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Analysis of asymptotic behavior of the Caputo–Fabrizio time-fractional diffusion equation

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ABSTRACT

We use the continuous time random walk to analyze the asymptotic behavior of the Caputo–Fabrizio time-fractional diffusion equation (CF-tFDE). By evaluating the diffusion limit of the Laplace–Fourier transform of the solution to the CF-tFDE, we prove that the long-time limiting governing equation satisfies an integer-order Fickian diffusion equation. Numerical experiments are presented to verify the theoretical results.

1. Introduction

Fractional diffusion equation was derived to accurately describe the power-law behavior that is observed in the diffusive transport of solutes in heterogeneous porous media, which cannot be accurately captured by the integer-order Fickian diffusion equation that is characterized by Gaussian-type exponentially decaying solutions [1–6]. However, time-fractional diffusion equation admits solutions with weak singularity near the initial time $t = 0$ [7,8].

The Caputo–Fabrizio time-fractional diffusion equation [9], which uses an exponentially decaying kernel in place of the weakly singular power-law kernel in the conventional Caputo time-fractional diffusion equation [10], was developed to reduce the singularity of its solution near the initial time $t = 0$ while retaining the nonlocal memory effect of the conventional time-fractional diffusion equation [11–13].

In this paper we use the continuous time random walk to analyze the asymptotic behavior of the initial-value problem of the Caputo–Fabrizio time-fractional diffusion equation

$$\begin{aligned} &\frac{\partial^\alpha p}{\partial t^\alpha}(x,t) = K\frac{\partial^2 p}{\partial x^2}(x,t), && -\infty < x < \infty,\ t > 0, \\ &p(x,0) = \delta(x), && -\infty < x < \infty, \end{aligned} \tag{1}$$

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where $\delta(x)$ is the Dirac delta function, K is the diffusion coefficient, $\frac{\partial^\alpha}{\partial t^\alpha}$ with $0 < \alpha < 1$ is the Caputo–Fabrizio time-fractional differential operator defined by

$$\begin{aligned}\frac{\partial^\alpha p}{\partial t^\alpha}(x,t) &:= \frac{M(\alpha)}{1-\alpha}\int_0^t \frac{\partial p}{\partial s}(x,s)\exp\left\{-\frac{\alpha(t-s)}{1-\alpha}\right\}ds \\ &= \frac{\partial p}{\partial t}(x,t) \star \frac{M(\alpha)}{1-\alpha}\exp\left\{-\frac{\alpha t}{1-\alpha}\right\},\end{aligned} \tag{2}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$ [9].

2. Asymptotic analysis using continuous time random walk

2.1. A continuous time random walk framework

We begin with a CTRW framework [5,14]. We consider a random motion of a particle that is located at the origin $x = 0$ initially at the time $t = 0$. Let $\phi(x,t)$ be the joint probability density function (pdf) for the particle to be located in the spatial interval $(x, x+dx)$ during the time period $(t, t+dt)$, and $\eta(x,t)$ be the pdf that the particle has just arrived in the spatial interval $(x, x+dx)$ at time t with probability $\eta(x,t)dx$. Then the CTRW master equation can be expressed as

$$\begin{aligned}\eta(x,t) &= \int_0^t\int_{-\infty}^{\infty}\eta(y,s)\phi(x-y,t-s)dyds + \delta(x)\delta(t) \\ &= \int_0^{\infty}\int_{-\infty}^{\infty}\eta(y,s)\phi(x-y,t-s)dyds + \delta(x)\delta(t),\end{aligned} \tag{3}$$

where the kernel $\phi(x,t)$ has been extended by zero to $t < 0$.

The jump length pdf $\lambda(x)$ and waiting time pdf $w(t)$ defined by

$$\lambda(x) := \int_0^\infty \phi(x,t)dt, \qquad w(t) := \int_{-\infty}^{\infty}\phi(x,t)dx$$

give the probability $\lambda(x)dx$ that the particle is located in the spatial interval $(x, x+dx)$ and the probability $w(t)dt$ that the waiting time is in the time interval $(t, t+dt)$, respectively. Then the cumulative probability $\psi(t)$ of no jump taking place during the time period $(0,t)$ is

$$\psi(t) := 1 - \int_0^t w(s)ds.$$

Let $p(x,t)$ be the pdf for the particle to be in the spatial interval $(x, x+dx)$ at time t, which is defined by

