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Analysis of a fractional viscoelastic Euler-Bernoulli beam and identification of its piecewise continuous polynomial order

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Abstract
We prove the well-posedness of a variable-order fractional viscoelastic Euler-Bernoulli beam and regularity estimates of its solution with low regularity assumption of its variable order. We further prove the unique identification of its variable order from the nonlinear manifold of piecewise continuous free-knot polynomials such that each function in the manifold may have different degrees on different pieces and may also vary with each individual function, with the time history of the responses measured on a space-time rectangular domain. Numerical experiments are carried out to identify the variable fractional order.

Keywords Fractional viscoelastic Euler-Bernoulli beam · Variable-order · Well-posedness and regularity · Identification of piecewise-continuous fractional order

Mathematics Subject Classification 35R11 · 35B65 · 35R30

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1 Introduction

Many modern materials are viscoelastic that exhibit anomalous power-law behavior in contrast to normal elastic exponential behavior [3, 5, 8, 13, 20, 33, 35, 36, 50]. Conventional elastic rheological relation based integer-order models, which are expressed as a combination of exponentially decaying functions [37, 40], do not yield accurate predictions of vibrations of viscoelastic materials especially over a wide range of parameters. A Scott-Blair element with a power-law relaxation modulus accurately describes the power-law behavior of viscoelastic material [11, 27, 34, 36, 59]. Furthermore, vibration of viscoelastic systems under long term external (cyclic) excitation could cause structural damage from micro scale and propagate to macro scale and eventually leads to material failure. The change of the material structure results in the change of its fractal dimension and so the fractional order, leading to variable-order fractional PDEs [14, 31, 43, 46, 47, 51, 55, 62].

In contrast to such material parameters as elastic modulus that can usually be measured a priori at least for undamaged structures, the variable order of a viscoelastic structure undergoing vibrations cannot be measured a priori and so needs to be estimated from the measurements, e.g, time history of responses that is measured at certain spatial locations. Extensive investigation has been conducted on the unique identification of fractional order (and possibly along with other parameters, e.g. diffusivity coefficient) in the context of time-fractional diffusion PDE [2, 7, 21–23, 25, 28, 29, 44, 48, 57]. The corresponding analysis on variable-order fractional PDEs is very challenging, e.g., due to the lack of closed-form solution representation in this context [31, 43, 46, 47, 54, 56]. It was proved that the fractional order of variable-order time-fractional PDEs can be uniquely identified from the admissible class of analytical functions [58, 60].

Unfortunately, in applications the fractional order is often not analytical but exhibits piecewise pattern [5, 31, 43, 46, 47], which motivates this work. In this paper we prove the well-posedness of a variable-order fractional viscoelastic Euler-Bernoulli beam and regularity estimates of its solution with low regularity assumption of its variable order. In particular, we prove that the variable order of the fractional viscoelastic Euler-Bernoulli beam can be uniquely identified from the nonlinear manifold of piecewise continuous free-knot polynomials, with the time history of the responses measured on a space-time rectangular domain. Finally, we use an adaptive Levenberg-Marquardt method to numerically invert the variable fractional order.

The rest of the paper is organized as follows: In §2 we recall the fractional viscoelastic Euler-Bernoulli beam and introduce notations and preliminary lemmas. In §3 we prove the well-posedness of the problem and the regularity of its solutions. In §4 we prove the unique identification of the variable fractional order from the nonlinear manifold of piecewise continuous free-knot polynomials such that each function in the manifold may have different degrees on different pieces and may also vary with each individual function, with the time history of the responses measured on a space-time rectangular domain. In §5 we carry out numerical experiments to invert the variable order by the Levenberg-Marquardt method. We address concluding remarks in the last section.

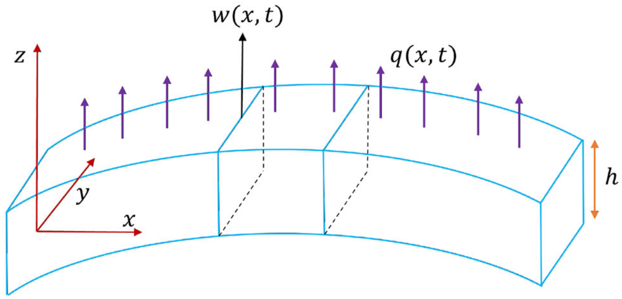

Fig. 1 A viscoelastic Euler-Bernoulli beam model with length l and thickness h that undergoes a transverse vibration under a distributed transverse load $q(x, t)$

2 Problem formulation and preliminaries

2.1 Model problem

We consider the small transverse vibrations of a straight isotropic viscoelastic beam under the following assumptions: (i) The beam has a straight centroidal axis (cf. Figure 1), with length l, cross sectional area A, and mass density ρ; (ii) the loading and support are symmetric about the x-z plane; (iii) cross sectional planes that are perpendicular to the centroidal axis of the undeformed beam remain planar after deformation and are perpendicular to the deflection curve of the deformed beam.

Given the distributed transverse load $q(x, t)$, let $w(x, t)$ be the transverse displacement of the centroidal beam axis and let $M(x, t)$ be the internal bending moment. The combination of Newton's second law and moment balance yields [6, 19, 26, 39]

$$\rho A \partial_t^2 w + \partial_x^2 M = q. \tag{2.1}$$

The Euler-Bernoulli assumptions lead to the following expressions for the infinitesimal strains [6, 19, 39, 40]

$$\begin{aligned} \varepsilon_{xx} &= \partial_x u_x(x, y, z, t) = -z\partial_x^2 w(x, t), \\ \varepsilon_{xz} &= (\partial_z u_x(x, y, z, t) + \partial_x u_z(x, y, z, t))/2 = 0, \end{aligned} \tag{2.2}$$

and all other strains vanish. Putting the fractional viscoelastic rheological relation [38, 46] with the infinitesimal strain (2.2)

$$\sigma_{xx}(x, z, t) = E_\alpha\, \partial_t^{\alpha(t)} \varepsilon_{xx}(x, z, t) = -E_\alpha z\, \partial_t^{\alpha(t)} \partial_x^2 w(x, t) \tag{2.3}$$

into the expression of the net bending moment $M(x, t)$ to obtain

$$M(x, t) = -\int_A z\sigma_{xx}(x, z, t) dA = E_\alpha I \partial_t^{\alpha(t)} \partial_x^2 w, \tag{2.4}$$

where $I = \int_A z^2 dA$ is the second-order moments about the y axis and $\partial_t^{\alpha(t)}$ is the variable-order fractional differential operator defined by [31, 42]

$$\partial_t^{\alpha(t)} g(t) := {}_0I_t^{1-\alpha(t)} \dot{g}(t), \quad {}_0I_t^{\alpha(t)} g(t) := \int_0^t \frac{g(s)}{\Gamma(\alpha(s))(t-s)^{1-\alpha(s)}} ds. \tag{2.5}$$

Incorporating the internal damping effect of the beam vibration $E_d I \partial_t \partial_x^2 w$ with $E_d > 0$ being the internal damping coefficient [4, 9, 18] yields a variable-order fractional model to describe the vibration of a viscoelastic Euler-Bernoulli beam under external excitation [19, 39]

$$\partial_t^2 w + K \partial_t \partial_x^4 w + K_\alpha \partial_t^{\alpha(t)} \partial_x^4 w = q/(\rho A), \quad (x,t) \in (0,l) \times (0,T]. \tag{2.6}$$

Here $K := E_d I/(\rho A)$ and $K_\alpha := E_\alpha I/(\rho A)$. We assume the beam to be simply supported, leading to the initial and boundary conditions

$$\begin{gathered} w(x,0) = w_0(x), \quad \partial_t w(x,0) = \check{w}_0(x), \quad x \in [0,l], \\ w(0,t) = w(l,t) = 0, \quad \partial_x^2 w(0,t) = \partial_x^2 w(l,t) = 0, \quad t \in [0,T]. \end{gathered} \tag{2.7}$$

Problem (2.6)-(2.7) can be reformulated in terms of $u = \partial_t w$ as follows

$$\begin{aligned} &\partial_t u + K \partial_x^4 u + K_\alpha \, {}_0I_t^{1-\alpha(t)} \partial_x^4 u = q/(\rho A), \quad (x,t) \in (0,l) \times (0,T], \\ &u(x,0) = \check{w}_0(x), \quad x \in [0,l], \\ &u(0,t) = u(l,t) = \partial_x^2 u(0,t) = \partial_x^2 u(l,t) = 0, \quad t \in [0,T]. \end{aligned} \tag{2.8}$$

2.2 Preliminaries

Let $C^m(\mathcal{I})$ and $C^\mu(\mathcal{I})$, with $m \in \mathbb{N}_0$, $0 \le \mu \le 1$ and $\mathcal{I} = [0,T]$ or $[0,l]$, be the spaces of continuous functions with continuous derivatives up to order m and Hölder continuous functions of index μ, respectively. Let $L^p(0,l)$, $1 \le p \le \infty$, be the Banach space of pth power Lebesgue integrable functions on $(0,l)$ and let $W^{m,p}(0,l)$ be the Sobolev space of L^p functions with mth weakly derivatives in $L^p(0,l)$. Let $H^m(0,l) = W^{m,2}(0,l)$. For a non-integer $s \ge 0$, the fractional Sobolev space $H^s(0,l)$ is defined by interpolation. All the spaces are equipped with standard norms [1, 12]. For a Banach space $\mathcal{X}$ equipped with the norm $\|\cdot\|_{\mathcal{X}}$, let $C^m([0,T];\mathcal{X})$ be the space of functions with continuous derivatives up to order m on $[0,T]$ belonging to $\mathcal{X}$ equipped with the norm

$$\begin{aligned} &C^m([0,T],\mathcal{X}) := \big\{ g : [0,T] \to \mathcal{X} : \|\partial_t^l g(\cdot,t)\|_{\mathcal{X}} \in C^l[0,T], \quad l = 0,1,\ldots,m \big\}, \\ &\qquad \|g\|_{C^m([0,T],\mathcal{X})} := \max_{0 \le l \le m} \max_{t \in [0,T]} \|\partial_t^l g(\cdot,t)\|_{\mathcal{X}}. \end{aligned}$$

Let $\{\lambda_i, \phi_i\}_{i=1}^\infty$ be the eigenvalues and eigenfunctions of $-\partial_x^2$ on $x \in (0,l)$. Here $\{\phi_i\}_{i=1}^\infty$ form an orthonormal basis in $L^2(0,l)$ and the corresponding eigenvalues

$\{\lambda_i\}_{i=1}^{\infty}$ form a positive nondecreasing sequence which tend to infinity [12]. For $s \geq 0$ we define the fractional Sobolev space [30, 41, 49]

$$\check{H}^s(0,l) := \left\{ v \in L^2(0,l) : |v|^2_{\check{H}^s} := \sum_{i=1}^{\infty} \lambda_i^s (v, \phi_i)^2 < \infty \right\}$$

equipped with the norm $\|v\|_{\check{H}^s} = (\|v\|^2_{L^2} + |v|^2_{\check{H}^s})^{1/2}$. $\check{H}^s(0,l)$ is the subspace of the fractional Sobolev space $H^s(0,l)$ with $\check{H}^0(0,l) = L^2(0,l)$ and $\check{H}^2(0,l) = H^2(0,l) \cap H_0^1(0,l)$.

