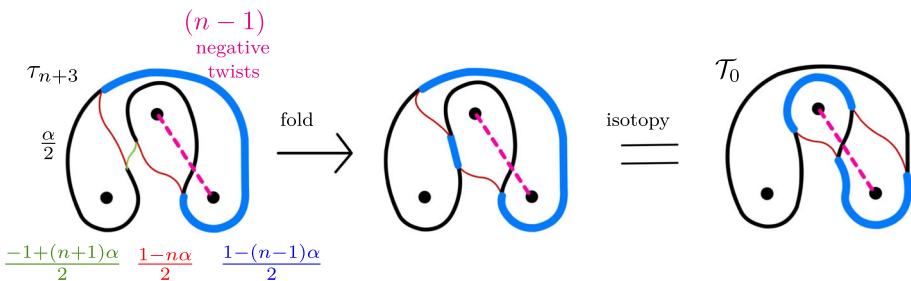
**Definition 6** We consider four types, Type I, I', II, and II', of triple-weight train tracks up to rotation as in Fig. 6. Edges with the same color have the same weight and the same thickness. Thickness of edges reflect the transverse measures. If we forget the thickness of the edges, I and II are the same and I' and II' are the same.

For instance, for train tracks of I and I', the measures of blue, black, and red edges satisfy  $\mu(\text{blue}) > \mu(\text{black}) > \mu(\text{red})$  and the switch condition  $\mu(\text{blue}) = \mu(\text{black}) + \mu(\text{red})$ . Types I and I' are mirror image to each other including the thickness data, and so are II and II'.

**Theorem 4** Train track types I, II, I', II' are related to each other by maximal splittings followed by a homeomorphism of the 3-punctured disk (i.e., a 3-braid) as shown in Fig. 6.

The first triple-weight train track  $\tau_{n+4}$  (Fig. 4) studied in Theorem 3 is Type I or II if we ignore the half twists  $\sigma_2^{-n} \in B_3$  and assuming the initial train track  $\tau_0$  is  $M_\alpha$  or  $W_\alpha$ . (This is because  $M_\alpha$  is a  $180^\circ$ -rotation of  $W_\alpha$ .) Similarly, if  $\tau_0 = M'_\alpha$  or  $W'_\alpha$  then the first triple-weight train track  $\tau_{n+4}$  is Type I' or II' if we forget the half twists  $\sigma_1^n \in B_3$ .

**Definition 7** It is convenient to define  $T_1 := \tau_{n+4}$  as it is the first triple-weight train track in our maximal splitting sequence and further define  $T_k := \tau_{n+4+k-1}$  for  $k \geq 1$ . Thus, given a



**Fig. 7** Obtaining  $T_0$  from  $\tau_{n+3}$

braid  $\beta$ , we obtain a maximal splitting sequence of triple-weight train tracks:

$$T_1 \rightarrow T_2 \rightarrow T_3 \rightarrow \dots$$

Folding is the inverse operation of a splitting.

**Definition 8** We fold the train track  $\tau_{n+3}$  and we call the resulting train track  $T_0$ . See Fig. 7. Note that  $T_0$  is a triple-weight train track. The three weights of  $T_0$  are

$$\frac{1-n\alpha}{2} \text{ (red)} < \frac{\alpha}{2} \text{ (black)} < \frac{1-(n-1)\alpha}{2} \text{ (blue)}.$$

We also note that  $T_0$  is Type I' (resp. Type I) in Fig. 6 up to the negative half twists  $\sigma_2^{-(n-1)} \in B_3$  (resp.  $\sigma_{n-1}$ ) if we start with the train track  $W_\alpha$  or  $M_\alpha$  (resp.  $W'_\alpha$  or  $M'_\alpha$ ).

An important observation is that applying maximal splitting to  $T_0$  results in  $T_1$ . So we can extend the above maximal splitting sequence to the following:

$$T_0 \rightarrow T_1 \rightarrow T_2 \rightarrow T_3 \rightarrow \dots$$

Before proving Theorem 4, we introduce new words (4-tuple, sign, 4-ratio of a triple-weight train track) in Definition 9 that are extensively used in the later discussion. Then we proceed to prove Theorem 4. In the proof, we find a nice relation between the types of 4-tuples and the types of triple-weight train tracks, which is stated as Corollary 2.

We have shown in Theorem 3 that the three weights of the train track  $T_1$  are

$$\frac{1-n\alpha}{2}, \frac{-1+(n+1)\alpha}{2}, \frac{\alpha}{2}. \quad (2)$$

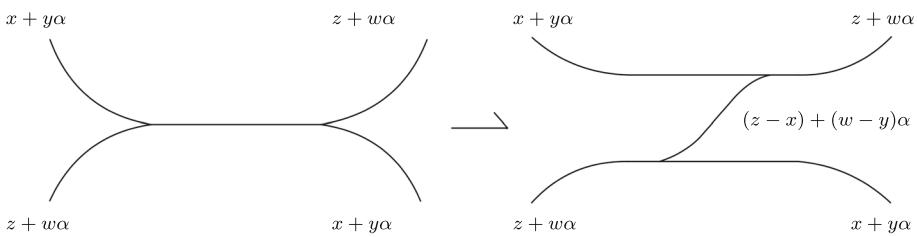
Therefore, every weight of a triple-weight train track  $T_k$  is in  $\frac{1}{2}\mathbb{Z} + \frac{1}{2}\alpha\mathbb{Z}$ .

**Definition 9** Suppose that the smallest weight of a triple-weight train track  $T$  is given by  $\frac{1}{2}x + \frac{1}{2}y\alpha$  and the second smallest weight is  $\frac{1}{2}z + \frac{1}{2}w\alpha$ . Thus  $x + y\alpha < z + w\alpha$ . We call

- $(x, y, z, w)$  the 4-tuple of the triple train track  $T$ ,
- $\text{sgn}(x) \in \{-1, 1\}$  the sign of the triple-weight train track  $T$  (we will show that  $x \neq 0$ ), and we denote  $\text{sgn}(T) = \text{sgn}(x)$ ,
- $\frac{x+y\alpha}{z+w\alpha}$  the 4-tuple ratio (or 4-ratio for short) of the triple-weight train track  $T$ .

Due to the switch condition of train tracks, the largest weight of the triple-weight train track is the sum of the other two, that is  $\frac{1}{2}(x+z) + \frac{1}{2}(y+w)\alpha$ . Thus the 4-tuple  $(x, y, z, w)$  carries all the weight data of the triple-weight train track.

We introduce a 4-tuple sequence  $\{(x_k, y_k, z_k, w_k) \mid k = 0, 1, \dots\}$  that plays an important role to prove the main results.



**Fig. 8** Maximal splitting of triple-weight train track

**Definition 10** For  $k = 0, 1, 2, \dots$ , let  $(x_k, y_k, z_k, w_k)$  denote the 4-tuple of  $\mathcal{T}_k$ .

For example, the 4-tuple of  $\mathcal{T}_0$  is

$$(x_0, y_0; z_0, w_0) := (1, -n; 0, 1). \quad (3)$$

Among the three weights of  $\mathcal{T}_1$  in (2),  $\frac{\alpha}{2}$  is the largest. Depending on the value of  $\alpha$  we have either

$$\frac{1}{2}(-1 + (n + 1)\alpha) < \frac{1}{2}(1 - n\alpha) \text{ or } \frac{1}{2}(1 - n\alpha) < \frac{1}{2}(-1 + (n + 1)\alpha).$$

In the former (resp. latter) case,

$$(x_1, y_1, z_1, w_1) = (-1, n + 1, 1, -n) \text{ (resp. } (1, -n, -1, n + 1))$$

and we say that the 4-tuple  $(x_1, y_1, z_1, w_1)$  is Type 1 (resp. Type 2). Combine this with the fact  $\frac{1}{n+1} < \alpha < \frac{1}{n}$ , and we obtain that the 4-tuple of  $\mathcal{T}_1$  is

$$(x_1, y_1, z_1, w_1) = \begin{cases} (-1, n + 1, 1, -n) & \text{if } \frac{1}{n+1} < \alpha < \frac{1}{n+\frac{1}{2}} \text{ (Type 1)} \\ (1, -n, -1, n + 1) & \text{if } \frac{1}{n+\frac{1}{2}} < \alpha < \frac{1}{n} \text{ (Type 2).} \end{cases}$$

**Proposition 2** In general, the 4-tuple  $(x_{k+1}, y_{k+1}, z_{k+1}, w_{k+1})$  can be computed from  $(x_k, y_k, z_k, w_k)$  as follows.

$$(x_{k+1}, y_{k+1}, z_{k+1}, w_{k+1}) := \begin{cases} (z_k - x_k, w_k - y_k, x_k, y_k) & \text{if } z_k - 2x_k < (2y_k - w_k)\alpha \text{ (Type 1)} \\ (x_k, y_k, z_k - x_k, w_k - y_k) & \text{if } (2y_k - w_k)\alpha < z_k - 2x_k \text{ (Type 2)} \end{cases}$$

For the first (resp. second) case, we say that the 4-tuple  $(x_{k+1}, y_{k+1}, z_{k+1}, w_{k+1})$  has Type 1 (resp. Type 2) and the maximal splitting  $\mathcal{T}_k \rightarrow \mathcal{T}_{k+1}$  has Type 1 (resp. Type 2), which we denote  $\mathcal{T}_k \xrightarrow{(*)} \mathcal{T}_{k+1}$  (resp.  $\mathcal{T}_k \xrightarrow{(**)} \mathcal{T}_{k+1}$ ).

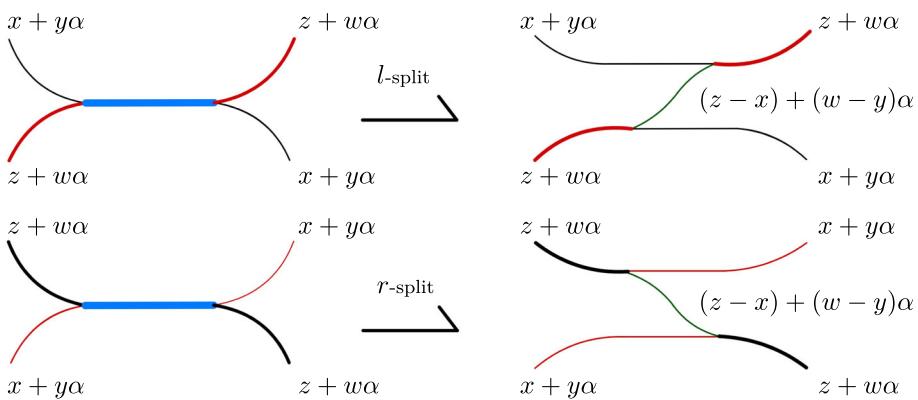
**Proof** Start with a triple-weight train track with  $(x, y, z, w)$ . See Fig. 8.

After a maximal splitting, the 4-tuple becomes  $(z - x, w - y, x, y)$  if  $(z - x) + (w - y)\alpha < x + y\alpha$ , equivalently  $z - 2x < (2y - w)\alpha$ . We call it Type 1.

Otherwise, we will have  $(x, y, z - x, w - y)$ . We call it Type 2.  $\square$

Before proving Theorem 4 we introduce two types of maximal splitting; *l-split* and *r-split*.

**Definition 11** Near an edge of a triple weight train track where maximal splitting occurs, there are two cases. When lower-left and upper-right have more weight than upper-left and lower-right, we call it an *l-split*. Otherwise we call it an *r-split*. See Fig. 9.



**Fig. 9** Two types of maximal splittings. (Top) is *l*-split and (Bottom) is *r*-split where  $x + y\alpha < z + w\alpha$

The terminologies *l*-split and *r*-split come from Mosher's article [13].

Now we prove Theorem 4.

**Proof of Theorem 4** Suppose that we have a train track  $\mathcal{T}$  with 4-tuple  $(x, y, z, w)$ . Assume that  $\mathcal{T}$  is Type I. The computation in Fig. 10 shows that the resulting train track  $\mathcal{T}^{spl}$  of the maximal splitting followed by the action of  $\sigma_1^{-1}$  is Type I' or II'.

Looking into the weights carefully with Proposition 2, we see that the 4-tuple of  $\mathcal{T}^{spl}$  is Type 1 (resp. Type 2) if and only if  $(z - x) + (w - y)\alpha < x + y\alpha$  (resp.  $>$ ) if and only if  $\mathcal{T}^{spl}$  followed by the action of  $\sigma_1^{-1}$  is Type I' (resp II'). This yields the arrows  $\xrightarrow{l,(*\sigma_1^{-1})}$  (resp.  $\xrightarrow{l,(\sigma_1^{-1})}$ ) in Fig. 6 coming out of the Type I train track.

Suppose next that we have the train track II with 4-tuple  $(x, y, z, w)$ . In Fig. 11, we show that the resulting train track after the maximal splitting followed by the action of  $\sigma_1^{-1}$  is either Type I or II. More precisely, after the maximal splitting we obtain a train track of Type I (resp. II) if and only if  $(z - x) + (w - y)\alpha < z + w\alpha$  (resp.  $>$ ) if and only if the new 4-tuple (equivalently, the splitting type) is Type 1 (resp. Type 2). This yields the arrows  $\xrightarrow{r,(*\sigma_1^{-1})}$  (resp.  $\xrightarrow{r,(\sigma_1^{-1})}$ ) in Fig. 6 coming out of the Type II train track (resp. coming back to itself).

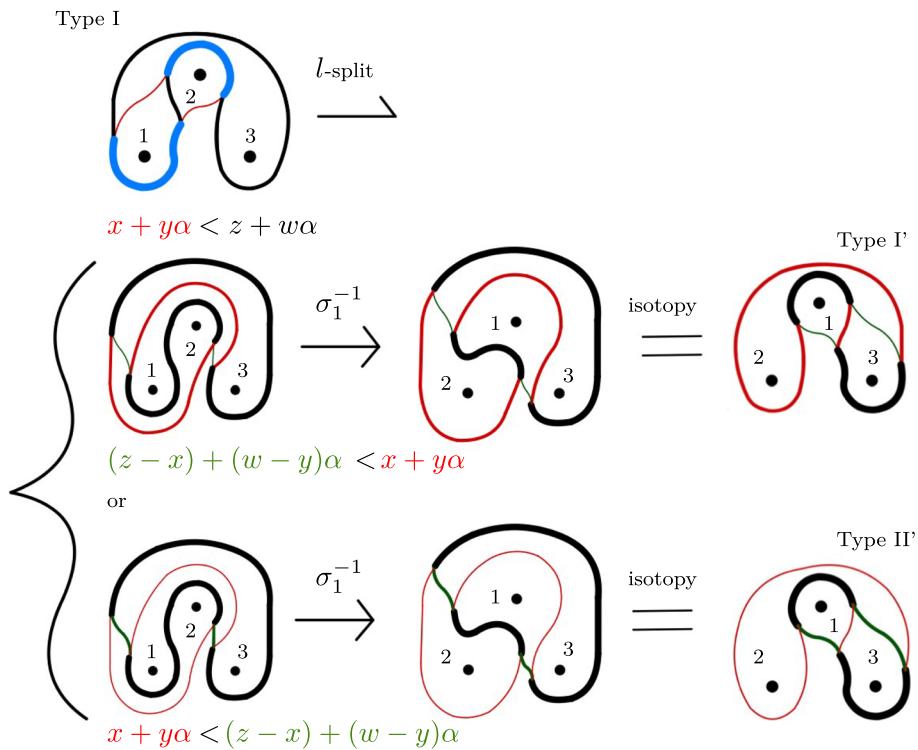
Recall Types I and I' are mirror to each other, and so are Types II and II'. This yields the rest of the maximal splitting and homeomorphism arrows and completes the maximal splitting diagram in Fig. 6.  $\square$

We extend the notion of Types I, II, I' and II' in Definition 6, and to each triple-weight train track  $\mathcal{T}_i$  we inductively assign one of the four types.

**Definition 12** As discussed in Definition 8 if we start with  $W_\alpha$  or  $M_\alpha$  (resp.  $W'_\alpha$  or  $M'_\alpha$ ) the triple-weight train track  $\mathcal{T}_0$  is Type I' (resp. I) denoted  $Type(\mathcal{T}_0) = I'$  (resp.  $Type(\mathcal{T}_0) = I$ ).

We inductively define  $Type(\mathcal{T}_{k+1})$  from  $Type(\mathcal{T}_k)$ . We need consider four cases. See Fig. 12.

- Assume that  $Type(\mathcal{T}_k) = I$  or  $I'$  and the 4-tuple of  $\mathcal{T}_{k+1}$  is Type 1. Then we define  $Type(\mathcal{T}_{k+1}) = I'$ .
- Assume that  $Type(\mathcal{T}_k) = I$  or  $I'$  and the 4-tuple of  $\mathcal{T}_{k+1}$  is Type 2. Then we define  $Type(\mathcal{T}_{k+1}) = II'$ .



