

Analysis and Petrov–Galerkin numerical approximation for variable coefficient two-sided fractional diffusion, advection, reaction equations

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ABSTRACT

In this paper we investigate the variable coefficient two-sided fractional diffusion, advection, reaction equations on a bounded interval. It is known that the fractional diffusion operator may lose coercivity due to the variable coefficient, which makes both the mathematical and numerical analysis challenging. To resolve this issue, we design appropriate test and trial functions to prove the inf-sup condition of the variable coefficient fractional diffusion, advection, reaction operators in suitable function spaces. Based on this property, we prove the well-posedness and regularity of the solutions, as well as analyze the Petrov–Galerkin approximation scheme for the proposed model. Numerical experiments are presented to substantiate the theoretical findings and to compare the behaviors of different models.

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1. Introduction

In deriving the modeling equations for many physical phenomena often considered is the conservation of a particular quantity within a subdomain O of the domain Ω . The diffusive flow (flux) into the O is given by $-\int_{\partial O} \mathbf{q} \cdot \mathbf{n} ds$, where ∂O denotes the boundary of O , $\mathbf{q}$ the flux, and $\mathbf{n}$ the outward pointing unit normal on ∂O . An application of the divergence theorem replaces the boundary integral with the volume integral $-\int_O \nabla \cdot \mathbf{q} dV$. In case of modeling a nonlocal phenomena, the flux may be modeled as either $-\mathcal{I}^{2-\alpha} k(x) \nabla u(x)$ or $-k(x) \mathcal{I}^{2-\alpha} \nabla u(x)$, where $\mathcal{I}^{2-\alpha} \cdot$ denotes a fractional order (nonlocal) integral operator. For a homogeneous medium, i.e., $k(x) = k_0 > 0$ a constant, one obtains a constant coefficient fractional order diffusion operator.

The Galerkin solution to the fractional order diffusion advection, reaction problem on a bounded interval in $\mathbb{R}^1$ was analyzed by Ervin and Roop in [1]. Their analysis was restricted to $k(x) = k_0 > 0$, for which they established existence and uniqueness of solution in $H_0^{\alpha/2}(\Omega)$. In [2] Wang and Yang gave a simple example (a piecewise constant $k(x)$, $k(x) > 0$) which demonstrated that the Galerkin formulation of the fractional order diffusion problem was not well-posed in $H_0^{\alpha/2}(\Omega)$ for a variable coefficient, $k(x)$. Specifically, they showed that the corresponding bilinear form was not coercive.

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In this paper we present the first well-posedness result for the weak (Petrov–Galerkin) formulation of the variable coefficient fractional order diffusion, advection, reaction equation. Specifically, we investigate the well-posedness, regularity and numerical approximation of the variable coefficient two-sided fractional diffusion, advection, reaction equation

$$\tilde{\mathcal{L}}_r^\alpha u(x) + b(x)Du(x) + c(x)u(x) = f(x), \quad x \in I, \quad (1.1)$$

$$\text{subject to } u(0) = u(1) = 0. \quad (1.2)$$

Here $I := (0, 1)$, $b \in W^{1,\infty}(I)$, $c, k \in L^\infty(I)$ with $k(x) \geq k_0 > 0$ represent advection, reaction, and material diffusivity coefficients, respectively, D denotes the usual differential operator, The fractional diffusion operator $\tilde{\mathcal{L}}_r^\alpha$ with $1 < \alpha < 2$ and $0 \leq r \leq 1$ has two possible forms $\hat{\mathcal{L}}_r^\alpha$ and $\check{\mathcal{L}}_r^\alpha$ according to the position of the variable diffusivity coefficient

$$\hat{\mathcal{L}}_r^\alpha u(x) := -D\left((rD^{-(2-\alpha)} + (1-r)D^{-(2-\alpha)*})k(x)Du(x)\right), \quad (1.3)$$

$$\check{\mathcal{L}}_r^\alpha u(x) := -D\left(k(x)(rD^{-(2-\alpha)} + (1-r)D^{-(2-\alpha)*})Du(x)\right), \quad (1.4)$$

where $D^{-(2-\alpha)}$ and $D^{-(2-\alpha)*}$ refer to the $(2-\alpha)$ -order left and right fractional integral operators, respectively, defined by

$$D^{-(2-\alpha)}u(x) := \frac{1}{\Gamma(2-\alpha)} \int_0^x \frac{1}{(x-s)^{\alpha-1}} u(s) ds, \quad (1.5)$$

$$D^{-(2-\alpha)*}u(x) := \frac{1}{\Gamma(2-\alpha)} \int_x^1 \frac{1}{(s-x)^{\alpha-1}} u(s) ds. \quad (1.6)$$

There are fruitful mathematical and numerical results for space-fractional differential equations in the literature [3–13]. The well-posedness of a constant diffusivity coefficient analogue of problem (1.1)–(1.2) and error estimates of its Galerkin finite element approximation was proved in [1]. It was shown in [2] that Galerkin weak formulation of the diffusive only version (i.e., $b = c = 0$) of problem (1.1)–(1.2) is not coercive and the corresponding Galerkin finite element approximation is not guaranteed to converge [14]. It was also shown in [6,15] that the solution to the diffusive only version of problem (1.1)–(1.2) or even its constant diffusivity coefficient analogue exhibits weak singularity near the end points of the interval, which makes the full regularity assumption on the true solution of fractional differential equations in many optimal-order error estimates of numerical approximations in the literature unrealistic. An optimal-order error estimate of a spectral Galerkin method in the L^2 norm was proved in [16] for a one-sided diffusive only version of problem (1.1)–(1.2) by assuming only the smoothness of the diffusivity coefficient and the right-hand side function. The complete regularity analysis of the diffusive only version of problem (1.1)–(1.2) was given in [17], which were subsequently extended to the constant diffusivity coefficient analogue of problem (1.1)–(1.2) in [18] (see also [19]).

Despite the aforementioned progress, the corresponding results for the variable coefficient fractional diffusion, advection, reaction equations are largely missing. There exists some recent work on Petrov–Galerkin approximations to two-sided fractional diffusion, reaction equations [20,21], one-sided fractional diffusion, advection, reaction equations [22–24], and two-sided fractional diffusion, advection, reaction equations [25], all with constant diffusivity coefficients. To the best of our knowledge, the only available result for the Petrov–Galerkin method applied to variable coefficient fractional diffusion problems is [2], in which the weak coercivity in the sense of inf-sup condition was proved for the one-sided variable coefficient fractional diffusion operator, i.e., $\check{\mathcal{L}}_r^\alpha$ with $r = 1$. Then a discontinuous Petrov–Galerkin method was applied to this model in [14] for numerical approximation. How to prove the weak coercivity for the two-sided case (1.1) remains untreated in the literature, which hinders the theoretical and numerical analysis.

In this paper we analyze the well-posedness, regularity and numerical approximation of problem (1.1)–(1.2) by designing appropriate test and trial functions to prove the weak coercivity of the Petrov–Galerkin weak formulation for problem (1.1)–(1.2). For different choices of $\tilde{\mathcal{L}}_r^\alpha$ we select different test and trial spaces for the sake of the proof. Particular attention has been paid in the choice of the weak formulations given, in order that the required regularity for $k(x)$, $b(x)$ and $c(x)$ are consistent with those for the usual (second-order) diffusion, advection, reaction problem. Error estimates of the Petrov–Galerkin spectral approximation scheme are proved in both weighted L^2 and energy norms, which provides theoretical supports for numerical computations.

The rest of the paper is organized as follows. In Section 2 we introduce definitions, notations, and useful results. In Section 3 we analyze properties of the Petrov–Galerkin weak formulation for (1.1)–(1.2), and establishes the existence and uniqueness of its solution. In Section 4 we prove the regularity of the solutions. The Petrov–Galerkin approximation scheme is proposed and analyzed in Section 5. Numerical experiments are presented in the last section to substantiate the theoretical findings and to compare the behaviors of different models.

2. Notation and properties

In this section, we present various notation and spaces to be used subsequently. We let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and use $y_n \sim n^p$ to denote that there exist constants c and $C > 0$ such that, as $n \rightarrow \infty$, $c n^p \leq |y_n| \leq C n^p$. Additionally, we use $a \lesssim b$ and $a \simeq b$ to denote that there exist constants C_1 and C_2 such that $a \leq C_2 b$, and $C_1 b \leq a \leq C_2 b$, respectively. For $t \in \mathbb{R}$, $\lfloor t \rfloor$ is used to denote the largest integer that is less than or equal to t , and $\lceil t \rceil$ is used to denote the smallest integer that is greater than or equal to t .

2.1. Jacobi polynomials

Jacobi polynomials have close relations to the fractional problems [17,26–28]. The classical Jacobi (orthogonal) polynomials $\{P_n^{(a,b)}(x)\}_{n \geq 0}$ for $n \geq 0$ and $a, b > -1$ are defined on $[-1, 1]$ (see [29,30]). As we are interested in the domain $I = (0, 1)$, we let $G_n^{(a,b)}(x) = P_n^{(a,b)}(2x - 1)$. Then

$$\int_0^1 (1-x)^a x^b G_j^{(a,b)}(x) G_k^{(a,b)}(x) dx = \begin{cases} 0, & k \neq j, \\ \|\|G_j^{(a,b)}\|\|^2, & k = j, \end{cases}$$

where $\|\|G_j^{(a,b)}\|\| = \|\|G_j^{(b,a)}\|\| = \left(\frac{1}{(2j+a+b+1)} \frac{\Gamma(j+a+1)\Gamma(j+b+1)}{\Gamma(j+1)\Gamma(j+a+b+1)} \right)^{1/2}$. (2.1)

From [31, equation (2.9)] we have for $1 \leq a \leq 2, b > 0, a - b > 0$,

$$\frac{1}{2} \leq \frac{\|\|G_j^{(a-b,b)}\|\|^2}{\|\|G_{j+1}^{(b-1,a-b-1)}\|\|^2} = \frac{j+1}{j+a} \leq 1, \quad j \geq 0. \quad (2.2)$$

From [28] it follows that

$$\frac{d^k}{dt^k} G_n^{(a,b)}(t) = \frac{\Gamma(n+k+a+b+1)}{\Gamma(n+a+b+1)} G_{n-k}^{(a+k,b+k)}(t). \quad (2.3)$$

Note that, from Stirling's formula, we have that

$$\lim_{n \rightarrow \infty} \frac{\Gamma(n+\sigma)}{\Gamma(n)n^\sigma} = 1, \text{ for } \sigma \in \mathbb{R}. \quad (2.4)$$

For compactness of notation, let $\omega^{(a,b)} = \omega^{(a,b)}(x) := (1-x)^a x^b$.