$$p(x,t) := \int_0^t \eta(x,s)\psi(t-s)ds. \tag{4}$$

Introduce the Laplace transform $\widetilde{w}(z)$, the Fourier transform $\widehat{\lambda}(k)$, and the Laplace–Fourier transform $\bar{p}(k,z)$ by

$$\begin{aligned}\widetilde{w}(z) &:= \mathcal{L}[w](z) = \int_0^\infty w(t)e^{-zt}dt, \\ \widehat{\lambda}(k) &:= \mathcal{F}[\lambda](k) = \int_{-\infty}^{\infty}\lambda(x)e^{-ikx}dx, \\ \bar{p}(k,z) &:= \mathcal{LF}[p](k,z) = \int_0^\infty e^{-zt}\int_{-\infty}^{\infty}p(x,t)e^{-ikx}dxdt.\end{aligned}$$

Eqs. (3)–(4) are transformed in phase planes as

$$\begin{aligned}\widetilde{\psi}(z) &= \frac{1}{z} - \frac{\widetilde{w}(z)}{z} = \frac{1-\widetilde{w}(z)}{z},\\ \widetilde{\eta}(x,z) &= \widetilde{\eta}(x,z) * \widetilde{\phi}(x,z) + \delta(x),\\ \bar{\eta}(k,z) &= \bar{\eta}(k,z)\bar{\phi}(k,z) + 1,\\ \widetilde{p}(x,z) &= \widetilde{\psi}(z)\widetilde{\eta}(x,z).\end{aligned} \tag{5}$$

We obtain from (5)

$$\bar{\eta}(k,z) = \frac{1}{1-\bar{\phi}(k,z)},$$

and, consequently, a transformed master equation in the phase plane in terms of the transformed quantity $\bar{p}(k,z)$

$$\bar{p}(k,z) = \widetilde{\psi}(z)\bar{\eta}(k,z) = \frac{1-\widetilde{w}(z)}{z}\frac{1}{1-\bar{\phi}(k,z)}. \tag{6}$$

Assume that the jump length pdf $\lambda(x)$ and the waiting time pdf $w(t)$ are independent

$$\phi(x,t) = \lambda(x)w(t).$$

If the waiting time pdf $w(t)$ has mean

$$\tau := \int_0^\infty t w(t)dt < \infty,$$

e.g., w is the pdf of a Poissonian distribution, then we have

$$\begin{aligned}\widetilde{w}(z) &= \int_0^\infty w(t)e^{-zt}dt = \int_0^\infty w(t)(1 - zt + \mathrm{o}(zt))dt\\ &= 1 - z\tau + \mathrm{o}(z\tau), \quad z \to 0.\end{aligned} \tag{7}$$

If the jump length pdf $\lambda(x)$ has zero mean and finite variance

$$\sigma^2 := \int_{-\infty}^\infty x^2\lambda(x)dx < \infty,$$

e.g., λ is the pdf of a Gaussian distribution, then

$$\begin{aligned}\widehat{\lambda}(k) &= \int_{-\infty}^\infty \lambda(x)e^{-ikx}dx = \int_{-\infty}^\infty \lambda(x)\Big(1 - ikx - \frac{k^2x^2}{2} + \mathrm{o}(k^2x^2)\Big)dx\\ &= 1 - \frac{\sigma^2k^2}{2} + \mathrm{o}(\sigma^2k^2), \quad k \to 0.\end{aligned} \tag{8}$$

We incorporate Eqs. (7) and (8) into (6) to find

$$\begin{aligned}\bar{p}(k,z) &= \frac{1-\widetilde{w}(z)}{z}\frac{1}{1-\bar{\phi}(k,z)} = \frac{1-\widetilde{w}(z)}{z}\frac{1}{1-\widetilde{w}(z)\widehat{\lambda}(k)}\\ &= \frac{\tau(1+\mathrm{o}(1))}{z\tau + \frac{\sigma^2k^2}{2} + \mathrm{o}(z\tau) + \mathrm{o}(k^2\sigma^2)}, \qquad k,z \to 0.\end{aligned} \tag{9}$$