**Fig. 10** A Type I train track admits  $l$ -split followed by action of  $\sigma_1^{-1}$  and it yields a train track of Type I' or II'

- Assume that  $\text{Type}(\mathcal{T}_k) = \text{II}$  and the 4-tuple of  $\mathcal{T}_{k+1}$  has Type 1. Then we define  $\text{Type}(\mathcal{T}_{k+1}) = \text{I}$ .
- Assume that  $\text{Type}(\mathcal{T}_k) = \text{II}$  and the 4-tuple of  $\mathcal{T}_{k+1}$  has Type 2. Then we define  $\text{Type}(\mathcal{T}_{k+1}) = \text{II}$ .

We observe the following relation between topological types (Type I, I', II, II') of triple-weight train tracks and types of 4-tuples (Type 1, 2).

**Corollary 2** *Type I and Type I' triple-weight train tracks are results of a Type I maximal splitting. Likewise, Type II and Type II' triple-weight train tracks are results of a Type 2 maximal splitting. In other words, a triple-weight train track is*

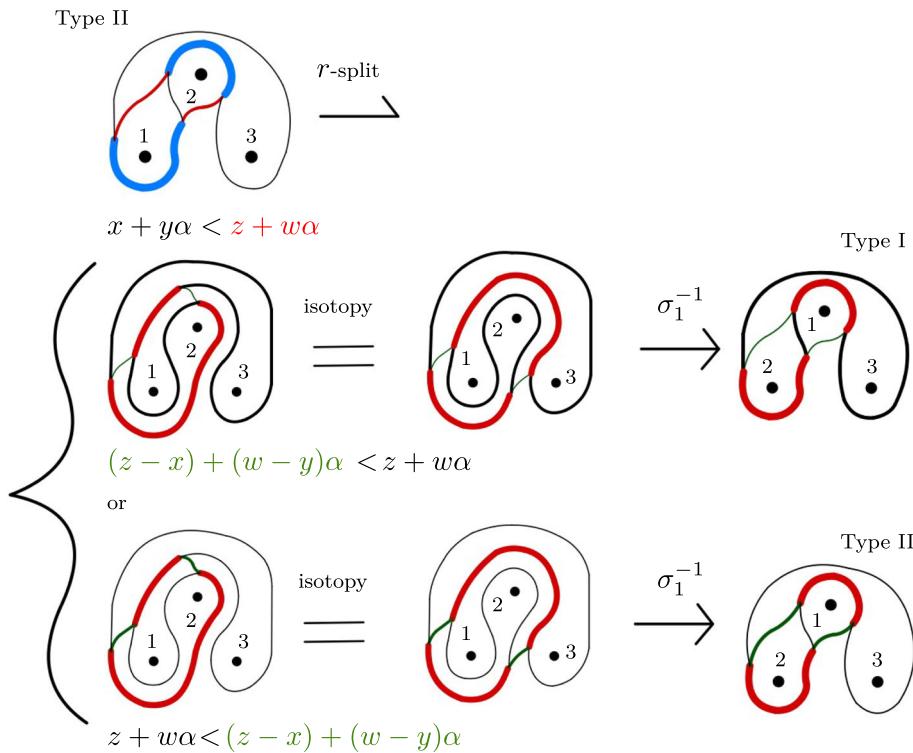
- *Type I or I' if and only if its 4-tuple is Type 1,*
- *Type II or II' if and only if its 4-tuple is Type 2.*

Here is an immediate consequence of Theorem 4.

**Corollary 3** *Type I and Type II' train tracks only admit  $l$ -splittings.*

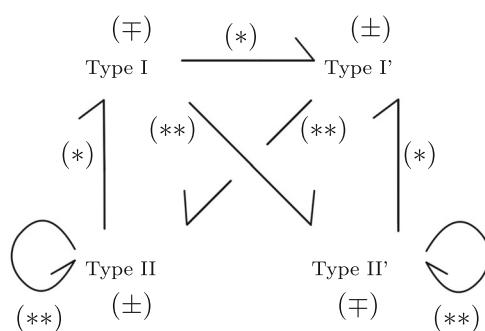
*Type I' and Type II train tracks only admit  $r$ -splittings.*

**Proof** Let  $X \in \{\text{I}, \text{II}, \text{I}', \text{II}'\}$ . For a Type X train track  $\mathcal{T}$  one can find a self homeomorphism of  $D_3$  that takes  $\mathcal{T}$  to the train track X (Fig. 6). By Theorem 4 we see that the train tracks I and II' only admit an  $l$ -splitting. We also see that the train tracks I' and II only admit an



**Fig. 11** A Type II train track admits an *r*-split followed by action of  $\sigma_1^{-1}$  which yields a train track of Type I or II

**Fig. 12** Four train track types I, II,  $\Gamma$ , and II'



*r*-splitting. The splitting type (*r* or *l*) only depends on nearby weights of the maximal weight edge. Since the homeomorphism preserves the local weight system, the statements follow.  $\square$

## 4 Triple-weight train track and Farey sequence

In this section, we establish a relationship between triple-weight train tracks and Farey sequences that we use to prove our main result Theorem 6.

Let  $\alpha \in (\frac{1}{n+1}, \frac{1}{n}) \setminus \mathbb{Q}$  be the MP-ratio of some pseudo-Anosov 3-braid. We will define an infinite set of intervals and then construct a map  $T$  which associates to each interval a 4-tuple of numbers. There is a unique infinite sequence  $I_{1,i_1} \supset I_{2,i_2} \supset I_{3,i_3} \supset \dots$  such that  $\bigcap_{k=1}^{\infty} I_{k,i_k} = \{\alpha\}$ . The goal of the section is to prove Theorem 5, where we relate the interval  $I_{k,i_k}$  and the 4-tuple  $(x_k, y_k, z_k, w_k)$  under the map  $T$ .

## 4.1 Farey sequence

First, we recall the *Farey sum*,  $\boxplus$ . Given two fractions,  $\frac{s}{t}$  and  $\frac{s'}{t'}$ , we define:

$$\frac{s}{t} \boxplus \frac{s'}{t'} := \frac{s+s'}{t+t'}.$$

Notice that if  $\frac{s}{t} < \frac{s'}{t'}$ , then  $\frac{s}{t} < \frac{s}{t} \boxplus \frac{s'}{t'} < \frac{s'}{t'}$ .

For  $k = 0, 1, 2, \dots$ , we inductively define an ordered set  $L_k$  of fractions with cardinality  $2^k + 1$ . The set  $L_k$  is called the *Farey sequence* of order  $k + 1$ . Define

$$L_0 = \{a_{0,1}, a_{0,2}\} = \left\{ \frac{0}{1}, \frac{1}{1} \right\}$$

where we are deliberately not simplifying the fractions. Assume  $L_k = \{a_{k,1}, a_{k,2}, \dots, a_{k,2^k+1}\}$  where

$$\frac{0}{1} = a_{k,1} < a_{k,2} < \dots < a_{k,2^k} < a_{k,2^k+1} = \frac{1}{1}.$$

Define

$$L_{k+1} = \{a_{k+1,1}, \dots, a_{k+1,2^{k+1}+1}\} = L_k \cup \{a_{k,i} \boxplus a_{k,i+1} \mid i = 1, 2, \dots, 2^k\}$$

as a set and then reorder them to have  $\frac{0}{1} = a_{k+1,1} < a_{k+1,2} < \dots < a_{k+1,2^{k+1}+1} = \frac{1}{1}$ .

### Example 1

$$\begin{aligned} L_1 &= \{a_{1,1}, a_{1,2}, a_{1,3}\} = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\} \\ L_2 &= \{a_{2,1}, \dots, a_{2,5}\} = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\} \\ L_3 &= \{a_{3,1}, \dots, a_{3,9}\} = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\} \end{aligned}$$

**Definition 13** Let  $n = \lfloor \frac{1}{\alpha} \rfloor$ . For  $k = 0, 1, \dots$  and  $i = 1, 2, \dots, 2^k$ , define the following open interval:

$$I_{k,i} := \left( \frac{1}{n+a_{k,i+1}}, \frac{1}{n+a_{k,i}} \right)$$

In particular, we have that  $I_{0,1} = \left( \frac{1}{n+1}, \frac{1}{n+0} \right) = \left( \frac{1}{n+1}, \frac{1}{n} \right)$  and

$$\bigsqcup_{i=1}^{2^k} I_{k,i} = \left[ \frac{1}{n+1}, \frac{1}{n} \right] \setminus \left\{ \frac{1}{n+a} \mid a \in L_k \right\}. \quad (4)$$

**Example 2** For  $k = 1$ ,

$$\begin{aligned} I_{1,2} \cup I_{1,1} &= \left( \frac{1}{n+\frac{1}{1}}, \frac{1}{n+\frac{1}{2}} \right) \cup \left( \frac{1}{n+\frac{1}{2}}, \frac{1}{n+\frac{0}{1}} \right) \\ &= \left[ \frac{1}{n+1}, \frac{1}{n} \right] \setminus \left\{ \frac{1}{n+\frac{1}{1}}, \frac{1}{n+\frac{1}{2}}, \frac{1}{n+\frac{0}{1}} \right\} \end{aligned}$$

For  $k = 2$ ,

$$\begin{aligned} I_{2,4} \cup I_{2,3} \cup I_{2,2} \cup I_{2,1} &= \left( \frac{1}{n+\frac{1}{1}}, \frac{1}{n+\frac{2}{3}} \right) \cup \left( \frac{1}{n+\frac{2}{3}}, \frac{1}{n+\frac{1}{2}} \right) \cup \left( \frac{1}{n+\frac{1}{2}}, \frac{1}{n+\frac{1}{3}} \right) \cup \left( \frac{1}{n+\frac{1}{3}}, \frac{1}{n+\frac{0}{1}} \right) \\ &= \left[ \frac{1}{n+1}, \frac{1}{n} \right] \setminus \left\{ \frac{1}{n+\frac{1}{1}}, \frac{1}{n+\frac{2}{3}}, \frac{1}{n+\frac{1}{2}}, \frac{1}{n+\frac{1}{3}}, \frac{1}{n+\frac{0}{1}} \right\} \end{aligned}$$

We observe that  $I_{1,2} \setminus \left\{ \frac{1}{n+\frac{2}{3}} \right\} = I_{2,4} \cup I_{2,3}$  and  $I_{1,1} \setminus \left\{ \frac{1}{n+\frac{1}{3}} \right\} = I_{2,2} \cup I_{2,1}$ .

In general, we have the following:

**Lemma 2** For any  $k = 0, 1, \dots$  and  $i = 1, 2, \dots, 2^k$ , the interval  $I_{k,i} = \left( \frac{1}{n+a_{k,i+1}}, \frac{1}{n+a_{k,i}} \right)$  at the  $k$ th stage splits into two at the  $(k+1)$ st stage.

$$\begin{aligned} I_{k,i} \setminus \left\{ \frac{1}{n+a_{k+1,2i}} \right\} &= I_{k+1,2i} \sqcup I_{k+1,2i-1} \\ &= \left( \frac{1}{n+a_{k+1,2i+1}}, \frac{1}{n+a_{k+1,2i}} \right) \sqcup \left( \frac{1}{n+a_{k+1,2i}}, \frac{1}{n+a_{k+1,2i-1}} \right) \end{aligned} \tag{5}$$

A standard fact about Farey sequence is that  $L_1 \subset L_2 \subset \dots \subset \cup_{k=1}^{\infty} L_k = \mathbb{Q} \cap [0, 1]$ , which gives the following.

**Lemma 3** Let  $\alpha \in \left( \frac{1}{n+1}, \frac{1}{n} \right) \setminus \mathbb{Q}$ . By Eq. 4, for every  $k = 0, 1, \dots$ , there exists an index  $i_k \in \{1, 2, 3, \dots, 2^k\}$  such that

$$\alpha \in I_{k,i_k} = \left( \frac{1}{n+a_{k,i_k+1}}, \frac{1}{n+a_{k,i_k}} \right). \tag{6}$$

Moreover, by Lemma 2 we obtain a sequence of nested intervals

$$I_{0,1} \supset I_{1,i_1} \supset I_{2,i_2} \supset I_{3,i_3} \supset \dots \supset \bigcap_{k=1}^{\infty} I_{k,i_k} = \{\alpha\}. \tag{7}$$

with  $i_k \in \{2i_{k-1}, 2i_{k-1} - 1\}$ . In particular, the sequence  $\{i_k\}_{k=1}^{\infty}$  is an invariant of the irrational number  $\frac{1}{n+1} < \alpha < \frac{1}{n}$ .

## 4.2 Computation of 4-tuples

The goal of this subsection is to prove Theorem 5 which allows us to compute the 4-tuple  $(x_k, y_k; z_k, w_k)$  of the triple-weight train track  $\mathcal{T}_k$  inductively. We do this using  $2 \times 2$  matrices.

**Definition 14** We inductively define a function  $T$  which associates to each interval  $I_{k,i}$  a  $2 \times 2$  matrix. Set

$$T(I_{0,1}) = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}. \quad (8)$$

Suppose that we have defined the function for  $I_{k,i}$  as

$$T(I_{k,i}) = \begin{pmatrix} x & z \\ y & w \end{pmatrix}. \quad (9)$$

Then we define the function for  $I_{k+1,2i-1}$  and  $I_{k+1,2i}$  as follows:

$$T(I_{k+1,2i}) = \begin{cases} \begin{pmatrix} x & z \\ y & w \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} z-x & x \\ w-y & y \end{pmatrix} & \text{if } x > 0 \\ \begin{pmatrix} x & z \\ y & w \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & z-x \\ y & w-y \end{pmatrix} & \text{if } x < 0 \end{cases} \quad (10)$$

$$T(I_{k+1,2i-1}) = \begin{cases} \begin{pmatrix} x & z \\ y & w \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & z-x \\ y & w-y \end{pmatrix} & \text{if } x > 0 \\ \begin{pmatrix} x & z \\ y & w \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} z-x & x \\ w-y & y \end{pmatrix} & \text{if } x < 0 \end{cases} \quad (11)$$

The next lemma shows that elements of the matrix  $T(I_{k,i})$  are in  $\mathbb{Z}n + \mathbb{Z}$  and contains all the information of the intervals  $I_{k,i}$ .

**Lemma 4** For each  $k, i$ , there exist  $\epsilon = \epsilon_{k,i} \in \{-1, 1\}$ ,  $a = a_{k,i} \in \mathbb{Z}_{>0}$  and  $b = b_{k,i}, c = c_{k,i}, d = d_{k,i} \in \mathbb{Z}_{\geq 0}$  such that

$$T(I_{k,i}) = \epsilon \begin{pmatrix} a & -b \\ -an - c & bn + d \end{pmatrix}. \quad (12)$$

Moreover, the non-negative integers  $a, b, c, d$  satisfy:

$$I_{k,i} = \begin{cases} \left( \frac{1}{n+\frac{c+d}{a+b}}, \frac{1}{n+\frac{c}{a}} \right) & \text{if } \epsilon = 1 \\ \left( \frac{1}{n+\frac{c}{a}}, \frac{1}{n+\frac{c+d}{a+b}} \right) & \text{if } \epsilon = -1 \end{cases} \quad (13)$$

and

$$\det T(I_{k,i}) = ad - bc = \text{sgn}(\epsilon). \quad (14)$$

We call  $\epsilon_{k,i}$  the sign of the interval  $I_{k,i}$ .

**Proof** (Induction on  $k$ ) When  $k = 0$ , the initial condition (8) gives  $\epsilon = 1, a = 1, b = 0, c = 0, d = 1$ , and  $I_{0,1} = \left( \frac{1}{n+\frac{1}{1}}, \frac{1}{n+\frac{0}{1}} \right)$ . Thus, all the assertions hold.

We assume that the assertions hold for  $I_{k,i}$  and will show that assertions (12) and (13) hold for both  $I_{k+1,2i}$  and  $I_{k+1,2i-1}$ .