Throughout this paper, we use Q, Q_i to denote positive constants and Q may assume different values at different occurrences. We finally refer the following lemmas for future use.

Lemma 1 *[53] Let $0 \leq D_0(t) \in L_{loc}[0,b)$ be nondecreasing and $D_1 \geq 0$ be a constant. If $0 \leq g(t) \in L_{loc}[0,b)$ satisfies $g(t) \leq D_0(t) + D_1 \, {}_0I_t^{\beta} g(t)$ for $t \in (0,b)$ and $0 < \beta < 1$, then $g(t) \leq D_0(t) E_{\beta,1}(D_1 \Gamma(\beta) t^{\beta})$ for $t \in (0,b)$, where $E_{p,q}(z)$ represents the Mittag-Leffler function [15].*

Lemma 2 *[24] The following equations hold*

$$\begin{aligned} {}_1F_1(1;\nu;t) &= \frac{\Gamma(\nu)}{\Gamma(\nu-1)} t^{-1}(1 + \mathcal{O}(t^{-1})), \ t \to -\infty, \ \nu \geq 1, \\ \int_0^t s^{\nu-1}(t-s)^{\mu-1} e^{-\beta s} ds &= B(\mu,\nu) t^{\mu+\nu-1} {}_1F_1(\nu; \mu+\nu; -\beta t) \end{aligned} \tag{2.9}$$

for β, μ, $\nu \in \mathbb{R}^+$, where B is the Beta function and ${}_1F_1$ represents the Kummer function.

3 Well-posedness and solution regularity

We prove well-posedness and solution regularity of models (2.6)–(2.7) and (2.8).

3.1 An auxiliary equation

We prove the well-posedness and solution regularity of the following variable-order fractional ordinary differential equation motivated by (3.23)

$$y'(t) + K\lambda^2 y(t) + K_\alpha \lambda^2 \, {}_0I_t^{1-\alpha(t)} y(t) = g(t), \quad t \in (0,T], \quad y(0) = \hat{y}_0. \tag{3.1}$$

Here λ, $g(t)$ and $\hat{y}_0$ are given data.

Lemma 3 *If $y \in C[0,T]$ and $0 \leq \alpha(t) < \alpha^* < 1$ for some upper bound α^*, then ${}_0I_t^{1-\alpha(t)} y \in C^{1-\alpha^*}[0,T]$.*

Proof Suppose first $\hbar \le t < t+\hbar \le T$, then the definition (2.5) gives that

$$\begin{aligned}
&{}_0I_t^{1-\alpha(t)}y(t+\hbar) - {}_0I_t^{1-\alpha(t)}y(t)\\
&= \int_0^{t+\hbar} \frac{y(s)ds}{\Gamma(1-\alpha(s))(t+\hbar-s)^{\alpha(s)}} - \int_0^t \frac{y(s)ds}{\Gamma(1-\alpha(s))(t-s)^{\alpha(s)}}\\
&= \int_t^{t+\hbar} \frac{y(s)ds}{\Gamma(1-\alpha(s))(t+\hbar-s)^{\alpha(s)}}\\
&\quad + \int_{t-\hbar}^{t} \frac{(t+\hbar-s)^{-\alpha(s)} - (t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} y(s)ds\\
&\quad + \int_0^{t-\hbar} \frac{(t+\hbar-s)^{-\alpha(s)} - (t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} y(s)ds := \sum_{i=1}^{3} I_i,
\end{aligned} \tag{3.2}$$

where I_1 could be bounded by the estimate (3.11)

$$\begin{aligned}
|I_1| &\le Q \int_t^{t+\hbar} |y(s)|(t+\hbar-s)^{-\alpha(s)}ds\\
&\le Q\|y\|_{C[0,T]} \int_t^{t+\hbar} (t+\hbar-s)^{-\alpha^*}ds \le Q\|y\|_{C[0,T]}\hbar^{1-\alpha^*}.
\end{aligned} \tag{3.3}$$

A similar argument yields

$$|I_2| \le Q \int_{t-\hbar}^{t} |y(s)|(t-s)^{-\alpha^*}ds \le Q\|y\|_{C[0,T]}\hbar^{1-\alpha^*}. \tag{3.4}$$

To bound I_3 in (3.2), we employ the fact that

$$\left|(t+\hbar-s)^{-\alpha(s)} - (t-s)^{-\alpha(s)}\right| \le |\alpha(s)|\hbar(t-s)^{-\alpha(s)-1} \le Q\hbar(t-s)^{-\alpha^*-1}$$

to obtain

$$|I_3| \le Q\hbar \int_0^{t-\hbar} |y(s)|(t-s)^{-\alpha^*-1}ds \le Q\|y\|_{C[0,T]}\hbar^{1-\alpha^*}. \tag{3.5}$$

Thus, the assertion follows from the estimates (3.3)–(3.5). When $t \le \hbar$, (3.2) becomes

$$\begin{aligned}
&{}_0I_t^{1-\alpha(t)}y(t+\hbar) - {}_0I_t^{1-\alpha(t)}y(t)\\
&= \int_t^{t+\hbar} \frac{y(s)ds}{\Gamma(1-\alpha(s))(t+\hbar-s)^{\alpha(s)}}\\
&\quad + \int_0^{t} \frac{(t+\hbar-s)^{-\alpha(s)} - (t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} y(s)ds := \sum_{i=1}^{2} \hat{I}_i,
\end{aligned} \tag{3.6}$$

where $|\hat{I}_1| \leq Q\|y\|_{C[0,T]}\hbar^{1-\alpha^*}$ from (3.3) and $\hat{I}_2$ could be estimated by

$$|\hat{I}_2| \leq Q\int_0^t |y(s)|(t-s)^{-\alpha^*}ds \leq Q\|y\|_{C[0,T]}t^{1-\alpha^*} \leq Q\|y\|_{C[0,T]}\hbar^{1-\alpha^*}, \quad (3.7)$$

which completes the proof of the lemma. □

Theorem 1 *If $g \in C[0,T]$ and $0 \leq \alpha(t) < \alpha^* < 1$ for some upper bound α^*, then the variable-order fractional ordinary differential equation (3.1) has a unique solution $y \in C^1[0,T]$ and*

$$\|y\|_{C[0,T]} \leq Q(\lambda^{-2}\|g\|_{C[0,T]} + |\hat{y}_0|), \quad \|y\|_{C^1[0,T]} \leq Q(\|g\|_{C[0,T]} + \lambda^2|\hat{y}_0|) \quad (3.8)$$

with $Q = Q(K, K_\alpha, \alpha^*, T)$.

Proof We integrate (3.1) multiplied by $\mathcal{R}(-t) := e^{K\lambda^2 t}$ to obtain a Volterra integral equation of the second kind in terms of $y(t)$

$$y(t) = -K_\alpha\lambda^2\mathcal{R}(t) * {}_0I_t^{1-\alpha(t)}y(t) + \mathcal{R}(t) * g(t) + \mathcal{R}(t)\hat{y}_0. \quad (3.9)$$

Here $*$ represents the symbol of convolution. Define an approximation sequence $\{y_n\}_{n=0}^\infty$ by $y_n(t) := -K_\alpha\lambda^2\mathcal{R}(t) * {}_0I_t^{1-\alpha(t)}y_{n-1}(t) + y_0(t)$ with $y_0(t) := \mathcal{R}(t) * g(t) + \mathcal{R}(t)\hat{y}_0$. We bound y_0 by $|y_0| \leq Q\|g\|_{C[0,T]}|\mathcal{R}(t) * 1| + |\hat{y}_0| \leq Q(\lambda^{-2}\|g\|_{C[0,T]} + |\hat{y}_0|) =: Q_0M$. Let $\triangle_n := y_n - y_{n-1}$ such that

$$\triangle_{n+1} = -K_\alpha\lambda^2\mathcal{R}(t) * {}_0I_t^{1-\alpha(t)}\triangle_n, \quad n \geq 1. \quad (3.10)$$

We interchange the order of integration on the right-hand side of (3.10) and utilize Lemma 2 and the estimate

$$(t-s)^{-\alpha(s)} = (t-s)^{-\alpha^*}(t-s)^{\alpha^*-\alpha(s)} \leq \max\{1, T\}(t-s)^{-\alpha^*} \quad (3.11)$$

to bound (3.10) for $n \geq 1$

$$
\begin{aligned}
\left|\triangle_{n+1}(t)\right| &= \left|\int_0^t \int_0^s \frac{K_\alpha \lambda^2 e^{-K\lambda^2(t-s)} \triangle_n(\theta)}{\Gamma(1-\alpha(\theta))(s-\theta)^{\alpha(\theta)}} d\theta ds\right| \\
&\leq \int_0^t \frac{K_\alpha \lambda^2 \left|\triangle_n(\theta)\right|}{\Gamma(1-\alpha(\theta))} \int_\theta^t \frac{e^{-K\lambda^2(t-s)}}{(s-\theta)^{\alpha(\theta)}} ds d\theta \\
&= \int_0^t \frac{K_\alpha \lambda^2 \left|\triangle_n(\theta)\right|}{\Gamma(1-\alpha(\theta))} \int_0^{t-\theta} \frac{e^{-K\lambda^2 y}}{(t-\theta-y)^{\alpha(\theta)}} dy d\theta \\
&= \int_0^t \frac{K_\alpha \lambda^2 \left|\triangle_n(\theta)\right|}{\Gamma(1-\alpha(\theta))} \big[B(1, 1-\alpha(\theta))(t-\theta)^{1-\alpha(\theta)} \\
&\quad \times {}_1F_1(1; 2-\alpha(\theta); -K\lambda^2(t-\theta))\big] d\theta \\
&\leq Q \int_0^t \frac{\left|\triangle_n(\theta)\right|(t-\theta)^{-\alpha(\theta)}}{\Gamma(1-\alpha(\theta))} d\theta \leq Q \int_0^t \frac{\left|\triangle_n(s)\right|(t-s)^{-\alpha^*}}{\Gamma(1-\alpha^*)} ds \\
&= Q_1\, {}_0I_t^{1-\alpha^*} \left|\triangle_n(t)\right|, \quad t \in [0, T].
\end{aligned}
\tag{3.12}
$$