A useful formula used in the analysis below is [28]

$$\frac{d^k}{dx^k} \omega^{(a+k,b+k)}(x) G_{n-k}^{(a+k,b+k)}(x) = \frac{(-1)^k n!}{(n-k)!} \omega^{(a,b)}(x) G_n^{(a,b)}(x), \text{ for } 0 \leq k \leq n. \quad (2.5)$$

Condition A. The parameters α, β, r and constant c_* satisfy: $1 < \alpha < 2, 0 \leq r \leq 1$

$$c_* = \frac{\sin(\pi\alpha)}{\sin(\pi(\alpha-\beta)) + \sin(\pi\beta)} < 0, \quad (2.6)$$

where $\beta, \alpha - 1 \leq \beta \leq 1$, is determined by

$$r = \frac{\sin(\pi\beta)}{\sin(\pi(\alpha-\beta)) + \sin(\pi\beta)}. \quad (2.7)$$

From [32], we have for $\mathcal{M}^* v(x) := (rD^{-(2-\alpha)*} + (1-r)D^{-(2-\alpha)})Dv(x)$

$$\mathcal{M}^* \omega^{(\beta, \alpha-\beta)}(x) G_k^{(\beta, \alpha-\beta)}(x) = \mu_k G_{k+1}^{(\alpha-\beta-1, \beta-1)}(x), \quad (2.8)$$

$$\text{where } \mu_k = c_*^* \frac{\Gamma(k+\alpha)}{\Gamma(k+1)}, \quad k = 0, 1, 2, \dots, \quad (2.9)$$

$$\text{Also, using (2.4), } |\mu_k| \sim (k+1)^{\alpha-1} \text{ as } k \rightarrow \infty. \quad (2.10)$$

Using (2.5) we have

$$D\omega^{(\alpha-\beta, \beta)}(x) G_n^{(\alpha-\beta, \beta)}(x) = -(n+1)\omega^{(\alpha-\beta-1, \beta-1)} G_{n+1}^{(\alpha-\beta-1, \beta-1)}(x), \quad n = 0, 1, 2, \dots \quad (2.11)$$

2.2. Function spaces

For $\sigma(x) > 0, x \in (0, 1)$, let

$$L_\sigma^2(I) := \left\{ f(x) : \int_0^1 \sigma(x) f(x)^2 dx < \infty \right\}. \quad (2.12)$$

Associated with $L_\sigma^2(0, 1)$ is the inner product, $(\cdot, \cdot)_\sigma$, and norm, $\|\cdot\|_\sigma$, defined by

$$(f, g)_\sigma := \int_0^1 \sigma(x) f(x) g(x) dx, \quad \text{and } \|f\|_\sigma := ((f, f)_\sigma)^{1/2}.$$

Without a subscript, $(\cdot, \cdot)$ denotes the usual $L^2(I)$ inner product. The set of orthogonal polynomials $\{G_j^{(a,b)}\}_{j=0}^\infty$ form an orthogonal basis for $L_{\omega^{(a,b)}}^2(I)$, and for $\tilde{G}_j^{(a,b)} := G_j^{(a,b)} / \|\|G_j^{(a,b)}\|\|$, $\{\tilde{G}_j^{(a,b)}\}_{j=0}^\infty$ form an orthonormal basis for $L_{\omega^{(a,b)}}^2(I)$.

The weighted Sobolev spaces $H_{(a,b)}^s(I)$ could be defined via two different but equivalent ways. In the first definition, for the non-negative integer s , define $H_{\omega(a,b)}^s(I)$ for $a, b > -1$ as [33,34]

$$H_{\omega(a,b)}^s(I) := \left\{ v : \|v\|_{s,\omega(a,b)}^2 := \sum_{j=0}^s \|D^j v\|_{\omega(a+j,b+j)}^2 < \infty \right\}. \quad (2.13)$$

The spaces $H_{\omega(a,b)}^s(I)$ for $s > 0 \notin \mathbb{N}$ are defined by the K -method of interpolation, while for $s < 0$ the spaces are defined by (weighted) L^2 duality. The second definition is based on the decay rate of the Jacobi coefficients of a function. For $v \in L_{\omega(a,b)}^2(I)$, the following expansion holds

$$v(x) = \sum_{j=0}^{\infty} v_j \widehat{G}_j^{(a,b)}(x), \quad v_j = \int_0^1 \omega^{(a,b)}(x) v(x) \widehat{G}_j^{(a,b)}(x) dx. \quad (2.14)$$

Then for $s, a, b \in \mathbb{R}$, $a, b > -1$, $L_{\omega(a,b)}^2(I) := L_{\omega(a,b)}^2(I)$, define

$$H_{(a,b)}^s(I) := \left\{ v : \sum_{j=0}^{\infty} (1+j^2)^s v_j^2 < \infty \right\}. \quad (2.15)$$

Theorem 2.1 ([18, Theorem 4.1]). *The spaces $H_{(a,b)}^s(I)$ and $H_{\omega(a,b)}^s(I)$ coincide, and their corresponding norms are equivalent.*

Lemma 2.1 ([18, Lemma 4.5]). *For $s, a, b \in \mathbb{R}$, $a, b > -1$, the differential operator D is a bounded mapping from $H_{(a,b)}^s(I)$ onto $H_{(a+1,b+1)}^{s-1}(I)$.*

For convenience, from hereon we use $H_{(a,b)}^s(I)$ to represent the spaces $H_{\omega(a,b)}^s(I)$ and $H_{(a,b)}^s(I)$.

Let S_N denote the space of polynomials of degree less than or equal to N . We define the weighted L^2 orthogonal projection $P_N : L_{\omega(a,b)}^2(I) \rightarrow S_N$ by the condition

$$(v - P_N v, \phi_N)_{\omega(a,b)} = 0, \quad \forall \phi_N \in S_N. \quad (2.16)$$

Note that $P_N v = \sum_{j=0}^N v_j \widehat{G}_j^{(a,b)}(x)$, where $v_j = \int_0^1 \omega^{(a,b)}(x) v(x) \widehat{G}_j^{(a,b)}(x) dx$.

Lemma 2.2 ([34, Theorem 2.1]). *For $\mu \in \mathbb{N}_0$, $a, b > -1$, and $v \in H_{\omega}^t(I)$, with $0 \leq \mu \leq t$, there exists a constant C , independent of N , a and b such that*

$$\|v - P_N v\|_{H_{\omega(a,b)}^{\mu}(I)} \leq C N^{\mu-t} \|v\|_{H_{\omega(a,b)}^t(I)}. \quad (2.17)$$

Remark. In [34] (2.17) is stated for $t \in \mathbb{N}_0$. The result extends to $t \in \mathbb{R}^+$ using interpolation.

Define the space $W_w^{k,\infty}(I)$ for $k \in \mathbb{N}_0$ and its associated norm as

$$W_w^{k,\infty}(I) := \{f : (1-x)^{j/2} x^{j/2} D^j f(x) \in L^{\infty}(I), j = 0, 1, \dots, k\}, \quad (2.18)$$

$$\|f\|_{W_w^{k,\infty}(I)} := \max_{0 \leq j \leq k} \|(1-x)^{j/2} x^{j/2} D^j f(x)\|_{L^{\infty}(I)}. \quad (2.19)$$

The subscript w denotes that $W_w^{k,\infty}(I)$ is a weaker space than $W^{k,\infty}(I)$.

Lemma 2.3 ([18, Lemma 7.1]). *Let $s \geq 0$, $\alpha, \beta > -1$, $k \geq s$, and $f \in W_w^{k,\infty}(I)$. For $g \in H_{(\alpha,\beta)}^s(I)$*

$$fg \in H_{(\alpha,\beta)}^s(I) \text{ and } \|fg\|_{H_{(\alpha,\beta)}^s(I)} \leq \|f\|_{W_w^{k,\infty}(I)} \|g\|_{H_{(\alpha,\beta)}^s(I)}. \quad (2.20)$$

For $k-1 < s < k$ the requirement that $f \in W_w^{k,\infty}(I)$ can be slightly relaxed to $f \in W_w^{k-1,\infty}(I)$ and $|f^{(k-1)}(x) - f^{(k-1)}(y)| \leq C|x-y|^\sigma$, for $\sigma > s - (k-1)$ (see the proof of Lemma 5.3 in [35]).

For compactness of notation, for α and r defined in (1.1) and β defined in (2.7) we introduce

$$\omega(x) := \omega^{(\alpha-\beta,\beta)}(x) = (1-x)^{\alpha-\beta} x^\beta, \quad \text{and} \quad \omega^*(x) := \omega^{(\beta,\alpha-\beta)}(x) = (1-x)^\beta x^{\alpha-\beta}. \quad (2.21)$$

The following lemma is helpful in establishing the error estimate between u and its approximation u_N . (See Corollary 5.1.)

Lemma 2.4 ([25, Lemma 4.3]). *Let $0 \leq \mu \leq 1$. For $\zeta \in H_{\omega}^{\mu}(I)$, then $z := \omega \zeta \in H_{\omega^{-1}}^{\mu}(I)$, with, for some $C > 0$,*

$$\|z\|_{H_{\omega^{-1}}^{\mu}(I)} \leq C \|\zeta\|_{H_{\omega}^{\mu}(I)}. \quad (2.22)$$

In the rest of the paper we use $\langle \cdot, \cdot \rangle_\omega$ to denote the weighted L^2 duality pairing between functions if $H_{(\alpha-\beta, \beta)}^{-s}(\mathbb{I})$ and $H_{(\alpha-\beta, \beta)}^s(\mathbb{I})$.

