Journal of Computational Mathematics
Vol.41, No.5, 2023, 817–841.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2112-m2021-0204

A NEW FINITE ELEMENT SPACE FOR EXPANDED MIXED FINITE ELEMENT METHOD *

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Abstract

In this article, we propose a new finite element space $\mathbf{\Lambda}_h$ for the expanded mixed finite element method (EMFEM) for second-order elliptic problems to guarantee its computing capability and reduce the computation cost. The new finite element space $\mathbf{\Lambda}_h$ is designed in such a way that the strong requirement $\mathbf{V}_h \subset \mathbf{\Lambda}_h$ in [9] is weakened to $\{\mathbf{v}_h \in \mathbf{V}_h; \mathrm{div}\mathbf{v}_h = 0\} \subset \mathbf{\Lambda}_h$ so that it needs fewer degrees of freedom than its classical counterpart. Furthermore, the new $\mathbf{\Lambda}_h$ coupled with the Raviart-Thomas space satisfies the inf-sup condition, which is crucial to the computation of mixed methods for its close relation to the behavior of the smallest nonzero eigenvalue of the stiff matrix, and thus the existence, uniqueness and optimal approximate capability of the EMFEM solution are proved for rectangular partitions in $\mathbb{R}^d, d = 2, 3$ and for triangular partitions in $\mathbb{R}^2$. Also, the solvability of the EMFEM for triangular partition in $\mathbb{R}^3$ can be directly proved without the inf-sup condition. Numerical experiments are conducted to confirm these theoretical findings.

Mathematics subject classification: 65N30.
Key words: New finite element space, Expanded mixed finite element, Minimum degrees of freedom, The inf-sup condition, Solvability, Optimal convergence.

1. Introduction

The expanded mixed finite element method (EMFEM) [9], first proposed for linear elliptic problems of second-order to generalize the classical mixed methods in the sense that the gradient as a newly introduced variable is explicitly approximated besides the unknown and flux, has achieved a significant success in applications to those diffusion processes within complex geometry and low permeability zones. Now the EMFEM has been extended successively to the quasi-linear elliptic problems [10, 21], the fourth order elliptic equations [10], parabolic problems [8, 16, 19, 24], hyperbolic problems [27], displacement in porous media [18, 28] and other physical models [6, 17, 20]. Recently, the EMFEM was found its application to the fractional-order diffusion equations [7, 26].

* Received July 18, 2021 / Revised version received November 12, 2021 / Accepted December 6, 2021 / Published online February 23, 2022 /
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Checking carefully the direct proof for the EMFEM's solvability (see pp. 487 in [9]), it is not difficult to find that the finite element space $\mathbf{\Lambda}_h$ for the gradient and the space $\mathbf{V}_h$ for the flux must satisfy the strong requirement $\mathbf{V}_h \subset \mathbf{\Lambda}_h$, which makes the selection of the test function $\boldsymbol{\mu}_h = \boldsymbol{\sigma}_h \in \mathbf{V}_h$ possible to ensure the existence and uniqueness of the solution to the EMFEM. As a consequence, the strong requirement $\mathbf{V}_h \subset \mathbf{\Lambda}_h$ excludes such the potential spaces as those $\mathbf{\Lambda}_h$ with lower space indices, especially the often used piecewise-constant spaces. This may confine the versatility of the EMFEM in applications and increase the computing burden. For example, if $\mathbf{V}_h$ and W_h are the the lowest order Raviart-Thomas space for triangular partitions, the best convergence rates for the unknown in $L^2(\Omega)$-norm, the gradient in $(L^2(\Omega))^d$-norm and the flux in $\mathbf{H}(\text{div};\Omega)$-norm are $\mathcal{O}(h)$ whatever the space index of $\mathbf{\Lambda}_h$ takes 1 or 0.

The main goals of this article are to: (1) Redesign the finite element space $\mathbf{\Lambda}_h$ in such a way that they contain as many full polynomials as W_h does in order to preserve the same approximate capability, and contain all the divergence-free vectors of $\mathbf{V}_h$ to ensure the solvability of the EMFEM for linear elliptic problems of second-order, and hence the strong requirement $\mathbf{V}_h \subset \mathbf{\Lambda}_h$ is weakened and thus designed $\mathbf{\Lambda}_h$ possesses minimum degrees of freedom. Specifically, for rectangular partitions, the degrees of freedom of $\mathbf{\Lambda}_h$ on an element E are $2k+1$ and $2(k+1)^2-1$ degrees of freedom less than those of $\mathbf{V}_h|_E$ for $d=2$ and $d=3$, respectively; for triangular partitions, $\mathbf{\Lambda}_h$ consists of all the piecewise polynomials of degree $\leq k$, which are $k+1$ and $\frac{1}{2}(k+1)(k+2)$ degrees of freedom less than that of $\mathbf{V}_h|_E$ for $d=2$ and $d=3$, respectively. And thus, the commonly used piecewise constant spaces are retrieved. (2) Prove that thus redesigned $\mathbf{\Lambda}_h$ combined with the Raviart-Thomas mixed space $\mathbf{V}_h \times W_h$ satisfies the coerciveness condition and the inf-sup condition. This finding is crucial to the computation of mixed methods since the inf-sup condition is closely related to the behavior of the smallest nonzero eigenvalue of the stiff matrix, the loss of which may leads extra artificial (nonphysical) constraints on the boundary conditions or locking phenomenon [3]. (3) Prove the existence, uniqueness and the same approximate capability of the EMFEM solution as the traditional mixed methods [9,10], by an application of the coerciveness and the inf-sup condition for rectangular partitions of $\Omega \subset \mathbb{R}^d, d=2,3$ and triangular partitions of $\Omega \subset \mathbb{R}^2$. (4) Present a direct proof as did in [9] for the solvability of the EMFEM on general partitions of $\mathbb{R}^d$. (5) Conduct numerical experiments to confirm the theoretical findings.

The rest of this article is organized as follows. In Section 2, we shall develop the weak form and the EMFEM for linear elliptic problems, prove the coerciveness for one of bilinear form and analyze the key points we will stress in the sequel. Section 3 is devoted to the rectangular partitions for $\Omega \subset \mathbb{R}^d$. we shall characterize the divergence-free vectors of $\mathbf{V}_h$ by decomposition techniques, redesign the space $\mathbf{\Lambda}_h$ with minimum degrees of freedom, then prove the validation of the inf-sup condition for the other bilinear form over the $\mathbf{\Lambda}_h$ and the Raviart-Thomas space. Section 4 is devoted to the triangular partitions. We shall use the inclusion of divergence-free vectors of $\mathbf{V}_h$ to redesign the $\mathbf{\Lambda}_h$ and give a direct proof for the solvability of the EMFEMs for $d=2$ and $d=3$. Further, we shall apply the discrete Helmholtz decomposition theory to prove the validation of the inf-sup condition over the newly defined space $\mathbf{\Lambda}_h$ and the $\mathbf{V}_h \times W_h$ for $d=2$, then derive the solvability again and the same approximate capability of the EFEMs solution as that of the traditional mixed methods have. In Section 5, numerical experiments are conducted to confirm our theoretical findings. The last section is for concluding remarks.

Through out this paper, we write vectors or vector spaces in boldface, use $(\cdot,\cdot)$ to denote the L^2-inner product, and use $\|\cdot\|$ to denote the L^2-norm or the Euclid norm in vector spaces. We also use $\|\cdot\|_H$ to denote the norm in Sobolev space H and $|\cdot|_H$ to denote its semi-norm.

2. Formulation of the Expanded Mixed Finite Element Method

To fix ideas, we take the following second-order elliptic problems as a model,

$$\begin{aligned} &(a) \quad -\nabla.(a\nabla u - \mathbf{b}u + \mathbf{c}) + du = f \quad && \mathbf{x} \in \Omega, \\ &(b) \quad u = 0 && \mathbf{x} \in \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^d, d = 2, 3$ with its boundary $\partial\Omega$, $\mathbf{x} = (x, y)$ and $\mathbf{x} = (x, y, z)$ for $d = 2, 3$ respectively; $a(\mathbf{x})$ is a uniformly positive definite and bounded tensor, $\mathbf{b}(\mathbf{x}), \mathbf{c}(\mathbf{x})$ and $d(\mathbf{x})$ are given vectors and scalar function, $f(\mathbf{x}) \in L^2(\Omega)$.

We let

$$W = L^2(\Omega), \quad \mathbf{\Lambda} = (L^2(\Omega))^d, \quad \mathbf{V} = \mathbf{H}(\text{div}; \Omega)$$

with their norms $\|\cdot\|_X$ as $X = W, \mathbf{\Lambda}$ and $\mathbf{V}$, respectively, and then rewrite (2.1) into the equivalent weak form: Find $(\boldsymbol{\sigma}, \boldsymbol{\lambda}, u) \in \mathbf{V} \times \mathbf{\Lambda} \times W$ such that

$$\begin{aligned} &(a) \quad (a\boldsymbol{\lambda}, \boldsymbol{\mu}) - (\boldsymbol{\sigma}, \boldsymbol{\mu}) + (\mathbf{b}u, \boldsymbol{\mu}) = (\mathbf{c}, \boldsymbol{\mu}), \quad && \boldsymbol{\mu} \in \mathbf{\Lambda}, \\ &(b) \quad (\boldsymbol{\lambda}, \mathbf{v}) - (u, \nabla \cdot \mathbf{v}) = 0, && \mathbf{v} \in \mathbf{V}, \\ &(c) \quad (\nabla \cdot \boldsymbol{\sigma}, w) + (du, w) = (f, w), && w \in W. \end{aligned} \tag{2.2}$$

Let $\mathbf{U} = W \times \mathbf{\Lambda}$ endowed with the usual product norm $\|\boldsymbol{\tau}\|_{\mathbf{U}}^2 = \|w\|^2 + \|\boldsymbol{\mu}\|^2$ for $\boldsymbol{\tau} = (w, \boldsymbol{\mu}) \in \mathbf{U}$, and introduce the bilinear forms $\mathbb{A}(\cdot, \cdot) : \mathbf{U} \times \mathbf{U} \to \mathbb{R}$ and $\mathbb{B}(\boldsymbol{\tau}, \mathbf{v}) : \mathbf{U} \times \mathbf{V} \to \mathbb{R}$,

$$\begin{aligned} &\mathbb{A}(\boldsymbol{\chi}, \boldsymbol{\tau}) = (a\boldsymbol{\lambda}, \boldsymbol{\mu}) + (\mathbf{b}u, \boldsymbol{\mu}) + (du, w), \quad && \boldsymbol{\chi} = (u, \boldsymbol{\lambda}), \boldsymbol{\tau} = (w, \boldsymbol{\mu}) \in \mathbf{U}, \\ &\mathbb{B}(\boldsymbol{\tau}, \mathbf{v}) = (\boldsymbol{\mu}, \mathbf{v}) - (w, \nabla \cdot \mathbf{v}), && \boldsymbol{\tau} = (w, \boldsymbol{\mu}) \in \mathbf{U}, \mathbf{v} \in \mathbf{V}. \end{aligned} \tag{2.3}$$

Then, (2.2) can be writren as the standard form: Find $(\boldsymbol{\chi}, \boldsymbol{\sigma}) \in \mathbf{U} \times \mathbf{V}$ such that

$$\begin{aligned} &(a) \quad \mathbb{A}(\boldsymbol{\chi}, \boldsymbol{\tau}) + \mathbb{B}(\boldsymbol{\tau}, \boldsymbol{\sigma}) = F(\boldsymbol{\tau}), \quad && \boldsymbol{\tau} \in \mathbf{U}, \\ &(b) \quad \mathbb{B}(\boldsymbol{\chi}, \mathbf{v}) = 0, && \mathbf{v} \in \mathbf{V}, \end{aligned} \tag{2.4}$$

with $F(\boldsymbol{\tau}) = (f, w) + (\mathbf{c}, \boldsymbol{\mu})$. We also let

$$\mathbf{Z} = \{\boldsymbol{\tau} \in \mathbf{U}; \mathbb{B}(\boldsymbol{\tau}, \mathbf{v}) = 0, \text{ for all } \mathbf{v} \in \mathbf{V}\}. \tag{2.5}$$

Since the bilinear form $\mathbb{B}(\cdot, \cdot)$ and the space $\mathbf{Z}$ are just the same as defined in [9], and $\mathbb{A}(\cdot, \cdot)$ is slightly different due to the adding of the extra terms $\mathbf{b}(\mathbf{x})$ and $d(\mathbf{x})$, the solvability can be proved in the same manner as done in Lemma 3.1-Lemma 3.3 and Theorem 3.4 of [9], we shall only present the conclusion under mild assumptions on the coefficients $a(\mathbf{x}), \mathbf{b}(\mathbf{x})$ and $d(\mathbf{x})$ as required in [14] without detailed proof here.