When  $\epsilon_{k,i} = 1$ , by the induction hypothesis we have  $I_{k,i} = \left( \frac{1}{n+\frac{c+d}{a+b}}, \frac{1}{n+\frac{c}{a}} \right)$ . By (10) and (11), we have

$$T(I_{k+1,2i}) \stackrel{(10)}{=} - \begin{pmatrix} a+b & -a \\ -(a+b)n - (c+d) & an + c \end{pmatrix} =: - \begin{pmatrix} a' & -b' \\ -a'n - c' & b'n + d' \end{pmatrix}$$

and

$$T(I_{k+1,2i-1}) \stackrel{(11)}{=} \begin{pmatrix} a & -(a+b) \\ -an - c & (a+b)n + c + d \end{pmatrix} =: \begin{pmatrix} a'' & -b'' \\ -a''n - c'' & b''n + d'' \end{pmatrix}.$$

We see that  $a', a'' \in \mathbb{Z}_{>0}$  and  $b', b'', c', c'', d, d'' \in \mathbb{Z}_{\geq 0}$ . Thus (12) is satisfied for  $k+1$  where  $\epsilon_{k+1,2i} = -1$  and  $\epsilon_{k+1,2i-1} = 1$ . Next we check (13) for  $k+1$ . By Lemma 2 and the induction hypothesis (13),

$$I_{k+1,2i} = \left( \frac{1}{n+\frac{c+d}{a+b}}, \frac{1}{n+\frac{c+d}{a+b} \boxplus \frac{c}{a}} \right) = \left( \frac{1}{n+\frac{c+d}{a+b}}, \frac{1}{n+\frac{2c+d}{2a+b}} \right) = \left( \frac{1}{n+\frac{c'}{a'}}, \frac{1}{n+\frac{c'+d'}{a'+b'}} \right)$$

and

$$I_{k+1,2i-1} = \left( \frac{1}{n+\frac{c+d}{a+b} \boxplus \frac{c}{a}}, \frac{1}{n+\frac{c}{a}} \right) = \left( \frac{1}{n+\frac{2c+d}{2a+b}}, \frac{1}{n+\frac{c}{a}} \right) = \left( \frac{1}{n+\frac{c''+d''}{a''+b''}}, \frac{1}{n+\frac{c''}{a''}} \right).$$

Thus the assertion (13) is true for  $I_{k+1,2i}$  and  $I_{k+1,2i-1}$ .

When  $\epsilon_{k,i} = -1$ , a similar argument works. Thus, (13) is proved for  $k+1$ .

Finally, for the last assertion (14) we observe that  $|\det T(I_{k,i})| = |ad - bc| = 1$  by the inductive definition of the function  $T$ . When  $\epsilon = 1$  by (13) we get  $\frac{c}{a} < \frac{c+d}{a+b}$ , which gives  $0 < ad - bc$  and thus  $ad - bc = 1$ . Similarly when  $\epsilon = -1$  we get  $ad - bc = -1$ .

Recall Lemma 2 which states that the interval  $I_{k,i}$  splits into  $I_{k+1,2i}$  and  $I_{k+1,2i-1}$ .

**Lemma 5** *The signs of the intervals  $I_{k+1,2i}$  and  $I_{k+1,2i-1}$  are*

$$\epsilon_{k+1,2i} = -1 \text{ and } \epsilon_{k+1,2i-1} = 1.$$

**Proof** By (9) and (12) we have  $x = \epsilon a$  and  $z = -\epsilon b$ . Knowing that  $a > 0$  we get  $\operatorname{sgn}(x) = \epsilon$ . We also see  $\operatorname{sgn}(z-x) = \operatorname{sgn}(-\epsilon(b+a)) = -\epsilon = -\operatorname{sgn}(x)$ . The second equation follows since  $a > 0$  and  $b \geq 0$ . The definition of  $T$  in (11) gives

$$\epsilon_{k+1,2i-1} = \begin{cases} \operatorname{sgn}(x) = 1 & \text{if } x > 0 \\ \operatorname{sgn}(z-x) = 1 & \text{if } x < 0. \end{cases}$$

A similar argument with (10) yields  $\epsilon_{k+1,2i} = -1$ .

Now, using the map  $T$  we can finally relate the nested intervals  $I_{k,i_k}$  that converge to  $\alpha$  (Lemma 3) and the 4-tuple  $(x_k, y_k; z_k, w_k)$  of the train track  $\mathcal{T}_k$  defined in Definition 10.

**Theorem 5** *For every  $k = 0, 1, \dots$ , the 4-tuple  $(x_k, y_k; z_k, w_k)$  of the train track  $\mathcal{T}_k$  satisfies*

$$\begin{pmatrix} x_k & z_k \\ y_k & w_k \end{pmatrix} = T(I_{k,i_k}) \stackrel{\text{Lem4}}{=} \epsilon_k \begin{pmatrix} a_k & -b_k \\ -a_k n - c_k & b_k n + d_k \end{pmatrix} \quad (15)$$

for  $\epsilon_k \in \{1, -1\}$  and some  $a_k \in \mathbb{Z}_{>0}$  and  $b_k, c_k, d_k \in \mathbb{Z}_{\geq 0}$  with  $a_k d_k - b_k c_k = \epsilon_k$ .

Note that  $\epsilon_k = \text{sgn}(\mathcal{T}_k)$ , the *sign* of the train track  $\mathcal{T}_k$  as in Definition 9.

**Proof** The proof is done by induction. When  $k = 0$ , the assertion holds by (3) and (8).

Assume that (15) holds for  $k$ . Recall that  $\bigcap_{l=1}^{\infty} I_{l,i_l} = \{\alpha\}$ . Since  $I_{k,i_k}$  splits into two intervals  $I_{k+1,2i_k}$  and  $I_{k+1,2i_k-1}$ , we have either  $I_{k+1,i_{k+1}} = I_{k+1,2i_k}$  or  $I_{k+1,i_{k+1}} = I_{k+1,2i_k-1}$ , equivalently  $i_{k+1} = 2i_k$  or  $2i_k - 1$ . There are four cases to consider depending on the sign of  $\epsilon_k$  and the type (see Proposition 2) of the 4-tuple.

Suppose that  $\epsilon_k = 1$  and  $(x_{k+1}, y_{k+1}, z_{k+1}, w_{k+1})$  has Type 1. By the induction hypothesis, we get

$$\epsilon_k(2(-a_k n - c_k) - (b_k n + d_k))\alpha \stackrel{(15)}{=} (2y_k - w_k)\alpha \stackrel{\text{Type 1}}{>} z_k - 2x_k \stackrel{(15)}{=} \epsilon_k(-b_k - 2a_k),$$

which yields  $\alpha < \frac{1}{n + \frac{2c_k + d_k}{2a_k + b_k}}$ . Since  $I_{k,i_k} = \left( \frac{1}{n + \frac{c_k + d_k}{a_k + b_k}}, \frac{1}{n + \frac{c_k}{a_k}} \right)$  by (13) and the assumption  $\epsilon_k = 1$ , we have

$$\alpha \in \left( \frac{1}{n + \frac{c_k + d_k}{a_k + b_k}}, \frac{1}{n + \frac{2c_k + d_k}{2a_k + b_k}} \right) \stackrel{(5)}{=} I_{k+1,2i_k}.$$

This gives

$$i_{k+1} = 2i_k. \quad (16)$$

The assertion can be verified as follows:

$$\begin{pmatrix} x_{k+1} & z_{k+1} \\ y_{k+1} & w_{k+1} \end{pmatrix} \stackrel{\text{Prop 2}}{=} \begin{pmatrix} z_k - x_k & x_k \\ w_k - y_k & y_k \end{pmatrix} \stackrel{(10)}{=} T(I_{k+1,2i_k}) = T(I_{k+1,i_{k+1}}).$$

Similarly, we can verify the assertion for the remaining three cases.  $\square$

The following technical corollary is deduced in the discussion of the above proof and it will be useful later.

**Corollary 4** *In the above proof of Theorem 5, we have observed that:*

1. If  $\epsilon_k = +1$  and  $(x_{k+1}, y_{k+1}, z_{k+1}, w_{k+1})$  has Type 1 then  $i_{k+1} = 2i_k$ .
2. If  $\epsilon_k = +1$  and  $(x_{k+1}, y_{k+1}, z_{k+1}, w_{k+1})$  has Type 2 then  $i_{k+1} = 2i_k - 1$ .
3. If  $\epsilon_k = -1$  and  $(x_{k+1}, y_{k+1}, z_{k+1}, w_{k+1})$  has Type 1 then  $i_{k+1} = 2i_k - 1$ .
4. If  $\epsilon_k = -1$  and  $(x_{k+1}, y_{k+1}, z_{k+1}, w_{k+1})$  has Type 2 then  $i_{k+1} = 2i_k$ .

With Lemma 5, we obtain:

- for Case (1) and (4)  $\epsilon_{k+1} = -1$ , and
- for Case (2) and (3)  $\epsilon_{k+1} = +1$ .

In particular, a Type 1 maximal splitting  $\xrightarrow{(*)}$  changes the sign of the train track and a Type 2 maximal splitting  $\xrightarrow{(**)}$  preserves the sign of the train track. Thus, in Fig. 6, we only see  $(\pm) \xrightarrow{(*)} (\mp)$  and  $(\pm) \xrightarrow{(**)} (\pm)$ .

Here is a corollary of Corollary 4.

**Corollary 5** *If  $\beta \in B_3$  starts with  $W_\alpha$  or  $M_\alpha$  then  $\text{Type}(\mathcal{T}_0) = I'$  and  $\text{sgn}(\mathcal{T}_0) = +1$ . In this case Type I' or Type II' train tracks that appear in the splitting sequence always have  $\text{sgn} = +1$ , and Type I and Type II train tracks have  $\text{sgn} = -1$ . For the  $(\pm)$  and  $(\mp)$  labellings in Fig. 6 we use the upper signs.*

On the other hand, if  $\beta \in B_3$  starts with  $W'_\alpha$  or  $M'_\alpha$  then  $\text{Type}(\mathcal{T}_0) = I$  and  $\text{sgn}(\mathcal{T}_0) = +1$ . In this case Type I or Type II' train tracks have  $\text{sgn} = +1$ , and Type I' and Type II train tracks have  $\text{sgn} = -1$ . For the  $(\pm)$  and  $(\mp)$  labellings in Fig. 6 we use the lower signs.

**Proof** The statement follows from the last part of Corollary 4, the definition of  $\mathcal{T}_0$  (Definition 8), and the definition of  $\text{Type}(\mathcal{T}_0)$  (Definition 12).  $\square$

**Corollary 6** If 3-braids  $\beta$  and  $\beta'$  are conjugate in  $B_3/Z(B_3)$  then

$$\text{Type}(\mathcal{T}_0) = \text{Type}(\mathcal{T}'_0) \in \{I, I'\}.$$

## 5 Nested Farey intervals

In this section we define nested intervals  $\{J_k\}_k$  in the Farey tessellation and an **LR**-sequence.

Let  $\beta$  be a pseudo-Anosov 3-braid with the MP-ratio  $\alpha$ . Let  $n = \lfloor 1/\alpha \rfloor$ . In Lemma 3, we have shown that  $\alpha$  is the intersection of nested intervals  $I_{1,i_1} \supset I_{2,i_2} \supset I_{3,i_3} \supset \dots$  where the interval  $I_{k,i_k} = \left( \frac{1}{n+a_{k,i_k}+1}, \frac{1}{n+a_{k,i_k}} \right)$ . We define  $J_k$  as a ‘reciprocal’ of  $I_{k,i_k}$ :

**Definition 15** Define an open interval

$$J_k := (a_{k,i_k}, a_{k,i_k+1})$$

for  $k = 0, 1, \dots$ . Note that  $J_0 = (\frac{0}{1}, \frac{1}{1})$  and  $J_k$  is a Farey interval since there is an arc in the Farey tessellation connecting the boundary points  $a_{k,i_k}$  and  $a_{k,i_k+1}$ .

The fact  $\alpha \in I_{k,i_k}$  is equivalent to  $\frac{1}{\alpha} - n \in J_k$  and the nested sequence (7) can be translated into

$$J_0 \supset J_1 \supset J_2 \supset J_3 \supset \dots \supset \bigcap_{k=1}^{\infty} J_k = \left\{ \frac{1}{\alpha} - n \right\}, \quad (17)$$

which we call the sequence of *nested Farey intervals* for the braid  $\beta$ .

**Definition 16** The interval  $J_k = (a_{k,i_k}, a_{k,i_k+1})$  splits into two Farey intervals;

- (L) the left subinterval  $(a_{k,i_k}, a_{k,i_k} \boxplus a_{k,i_k+1})$  that is reciprocal to  $I_{k+1,2i_k-1}$ , and
- (R) the right subinterval  $(a_{k,i_k} \boxplus a_{k,i_k+1}, a_{k,i_k+1})$  that is reciprocal to  $I_{k+1,2i_k}$ .

For each  $k = 0, 1, \dots$ , the interval  $J_{k+1}$  is exactly the left or right subinterval. We associate a letter **L** or **R** to each interval  $J_{k+1}$  depending on the left or right subinterval status. The nested interval sequence (17) can be encoded into a sequence in **L** and **R**. We call it the **LR-sequence** for  $\beta$ .

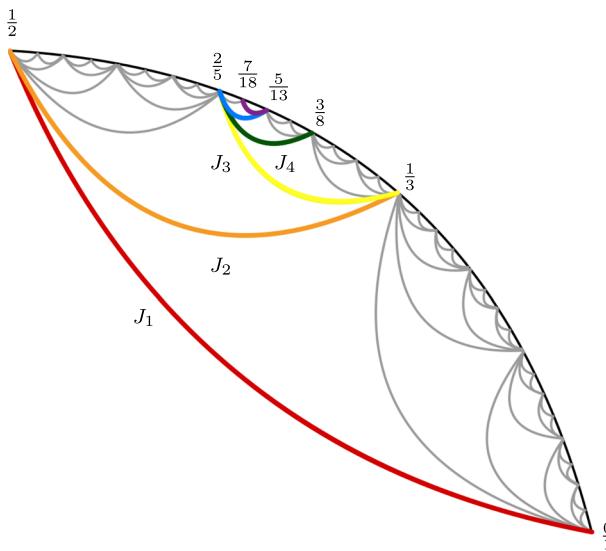
**Example 3** Let  $\beta = \sigma_1^4 \sigma_2 \sigma_1^3 \sigma_2^4$  which is a pseudo-Anosov 3-braid. The MP-ratio is  $\alpha = (19 - \sqrt{221})/14 \approx 0.2952 \dots$  and  $n = \lfloor 1/\alpha \rfloor = 3$ . Here is an estimate of  $1/\alpha - n$ :

$$\frac{0}{1} < \frac{1}{3} < \frac{3}{8} < \frac{5}{13} < \dots < \frac{1}{\alpha} - n < \dots < \frac{7}{18} < \frac{2}{5} < \frac{1}{2} < \frac{1}{1}$$

This gives the nested Farey intervals that converges to  $1/\alpha - n$  (see Fig. 13)

$$\begin{aligned} J_0 &= (0/1, 1/1) \supset J_1 = (0/1, 1/2) \supset J_2 = (1/3, 1/2) \supset J_3 = (1/3, 2/5) \\ &\supset J_4 = (3/8, 2/5) \supset J_5 = (5/13, 2/5) \supset J_6 = (5/13, 7/18) \supset \dots \end{aligned}$$

and its associated **LR**-sequence; **L, R, L, R, R, L, …**. To see this, we note that  $J_1$  is the left subinterval of  $J_0$ ; thus the first letter of the sequence is **L**. Likewise,  $J_2$  is the right subsequence of  $J_1$ ; thus the second letter of the sequence is **R**.



**Fig. 13** (Example 3) Nested Farey intervals for  $\beta = \sigma_1^4 \sigma_2 \sigma_1^3 \sigma_2^4$ ;  $J_1 = (0/1, 1/2)$  red,  $J_2 = (1/3, 1/2)$  orange,  $J_3 = (1/3, 2/5)$  yellow,  $J_4 = (3/8, 2/5)$  green,  $J_5 = (5/13, 2/5)$  blue,  $J_6 = (5/13, 7/18)$  purple. (Color figure online)

**Example 4** Let  $\beta = \sigma_1^4 \sigma_2 \sigma_1^3 \sigma_2^4$  and  $\beta' = \sigma_1^{-4} \sigma_2^{-4} \sigma_1^{-3} \sigma_2^{-1}$  be pseudo-Anosov 3-braids. It is interesting that  $\beta$  is conjugate to  $\beta' \Delta^8$  in  $B_3$ . Therefore,  $\beta$  and  $\beta'$  are conjugate in the quotient group  $B_3/Z(B_3) \cong \mathcal{MCG}(D_3)/Z$ . It is also interesting that  $\beta$  is conjugate to the mirror image of the negative flype (Definition 17) of  $\beta'$ , where  $\beta'' := \text{flype}(\beta') = \sigma_1^{-4} \sigma_2^{-1} \sigma_1^{-3} \sigma_2^{-4}$  and  $\text{mirror}(\beta'') = \sigma_2^4 \sigma_1 \sigma_2^3 \sigma_1^4$ .

By Theorem 1 of Agol, they must have combinatorially isomorphic Agol cycles. Indeed, their I-II-I'-II'-sequences and **LR**-sequences are the same up to shift.