We utilize the estimate $|y_0| \leq Q_0 M$ and the same estimate as (3.12) to obtain

$$
\left|\triangle_1(t)\right| = |K_\alpha \lambda^2 \mathcal{R}(t) * {}_0I_t^{1-\alpha(t)} y_0| \leq Q_1\, {}_0I_t^{1-\alpha^*} |y_0| \leq Q_0 Q_1 M_0 I_t^{1-\alpha^*} 1,
$$

and we assume that the generalized form of this equation holds for $1 \leq n \leq n^*$ for some $n^* \geq 1$

$$
\left|\triangle_n(t)\right| \leq Q_0 Q_1^n M_0 I_t^{n(1-\alpha^*)} 1, \quad t \in [0, T]. \tag{3.13}
$$

We combine (3.13) with (3.12) and the semigroup property of the fractional integral operator to arrive at

$$
\left|\triangle_{n^*+1}(t)\right| \leq Q_0 Q_1^{n+1} M_0 I_t^{1-\alpha^*} ({}_0I_t^{n(1-\alpha^*)} 1) = Q_0 Q_1^{n+1} M_0 I_t^{(n+1)(1-\alpha^*)} 1.
$$

By mathematical induction, (3.13) holds for $n \in \mathbb{N}$ by mathematical induction. The series defined by the right-hand side of (3.13) could be bounded as

$$
\sum_{n=1}^{\infty} (Q_1\, {}_0I_t^{1-\alpha^*})^n 1 = Q_1 (I - Q_1\, {}_0I_t^{1-\alpha^*})^{-1}\, {}_0I_t^{1-\alpha^*} 1 < \infty \tag{3.14}
$$

due to boundedness of $(I - Q_1\, {}_0I_t^{1-\alpha^*})^{-1}$ [10, 15–17]. As each $y_n \in C[0, T]$, the series on the left-hand side of (3.13) converge uniformly to its limiting function

$$
y(t) := \lim_{n\to\infty} y_n(t) = \lim_{n\to\infty} \sum_{m=1}^{n} \triangle_m + y_0(t) \in C[0, T]. \tag{3.15}
$$

We pass the limit on both sides of $y_n(t) := -K_\alpha \lambda^2 \mathcal{R}(t) * {}_0I_t^{1-\alpha(t)} y_{n-1}(t) + y_0(t)$ to conclude that y is a continuous solution of equation (3.9) with its estimate in (3.8) following from (3.13)–(3.15). Let $\bar{y} \in C[0, T]$ be another solution to (3.9), then we apply similar techniques in (3.12) to bound $e(t) := y(t) - \bar{y}(t)$ by

$$|e(t)| = |K_\alpha \lambda^2 \mathcal{R}(t) * {}_0I_t^{1-\alpha(t)} e(t)| \leq Q_1 \, {}_0I_t^{1-\alpha^*} |e(t)|. \tag{3.16}$$

We apply the Gronwall's inequality in Lemma 1 to conclude $e(t) \equiv 0$ such that the integral equation (3.9) (and thus the differential equation (3.1)) admits a unique solution $y \in C[0, T]$.

To bound y', we differentiate (3.9) with respect to t and apply $\mathcal{R}'(t) = -K\lambda^2 \mathcal{R}(t)$ to obtain

$$\begin{aligned} y'(t) = -K_\alpha \lambda^2 {}_0I_t^{1-\alpha(t)} y(t) + K_\alpha K \lambda^4 \mathcal{R}(t) * {}_0I_t^{1-\alpha(t)} y(t) \\ - K\lambda^2 \mathcal{R}(t) * g(t) + g(t) - K\lambda^2 \mathcal{R}(t) \hat{y}_0. \end{aligned} \tag{3.17}$$

We note from Lemma 3 that the first term on the right hand side of (3.17) belongs to $C^{1-\alpha^*}[0, T]$. In addition, it is clear that the remaining terms on the right hand side of (3.17) belong to $C[0, T]$ and thus $y' \in C[0, T]$. We apply (3.11)–(3.12) and the Young's convolution inequality to get

$$\begin{aligned} \left| K_\alpha \lambda^2 {}_0I_t^{1-\alpha(t)} y(t) \right| &\leq Q\lambda^2 {}_0I_t^{1-\alpha^*} |y(t)| \leq Q\lambda^2 \|y\|_{C[0,T]}, \\ |K\lambda^2 \mathcal{R}(t) * g(t)| &\leq Q\lambda^2 \|g\|_{C[0,T]} |\mathcal{R}(t) * 1| \leq Q \|g\|_{C[0,T]}, \\ \left| K_\alpha K \lambda^4 \mathcal{R}(t) * {}_0I_t^{1-\alpha(t)} y(t) \right| &\leq Q_1 K \lambda^2 \, {}_0I_t^{1-\alpha^*} |y(t)| \leq Q\lambda^2 \|y\|_{C[0,T]}. \end{aligned} \tag{3.18}$$

We incorporate (3.17) with the estimates in (3.18) and the first estimate in (3.8) to conclude that

$$\|y'\|_{C[0,T]} \leq Q(\lambda^2 \|y\|_{C[0,T]} + \|g\|_{C[0,T]} + \lambda^2 |\hat{y}_0|) \leq Q(\|g\|_{C[0,T]} + \lambda^2 |\hat{y}_0|), \tag{3.19}$$

which completes the proof.

□

3.2 Analysis of problems (2.6)–(2.7) and (2.8)

We prove the well-posedness and regularity estimates of models (2.6)–(2.7) and (2.8) based on the previous theorem.

Theorem 2 *If $0 \leq \alpha(t) < \alpha^* < 1$, $q \in H^\kappa(0, T; \check{H}^\gamma(0, l))$ and $\check{w}_0 \in \check{H}^{4+\gamma}(0, l)$ for $\kappa, \gamma > 1/2$, then the reduced problem (2.8) has a unique solution $u \in$*

$C([0,T];\check{H}^{4+\gamma}(0,l))\cap C^1([0,T];\check{H}^{\gamma}(0,l))$ and

$$\begin{aligned}&\|u\|_{C([0,T];\check{H}^{4+s}(0,l))}+\|u\|_{C^1([0,T];\check{H}^{s}(0,l))}\\&\leq Q\big(\|q\|_{H^\kappa(0,T;\check{H}^s(0,l))}+\|\check{w}_0\|_{\check{H}^{4+s}(0,l)}\big),\end{aligned}\tag{3.20}$$

for $0\leq s\leq\gamma$ and $Q=Q(\rho,A,K,K_\alpha,\alpha^*,T,\kappa)$. If further $w_0\in\check{H}^{4+\gamma}(0,l)$, the original problem (2.6)–(2.7) has a unique solution $w\in C^1([0,T];\check{H}^{4+\gamma}(0,l))\cap C^2([0,T];\check{H}^{\gamma}(0,l))$ for $0\leq s\leq\gamma$ and

$$\begin{aligned}&\|w\|_{C^1([0,T];\check{H}^{4+s}(0,l))}+\|w\|_{C^2([0,T];\check{H}^{s}(0,l))}\\&\leq Q\big(\|q\|_{H^\kappa(0,T;\check{H}^s(0,l))}+\|\check{w}_0\|_{\check{H}^{4+s}(0,l)}+\|w_0\|_{\check{H}^{4+s}(0,l)}\big).\end{aligned}\tag{3.21}$$

Proof We express u and q in (2.8) in terms of $\{\phi_i\}_{i=1}^\infty$ with the corresponding Fourier coefficients for $1\leq i\leq\infty$ [32, 41, 45, 52]

$$u_i(t):=(u(\cdot,t),\phi_i),\quad q_i(t):=(q(\cdot,t),\phi_i),\quad t\in[0,T],\tag{3.22}$$

which, according to the spectral expansion of (2.8), satisfy

$$\begin{aligned}u_i'(t)+K\lambda_i^2u_i(t)+K_\alpha\lambda_i^2{}_0I_t^{1-\alpha(t)}u_i(t)&=q_i/(\rho A),\\u_i(0)=\check{w}_{0,i}:=(\check{w}_0,\phi_i),\quad i&\geq 1.\end{aligned}\tag{3.23}$$

The (3.23) corresponds to the problem (3.1) with $y(t)=u_i(t)$, $y_0=\check{w}_{0,i}$, $\lambda=\lambda_i$ and $g=q_i/(\rho A)$ such that by Theorem 1, the problem (3.23) has a unique solution $u_i\in C^1[0,T]$ with the stability estimates as (3.8).