Lemma 2.1 (Lemma 3.1, [9]). $\boldsymbol{\tau} = (w, \boldsymbol{\mu}) \in \mathbf{Z}$ *if and only if*

$$\boldsymbol{\mu} = -\nabla u \quad \text{and} \quad w \in H_0^1(\Omega) := \{w \in H^1(\Omega); w|_{\partial\Omega} = 0\}.$$

Theorem 2.2. *Assume that there exist positive constant α and γ with $0 < \gamma < 1$ such that the coefficients $a(\mathbf{x}), \mathbf{b}(\mathbf{x}) \in C^1(\overline{\Omega})$ and $d(\mathbf{x}) \in C^0(\overline{\Omega})$ satisfy, for $\mathbf{x} \in \Omega$,*

$$d(\mathbf{x}) \geq 0, \quad \mathbf{X}^T a(\mathbf{x}) \mathbf{X} \geq \alpha \|\mathbf{X}\|^2, \mathbf{X} \in R^d \quad \text{and} \quad |\mathbf{b}(\mathbf{x})|^2 \leq 4\gamma\alpha d(\mathbf{x}). \tag{2.6}$$

Then, there exists a unique solution $(\boldsymbol{\sigma}, \boldsymbol{\lambda}, u) \in \mathbf{V} \times \mathbf{\Lambda} \times W$ to the weak form (2.4) and thus to (2.2).

Proof. Selecting γ such that $0 < \gamma < 1$ and $|\mathbf{b}|^2 \le 4\alpha\gamma d$, then

$$
\begin{aligned}
&\alpha|\nabla w|^2 - |\mathbf{b}||w||\nabla w| + dw^2 \\
&= \alpha(1-\gamma)|\nabla w|^2 + \left(\sqrt{\alpha\gamma}|\nabla w| - \frac{|b|}{2\sqrt{\alpha\gamma}}|w|\right)^2 + \left(d - \frac{|\mathbf{b}|^2}{4\alpha\gamma}\right)|w|^2 \\
&\ge \alpha(1-\gamma)|\nabla w|^2.
\end{aligned}
$$

Thus, we apply Lemma 2.1 and the norm equivalence in $H_0^1(\Omega)$ to derive

$$
\begin{aligned}
\mathbb{A}(\boldsymbol{\tau},\boldsymbol{\tau}) =&(a\boldsymbol{\mu},\boldsymbol{\mu}) + (\mathbf{b}w,\boldsymbol{\mu}) + (dw,w) \\
=&(a\nabla w,\nabla w) + (\mathbf{b}w,\nabla w) + (dw,w) \\
\ge& \int_\Omega \{\alpha|\nabla w|^2 - |b||w||\nabla w| + dw^2\}d\mathbf{x} \\
\ge&\alpha(1-\gamma)\|\nabla w\|^2 \\
\ge&C\alpha(1-\gamma)\Big(\|\nabla w\|^2 + \|w\|^2\Big) \\
=&C\alpha(1-\gamma)\|\boldsymbol{\tau}\|_U^2.
\end{aligned}
$$

which shows that the bilinear form $\mathbb{A}(\cdot,\cdot)$ is coercive over $\mathbf{Z}$.

On the other hand, the bilinear form $\mathbb{B}(\cdot,\cdot)$ is the same as in [9] and also meets the inf-sup condition. Thus, an application of the Br*é*zzi's theorem [4] ensures the existence and the uniqueness of the solution to the weak form (2.4) or (2.2).

We shall denote by $\mathcal{E}_h$ a regular partition of $\Omega = \cup_{E\in\mathcal{E}_h} E$ into triangles or into rectangles (tetrahedra or cuboid for three dimensional domain). We then introduce the Raviart-Thomas mixed finite element spaces $\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ with the space index $k \ge 0$ (see [3, 9, 22, 23]) to approximate the two variables $\boldsymbol{\sigma}$ and u. Once the space $\boldsymbol{\Lambda}_h \subset \boldsymbol{\Lambda}$ is defined for the third variable $\boldsymbol{\lambda}$, the EMFEM can be defined as to find $(\boldsymbol{\sigma}_h, \boldsymbol{\lambda}_h, u_h) \in \mathbf{V}_h \times \boldsymbol{\Lambda}_h \times W_h$ such that [9]

$$
\begin{aligned}
&(a) \quad (a\boldsymbol{\lambda}_h,\boldsymbol{\mu}_h) - (\boldsymbol{\sigma}_h,\boldsymbol{\mu}_h) + (\mathbf{b}u_h,\boldsymbol{\mu}_h) = (\mathbf{c},\boldsymbol{\mu}_h), && \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h, \\
&(b) \quad (\boldsymbol{\lambda}_h,\mathbf{v}_h) - (u_h,\nabla\cdot\mathbf{v}_h) = 0, && \mathbf{v}_h \in \mathbf{V}_h, \\
&(c) \quad (\nabla\cdot\boldsymbol{\sigma}_h, w) + (du_h,w_h) = (f,w_h), && w_h \in W_h.
\end{aligned}
\tag{2.7}
$$

We let $\mathbf{U}_h = \boldsymbol{\Lambda}_h \times W_h$ and use the bilinear forms $\mathbb{A}(\cdot,\cdot)$ and $\mathbb{B}(\cdot,\cdot)$ to rewrite (2.7) as to find $(\boldsymbol{\sigma}_h, \boldsymbol{\chi}_h) \in \mathbf{V}_h \times \mathbf{U}_h$ with $\boldsymbol{\chi}_h = (\boldsymbol{\lambda}_h, u_h)$ such that

$$
\begin{aligned}
&(a) \quad \mathbb{A}(\boldsymbol{\chi}_h,\boldsymbol{\tau}_h) + \mathbb{B}(\boldsymbol{\tau}_h,\boldsymbol{\sigma}_h) = F(\boldsymbol{\tau}_h), && \boldsymbol{\tau}_h = (\boldsymbol{\mu}_h, w_h) \in \mathbf{U}, \\
&(b) \quad \mathbb{B}(\boldsymbol{\chi}_h,\mathbf{v}_h) = 0, && \mathbf{v}_h \in \mathbf{V}_h.
\end{aligned}
\tag{2.8}
$$

We then define the discrete version $\mathbf{Z}_h$ of $\mathbf{Z}$ by

$$
\mathbf{Z}_h = \{\boldsymbol{\tau}_h = (w_h,\boldsymbol{\mu}_h) \in \mathbf{U}; \mathbb{B}(\boldsymbol{\tau}_h,\mathbf{v}_h) = 0, \text{ for } \forall\mathbf{v}_h \in \mathbf{V}_h\}. \tag{2.9}
$$

It should be pointed out that $\mathbf{Z}_h$ is not a subset of $\mathbf{Z}$, and thus, the coerciveness of the bilinear form $\mathbb{A}(\cdot,\cdot)$ over $\mathbf{Z}_h$ can not be inferred from Theorem 2.2 as a corollary. Here, we shall apply the approximating property of the space $\mathbf{V}_h$ and give a new proof for the coerciveness. This proof does not rely much on the choice of the space $\boldsymbol{\Lambda}_h$.

Theorem 2.3. *Assume that the space-step h is sufficiently small and the coefficients $a(\mathbf{x}), \mathbf{b}(\mathbf{x})$ and $d(\mathbf{x})$ satisfy the conditions (2.6). Then, the bilinear form $\mathbb{A}(\cdot,\cdot)$ satisfies the following coerciveness over the discrete space $\mathbf{Z}_h$ in the sense, for $\boldsymbol{\tau}_h = (w_h, \boldsymbol{\mu}_h) \in \mathbf{Z}_h$,*

$$\mathbb{A}(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \geq \alpha_1 \|\boldsymbol{\tau}_h\|_U^2 \tag{2.10}$$

with $\alpha_1 = \min\{\frac{\alpha(1-\gamma)}{2}, \frac{\alpha(1-\gamma)}{2C(1+h)}\}$ for some generic constant C independent of h.

Proof. We begin by proving the norm equivalence in $\mathbf{Z}_h$ through a duality argument. Let $u_w \in H_0^1(\Omega) \cap H^2(\Omega)$ solves the equation

$$-\nabla \cdot (\nabla u_w) = w_h,$$

then, there exist a generic constant $C > 0$ such that $\|u_w\|_{H^2(\Omega)} \leq C\|w_h\|$.

Consequently, for $\boldsymbol{\tau}_h = (w_h, \boldsymbol{\mu}_h) \in \mathbf{Z}_h$, we apply the property of the projection Π_h of mixed finite element spaces [2, 3, 15, 22, 23] to manipulate,

$$\begin{aligned}
\|w_h\|^2 =&(w_h, w_h) = -(w_h, \nabla \cdot (\nabla u_w)) \\
=&-(w_h, \nabla \cdot \Pi_h(\nabla u_w)) \\
=&(\mu_h, \Pi_h(\nabla u_w)) \\
=&(\mu_h, \Pi_h(\nabla u_w) - \nabla u_w) + (\boldsymbol{\mu}_h, \nabla u_w) \\
\leq&\|\mu_h\|\|\Pi_h(\nabla u_w) - \nabla u_w\| + \|\boldsymbol{\mu}_h\|\|\nabla u_w\| \\
\leq&(1 + Ch)\|\boldsymbol{\mu}_h\|\|u_w\|_{H^2(\Omega)} \\
\leq&C(1 + h)\|\boldsymbol{\mu}_h\|\|w_h\|,
\end{aligned}$$

and hence,

$$\|w_h\| \leq C(1 + h)\|\boldsymbol{\mu}_h\|,$$

which implies the equivalence between the norms $\|\boldsymbol{\mu}_h\|$ and $\{\|w_h\|^2 + \|\boldsymbol{\mu}_h\|^2\}^{\frac{1}{2}}$ over $\mathbf{Z}_h$.

Analogues to the proof of Theorem 2.2, the coerciveness of the bilinear $\mathbb{A}(\cdot,\cdot)$ over $\mathbf{Z}_h$ is derived in the following way,

$$\begin{aligned}
\mathbb{A}(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) =&(a\boldsymbol{\mu}_h, \boldsymbol{\mu}_h) + (\mathbf{b}w_h, \boldsymbol{\mu}_h) + (dw_h, w_h) \\
\geq&\alpha\|\boldsymbol{\mu}_h\|^2 - (|\mathbf{b}||w_h|, |\boldsymbol{\mu}_h|) + (dw_h, w_h) \\
=&\alpha(1-\gamma)\|\boldsymbol{\mu}_h\|^2 + \left\|\sqrt{\alpha\gamma}|\boldsymbol{\mu}_h| - \tfrac{|\mathbf{b}|}{2\sqrt{\alpha\gamma}}w_h\right\|^2 + \left\|\sqrt{d - \tfrac{|\mathbf{b}^2|}{4\alpha\gamma}}w_h\right\|^2 \\
\geq&\alpha(1-\gamma)\|\boldsymbol{\mu}_h\|^2 \\
\geq&\frac{\alpha(1-\gamma)}{2}\|\boldsymbol{\mu}_h\|^2 + \frac{\alpha(1-\gamma)}{2C(1+h)}\|w_h\|^2 \\
\geq&\min\left\{\frac{\alpha(1-\gamma)}{2}, \frac{\alpha(1-\gamma)}{2C(1+h)}\right\}\|\boldsymbol{\tau}_h\|^2.
\end{aligned}$$

Thus, the proof is completed by taking $\alpha_1 = \min\{\frac{\alpha(1-\gamma)}{2}, \frac{\alpha(1-\gamma)}{2C(1+h)}\}$. $\square$

Remark 2.4. As mentioned in the previous section, the space $\boldsymbol{\Lambda}_h$ should be carefully chosen so that it possesses as few degrees of freedom as possible, as well as preserves its optimal approximation. This can be achieved by only requiring $\{\mathbf{v}_h \in \mathbf{V}_h; \mathrm{div}\mathbf{v}_h = 0\} \subset \boldsymbol{\Lambda}_h$ in place of $\mathbf{V}_h \subset \boldsymbol{\Lambda}_h$ in next two sections. In this way, those lower-order finite element spaces $\boldsymbol{\Lambda}_h$ can be retrieved to approximate $\boldsymbol{\lambda}_h$ without losing accuracy, including the piecewise constant vector spaces.