3. Analysis of weak formulation

We mainly consider model (1.1)–(1.2) with $\tilde{\mathcal{L}}_r^\alpha = \mathcal{L}_r^\alpha$, and briefly address the case of $\tilde{\mathcal{L}}_r^\alpha = \hat{\mathcal{L}}_r^\alpha$. (Since the latter is the adjoint of the former, the proofs are similar.)

Weak Formulation

Given $f \in H_{\omega^*}^{-(\alpha-1)}(\mathbb{I})$, $0 < k_0 \leq k(x) \in L^\infty(\mathbb{I})$, $b \in W_w^{1,\infty}(\mathbb{I})$ and $c \in L^\infty(\mathbb{I})$, determine $\phi \in H_\omega^1(\mathbb{I})$ such that $u(x) = \omega(x)\phi(x)$ satisfies

$$\langle \mathcal{L}_r^\alpha u + bDu + cu, \psi \rangle_{\omega^*} = \langle f, \psi \rangle_{\omega^*}, \quad \forall \psi \in H_{\omega^*}^{\alpha-1}(\mathbb{I}). \quad (3.1)$$

For simplicity of the notations, let $B : H_\omega^1(\mathbb{I}) \times H_{\omega^*}^{\alpha-1}(\mathbb{I}) \rightarrow \mathbb{R}$ and $F : H_{\omega^*}^{\alpha-1}(\mathbb{I}) \rightarrow \mathbb{R}$ be defined by

$$B(\phi, \psi) := \langle \mathcal{L}_r^\alpha u + bDu + cu, \psi \rangle_{\omega^*} =: B_0(\phi, \psi) + B_1(\phi, \psi) + B_2(\phi, \psi), \quad (3.2)$$

$$F(\psi) := \langle f, \psi \rangle_{\omega^*}. \quad (3.3)$$

We recall the Banach-Nečas-Babuška theorem to support the subsequent analysis.

Theorem 3.1 ([36, Pg. 85, Theorem 2.6]). Let H_1 and H_2 denote two real Hilbert spaces, $B(\cdot, \cdot) : H_1 \times H_2 \rightarrow \mathbb{R}$ a bilinear form, and $F : H_2 \rightarrow \mathbb{R}$ a bounded linear functional on H_2 . Suppose there are constants $C_1 < \infty$ and $C_2 > 0$ such that

$$(i) |B(w, v)| \leq C_1 \|w\|_{H_1} \|v\|_{H_2}, \quad \text{for all } w \in H_1, v \in H_2, \quad (3.4)$$

$$(ii) \sup_{0 \neq v \in H_2} \frac{|B(w, v)|}{\|v\|_{H_2}} \geq C_2 \|w\|_{H_1}, \quad \text{for all } w \in H_1, \quad (3.5)$$

$$(iii) \sup_{w \in H_1} |B(w, v)| > 0, \quad \text{for all } v \in H_2, v \neq 0. \quad (3.6)$$

Then there exists a unique solution $w_0 \in H_1$ satisfying $B(w_0, v) = F(v)$ for all $v \in H_2$. Further, $\|w_0\|_{H_1} \leq \frac{1}{C_2} \|F\|_{H_2}$.

3.1. Property (3.4) – Continuity of $B(\cdot, \cdot)$

In this section we establish property (3.4).

Continuity of $B_0(\cdot, \cdot)$

To establish the continuity of $B_0(\cdot, \cdot)$, we begin by choosing $\phi(x) = \sum_{i=0}^\infty \phi_i \widehat{G}_i^{(\alpha-\beta, \beta)}(x) \in H_\omega^1(\mathbb{I})$. Then,

$$\|\phi\|_{H_\omega^1(\mathbb{I})}^2 = \sum_{i=0}^\infty (1+i^2) \phi_i^2 < \infty. \quad (3.7)$$

Using (2.11),

$$D\omega\phi(x) = -\omega^{(\alpha-\beta-1, \beta-1)} \sum_{i=0}^\infty (1+i) \phi_i \frac{\|\widehat{G}_{i+1}^{(\alpha-\beta-1, \beta-1)}\|}{\|\widehat{G}_i^{(\alpha-\beta, \beta)}\|} \widehat{G}_{i+1}^{(\alpha-\beta-1, \beta-1)}(x) := -\omega^{(\alpha-\beta-1, \beta-1)} \eta(x). \quad (3.8)$$

Next, using (2.2),

$$\|\eta\|_{L_{\omega^{(\alpha-\beta-1, \beta-1)}}^2}^2 \simeq \sum_{i=0}^\infty ((1+i)\phi_i)^2 \simeq \sum_{i=0}^\infty (1+i^2)\phi_i^2 \simeq \|\phi\|_{H_\omega^1(\mathbb{I})}^2. \quad (3.9)$$

For $\psi(x) = \sum_{j=0}^\infty \psi_j \widehat{G}_j^{(\beta, \alpha-\beta)}(x) \in H_{\omega^*}^{\alpha-1}(\mathbb{I})$,

$$\|\psi\|_{H_{\omega^*}^{\alpha-1}(\mathbb{I})}^2 = \sum_{j=0}^\infty (1+j^2)^{\alpha-1} \psi_j^2 < \infty. \quad (3.10)$$

Using (2.8),

$$\mathcal{M}^* \omega^* \psi(x) = \sum_{j=0}^\infty \mu_j \psi_j \frac{\|\widehat{G}_{j+1}^{(\alpha-\beta-1, \beta-1)}\|}{\|\widehat{G}_j^{(\beta, \alpha-\beta)}\|} \widehat{G}_{j+1}^{(\alpha-\beta-1, \beta-1)}(x).$$

Again using (2.2),

$$\|\mathcal{M}^* \omega^* \psi\|_{L^2_{\omega(\alpha-\beta-1, \beta-1)}}^2 \simeq \sum_{j=0}^{\infty} (\mu_j \psi_j)^2 \simeq \sum_{j=0}^{\infty} (1+j^2)^{\alpha-1} \psi_j^2 \simeq \|\psi\|_{H_{\omega}^{\alpha-1}(\mathbb{I})}^2. \quad (3.11)$$

Combining the above pieces we have that

$$\begin{aligned} |B_0(\phi, \psi)| &= |(k(x) D\omega\phi(x), \mathcal{M}^* \omega^* \psi(x))| \\ &= |(-k(x) \eta(x), \mathcal{M}^* \omega^* \psi(x))_{\omega(\alpha-\beta-1, \beta-1)}| \quad (\text{using (3.8)}) \\ &\leq \|k\|_{L^\infty(\mathbb{I})} \|\eta\|_{L^2_{\omega(\alpha-\beta-1, \beta-1)}} \|\mathcal{M}^* \omega^* \psi\|_{L^2_{\omega(\alpha-\beta-1, \beta-1)}} \\ &\lesssim \|k\|_{L^\infty(\mathbb{I})} \|\phi\|_{H_{\omega^*}^1(\mathbb{I})} \|\psi\|_{H_{\omega}^{\alpha-1}(\mathbb{I})} \quad (\text{using (3.9) and (3.11)}). \end{aligned} \quad (3.12)$$

Continuity of $B_1(\cdot, \cdot)$

For $\phi \in H_{\omega}^1(\mathbb{I})$, an application of Theorem 5.1 in [18] (see also the Proof of Lemma 3.1 in [25]) establishes that $\omega\phi \in H_{(\beta-1, \alpha-\beta-1)}^{2-\alpha}(\mathbb{I})$ with $\|\omega\phi\|_{H_{(\beta-1, \alpha-\beta-1)}^{2-\alpha}(\mathbb{I})} \lesssim \|\phi\|_{H_{\omega}^1(\mathbb{I})}$. Then, using Lemma 2.1, $D\omega\phi \in H_{\omega^*}^{-(\alpha-1)}(\mathbb{I})$ with $\|D\omega\phi\|_{H_{\omega^*}^{-(\alpha-1)}(\mathbb{I})} \lesssim \|\omega\phi\|_{H_{(\beta-1, \alpha-\beta-1)}^{2-\alpha}(\mathbb{I})} \lesssim \|\phi\|_{H_{\omega}^1(\mathbb{I})}$.

Thus,

$$\begin{aligned} |B_1(\phi, \psi)| &= |\langle b Du, \psi \rangle_{\omega^*}| \\ &\leq \|D\omega\phi\|_{H_{\omega^*}^{-(\alpha-1)}(\mathbb{I})} \|b \psi\|_{H_{\omega^*}^{\alpha-1}(\mathbb{I})} \lesssim \|b\|_{W_w^{1,\infty}(\mathbb{I})} \|\phi\|_{H_{\omega}^1(\mathbb{I})} \|\psi\|_{H_{\omega^*}^{\alpha-1}(\mathbb{I})}. \end{aligned} \quad (3.13)$$

Continuity of $B_2(\cdot, \cdot)$

To establish the continuity of $B_2(\cdot, \cdot)$, note that

$$\begin{aligned} |B_2(\phi, \psi)| &= |\langle cu, \psi \rangle_{\omega^*}| = \left| \int_{\mathbb{I}} \omega^*(x) c(x) \omega(x) \phi(x) \psi(x) dx \right| \\ &\leq \|c\|_{L^\infty(\mathbb{I})} \|\omega^{1/2} \omega^{*1/2}\|_{L^\infty(\mathbb{I})} \int_{\mathbb{I}} |\omega^{1/2}(x) \phi(x)| |\omega^{*1/2}(x) \psi(x)| dx \leq \|c\|_{L^\infty(\mathbb{I})} \|\phi\|_{L_{\omega}^2(\mathbb{I})} \|\psi\|_{L_{\omega^*}^2(\mathbb{I})} \\ &\leq \|c\|_{L^\infty(\mathbb{I})} \|\phi\|_{H_{\omega}^1(\mathbb{I})} \|\psi\|_{H_{\omega^*}^{\alpha-1}(\mathbb{I})}. \end{aligned} \quad (3.14)$$

Combining (3.12), (3.13), and (3.14) we obtain the following lemma.