For any $k,n\in\mathbb{N}$ and $S_n(x,t):=\sum_{i=1}^n u_i(t)\phi_i(x)$, we use Sobolev embedding theorem and the estimates in (3.8) to conclude that for $n\to\infty$

$$\begin{aligned}\big\|S_{n+k}'-S_n'\big\|^2_{C([0,T];C[0,l])}&\leq Q\big\|S_{n+k}'-S_n'\big\|^2_{C([0,T];\check{H}^\gamma(0,l))}\\ s\qquad&=Q\Big\|\sum_{i=n+1}^{n+k}\lambda_i^\gamma\big(u_i'(t)\big)^2\Big\|_{C[0,T]}\leq Q\sum_{i=n+1}^{n+k}\lambda_i^\gamma\|u_i\|^2_{C^1[0,T]}\\&\leq Q\sum_{i=n+1}^{n+k}(\lambda_i^\gamma\|q_i\|^2_{H^\kappa(0,T)}+\lambda_i^{4+\gamma}\check{w}_{0,i}^2)\to 0,\end{aligned}\tag{3.24}$$

namely, S_n' converges in $C([0,T];C[0,l])$ such that the interchange of the differentiation with the summation (i.e. the Fourier expansions of u) is justified. Consequently, u defined by $u:=\sum_{i=1}^\infty u_i(t)\phi_i(x)$ is the solution to the problem (2.8) with the stability estimate

$$\|\partial_t u\|^2_{C([0,T];\check{H}^s(0,l))} \le \sum_{i=1}^{\infty} \lambda_i^s \|u_i\|^2_{C^1[0,T]} \le Q \sum_{i=1}^{\infty} \big(\lambda_i^s \|q_i\|^2_{C[0,T]} + \lambda_i^{4+s} |\check{w}_{0,i}|^2\big)$$
$$= Q\big(\|q\|^2_{H^\kappa(0,T;\check{H}^s(0,l))} + \|\check{w}_0\|^2_{\check{H}^{4+s}(0,l)}\big), \qquad 0 \le s \le \gamma.$$

The estimate of u could be performed similarly via the estimates in (3.8)

$$\|u\|^2_{C([0,T];\check{H}^{4+s}(0,l))} \le \sum_{i=1}^{\infty} \lambda_i^{4+s} \|u_i\|^2_{C[0,T]} \le Q \sum_{i=1}^{\infty} \big(\lambda_i^s \|q_i\|^2_{C[0,T]} + \lambda_i^{4+s} |\check{w}_{0,i}|^2\big)$$
$$\le Q\big(\|q\|^2_{H^\kappa(0,T;\check{H}^s(0,l))} + \|\check{w}_0\|^2_{\check{H}^{4+s}(0,l)}\big), \qquad 0 \le s \le \gamma.$$

We incorporate the above two estimates to obtain (3.20), and the uniqueness of the solutions follows from that for the ordinary differential equations (3.23). Finally, we conclude that (2.6)–(2.7) has a unique solution $w \in C^1([0,T];\check{H}^{4+\gamma}(0,l)) \cap C^2([0,T];\check{H}^{\gamma}(0,l))$ with the stability estimate (3.21) obtained directly from (3.20), which completes the proof.

□

4 Global unique determination of variable fractional order

We prove the global uniqueness of the inverse problem of determining the variable fractional order in the variable-order time-fractional viscoelastic Euler-Bernoulli beam model

$$\begin{aligned} \partial_t^2 w + K\partial_t\partial_x^4 w + K_\alpha \partial_t^{\alpha(t)}\partial_x^4 w = 0, \quad (x,t) \in (0,l)\times(0,T], \\ w(x,0) = w_0(x), \quad \partial_t w(x,0) = \check{w}_0(x), \quad x \in [0,l], \\ sw(0,t) = w(l,t) = 0, \qquad \partial_x^2 w(0,t) = \partial_x^2 w(l,t) = 0, \quad t \in [0,T], \end{aligned} \tag{4.1}$$

based on the observation data $w(x,t)$ measured on a space-time rectangular domain. Based on several experimental results [31, 43, 46, 47], which demonstrate that it usually suffices to consider a (piecewise) constant or linear variable fractional order in practical applications, we choose the following admissible set while studying the inverse problem

$$\mathcal{A} := \{\alpha(t) : \alpha(t) \text{ is a piecewise polynomial function on } [0,T]\}. \tag{4.2}$$

At possible discontinuous points $\{t_i^d\}_{i=1}^{N_d} \subset [0,T]$ of $\alpha(t)$, the values of $\alpha(t)$ are chosen as its left limits. The requirement of this admissible set is much weaker than those in the literature, which constrain the variable order in, e.g., the space of analytic functions that may not be practical in real problems [58, 60].

Lemma 4 *Suppose* $0 \leq \alpha(t) < \alpha^* < 1$, $w_0, \check{w}_0 \in \check{H}^{4+\gamma}(0,l)$ *for* $\gamma > 1/2$ *and* $\partial_t w(x,t_0) \not\equiv 0$ *on* $[0,l]$ *for each* $t_0 \in [0,T)$*. Then for each* $t_0 \in [0,T)$*, there exists an open spatial interval* $\Lambda_{t_0} \subset (0,l)$ *and a positive constant* σ_{t_0} *such that* $|\partial_t \partial_x^4 w(x,t_0)| \geq \sigma_{t_0}$ *for* $x \in \Lambda_{t_0}$.

Proof By Theorem 2, the given assumptions imply that $w \in C^1([0,T]; \check{H}^{4+\gamma}) \cap C^2([0,T]; \check{H}^{\gamma})$. We apply the Sobolev embedding $H^{r+1/2+\varepsilon}(0,l) \hookrightarrow C^r[0,l]$ for any $\varepsilon > 0$ and $0 \leq r \in \mathbb{N}$ [1] and $\check{H}^s(0,l) \subset H^s(0,l)$ for $s \geq 0$ to conclude that $w \in C^2\big([0,T]; C[0,l]\big) \cap C^1\big([0,T]; C^4[0,l]\big)$.

Then we intend to prove by contradiction that for each $t_0 \in [0,T)$, there exists an $x_{t_0} \in (0,l)$ such that $\partial_t \partial_x^4 w(x_{t_0}, t_0) \neq 0$. If not, we have $\partial_t \partial_x^4 w(x,t_0) = \partial_x^2[\partial_t \partial_x^2 w(x,t_0)] = 0$ on $(0,l)$ (and thus on $[0,l]$ by the continuity of $\partial_t \partial_x^4 w$) for some $t_0 \in [0,T)$, which, together with the smoothness of w, implies

$$\partial_t \partial_x^2 w(x,t_0) = a_1 x + b_1 \text{ for some } a_1, b_1 \in \mathbb{R}, \quad x \in [0,l]. \tag{4.3}$$

The boundary conditions in (2.8) lead to $\partial_t \partial_x^2 w(0,t_0) = \partial_x^2 u(0,t_0) = 0$ and $\partial_t \partial_x^2 w(l,t_0) = \partial_x^2 u(l,t_0) = 0$, which gives $\partial_t \partial_x^2 w(x,t_0) \equiv 0$ on $[0,l]$. We then incorporate $\partial_t w(0,t_0) = u(0,t_0) = 0$ and $\partial_t w(l,t_0) = u(l,t_0) = 0$ from (2.8) to further prove $\partial_t w(x,t_0) \equiv 0$ on $[0,l]$, which contradicts to the assumption of this lemma and thus proves the existence of $x_{t_0} \in (0,l)$ such that $\partial_t \partial_x^4 w(x_{t_0}, t_0) \neq 0$. Based on this result and the continuity of $\partial_t \partial_x^4 w$, the conclusion of this lemma could be reached by choosing Λ_{t_0} as a sufficiently small open neighborhood of x_{t_0}.

$\square$

Remark 1 Lemma 4 ensures the existence of an open set $\Lambda \subseteq (0,l)$, e.g. $\Lambda := \cup_{t_0 \in [0,T)} \Lambda_{t_0}$, such that for each $t_0 \in [0,T)$, there exists an open subset $\Lambda_{t_0} \subset \Lambda$ on which $\partial_t \partial_x^4 w(x,t_0)$ is strictly bounded away from 0.

We next prove the main result of this section in the following theorem.

Theorem 3 *Suppose the assumptions in Lemma 4 hold. Then the variable order* $\alpha(t)$ *in the time-fractional viscoelastic Euler-Bernoulli beam model (4.1) could be determined uniquely in the admissible set* $\mathcal{A}$ *on* $[0,T]$*, given the observation data on a space-time rectangular domain* $\Lambda \times [0,T]$ *where* Λ *is such a set as described in Remark 1.*

More precisely, let $\check{w}(t)$ *be the solution to model (4.1) with* $\alpha(t) \in \mathcal{A}$ *replaced by some variable order* $\check{\alpha}(t) \in \mathcal{A}$*. Then* $w(x,t) = \check{w}(x,t)$ *for* $(x,t) \in \Lambda \times [0,T]$ *implies* $\alpha(t) = \check{\alpha}(t)$ *for* $t \in [0,T]$.

Proof Let $\{t_i^{d,\alpha}\}_{i=1}^{N_d^\alpha}$, $\{t_i^{d,\check{\alpha}}\}_{i=1}^{N_d^{\check{\alpha}}} \subset (0,T)$ be the sets of discontinuous points of $\alpha(t)$ and $\check{\alpha}(t)$ on $[0,T]$, respectively. Under the assumptions, we have proved in Lemma 4 that $w, \check{w} \in C^2([0,T]; C[0,l]) \cap C^1([0,T]; C^4[0,l])$. Since $w(x,t) = \check{w}(x,t)$ for $(x,t) \in \Lambda \times [0,T]$, the difference of their equations leads to

$$(\partial_t^{\alpha(t)} \partial_x^4 - \partial_t^{\check{\alpha}(t)} \partial_x^4) w(x,t) = 0, \quad (x,t) \in \Lambda \times [0,T]. \tag{4.4}$$

By the properties of Λ, there exists an open subset $\Lambda_0 \subset \Lambda$ and a positive constant σ_0 such that $|\partial_t \partial_x^4 w(x,0)| \geq \sigma_0$ on Λ_0. Without loss of generality, we may assume that $\partial_t \partial_x^4 w(x,0) \geq \sigma_0$ on Λ_0. By the continuity of $\partial_t \partial_x^4 w$ in time, there exists a small time interval $[0,\tau]$ with $\tau < \min\{t_1^{d,\alpha}, t_1^{d,\check{\alpha}}\}$ such that

$$\partial_t \partial_x^4 w(x,t) \geq \frac{\sigma_0}{2} > 0, \quad \forall (x,t) \in \Lambda_0 \times [0,\tau]. \tag{4.5}$$

We intend to prove $\alpha(0) = \check{\alpha}(0)$ by contradiction. If not, we assume $\alpha(0) > \check{\alpha}(0)$ without loss of generality, and the continuity of $\alpha(t)$ and $\check{\alpha}(t)$ over $[0,\tau]$ implies that there exists a positive $0 < \varepsilon_0 \leq \tau$ such that $\alpha(s) - \check{\alpha}(s) \geq \nu$ on $[0,\varepsilon_0]$ for some $\nu > 0$. We then use (2.5) to reformulate (4.4) as follows

$$\begin{aligned}
&\big({}_0I_t^{1-\alpha(t)} - {}_0I_t^{1-\check{\alpha}(t)}\big)\partial_t \partial_x^4 w(x,t) \\
&= \int_0^t \left[\frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} - \frac{(t-s)^{-\check{\alpha}(s)}}{\Gamma(1-\check{\alpha}(s))}\right]\partial_s \partial_x^4 w(x,s)ds \\
&= \int_0^t (t-s)^{-\alpha(s)}\left[\frac{1}{\Gamma(1-\alpha(s))} - \frac{(t-s)^{\alpha(s)-\check{\alpha}(s)}}{\Gamma(1-\check{\alpha}(s))}\right]\partial_s \partial_x^4 w(x,s)ds = 0, \\
&\forall (x,t) \in \Lambda_0 \times (0,\varepsilon_0].
\end{aligned} \tag{4.6}$$