3. New $\mathbf{\Lambda}_h$ with Minimum Degrees of Freedom for Rectangular Partition

In this section, we shall carefully analyze the structure of those divergence free vectors $\mathbf{v}_h \in \mathbf{V}_h$ for rectangular partition for $\Omega \subset \mathbb{R}^d, d = 2,3$, then define some new $\mathbf{\Lambda}_h s$ with minimum degrees of freedom in the sense that they possess the same approximate accuracy as that of the space W_h. Further, we shall show that these $\mathbf{\Lambda}_h s$ combined with the Raviart-Thomas spaces satisfy the inf-sup condition, and thus prove the solvability of the EMFEM (2.7).

3.1. Two Dimensional Case

To simplify exposition, we let $\mathcal{E}_h$ be a regular family of rectangles of $\Omega \subset \mathbb{R}^2$ with their edges parallel to x or y-axis, where $h > 0$ is a parameter representative of the diameter of the elements. For each non-negative integer $k \geq 0$, the Raviart-Thomas space of index k is given by [3, 22, 23]

$$\mathbf{V}_h = \Big\{ \mathbf{v}_h \in \mathbf{H}(\mathrm{div};\Omega);\ \mathbf{v}_h|_E \in P_{k+1,k}(E) \times P_{k,k+1}(E) \text{ for all } E \in \mathcal{E}_h \Big\}$$

and

$$W_h = \{ w_h \in L^2(\Omega); w_h|_E \in P_{k,k}(E) \text{ for all } E \in \mathcal{E}_h \}$$

with

$$P_{k_1,k_2}(E) = \Big\{ p(x,y); p(x,y) = \sum_{0\leq i\leq k_1, 0\leq j\leq k_2} a_{ij} x^i y^j \Big\},$$

the space of polynomials of degree $\leq k_1$ in x and $\leq k_2$ in y. Obviously

$$\dim \mathbf{V}_h|_E = 2(k+1)^2 + 2(k+1) = 2(k+1)(k+2).$$

We denote by $q_k(x)$ and $p_k(y)$ the polynomials of degree $\leq k$ with respect to variable x and y respectively, then any polynomials $(p_{k,k}(x,y), q_{k,k}(x,y)) \in (P_{k,k}(E))^2$ can be recast as

$$p_{k,k}(x,y) = p_k(y) + \sum_{i=0}^{k-1}\sum_{j=0}^{k} a_{i,j} x^{i+1} y^j, \qquad q_{k,k}(x,y) = q_k(x) + \sum_{i=0}^{k}\sum_{j=0}^{k-1} b_{i,j} x^i y^{j+1}. \tag{3.1}$$

Lemma 3.1. *Let $k \geq 0$ be an integer. Then, any vector $\mathbf{v}_h \in \mathbf{V}_h$ on an element E can be decomposed as*

$$\begin{aligned} \mathbf{v}_h|_E = &(p_k(y), q_k(x)) + \Big(-\sum_{i=0}^{k}\sum_{j=0}^{k} \frac{j+1}{i+1} b_{i,j} x^{i+1} y^j, \sum_{i=0}^{k}\sum_{j=0}^{k} b_{i,j} x^i y^{j+1} \Big) \\ &+ \Big(\sum_{i=0}^{k}\sum_{j=0}^{k} \{ a_{i,j} + \frac{j+1}{i+1} b_{i,j} \} x^{i+1} y^j, 0 \Big) \\ =:& \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \boldsymbol{\mu}_3, \end{aligned} \tag{3.2}$$

with

$$\mu_3 = \sum_{i=0}^{k}\sum_{j=0}^{k} \Big\{ a_{i,j} + \frac{j+1}{i+1} b_{i,j} \Big\} x^{i+1} y^j.$$

Further there hold,

$$\mathrm{div}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) = 0, \qquad \mathrm{div}\mathbf{v}_h|_E = \mathrm{div}\boldsymbol{\mu}_3 = \tfrac{\partial \mu_3}{\partial x}. \tag{3.3}$$

Proof. Recalling the definition of the space $\mathbf{V}_h$, we then rewrite $\mathbf{v}_h$ on an element E as

$$\begin{aligned}
\mathbf{v}_h|_E &= \left(p_k(y) + \sum_{i=0}^{k} x^{i+1} \sum_{j=0}^{k} a_{ij} y^j,\, q_k(x) + \sum_{i=0}^{k} x^i \sum_{j=0}^{k} b_{ij} y^{j+1}\right) \\
&= \left(p_k(y) + \sum_{i=0}^{k}\sum_{j=0}^{k} a_{i,j} x^{i+1} y^j,\, q_k(x) + \sum_{i=0}^{k}\sum_{j=0}^{k} b_{i,j} x^i y^{j+1}\right) \\
&= (p_k(y), q_k(x)) + \left(-\sum_{i=0}^{k}\sum_{j=0}^{k} \frac{j+1}{i+1} b_{i,j} x^{i+1} y^j,\, \sum_{i=0}^{k}\sum_{j=0}^{k} b_{i,j} x^i y^{j+1}\right) \\
&\quad + \left(\sum_{i=0}^{k}\sum_{j=0}^{k} \{a_{i,j} + \frac{j+1}{i+1} b_{i,j}\} x^{i+1} y^j,\, 0\right) \\
&:= \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \boldsymbol{\mu}_3.
\end{aligned}$$

It can be easily checked that the vector

$$\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 = (p_k(y), q_k(x)) + \left(-\sum_{i=0}^{k}\sum_{j=0}^{k} \frac{j+1}{i+1} b_{i,j} x^{i+1} y^j,\, \sum_{i=0}^{k}\sum_{j=0}^{k} b_{i,j} x^i y^{j+1}\right)$$

is divergence free and possesses $(k+1)(k+3)$ degrees of freedom, i.e., the first equality of (3.3) is true. By a direct differentiation we may easily verify that the vector $\boldsymbol{\mu}_3$ satisfies the second equality of (3.3) and possesses $(k+1)^2$ degrees of freedom. That completes the proof. $\square$

From this Lemma, we may draw the following corollary to characterize the divergence free vector $\mathbf{v}_h \in \mathbf{V}_h$.

Corollary 3.2. $\mathbf{v}_h \in \mathbf{V}_h$ *is divergence free if and only if*

$$a_{i,j} = -\frac{j+1}{i+1} b_{i,j}, \quad i = 0, 1, \cdots, k; j = 0, 1, \cdots, k. \tag{3.4}$$

Further, $\mathbf{v}_h$ *can be characterized as*

$$\mathbf{v}_h = (p_{k,k}, q_{k,k}) + \left(-\sum_{j=0}^{k} \frac{j+1}{k+1} b_{k,j} x^{k+1} y^j,\, -\sum_{i=0}^{k} \frac{i+1}{k+1} a_{i,k} x^i y^{k+1}\right), \tag{3.5}$$

by a pair of polynomials $p_{k,k}$ *and* $q_{k,k}$ *with their coefficients* $a_{i,j}$ *for* $i = 0, 1, \cdots, k-1; j = 0, 1, \cdots, k$ *and* $b_{i,j}$ *for* $i = 0, 1, \cdots, k; j = 0, 1, \cdots, k-1$. *In* (3.5), *only* $a_{k,k} = -b_{k,k}$ *is undetermined by the coefficients of* $p_{k,k}$ *and* $q_{k,k}$.

Proof. Noting that

$$\mathbf{v}_h = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \boldsymbol{\mu}_3, \quad \mathrm{div}\mathbf{v}_h = 0 \quad \text{and} \quad \mathrm{div}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) = 0,$$

we immediately obtain

$$\mathrm{div}\boldsymbol{\mu}_3 = \frac{\partial \mu_3}{\partial x} = \frac{\partial}{\partial x}\left(\sum_{i=0}^{k}\sum_{j=0}^{k} \{a_{i,j} + \frac{j+1}{i+1} b_{i,j}\} x^{i+1} y^j\right)$$

$$=\sum_{i=0}^{k}\sum_{j=0}^{k}\{(i+1)a_{i,j}+(j+1)b_{i,j}x^iy^j\}=0.$$

This is equivalent to

$$(i+1)a_{i,j}+(j+1)b_{i,j}=0 \quad \text{or} \quad a_{i,j}=-\frac{j+1}{i+1}b_{i,j}, \quad \text{for} \quad i,j=0,1,\cdots,k.$$

For the second conclusion (3.5), we check carefully the structure of the divergence free vector $\boldsymbol{\mu}_1+\boldsymbol{\mu}_2$ and find that

$$\begin{aligned}\boldsymbol{\mu}_1+\boldsymbol{\mu}_2=&(p_k(y),q_k(x))+\left(-\sum_{i=0}^{k}\sum_{j=0}^{k}\frac{j+1}{i+1}b_{i,j}x^{i+1}y^j,\sum_{i=0}^{k}\sum_{j=0}^{k}b_{i,j}x^iy^{j+1}\right)\\=&(p_k(y),q_k(x))+\left(-\sum_{i=0}^{k-1}\sum_{j=0}^{k}\frac{j+1}{i+1}b_{i,j}x^{i+1}y^j,\sum_{i=0}^{k}\sum_{j=0}^{k-1}b_{i,j}x^iy^{j+1}\right)\\&+\left(-\sum_{j=0}^{k}\frac{j+1}{k+1}b_{k,j}x^{k+1}y^j,\sum_{i=0}^{k}b_{i,k}x^iy^{k+1}\right).\end{aligned}\tag{3.6}$$

Applying (3.4) and noting $a_{k,k}=-b_{k,k}$, we obtain

$$\begin{aligned}\boldsymbol{\mu}_1+\boldsymbol{\mu}_2=&(p_k(y),q_k(x))+\left(\sum_{i=0}^{k-1}\sum_{j=0}^{k}a_{ij}x^{i+1}y^j,\sum_{i=0}^{k}\sum_{j=0}^{k-1}b_{i,j}x^iy^{j+1}\right)\\&+\left(-\sum_{j=0}^{k}\frac{j+1}{k+1}b_{k,j}x^{k+1}y^j,-\sum_{i=0}^{k}\frac{i+1}{k+1}a_{i,k}x^iy^{k+1}\right)\\=&(p_{k,k},q_{k,k})-\left(\sum_{j=0}^{k}\frac{j+1}{k+1}b_{k,j}x^{k+1}y^j,\sum_{i=0}^{k}\frac{i+1}{k+1}a_{i,k}x^iy^{k+1}\right).\end{aligned}$$

This implies that the divergence-free vector $\boldsymbol{\mu}_1+\boldsymbol{\mu}_2$ can be expressed as the sum of $(p_{k,k},q_{k,k})\in(P_{k,k}(E))^2$ and the vector

$$-\left(\sum_{j=0}^{k}\frac{j+1}{k+1}b_{k,j}x^{k+1}y^j,\sum_{i=0}^{k}\frac{i+1}{k+1}a_{i,k}x^iy^{k+1}\right)$$

whose $2k$ coefficients $b_{k,j}, j=0,1,\cdots,k-1$ and $a_{i,k}, i=0,1,\cdots,k-1$ are determined by those coefficients of the first summand $(p_{k,k},q_{k,k})$ and only one coefficient $a_{k,k}=-b_{k,k}$ is undetermined by $(p_{k,k},q_{k,k})$.