Lemma 3.1. For $k \in L^\infty(\mathbb{I})$, $b \in W_w^{1,\infty}(\mathbb{I})$ and $c \in L^\infty(\mathbb{I})$, there exists a constant $C_1 > 0$ such that $|B(\phi, \psi)| \leq C_1 \|\phi\|_{H_{\omega}^1(\mathbb{I})} \|\psi\|_{H_{\omega^*}^{\alpha-1}(\mathbb{I})}$ for all $\phi \in H_{\omega}^1(\mathbb{I})$ and $\psi \in H_{\omega^*}^{\alpha-1}(\mathbb{I})$. ■

3.2. Property (3.5) – Positivity of $B(\phi, \cdot)$

To establish the positivity of $B(\phi, \cdot)$, i.e. property (3.5), we need to show that there exists $C_2 > 0$ such that for $\phi \in H_{\omega}^1(\mathbb{I})$

$$\sup_{\psi \in H_{\omega^*}^1(\mathbb{I})} B(\phi, \psi) \geq C_2 \|\phi\|_{H_{\omega}^1(\mathbb{I})} \|\psi\|_{H_{\omega^*}^{\alpha-1}(\mathbb{I})}.$$

In comparison with the case $k = 1$ discussed in [25], given $\phi \in H_{\omega}^1(\mathbb{I})$ the appropriate choice of $\psi \in H_{\omega^*}^{\alpha-1}(\mathbb{I})$ for $k(x) \neq \text{constant}$ is more restrictive. Specifically, in this case we need ψ such that

$$D(\omega\phi(x)) \mathcal{M}^*(\omega^* \psi(x)) > 0$$

for x a.e. in $\mathbb{I}$. This then allows $k(x)$ to be taken out of the inner product and the orthogonality property of the Jacobi polynomials to be used.

To achieve this we choose ϕ and ψ such that $\omega(\alpha-\beta-1, \beta-1) D(\omega\phi(x)) = \mathcal{M}^*(\omega^* \psi(x))$. Let

$$\phi(x) = \sum_{i=0}^{\infty} \phi_i \widehat{G}_i^{(\alpha-\beta, \beta)}(x) \in H_{\omega}^1(\mathbb{I}),$$

and define $\Phi_i := -\phi_i / \mu_i$ such that

$$\phi(x) = - \sum_{i=0}^{\infty} \mu_i \Phi_i \widehat{G}_i^{(\alpha-\beta, \beta)}(x), \quad (3.15)$$

$$\text{and } \psi(x) = \sum_{j=0}^{\infty} (j+1) \Phi_j \widehat{G}_j^{(\beta, \alpha-\beta)}(x). \quad (3.16)$$

As $\phi \in H_{\omega}^1(I)$, using (2.10),

$$\|\phi\|_{H_{\omega}^1(I)}^2 = \sum_{i=0}^{\infty} (1+i^2)^1 (\mu_i \Phi_i)^2 \simeq \sum_{i=0}^{\infty} (1+i^2)^1 (1+i)^{2(\alpha-1)} \Phi_i^2 \simeq \sum_{i=0}^{\infty} (1+i^2)^{\alpha} \Phi_i^2 < \infty.$$

Then,

$$\|\psi\|_{H_{\omega^*}^{\alpha-1}(I)}^2 = \sum_{j=0}^{\infty} (1+j^2)^{\alpha-1} ((j+1)\Phi_j)^2 \simeq \sum_{j=0}^{\infty} (1+j^2)^{\alpha} \Phi_j^2 < \infty,$$

i.e., $\psi \in H_{\omega^*}^{\alpha-1}(I)$ and $\|\psi\|_{H_{\omega^*}^{\alpha-1}(I)} \simeq \|\phi\|_{H_{\omega}^1(I)}$.

Using (2.8) and (2.11)

$$\begin{aligned} B_0(\phi, \psi) &= (k(x)(D\omega\phi(x)), \mathcal{M}^*(\omega^*\psi(x))) \\ &= \left(k(x)\omega^{(\alpha-\beta-1, \beta-1)} \sum_{i=0}^{\infty} (i+1)\mu_i \Phi_i \frac{\|G_{i+1}^{(\alpha-\beta-1, \beta-1)}\|}{\|G_i^{(\alpha-\beta, \beta)}\|} \widehat{G}_{i+1}^{(\alpha-\beta-1, \beta-1)}(x), \right. \\ &\quad \left. \sum_{j=0}^{\infty} (j+1)\mu_j \Phi_j \frac{\|G_{j+1}^{(\alpha-\beta-1, \beta-1)}\|}{\|G_j^{(\beta, \alpha-\beta)}\|} \widehat{G}_{j+1}^{(\alpha-\beta-1, \beta-1)}(x) \right). \end{aligned}$$

As $k(x)$ and $\omega^{(\alpha-\beta-1, \beta-1)}(x) \geq 0$ for all $x \in I$, $\|G_i^{(\alpha-\beta, \beta)}\| = \|G_i^{(\beta, \alpha-\beta)}\|$, and using (2.2),

$$\begin{aligned} B_0(\phi, \psi) &\gtrsim k_0 \left(\omega^{(\alpha-\beta-1, \beta-1)} \sum_{i=0}^{\infty} (i+1)\mu_i \Phi_i \widehat{G}_{i+1}^{(\alpha-\beta-1, \beta-1)}(x), \sum_{j=0}^{\infty} (j+1)\mu_j \Phi_j \widehat{G}_{j+1}^{(\alpha-\beta-1, \beta-1)}(x) \right) \\ &= k_0 \sum_{i=0}^{\infty} (i+1)^2 \mu_i^2 \Phi_i^2 \simeq k_0 \sum_{i=0}^{\infty} (1+i^2) \Phi_i^2 \simeq k_0 \|\phi\|_{H_{\omega}^1(I)}^2 \\ &\simeq k_0 \|\phi\|_{H_{\omega}^1(I)} \|\psi\|_{H_{\omega^*}^{\alpha-1}(I)}. \end{aligned} \quad (3.17)$$

Combining (3.17), (3.13), and (3.14) we obtain the following lemma.

Lemma 3.2. For $0 < k_0 \leq k(x) \in L^{\infty}(I)$, and $b \in W_w^{1,\infty}(I)$, $c \in L^{\infty}(I)$ sufficiently small (with respect to k_0), there exists a constant $C_2 > 0$ such that

$$\sup_{\psi \in H_{\omega^*}^{\alpha-1}(I)} B(\phi, \psi) \geq C_2 \|\phi\|_{H_{\omega}^1(I)} \|\psi\|_{H_{\omega^*}^{\alpha-1}(I)} \text{ for all } \phi \in H_{\omega}^1(I). \quad (3.18)$$

Proof. Let $\phi \in H_{\omega}^1(I)$ be given by (3.15) and choose $\psi \in H_{\omega^*}^{\alpha-1}(I)$ to be given by (3.16). Then, from (3.17), (3.13), and (3.14) there exists $c_0, c_1 > 0$ such that

$$\begin{aligned} B(\phi, \psi) &\geq B_0(\phi, \psi) - |B_1(\phi, \psi) - B_2(\phi, \psi)| \\ &\geq c_0 k_0 \|\phi\|_{H_{\omega}^1(I)} \|\psi\|_{H_{\omega^*}^{\alpha-1}(I)} - c_1 \|b\|_{W_w^{1,\infty}(I)} \|\phi\|_{H_{\omega}^1(I)} \|\psi\|_{H_{\omega^*}^{\alpha-1}(I)} - \|c\|_{L^{\infty}(I)} \|\phi\|_{H_{\omega}^1(I)} \|\psi\|_{H_{\omega^*}^{\alpha-1}(I)} \\ &\geq C_2 \|\phi\|_{H_{\omega}^1(I)} \|\psi\|_{H_{\omega^*}^{\alpha-1}(I)}, \end{aligned}$$

for $b \in W_w^{1,\infty}(I)$ and $c \in L^{\infty}(I)$ sufficiently small. The stated result (3.18) then follows. ■

3.3. Property (3.6) – Positivity of $B(\cdot, \psi)$

The argument to establish (3.6) is almost verbatim of that used to show (3.5).

Lemma 3.3. For $0 < k_0 \leq k(x) \in L^{\infty}(I)$, and $b \in W_w^{1,\infty}(I)$, $c \in L^{\infty}(I)$ sufficiently small (with respect to k_0), there exists a constant $C_2 > 0$ such that

$$\sup_{\phi \in H_{\omega}^1(I)} B(\phi, \psi) \geq C_2 \|\phi\|_{H_{\omega}^1(I)} \|\psi\|_{H_{\omega^*}^{\alpha-1}(I)} \text{ for all } \psi \in H_{\omega^*}^{\alpha-1}(I). \quad (3.19)$$

Proof. Let $\psi \in H_{\omega^*}^{\alpha-1}(I)$ be given by (3.16) and choose $\phi \in H_{\omega}^1(I)$ to be given by (3.15).

Then, as in the proof of Lemma 3.2, we obtain $B(\phi, \psi) \geq C_2 \|\phi\|_{H_{\omega}^1(I)} \|\psi\|_{H_{\omega^*}^{\alpha-1}(I)}$, for $b \in W_w^{1,\infty}(I)$ and $c \in L^{\infty}(I)$ sufficiently small. ■

Combining the above we obtain the following existence and uniqueness result for the solution of (3.1).

Theorem 3.2. For $f \in H_{\omega^*}^{-(\alpha-1)}(I)$, $0 < k_0 \leq k(x) \in L^\infty(I)$, and $b \in W_w^{1,\infty}(I)$, $c \in L^\infty(I)$ sufficiently small (with respect to k_0), there exists a unique $u(x) = \omega \phi(x)$ satisfying (3.1) with $\phi \in H_\omega^1(I)$, and $\|\phi\|_{H_\omega^1(I)} \leq \frac{1}{c_2} \|f\|_{H_{\omega^*}^{-(\alpha-1)}(I)}$.