To derive the long-time limiting governing equation that is satisfied by $p(x,t)$ in the physical space, we look at the lowest order terms in (9) as $(k,z) \to (0,0)$ to obtain the diffusion limit satisfied by $\bar{p}(k,z)$ in the phase space as

$$\bar{p}(k,z) = \frac{\tau}{z\tau + \frac{\sigma^2k^2}{2}} = \frac{1}{z + K_1k^2}, \qquad k,z \to 0 \tag{10}$$

with K_1 defined by

$$K_1 := \frac{\sigma^2}{2\tau}.$$

Eq. (10) can be written as

$$z\overline{p}(k,z) - 1 = -K_1 k^2 \overline{p}(k,z). \tag{11}$$

The inverse Laplace transform of (11) yields

$$\begin{aligned} \frac{\partial \hat{p}}{\partial t}(k,t) - \hat{p}(k,0) &= -K_1 k^2 \hat{p}(k,t), \\ \hat{p}(k,0) &= 1. \end{aligned} \tag{12}$$

Apply the inverse Fourier transform to (12) to arrive at the initial-value problem of the classical Fickian diffusion equation

$$\begin{aligned} &\frac{\partial p}{\partial t}(x,t) = K_1 \frac{\partial^2 p}{\partial x^2}(x,t), && -\infty < x < \infty,\ t > 0 \\ &p(x,0) = \delta(x), && -\infty < x < \infty. \end{aligned} \tag{13}$$

2.2. Analysis of the time-fractional diffusion equation (1)–(2)

We are now in the position to applying the CTRW framework to analyze problem (1)–(2). Note that physically, the solution p to problem (1)–(2) may be viewed as the pdf that describes the subdiffusive transport of a particle, which is located at the origin at the initial time $t = 0$, has its movement governed by the CF-tFDE (1), and stays at the spatial interval $(x, x+dx)$ at time t with a probability $p(x,t)dx$. In other words, the solution $p(x,t)$ to problem (1) has the same physical meaning as the solution $p(x,t)$ to problem (13) except that the random motion of the particle in the latter is governed by a Fickian diffusive process. Hence, we use the CTRW approach in Section 2.1 to analyze problem (1)–(2).

We now take the Laplace–Fourier transform of $\frac{\partial^2 p}{\partial x^2}$ in Eq. (1) to obtain

$$\mathcal{LF}\left[\frac{\partial^2 p}{\partial x^2}\right](k,z) = -k^2 \overline{p}(k,z) \tag{14}$$

We use Eq. (2) to evaluate the Laplace–Fourier transform of $\frac{\partial^\alpha p}{\partial t^\alpha}$ in Eq. (2) by

$$\begin{aligned} \mathcal{L}\left[\frac{\partial^\alpha p}{\partial t^\alpha}(x,t)\right] &= \mathcal{L}\left[\frac{\partial p}{\partial t}(x,t) \star \frac{M(\alpha)}{1-\alpha} \exp\left\{-\frac{\alpha t}{1-\alpha}\right\}\right] \\ &= \mathcal{L}\left[\frac{\partial p}{\partial t}(x,t)\right] \frac{M(\alpha)}{1-\alpha} \mathcal{L}\left[\exp\left\{-\frac{\alpha t}{1-\alpha}\right\}\right] \\ &= \frac{M(\alpha)}{1-\alpha} \frac{1}{z + \frac{\alpha}{1-\alpha}} \big(z\widetilde{p}(x,z) - \delta(x)\big), \end{aligned}$$

and

$$\mathcal{LF}\left[\frac{\partial^\alpha p}{\partial t^\alpha}\right] = \frac{M(\alpha)}{1-\alpha} \frac{1}{z + \frac{\alpha}{1-\alpha}} \big(z\overline{p}(x,z) - 1\big). \tag{15}$$

Incorporate Eqs. (14) and (15) into Eq. (1) to deduce

$$\frac{M(\alpha)}{1-\alpha} \frac{1}{z + \frac{\alpha}{1-\alpha}} \big(z\overline{p}(x,z) - 1\big) = -Kk^2 \overline{p}(k,z). \tag{16}$$