Consequently, for all $(p_{k,k},q_{k,k})\in(P_{k,k}(E)^2$ given by

$$p_{k,k}(x,y)=p_k(y)+\sum_{i=0}^{k-1}\sum_{j=0}^{k}a_{i,j}x^{i+1}y^j, \qquad q_{k,k}(x,y)=q_k(x)+\sum_{i=0}^{k}\sum_{j=0}^{k-1}b_{i,j}x^iy^{j+1},$$

we may define the finite element space

$$\boldsymbol{\Lambda}_h=\Bigg\{\boldsymbol{\lambda}_h\in(L^2(\Omega))^n;\boldsymbol{\lambda}_h|_E=(p_{k,k},q_{k,k})$$

$$-\left(\sum_{j=0}^{k-1}\frac{j+1}{k+1}b_{k,j}x^{k+1}y^j,\sum_{i=0}^{k-1}\frac{i+1}{k+1}a_{i,k}x^iy^{k+1}\right)+b_{k,k}x^ky^k(-x,y)\Bigg\}. \tag{3.7}$$

It is easily verified that

$$\dim\mathbf{\Lambda}_h|_E=2(k+1)^2+1.$$

Remark 3.3. The dimension of $\mathbf{\Lambda}_h|_E$ is $2(k+1)^2+1$, which is $2k+1$ degrees of freedom less than that of $\mathbf{V}_h|_E$, as well as $2k+5$ degrees of freedom less than that of $(P_{k+1,k+1}(E))^2$. Obviously, $(P_{k,k}(E))^2$ and the subspace $\{\mathbf{v}_h\in\mathbf{V}_h;\mathrm{div}\mathbf{v}_h=0\}$ of $\mathbf{V}_h|_E$ are contained in $\mathbf{\Lambda}_h|_E$.

Example 3.4. Let $k=0$. Then,

$$\mathbf{\Lambda}_h=\Big\{\boldsymbol{\mu}_h\in(L^2(\Omega))^2;\boldsymbol{\mu}_h|_E=(a_0,b_0)+b_{0,0}(-x,y),E\in\mathcal{E}_h\Big\}$$

with $\dim\mathbf{\Lambda}_h|_E=3$, which is of $2k+1=1$ freedom of degree less than that of $\mathbf{V}_h|_E$.

Let $k=1$. We write

$$p_{1,1}(x,y)=p_1(y)+x(a_{0,0}+a_{0,1}y),\qquad q_{1,1}(x,y)=q_1(x)+y(b_{0,0}+b_{1,0}x).$$

Then,

$$\mathbf{\Lambda}_h=\Big\{\boldsymbol{\mu}_h\in(L^2(\Omega))^2;\boldsymbol{\mu}_h|_E=(p_{1,1},q_{1,1})-\frac{1}{2}(b_{1,0}x^2,a_{0,1}y^2)+b_{1,1}xy(-x,y),E\in\mathcal{E}_h\Big\}$$

in which, only $b_{1,1}$ is undetermined by the coefficients of $p_{1,1}$ and $q_{1,1}$. Evidently, $\dim\mathbf{\Lambda}_h|_E=9$, which has $2k+1=3$ degrees of freedom less than that of $\mathbf{V}_h|_E$.

As expected, thus redesigned $\mathbf{\Lambda}_h$ combinning with $\mathbf{V}_h\times W_h$ satisfies the inf-sup condition, and hence ensures the solvabilty of the EMFEM.

Theorem 3.5. *Assume that $k\ge 0$ is an integer and the finite element space $\mathbf{\Lambda}_h$ is defined by* (3.7)*. Then, the mixed finite element space $\mathbf{V}_h\times\mathbf{\Lambda}_h\times W_h$ satisfies the inf-sup condition, that is, for sufficiently small $h>0$,*

$$\inf_{\mathbf{v}_h\in\mathbf{V}_h}\sup_{\boldsymbol{\tau}_h\in\mathbf{U}_h}\frac{\mathbb{B}(\boldsymbol{\tau}_h,v_h)}{\|\boldsymbol{\tau}_h\|_U\|\mathbf{v}_h\|_{H(\mathrm{div};\Omega)}}\ge\frac{1}{2\sqrt{2}}. \tag{3.8}$$

Proof. For any $\mathbf{v}_h\in\mathbf{V}_h$, we have the decomposition (3.2) on an element $E\in\mathcal{E}_h$ as Lemma 3.1 stated,

$$\mathbf{v}_h|_E=\boldsymbol{\mu}_1+\boldsymbol{\mu}_2+\boldsymbol{\mu}_3$$

with $\boldsymbol{\mu}_3=(\mu_3,0)$. For each fixed y, we let $P_h^x:L^2(E)\to P_k(E)$ denote the orthogonal projection operator in x-direction. Then, $P_h^x\mu_3$ can be written as

$$P_h^x\mu_3=\sum_{i,j=0}^{k}c_{i,j}x^iy^j\in P_{k,k}(E), \tag{3.9}$$

and further, the standard scaling argument and (3.3) ensure that the following projection estimate holds,

$$\|\mu_3-P_h^x\mu_3\|\le Ch\left\|\frac{\partial\mu_3}{\partial x}\right\|=Ch\|\mathrm{div}\mathbf{v}_h\|. \tag{3.10}$$

Noticing that $\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + (P_h^x \mu_3, 0)$ belongs to $\boldsymbol{\Lambda}_h$, hence we can take

$$\boldsymbol{\mu}_h|_E = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + (P_h^x \mu_3, 0)$$

and rewrite the decomposition of $\mathbf{v}_h|_E$ as

$$\mathbf{v}_h|_E = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \boldsymbol{\mu}_3 = \boldsymbol{\mu}_h|_E + (\mu_3 - P_h^x \mu_3, 0), \tag{3.11}$$

and thus,

$$\|\mathbf{v}_h\|_E^2 = (\boldsymbol{\mu}_h, \boldsymbol{\mu}_h)_E + 2(\boldsymbol{\mu}_h, (\mu_3 - P_h^x \mu_3, 0))_E + \|(\mu_3 - P_h^x \mu_3, 0)\|_E^2.$$

Using the Hölder inequality and the projection estimate (3.10), we obtain, for sufficiently small $h > 0$,

$$\frac{1}{2}\|\boldsymbol{\mu}_h\|_E^2 - Ch^2\|\mathrm{div}\mathbf{v}_h\|_E^2 \le \|\mathbf{v}_h\|_E^2 \le 2\|\boldsymbol{\mu}_h\|_E^2 + Ch^2\|\mathrm{div}\mathbf{v}_h\|_E^2,$$

or equivalently,

$$\begin{aligned}&\frac{1}{2}\|\boldsymbol{\mu}_h\|_E^2 + (1 - Ch^2)\|\mathrm{div}\mathbf{v}_h\|_E^2 \\ \le&\|\mathbf{v}_h\|_E^2 + \|\mathrm{div}\mathbf{v}_h\|_E^2 \le \frac{1}{2}\|\boldsymbol{\mu}_h\|_E^2 + (1 + Ch^2)\|\mathrm{div}\mathbf{v}_h\|_E^2.\end{aligned} \tag{3.12}$$

Analogously, we derive the estimate for $(\boldsymbol{\mu}_h, \mathbf{v}_h)_E$,

$$(\boldsymbol{\mu}_h, \mathbf{v}_h)_E = (\boldsymbol{\mu}_h, \boldsymbol{\mu}_h)_E + (\boldsymbol{\mu}_h, (\mu_3 - P_h^x \mu_3, 0))_E \ge \frac{1}{2}\|\boldsymbol{\mu}_h\|_E^2 - Ch^2\|\mathrm{div}\mathbf{v}_h\|_E^2. \tag{3.13}$$

Combining these estimates, then taking $w_h = -\mathrm{div}\mathbf{v}_h$ and selecting $h > 0$ small enough such that $1 - Ch^2 \ge \frac{1}{2}$, we obtain

$$\begin{aligned}&(\boldsymbol{\mu}_h, v_h)|_E - (w_h, \mathrm{div}\mathbf{v}_h)|_E = (\boldsymbol{\mu}_h, \mathbf{v}_h)_E + \|\mathrm{div}\mathbf{v}_h\|_E^2 \\ \ge&\frac{1}{2}\|\boldsymbol{\mu}_h\|_E^2 + (1 - Ch^2)\|\mathrm{div}\mathbf{v}_h\|_E^2 \\ \ge&\frac{1}{2}\{\|\mathbf{v}_h\|_E^2 + \|\mathrm{div}\mathbf{v}_h\|_E^2\} = \frac{1}{2}\|\mathbf{v}_h\|_{\mathbf{H}(\mathrm{div};E)}^2,\end{aligned} \tag{3.14}$$

and

$$\|\boldsymbol{\mu}_h\|_E^2 + \|w_h\|_E^2 = \|\boldsymbol{\mu}_h\|_E^2 + \|\mathrm{div}\mathbf{v}_h\|_E^2 \le 2\{\|\mathbf{v}_h\|_E^2 + \|\mathrm{div}\mathbf{v}_h\|_E^2\}. \tag{3.15}$$

Consequently, (3.14) and (3.15) imply that, for all $\boldsymbol{\tau}_h = (\widetilde{\boldsymbol{\mu}}_h, \widetilde{w}_h) \in \mathbf{U}_h = \boldsymbol{\Lambda}_h \times W_h$,

$$\begin{aligned}&\inf_{\mathbf{v}_h \in \mathbf{V}_h} \sup_{\boldsymbol{\tau}_h \in \mathbf{U}_h} \frac{\mathbb{B}(\boldsymbol{\tau}_h, v_h)}{\|\boldsymbol{\tau}_h\|_U \|\mathbf{v}_h\|_{H(\mathrm{div};\Omega)}} \\ =&\inf_{\mathbf{v}_h \in \mathbf{V}_h} \sup_{(\widetilde{\boldsymbol{\mu}}_h, \widetilde{w}_h) \in \boldsymbol{\Lambda}_h \times W_h} \frac{(\widetilde{\boldsymbol{\mu}}_h, \mathbf{v}_h) - (\widetilde{w}_h, \nabla \cdot \mathbf{v}_h)}{\sqrt{\|\widetilde{\boldsymbol{\mu}}_h\|^2 + \|\widetilde{w}_h\|^2}\|\mathbf{v}_h\|_{H(\mathrm{div};\Omega)}} \\ \ge&\inf_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(\boldsymbol{\mu}_h, \mathbf{v}_h) - (w_h, \nabla \cdot \mathbf{v}_h)}{\sqrt{\|\boldsymbol{\mu}_h\|^2 + \|w_h\|^2}\|\mathbf{v}_h\|_{H(\mathrm{div};\Omega)}} \\ \ge&\inf_{\mathbf{v}_h \in \mathbf{V}_h} \frac{\frac{1}{2}\|\mathbf{v}_h\|_{H(\mathrm{div};\Omega)}^2}{\sqrt{2}\|\mathbf{v}_h\|_{H(\mathrm{div};\Omega)}^2} = \frac{1}{2\sqrt{2}},\end{aligned}$$

which completes the proof. $\square$

3.2. Three dimensional Case

Analogues to the previous subsection, we let $\mathcal{E}_h$ be a rectangular partition for $\Omega \subset \mathbb{R}^3$ with its space parameter $h > 0$ and define the Raviart-Thomas space on a cuboid E [3, 23],

$$\mathbf{V}_h = \Big\{\mathbf{v}_h \in \mathbf{H}(\mathrm{div};\Omega); \mathbf{v}_h|_E \in P_{k+1,k,k}(E) \times P_{k,k+1,k}(E) \times P_{k,k,k+1}(E), E \in \mathcal{E}_h\Big\},$$
$$W_h = \Big\{w_h \in L^2(\Omega); w_h|_E \in P_{k,k,k}(E), E \in \mathcal{E}_h\Big\}$$

with $P_{k_1,k_2,k_3}(E)$ being the space of polynomials of degree $\le k_1$ in x, $\le k_2$ in y and $\le k_3$ in z. Obviously

$$\mathrm{dim}\mathbf{V}_h|_E = 3(k+1)^3 + 3(k+1)^2.$$

We denote by $p_{k,k}(y,z), q_{k,k}(z,x)$, and $r_{k,k}(x,y)$ the polynomials of degree $\le k$ in each direction with respect to two variables (x,y), (y,z) and (z,x) respectively, then recast $p_{k,k,k}$, $q_{k,k,k}$ and $r_{k,k,k}$ as

$$p_{k,k,k}(x,y,z) = p_{k,k}(y,z) + \sum_{i=0}^{k-1}\sum_{j,l=0}^{k} a_{i,j,l}x^{i+1}y^j z^l, \tag{3.16a}$$

$$q_{k,k,k}(x,y,z) = q_{k,k}(z,x) + \sum_{i,l=0}^{k}\sum_{j=0}^{k-1} b_{i,j,l}x^i y^{j+1} z^l, \tag{3.16b}$$

$$xr_{k,k,k}(x,y,z) = r_{k,k}(x,y) + \sum_{i,j=0}^{k}\sum_{l=0}^{k-1} c_{i,j,l}x^i y^j z^{l+1}. \tag{3.16c}$$