Proof. Note that F defined by (3.3) satisfies

$$\|F\| = \sup_{0 \neq \psi \in H_{\omega^*}^{\alpha-1}(I)} \frac{|F(\psi)|}{\|\psi\|_{H_{\omega^*}^{\alpha-1}(I)}} = \sup_{0 \neq \psi \in H_{\omega^*}^{\alpha-1}(I)} \frac{|\langle f, \psi \rangle_{\omega^*}|}{\|\psi\|_{H_{\omega^*}^{\alpha-1}(I)}} \leq \sup_{0 \neq \psi \in H_{\omega^*}^{\alpha-1}(I)} \frac{\|f\|_{H_{\omega^*}^{-(\alpha-1)}(I)} \|\psi\|_{H_{\omega^*}^{\alpha-1}(I)}}{\|\psi\|_{H_{\omega^*}^{\alpha-1}(I)}} = \|f\|_{H_{\omega^*}^{-(\alpha-1)}(I)}.$$

Hence, F defines a bounded linear functional. The existence, uniqueness and bound for ϕ then follows from combining Lemmas 3.1–3.3 with Theorem 3.1. ■

3.4. The case of $\dot{\mathcal{L}}_r^\alpha$.

For model (1.1) with $\tilde{\mathcal{L}}_r^\alpha = \dot{\mathcal{L}}_r^\alpha$ we have the following problem: Given $f \in H_{\omega^*}^{-1}(I)$, $0 < k_0 \leq k(x) \in L^\infty(I)$, $b \in W_w^{1,\infty}(I)$ and $c \in L^\infty(I)$, determine $\phi \in H_\omega^{\alpha-1}(I)$ such that $u(x) = \omega(x) \phi(x)$ satisfies

$$\langle \dot{\mathcal{L}}_r^\alpha u + bDu + cu, \psi \rangle_{\omega^*} = \langle f, \psi \rangle_{\omega^*}, \quad \forall \psi \in H_{\omega^*}^1(I). \quad (3.20)$$

The analysis for the existence and uniqueness of solution to (3.20) follows similarly to that presented for (3.1). In place of (2.8) the following result is used. From [32], with $\mathcal{M}u(x) := (rD^{-(2-\alpha)} + (1-r)D^{-(2-\alpha)*})Du(x)$

$$\mathcal{M}\omega(x)G_k^{(\alpha-\beta, \beta)}(x) = \mu_k G_{k+1}^{(\beta-1, \alpha-\beta-1)}(x), \quad (3.21)$$

$$\text{where } \mu_k = c_*^* \frac{\Gamma(k+\alpha)}{\Gamma(k+1)}, \quad k = 0, 1, 2, \dots, \quad (3.22)$$

In summary we have the following result.

Theorem 3.3. For $f \in H_{\omega^*}^{-1}(I)$, $0 < k_0 \leq k(x) \in L^\infty(I)$, and $b \in W_w^{1,\infty}(I)$, $c \in L^\infty(I)$ sufficiently small (with respect to k_0), there exists a unique $u(x) = \omega \phi(x)$ satisfying (3.20) with $\phi \in H_\omega^{\alpha-1}(I)$, and $\|\phi\|_{H_\omega^{\alpha-1}(I)} \leq \frac{1}{c_2} \|f\|_{H_{\omega^*}^{-1}(I)}$.

4. Regularity of the solutions

In this section we investigate the regularity of the solution to (1.1)–(1.2). The first step is to characterize the mapping properties of $(\tilde{\mathcal{L}}_r^\alpha)^{-1}$. With this in hand the second step employs a bootstrapping argument to determine the regularity of the solution. We again give the detailed analysis for $\dot{\mathcal{L}}_r^\alpha$ and then give the corresponding results for $\dot{\mathcal{L}}_r^\alpha$.

4.1. Mapping properties of $\dot{\mathcal{L}}_r^\alpha$.

The following two lemmas are helpful in determining the mapping properties of $\dot{\mathcal{L}}_r^\alpha$. We firstly introduce the subspace of $H_{(a,b)}^t(I)$, ${}_0H_{(a,b)}^t(I)$, defined by

$${}_0H_{(a,b)}^t(I) := \{h \in H_{(a,b)}^t(I) : \langle h, 1 \rangle_{\omega(a,b)} = 0\}.$$

In words, ${}_0H_{(a,b)}^t(I)$ represents those functions in $H_{(a,b)}^t(I)$ that when expanded in the basis $\{\widehat{G}_i^{(a,b)}(x)\}_{i=0}^\infty$ have coefficient of $\widehat{G}_0^{(a,b)}(x)$ equal to 0.

Lemma 4.1. For $1 < \alpha < 2$, $0 \leq r \leq 1$, let β be determined by Condition A and let $\mathcal{N} : \omega^{(\alpha-\beta-1, \beta-1)}(x) \otimes {}_0H_{(\alpha-\beta-1, \beta-1)}^s(I) \rightarrow H_{(\beta, \alpha-\beta)}^{s-(\alpha-1)}(I)$ be defined by

$$\mathcal{N} \omega^{(\alpha-\beta-1, \beta-1)}(x) \psi(x) := -D(rD^{-(2-\alpha)} + (1-r)D^{-(2-\alpha)*})\omega^{(\alpha-\beta-1, \beta-1)}(x) \psi(x). \quad (4.1)$$

Then the mapping $\mathcal{N}$ is bijective, continuous and has a continuous inverse.

Proof. We begin by showing that $\mathcal{N}$ is well defined.

From [32, Lemma 2.2], we have for $k = 0, 1, 2, \dots$

$$(rD^{-(2-\alpha)} + (1-r)D^{-(2-\alpha)*})\omega^{(\alpha-\beta-1, \beta-1)}(x)G_k^{(\alpha-\beta-1, \beta-1)}(x) = \sigma_k G_k^{(\beta-1, \alpha-\beta-1)}(x), \quad (4.2)$$

$$\text{where } \sigma_k := -c_*^* \frac{\Gamma(k+\alpha-1)}{\Gamma(k+1)}. \quad (4.3)$$

Thus, using (2.3),

$$-\mathcal{N}\omega^{(\alpha-\beta-1, \beta-1)}(x)G_k^{(\alpha-\beta-1, \beta-1)}(x) = -\mu_k G_{k-1}^{(\beta, \alpha-\beta)}(x), \quad k = 1, 2, \dots,$$

where μ_k is given by (2.9).

Now, for $\psi(x) = \sum_{i=1}^{\infty} \psi_i \widehat{G}_i^{(\alpha-\beta-1, \beta-1)}(x) \in {}_0H_{(\alpha-\beta-1, \beta-1)}^s(I)$,

$$\|\psi\|_{H_{(\alpha-\beta-1, \beta-1)}^s(I)}^2 = \sum_{i=1}^{\infty} (1+i^2)^s \psi_i^2 < \infty.$$

Note that $\mathcal{N}\omega^{(\alpha-\beta-1, \beta-1)}(x)\psi(x) = \sum_{i=1}^{\infty} \mu_i \psi_i \frac{\|G_{i-1}^{(\beta, \alpha-\beta)}\|}{\|G_i^{(\alpha-\beta-1, \beta-1)}\|} \widehat{G}_{i-1}^{(\beta, \alpha-\beta)}(x)$, and using (2.2) and (2.10),

$$\begin{aligned} \|\mathcal{N}\omega^{(\alpha-\beta-1, \beta-1)}(x)\psi(x)\|_{H_{(\alpha-\beta-1, \beta-1)}^{s-(\alpha-1)}(I)}^2 &\simeq \sum_{i=1}^{\infty} (1+(i-1)^2)^{s-(\alpha-1)} (\mu_i \psi_i)^2 \simeq \sum_{i=1}^{\infty} (1+i^2)^{s-(\alpha-1)} (1+i)^{2(\alpha-1)} \psi_i^2 \\ &\simeq \sum_{i=1}^{\infty} (1+i^2)^s \psi_i^2 = \|\psi\|_{H_{(\alpha-\beta-1, \beta-1)}^s(I)}^2 < \infty. \end{aligned}$$

Hence it follows that $\mathcal{N}$ is a continuous, bijective mapping from $\omega^{(\alpha-\beta-1, \beta-1)}(x) \otimes {}_0H_{(\alpha-\beta-1, \beta-1)}^s(I)$ onto $H_{(\beta, \alpha-\beta)}^{s-(\alpha-1)}(I)$ with a continuous inverse. ■

The second lemma involves the differentiation operator, D , and the identity given by (2.3).

Lemma 4.2. Let $a, b > 0$. Then,

$$D : \omega^{(a,b)}(x) \otimes H_{(a,b)}^t(I) \rightarrow \omega^{(a-1, b-1)}(x) \otimes {}_0H_{(a-1, b-1)}^{t-1}(I) \quad (4.4)$$

is bijective, continuous and has a continuous inverse.

Proof. We begin by showing that D , with domain and range space specified in (4.4) is well defined.

Let $\phi(x) = \sum_{i=0}^{\infty} \phi_i \widehat{G}_i^{(a,b)}(x) \in H_{(a,b)}^t(I)$. Then,

$$\|\phi\|_{H_{(a,b)}^t(I)}^2 = \sum_{i=0}^{\infty} (1+i^2)^t \phi_i^2 < \infty.$$

Using (2.5),

$$D\omega^{(a,b)}(x)\phi(x) = \omega^{(a-1, b-1)}(x) \sum_{i=0}^{\infty} -(i+1)\phi_i \frac{\|G_{i+1}^{(a-1, b-1)}\|}{\|G_i^{(a,b)}\|} \widehat{G}_{i+1}^{(a-1, b-1)}(x) := \omega^{(a-1, b-1)}(x)\psi(x).$$

Then, using (2.2),

$$\|\psi\|_{H_{(a-1, b-1)}^{t-1}(I)}^2 \simeq \sum_{i=0}^{\infty} (1+i^2)^{t-1} ((i+1)\phi_i)^2 \simeq \sum_{i=0}^{\infty} (1+i^2)^t \phi_i^2 \simeq \|\phi\|_{H_{(a,b)}^t(I)}^2.$$

Hence it follows that D is a continuous, bijective mapping from $\omega^{(a,b)}(x) \otimes H_{(a,b)}^t(I)$ onto $\omega^{(a-1, b-1)}(x) \otimes {}_0H_{(a-1, b-1)}^{t-1}(I)$ with a continuous inverse. ■

Recall, by the imbedding of the Sobolev spaces, $H_{(a,b)}^s(I) \subset H_{(a,b)}^{s'}(I)$ for $s' < s$.