We follow the derivation of Eq. (10) to evaluate the diffusion limit satisfied by $\overline{p}(k,z)$ in the phase space as $(k,z) \to (0,0)$ by looking at the lowest order terms in (16)

$$z\overline{p}(x,z) - 1 = -K_2 k^2 \overline{p}(k,z)$$

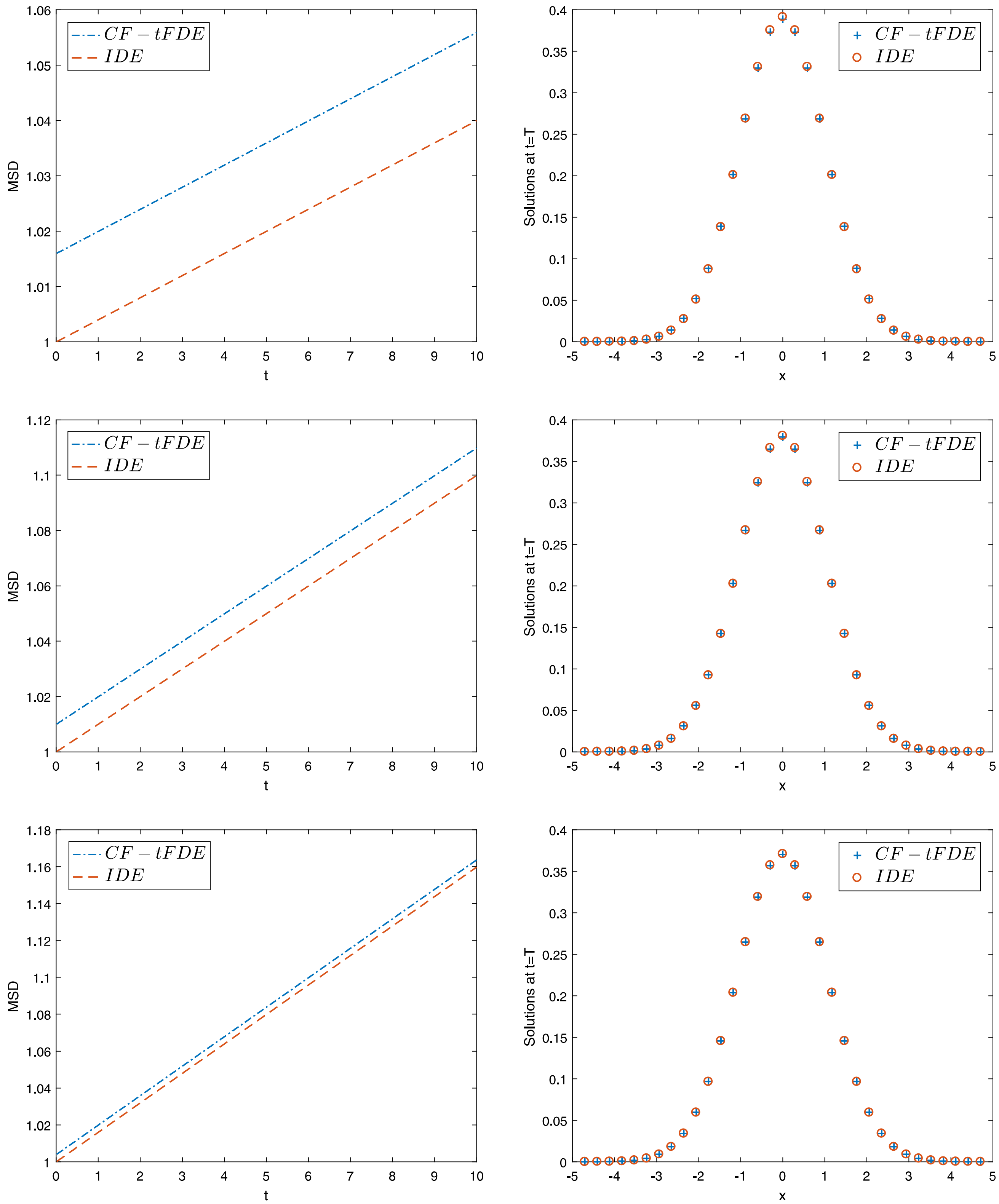

Fig. 1. Left column: The mean-square displacement (MSD) of the solutions to the CF-tFDE (1), in comparison to the MSD of the solutions to the integer-order diffusion equation (IDE) (17); Right column: The solutions to the CF-tFDE (1) and IDE (17) with different values of $\alpha = 0.2$ (first row), 0.5(second row), and 0.8 (third row), respectively.