Lemma 3.6. *Let $k \ge 0$ be an integer. Then, any vector $\mathbf{v}_h \in \mathbf{V}_h$ on an element E can be decomposed as*

$$\begin{aligned}\mathbf{v}_h|_E =&(p_{k,k}(y,z), q_{k,k}(z,x), r_{k,k}(x,y))\\ &+\Bigg(-\sum_{i,j,l=0}^{k}(\frac{j+1}{i+1}b_{ijl} + \frac{l+1}{i+1}c_{ijl})x^{i+1}y^j z^l, \sum_{i,j,l=0}^{k} b_{i,j,l}x^i y^{j+1}z^l, \sum_{i,j,l=0}^{k} c_{i,j,l}x^i y^j z^{l+1}\Bigg)\\ &+\Bigg(\sum_{i,j,l=0}^{k}(a_{i,j,l} + \frac{j+1}{i+1}b_{i,j,l} + \frac{l+1}{i+1}c_{i,j,l})x^{i+1}y^j z^l, 0, 0\Bigg)\\ =:&\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \boldsymbol{\mu}_3\end{aligned} \tag{3.17}$$

with

$$\boldsymbol{\mu}_3 = (\mu_3, 0, 0) \quad \text{and} \quad \mu_3 = \sum_{i,j,l=0}^{k}\Big(a_{i,j,l} + \frac{j+1}{i+1}b_{i,j,l} + \frac{l+1}{i+1}c_{i,j,l}\Big)x^{i+1}y^j z^l.$$

Further there hold,

$$\mathrm{div}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) = 0, \quad \frac{\partial \mu_3}{\partial x} = \mathrm{div}\boldsymbol{\mu}_3 = \mathrm{div}\mathbf{v}_h|_E. \tag{3.18}$$

Proof. Recalling the definition of the space $\mathbf{V}_h$, we then recast $\mathbf{v}_h$ on an element E as

$$(p_{k,k}(y,z), q_{k,k}(z,x), r_{k,k}(x,y))$$

$$
\begin{aligned}
&+\left(\sum_{i=0}^{k}\sum_{j,l=0}^{k} a_{i,j,l}x^{i+1}y^jz^l,\ \sum_{i,j,l=0}^{k} b_{i,j,l}x^iy^{j+1}z^l,\ \sum_{i,j,l=0}^{k} c_{i,j,l}x^iy^jz^{l+1}\right)\\
=&(p_{k,k}(y,z), q_{k,k}(z,x), r_{k,k}(x,y))\\
&+\left(\sum_{i,j,l=0}^{k} a_{i,j,l}x^{i+1}y^jz^l,\ \sum_{i,j,l=0}^{k} b_{i,j,l}x^iy^{j+1}z^l,\ \sum_{i,j,l=0}^{k} c_{i,j,l}x^iy^jz^{l+1}\right)\\
=&(p_{k,k}(y,z), q_{k,k}(z,x), r_{k,k}(x,y))\\
&+\left(-\sum_{i,j,l=0}^{k}(\frac{j+1}{i+1}b_{i,j,l}+\frac{l+1}{i+1}c_{i,j,l})x^{i+1}y^jz^l,\ \sum_{i,j,l=0}^{k} b_{i,j,l}x^iy^{j+1}z^l,\ \sum_{i,j,l=0}^{k} c_{i,j,l}x^iy^jz^{l+1}\right)\\
&+\left(\sum_{i,j,l=0}^{k}(a_{i,j,l}+\frac{j+1}{i+1}b_{i,j,l}+\frac{l+1}{i+1}c_{i,j,l})x^{i+1}y^jz^l, 0, 0\right)\\
=&\boldsymbol{\mu}_1+\boldsymbol{\mu}_2+\boldsymbol{\mu}_3.
\end{aligned}
$$

It can be easily checked that the vector

$$
\begin{aligned}
\boldsymbol{\mu}_1+\boldsymbol{\mu}_2 =& (p_{k,k}(y,z), q_{k,k}(z,x), r_{k,k}(x,y))\\
&+\left(-\sum_{i,j,l=0}^{k}(\frac{j+1}{i+1}b_{i,j,l}+\frac{l+1}{i+1}c_{i,j,l})x^{i+1}y^jz^l,\ \sum_{i,j,l=0}^{k} b_{i,j,l}x^iy^{j+1}z^l,\ \sum_{i,j,l=0}^{k} c_{i,j,l}x^iy^jz^{l+1}\right)
\end{aligned}
$$

is divergence free and possesses $2(k+1)^3+3(k+1)^2$ degrees of freedom, and thus satisfies the first equality of (3.18). The vector

$$
\boldsymbol{\mu}_3=\left(\sum_{i,j,l=0}^{k}(a_{i,j,l}+\frac{j+1}{i+1}b_{i,j,l}+\frac{l+1}{i+1}c_{i,j,l})x^{i+1}y^jz^l, 0, 0\right)=(\mu_3, 0)
$$

possesses $(k+1)^3$ degrees of freedom and satisfies the second equality of (3.18). That completes the proof. $\square$

From Lemma 3.6, we may draw the following corollary to characterize the divergence free vector $\mathbf{v}_h \in \mathbf{V}_h$.

Corollary 3.7. $\mathbf{v}_h \in \mathbf{V}_h$ *is divergence free if and only if*

$$
a_{i,j,l}+\frac{j+1}{i+1}b_{i,j,l}+\frac{l+1}{i+1}c_{i,j,l}=0, \qquad i,j,k=0,1,\cdots,k. \tag{3.19}
$$

Further, $\mathbf{v}_h$ *can be characterized as*

$$
\begin{aligned}
\mathbf{v}_h|_E = (p_{k,k,k}, q_{k,k,k}, r_{k,k,k}) + \Bigg(&-\sum_{j,l=0}^{k}(\frac{j+1}{k+1}b_{k,j,l}+\frac{l+1}{k+1}c_{k,j,l})x^{k+1}y^jz^l,\\
&-\sum_{i,l=0}^{k}\frac{j+1}{k+1}(a_{i,k,l}+\frac{l+1}{i+1}c_{i,k,l})x^iy^{k+1}z^l,\ \sum_{i,j=0}^{k} c_{i,j,k}x^iy^jz^{k+1}\Bigg)
\end{aligned} \tag{3.20}
$$

by a triple $(p_{k,k,k}, q_{k,k,k}, r_{k,k,k})$ *with their coefficients* $a_{i,j,l}, i=0,1,\cdots,k-1, l,j=0,1,\cdots,k$; $b_{i,j,l}, i,l=0,1,\cdots,k, j=0,1,\cdots,k-1$ *and* $c_{i,j,l}, i,j=0,1,\cdots,k, l=0,1,\cdots,k-1$.

In (3.20)*, only those* $(k+1)^2$ *coefficients* $c_{i,j,k}$ *for* $i,j=0,1,\cdots,k$ *and one of the coefficient* $a_{k,k,k}$ *or* $b_{k,k,k}$ *satisfying* $a_{k,k,k}+b_{k,k,k}+c_{k,k,k}=0$ *are undetermined by the coefficients of* $(p_{k,k,k},q_{k,k,k},r_{k,k,k})$.

Proof. Noting that

$$\mathbf{v}_h=\boldsymbol{\mu}_1+\boldsymbol{\mu}_2+\boldsymbol{\mu}_3,\quad \mathrm{div}(\boldsymbol{\mu}_1+\boldsymbol{\mu}_2)=0\quad\text{and}\quad \mathrm{div}\mathbf{v}_h=0,$$

we immediately obtain

$$\mathrm{div}\boldsymbol{\mu}_3=\frac{\partial\mu_3}{\partial x}=\frac{\partial}{\partial x}\sum_{i,j,l=0}^{k}\left(a_{i,j,l}+\frac{j+1}{i+1}b_{i,j,l}+\frac{l+1}{i+1}c_{i,j,l}\right)x^{i+1}y^jz^l=0.$$

This is equivalent to

$$a_{i,j,l}+\frac{j+1}{i+1}b_{i,j,l}+\frac{l+1}{i+1}c_{i,j,l}=0,\quad i,j,l=0,1,\cdots,k,$$

from which the conclusion (3.19) follows directly.

For the second conclusion (3.20), we check carefully the structure of the divergence free vector $\boldsymbol{\mu}_1+\boldsymbol{\mu}_2$ and find that

$$\begin{aligned}&\boldsymbol{\mu}_1+\boldsymbol{\mu}_2\\
=&(p_{k,k}(y,z),q_{k,k}(z,x),r_{k,k}(x,y))\\
&\quad+(-\sum_{i,j,l=0}^{k}(\frac{j+1}{i+1}b_{i,j,l}+\frac{l+1}{i+1}c_{i,j,l})x^{i+1}y^jz^l,\sum_{i,j,l=0}^{k}b_{i,j,l}x^iy^{j+1}z^l,\sum_{i,j,l=0}^{k}c_{i,j,l}x^iy^jz^{l+1})\\
=&(p_{k,k}(y,z),q_{k,k}(z,x),r_{k,k}(x,y))\\
&\quad+(-\sum_{i=0}^{k-1}\sum_{j,l=0}^{k}(\frac{j+1}{i+1}b_{i,j,l}+\frac{l+1}{i+1}c_{i,j,l})x^{i+1}y^jz^l,\sum_{j=0}^{k-1}\sum_{i,l=0}^{k}b_{i,j,l}x^iy^{j+1}z^l,\sum_{l=0}^{k-1}\sum_{i,j=0}^{k}c_{i,j,l}x^iy^jz^{l+1})\\
&\quad+(-\sum_{j,l=0}^{k}(\frac{j+1}{k+1}b_{k,j,l}+\frac{l+1}{k+1}c_{k,j,l})x^{k+1}y^jz^l,\sum_{i,l=0}^{k}b_{i,k,l}x^iy^{k+1}z^l,\sum_{i,j=0}^{k}c_{i,j,k}x^iy^jz^{k+1}).
\end{aligned}\tag{3.21}$$

We then apply (3.19) to obtain

$$\begin{aligned}\boldsymbol{\mu}_1+\boldsymbol{\mu}_2=&(p_{k,k}(y,z),q_{k,k}(z,x),r_{k,k}(x,y))\\
&+\Bigg(-\sum_{i=0}^{k-1}\sum_{j,l=0}^{k}(\frac{j+1}{i+1}b_{i,j,l}+\frac{l+1}{i+1}c_{i,j,l})x^{i+1}y^jz^l,\sum_{j=0}^{k-1}\sum_{i,l=0}^{k}b_{i,j,l}x^iy^{j+1}z^l,\sum_{l=0}^{k-1}\sum_{i,j=0}^{k}c_{i,j,l}x^iy^jz^{l+1}\Bigg)\\
&+\Bigg(-\sum_{j,l=0}^{k}(\frac{j+1}{k+1}b_{k,j,l}+\frac{l+1}{k+1}c_{k,j,l})x^{k+1}y^jz^l,\\
&\quad-\sum_{i,l=0}^{k}\frac{j+1}{k+1}(a_{i,k,l}+\frac{l+1}{i+1})c_{i,k,l}x^iy^{k+1}z^l,\sum_{i,j=0}^{k}c_{i,j,k}x^iy^jz^{k+1}\Bigg).\end{aligned}$$

This implies that the divergence-free vector $\boldsymbol{\mu}_1+\boldsymbol{\mu}_2$ can be expressed as the sum of a vector

$(p_{k,k,k}, q_{k,k,k}, r_{k,k,k}) \in (P_{k,k,k}(E))^3$ with the known coefficients

$$\begin{aligned} &a_{i,j,l} \quad \text{for } i = 0, 1, \cdots, k-1 \text{ and } j, l = 0, \cdots, k; \\ &b_{i,j,l} \quad \text{for } j = 0, 1, \cdots, k-1 \text{ and } i, l = 0, \cdots, k; \\ &c_{i,j,l} \quad \text{for } l = 0, 1, \cdots, k-1 \text{ and } i, j = 0, \cdots, k \end{aligned}$$

and the vector

$$\Big(- \sum_{j,l=0}^{k} (\frac{j+1}{k+1} b_{k,j,l} + \frac{l+1}{k+1} c_{k,j,l}) x^{k+1} y^j z^l, \\ - \sum_{i,l=0}^{k} \frac{j+1}{k+1} (a_{i,k,l} + \frac{l+1}{i+1} c_{i,k,l}) x^i y^{k+1} z^l, \sum_{i,j=0}^{k} c_{i,j,k} x^i y^j z^{k+1} \Big).$$