For $f \in H_{(\beta, \alpha-\beta)}^s(I)$, $s \geq -1$, from Theorem 3.2 we have that for $0 < k_0 \leq k(x) \in L^\infty(I)$ there exists a unique solution $u(x) = \omega\phi(x)$ satisfying

$$\hat{\mathcal{L}}_r^\alpha \omega\phi(x) = \mathcal{N}k(x)D\omega\phi(x) = f(x).$$

As $f \in H_{(\beta, \alpha-\beta)}^{s'}(I)$, by the mapping properties of $\mathcal{N}$, there exists a unique $\psi \in {}_0H_{(\alpha-\beta-1, \beta-1)}^{s'+(\alpha-1)}(I)$ such that

$$k(x)D\omega\phi(x) = \omega^{(\alpha-\beta-1, \beta-1)}(x)\psi(x).$$

Assuming that $\phi(x) \in H_{(\alpha-\beta, \beta)}^t(I)$ for some $t \geq \alpha-1$, from the mapping properties of D given by Lemma 4.2, there exists $\tilde{\phi}(x) \in {}_0H_{(\alpha-\beta-1, \beta-1)}^{t-1}(I)$ such that

$$k(x)\tilde{\phi}(x) = \psi(x) \in H_{(\alpha-\beta-1, \beta-1)}^{s'+(\alpha-1)}(I) \text{ for } s' \leq s.$$

The regularity of $k(x)\tilde{\phi}(x)$ and $\psi(x) \in H_{(\alpha-\beta-1, \beta-1)}^{s'+(\alpha-1)}(I)$ must match. Specifically, if $k(x)$ is not sufficiently smooth $\psi(x)$ must be interpreted to line in some $H_{(\alpha-\beta-1, \beta-1)}^{s'+(\alpha-1)}(I)$ space for $s' < s$ in order to match the regularity of $k(x)\tilde{\phi}(x)$.

In view of Lemma 2.3 and the above discussion we have the following theorem.

Theorem 4.1. For $f \in H_{\omega^*}^s(I)$, $-(\alpha - 1) \leq s$, $0 < k_0 \leq k(x) \in W_w^{[s+\alpha-1],\infty}(I)$, there exists a unique $u(x) = \omega \phi(x)$ satisfying $\hat{\mathcal{L}}_r^\alpha \omega \phi(x) = f(x)$ with $\phi \in H_{(\alpha-\beta, \beta)}^{s+\alpha}(I)$.

As shown in [18,19], the presence of an advection or reaction term limits the regularity of the solution to (1.1), (1.2). The regularity of the solution of (1.1), (1.2) for $k(x) = 1$ was given by Theorems 5.2 and 5.3 in [18] (see also [25]). Combining that result with Theorems 3.2 and 4.1 we have the following corollary.

For $\epsilon > 0$ arbitrary, introduce $\tilde{s}$ defined by

$$\tilde{s} := \begin{cases} \min\{s, \alpha + (\alpha - \beta) + 1 - \epsilon, \alpha + \beta + 1 - \epsilon\}, & \text{if } b = 0, \\ \min\{s, \alpha + (\alpha - \beta) - 1 - \epsilon, \alpha + \beta - 1 - \epsilon\}, & \text{if } b \neq 0. \end{cases} \quad (4.5)$$

Corollary 4.1. For $f \in H_{\omega^*}^s(I)$, $-(\alpha - 1) \leq s$, β be determined by Condition A, $0 < k_0 \leq k(x) \in W_w^{[s+\alpha-1],\infty}(I)$, $b(x) \in W_w^{\max\{1, \lceil \tilde{s} \rceil\}, \infty}(I)$ and $c(x) \in W_w^{\lceil \tilde{s} \rceil, \infty}(I)$ sufficiently small (with respect to k_0), there exists a unique $u(x) = \omega \phi(x)$ satisfying (3.1) with $\phi \in H_{(\alpha-\beta, \beta)}^{\alpha+\tilde{s}}(I)$. In addition, there exists $C > 0$ such that

$$\|\phi\|_{H_{(\alpha-\beta, \beta)}^{\alpha+\tilde{s}}(I)} \leq C \|f\|_{H_{\omega^*}^{\tilde{s}}(I)}. \quad (4.6)$$

Remark. The norm estimate (4.6) follows from that at each of the (finite number of) steps in the boot strapping argument used to establish the regularity of ϕ the terms on the right hand side are bounded by a constant times $\|f\|_{H_{\omega^*}^{\tilde{s}}(I)}$.

4.2. Mapping properties of $\hat{\mathcal{L}}_r^\alpha$.

The mapping properties for $\hat{\mathcal{L}}_r^\alpha$ are obtained in a similar manner to $\hat{\mathcal{L}}_r^\alpha$. In place of Lemmas 4.1 and 4.2 we have the following two lemmas.

Lemma 4.3. Let $a, b > -1$. Then,

$$D : {}_0H_{(a,b)}^s(I) \rightarrow H_{(a+1,b+1)}^{s-1}(I) \quad (4.7)$$

is bijective, continuous and has a continuous inverse.

Proof. The proof uses property (2.3), and it similar to that for Lemma 4.2. ■

Lemma 4.4. For $1 < \alpha < 2$, $0 \leq r \leq 1$, let β be determined by Condition A and let $\mathcal{M}u(x) := (rD^{-(2-\alpha)} + (1-r)D^{-(2-\alpha)*})Du(x)$. Then, $\mathcal{M} : \omega^{(\alpha-\beta, \beta)}(x) \otimes H_{(\alpha-\beta, \beta)}^s(I) \rightarrow {}_0H_{(\beta-1, \alpha-\beta-1)}^{s-(\alpha-1)}(I)$ is bijective, continuous and has a continuous inverse.

Proof. The proof uses (3.21), (3.22) and is similar to that for Lemma 4.1. ■

Noting that $\hat{\mathcal{L}}_r^\alpha \omega \phi(x) = -Dk(x)\mathcal{M}\omega \phi(x)$, a similar argument as was used above for $\hat{\mathcal{L}}_r^\alpha$ results in the following theorem and corollary.

Theorem 4.2. For $f \in H_{\omega^*}^s(I)$, $-1 \leq s$, $0 < k_0 \leq k(x) \in W_w^{[s+1],\infty}(I)$, there exists a unique $u(x) = \omega \phi(x)$ satisfying $\hat{\mathcal{L}}_r^\alpha \omega \phi(x) = f(x)$ with $\phi \in H_{\omega^*}^{s+\alpha}(I)$.

Corollary 4.2. For $f \in H_{\omega^*}^s(I)$, $-1 \leq s$, β be determined by Condition A, $0 < k_0 \leq k(x) \in W_w^{[s+1],\infty}(I)$, $b(x) \in W_w^{\max\{1, \lceil \tilde{s} \rceil\}, \infty}(I)$ and $c(x) \in W_w^{\lceil \tilde{s} \rceil, \infty}(I)$ sufficiently small (with respect to k_0), there exists a unique $u(x) = \omega \phi(x)$ satisfying (3.1) with $\phi \in H_{(\alpha-\beta, \beta)}^{\alpha+\tilde{s}}(I)$. In addition, there exists $C > 0$ such that

$$\|\phi\|_{H_{(\alpha-\beta, \beta)}^{\alpha+\tilde{s}}(I)} \leq C \|f\|_{H_{\omega^*}^{\tilde{s}}(I)}. \quad (4.8)$$

5. Numerical approximation and analysis

In this section, we prove error estimates in both weighted L^2 and energy norms for a numerical approximation of (1.1)–(1.2). Building on the analysis of the bi-linear form, $B(\cdot, \cdot)$, in Section 3, we follow the ideas in [25, Section 4] to perform numerical analysis for the different trial and test spaces used here. We emphasize that the novel analysis on the inf-sup condition for $B(\cdot, \cdot)$ in Section 3, which resolves the loss of coercivity caused by the variable coefficient $k(x)$, is critical in the error estimates. This distinguishes the current work, by extending the analysis of Petrov–Galerkin spectral approximation method of constant coefficient problems to the variable coefficient case.

As we have done above, we give the analysis for the diffusion operator $\hat{\mathcal{L}}_r^\alpha$ and then the corresponding result for $\hat{\mathcal{L}}_r^\alpha$.

5.1. Approximation of (3.1)

Let $X_N := \text{span}\{\widehat{G}_j^{(\alpha-\beta, \beta)}\}_{j=0}^N \subset H_\omega^1(I)$, and $Y_N := \text{span}\{\widehat{G}_j^{(\beta, \alpha-\beta)}\}_{j=0}^N \subset H_{\omega^*}^{\alpha-1}(I)$. Then a Petrov–Galerkin spectral approximation to (3.1) is: Given $f \in H_{\omega^*}^{-(\alpha-1)}(I)$, determine $\phi_N \in X_N$ such that $u_N(x) = \omega(x)\phi_N(x)$ satisfies

$$B(\phi_N, \psi_N) = \langle f, \psi_N \rangle_{\omega^*}, \quad \forall \psi_N \in Y_N. \quad (5.1)$$

We first give the well posedness of (5.1).