The I-II-I'-II'-sequences of  $\beta$  and  $\beta'$  are

$$\text{II}', \text{I}', \text{I}, \text{I}', \text{II}, \text{I}, \text{II}', \text{I}', \text{I}, \text{I}', \text{II}, \text{I}, \text{II}', \text{I}', \text{I}, \text{I}', \text{II}, \text{I}, \text{II}', \text{I}', \text{I}, \text{I}', \text{II}, \text{I}, \dots$$

and

$$\text{I}', \text{I}, \text{II}', \text{I}', \text{I}, \text{I}', \text{II}, \text{I}, \text{II}', \text{I}', \text{I}, \text{I}', \text{II}, \text{I}, \text{II}', \text{I}', \text{I}, \text{I}', \text{II}, \text{I}, \dots,$$

respectively. The computations were completed with the use of MatLab. Removing the beginning  $\text{I}'$ ,  $\text{I}$  from the latter sequence, the two I-II-I'-II'-sequences become identical.

The **LR**-sequences of  $\beta$  and  $\beta'$  are

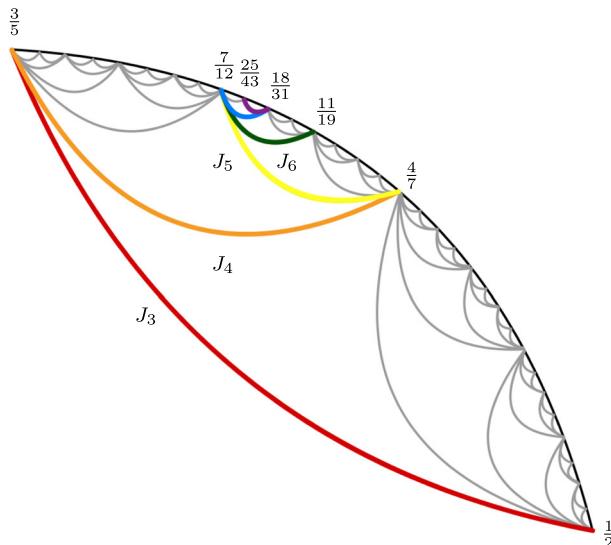
$$\text{L}, \text{R}, \text{L}, \text{R}, \text{R}, \text{L}, \text{L}, \text{R}, \text{L}, \text{R}, \text{R}, \text{L}, \text{L}, \text{R}, \text{L}, \text{R}, \text{R}, \text{L}, \text{L}, \text{R}, \text{R}, \text{L}, \text{R}, \text{R}, \text{L}, \dots$$

and

$$\text{R}, \text{L}, \text{L}, \text{R}, \text{L}, \text{R}, \text{R}, \text{L}, \text{L}, \text{R}, \text{L}, \text{R}, \text{R}, \text{L}, \text{L}, \text{R}, \text{L}, \text{R}, \text{R}, \text{L}, \text{L}, \text{R}, \text{R}, \text{L}, \text{R}, \text{R}, \text{L}, \dots,$$

respectively. Again, removing the beginning **R**, **L** from the latter sequence, the two **LR**-sequences are identical.

In fact,  $\beta'$  has  $\alpha' = (37 + \sqrt{221})/82 \approx 0.6325130335 \dots$  and  $n' = \lfloor 1/\alpha' \rfloor = 1$  and nested Farey intervals  $J_0 = (\frac{0}{1}, \frac{1}{1}) \supset J_1 = (\frac{1}{2}, \frac{1}{1}) \supset J_2 = (\frac{1}{2}, \frac{2}{3}) \supset J_3 = (\frac{1}{2}, \frac{3}{5}) \supset J_4 = (\frac{4}{7}, \frac{3}{5}) \supset \dots$



**Fig. 14** (Example 4) Nested Farey intervals for  $\beta' = \sigma_1^{-4} \sigma_2^{-4} \sigma_1^{-3} \sigma_2^{-1}$

$J_5 = (\frac{4}{7}, \frac{7}{12}) \supset J_6 = (\frac{11}{19}, \frac{7}{12}) \supset J_7 = (\frac{18}{31}, \frac{7}{12}) \supset J_8 = (\frac{18}{31}, \frac{25}{43}) \supset \dots$  that are converging to  $\frac{1}{\alpha'} - n'$ . In Fig. 14 the intervals  $J_3, \dots, J_8$  are highlighted.

Up to zooming in, the pictures in Figs. 13 and 14 are the same.

In Margalit's slide [11] the Agol cycle of  $\sigma_1 \sigma_2^{-1}$  is shown. We study it from our viewpoint here.

**Example 5** Let  $\beta = \sigma_1 \sigma_2^{-1}$  which is a pseudo-Anosov 3-braid. The dilatation is  $\lambda = \frac{3+\sqrt{5}}{2}$  with MP-ratio  $\alpha = \frac{-1+\sqrt{5}}{2} \approx 0.618 \dots$ ,  $n = \lfloor 1/\alpha \rfloor = 1$  and the initial train track is  $W_\alpha$ . We would like to find the Agol cycle for  $\beta$ . The 4-tuple associated to  $\mathcal{T}_0$  is  $(1, -n; 0, 1) = (1, -1; 0, 1)$  and is of Type I' as defined in Fig. 6. We perform two maximal splittings in Fig. 15 to find  $\mathcal{T}_1$  and  $\mathcal{T}_2$ . We verify that  $\beta(\mathcal{T}_0) = \mathcal{T}_2$  by showing that  $\beta^{-1}(\mathcal{T}_3) = \mathcal{T}_1$  in Fig. 16. Thus,

$$(\mathcal{T}_0, \mu_0) \rightarrow (\mathcal{T}_1, \mu_1) \rightarrow (\mathcal{T}_2, \mu_2) = (\beta(\mathcal{T}_0), \lambda^{-1} \beta_*(\mu_0))$$

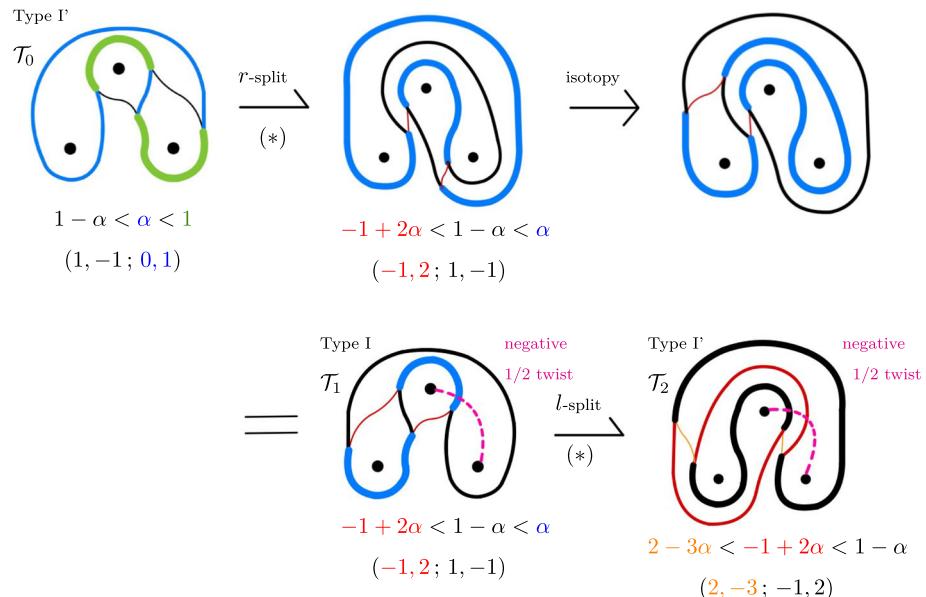
is the Agol cycle for  $\beta$ .

One can verify that the 4-tuples of  $\sigma_1 \sigma_2^{-1}$  satisfy

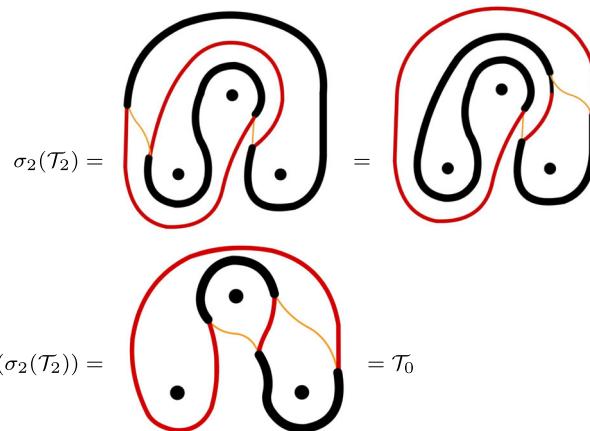
$$(x_k, y_k, ; z_k, w_k) = (-1)^k (F_{k+1}, -F_{k+2}, ; -F_k, F_{k+1})$$

where  $\{F_k\}_{k=0}^\infty = \{0, 1, 1, 2, 3, 5, \dots\}$  is the Fibonacci sequence. Therefore, all the 4-tuples are Type 1. This gives that the I-II-I'-II' sequence for  $\sigma_1 \sigma_2^{-1}$  is I', I, I', I, I', I, I', I,  $\dots$ .

Let  $a_k := \frac{F_n}{F_{n+1}}$ . The MP-ratio  $\alpha$  is the reciprocal of the golden ratio:  $\lim_{k \rightarrow \infty} \frac{F_n}{F_{n+1}} = \frac{-1+\sqrt{5}}{2} = \alpha$ . Since  $\frac{F_n}{F_{n+1}} = \frac{F_{n-2}}{F_{n-1}} \boxplus \frac{F_{n-1}}{F_n}$  in the Farey tessellation  $a_n$  is between  $a_{n-2}$  and  $a_{n-1}$ . Define a nested sequence of intervals  $J_k = (a_{k+1}, a_k)$  if  $k$  is odd and  $J_k = (a_k, a_{k+1})$  if  $k$  is even. For instance  $J_0 = (\frac{0}{1}, \frac{1}{1}) \supset J_1 = (\frac{1}{2}, \frac{1}{1}) \supset J_2 = (\frac{1}{2}, \frac{2}{3}) \supset J_3 = (\frac{3}{5}, \frac{2}{3}) \supset J_4 = (\frac{3}{5}, \frac{5}{8}) \supset \dots$ . Clearly,  $J_{k+1}$  is the left half of  $J_k$  if and only if  $k$  is odd. We also see  $\cap_{k=0}^\infty J_k = \alpha$ . Therefore, the **LR**-sequences of  $\sigma_1 \sigma_2^{-1}$  is **L, R, L, R, L, R, L, R,  $\dots$**



**Fig. 15** An Agol cycle for  $\beta = \sigma_1 \sigma_2^{-1}$



**Fig. 16** Verifying  $\beta^{-1}(\mathcal{T}_3) = \mathcal{T}_1$  for  $\beta = \sigma_1 \sigma_2^{-1}$

## 6 Necessary conditions for combinatorially isomorphic Agol cycles

We are ready to state the main theorem that gives a number of equivalent conditions for combinatorially isomorphic Agol cycles. Those conditions have different characteristics: topological, number theoretic, and numerical.

**Theorem 6** *Let  $\beta$  and  $\beta'$  be pseudo-Anosov 3-braids with the same dilatation. The following conditions are equivalent. (Equivalence of (1) and (2) is shown in Theorem 2.)*

1.  $\beta$  and  $\beta'$  are conjugate in  $B_3/Z(B_3)$ .
2.  $\beta$  and  $\beta'$  have combinatorially isomorphic Agol cycles.

3. (Topological condition) *There exist  $l$  and  $m \in \mathbb{N}$  such that  $\text{sgn}(\mathcal{T}_l) = \text{sgn}(\mathcal{T}'_m)$  and the triple-weight train track sequences*

$$\mathcal{T}_l \rightharpoonup \mathcal{T}_{l+1} \rightharpoonup \mathcal{T}_{l+2} \rightharpoonup \cdots \quad (\text{for } \beta)$$

and

$$\mathcal{T}'_m \rightharpoonup \mathcal{T}'_{m+1} \rightharpoonup \mathcal{T}'_{m+2} \rightharpoonup \cdots \quad (\text{for } \beta')$$

give the same periodic I-II-I'-II'-sequence.

4. (Number theoretic condition) *There exist  $l$  and  $m \in \mathbb{N}$  such that  $\text{sgn}(\mathcal{T}_l) = \text{sgn}(\mathcal{T}'_m)$  and the nested Farey interval sequences  $J_l \supset J_{l+1} \supset J_{l+2} \supset \cdots$  for  $\beta$  and  $J'_m \supset J'_{m+1} \supset J'_{m+2} \supset \cdots$  for  $\beta'$  give the same periodic **LR**-sequences.*

5. (Numerical condition) *There exist  $l$  and  $m \in \mathbb{N}$  such that  $\text{sgn}(\mathcal{T}_l) = \text{sgn}(\mathcal{T}'_m)$  and the 4-ratios of  $\mathcal{T}_l$  and  $\mathcal{T}'_m$  are the same; namely,*

$$\frac{x_l + y_l\alpha}{z_l + w_l\alpha} = \frac{x'_m + y'_m\alpha'}{z'_m + w'_m\alpha'}.$$

The same **LR**-sequence condition in (4) implies the existence of integers  $A, B, C, D$  with  $AD - BC = 1$  or  $-1$  (if  $\text{sgn}(\mathcal{T}_l) = -\text{sgn}(\mathcal{T}'_m)$  we have  $AD - BC = -1$ ) such that

$$\frac{1}{\alpha} - n = \frac{A(\frac{1}{\alpha'} - n') + B}{C(\frac{1}{\alpha'} - n') + D}.$$

In particular, when  $l = m = 1$  we obtain

$$\frac{1}{\alpha} - n = \frac{1}{\alpha'} - n'.$$

More detail and precise expression of  $A, B, C, D$  are given in the proof of the theorem.

A feature of condition (5) is that it is not seeing the entire infinite sequences like (3) and (4), but rather focusing on two particular train tracks  $\mathcal{T}_l$  and  $\mathcal{T}'_m$ .

**Proof of Theorem 6** The equivalence of (1) and (2) is due to Agol [1]. See also Margalit's talk slides [11].

(2)  $\Rightarrow$  (3): Suppose that an orientation preserving homeomorphism  $\phi : D_3 \rightarrow D_3$  satisfies  $\phi(\mathcal{T}_{l+t}) = \mathcal{T}'_{m+t}$  and  $\phi_*(\mu_{l+t}) = \mu'_{m+t}$  for all  $t \geq 1$ . This implies that 4-tuples of  $\mathcal{T}_{l+t}$  and  $\mathcal{T}'_{m+t}$  have the same type (Type 1 or Type 2). Since a homeomorphism and an  $r$ -/ $l$ -splitting commute,  $\mathcal{T}_{l+t}$  and  $\mathcal{T}'_{m+t}$  admit the same type ( $r$  or  $l$ ) of splitting.

The chart in Fig. 6 shows that

- an  $r$ -splitting of Type 2 ( $\xrightarrow{r,(\ast\ast)}$ ) yields a Type II train track.
- an  $l$ -splitting of Type 2 ( $\xrightarrow{l,(\ast\ast)}$ ) yields a Type II' train track.
- an  $r$ -splitting of Type 1 ( $\xrightarrow{r,(\ast)}$ ) yields a Type I train track.
- an  $l$ -splitting of Type 1 ( $\xrightarrow{l,(\ast)}$ ) yields a Type I' train track.

As a consequence,  $\text{Type}(\mathcal{T}_{l+t}) = \text{Type}(\mathcal{T}'_{m+t}) \in \{\text{I}, \text{II}, \text{I}', \text{II}'\}$  for all  $t \geq 1$ .

Recall that our maximal splitting sequence always starts from one of the train tracks  $M_\alpha, M'_\alpha, W_\alpha, W'_\alpha$  (Proposition 1). Up to rotation,  $M_\alpha = W_\alpha$  and  $M'_\alpha = W'_\alpha$ , and the two are related by taking mirror image.

Assume to the contrary that  $\text{sgn}(\mathcal{T}_{l+t}) = -\text{sgn}(\mathcal{T}'_{m+t})$  for some  $t$ . Since  $\text{Type}(\mathcal{T}_{l+t}) = \text{Type}(\mathcal{T}'_{m+t})$  Corollary 5 implies that  $\beta$  starts with  $M$  or  $W$  if and only if  $\beta'$  starts with  $W'$

or  $M'$ . This means  $\beta$  and  $\beta'$  are not conjugate to each other, which is a contradiction. This concludes  $\text{sgn}(\mathcal{T}_{l+t}) = \text{sgn}(\mathcal{T}'_{m+t})$  for all  $t \geq 1$ .