If we leave the $(k+1)^2$ coefficients $c_{i,j,k}$ for $i, j = 0, 1 \cdots, k$ as the degrees of freedom, then the $2(k+1)^2$ coefficients of the second summand

$$\begin{aligned} &- (\frac{j+1}{k+1} b_{k,j,l} + \frac{l+1}{k+1} c_{k,j,l}), \quad j, l = 0, 1 \cdots, k \quad \text{and} \\ &- \frac{j+1}{k+1} (a_{i,k,l} + \frac{l+1}{i+1} c_{i,k,l}), \quad i, l = 0, 1, \cdots, k \end{aligned}$$

are determined by those coefficients of the first summand $(p_{k,k,k}, q_{k,k,k}, r_{k,k,k})$, except $a_{k,k,k}$ and $b_{k,k,k}$ which satisfy the constraint $a_{k,k,k} = -(b_{k,k,k} + c_{k,k,k})$. Thus, the degrees of freedom of the second summand is $(k+1)^2 + 1$. This completes the proof. □

Consequently, for all $(p_{k,k,k}, q_{k,k,k}, r_{k,k,k}) \in (P_{k,k,k}(E))^3$ given by (3.16) we may define the finite element space

$$\mathbf{\Lambda}_h = \Big\{ \boldsymbol{\lambda}_h \in (L^2(\Omega))^n; \boldsymbol{\lambda}_h|_E = (p_{k,k,k}, q_{k,k,k}, r_{k,k,k}) + \Big(- \sum_{j,l=0}^{k} (\frac{j+1}{k+1} b_{k,j,l} + \frac{l+1}{k+1} c_{k,j,l}) x^{k+1} y^j z^l, \\ - \sum_{i,l=0}^{k} \frac{j+1}{k+1} (a_{i,k,l} + \frac{l+1}{i+1} c_{i,k,l}) x^i y^{k+1} z^l, \sum_{i,j=0}^{k} c_{i,j,k} x^i y^j z^{k+1} \Big) \Big\}. \tag{3.22}$$

As stated in Corollary 3.7, only the $(k+1)^2$ coefficients $c_{i,j,k}, i, j = 0, 1, \cdots, k$ and one of $a_{k,k,k}$ or $b_{k,k,k}$ satisfying $a_{k,k,k} + b_{k,k,k} + c_{k,k,k} = 0$ are undetermined by the triple $(p_{k,k,k}, q_{k,k,k}, r_{k,k,k})$. Thus,

$$\dim \mathbf{\Lambda}_h|_E = 3(k+1)^3 + (k+1)^2 + 1,$$

which is $2(k+1)^2 - 1$ degrees of freedom less than that of $\mathbf{V}_h|_E$.

Example 3.8. Let $k = 0$. Then,

$$\mathbf{\Lambda}_h = \{ \boldsymbol{\mu}_h \in (L^2(\Omega))^3; \boldsymbol{\mu}_h|_E = (a_0, b_0, c_0) + (-(b_1 + c_1)x, b_1 y, c_1 z), E \in \mathcal{E}_h \}$$

with b_1 and c_1 undetermined. Evidently, $\dim \mathbf{\Lambda}_h|_E = 5$, which is of $2(k+1)^2 - 1 = 1$ freedom of degree less than that of $\mathbf{V}_h|_E$.

Let $k=1$. We may write

$$p_{1,1,1}(x,y,z)=p_{1,1}(y,z)+x(a_{0,0,0}+a_{0,1,0}y+a_{0,0,1}z+a_{0,1,1}yz),$$
$$q_{1,1,1}(x,y,z)=q_{1,1}(z,x)+y(b_{0,0,0}+b_{1,0,0}x+b_{0,0,1}z+b_{1,0,1}xz),$$
$$r_{1,1,1}(x,y,z)=r_{1,1}(x,y)+z(c_{0,0,0}+c_{1,0,0}x+c_{0,1,0}y+c_{1,1,0}xy),$$

Then,

$$\mathbf{\Lambda}_h=\Big\{\boldsymbol{\mu}_h\in(L^2(\Omega))^2;\boldsymbol{\mu}_h|_E=(p_{1,1,1},q_{1,1,1},r_{1,1,1})+(\mu_1,\mu_2,\mu_3),E\in\mathcal{E}_h\Big\},$$

where

$$\mu_1=-\frac{1}{2}(b_{1,0,0}+c_{1,0,0})x^2-(\frac{1}{2}b_{1,0,1}+c_{1,0,1})x^2z-(b_{1,1,0}+\frac{1}{2}c_{1,1,0})x^2y-(b_{1,1,1}+c_{1,1,1})x^2yz,$$
$$\mu_2=-\frac{1}{2}(a_{0,1,0}+c_{0,1,0})y^2-\frac{1}{2}(a_{0,1,1}+2c_{0,1,1})y^2z-\frac{1}{2}(a_{1,1,0}+\frac{1}{2}c_{1,1,0})xy^2-\frac{1}{2}(a_{1,1,1}+c_{1,1,1})xy^2z,$$
$$\mu_3==c_{0,0,1}z^2+c_{0,1,1}yz^2+c_{1,0,1}xz^2+c_{1,1,1}xyz^2.$$

From this expression, we clearly see that the 5 independent degrees of freedom are $c_{0,0,1},c_{0,1,1}$, $c_{1,0,1},c_{1,1,1}$ and one of $b_{1,1,1}$ or $a_{1,1,1}$ satisfying $a_{1,1,1}+b_{1,1,1}+c_{1,1,1}=0$. Therefore, $\dim\mathbf{\Lambda}_h|_E=3(k+1)^3+(k+1)^2+1=29$, which has $2(k+1)^2-1=7$ degrees of freedom less that that of $\mathbf{V}_h|_E$.

As expected, one can prove, analogues to the proof of Lemma 3.5 for two dimensional case, that thus selected $\mathbf{\Lambda}_h$ combinning with $\mathbf{V}_h\times W_h$ satisfies the inf-sup condition, and hence ensures the solvabilty of the EMFEM.

Theorem 3.9. *Assume that $k\geq 0$ is an integer and the finite element space $\mathbf{\Lambda}_h$ is defined by* (3.7)*. Then, the mixed finite element space $\mathbf{V}_h\times\mathbf{\Lambda}_h\times W_h$ satisfies the inf-sup condition, that is, for sufficiently small $h>0$,*

$$\inf_{\mathbf{v}_h\in\mathbf{V}_h}\sup_{\boldsymbol{\tau}_h\in\mathbf{U}_h}\frac{\mathbb{B}(\boldsymbol{\tau}_h,v_h)}{\|\boldsymbol{\tau}_h\|_U\|\mathbf{v}_h\|_{H(\mathrm{div};\Omega)}}\geq\frac{1}{2\sqrt{2}}. \tag{3.23}$$

Proof. The proof is completely analogues to the proof of Theorem 3.5, and thus omitted.□

3.3. Main Conclusions

In this subsection, we shall present the solvability of the EMFEM (2.7) or (2.8), and show that thus selected $\mathbf{\Lambda}_h$ possesses the same approximate capability as the traditional EMFEM does. For this purpose, we combine Theorem 2.3 for the coerciveness of $\mathbb{A}(\cdot,\cdot)$ over $\mathbf{Z}_h$, Theorem 3.5 and Theorem 3.9 for the inf-sup conditions of $\mathbb{B}(\cdot,\cdot)$ over $\mathbf{V}_h\times\mathbf{U}_h$, then apply the Br*é*zzi theorem (see Theorem 1 in [4], Theorem 2 in [23], Theorem 1.2 and Proposition 2.4 in [3]), to obtain the following theorem.

Theorem 3.10. *Assume that the assumptions* (2.6) *on the coefficients $a(\mathbf{x}),\mathbf{b}(\mathbf{x})$ and $\mathbf{c}(\mathbf{x})$ are valid, $(\boldsymbol{\sigma},\boldsymbol{\chi})$ with $\boldsymbol{\chi}=(\boldsymbol{\lambda},u)$ is the solution to* (2.2) *or* (2.4)*. Then, there exists a unique solution $(\boldsymbol{\sigma}_h,\boldsymbol{\chi}_h)\in\mathbf{V}_h\times\mathbf{U}_h$ with $\boldsymbol{\chi}_h=(\boldsymbol{\lambda}_h,u_h)$ to the EMFEM* (2.7) *or* (2.8)*. Further, there exists a positive constant C_1 independent of the space parameter h such that the following approximate property holds, for sufficiently small $h>0$,*

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{H}(\mathrm{div};\Omega)}+\|\boldsymbol{\chi}-\boldsymbol{\chi}_h\|\leq C_1\Big\{\inf_{\mathbf{v}_h\in\mathbf{V}_h}\|\boldsymbol{\sigma}-\mathbf{v}_h\|_{\mathbf{H}(\mathrm{div};\Omega)}+\inf_{\boldsymbol{\mu}_h\in\mathbf{U}_h}\|\boldsymbol{\chi}-\boldsymbol{\mu}_h\|\Big\}. \tag{3.24}$$

We conclude this section by two remarks, of which one concerns an explanation for minimum degrees of freedom of $\mathbf{\Lambda}_h$ and the other is related to an alternative proof for the solvability of the EMFEMs.

Remark 3.11. $\mathbf{\Lambda}_h|_E$ contains $(P_{k,k})^2$ or $(P_{k,k,k})^3$ as its subset so that it can keep the same approximate capability as $W_h|_E$ does. On the other hand, $\mathbf{\Lambda}_h|_E$ also contains the divergence-free vector space

$$\{\mathbf{v}_h \in \mathbf{V}_h; \mathrm{div}\mathbf{v}_h = 0\}$$

so that it can ensure the validity of the inf-sup condition, and thus the solvability of the EMFEM (2.7) or (2.8). In this sense, we say that thus selected $\mathbf{\Lambda}_h|_E$ possesses minimum degrees of freedom.

Remark 3.12. If we do not pursue sedulously the presence of inf-sup condition, we may give an alternative proof for the solvability of the EMFEM (2.7) as did in [9], which looks much more compact and direct, see the detailed proof of Theorem 4.2 in the next section.

4. New $\mathbf{\Lambda}_h$ with Minimum Degrees of Freedom for Triangular Partition

In this section, we shall choose $\mathbf{\Lambda}_h$ as the space of the piecewise polynomials of degree $\leq k$ for triangular partition of $\Omega \subset \mathbb{R}^d, d = 2, 3$, then prove the solvability of the EMFEMs (2.7) or (2.8) without using the Brézzi inf-sup theorem. Further, we prove, in two dimensional case, thus selected $\mathbf{\Lambda}_h$ combined with the standard Raviart-Thomas space $\mathbf{V}_h \times W_h$ still satisfies the inf-sup condition.

We let $\mathcal{E}_h$ be a regular family of triangulations of $\Omega \subset \mathbb{R}^d$ with $h > 0$ representing the diameter of the elements. For each non-negative integer k, the Raviart-Thomas space of index k is given by [3, 22, 23]

$$\begin{aligned}
&\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{H}(\mathrm{div}; \Omega); \mathbf{v}_h|_E \in (P_k(E))^d + \mathbf{x}P_k(E) \text{ for all } E \in \mathcal{E}_h\},\\
&W_h = \{w_h \in L^2(\Omega); w_h|_E \in P_k(E) \text{ for all } E \in \mathcal{E}_h\}.
\end{aligned}$$

Obviously, we have

$$\mathrm{dim}\mathbf{V}_h|_E = \begin{cases} (k+1)(k+3), & d = 2,\\ \dfrac{1}{2}(k+1)(k+2)(k+4), & d = 3. \end{cases}$$

4.1. New $\mathbf{\Lambda}_h$ and a Direct Proof for Solvability

Recalling the construction of $\mathbf{V}_h \times W_h$ in [23] for $d = 2$, [22] for $d = 3$ and Corollary 3.1 of [3], we know that

$$\{\mathbf{v}_h|_E \in \mathbf{V}_h|_E; \mathrm{div}\mathbf{v}_h|_E = 0\} \subset ((P_k(E))^d.$$

Therefore we may redesign the third finite element space $\mathbf{\Lambda}_h$ as

$$\mathbf{\Lambda}_h = \{\boldsymbol{\mu}_h \in L^2(\Omega)^d; \boldsymbol{\mu}_h|_E \in ((P_k(E))^d, \text{for } E \in \mathcal{E}_h\}. \tag{4.1}$$

Certainly, thus redesigned $\mathbf{\Lambda}_h$ contains all divergence free vectors of $\mathbf{V}_h$ as its subset, and

$$\dim \mathbf{\Lambda}_h|_E = \begin{cases} (k+1)(k+2), & d=2, \\ \dfrac{1}{2}(k+1)(k+2)(k+3), & d=3, \end{cases}$$

which has $k+1$ and $\frac{1}{2}(k+1)(k+2)$ degrees of freedom less than that of $\mathbf{V}_h|_E$ for $d=2$ and $d=3$, respectively.