with K_2 given by

$$K_2 := \frac{K\alpha}{M(\alpha)}.$$

The same derivation as Eq. (13) finds the long-time limiting governing equation that is satisfied by $p(x,t)$ in the physical space

$$\begin{aligned} &\frac{\partial p}{\partial t}(x,t) = K_2 \frac{\partial^2 p}{\partial x^2}(x,t), \quad -\infty < x < \infty,\ t > 0 \\ &p(x,0) = \delta(x), \qquad\qquad\qquad -\infty < x < \infty. \end{aligned} \tag{17}$$

3. Numerical investigations

In this section we carry out numerical experiments to substantiate the theoretical results in the previous section. It is well known that the mean square displacement (MSD) of the solution p to the classical integer-order diffusion equation (IDE) (17) obeys the following relation asymptotically

$$< x^2(t) > := \int_{-\infty}^{\infty} x^2 p(x,t) dx \sim t, \tag{18}$$

while the MSD of the solution p to the conventional time-fractional diffusion equation of order $0 < \alpha < 1$ obeys the relation

$$< x^2(t) > \sim t^{\alpha}. \tag{19}$$

We now evaluate the MSD of the solution to the CF-tFDE (1) by setting $M(\alpha) \equiv 1$, the diffusivity $K = 0.01$, and $T = 10$, so that the diffusivity in (17) is $K_2 = K\alpha$. We plot the MSD of the solutions to the CF-tFDE (1) and the solutions at the final time $T = 10$ with $\alpha = 0.2, 0.5$ and 0.8 respectively, in comparison with the MSD of the solution to the integer-order diffusion equation and the solution at time $T = 10$ in Fig. 1.

We observe that the MSD of the solutions to the CF-tFDE (17) increases linearly with respect to the time t and the solutions to (1) coincide with those to the integer-order diffusion Eq. (17). In summary, these results verify the analysis in the previous section. Although it is formally nonlocal, the CF-fFDE (1) behaves like the integer-order diffusion equation asymptotically.

Data availability

No data was used for the research described in the article.

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References

[1] D. del Castillo-Negrete, Front propagation in reaction–diffusion systems with anomalous diffusion, Bol. Soc. Mat. Mex. 20 (2014) 87–105.
[2] M. Dentz, A. Cortis, H. Scher, B. Berkowitz, Time behavior of solute transport in heterogeneous media: transition from anomalous to normal transport, Adv. Water Resour. 27 (2004) 155–173.
[3] C. Li, M. Cai, Theory and Numerical Approximations of Fractional Integrals and Derivatives, SIAM, 2020.

[4] B. Jin, Z. Zhou, Recovering the potential and order in one-dimensional time-fractional diffusion with unknown initial condition and source, Inverse Problems 37 (2021) 105009.
[5] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, Phys. Rep. 339 (2000) 1–77.
[6] M. Samiee, A. Akhavan-Safaei, M. Zayernouri, A fractional subgrid-scale model for turbulent flows: Theoretical formulation and a priori study, Physs. Fluids 32 (2020) 055102.
[7] K. Sakamoto, M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems, J. Math. Anal. Appl. 382 (2011) 426–447.
[8] M. Stynes, E. O'Riordan, J.L. Gracia, Error analysis of a finite difference method on graded mesh for a time-fractional diffusion equation, SIAM J. Numer. Anal. 55 (2017) 1057–1079.
[9] M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, Progr. Fract. Differ. Appl. 1 (2015) 73–85.
[10] I. Podlubny, Fractional Differential Equations, Academic Press, 1999.
[11] A. Atangana, D. Baleanu, Caputo–Fabrizio derivative applied to groundwater flow within confined aquifer, J. Engrg. Mech. 143 (2017) D4016005.
[12] Z. Cui, Solutions of some typical nonlinear differential equations with Caputo–Fabrizio fractional derivative, AIMS Math. (2022) 14139–14153.
[13] X. Zheng, H. Wang, H. Fu, Well-posedness of fractional differential equations with variable-order Caputo–Fabrizio derivative, Chaos Solitons Fractals 138 (2020) 109966.
[14] M.M. Meerschaert, A. Sikorskii, Stochastic Models for Fractional Calculus, De Gruyter, 2019.