(3)  $\Rightarrow$  (4): We need to show for all  $t = 0, 1, 2, \dots$ ,  $J_{l+t}$  and  $J'_{m+t}$  correspond to the same letter, either **L** or **R**.

We first check the base case ( $t = 0$ ): By Lemma 5 and Definition 16, the condition  $\epsilon_l = \text{sgn}(\mathcal{T}_l) = \text{sgn}(\mathcal{T}'_m) = \epsilon'_m$  implies that the Farey intervals  $J_l$  and  $J'_m$  correspond to the same letter, either **L** or **R**.

To see the next stage ( $t = 1$ ), we first note that by the assumption (3) and Corollary 2, the 4-tuples of  $\mathcal{T}_{l+1}$  and  $\mathcal{T}'_{m+1}$  have the same type (Type 1 or Type 2). Then by Corollary 4,  $\epsilon_l = \epsilon'_m$  implies that  $\epsilon_{l+1} = \epsilon'_{m+1}$ . The argument for the base case applies here and we can conclude that  $J_{l+1}$  and  $J'_{m+1}$  correspond to the same letter, either **L** or **R**.

Inductively, for all  $t = 2, 3, \dots$ , we can show that  $J_{l+t}$  and  $J'_{m+t}$  correspond to the same letter, either **L** or **R**.

(4)  $\Rightarrow$  (5): We first note that condition (4) is equivalent to the existence of a  $2 \times 2$  matrix  $N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2, \mathbb{Z})$  that takes the Farey interval  $J_{l+t} = (\frac{u_t}{v_t}, \frac{w_t}{s_t})$  to  $J'_{m+t} = (\frac{u'_t}{v'_t}, \frac{w'_t}{s'_t})$  simultaneously; which means  $N \begin{pmatrix} u_t & w_t \\ v_t & s_t \end{pmatrix} = \begin{pmatrix} u'_t & w'_t \\ v'_t & s'_t \end{pmatrix}$  for each  $t = 0, 1, \dots$ . Since the nested sequences have convergences  $\cap_{k=1}^{\infty} \{J_k\} = \{\frac{1}{\alpha} - n\}$  and  $\cap_{k=1}^{\infty} \{J'_k\} = \{\frac{1}{\alpha'} - n'\}$ , it is equivalent to

$$\frac{1}{\alpha} - n = \frac{A(\frac{1}{\alpha'} - n') + B}{C(\frac{1}{\alpha'} - n') + D}.$$

In other words, the following vectors are parallel (the symbol  $\parallel$  stands for parallel):

$$\begin{pmatrix} \frac{1}{\alpha} - n \\ 1 \end{pmatrix} \parallel N \begin{pmatrix} \frac{1}{\alpha'} - n' \\ 1 \end{pmatrix} \quad (18)$$

The matrix  $N$  can be explicitly computed as follows: By Lemma 4, the Farey numbers  $\frac{u_t}{v_t}, \frac{w_t}{s_t}, \frac{u'_t}{v'_t}, \frac{w'_t}{s'_t}$  for  $t = 0$  satisfy one of the two cases (here we use the assumption  $\epsilon_l = \epsilon'_m$ ):

- $\frac{u_0}{v_0} = \frac{c_l}{a_l}, \quad \frac{w_0}{s_0} = \frac{c_l + d_l}{a_l + b_l}, \quad \frac{u'_0}{v'_0} = \frac{c'_m}{a'_m}, \quad \frac{w'_0}{s'_0} = \frac{c'_m + d'_m}{a'_m + b'_m} \quad \text{if } \epsilon_l = \epsilon_m = 1,$
- $\frac{u_0}{v_0} = \frac{c_l + d_l}{a_l + b_l}, \quad \frac{w_0}{s_0} = \frac{c_l}{a_l}, \quad \frac{u'_0}{v'_0} = \frac{c'_m + d'_m}{a'_m + b'_m}, \quad \frac{w'_0}{s'_0} = \frac{c'_m}{a'_m} \quad \text{if } \epsilon_l = \epsilon_m = -1.$

In either case,

$$N = \begin{pmatrix} c_l & d_l \\ a_l & b_l \end{pmatrix} \begin{pmatrix} c'_m & d'_m \\ a'_m & b'_m \end{pmatrix}^{-1}$$

and  $\det N = (-\epsilon_l)(-\epsilon'_m) = 1$  since  $\epsilon_l = \epsilon_m$ .

Next, we note that

$$\begin{pmatrix} a_l & -c_l \\ -b_l & d_l \end{pmatrix}^{-1} \begin{pmatrix} a'_m & -c'_m \\ -b'_m & d'_m \end{pmatrix} = \begin{pmatrix} c_l & d_l \\ a_l & b_l \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c'_m & d'_m \\ a'_m & b'_m \end{pmatrix}^{-1} = -N.$$

By (18) we have

$$\begin{pmatrix} a_l & -c_l \\ -b_l & d_l \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha} - n \\ 1 \end{pmatrix} \parallel \begin{pmatrix} a'_m & -c'_m \\ -b'_m & d'_m \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha'} - n' \\ 1 \end{pmatrix}. \quad (19)$$

Using Theorem 5, we can rewrite the 4-ratio of  $\mathcal{T}_l$  as

$$\frac{x_l + y_l\alpha}{z_l + w_l\alpha} = \frac{a_l - (a_l n + c_l)\alpha}{-b_l + (b_l n + d_l)\alpha} = \frac{a_l(\frac{1}{\alpha} - n) - c_l}{-b_l(\frac{1}{\alpha} - n) + d_l},$$

which is equal to the slope of the left hand side vector of (19). Likewise the 4-ratio of  $\mathcal{T}'_m$  is equal to the slope of the right hand side vector of (19). Therefore, we conclude the 4-ratios of  $\mathcal{T}_l$  and  $\mathcal{T}'_m$  are the same:  $\frac{x_l + y_l\alpha}{z_l + w_l\alpha} = \frac{x'_m + y'_m\alpha'}{z'_m + w'_m\alpha'}$ .

(5)  $\Rightarrow$  (2): It follows from Theorem 7 below.

## 7 Mirror combinatorially isomorphic Agol cycles

In Sect. 6, we studied equivalent conditions of combinatorially isomorphic Agol cycles. In this section we explore generalization of the statement (5)  $\Rightarrow$  (2) in Theorem 6. Our goal is to prove: if  $\mathcal{T}_l$  and  $\mathcal{T}'_m$  have the same 4-ratio for some  $l$  and  $m$ , then Agol cycles of  $\beta$  and  $\beta'$  are combinatorially isomorphic or mirror combinatorially isomorphic.

Let  $\mathcal{T}$  be a triple-weight train track and  $\text{Type}(\mathcal{T}) \in \{\text{I}, \text{II}, \text{I}', \text{II}'\}$  denote the homeomorphism type (Type I, II, I', and II') as introduced in Fig. 6.

**Theorem 7** *Let  $\beta$  and  $\beta'$  be pseudo-Anosov 3-braids with the same dilatation. Suppose that there exist  $l$  and  $m$  such that triple-weight train tracks  $\mathcal{T}_l$  and  $\mathcal{T}'_m$  have the same 4-ratio; i.e.,  $\frac{x_l + y_l\alpha}{z_l + w_l\alpha} = \frac{x'_m + y'_m\alpha'}{z'_m + w'_m\alpha'}$ . Then one of the following cases occur:*

1.  $\text{Type}(\mathcal{T}_{l+1}) = \text{Type}(\mathcal{T}'_{m+1})$
2.  $\text{Type}(\mathcal{T}_{l+1}) = (\text{Type}(\mathcal{T}'_{m+1}))'$

*In (1) the Agol cycles are combinatorially isomorphic. That is, there exists an orientation-preserving homeomorphism  $\phi : D_3 \rightarrow D_3$  and a constant  $a > 0$  such that*

$$\beta = \phi^{-1} \circ \beta' \circ \phi$$

*and*

$$(\mathcal{T}'_{m+t}, a\mu'_{m+t}) = \phi(\mathcal{T}_{l+t}, \mu_{l+t})$$

*for all  $t \geq 1$ .*

*In (2) the Agol cycles are mirror combinatorially isomorphic. Namely, there exists an orientation-reversing homeomorphism  $\psi : D_3 \rightarrow D_3$  and a constant  $a > 0$  such that*

$$\beta = \psi^{-1} \circ \beta' \circ \psi$$

*and*

$$(\mathcal{T}'_{m+t}, a\mu'_{m+t}) = \psi(\mathcal{T}_{l+t}, \mu_{l+t})$$

*for all  $t \geq 1$ .*

**Example 6** Related to Example 4, let  $\beta = \sigma_1^4 \sigma_2 \sigma_1^3 \sigma_2^4$ ,  $\beta' = \sigma_1^{-4} \sigma_2^{-4} \sigma_1^{-3} \sigma_2^{-1}$ , and  $\beta'' = \sigma_1^{-4} \sigma_2^{-1} \sigma_1^{-3} \sigma_2^{-4}$ . The pair  $(\beta, \beta')$  falls into Case (1) of Theorem 7. and the pair  $(\beta'', \beta')$  falls into Case (2). It is interesting to point out that  $\beta''$  and  $\beta'$  are related by a non-degenerate flype. Thus they belong to distinct conjugacy classes and their Agol cycles are mirror combinatorially isomorphic.

Below is a lemma for Theorem 7.

**Lemma 6** Suppose that  $\mathcal{T}_l$  and  $\mathcal{T}'_m$  have the same 4-ratio. Then for all  $t \geq 1$ , the subsequent train tracks  $\mathcal{T}_{l+t}$  and  $\mathcal{T}'_{m+t}$  have the same 4-ratio and their 4-tuples have the same type (either Type 1 or Type 2).

**Proof of Lemma 6** Define  $A, B, C, D$  by

$$A + B\alpha + C\alpha' + D\alpha\alpha' = (x_l + y_l\alpha)(z'_m + w'_m\alpha') - (z_l + w_l\alpha)(x'_m + y'_m\alpha').$$

Since  $\mathcal{T}_l$  and  $\mathcal{T}'_m$  have the same 4-ratio

$$A + B\alpha + C\alpha' + D\alpha\alpha' = 0.$$

Using Theorem 5 we may describe  $A, B, C, D$  as follows.

$$\begin{aligned} A &= \epsilon_l \epsilon'_m (-a_l b'_m + b_l a'_m) \\ B &= \epsilon_l \epsilon'_m ((a_l b'_m - b_l a'_m)n + (c_l b'_m - d_l a'_m)) \\ C &= \epsilon_l \epsilon'_m ((a_l b'_m - b_l a'_m)n' + (a_l d'_m - b_l c'_m)) \\ D &= \epsilon_l \epsilon'_m ((a_l b'_m - b_l a'_m)nn' + (a_l d'_m - b_l c'_m)n + (c_l b'_m - d_l a'_m)n' + (c_l d'_m - d_l c'_m)) \end{aligned}$$

We have:

$$\begin{aligned} C\alpha' + A &= -\alpha(D\alpha' + B) \\ \frac{1}{\alpha} &= -\frac{D\alpha' + B}{C\alpha' + A} \\ \frac{1}{\alpha} - n &= -\frac{(D + nC)\alpha' + (B + nA)}{C\alpha' + A} \\ &= \frac{-(B + nA)(\frac{1}{\alpha'} - n') - (D + nC) - n'(B + nA)}{A(\frac{1}{\alpha'} - n') + C + An'} \\ &= \frac{\epsilon_l \epsilon'_m (d_l a'_m - c_l b'_m)(\frac{1}{\alpha'} - n') + \epsilon_l \epsilon'_m (c_l d'_m - d_l c'_m)}{\epsilon_l \epsilon'_m (-a_l b'_m + b_l a'_m)(\frac{1}{\alpha'} - n') + \epsilon_l \epsilon'_m (a_l d'_m - b_l c'_m)} \end{aligned}$$

From the last fraction, we define a matrix

$$N = \epsilon_l \epsilon'_m \begin{pmatrix} d_l a'_m - c_l b'_m & -d_l c'_m + c_l d'_m \\ b_l a'_m - a_l b'_m & -b_l c'_m + a_l d'_m \end{pmatrix} = \epsilon_l \begin{pmatrix} c_l & d_l \\ a_l & b_l \end{pmatrix} \begin{pmatrix} c'_m & d'_m \\ a'_m & b'_m \end{pmatrix}^{-1}$$

so that the following vectors are parallel:

$$\begin{pmatrix} \frac{1}{\alpha} - n \\ 1 \end{pmatrix} \parallel N \begin{pmatrix} \frac{1}{\alpha'} - n' \\ 1 \end{pmatrix} \quad (20)$$

For later use, we note that:

$$N \begin{pmatrix} c'_m & d'_m \\ a'_m & b'_m \end{pmatrix} = \epsilon_l \begin{pmatrix} c_l & d_l \\ a_l & b_l \end{pmatrix}, \quad (21)$$

and the determinant of  $N$  is either 1 or -1.

$$\det N = (b_l c_l - a_l d_l)(b'_m c'_m - a'_m d'_m)^{-1} \stackrel{(14)}{=} \epsilon_l \epsilon'_m \in \{-1, 1\} \quad (22)$$

By Lemma 3,  $\alpha \in \bigcap_{k=0}^{\infty} I_{k,i_k}$  and  $\alpha' \in \bigcap_{k=0}^{\infty} I'_{k,i'_k}$ . By Lemma 4, we have

$$I_{l,i_l} = \begin{cases} \left( \frac{1}{n + \frac{c_l+d_l}{a_l+b_l}}, \frac{1}{n + \frac{c_l}{a_l}} \right) & \text{thus } \frac{c_l}{a_l} < \frac{1}{\alpha} - n < \frac{c_l+d_l}{a_l+b_l} \text{ if } \epsilon_l = 1 \\ \left( \frac{1}{n + \frac{c_l}{a_l}}, \frac{1}{n + \frac{c_l+d_l}{a_l+b_l}} \right) & \text{thus } \frac{c_l+d_l}{a_l+b_l} < \frac{1}{\alpha} - n < \frac{c_l}{a_l} \text{ if } \epsilon_l = -1 \end{cases} \quad (23)$$

and

$$I'_{m,i'_m} = \begin{cases} \left( \frac{1}{n' + \frac{c'_m+d'_m}{a'_m+b'_m}}, \frac{1}{n' + \frac{c'_m}{a'_m}} \right) & \text{thus } \frac{c'_m}{a'_m} < \frac{1}{\alpha'} - n' < \frac{c'_m+d'_m}{a'_m+b'_m} \text{ if } \epsilon'_m = 1 \\ \left( \frac{1}{n' + \frac{c'_m}{a'_m}}, \frac{1}{n' + \frac{c'_m+d'_m}{a'_m+b'_m}} \right) & \text{thus } \frac{c'_m+d'_m}{a'_m+b'_m} < \frac{1}{\alpha'} - n' < \frac{c'_m}{a'_m} \text{ if } \epsilon'_m = -1. \end{cases} \quad (24)$$

By Lemma 2,  $I_{l,i_l} = \left( \frac{1}{n + \frac{p_l}{q_l}}, \frac{1}{n + \frac{r_l}{s_l}} \right)$  splits into  $I_{l+1,2i_l} = \left( \frac{1}{n + \frac{p_l}{q_l}}, \frac{1}{n + \frac{p_l+r_l}{q_l+s_l}} \right)$  and  $I_{l+1,2i_l-1} = \left( \frac{1}{n + \frac{p_l+r_l}{q_l+s_l}}, \frac{1}{n + \frac{r_l}{s_l}} \right)$ . Thus  $i_{l+1} = 2i_l$  or  $2i_l - 1$ . We observe

- If  $i_{l+1} = 2i_l$  then  $\frac{p_l}{q_l} \boxplus \frac{r_l}{s_l} = \frac{p_l+r_l}{q_l+s_l} < \frac{1}{\alpha} - n < \frac{p_l}{q_l}$ .
- If  $i_{l+1} = 2i_l - 1$  then  $\frac{r_l}{s_l} < \frac{1}{\alpha} - n < \frac{p_l}{q_l} \boxplus \frac{r_l}{s_l} = \frac{p_l+r_l}{q_l+s_l}$ .

Similarly,  $i'_{m+1} = 2i'_m$  or  $2i'_m - 1$ .

- If  $i'_{m+1} = 2i'_m$  then  $\frac{p'_m+r'_m}{q'_m+s'_m} < \frac{1}{\alpha'} - n' < \frac{p'_m}{q'_m}$ .
- If  $i'_{m+1} = 2i'_m - 1$  then  $\frac{r'_m}{s'_m} < \frac{1}{\alpha'} - n' < \frac{p'_m+r'_m}{q'_m+s'_m}$ .