Example 4.1. Let $k=0$. Then, for $d=2$,

$$\mathbf{\Lambda}_h|_E = \{(a_0, b_0); a_0, b_0 \in \mathbb{R}\}$$

and for $d=3$,

$$\mathbf{\Lambda}_h|_E = \{(a_0, b_0, c_0); a_0, b_0, c_0 \in \mathbb{R}\},$$

which have 1 degree of freedom less than those of $\mathbf{V}_h|_E$, respectively. Thus, the piecewise constant spaces are retrieved in applications.

As mentioned in Remark 3.12, we next present the solvability of the EMFEM (2.7) without applying the inf-sup theorem.

Theorem 4.2. *Assume that $k \geq 0$ and $W_h \times \mathbf{\Lambda}_h \times \mathbf{V}_h$ are defined as above. Then, there exists a unique solution $(u_h, \boldsymbol{\lambda}_h, \boldsymbol{\sigma}_h) \in W_h \times \mathbf{\Lambda}_h \times \mathbf{V}_h$ to the EMFEMs (2.7).*

Proof. It suffices to prove that there exists only zero solution to (2.7) if $\mathbf{c}=0, f=0$. For this purpose, we take $\boldsymbol{\mu}_h = \boldsymbol{\lambda}_h$ in (2.7a), $\mathbf{v}_h = \boldsymbol{\sigma}_h$ in (2.7b) and $w = u_h$ in (2.7c), then add to give,

$$(a\boldsymbol{\lambda}_h, \boldsymbol{\lambda}_h) + (du_h, u_h) + (\mathbf{b}u_h, \boldsymbol{\sigma}_h) = 0.$$

Then, applying Lemma 2.3 and the assumptions on the coefficients $a, \mathbf{b}$ and d we derive

$$0 = (a\boldsymbol{\lambda}_h, \boldsymbol{\lambda}_h) + (du_h, u_h) + (\mathbf{b}u_h, \boldsymbol{\sigma}_h) \geq C_0\{\|u_h\|^2 + \|\boldsymbol{\lambda}_h\|^2\},$$

which implies

$$u_h = 0, \qquad \boldsymbol{\lambda}_h = 0.$$

Kicking $u_h = 0$ back to (2.7c), we obtain that $\boldsymbol{\sigma}_h$ satisfies, by Corollary 3.1 in [3] and (2.7a),

$$\nabla \cdot \boldsymbol{\sigma}_h = 0, \quad \boldsymbol{\sigma}_h|_E \in (P_k(E))^d \quad \text{and} \quad (\boldsymbol{\sigma}_h, \boldsymbol{\mu}_h) = 0, \forall \boldsymbol{\mu}_h \in \mathbf{\Lambda}_h.$$

Therefore, recalling the definition of $\mathbf{\Lambda}_h$, we then can take $\boldsymbol{\mu}_h = \boldsymbol{\sigma}_h$ to force $\boldsymbol{\sigma}_h = 0$. That completes the proof. $\square$

Remark 4.3. We have obtained the solvability of the EMFEMs without the inf-sup condition and can conduct numerical analysis as in [9] to derive an optimal convergence result. However, as pointed out in [3]: the inf-sup condition is closely related to the behavior of the smallest nonzero eigenvalue. This eigenvalue is nothing but the positive constant in the inf-sup condition and must remain bounded away from zero when the dimensions of the spaces increase. The loss of the inf-sup condition may leads extra artificial (nonphysical) constraints on the boundary conditions or locking phenomenon.

Therefore, we should further discuss the validity of the inf-sup condition on the newly defined space $\mathbf{V}_h \times \mathbf{U}_h$ or $\mathbf{V}_h \times \mathbf{\Lambda}_h \times W_h$ for triangular partitions.

4.2. The Inf-Sup Condition for Two Dimensional Space

We begin this subsection by defining the space over the domain $\Omega \subset \mathbb{R}^2$,

$$S_h = \{s_h \in H^1(\Omega); s_h|_E \in P_{k+1}(E) \text{ for all } E \in \mathcal{E}_h\},$$

and the discrete gradient operator $\text{grad}_h : W_h \to \mathbf{V}_h$ by the equation [5, 13]

$$(\text{grad}_h w_h, \mathbf{v}_h) = -(w_h, \text{div}\mathbf{v}_h), \qquad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{4.2}$$

We immediately have the discrete Helmholtz decomposition [5, 13]

$$\mathbf{V}_h = \text{grad}_h W_h \oplus \text{curl} S_h, \tag{4.3}$$

where the decomposition is orthogonal with respect to both the $L^2(\Omega)\times L^2(\Omega)$ and the $\mathbf{H}(\text{div};\Omega)$ inner products.

To prove that the inf-sup condition is valid over the space $\mathbf{V}_h \times \mathbf{U}_h$, we first give an estimate for w_h bounded by $\text{grad}_h w_h$.

Lemma 4.4. *Let* $\text{grad}_h \overline{w}_h$ *be defined via the relation*

$$(\text{grad}_h \overline{w}_h, \mathbf{v}_h) = -(\overline{w}_h, \text{div}\mathbf{v}_h), \forall \mathbf{v}_h \in \mathbf{V}_h.$$

Then, there exists a positive constant C_0 *independent of* h *such that*

$$\|\overline{w}_h\| \le C_0 \|\text{grad}_h \overline{w}_h\|.$$

Proof. Letting $(\boldsymbol{\xi}_h, \eta_h) \in \mathbf{V}_h \times W_h$ be the mixed finite element solution to the equation

$$\begin{aligned} &\Delta\eta = \overline{w}_h && \text{in } \Omega, \\ &\eta = 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.4}$$

then, we have

$$\begin{aligned} &(\boldsymbol{\xi}_h, \mathbf{v}_h) = -(\eta_h, \text{div}\mathbf{v}_h), \forall \mathbf{v}_h \in \mathbf{V}_h, \\ &(\text{div}\boldsymbol{\xi}_h, w_h) = (\overline{w}_h, w_h), \forall w_h \in W_h, \end{aligned} \tag{4.5}$$

An equivalent formulation is [13]

$$\boldsymbol{\xi}_h = \text{grad}_h \eta_h, \qquad \text{div}\boldsymbol{\xi}_h = \overline{w}_h. \tag{4.6}$$

Further, using a standard stability analysis, we have

$$\|\boldsymbol{\xi}_h\|_{\mathbf{H}(\text{div};\Omega)} + \|\eta_h\| \le C_0 \|\overline{w}_h\|. \tag{4.7}$$

Take $\mathbf{v}_h = \boldsymbol{\xi}_h$ in the definition of $\text{grad}_h \overline{w}_h$ and use (4.6) and (4.7) to derive

$$\begin{aligned} &(\overline{w}_h, \overline{w}_h) = -(\text{grad}_h \overline{w}_h, \boldsymbol{\xi}_h) \\ \le& \|\text{grad}_h \overline{w}_h\| \|\boldsymbol{\xi}_h\| \le C_0 \|\text{grad}_h \overline{w}_h\| \|\overline{w}_h\| \end{aligned} \tag{4.8}$$

which implies the desired estimates. □

With the help of this lemma, we proceed with the validity of the inf-sup condition on $\mathbf{V}_h \times \mathbf{U}_h$.

Theorem 4.5. *Let $\mathbf{U}_h = \boldsymbol{\Lambda}_h \times W_h$. Then, the mixed finite element space $\mathbf{V}_h \times \mathbf{U}_h$ satisfies the inf-sup condition*

$$\inf_{\mathbf{v}_h\in\mathbf{V}_h}\sup_{\boldsymbol{\tau}_h\in\mathbf{U}_h}\frac{\mathbb{B}(\boldsymbol{\tau}_h,\mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{U}}\|\mathbf{v}_h\|_{\mathbf{V}}}\geq\alpha_2, \tag{4.9}$$

where $\alpha_2 = \frac{1}{\sqrt{\max\{2,C_0\}}}$ for the positive constant C_0 given in Lemma 4.4.

Proof. For any $\overline{\mathbf{v}}_h \in \mathbf{V}_h$ with its Helmholtz decomposition $\overline{\mathbf{v}}_h = \mathrm{grad}_h\overline{w}_h + \mathrm{curl}\overline{s}_h$, then by $\mathrm{divcurl}\overline{s}_h = 0$ and the orthogonal decomposition of $\mathbf{v}_h$, we have

$$\begin{aligned}
&\mathrm{div}\overline{\mathbf{v}}_h = \mathrm{divgrad}_h\overline{w}_h,\\
&(\mathrm{curl}\overline{s}_h,\overline{\mathbf{v}}_h) = (\mathrm{curl}\overline{s}_h,\mathrm{grad}_h\overline{w}_h + \mathrm{curl}\overline{s}_h) = (\mathrm{curl}\overline{s}_h,\mathrm{curl}\overline{s}_h)\\
&(\mathrm{grad}_h\overline{w}_h,\mathrm{grad}_h\overline{w}_h) = -(\overline{w}_h,\mathrm{divgrad}_h\overline{w}_h).
\end{aligned}$$

We take $\boldsymbol{\mu}_h = \mathrm{curl}\overline{s}_h$ and $w_h = \overline{w}_h - \mathrm{div}\overline{\mathbf{v}}_h$, then manipulate in the following way

$$\begin{aligned}
&(\boldsymbol{\mu}_h,\overline{\mathbf{v}}_h) - (w_h,\mathrm{div}\overline{\mathbf{v}}_h)\\
=&(\boldsymbol{\mu}_h,\boldsymbol{\mu}_h) - (w_h,\mathrm{div}\overline{\mathbf{v}}_h) && \text{(by the orthogonality)}\\
=&(\boldsymbol{\mu}_h,\boldsymbol{\mu}_h)+(\mathrm{grad}_h\overline{w}_h,\mathrm{grad}_h\overline{w}_h)-(\mathrm{grad}_h\overline{w}_h,\mathrm{grad}_h\overline{w}_h)-(w_h,\mathrm{div}\overline{\mathbf{v}}_h)\\
=&(\boldsymbol{\mu}_h,\boldsymbol{\mu}_h) + (\mathrm{grad}_h\overline{w}_h,\mathrm{grad}_h\overline{w}_h) + (\overline{w}_h,\mathrm{div}\overline{\mathbf{v}}_h) - (w_h,\mathrm{div}\overline{\mathbf{v}}_h) && \text{(by the definition (4.2))}\\
=&(\boldsymbol{\mu}_h,\boldsymbol{\mu}_h) + (\mathrm{grad}_h\overline{w}_h,\mathrm{grad}_h\overline{w}_h) - (w_h - \overline{w}_h,\mathrm{div}\overline{\mathbf{v}}_h)\\
=&(\boldsymbol{\mu}_h,\boldsymbol{\mu}_h) + (\mathrm{grad}_h\overline{w}_h,\mathrm{grad}_h\overline{w}_h) + (\mathrm{div}\overline{\mathbf{v}}_h,\mathrm{div}\overline{\mathbf{v}}_h)\\
=&(\overline{\mathbf{v}}_h,\overline{\mathbf{v}}_h) + (\mathrm{div}\overline{\mathbf{v}}_h,\mathrm{div}\overline{\mathbf{v}}_h) && \text{(by the orthogonality)}\\
=&\|\overline{\mathbf{v}}_h\|^2_{\mathbf{H}(\mathrm{div};\Omega)}, && \text{(the norm of } \mathbf{H}(\mathrm{div};\Omega))
\end{aligned}$$