Theorem 5.1. For $f \in H_{\omega^*}^{-(\alpha-1)}(I)$, $0 < k_0 \leq k(x) \in L^\infty(I)$, and $b \in W_w^{1,\infty}(I)$, $c \in L^\infty(I)$ sufficiently small (with respect to k_0), there exists a unique $u_N(x) = \omega\phi_N(x)$ satisfying (5.1) with $\phi_N \in X_N$, and $\|\phi_N\|_{H_\omega^1(I)} \leq \frac{1}{C_3} \|f\|_{H_{\omega^*}^{-(\alpha-1)}(I)}$, for some positive constant C_3 .

Proof. Similar to the proof of Lemma 3.2, given $\phi_N \in X_N$ we can construct a $\psi_N \in Y_N$ such that there exists $C_3 > 0$ such that

$$\sup_{0 \neq \psi_N \in Y_N} \frac{|B(\phi_N, \psi_N)|}{\|\psi_N\|_{H_{\omega^*}^{\alpha-1}(I)}} \geq C_3 \|\phi_N\|_{H_\omega^1(I)}, \quad \forall \phi_N \in X_N. \quad (5.2)$$

The proof then follows in an analogous manner to that of Theorem 3.2. ■

Next we establish the error bound of $\phi - \phi_N$ in both the weighted L^2 and energy norms in the following theorem.

Lemma 5.1. Let $f \in H_{\omega^*}^s(I)$ for $s \geq -(\alpha-1)$, $0 < k_0 \leq k(x) \in W_w^{[s+\alpha-1],\infty}(I)$, $b(x) \in W_w^{\max[1, \lceil \tilde{s} \rceil], \infty}(I)$ and $c(x) \in W_w^{[\tilde{s}], \infty}(I)$ sufficiently small (with respect to k_0). Then there exists $C > 0$ such that

$$\|\phi - \phi_N\|_{H_\omega^1(I)} \leq C N^{-(\tilde{s}+\alpha-1)} \|\phi\|_{H_{\omega^*}^{\tilde{s}+\alpha}(I)} \leq C N^{-(\tilde{s}+\alpha-1)} \|f\|_{H_{\omega^*}^{\tilde{s}}(I)}, \quad (5.3)$$

$$\|\phi - \phi_N\|_{L_\omega^2(I)} \leq C N^{-(\tilde{s}+\alpha)} \|\phi\|_{H_{\omega^*}^{\tilde{s}+\alpha}(I)} \leq C N^{-(\tilde{s}+\alpha)} \|f\|_{H_{\omega^*}^{\tilde{s}}(I)}. \quad (5.4)$$

Proof. We apply (5.2) to obtain for $\zeta_N \in X_N$

$$\begin{aligned} C_3 \|\phi - \zeta_N\|_{H_\omega^1(I)} &\leq \sup_{\substack{\psi_N \in Y_N \\ \psi_N \neq 0}} \frac{|B(\phi - \zeta_N, \psi_N)|}{\|\psi_N\|_{H_{\omega^*}^{\alpha-1}(I)}} = \sup_{\substack{\psi_N \in Y_N \\ \psi_N \neq 0}} \frac{|B(\phi - \zeta_N, \psi_N)|}{\|\psi_N\|_{H_{\omega^*}^{\alpha-1}(I)}} \\ &\leq \sup_{\substack{\psi_N \in Y_N \\ \psi_N \neq 0}} \frac{C_1 \|\phi - \zeta_N\|_{H_\omega^1(I)} \|\psi_N\|_{H_{\omega^*}^{\alpha-1}(I)}}{\|\psi_N\|_{H_{\omega^*}^{\alpha-1}(I)}} = C_1 \|\phi - \zeta_N\|_{H_\omega^1(I)}. \end{aligned} \quad (5.5)$$

Then, with $\zeta_N = P_N \phi \in X_N$, using Lemma 2.2,

$$\begin{aligned} \|\phi - \phi_N\|_{H_\omega^1(I)} &\leq \|\phi - \zeta_N\|_{H_\omega^1(I)} + \|\zeta_N - \phi_N\|_{H_\omega^1(I)} \leq C \|\phi - \zeta_N\|_{H_\omega^1(I)} \leq C N^{-(\tilde{s}+\alpha-1)} \|\phi\|_{H_{\omega^*}^{\tilde{s}+\alpha}(I)} \\ &\leq C N^{-(\tilde{s}+\alpha-1)} \|f\|_{H_{\omega^*}^{\tilde{s}}(I)} \quad (\text{using Corollary 4.1}). \end{aligned}$$

To obtain the estimate for $\|\phi - \phi_N\|_{L_\omega^2(I)}$ we use an Aubin–Nitsche type argument. Introduce the following adjoint problem. Determine $\psi(x) \in H_{\omega^*}^{\alpha-1}(I)$ satisfying

$$\mathcal{L}_{1-r}^\alpha(\omega^* \psi)(x) - b(x)D(\omega^* \psi)(x) + (c(x) - Db(x))(\omega^* \psi)(x) = (\phi - \phi_N)(x), \quad x \in I.$$

As $\phi - \phi_N \in L_\omega^2(I) = H_\omega^0(I)$, from (4.6) we have

$$\|\psi\|_{H_{\omega^*}^\alpha(I)} \leq C \|\phi - \phi_N\|_{L_\omega^2(I)}. \quad (5.6)$$

Then, with $\zeta_N = P_N \psi \in X_N$,

$$\begin{aligned} \|\phi - \phi_N\|_{L_\omega^2(I)}^2 &= ((\phi - \phi_N), (\phi - \phi_N))_\omega \\ &= (\phi - \phi_N, \mathcal{L}_{1-r}^\alpha \omega^* \psi - bD\omega^* \psi + (c - Db)\omega^* \psi)_\omega \\ &= B((\phi - \phi_N), \psi) = B((\phi - \phi_N), \psi - \zeta_N) \quad (\text{using Galerkin orthogonality}) \\ &\leq C_1 \|\phi - \phi_N\|_{H_\omega^1(I)} \|\psi - \zeta_N\|_{H_{\omega^*}^{\alpha-1}(I)} \\ &\leq C N^{-(\tilde{s}+\alpha-1)} \|\phi\|_{H_{\omega^*}^{\tilde{s}+\alpha}(I)} N^{-1} \|\psi\|_{H_{\omega^*}^\alpha(I)} \quad (\text{using (5.3) and Lemma 2.2}) \\ &\leq C N^{-(\tilde{s}+\alpha)} \|\phi\|_{H_{\omega^*}^{\tilde{s}+\alpha}(I)} \|\phi - \phi_N\|_{L_\omega^2(I)} \quad (\text{using (5.6)}), \end{aligned}$$

from which (5.4) follows. ■

Table 6.1
Accuracy test for case (a).

N	$\ u - u_N\ _{L^2_{\omega^{-1}}}$	κ	$\ u - u_N\ _{H^1_{\omega^{-1}}}$	κ
8	6.50E-03		6.81E-02	
10	4.19E-03	1.96	5.31E-02	1.12
12	2.91E-03	2.01	4.28E-02	1.18
14	2.12E-03	2.04	3.54E-02	1.24
16	1.62E-03	2.05	2.97E-02	1.30
Pred.		2.25		1.25

The weighted L^2 and energy estimates of $u - u_N$ immediately follow from [Lemma 5.1](#) and [Lemma 2.4](#).

Corollary 5.1. Let $f \in H^s_{\omega^*}(\mathcal{I})$ for $s \geq -(\alpha - 1)$, $0 < k_0 \leq k(x) \in W^{[s+\alpha-1], \infty}_w(\mathcal{I})$, $b(x) \in W^{\max\{1, [\tilde{s}]\}, \infty}_w(\mathcal{I})$ and $c(x) \in W^{[\tilde{s}], \infty}_w(\mathcal{I})$ sufficiently small (with respect to k_0). Then there exists $C > 0$ such that

$$\|u - u_N\|_{H^1_{\omega^{-1}}(\mathcal{I})} \leq C N^{-(\tilde{s} + \alpha - 1)} \|f\|_{H^{\tilde{s}}_{\omega^*}(\mathcal{I})}, \quad (5.7)$$

$$\|u - u_N\|_{L^2_{\omega^{-1}}(\mathcal{I})} \leq C N^{-(\tilde{s} + \alpha)} \|f\|_{H^{\tilde{s}}_{\omega^*}(\mathcal{I})}. \quad (5.8)$$

5.2. Approximation of (3.20)

The main difference between the problems involving the different fractional diffusion operators, $\hat{\mathcal{L}}^\alpha_r$ and $\hat{\mathcal{L}}^\alpha_r$ is the functional spaces used.

Let $X_N := \text{span}\{\hat{G}_j^{(\alpha-\beta, \beta)}\}_{j=0}^N \subset H^{\alpha-1}_{\omega}(\mathcal{I})$, and $Y_N := \text{span}\{\hat{G}_j^{(\beta, \alpha-\beta)}\}_{j=0}^N \subset H^1_{\omega^*}(\mathcal{I})$. Then a Petrov–Galerkin spectral approximation to (3.1) is: Given $f \in H^s_{\omega^*}(\mathcal{I})$, determine $\phi_N \in X_N$ such that $u_N(x) = \omega(x)\phi_N(x)$ satisfies

$$B(\phi_N, \psi_N) = \langle f, \psi_N \rangle_{\omega^*}, \quad \forall \psi_N \in Y_N. \quad (5.9)$$

Analogous to [Theorem 5.1](#), [Lemma 5.1](#) and [Corollary 5.1](#) we have the following.