**Claim 1** The indices  $i_{l+1}$  and  $i'_{m+1}$  obey the following rule.

- If  $\epsilon_l \epsilon'_m = 1$  then  $i_{l+1} = 2i_l$  if and only if  $i'_{m+1} = 2i'_m$ .
- If  $\epsilon_l \epsilon'_m = -1$  then  $i_{l+1} = 2i_l - 1$  if and only if  $i'_{m+1} = 2i'_m$ .

**Proof of Claim 1** We have four cases to consider.

**(Case 1:**  $\epsilon_l = \epsilon'_m = 1$ ) By (24) with  $\epsilon'_m = 1$ , we note that  $\frac{p'_m}{q'_m} = \frac{c'_m+d'_m}{a'_m+b'_m}$  and  $\frac{r'_m}{s'_m} = \frac{c'_m}{a'_m}$ . Thus, we have  $i'_{m+1} = 2i'_m$  if and only if

$$\frac{2c'_m+d'_m}{2a'_m+b'_m} = \frac{p'_m}{q'_m} \boxplus \frac{r'_m}{s'_m} < \frac{1}{\alpha'} - n' < \frac{p'_m}{q'_m} = \frac{c'_m+d'_m}{a'_m+b'_m}.$$

Since  $\det N = 1 > 0$  by (22) the slopes of the following three vectors satisfy

$$\text{slope} \left( N \begin{pmatrix} 2c'_m+d'_m \\ 2a'_m+b'_m \end{pmatrix} \right) < \text{slope} \left( N \begin{pmatrix} \frac{1}{\alpha'} - n' \\ 1 \end{pmatrix} \right) < \text{slope} \left( N \begin{pmatrix} c'_m+d'_m \\ a'_m+b'_m \end{pmatrix} \right).$$

By (20) and (21), we obtain

$$\frac{2c_l+d_l}{2a_l+b_l} < \frac{1}{\alpha} - n < \frac{c_l+d_l}{a_l+b_l}.$$

By (23), with  $\epsilon_l = 1$  it is equivalent to  $i_{l+1} = 2i_l$ .

**(Case 2:**  $\epsilon_l = \epsilon'_m = -1$ ) We have  $i'_{m+1} = 2i'_m$  if and only if  $\frac{2c'_m+d'_m}{2a'_m+b'_m} < \frac{1}{\alpha'} - n' < \frac{c'_m}{a'_m}$ . Since  $\det N > 0$  we obtain  $\frac{2c_l+d_l}{2a_l+b_l} < \frac{1}{\alpha} - n < \frac{c_l}{a_l}$ . It is equivalent to  $i_{l+1} = 2i_l$ .

**(Case 3:**  $\epsilon_l = -\epsilon'_m = 1$ ) We have  $i'_{m+1} = 2i'_m$  if and only if  $\frac{2c'_m+d'_m}{2a'_m+b'_m} < \frac{1}{\alpha'} - n' < \frac{c'_m}{a'_m}$ . Since  $\det N < 0$  we obtain  $\frac{c_l}{a_l} < \frac{1}{\alpha} - n < \frac{2c_l+d_l}{2a_l+b_l}$ . It is equivalent to  $i_{l+1} = 2i_l - 1$ .

**(Case 4:**  $-\epsilon_l = \epsilon'_m = 1$ ) We have  $i'_{m+1} = 2i'_m$  if and only if  $\frac{2c'_m+d'_m}{2a'_m+b'_m} < \frac{1}{\alpha'} - n' < \frac{c'_m+d'_m}{a'_m+b'_m}$ . Since  $\det N < 0$  we have  $\frac{c_l+d_l}{a_l+b_l} < \frac{1}{\alpha} - n < \frac{2c_l+d_l}{2a_l+b_l}$ . It is equivalent to  $i_{l+1} = 2i_l - 1$ .  $\square$

It is interesting to note that Claim 1 and Lemma 5 imply that the product of the signs is preserved:

$$\epsilon_{l+1}\epsilon'_{m+1} = \epsilon_l\epsilon'_m. \quad (25)$$

**Claim 2** The 4-tuples of  $\mathcal{T}_{l+1}$  and  $\mathcal{T}'_{m+1}$  have the same type (Type 1 or Type 2) as stated in the following table.

$(\text{sgn}(\epsilon_l), \text{sgn}(\epsilon'_m))$	Case 1 (+, +)	Case 2 (-, -)	Case 3 (+, -)	Case 4 (-, +)
$i_{l+1} = 2i_l$	Type 1	Type 2	Type 1	Type 2
$i_{l+1} = 2i_l - 1$	Type 2	Type 1	Type 2	Type 1

**Proof** All the eight cases can be checked similarly. For example, we check the claim for Case 3 where  $i_{l+1} = 2i_l - 1$ . By Corollary 4-(2) the 4-tuple of  $\mathcal{T}_{l+1}$  is Type 2. Next, Claim 1 states  $i'_{m+1} = 2i'_m$ . By Corollary 4-(4), the 4-tuple of  $\mathcal{T}'_{m+1}$  is Type 2.  $\square$

We will continue the proof of Lemma 6 and show that  $\mathcal{T}_{l+1}$  and  $\mathcal{T}'_{m+1}$  have the same 4-ratio, which by induction concludes Lemma 6.

Since  $\mathcal{T}_l$  and  $\mathcal{T}'_m$  have the same 4-ratio

$$(x_l + y_l\alpha)(z'_m + w'_m\alpha') - (z_l + w_l\alpha)(x'_m + y'_m\alpha') = 0. \quad (26)$$

Having Claim 2 proved, both  $\mathcal{T}_{l+1}$  and  $\mathcal{T}'_{m+1}$  have the same 4-tuple type, say Type 1. By Proposition 2, their 4-tuples are:

$$(x_{l+1}, y_{l+1}, z_{l+1}, w_{l+1}) = (z_l - x_l, w_l - y_l, x_l, y_l) \\ \text{and } (x'_{m+1}, y'_{m+1}, z'_{m+1}, w'_{m+1}) = (z'_m - x'_m, w'_m - y'_m, x'_m, y'_m). \quad (27)$$

We obtain  $\frac{x_{l+1}+y_{l+1}\alpha}{z_{l+1}+w_{l+1}\alpha} = \frac{x'_{m+1}+y'_{m+1}\alpha'}{z'_{m+1}+w'_{m+1}\alpha'}$  since

$$(x_{l+1} + y_{l+1}\alpha)(z'_{m+1} + w'_{m+1}\alpha') - (z_{l+1} + w_{l+1}\alpha)(x'_{m+1} + y'_{m+1}\alpha') \\ \stackrel{(27)}{=} -(x_l + y_l\alpha)(z'_m + w'_m\alpha') + (z_l + w_l\alpha)(x'_m + y'_m\alpha') \stackrel{(26)}{=} 0.$$

Thus,  $\mathcal{T}_{l+1}$  and  $\mathcal{T}'_{m+1}$  have the same 4-ratio. When both the 4-tuples of  $\mathcal{T}_{l+1}$  and  $\mathcal{T}'_{m+1}$  are Type 2, a similar argument holds.  $\square$

This concludes Lemma 6.  $\square$

Finally we are ready to prove Theorem 7.

**Proof of Theorem 7** Assume that triple-weight train tracks  $\mathcal{T}_l$  and  $\mathcal{T}'_m$  have the same sign  $\text{sgn}(\mathcal{T}_{l+1}) = \text{sgn}(\mathcal{T}'_{m+1})$  and the same 4-ratio; i.e.,  $\frac{x_l+y_l\alpha}{z_l+w_l\alpha} = \frac{x'_m+y'_m\alpha'}{z'_m+w'_m\alpha'}$ . By Lemma 6,  $\mathcal{T}_{l+t}$  and  $\mathcal{T}'_{m+t}$  have the same 4-tuple types (Type 1 or Type 2) for all  $t \geq 1$ .

By Corollary 2 their topological types (I, II, I', II') satisfy either

1.  $\text{Type}(\mathcal{T}_{l+1}) = \text{Type}(\mathcal{T}'_{m+1})$  or
2.  $\text{Type}(\mathcal{T}_{l+1}) = \text{Type}(\mathcal{T}'_{m+1})'$ .

The system of Types I, II, I', II' is closed under maximal splitting operations and follow the rule as described in Fig. 6. Notice the diagram in Fig. 6 is mirror symmetric with respect to a vertical axis. Thus, in Case (1) the I-II-I'-II'-sequences corresponding to  $\mathcal{T}_{l+1} \rightarrow \mathcal{T}_{l+2} \rightarrow \dots$  and  $\mathcal{T}'_{m+1} \rightarrow \mathcal{T}'_{m+2} \rightarrow \dots$  are exactly the same, and in Case (2) exactly the same up to simultaneously putting '. Namely,

1.  $\text{Type}(\mathcal{T}_{l+t}) = \text{Type}(\mathcal{T}'_{m+t})$  for all  $t \geq 1$  or
2.  $\text{Type}(\mathcal{T}_{l+t}) = (\text{Type}(\mathcal{T}'_{m+t}))'$  for all  $t \geq 1$ .

For Case (1) let  $\phi \in \text{Homeo}^+(D_3)$  be an orientation preserving homeomorphism such that  $\phi(\mathcal{T}_l) = \mathcal{T}'_m$ . By Corollary 3 both  $\mathcal{T}_l$  and  $\mathcal{T}'_m$  admit the same type of maximal splitting ( $r$ -split or  $l$ -split). Since a homeomorphism and a splitting operation commute, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{T}_l & \xrightarrow{l} & \mathcal{T}_{l+1} \\ \phi \downarrow & & \phi \downarrow \\ \mathcal{T}'_m & \xrightarrow{l} & \mathcal{T}'_{m+1} \end{array} \quad \text{or} \quad \begin{array}{ccc} \mathcal{T}_l & \xrightarrow{r} & \mathcal{T}_{l+1} \\ \phi \downarrow & & \phi \downarrow \\ \mathcal{T}'_m & \xrightarrow{r} & \mathcal{T}'_{m+1} \end{array}$$

In particular  $\phi(\mathcal{T}_{l+1}) = \mathcal{T}'_{m+1}$ . Repeating the same argument we obtain

$$\phi(\mathcal{T}_{l+t}) = \mathcal{T}'_{m+t} \text{ for all } t \geq 1.$$

Lemma 6 also states that  $\mathcal{T}_{l+t}$  and  $\mathcal{T}'_{m+t}$  have the same 4-ratios. Therefore, for some constant  $a > 0$ , the induced map  $\phi_*$  yields

$$\phi_*\mu(\mathcal{T}_{l+t}) = a\mu'(\mathcal{T}'_{m+t}) \text{ for all } t \geq 1.$$

Since  $\beta$  and  $\beta'$  have the same dilatation, their Agol cycles have the same length (call it  $p$ ). Therefore,  $\beta$  and  $\beta'$  have equivalent Agol-cycles. We have a diagram

$$\begin{array}{ccc} (\mathcal{T}_l, \mu_l) & \xrightarrow{\beta} & (\mathcal{T}_{l+p}, \mu_{l+p}) \\ \phi \downarrow & & \phi \downarrow \\ (\mathcal{T}'_m, \mu'_m) & \xrightarrow{\beta'} & (\mathcal{T}'_{m+p}, \mu'_{m+p}) \end{array}$$

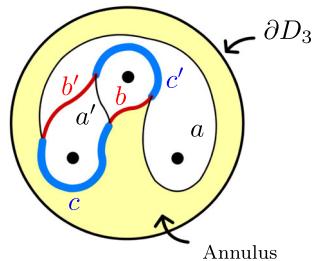
Let  $\Phi := \phi^{-1} \circ (\beta')^{-1} \circ \phi \circ \beta$ . We see:

- (a)  $\Phi$  is an orientation preserving homeomorphism.
- (b)  $\Phi$  restricted to the boundary  $\partial D_3$  is the identity map.
- (c)  $\Phi(\mathcal{T}_l) = \mathcal{T}_l$ .
- (d)  $\Phi$  also preserves the thickness data; namely,  $\Phi(a \cup a') = a \cup a'$ ,  $\Phi(b \cup b') = b \cup b'$  and  $\Phi(c \cup c') = c \cup c'$ . Here,  $a, a', b, b', c, c'$  are thickened edges as in Fig. 17. Edges  $a, a'$  have the same weight, as do  $b, b'$  and  $c, c'$ .

It is impossible that  $\Phi(a) = a'$  because the circle  $a \cup b \cup c$  and  $\partial D_3$  bound an annulus, which is fixed by  $\Phi$ . Therefore by (d)  $\Phi(a) = a$  and  $\Phi(a') = a'$ . Likewise,  $\Phi(b) = b$ ,  $\Phi(b') = b'$ ,  $\Phi(c) = c$  and  $\Phi(c') = c'$ . This yields

$$\Phi_{\mathcal{T}_l} = id_{\mathcal{T}_l}.$$

The set  $D_3 \setminus \mathcal{T}_l$  is disjoint union of the annulus along the boundary  $\partial D_3$  and three 1-punctured monogons (one for each puncture of  $D_3$ ). By the Alexander trick (see [8] for example) we

**Fig. 17**  $\mathcal{T}_l \subset D_3$ 

conclude  $\Phi = id_{D_3}$ , which means

$$\beta = \phi^{-1} \circ \beta' \circ \phi.$$

Similarly, for Case (2) there exist constant  $a > 0$  and orientation preserving homeomorphism  $\phi \in \text{Homeo}^+(D_3)$  such that for all  $t \geq 1$

$$\phi(\mathcal{T}_{l+t}) = \text{mirror}(\mathcal{T}'_{m+t})$$

and

$$(\text{mirror} \circ \phi)_* \mu(\mathcal{T}_{l+t}) = a \mu'(\mathcal{T}'_{m+t}).$$

This implies that  $\beta$  and  $\beta'$  have mirror equivalent Agol cycles. Furthermore we can also show that  $\beta = \phi^{-1} \circ \text{mirror}^{-1} \circ \beta' \circ \text{mirror} \circ \phi$ .

## 8 Dilatation is preserved under 3-braid flypes

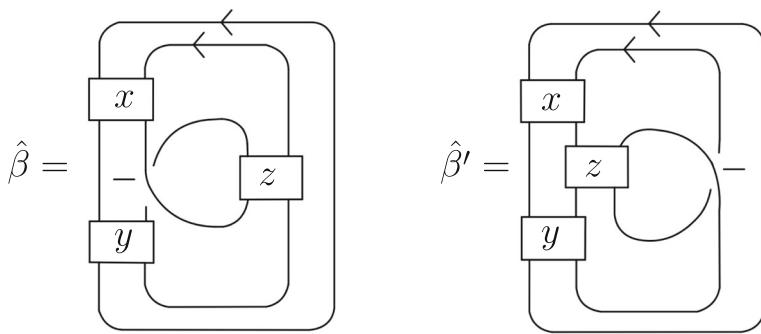
The goal of this section is to prove Theorem 10, which states that the dilatation is preserved under flype moves. Thus, dilatation is not as strong invariant as Agol cycle. However, computation of a dilatation is easier than that of Agol cycle in general. In fact, in the proof of Lemma 9 we describe how to compute the transition matrix of a given 3-braid, from which the dilatation can be explicitly computed as the eigenvalue  $> 1$ .

### 8.1 Birman–Menasco’s classification of links of braid index 3

We briefly review well-known facts on 3-braid flypes that are relevant to this paper.

**Definition 17** Let  $\varepsilon = \pm 1$  and  $x, y, z \in \mathbb{Z}$ . If  $\beta = \sigma_1^x \sigma_2^y \sigma_1^z \sigma_2^y$  and  $\beta' = \sigma_1^x \sigma_2^z \sigma_1^y \sigma_2^{\varepsilon}$ , then we say that  $\beta$  and  $\beta'$  are related by an  $\varepsilon$ -flype. See Fig. 18. If  $\beta$  and  $\beta'$  are conjugate (resp. not conjugate), then the flype is called *degenerate* (resp. *non-degenerate*).

Non-degenerate flypes play a significant role in low-dimensional topology. Flypes are used in the classification of 3-braids in Birman and Menasco’s work [3], in Markov’s Theorem without Stabilization [4], and the classification of transversally simple knots [5], [6]. Many other transversally simple knots admit a negative flype like those found by Etnyre and Honda [7] (cf. Matsuda and Menasco [12]) and Ng, Ozsváth, and Thurston [15]. In the Tait flype conjecture, Thistlethwaite and Menasco proved that two reduced alternating diagrams of an alternating link are related by a sequence of flypes [14].



**Fig. 18** (Definition 17) The braid closures  $\hat{\beta}$  and  $\hat{\beta}'$  obtained from a negative flype

It is easy to see that a flype move preserves the topological link type of the braid closure. On the contrary, the following Birman and Menasco's theorem states that a flype changes the conjugacy class in general.