Since $\alpha(t)$ is bounded away from 1, there exists an $0 < \varepsilon_1 \leq \varepsilon_0$ and a positive constant Q_0 such that

$$\frac{1}{\Gamma(1-\alpha(s))} \geq 2Q_0, \quad \forall 0 < s < t, \quad t \in (0,\varepsilon_1]. \tag{4.7}$$

As $-\ln(t-s) \to \infty$ as $t \to 0^+$ and $\alpha(s) - \check{\alpha}(s) \geq \nu$ on $[0,\varepsilon_0]$, we conclude that $(t-s)^{\alpha(s)-\check{\alpha}(s)} = e^{(\alpha(s)-\check{\alpha}(s))\ln(t-s)} \to 0$ as $t \to 0^+$. Therefore, for the Q_0 given in (4.7), there exists an $0 < \varepsilon_2 \leq \varepsilon_1$ such that

$$\frac{(t-s)^{\alpha(s)-\check{\alpha}(s)}}{\Gamma(1-\check{\alpha}(s))} \leq Q_0, \quad \forall 0 < s < t, \quad t \in (0,\varepsilon_2]. \tag{4.8}$$

We incorporate the preceding estimates (4.7)–(4.8) to give a lower bound for the terms in the bracket of (4.6) by

$$\frac{1}{\Gamma(1-\alpha(s))} - \frac{(t-s)^{\alpha(s)-\check{\alpha}(s)}}{\Gamma(1-\check{\alpha}(s))} \geq Q_0, \quad \forall 0 < s < t, \quad t \in (0,\varepsilon_2], \tag{4.9}$$

which, together with (4.5), yields

$$
\begin{aligned}
&(t-s)^{-\alpha(s)}\left[\frac{1}{\Gamma(1-\alpha(s))}-\frac{(t-s)^{\alpha(s)-\check{\alpha}(s)}}{\Gamma(1-\check{\alpha}(s))}\right]\partial_s\partial_x^4 w(x,s) \\
&\quad\geq \frac{1}{2}\sigma_0 Q_0 (t-s)^{-\alpha(s)}, \quad \forall 0<s<t, \quad (x,t)\in\Lambda_0\times(0,\varepsilon_2].
\end{aligned} \tag{4.10}
$$

We invoke this estimate in (4.6) to find the contradiction, which implies $\alpha(0)=\check{\alpha}(0)$.

Define the set $t_* := \sup\{t : \alpha(s)=\check{\alpha}(s),\ \forall s\in[0,t]\}\in[0,T]$ such that $\alpha(s)=\check{\alpha}(s)$ on $[0,t_*]$. Such closed interval exists due to $\alpha(0)=\check{\alpha}(0)$ and the assumption that both variable orders takes their values as their left limits at possible discontinuous points. As $\alpha,\check{\alpha}\in\mathcal{A}$, $\alpha(s)-\check{\alpha}(s)$ is a piecewise polynomial function with possible discontinuous points $\{t_i^d\}_{i=1}^{N_d}\subseteq\{t_i^{d,\alpha}\}_{i=1}^{N_d^\alpha}\cup\{t_i^{d,\check{\alpha}}\}_{i=1}^{N_d^{\check{\alpha}}}$ on $[0,T]$. To prove $t_*=T$ by contradiction, we suppose $t_*<T$. We consider the case $t_*=t_{i^*}^d$ for some $1\leq i^*\leq N_d$, and the case that $t_*\notin\{t_i^d\}_{i=1}^{N_d}$ could be proved in a similar manner and is thus omitted.

Since $\alpha(s)-\check{\alpha}(s)$ is a piecewise polynomial function, there exists an $0<\varepsilon<\min\{t_{i^*+1}^d-t_{i^*}^d, T-t_{N_d}^d\}$ such that $\alpha(s)-\check{\alpha}(s)$ has finite zero points on $(t_*,t_*+\varepsilon]$ and there exists an $0<\varepsilon_{t_*,1}\leq\varepsilon$ such that $\alpha(s)-\check{\alpha}(s)\neq 0$ on $(t_*,t_*+\varepsilon_{t_*,1}]$. Without loss of generality, we may assume that $\alpha(s)-\check{\alpha}(s)>0$ on $(t_*,t_*+\varepsilon_{t_*,1}]$. By the properties of Λ, there exists an open subset $\Lambda_{t_*}\subset\Lambda$ and a positive constant σ_{t_*} such that $|\partial_t\partial_x^4 w(x,t_*)|\geq\sigma_{t_*}$ on Λ_{t_*}. Without loss of generality, we may assume that $\partial_t\partial_x^4 w(x,t_*)\geq\sigma_{t_*}$ on Λ_{t_*}. By the continuity of $\partial_t\partial_x^4 w$ in time, there exists a small time interval $[t_*,t_*+\tau_{t_*}]$ with $0<\tau_{t_*}<\varepsilon_{t_*,1}$ such that $\partial_t\partial_x^4 w(x,t)\geq\sigma_{t_*}/2$ on $\Lambda_{t_*}\times[t_*,t_*+\tau_{t_*}]$.

Since $\alpha(t)=\check{\alpha}(t)$ for $t\in[0,t_*]$, we apply the mean value theorem to reformulate equation (4.4) to obtain

$$
\begin{aligned}
0&=\left({}_0I_t^{1-\alpha(t)}-{}_0I_t^{1-\check{\alpha}(t)}\right)\partial_t\partial_x^4 w(x,t)\\
&=\int_0^t\left[\frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))}-\frac{(t-s)^{-\check{\alpha}(s)}}{\Gamma(1-\check{\alpha}(s))}\right]\partial_s\partial_x^4 w(x,s)ds\\
&=\int_{t_*}^t\left[\frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))}-\frac{(t-s)^{-\check{\alpha}(s)}}{\Gamma(1-\check{\alpha}(s))}\right]\partial_s\partial_x^4 w(x,s)ds\\
&=\int_{t_*}^t\frac{(t-s)^{-\bar{\alpha}(s)}}{\Gamma(1-\bar{\alpha}(s))}\big(\psi(1-\bar{\alpha}(s))-\ln(t-s)\big)(\alpha(s)-\check{\alpha}(s))\partial_s\partial_x^4 w(x,s)ds,\\
&\forall(x,t)\in\Lambda_{t_*}\times(t_*,t_*+\tau_{t_*}],
\end{aligned} \tag{4.11}
$$

where $\psi(x):=\frac{d}{dx}\ln\Gamma(x)$ is the polygamma function and $\bar{\alpha}(s)$ lies in between $\alpha(s)$ and $\check{\alpha}(s)$ for $t_*<s\leq t_*+\tau_{t_*}$. Since $0\leq\bar{\alpha}(s)\leq\alpha^*<1$ and $-\ln(t-s)\to\infty$ as $t\to t_*^+$, there exists some positive Q_1 and $0<\varepsilon_3<\tau_{t_*}$ such that

$$
\psi(1-\bar{\alpha}(s))-\ln(t-s)\geq Q_1, \quad \forall t_*<s<t, \quad t\in(t_*,t_*+\varepsilon_3], \tag{4.12}
$$

which, together with the preceding estimates yields that for $(x, t) \in \Lambda_{t_*} \times (t_*, t_* + \varepsilon_3]$

$$\frac{(t-s)^{-\bar{\alpha}(s)}}{\Gamma(1-\bar{\alpha}(s))}\big(\psi(1-\bar{\alpha}(s)) - \ln(t-s)\big)(\alpha(s) - \check{\alpha}(s))\partial_s \partial_x^4 w(x,s)$$
$$\geq \frac{\sigma_{t_*} Q_1 (t-s)^{-\bar{\alpha}(s)}(\alpha(s) - \check{\alpha}(s))}{2\Gamma(1-\bar{\alpha}(s))} > 0, \quad \forall t_* < s < t. \tag{4.13}$$

Thus we apply this with (4.11) to similarly claim $\alpha(t) = \check{\alpha}(t)$ on $(t_*, t_* + \varepsilon_3]$, and thus on $[0, t_* + \varepsilon_3]$. This contradicts the definition of t_*, which implies $t_* = T$ and thus completes the proof.

□

Remark 2 To explain the motivation of imposing the assumption on $\partial_t w$ in Lemma 4 and Theorem 3, we note that the condition "$\partial_t w(x, t) \not\equiv 0$ on $[0, l] \times \Phi$ for each non-empty open sub-interval Φ of $[0, T)$" is necessary for unique determination of the variable fractional order $\alpha(t)$ in (4.1). This is due to the fact in the fractional operators (2.5), both the function $g(s)$ and the variable fractional order $\alpha(s)$ have the same variable s such that if $g(t) \equiv 0$ on Φ for some Φ mentioned above, then we could slightly change α inside Φ without affecting anything since the variable order $\alpha(s)$ for $s \in \Phi$ has impacts only for the sub-integral over Φ that is always 0. Therefore, it is impossible to determine the variable fractional order uniquely over Φ in this case. We apply this argument to the fractional operator in (4.1) and use $\partial_t w(x, t) \equiv 0$ over $[0, l]$ implies $\partial_t \partial_x^4 w(x, t) \equiv 0$ to reach the aforementioned necessary condition.

5 Numerical inversion of variable fractional order

We incorporate the Levenberg-Marquardt method with the fully-discrete finite element method to numerically infer the variable order in model (4.1), which are implemented in several experiments for illustration.