Corollary 5.2. For $f \in H^s_{\omega^*}(\mathcal{I})$, $-1 \leq s$, $0 < k_0 \leq k(x) \in W^{[s+1], \infty}_w(\mathcal{I})$, $b(x) \in W^{\max\{1, [\tilde{s}]\}, \infty}_w(\mathcal{I})$ and $c(x) \in W^{[\tilde{s}], \infty}_w(\mathcal{I})$ sufficiently small (with respect to k_0), there exists a unique $u_N(x) = \omega\phi_N(x)$ satisfying (5.9) with $\phi_N \in X_N$, and $\|\phi_N\|_{H^{\alpha-1}_{\omega}(\mathcal{I})} \leq \frac{1}{C_3} \|f\|_{H^{-1}_{\omega^*}(\mathcal{I})}$, for some positive constant C_3 . In addition there exists $C > 0$ such that

$$\|\phi - \phi_N\|_{H^{\alpha-1}_{\omega}(\mathcal{I})} \leq C N^{-(\tilde{s} + 1)} \|\phi\|_{H^{\tilde{s} + \alpha}_{\omega}(\mathcal{I})} \leq C N^{-(\tilde{s} + 1)} \|f\|_{H^{\tilde{s}}_{\omega^*}(\mathcal{I})}, \quad (5.10)$$

$$\|\phi - \phi_N\|_{L^2_{\omega}(\mathcal{I})} \leq C N^{-(\tilde{s} + \alpha)} \|\phi\|_{H^{\tilde{s} + \alpha}_{\omega}(\mathcal{I})} \leq C N^{-(\tilde{s} + \alpha)} \|f\|_{H^{\tilde{s}}_{\omega^*}(\mathcal{I})}, \quad (5.11)$$

$$\|u - u_N\|_{H^{\alpha-1}_{\omega^{-1}}(\mathcal{I})} \leq C N^{-(\tilde{s} + 1)} \|f\|_{H^{\tilde{s}}_{\omega^*}(\mathcal{I})}, \quad (5.12)$$

$$\|u - u_N\|_{L^2_{\omega^{-1}}(\mathcal{I})} \leq C N^{-(\tilde{s} + \alpha)} \|f\|_{H^{\tilde{s}}_{\omega^*}(\mathcal{I})}. \quad (5.13)$$

6. Model discussion and numerical experiments

In this section we carry out numerical experiments to demonstrate the accuracy and performance of the scheme and to study the behavior of the two models.

6.1. Accuracy test

In this subsection we test the accuracy of the scheme (5.1) under different parameters

- (a) $\alpha = 1.30$, $r = 0.50$, $k(x) = 1 + 2x$;
- (b) $\alpha = 1.60$, $r = 0.40$, $k(x) = 1 - 0.3 \sin x$.

The other parameters are selected as those in [25, Experiment 2]: $b(x) = e^x$, $c(x) = 5 + \sin(x)$, and $f(x) = 1$. Then we can use the results in [Corollary 5.1](#) to predict the convergence rates for the errors under both the weighted L^2 and H^1 norms. The numerical convergence rates are presented in [Tables 6.1–6.2](#), which are in good agreement with the predicted rates.

6.2. Model comparison

In this subsection we carry out numerical experiments to study the behavior of the two models.

Table 6.2
Accuracy test for case (b).

N	$\ u - u_N\ _{L^2_{\omega^{-1}}}$	κ	$\ u - u_N\ _{H^1_{\omega^{-1}}}$	κ
8	3.37E-03		3.61E-02	
10	1.70E-03	3.06	2.21E-02	2.19
12	9.70E-04	3.08	1.49E-02	2.19
14	6.09E-04	3.03	1.07E-02	2.15
16	4.10E-04	2.96	8.04E-03	2.11
Pred.		2.95		1.95

We compare the solutions for model (1.1)–(1.2) involving both $\mathcal{L}_r^\alpha(\cdot)$ and $\dot{\mathcal{L}}_r^\alpha(\cdot)$. Let $\alpha = 1.4$, $r = 0.4$, $b(x)$, $c(x)$ and $f(x)$ are given as before, and we take the practical diffusivity coefficient $k(x)$ as follows

$$k_1(x) = \begin{cases} 2, & \text{for } 0 \leq x < 1/2; \\ 1, & \text{for } 1/2 \leq x \leq 1, \end{cases} \quad k_2(x) = \begin{cases} 1, & \text{for } 0 \leq x < 1/2; \\ 2, & \text{for } 1/2 \leq x \leq 1, \end{cases} \quad (6.1)$$

and present the numerical results in Fig. 6.1.

To better understand these results we note that the operator $\dot{\mathcal{L}}_r^\alpha u(x)$ in (1.4) can be expressed as a local mass balance with a two-sided fractional Fick's law

$$\dot{\mathcal{L}}_r^\alpha u(x) = D\dot{\mathcal{F}}^\alpha(x), \quad \text{with } \dot{\mathcal{F}}^\alpha(x) := -k(x)(rD^{-(2-\alpha)} + (1-r)D^{-(2-\alpha)*})Du(x). \quad (6.2)$$

Hence, the model (1.1) and (6.2) is a nonlocal extension of the classical local mass balance with a local Fick's law. Like its second-order analogue, the interface conditions for the strong solutions of problem (1.1), (1.2), and (1.4) now take the form

$$u((1/2)_-) = u((1/2)_+), \quad \dot{\mathcal{F}}^\alpha((1/2)_-) = \dot{\mathcal{F}}^\alpha((1/2)_+). \quad (6.3)$$

The combination of the continuity of the fractional diffusive flux in (6.3) and the jump discontinuity of the material diffusivity coefficients $k_1(x)$ and $k_2(x)$ in (6.1) across the interface, and the fact that $\dot{\mathcal{F}}^\alpha(x)$ depends only on $k(x)$ enforce that the two-sided fractional derivative of u has a jump discontinuity across the interface $x = 1/2$, which is proportional to the jump size of the diffusivity coefficient k_1 and k_2 across the interface, although u is continuous across the interface. The continuity of u and the jumps of the fractional derivative of u across the interface can be observed clearly in Fig. 6.1.

On the other hand, the operator $\mathcal{L}_r^\alpha u(x)$ in (1.3) can be expressed as a local mass balance with a nonlocal Fick's law, in which the flux at a point is a weighted average of local fluxes throughout the domain

$$\mathcal{L}_r^\alpha u(x) = D\mathcal{F}^\alpha(x), \quad \text{with } \mathcal{F}^\alpha(x) := (rD^{-(2-\alpha)} + (1-r)D^{-(2-\alpha)*})(-k(x)Du(x)). \quad (6.4)$$

In this case the interface conditions

$$u((1/2)_-) = u((1/2)_+), \quad \mathcal{F}^\alpha((1/2)_-) = \mathcal{F}^\alpha((1/2)_+) \quad (6.5)$$

assume the same form as (6.3) but with a different flux function $\mathcal{F}^\alpha$ defined in (6.4). In contrast to $\dot{\mathcal{F}}^\alpha$, at each $x \in (0, 1)$ $\mathcal{F}^\alpha(x)$ depends on $k(y)$ for all the $y \in (0, 1)$. In particular, the continuity of u and the flux $\mathcal{F}^\alpha$ across the interface specified in (6.5) imply

$$\begin{aligned} \mathcal{F}^\alpha((1/2)_-) &= -\left\{ \frac{r}{\Gamma(2-\alpha)} \int_0^{1/2} \frac{k(s)u'(s)}{(\frac{1}{2}-s)^{\alpha-1}} ds + \frac{1-r}{\Gamma(2-\alpha)} \int_{1/2}^1 \frac{k(s)u'(s)}{(s-\frac{1}{2})^{\alpha-1}} ds \right\} \\ &= \mathcal{F}^\alpha((1/2)_+) \end{aligned} \quad (6.6)$$

which holds naturally without enforcing a jump discontinuity of the derivative of u across the interface, as shown in Fig. 6.1.

7. Conclusions

In this paper we prove the well-posedness of the variable coefficient two-sided fractional diffusion, advection, reaction equations on a bounded interval, which remains untreated in the literature. Additionally we establish the regularity of the solution in terms of the regularity of the coefficient functions and the right hand side function. A Petrov–Galerkin scheme for the approximation of the solution is proposed and analyzed. A main contribution of this work lies in designing appropriate test and trial functions to prove the inf-sup condition of the variable coefficient fractional diffusion, advection, reaction operators in suitable function spaces, upon which the subsequent mathematical and numerical analysis rely. Numerical experiments are presented to substantiate the theoretical findings and to compare the behaviors of different models.

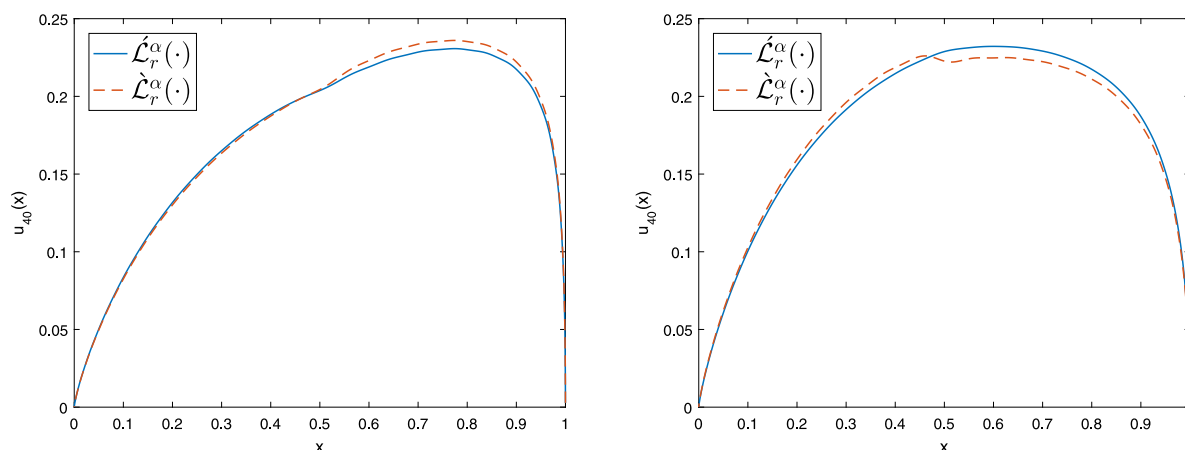


Fig. 6.1. Plots of the reference solution $u_{40}(x)$ under (left) $k_1(x)$ and (right) $k_2(x)$.

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All data generated or analyzed during this study are included in this published article.

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