**Theorem 8** [3] *Let  $\mathcal{L}$  be a link type of braid index three. Then one of the following holds:*

1. *There exists a unique conjugacy class of 3-braid representatives of  $\mathcal{L}$ .*
2. *There exists two conjugacy classes of 3-braid representatives of  $\mathcal{L}$ .*

*Case (2) happens if and only if  $\mathcal{L}$  has a 3-braid representative that admits a non-degenerate flype.*

Moreover, Ko and Lee determine all the non-degenerate flypes.

**Theorem 9** [10, Theorem 5] *The 3-braids  $\beta = \sigma_1^x \sigma_2^\varepsilon \sigma_1^y \sigma_2^z$  and  $\beta' = \sigma_1^x \sigma_2^z \sigma_1^y \sigma_2^\varepsilon$  have distinct conjugacy classes if and only if*

- Neither  $x$  nor  $y$  is equal to 0,  $\varepsilon$ ,  $2\varepsilon$  or  $z + \varepsilon$ ,
- $x \neq y$ , and
- $|z| \geq 2$ .

## 8.2 Dilatation is preserved under flypes

Now we state the main result of this section.

**Theorem 10** *There are infinitely many integers  $x, y$  and  $z$ , such that the braids  $\beta = \sigma_1^x \sigma_2^{-1} \sigma_1^y \sigma_2^z$  and  $\beta' = \sigma_1^x \sigma_2^z \sigma_1^y \sigma_2^{-1}$  belong to distinct conjugacy classes (suggested by Theorem 9) but have the same dilatation*

$$\lambda = \frac{1}{2}(\gamma + \sqrt{\gamma^2 - 4})$$

where

$$\gamma = \gamma(x, y, z) = \text{sgn}(xyz)(-2 - x - y + xz + yz + xyz). \quad (28)$$

The theorem is an immediate consequence of Theorem 9 and Lemma 9 below.

We need three lemmas to prove Theorem 10. We start by analyzing behavior of train tracks under the braid generators  $\sigma_1$  and  $\sigma_2$ , and compute transition matrices.

**Lemma 7** Measured train tracks are affected by  $\sigma_1^{\pm 1}$  and  $\sigma_2^{\pm 1}$  as shown in Figs. 19 and 20. The matrices between train track types are defined below. They designate how the labels change after the application of  $\sigma_i^{\pm 1}$ .

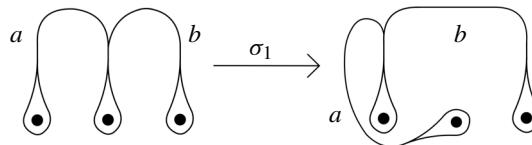
$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

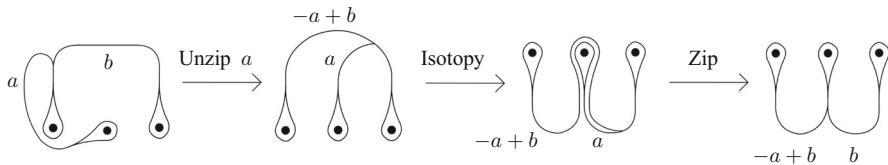
Note  $T B^{-1} = B T$  and  $V A^{-1} = A V$ .

**Proof** In this proof, we unzip and zip the edges of the train track, which is an operation where we separate or condense, respectively, the edges according to the assigned weights. More details about this operation can be found in [8].

We consider the case of  $\sigma_1$  acting on  $M(a, b)$ .

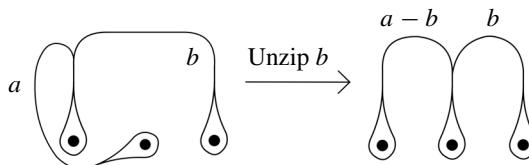


If  $a < b$ , then  $2a < a + b$  and we unzip the train track along  $a$ .



Notice that the labels have changed by an application of the matrix  $T$ .

If  $a > b$ , then  $a + b > 2b$  and we unzip the train track along  $b$ .



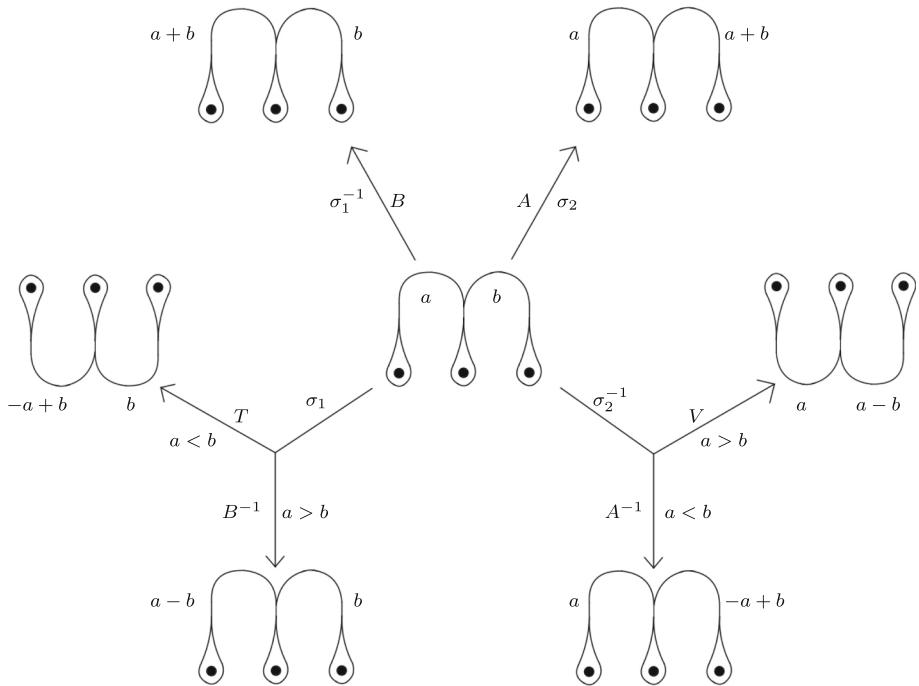
Notice that the labels have changed by an application of the matrix  $B^{-1}$ .

All other cases follow similarly.  $\square$

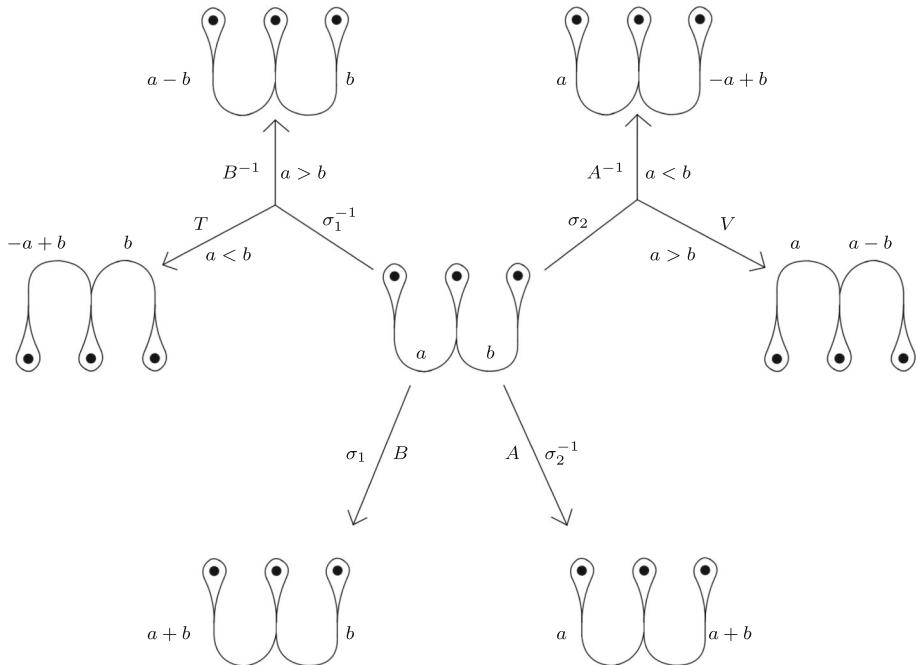
**Lemma 8** Based on Lemma 7, we obtain four commutative diagrams.

- The action of  $\sigma_1$  is shown in the following commutative diagram. For  $x \gg 0$ , Type M converges to Type W under  $\sigma_1$  and the change in weights on the train track is represented by  $B^{x-1}T$ .

$$\begin{array}{ccccccc}
 M(a, b) & \xrightarrow[B^{-1}]{a>b} & M(a-b, b) & \xrightarrow[B^{-1}]{a>2b} & M(a-2b, b) & \xrightarrow[B^{-1}]{a>3b} & \dots \dots \\
 \searrow_{a < b} & & \searrow_{a < 2b} & & \searrow_{a < 3b} & & \\
 & & W(-a+b, b) & \xrightarrow[B]{a < 2b} & W(-a+2b, b) & \xrightarrow[B]{a < 3b} & \dots \dots
 \end{array}$$



**Fig. 19** (Lemma 7) Action of  $\sigma_1^{\pm 1}$  and  $\sigma_2^{\pm 1}$  on Type M



**Fig. 20** (Lemma 7) Action of  $\sigma_1^{\pm 1}$  and  $\sigma_2^{\pm 1}$  on Type W

- The action of  $\sigma_1^{-1}$  is shown in the following commutative diagram. For  $x \gg 0$ , Type W converges to Type M under  $\sigma_1^{-x}$  and the change in weights on the train track is represented by  $B^{x-1}T$ .

$$\begin{array}{ccccc}
 W(a, b) & \xrightarrow[B^{-1}]{a < b} & W(a-b, b) & \xrightarrow[B^{-1}]{a > 2b} & W(a-2b, b) \xrightarrow[B^{-1}]{a > 3b} \dots \\
 & \searrow \begin{smallmatrix} a < b \\ T \end{smallmatrix} & \searrow \begin{smallmatrix} a < 2b \\ B \end{smallmatrix} & \searrow \begin{smallmatrix} a < 3b \\ T \end{smallmatrix} & \\
 & M(-a+b, b) & \xrightarrow[a < 2b]{B} M(-a+2b, b) & \xrightarrow[a < 3b]{B} M(-a+3b, b) & \dots
 \end{array}$$

- The action of  $\sigma_2$  is shown in the following commutative diagram. For  $x \gg 0$ , Type W converges to Type M under  $\sigma_2$  and the change in weights on the train track is represented by  $A^{x-1}V$ .

$$\begin{array}{ccccc}
 W(a, b) & \xrightarrow[A^{-1}]{a < b} & W(a, -a+b) & \xrightarrow[A^{-1}]{2a < b} & W(a, -2a+b) \xrightarrow[A^{-1}]{3a < b} \dots \\
 & \searrow \begin{smallmatrix} a < b \\ V \end{smallmatrix} & \searrow \begin{smallmatrix} 2a < b \\ A \end{smallmatrix} & \searrow \begin{smallmatrix} 3a < b \\ V \end{smallmatrix} & \\
 & M(a, a-b) & \xrightarrow[2a > b]{A} M(a, 2a-b) & \xrightarrow[3a > b]{A} M(a, 2a-b) & \dots
 \end{array}$$

- The action of  $\sigma_2^{-1}$  is shown in the following commutative diagram. For  $x \gg 0$ , Type M converges to Type W under  $\sigma_2^{-x}$  and the change in weights on the train track is represented by  $A^{x-1}V$ .

$$\begin{array}{ccccc}
 M(a, b) & \xrightarrow[A^{-1}]{a < b} & M(a, -a+b) & \xrightarrow[A^{-1}]{2a < b} & M(a, -2a+b) \xrightarrow[A^{-1}]{3a < b} \dots \\
 & \searrow \begin{smallmatrix} a < b \\ V \end{smallmatrix} & \searrow \begin{smallmatrix} 2a > b \\ A \end{smallmatrix} & \searrow \begin{smallmatrix} 3a > b \\ A \end{smallmatrix} & \\
 & W(a, a-b) & \xrightarrow[2a > b]{A} W(a, 2a-b) & \xrightarrow[3a > b]{A} W(a, 2a-b) & \dots
 \end{array}$$

Let  $\beta \in B_3 \simeq \mathcal{MCG}(D_3)$  be a pseudo-Anosov 3-braid. Suppose that  $\beta$  has an invariant train track of Type X, where X = M or W. That is, if we apply  $\beta$  to a Type X train track, then we return to the same type of train track. A *transition matrix*  $M$  tells how the weights of the train track edges have changed after applying  $\beta$ . The key idea of the upcoming Lemma 9 is to compute the type of  $\beta$  and the transition matrix by applying Lemmas 7 and 8.

**Lemma 9** *Let  $\beta = \sigma_1^x \sigma_2^{-1} \sigma_1^y \sigma_2^z$  and  $\beta' = \sigma_1^x \sigma_2^z \sigma_1^y \sigma_2^{-1}$  be pseudo-Anosov 3-braids related by a negative flype. For large  $|x|, |y|$  and  $|z|$ , the transition matrices  $M$  and  $M'$  associated to  $\beta$  and  $\beta'$  respectively are the following:*

$$\begin{aligned}
 M &= \text{sgn}(xyz) \begin{pmatrix} -1-y & \text{sgn}(z)(x+y+xy) \\ \text{sgn}(z)(1-z-yz) & -1-x+xz+yz+xyz \end{pmatrix} \\
 M' &= \text{sgn}(xyz) \begin{pmatrix} -1+yz & -x-y+xyz \\ -1+z+yz & -1-x-y+xz+xyz \end{pmatrix}
 \end{aligned}$$

Furthermore, we have  $\det(M) = \det(M') = 1$  and

$$\text{tr}(M) = \text{tr}(M') = \gamma(x, y, z) = \text{sgn}(xyz)(-2 - x - y + xz + yz + xyz).$$

In the proof of Lemma 9 we separate all the applicable 3-braids into eight cases, depending on the sign of  $x, y$ , and  $z$ :

	$x$	$y$	$z$
Case 1	–	–	–
Case 2	–	–	+
Case 3	–	+	–
Case 4	–	+	+
Case 5	+	–	–
Case 6	+	–	+
Case 7	+	+	–
Case 8	+	+	+

**Remark 1** There is a bijection between the set of Case 3 braids and the set of Case 5 braids. Suppose  $x, y, z > 0$ . Then

$$(\text{Case 3}) \sigma_1^{-x} \sigma_2^{-1} \sigma_1^y \sigma_2^{-z} \stackrel{\text{conj}}{\sim} \sigma_2^{-z} \sigma_1^{-x} \sigma_2^{-1} \sigma_1^y \stackrel{\text{rev}}{\sim} \sigma_1^y \sigma_2^{-1} \sigma_1^{-x} \sigma_2^{-z} \text{ (Case 5)}$$

where  $\stackrel{\text{conj}}{\sim}$  means conjugation and  $\stackrel{\text{rev}}{\sim}$  means reverse orientation or read the word backward. Thus swapping  $x$  and  $y$  gives a bijection between the two sets.

Similarly, we can find a bijection between the sets for Case 4 and Case 6 braids by swapping  $x$  and  $y$ .

**Proof of Lemma 9** We calculate the matrices  $M_i$  and  $M'_i$  associated to  $\beta$  and  $\beta'$  respectively for each Case  $i$ . Below we manipulate the braid word of  $\beta$  and  $\beta'$  in each case to force  $x, y$ , and  $z$  to be positive in our calculations.