5.1 Fully-discrete finite element method for (4.1)

We reformulate the problem (4.1) as a first-order system (2.8) as follows

$$\partial_t u + K\partial_x^4 u + K_\alpha\, {}_0I_t^{1-\alpha(t)} \partial_x^4 u = 0, \qquad u = \partial_t w. \tag{5.1}$$

Let $t_n := n\tau$ for $n = 0, 1, \ldots, N$ with $\tau := T/N$ be a uniform partition on $[0, T]$, $u_n := u(x, t_n)$, $w_n := w(x, t_n)$ and $\alpha_n := \alpha(t_n)$. Then we discretize $\partial_t u$, $\partial_t w$ and

${}_0I_t^{1-\alpha(t)}\partial_x^4 u$ at $t = t_n$ for $1 \le n \le N$ by

$$
\begin{aligned}
&\partial_t u(x,t_n) \approx \frac{u_n - u_{n-1}}{\tau} := \delta_\tau u_n, \quad \partial_t w(x,t_n) \approx \frac{w_n - w_{n-1}}{\tau} := \delta_\tau w_n,\\
&{}_0I_t^{1-\alpha(t)}\partial_x^4 u(x,t)\big|_{t=t_n} = \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} \frac{\partial_x^4 u(x,s)ds}{\Gamma(1-\alpha(s))(t_n-s)^{\alpha(s)}}\\
&\approx \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} \frac{\partial_x^4 u_k ds}{\Gamma(1-\alpha_k)(t_n-s)^{\alpha_k}} = \sum_{k=1}^{n} b_{n,k}\partial_x^4 u_k := I_\tau^{1-\alpha_n}\partial_x^4 u_n,
\end{aligned}
\tag{5.2}
$$

with

$$
b_{n,k} := \frac{1}{\Gamma(1-\alpha_k)}\int_{t_{k-1}}^{t_k}\frac{ds}{(t_n-s)^{\alpha_k}} = \frac{(t_n-t_{k-1})^{1-\alpha_k} - (t_n-t_k)^{1-\alpha_k}}{\Gamma(2-\alpha_k)} \tag{5.3}
$$

for $1 \le k \le n$.

Let $S_h \subset \check{H}^2(0,l)$ be the continuously differentiable piecewise cubic Hermite finite element space on a quasi-uniform partition on $[0,l]$ with the partition diameter h. Define the Ritz projection $\Pi_h : \check{H}^2 \to S_h$ [49] by

$$
(\partial_x^2\Pi_h g, \partial_x^2\chi) = (\partial_x^2 g, \partial_x^2\chi), \quad \forall \chi \in S_h. \tag{5.4}
$$

We incorporate the preceding discretizations into (5.1) multiplied by $\chi \in S_h$ and then integrate the resulting equation on $(0,l)$ to derive the finite element scheme for (5.1): Find $U_n, W_n \in S_h$ for $n = 1, \ldots, N$ such that for any $\chi \in S_h$

$$
\begin{aligned}
&\big(\delta_\tau U_n, \chi\big) + K\big(\partial_x^2 U_n, \partial_x^2\chi\big) + K_\alpha(I_\tau^{1-\alpha_n}\partial_x^2 U_n, \partial_x^2\chi) = 0,\\
&U_n = \delta_\tau W_n, \quad U_0 := \Pi_h\check{w}_0, \quad W_0 := w_0.
\end{aligned}
\tag{5.5}
$$

5.2 A free-knot partitioned Levenberg-Marquardt method

Given some observation data $\{w_{hist}(x_i,t_n)\}_{i,n=1}^{J,N}$ on the time interval $[0,T]$ measured at certain spatial locations $0 < x_1 < x_2 < \cdots < x_J < l$ where J denotes the number of sensors, we intend to develop a algorithm to numerically invert $\alpha(t)$. Concerning the possible discontinuities of variable order (cf. the definition of the admissible set $\mathcal{A}$), we split the temporal interval $[0,T]$ into I subintervals with possible discontinuity nodes $\boldsymbol{P} = \{P_1, \cdots, P_{I-1}\}$ such that $0 =: P_0 < P_1 < P_2 < \cdots < P_I := T$. The approximation $\alpha_I(t)$ to $\alpha(t)$ is selected as a piecewise linear function as follows

$$
\alpha_I(t) := \sum_{m=1}^{I} L_{\boldsymbol{\alpha},m}(t)\mathbb{1}_{(P_{m-1},P_m]}, \quad \boldsymbol{\alpha} := [\alpha_1, \beta_1, \cdots, \alpha_I, \beta_I] \in \mathbb{R}^{2I}, \tag{5.6}
$$

where $L_{\boldsymbol{\alpha},m}(t)$ refers to the linear function on the subinterval $[P_{m-1}, P_m]$ with $L_{\boldsymbol{\alpha},m}(P_{m-1}) = \alpha_m$ and $L_{\boldsymbol{\alpha},m}(P_m) = \beta_m$ for $1 \le m \le I$. Let $\{w_{pred}(x_i,t_n,\boldsymbol{p})\}_{i,n=1}^{J,N}$

be the numerical solution of the fully-discrete finite element scheme (5.5) with the variable order $\alpha(t)$ replaced by $\alpha_I(t)$, we aim at finding the optimal parameters $\boldsymbol{p}_*$ over $\boldsymbol{p} = [\boldsymbol{P}, \boldsymbol{\alpha}] \in \mathbb{R}^{3I-1}$ to minimize the cost functional defined as

$$\mathcal{F}(\boldsymbol{p}) := \frac{\tau}{2} \sum_{i=1}^{J} \sum_{n=1}^{N} \left(w_{pred}(x_i, t_n; \boldsymbol{p}) - w_{hist}(x_i, t_n) \right)^2$$

by the Levenberg-Marquardt algorithm in the iterative procedure

$$\boldsymbol{p}_{j+1} := \boldsymbol{p}_j - \left(\boldsymbol{J}_j^\top \boldsymbol{J}_j + \beta_j \boldsymbol{I}_{3I-1} \right)^{-1} \boldsymbol{J}_j^\top \boldsymbol{r}_j. \tag{5.7}$$

Here β_j is the regularization parameter, the residual vector $\boldsymbol{r}_j \in \mathbb{R}^{JN}$ is evaluated by

$$\boldsymbol{r}_j := \Big[w_{pred}(x_i, t_n; \boldsymbol{p}) - w_{hist}(x_i, t_n) \Big]_{i,n=1}^{J,N}, \tag{5.8}$$

and the Jacobian matrix $\boldsymbol{J}_j$ of order $JN \times (3I-1)$ is evaluated by

$$\boldsymbol{J}_j := \left[\frac{w_{pred}(x_i, t_n; \boldsymbol{p} + \delta \boldsymbol{e}_k) - w_{pred}(x_i, t_n; \boldsymbol{p})}{\delta} \right]_{i,n,k}^{J,N,3I-1}, \tag{5.9}$$

where $\delta > 0$ the numerical differentiation step size and $\boldsymbol{e}_k \in \mathbb{R}^{3I-1}$ is the unit vector in the k-th coordinate direction for $k = 1, 2, \cdots, 3I-1$. We summarize the above parameter identification method in Algorithm 1.

Algorithm 1: A Levenberg-Marquardt Algorithm

1. Given the observation data $\{w_{hist}(x_i, t_n)\}_{i,n=1}^{J,N}$ for model (4.1), the parameters $\gamma, \nu \in (0, 1)$, $\beta_0 > 0$, $0 < \delta \ll 1$, $\boldsymbol{p}_0$, TOL> 0, and $j := 0$.
2. Solve the scheme (5.5) with $\alpha(t)$ replaced by $\alpha_I(t)$ given in (5.6).
3. Use formula (5.8)-(5.9) to numerically evaluate Jacobian $\boldsymbol{J}_j$ and $\boldsymbol{J}_j^\top \boldsymbol{r}_j$.
4. If $\|\boldsymbol{J}_j^\top \boldsymbol{r}_j\| <$TOL, then stop and let $\boldsymbol{p}_* := \boldsymbol{p}_j$.
5. Compute the search direction $\boldsymbol{d}_j := -(\boldsymbol{J}_j^\top \boldsymbol{J}_j + \beta_j \boldsymbol{I}_{3I-1})^{-1} \boldsymbol{J}_j^\top \boldsymbol{r}_j$.
6. Determine the search step γ^m by the Armijo rule: find the smallest nonnegative integer m such that $\mathcal{F}\left(\boldsymbol{p}_j + \gamma^m \boldsymbol{d}_j\right) \leq \mathcal{F}\left(\boldsymbol{p}_j\right) + \nu\gamma^m \boldsymbol{d}_j \mathbf{J}_j^\top \mathbf{r}_j$.
7. Update $\boldsymbol{p}_{j+1} := \boldsymbol{p}_j + \gamma^m \boldsymbol{d}_j$, $\beta_{j+1} := \beta_j/2$. Let $j := j+1$ and go to Step 2.

5.3 Numerical investigation

We carry out numerical experiments to investigate the performance of the proposed Levenberg-Marquardt method to numerically evaluate the variable fractional order in the model (4.1) for a Euler-Bernoulli beam of length $l = 1$m, width 0.1m and height 0.01m, given observation data on $[0, T] = [0, 0.5]$s measured at certain spatial locations $\{x_i = (5-i)/8\}_{i=1}^{J}$. The Euler-Bernoulli beam is made of a widely used superalloy, i.e., Inconel alloy 718 material [63] with $\rho = 8192$ kg/m^3, $E_\alpha = 200$ GPa

Table 1 Errors $\alpha - \alpha_I$ in Example 1

J	I	Itr	CPU time	$\|\alpha - \alpha_I\|_{L_\infty}$	$\|\alpha - \alpha_I\|_{L_2}$	$\|\alpha - \alpha_I\|_{L_1}$
1	2	43	1m 18s	9.01E-02	2.86E-02	1.61E-02
	4	43	4m 2s	4.55E-02	1.49E-02	8.55E-03
	6	43	8m 18s	3.60E-02	1.20E-02	6.89E-03
	8	43	14m 9s	2.58E-02	8.53E-03	4.87E-03
2	2	100	10m 42s	8.74E-02	2.60E-02	1.46E-02
	4	42	3m 58s	4.55E-02	1.49E-02	8.55E-03
	6	100	40m 21s	2.33E-02	8.50E-03	4.88E-03
	8	100	0m 18s	1.91E-02	6.70E-03	3.84E-03
3	2	100	11m 14s	8.56E-02	2.39E-02	1.32E-02
	4	44	4m 9s	4.55E-02	1.49E-02	8.55E-03
	6	44	8m 29s	3.60E-02	1.20E-02	6.89E-03
	8	100	59m 21s	1.91E-02	6.52E-03	3.81E-03
4	2	100	15m 40s	8.67E-02	2.76E-02	1.56E-02
	4	42	3m 58s	4.55E-02	1.49E-02	8.55E-03
	6	42	8m 6s	3.60E-02	1.20E-02	6.89E-03
	8	100	58m 40s	1.77E-02	6.16E-03	3.54E-03

and $E_d = 1 \times 10^{-5}E$. The model (4.1) is simulated via the finite element scheme (5.5) with $N = 256$ and the uniform spatial partition $h = 1/8$.

Example 1: Inverting a smooth variable fractional order.

Let $w_0(x) = 0$, $\check{w}_0(x) = x^3(1-x)^3$ and $\alpha(t) = \frac{14}{15}t^3 + \frac{2}{5}(t - \frac{1}{5})^2 + 0.4$ in problem (4.1). In the Levenberg-Marquardt algorithm, we set $\gamma = 0.75$, $\nu = 0.25$, $\delta = 10^{-6}$, $\text{TOL} = 10^{-17}$, $\boldsymbol{p}_0 = [T/I, 2T/I, \ldots, (I-1)T/I, 0.5, 0.5, \ldots, 0.5] \in \mathbb{R}^{3I-1}$ and maximum iteration number $\text{Itr}_{\text{max}} = 100$. We present the errors $\alpha - \alpha_I$ under L_1, L_2 and L_∞ norms in Table 1, and plot $\alpha(t)$ and $\alpha_I(t)$ as well as the corresponding cost functionals in Fig. 2. From these results we find that the free-knot partitioned Levenberg-Marquardt method generates an accurate and convergent numerical inversion $\alpha_I(t)$ to $\alpha(t)$.

Example 2 : Inverting a continuous and piecewise smooth variable fractional order.

We consider a continuous and piecewise smooth variable order

$$\alpha(t) = \begin{cases} \frac{3}{10} - \frac{2}{5}t, & t \in [0, T/2], \\ \frac{59}{320} + (t - \frac{3T}{4})^2, & t \in (T/2, T] \end{cases}$$

in problem (4.1). Let $\check{w}_0(x) = 0.1\sin(\pi x)$ and the other data are the same as those in Example 1. We present the numerical inversion results in Fig. 3, which again demonstrate the effectiveness of the proposed method.

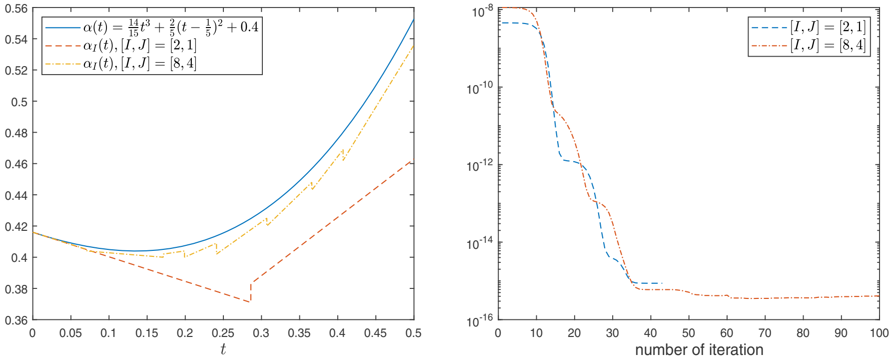

Fig. 2 Plots of $\alpha(t)$ and $\alpha_I(t)$ (left) and the cost functionals (right) with different values of $[I, J]$ in Example 1

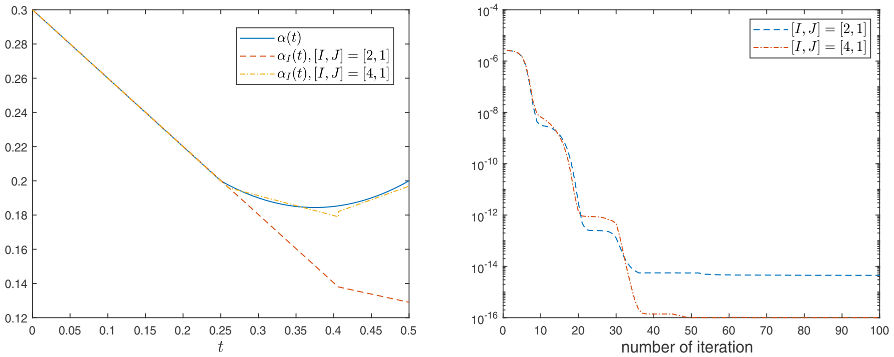

Fig. 3 Plots of $\alpha(t)$ and $\alpha_I(t)$ (left) and the cost functionals (right) with different values of $[I, J]$ in Example 2

Example 3: Inverting a discontinuous variable fractional order. We consider a piecewise smooth variable order

$$\alpha(t)=\begin{cases}\frac{13}{25}+2t^3+\frac{1}{5}t^2/T, & t\in[0,T/4],\\ \frac{1}{2}t+\frac{1}{2}(1-t^5/T), & t\in(T/4,T],\end{cases}$$

which is discontinuous at $t=T/4$. Let $\check{w}_0(x)=x^3(1-x)^3$ and the other data are the same as those in Example 1. We present the numerical inversion results in Fig. 4, which demonstrate that the proposed algorithm also works well for the model with discontinuous variable fractional orders.

Example 4 : Inverting a discontinuous variable fractional order with fixed P.

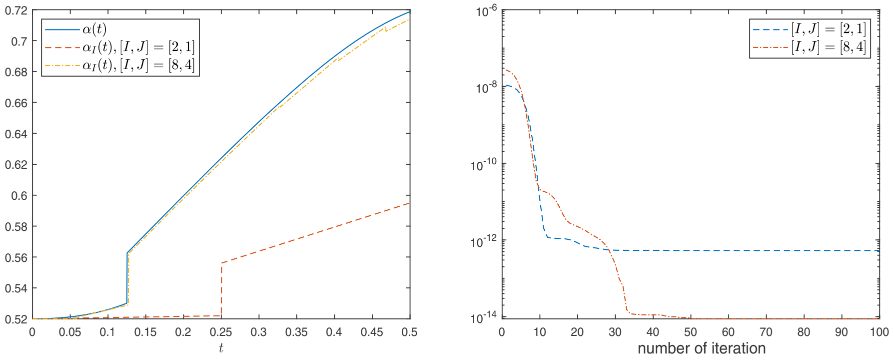

Fig. 4 Plots of $\alpha(t)$ and $\alpha_I(t)$ (left) and the cost functionals (right) with different values of $[I, J]$ in Example 3

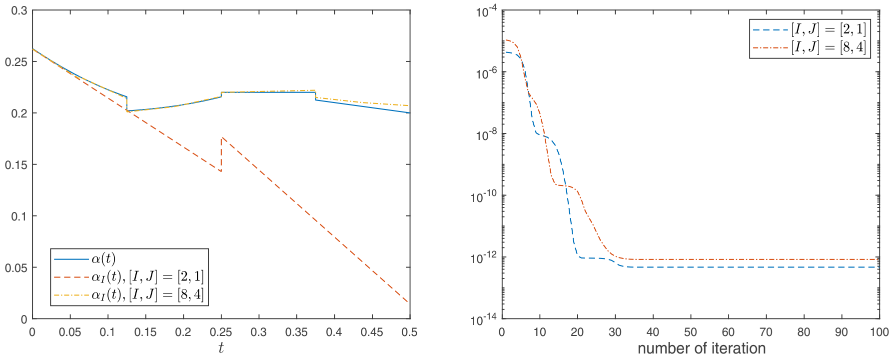

Fig. 5 Plots of $\alpha(t)$ and $\alpha_I(t)$ (left) and the cost functionals (right) with different values of $[I, J]$ in Example 4

We consider a discontinuous variable fractional order

$$\alpha(t) = \begin{cases} 0.2 + (x - T/2)^2, & t \in [0, T/4], \\ 0.2 + t^3, & t \in (T/4, T/2], \\ 0.22, & t \in (T/2, 3T/4], \\ 0.25 - 0.1t, & t \in (3T/4, T]. \end{cases}$$

In this example, we fix $\boldsymbol{P} = \{T/I, 2T/I, \ldots, (I-1)T/I\}$ and then follow the Algorithm 1 to progressively update $\boldsymbol{p} = \boldsymbol{\alpha} \in \mathbb{R}^{2I}$ with $\boldsymbol{p}_0 = [0.5, 0.5, \ldots, 0.5]$. Let $\check{w}_0(x) = x^3(1-x)^3$ and the other data are the same as those in Example 1. Numerical results are presented in Fig. 5, which indicates that the proposed method provides a satisfactory approximation of $\alpha(t)$ even in the case that the $\boldsymbol{P}$ is fixed.

6 Concluding remarks

In this paper, we analyze the forward and inverse problems of a variable-order viscoelastic Euler-Bernoulli beam. The well-posedness and the solution regularity of the proposed model are proved, based on which we further prove the uniqueness of the inverse problem of determining a piecewise polynomial variable-order contained in the proposed model, with the measurements of the unknown solutions on a space-time rectangular domain. Several numerical experiments are conducted to identify the variable fractional order.

A potential extension of the current work is to consider the corresponding inverse problem for model (4.1) with other definitions of variable-order fractional derivative such as [52, 61]

$$\bar{\partial}_t^{\alpha(t)} g(t) := {}_0\bar{I}_t^{1-\alpha(t)} \dot{g}(t), \quad {}_0\bar{I}_t^{\alpha(t)} g(t) := \int_0^t \frac{g(s)}{\Gamma(\alpha(t))(t-s)^{1-\alpha(t)}} ds. \tag{6.1}$$

We note from the proof of Theorem 3 that the derivation of the third equality in (4.11) implicitly employs the hidden-memory feature of the variable-order operator (2.5) such that $\alpha = \check{\alpha}$ on $[0, t_*]$ yields

$$\int_0^{t_*} \left[\frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} - \frac{(t-s)^{-\check{\alpha}(s)}}{\Gamma(1-\check{\alpha}(s))} \right] \partial_s \partial_x^4 w(x,s) ds = 0, \ \ t \in (t_*, t_* + \tau_{t_*}], \tag{6.2}$$

which is in general not true for the case of (6.1) since the variable order in (6.1) assumes its current value $\alpha(t)$ on the entire interval $[0, t]$. Thus (4.11) and the subsequent proof may not hold in this case that requires further consideration.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Adams, R.A., Fournier, J.J.: F: Sobolev Spaces. Elsevier, New York (1975)
2. Ashurov, R., Umarov, S.: Determination of the order of fractional derivative for subdiffusion equations. Fract. Calc. Appl. Anal. **23**(6), 1647–1662 (2020). https://doi.org/10.1515/fca-2020-0081
3. Bagley, R.L.: Power law and fractional calculus model of viscoelasticity. AIAA J. **27**(10), 1412–1417 (1989)
4. Banks, H.T., Inman, D.J.: On damping mechanisms in beams. J. Appl. Mech. **58**(3), 716–723 (1991)
5. Bonfanti, A., Kaplan, J.L., Charras, G., Kabla, A.: Fractional viscoelastic models for power-law materials. Soft Matter. **16**(26), 6002–6020 (2020)