**Case 1:** ( $\beta = \sigma_1^{-x} \sigma_2^{-1} \sigma_1^{-y} \sigma_2^{-z}$  and  $\beta' = \sigma_1^{-x} \sigma_2^{-z} \sigma_1^{-y} \sigma_2^{-1}$ ) We use Lemmas 7 and 8 in the calculations for  $\beta$  and  $\beta'$ .

$$\beta : W \xrightarrow[\sigma_1^{-x}]{} M \xrightarrow[\sigma_2^{-1}]{} W \xrightarrow[\sigma_1^{-y}]{} M \xrightarrow[\sigma_2^{-z}]{} W \quad (\text{Type W})$$

$$\beta' : W \xrightarrow[\sigma_1^{-x}]{} M \xrightarrow[\sigma_2^{-z}]{} W \xrightarrow[\sigma_1^{-y}]{} M \xrightarrow[\sigma_2^{-1}]{} W \quad (\text{Type W})$$

Both  $\beta$  and  $\beta'$  have Type W invariant measured train tracks. Transition matrices are:

$$\begin{aligned} M_1 &= A^{z-1} V B^{y-1} T V B^{x-1} T \\ &= \begin{pmatrix} 1-y & -x-y+xy \\ 1+z-yz & 1-x-xz-yz+xyz \end{pmatrix} \end{aligned} \quad (29)$$

$$\begin{aligned} M'_1 &= V B^{y-1} T A^{z-1} V B^{x-1} T \\ &= \begin{pmatrix} 1-yz & -x-y+xyz \\ 1+z-yz & 1-x-y-xz+xyz \end{pmatrix} \end{aligned} \quad (30)$$

Below we turn  $M_1$  and  $M'_1$  into Perron-Frobenius by taking conjugates of the original matrices. When  $x \geq 3, y \geq 3$ , and  $z \geq 2$ , we can verify that  $A^{-1} M_1 A$  is a non-negative integral (i.e., Perron-Frobenius) matrix. Here are the computations. The  $(1, 1)$  element of the matrix  $A^{-1} M_1 A$  is

$$xy - x - 2y + 1 = (x-2)(y-1) - 1 \geq 0.$$

The  $(1, 2)$  element is

$$(x-1)(y-1) - 1 \geq 0.$$

The  $(2, 1)$  element is

$$(x-2)(z-1) \left( (y-1) - \frac{1}{x-2} - \frac{1}{z-1} \right) \geq 0.$$

The  $(2, 2)$  element is

$$(x-1)(z-1) \left( (y-1) - \frac{1}{x-1} - \frac{1}{z-1} \right) \geq 0.$$

If  $x \geq 3$ ,  $y \geq 3$ , and  $z \geq 2$ , the matrix  $C^{-1}M'_1C$  where  $C := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$  is also Perron-Frobenius. Indeed, the  $(1, 1)$  element is

$$z(x-2) - 1 \geq 0.$$

The  $(1, 2)$  element is

$$z(x-1) - 1 \geq 0.$$

The  $(2, 1)$  element is

$$(x-2)(y-2) \left( z - \frac{1}{y-2} - \frac{1}{x-2} \right) \geq 0.$$

The  $(2, 2)$  element is

$$(x-1)(y-2) \left( z - \frac{1}{y-2} - \frac{1}{x-1} \right) \geq 0.$$

**Case 2:** ( $\beta = \sigma_1^{-x} \sigma_2^{-1} \sigma_1^{-y} \sigma_2^z$  and  $\beta' = \sigma_1^{-x} \sigma_2^z \sigma_1^{-y} \sigma_2^{-1}$ ) We have:

$$\beta : M \xrightarrow[\sigma_1^{-x}]{B^x} M \xrightarrow[\sigma_2^{-1}]{V} W \xrightarrow[\sigma_1^{-y}]{B^{y-1}T} M \xrightarrow[\sigma_2^z]{A^z} M \quad (\text{Type M})$$

$$\beta' : W \xrightarrow[\sigma_1^{-x}]{B^{x-1}T} M \xrightarrow[\sigma_2^z]{A^z} M \xrightarrow[\sigma_1^{-y}]{B^y} M \xrightarrow[\sigma_2^{-1}]{V} W \quad (\text{Type W})$$

$$M_2 = A^z B^{y-2} T B^{-1} V B^x$$

$$= \begin{pmatrix} -1+y & -x-y+xy \\ 1-z+yz & -1+x-xz-yz+xyz \end{pmatrix}$$

$$M'_2 = V B^y A^z B^{x-1} T$$

$$= \begin{pmatrix} -1-yz & x+y+xyz \\ -1+z-yz & -1+x+y-xz+xyz \end{pmatrix}$$

For  $x, y, z \gg 1$ , both  $M_2$  and  $C^{-1}M'_2C$  are Perron-Frobenius.

**Case 3:** ( $\beta = \sigma_1^{-x} \sigma_2^{-1} \sigma_1^y \sigma_2^{-z}$  and  $\beta' = \sigma_1^{-x} \sigma_2^{-z} \sigma_1^y \sigma_2^{-1}$ ) We have:

$$\beta : W \xrightarrow[\sigma_1^{-x}]{B^{x-1}T} M \xrightarrow[\sigma_2^{-1}]{V} W \xrightarrow[\sigma_1^y]{B^y} W \xrightarrow[\sigma_2^{-z}]{A^z} W \quad (\text{Type W})$$

$$\beta' : W \xrightarrow[\sigma_1^{-x}]{B^{x-1}T} M \xrightarrow[\sigma_2^{-z}]{A^{z-1}V} W \xrightarrow[\sigma_1^y]{B^y} W \xrightarrow[\sigma_2^{-1}]{A} W \quad (\text{Type W})$$

$$M_3 = A^z B^y V B^{x-1} T$$

$$= \begin{pmatrix} -1-y & x-y+xy \\ -1-z-yz & -1+x+xz-yz+xyz \end{pmatrix}$$

$$M'_3 = A B^y A^{z-1} V B^{x-1} T$$

$$= \begin{pmatrix} -1-yz & x-y+xyz \\ -1-z-yz & -1+x-y+xz+xyz \end{pmatrix}$$

For  $x, y, z \gg 1$ , both  $A^{-1}M_3A$  and  $A^{-1}M'_3A$  are Perron-Frobenius.

**Case 4:** ( $\beta = \sigma_1^{-x}\sigma_2^{-1}\sigma_1^y\sigma_2^z$  and  $\beta' = \sigma_1^{-x}\sigma_2^z\sigma_1^y\sigma_2^{-1}$ ) We have:

$$\beta : M \xrightarrow[\sigma_1^{-x}]{B^x} M \xrightarrow[\sigma_2^{-1}]{V} W \xrightarrow[\sigma_1^y]{B^y} W \xrightarrow[\sigma_2]{V} M \xrightarrow[\sigma_2^{z-1}]{A^{z-1}} M \quad (\text{Type M})$$

$$\beta' : W \xrightarrow[\sigma_1^{-x}]{B^{x-1}T} M \xrightarrow[\sigma_2^z]{A^z} M \xrightarrow[\sigma_1^y]{T} W \xrightarrow[\sigma_1^{y-1}]{B^{y-1}} W \xrightarrow[\sigma_2^{-1}]{A} W \quad (\text{Type W})$$

$$\begin{aligned} M_4 &= A^{z-1}V B^y V B^x \\ &= \begin{pmatrix} 1+y & x-y+xy \\ -1+z+yz & 1-x+xz-yz+xyz \end{pmatrix} \\ M'_4 &= AB^{y-1}TA^zB^{x-1}T \\ &= \begin{pmatrix} 1-yz & -x+y+xyz \\ 1-z-yz & 1-x+y+xz+xyz \end{pmatrix} \end{aligned}$$

For  $x, y, z \gg 1$ , both  $M_4$  and  $A^{-1}M'_4A$  are Perron-Frobenius.

**Case 5:** ( $\beta = \sigma_1^x\sigma_2^{-1}\sigma_1^{-y}\sigma_2^{-z}$  and  $\beta' = \sigma_1^x\sigma_2^{-z}\sigma_1^{-y}\sigma_2^{-1}$ ) We have:

$$\beta : W \xrightarrow[\sigma_1^x]{B^x} W \xrightarrow[\sigma_2^{-1}]{A} W \xrightarrow[\sigma_1^{-1}]{T} M \xrightarrow[\sigma_1^{-y+1}]{B^{y-1}} M \xrightarrow[\sigma_2^{-1}]{V} W \xrightarrow[\sigma_2^{-z+1}]{A^{z-1}} W \quad (\text{Type W})$$

$$\beta' : W \xrightarrow[\sigma_1^x]{B^x} W \xrightarrow[\sigma_2^{-z}]{A^z} W \xrightarrow[\sigma_1^{-1}]{T} M \xrightarrow[\sigma_1^{-y+1}]{B^{y-1}} M \xrightarrow[\sigma_2^{-1}]{V} W \quad (\text{Type W})$$

$$\begin{aligned} M_5 &= A^{z-1}V B^{y-1}T A B^x \\ &= \begin{pmatrix} -1+y & -x+y+xy \\ -1-z+yz & -1-x-xz+yz+xyz \end{pmatrix} \\ M'_5 &= V B^{y-1}T A^z B^x \\ &= \begin{pmatrix} -1+yz & -x+y+xyz \\ -1-z+yz & -1-x+y-xz+xyz \end{pmatrix} \end{aligned}$$

For  $x, y, z \gg 1$ , both  $M_5$  and  $M'_5$  are Perron-Frobenius.

**Case 6:** ( $\beta = \sigma_1^x\sigma_2^{-1}\sigma_1^{-y}\sigma_2^z$  and  $\beta' = \sigma_1^x\sigma_2^z\sigma_1^{-y}\sigma_2^{-1}$ ) We have:

$$\beta : M \xrightarrow[\sigma_1^x]{B^{x-1}T} W \xrightarrow[\sigma_2^{-1}]{A} W \xrightarrow[\sigma_1^{-y}]{B^{y-1}T} M \xrightarrow[\sigma_2^z]{A^z} M \quad (\text{Type M})$$

$$\beta' : W \xrightarrow[\sigma_1^x]{B^x} W \xrightarrow[\sigma_2]{V} M \xrightarrow[\sigma_2^{z-1}]{A^{z-1}} M \xrightarrow[\sigma_1^{-y}]{B^y} M \xrightarrow[\sigma_2^{-1}]{V} W \quad (\text{Type W})$$

$$\begin{aligned} M_6 &= A^z B^{y-1}T A B^{x-1}T \\ &= \begin{pmatrix} 1-y & -x+y+xy \\ -1+z-yz & 1+x-xz+yz+xyz \end{pmatrix} \end{aligned}$$

$$\begin{aligned} M'_6 &= VB^y A^{z-1} VB^x \\ &= \begin{pmatrix} 1+yz & x-y+xyz \\ 1-z+yz & 1+x-y-xz+xyz \end{pmatrix} \end{aligned}$$

For  $x, y, z \gg 1$ , both  $A^{-1}M_6A$  and  $M'_6$  are Perron-Frobenius.

**Case 7:** ( $\beta = \sigma_1^x \sigma_2^{-1} \sigma_1^y \sigma_2^{-z}$  and  $\beta' = \sigma_1^x \sigma_2^{-z} \sigma_1^y \sigma_2^{-1}$ ) We have:

$$\beta : W \xrightarrow[\sigma_1^x]{B^x} W \xrightarrow[\sigma_2^{-1}]{A} W \xrightarrow[\sigma_1^y]{B^y} W \xrightarrow[\sigma_2^{-z}]{A^z} W \quad (\text{Type W})$$

$$\beta' : W \xrightarrow[\sigma_1^x]{B^x} W \xrightarrow[\sigma_2^{-z}]{A^z} W \xrightarrow[\sigma_1^y]{B^y} W \xrightarrow[\sigma_2^{-1}]{A} W \quad (\text{Type W})$$

$$M_7 = A^z B^y A B^x = \begin{pmatrix} 1+y & x+y+xy \\ 1+z+yz & 1+x+xz+yz+xyz \end{pmatrix}$$

$$M'_7 = A B^y A^z B^x = \begin{pmatrix} 1+yz & x+y+xyz \\ 1+z+yz & 1+x+y+xz+xyz \end{pmatrix}$$

Both  $M_7$  and  $M'_7$  are Perron-Frobenius.

**Case 8:** ( $\beta = \sigma_1^x \sigma_2^{-1} \sigma_1^y \sigma_2^z$  and  $\beta' = \sigma_1^x \sigma_2^z \sigma_1^y \sigma_2^{-1}$ ) We have:

$$\beta : M \xrightarrow[\sigma_1^x]{B^{x-1}T} W \xrightarrow[\sigma_2^{-1}]{A} W \xrightarrow[\sigma_1^y]{B^y} W \xrightarrow[\sigma_2]{V} M \xrightarrow[\sigma_2^{z-1}]{A^{z-1}} M \quad (\text{Type M})$$

$$\beta' : W \xrightarrow[\sigma_1^x]{B^x} W \xrightarrow[\sigma_2]{V} M \xrightarrow[\sigma_2^{z-1}]{A^{z-1}} M \xrightarrow[\sigma_1]{T} W \xrightarrow[\sigma_1^{y-1}]{B^{y-1}} W \xrightarrow[\sigma_2^{-1}]{A} W \quad (\text{Type W})$$

$$\begin{aligned} M_8 &= A^{z-1} V B^y A B^{x-1} T \\ &= \begin{pmatrix} -1-y & x+y+xy \\ 1-z-yz & -1-x+xz+yz+xyz \end{pmatrix} \end{aligned}$$

$$\begin{aligned} M'_8 &= A B^{y-1} T A^{z-1} V B^x \\ &= \begin{pmatrix} -1+yz & -x-y+xyz \\ -1+z+yz & -1-x-y+xz+xyz \end{pmatrix} \end{aligned}$$

For  $x, y, z \gg 1$ , both  $A^{-1}M_8A$  and  $M'_8$  are Perron-Frobenius.  $\square$

**Remark 2** In Proposition 9, we assume that  $x, y, z \gg 1$ . More precisely in Case 1, for example, we require  $(x-1) - \frac{1}{(y-1)-\frac{1}{z}} > \alpha$ .

Let  $a$  and  $b$  be positive integers. After scaling, we can identify  $W(a, b)$  with  $W(\alpha, 1)$  for some  $\alpha = \frac{a}{b}$ . We start with  $W(\alpha, 1)$  and apply  $\sigma_1^{-x}$ . If  $x > \alpha$ , we obtain

$$W(\alpha, 1) \xrightarrow{\sigma_1^{-x}} M(x-\alpha, 1).$$

If  $x-1 > \alpha$ , we then obtain

$$M(x-\alpha, 1) \xrightarrow{\sigma_2^{-1}} W(x-\alpha, x-1-\alpha).$$

If  $(x - 1) - \frac{1}{y-1} > \alpha$ , we obtain

$$W(x - \alpha, x - 1 - \alpha) \xrightarrow{\sigma_1^{-y}} M((y - 1)(x - 1) - 1 - \alpha(y - 1), (x - 1) - \alpha).$$

If  $(x - 1) - \frac{1}{(y-1)-\frac{1}{z}} > \alpha$ , we further obtain

$$\begin{aligned} & M((y - 1)(x - 1) - 1 - \alpha(y - 1), (x - 1) - \alpha) \\ & \xrightarrow{\sigma_2^{-z}} W((y - 1)(x - 1) - 1 - \alpha(y - 1), (x - 1)(zy - z - 1) - z - \alpha(zy - z - 1)) \\ & = W((1 - y)\alpha - x - y + xy, (1 + z - yz)\alpha + (1 - x - xz - yz + xyz)) \end{aligned} \quad (31)$$

The expression in (31) matches with the matrix  $M_1$  in (29). Therefore, as long as

$$(x - 1) - \frac{1}{(y - 1) - \frac{1}{z}} > \alpha \quad (32)$$

holds, the transition matrix  $M_1$  is valid.

**Remark 3** With regard to Theorem 10, the referee kindly shared with the authors a very short proof to show that braids  $\beta = \sigma_1^x \sigma_2^\epsilon \sigma_1^y \sigma_2^z$  and  $\beta' = \text{flype}(\beta) = \sigma_1^x \sigma_2^z \sigma_1^y \sigma_2^\epsilon$  related by a flype have the same dilatation, without explicitly computing the dilatation. Here is the referee's argument:

Clearly the inverse  $\beta^{-1} = \sigma_2^{-z} \sigma_1^{-y} \sigma_2^{-\epsilon} \sigma_1^{-x}$  and  $\beta$  have the same dilatation.

Consider the *reverse*  $\beta^{\text{rev}}$  of the braid  $\beta$ :

$$\beta^{\text{rev}} = \sigma_2^z \sigma_1^y \sigma_2^\epsilon \sigma_1^x,$$

which is the braid obtained from  $\beta$  by reading the word of  $\beta$  from the right.

Since  $\beta^{-1}$  and  $\beta^{\text{rev}}$  are related by mirror image and conjugate,  $\beta^{-1}$  and  $\beta^{\text{rev}}$  have the same dilatation.

Since  $\beta^{\text{rev}}$  and  $\beta'$  are conjugate  $\beta^{\text{rev}}$  and  $\beta'$  have the same dilatation.

This shows  $\beta$  and  $\beta'$  have the same dilatation.

**Acknowledgements** The authors would like to thank Dan Margalit for useful conversation and talk slides, Ian Agol for interesting conversation, Eiko Kin for inspiring comments especially on Fig. 6, Theorem 6, and Theorem 7, the anonymous referee for numerous helpful suggestions to improve the entire paper, and Manuel Albrizzio for the instructive initial draft of the Matlab code to predict Agol cycles. EA was partially supported by the Ford Foundation and the Alfred P. Sloan Foundation and KK was partially supported by NSF DMS2005450.

**Author Contributions** E.A. and K.K. wrote the main manuscript text and E.A. prepared all figures. All authors reviewed the manuscript.

## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

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