6. Bottega, W.J.: Engineering Vibrations. CRC Press, Boca Raton (2013)
7. Cheng, J., Nakagawa, J., Yamamoto, M., Yamazaki, T.: Uniqueness in an inverse problem for a one dimensional fractional diffusion equation. Inverse Probl. **25**(11), 115002 (2009)
8. Christensen, R.: Theory of Viscoelasticity: An Introduction. Academic Press, New York (1982)
9. Clough, R.W., Penzien, J.: Dynamics of Structures. John Wiley and Sons, New York (1975)
10. Diethelm, K., Ford, N.J.: The Analysis of Fractional Differential Equations. Ser. Lecture Notes in Mathematics. Springer-Verlag, Berlin (2010)
11. Eldred, L.B., Baker, W.P., Palazotto, A.N.: Kelvin-Voigt versus fractional derivative model as constitutive relations for viscoelastic materials. AIAA J. **33**(3), 547–550 (1995)
12. Evans, L.C.: Partial Differential Equations. American Math. Soc, Providence, Rhode Island (1998)
13. Fritsch, A., Höckel, M., Kiessling, T., Nnetu, K.D., Wetzel, F., Zink, M., Käs, J.A.: Are biomechanical changes necessary for tumour progression? Nat. Phys. **6**(10), 730–732 (2010)
14. Garrappa, R., Giusti, A., Mainardi, F.: Variable-order fractional calculus: a change of perspective. Commun. Nonlinear Sci. Numer. Simul. **102**, 105904 (2021)
15. Gorenflo, R., Kilbas, A.A., Mainardi, F., Rogosin, S.V.: Mittag-Leffler Functions. Related Topics and Applications. Springer-Verlag, Berlin (2014)
16. Gorenflo, R., Vessella, S.: Abel Integral Equations. Lecture Notes in Mathematics. Springer, Berlin (1991)
17. Hackbusch, W.: Integral Equations: Theory and Numerical Treatment. International Series of Numerical Mathematics. Birkhäuser Verlag, Basel (1995)
18. Hagedorn, P., Spelsberg-Korspeter. G.: Active and Passive Vibration Control of Structures. Springer, New York (2014)
19. Inman, D.J.: Engineering Vibrations. Pearson, New Jersey (2014)
20. Jaiseenkar, A., McKinley, G.H.: Power-law rheology in the bulk and at the interface: quasi-properties and fractional constitutive equations. Proc. R. Soc. A. **469**(2149), 20120284 (2013)
21. Janno, J.: Determination of the order of fractional derivative and a kernel in an inverse problem for a generalized time fractional diffusion equation. Elect J. Differential Eqs. **2016**(199), 1–28 (2016)
22. Jin, B., Rundell, W.: An inverse problem for a one-dimensional time-fractional diffusion problem. Inverse Probl. **28**, 7501075028 (2012)
23. Kian, Y., Oksanen, L., Soccorsi, E., Yamamoto, M.: Global uniqueness in an inverse problem for time fractional diffusion equations. J. Differ. Equ. **264**, 1146–1170 (2018)
24. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
25. Li, G., Zhang, D., Jia, X., Yamamoto, M.: Simultaneous inversion for the space-dependent diffusion coefficient and the fractional order in the time-fractional diffusion equation. Inverse Probl. **29**, 065014 (2013)
26. Li, Y., Wang, H.: A finite element approximation to a viscoelastic Euler-Bernoulli beam with internal damping. Math. Comput. Simul. **212**, 138–158 (2023)
27. Li, Y., Wang, H., Zheng, X.: A viscoelastic Timoshenko beam model: regularity and numerical approximation. J. Sci. Comput. **95**, 57 (2023)
28. Li, Z., Liu, Y., Yamamoto, M.: Inverse problems of determining parameters of the fractional partial differential equations. In: Kochubei, A., Luchko, Y. (eds.) Handbook of Fractional Calculus with Applications, Volume 2: Fractional Differential Equations, pp. 431-442. De Gruyter, Berlin (2019). https://doi.org/10.1515/9783110571660-019
29. Li, Z., Yamamoto, M.: Uniqueness for inverse problems of determining orders of multi-term time-fractional derivatives of diffusion equation. Appl. Anal. **94**(3), 570–579 (2015)
30. Lord, G.J., Powell, C.E., Shardlow, T.: An Introduction to Computational Stochastic PDEs. Cambridge University Press, (2014)
31. Lorenzo, C.F., Hartley, T.T.: Variable order and distributed order fractional operators. Nonlinear Dyn. **29**, 57–98 (2002)
32. Luchko, Y.: Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation. Fract. Calc. Appl. Anal. **15**(1), 141–160 (2012). https://doi.org/10.2478/s13540-012-0010-7
33. Mainardi, F.: Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models. Imperial College Press, London (2010)
34. Mainardi, F., Gorenflo, R.: Time-fractional derivatives in relaxation processes: a tutorial survey. Fract. Calc. Appl. Anal. **10**, 269–308 (2007). https://doi.org/10.48550/arXiv.0801.4914
35. Nutting, P.G.: A new general law of deformation. J. Franklin Inst. **191**(5), 679–685 (1921)

36. Perdikaris, P., Karniadakis, G.E.: Fractional-order viscoelasticity in one-dimensional blood flow models. Ann. Biomed. Eng. **42**, 1012–1023 (2014)
37. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
38. Ramirez, L.E.S., Coimbra, C.F.M.: A variable order constitutive relation for viscoelasticity. Ann. Phys. **519**(7–8), 543–552 (2007)
39. Rao, S.S.: Vibration of Continuous Systems. John Wiley Sons, New Jersey (2019)
40. Reddy, J.N.: An Introduction to Continuum Mechanics. Cambridge University Press, New York (2013)
41. Sakamoto, K., Yamamoto, M.: Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. J. Math. Anal. Appl. **382**(1), 426–447 (2011)
42. Samko, S.G., Ross, B.: Integration and differentiation to a variable fractional order. Integral Transform Spec. Funct. **1**(4), 277–300 (1993)
43. Sheng, H., Sun, H., Coopmans, C., Chen, Y., Bohannan, G.W.: A Physical experimental study of variable-order fractional integrator and differentiator. Eur. Phys. J. Spec. Top. **193**, 93–104 (2011)
44. Slodička, M.: Some direct and inverse source problems in nonlinear evolutionary PDEs with Volterra operators. Inverse Probl. **38**, 124001 (2022)
45. Stynes, M., O'Riordan, E., Gracia, J.L.: Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation. SIAM J. Numer. Anal. **55**(2), 1057–1079 (2017)
46. Sun, H., Chang, A., Zhang, Y., Chen, W.: A review on variable-order fractional differential equations: mathematical foundations, physical models, numerical methods and applications. Fract. Calc. Appl. Anal. **22**(1), 27–59 (2019). https://doi.org/10.1515/fca-2019-0003
47. Sun, H., Zhang, Y., Chen, W., Reeves, D.M.: Use of a variable-index fractional-derivative model to capture transient dispersion in heterogeneous media. J. Contam. Hydrol. **157**, 47–58 (2014)
48. Tatar, S., Ulusoy, S.: A uniqueness result for an inverse problem in a space-time fractional diffusion equation. Electron. J. Differ. Equ. **258**, 1–9 (2013)
49. Thomée, V.: Galerkin Finite Element Methods for Parabolic Problems. Lecture Notes in Math. Springer-Verlag, New York (1984)
50. Suzuki, J.L., Kharazmi, E., Varghaei, P., Naghibolhosseini, M.: Zayernouri, M: Anomalous nonlinear dynamics behavior of fractional viscoelastic beams. J. Comput. Nonlinear Dyn. **16**(11), 111005 (2021)
51. Van Bockstal, K., Hendy, A.S., Zaky, M.A.: Space-dependent variable-order time-fractional wave equation: existence and uniqueness of its weak solution. Quaest. Math. **0**(0), 1-21 (2022)
52. Wang, H., Zheng, X.: Wellposedness and regularity of the variable-order time-fractional diffusion equations. J. Math. Anal. Appl. **475**(2), 1778–1802 (2019)
53. Ye, H., Gao, J., Ding, Y.: A generalized Gronwall inequality and its application to a fractional differential equation. J. Math. Anal. Appl. **328**(2), 1075–1081 (2007)
54. Zaky, M.A., Van Bockstal, K., Taha, T.R., Suragan, D., Hendy, A.S.: An L1 type difference/Galerkin spectral scheme for variable-order time-fractional nonlinear diffusion-reaction equations with fixed delay. J. Comput. Appl. Math. **420**, 114832 (2023)
55. Zeng, F., Zhang, Z., Karniadakis, G.E.: A generalized spectral collocation method with tunable accuracy for variable-order fractional differential equations. SIAM J Sci. Comp. **37**(6), A2710–A2732 (2015)
56. Zhang, M., Liu, J.: On the simultaneous reconstruction of boundary Robin coefficient and internal source in a slow diffusion system. Inverse Probl. **37**, 075008 (2021)
57. Zheng, G., Wei, T.: A new regularization method for the time fractional inverse advection-dispersion problem. SIAM J. Numer. Anal. **49**(5), 1972–1990 (2011)
58. Zheng, X., Li, Y., Cheng, J., Wang, H.: Inverting the variable fractional order in a variable-order space-fractional diffusion equation with variable diffusivity: analysis and simulation. J. Inverse Ill-Posed Probl. **29**(2), 219–231 (2021)
59. Zheng, X., Li, Y.: Wang, H: A viscoelastic Timoshenko beam: Model development, analysis, and investigation. J. Math. Phys. **63**(6), 061509 (2022)
60. Zheng, X.: Wang, H: Uniquely identifying the variable order of time-fractional partial differential equations on general multi-dimensional domains. Inverse Probl. Sci. Engrg. **29**(10), 1401–1411 (2021)
61. Zheng, X.: Wang, H: Optimal-order error estimates of finite element approximations to variable-order time-fractional diffusion equations without regularity assumptions of the true solutions. IMA J. Numer. Anal. **41**(2), 1522–1545 (2021)
62. Zhuang, P., Liu, F., Anh, V., Turner, I.: Numerical methods for the variable-order fractional advection-diffusion equation with a nonlinear source term. SIAM J. Numer. Anal. **47**(3), 1760–1781 (2009)
63. Inconel alloy 718, Publication Number SMC-045 Copyright Special Metals Corporation. www.specialmetals.com (2007)

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