and

$$\begin{aligned}
&(\boldsymbol{\mu}_h,\boldsymbol{\mu}_h) + (w_h,w_h)\\
=&(\boldsymbol{\mu}_h,\boldsymbol{\mu}_h) + (\overline{w}_h - \mathrm{div}\overline{\mathbf{v}}_h,\overline{w}_h - \mathrm{div}\overline{\mathbf{v}}_h)\\
\leq&2\{(\boldsymbol{\mu}_h,\boldsymbol{\mu}_h) + (\overline{w}_h,\overline{w}_h) + (\mathrm{div}\overline{\mathbf{v}}_h,\mathrm{div}\overline{\mathbf{v}}_h)\}\\
\leq&\max\{2,C_0\}\{(\boldsymbol{\mu}_h,\boldsymbol{\mu}_h) + (\mathrm{grad}_h\overline{w}_h,\mathrm{grad}_h\overline{w}_h) + (\mathrm{div}\overline{\mathbf{v}}_h,\mathrm{div}\overline{\mathbf{v}}_h)\} && \text{(by Lemma 4.4)}\\
=&\max\{2,C_0\}\|\overline{\mathbf{v}}_h\|^2_{\mathbf{H}(\mathrm{div};\Omega)} && \text{(by the orthogonality),}
\end{aligned}$$

where Lemma 4.4 is used to bound $\overline{w}_h$. Thus,

$$\begin{aligned}
&\inf_{\overline{\mathbf{v}}_h\in\mathbf{V}_h}\sup_{\boldsymbol{\tau}_h\in\mathbf{U}_h}\frac{\mathbb{B}(\boldsymbol{\tau}_h,\mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{U}}\|\overline{\mathbf{v}}_h\|_{\mathbf{H}(\mathrm{div};\Omega)}}\\
=&\inf_{\overline{\mathbf{v}}_h\in\mathbf{V}_h}\sup_{\boldsymbol{\tau}_h\in\mathbf{U}_h}\frac{(\boldsymbol{\mu}_h,\mathbf{v}_h) - (w_h,\nabla\cdot\mathbf{v}_h)}{\sqrt{\|\boldsymbol{\mu}_h\|^2 + \|w_h\|^2}\|\overline{\mathbf{v}}_h\|_{\mathbf{H}(\mathrm{div};\Omega)}}\\
\geq&\inf_{\overline{\mathbf{v}}_h\in\mathbf{V}_h}\frac{\|\overline{\mathbf{v}}_h\|^2_{\mathbf{H}(\mathrm{div};\Omega)}}{\sqrt{\max\{2,C_0\}}\|\overline{\mathbf{v}}_h\|^2_{\mathbf{H}(\mathrm{div};\Omega)}}
\end{aligned}$$

$$=\frac{1}{\sqrt{\max\{2,C_0\}}}, \tag{4.10}$$

which shows the inf-sup condition is valid over $\mathbf{V}_h \times \mathbf{U}_h$ by taking $\alpha_2 = \frac{1}{\sqrt{\max\{2,C_0\}}}$. □

Remark 4.6. If the decomposition (4.3) of $\mathbf{V}_h$ for 3-dimension case still holds, we would analogously prove the inf-sup condition as stated in Theorem 4.5.

4.3. Main Conclusions

In this subsection, we shall present the solvability of the EMFEM (2.7) or (2.8) for the triangular partitions, and show that thus redesigned $\mathbf{\Lambda}_h$ possesses the same approximate capability as the traditional EMFEMs do, by combining Theorem 2.3 and Theorem 4.5 with the Br*ézzi* theorem [3, 4, 23].

Theorem 4.7. *Let* $\Omega \subset \mathbb{R}^2$. *Under the same assumptions as in Theorem* 3.10*, the EMFEMs* (2.7) *permits a unique solution* $(\boldsymbol{\sigma}_h, \boldsymbol{\chi}_h) \in \mathbf{V}_h \times \mathbf{U}_h$ *with* $\boldsymbol{\chi}_h = (\boldsymbol{\lambda}_h, u_h)$ *satisfying the following approximate property,*

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{H}(\mathrm{div};\Omega)} + \|\boldsymbol{\chi}-\boldsymbol{\chi}_h\| \le C_1\{\inf_{\mathbf{v}_h\in\mathbf{V}_h}\|\boldsymbol{\sigma}-\mathbf{v}_h\|_{\mathbf{H}(\mathrm{div};\Omega)} + \inf_{\boldsymbol{\mu}_h\in\mathbf{U}_h}\|\boldsymbol{\chi}-\boldsymbol{\mu}_h\|\}. \tag{4.11}$$

Evidently, thus selected $\mathbf{\Lambda}_h$ can keep the same approximate capability as W_h does and possesses minimum degrees of freedom in the sense of Remark 3.11.

5. Numerical Experiments

In this section, we shall carry out three numerical examples to test the validation of the EMFEM, of which the first two are for elliptic equations of second-order in 2-D with triangular and rectangular partitions respectively and the last one is for fractional differential equation in 1-D. All the numerical experiments are implemented by Matlab R2014a on a Lenovo-PC with Intel(R) Core(TM) i7-4720HQ of 2.60GHz CPU and 8 GB RAM.

Example 5.1 (Triangular partitions). In this example we set $\overline{\Omega} = [0,1]\times[0,1], a(\mathbf{x}) = 1 + x_1^2 + x_2^2$ and $c(\mathbf{x}) = d(\mathbf{x}) = 0$. The right hand term f can be calculated by inserting above information into the governing equation.

In this example we consider two cases, i.e., pure diffusion and advection diffusion. The discrete space $\mathbf{V}_h \times W_h$ are chosen as the lowest order Raviart-Thomas mixed finite element spaces, while the space $\mathbf{\Lambda}_h \subset \left(L^2(\Omega)\right)^2$ is defined as the piecewise constant vector space with local basis function $\phi_{\mathbf{1}} = (1,0)^T$ and $\phi_{\mathbf{2}} = (0,1)^T$ in each element.

(a) Firstly we set $\mathbf{b} = [0,0]^T$. The governing equation is pure diffusion. The exact solution is given as follows

$$u(\mathbf{x}) = \sin(\pi x_1)\sin(\pi x_2).$$

The errors of $u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h$ are presented in Table 5.1 with different mesh size. We can see that the convergence rates are in agreement with the theoretical findings.

Table 5.1: Errors and convergence rates of $u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h$.

h	$\|u - u_h\|$	rate	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|$	rate	$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|$	rate
$\frac{1}{10}$	5.220E-2		3.657E-1		2.617E-1	
$\frac{1}{20}$	2.620E-2	0.9945	1.832E-1	0.9972	1.312E-1	0.9961
$\frac{1}{30}$	1.740E-2	1.0094	1.222E-1	0.9987	8.750E-2	0.9991
$\frac{1}{40}$	1.310E-2	0.9867	9.160E-2	1.0019	6.560E-2	1.0013
$\frac{1}{50}$	1.050E-2	0.9915	7.330E-2	0.9988	5.250E-2	0.9983

Table 5.2: Errors and convergence rates of $u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h$.

h	$\|u - u_h\|$	rate	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|$	rate	$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|$	rate
$\frac{1}{10}$	1.037E-1		1.4016		1.0289	
$\frac{1}{20}$	5.220E-2	0.9903	7.028E-1	0.9959	5.184E-1	0.9890
$\frac{1}{30}$	3.490E-2	0.9929	4.687E-1	0.9991	3.461E-1	0.9964
$\frac{1}{40}$	2.620E-2	0.9967	3.516E-1	0.9993	2.597E-1	0.9983
$\frac{1}{50}$	2.090E-2	1.0128	2.813E-1	0.9997	2.078E-1	0.9991

(b) Secondly we choose $\mathbf{b} = [1, 1]^T$. The governing equation is advection diffusion. The exact solutions is defined by

$$u(\mathbf{x}) = \sin(2\pi x_1) \sin(2\pi x_2)$$

The errors of $u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h$ are listed in Table 5.2 with different mesh size. It is easy to see that the convergence rates are optimal.

Example 5.2 (Rectangular partitions). In this example we set $\overline{\Omega} = [0, 1] \times [0, 1], a(\mathbf{x}) = 1 + x_1 x_2$ and $c(\mathbf{x}) = d(\mathbf{x}) = 0$. The right hand term f can be calculated by inserting above information into the governing equation.

In this example we still consider two cases, i.e., pure diffusion and advection diffusion. The discrete space $\mathbf{V}_h \times W_h$ are chosen as the lowest order Raviart-Thomas mixed finite element spaces, while the space $\boldsymbol{\Lambda}_h \subset \left(L^2(\Omega)\right)^2$ is defined as Example 2.3 in Section 2. For $k = 0$ the dimension of the local space is 3 in each element. Let $\psi_k, k = 1, 2, 3, 4$, denote the local basis functions of the vector space $\mathbf{V}_h$ in the lowest order Raviart-Thomas mixed finite element spaces. Then the local basis functions $\phi_i, i = 1, 2, 3$, of $\boldsymbol{\Lambda}_h$ are constructed as $\phi_i = \psi_i - \psi_4$.

(a) Firstly we set $\mathbf{b} = [0, 0]^T$. The governing equation is pure diffusion. The exact solution is defined below

$$u(\mathbf{x}) = x_1(1 - x_1)x_2(1 - x_2).$$

The errors of $u - u_h, \sigma - \sigma_h, \lambda - \lambda_h$ are displayed in Table 5.3 with different mesh size. We can observe that the convergence rates are in agreement with the theoretical findings.

(b) Secondly we choose $\mathbf{b} = [1, 1]^T$. The governing equation is advection diffusion. The exact solution is chosen as follows

$$u(\mathbf{x}) = \sin(3\pi x_1) \sin(3\pi x_2)$$

The errors of $u - u_h, \sigma - \sigma_h, \lambda - \lambda_h$ are listed in Table 5.4 with different mesh size. Again we can find that the convergence rates are optimal.

Table 5.3: Error of $u-u_h, \sigma-\sigma_h, \lambda-\lambda_h$.

h	$\|u-u_h\|$	rate	$\|\sigma-\sigma_h\|$	rate	$\|\lambda-\lambda_h\|$	rate
$\frac{1}{10}$	4.280E-3		1.747E-2		1.967E-2	
$\frac{1}{20}$	2.150E-3	0.9932	8.760E-3	0.9958	9.850E-3	0.9978
$\frac{1}{30}$	1.430E-3	1.0057	5.840E-3	1.0000	6.570E-3	0.9987
$\frac{1}{40}$	1.080E-3	0.9757	4.380E-3	1.0000	4.930E-3	0.9982
$\frac{1}{50}$	8.604E-04	1.0182	3.510E-3	0.9923	3.940E-3	1.0045

Table 5.4: Error of $u-u_h, \sigma-\sigma_h, \lambda-\lambda_h$.

h	$\|u-u_h\|$	rate	$\|\sigma-\sigma_h\|$	rate	$\|\lambda-\lambda_h\|$	rate
$\frac{1}{10}$	1.896E-1		2.3021		2.4831	
$\frac{1}{20}$	9.580E-2	0.9848	1.1561	0.9936	1.2727	0.9642
$\frac{1}{30}$	6.400E-2	0.9948	7.714E-1	0.9978	8.524E-1	0.9885
$\frac{1}{40}$	4.810E-2	0.9927	5.788E-1	0.9984	6.403E-1	0.9945
$\frac{1}{50}$	3.840E-2	1.0093	4.631E-1	0.9994	5.127E-1	0.9959

Example 5.3 (Fractional diffusion model). Let $\Omega=[0,1], T=1$. Consider the following fractional diffusion equation of order $2-\beta$ for $0<\beta<1$,

$$\begin{aligned}
&\frac{\partial u}{\partial t}-\frac{d}{dx}\frac{1}{\Gamma(\beta)}\int_0^x \frac{1}{(x-s)^{1-\beta}}\frac{du}{dx}(s)ds+3\frac{du}{dx}=f(x) && \text{in } \Omega\times[0,1],\\
&u(x)=0, && x=0 \text{ and } 1,\\
&u(x,0)=x^2(1-x)^2, && \text{in } \Omega. \qquad (5.1)
\end{aligned}$$

The exact solution is prescribed to be $u(x,t)=x^2(1-x)^2e^t$, and therefore $\boldsymbol{\lambda},\boldsymbol{\sigma}$ and the source term f can be calculated accordingly from the governing equation,

$$\begin{aligned}
\boldsymbol{\lambda}(x,t)&=(2x-6x^2+4x^3)e^t,\\
\boldsymbol{\sigma}(x,t)&=-\Big\{24\tfrac{\theta x^{3+\beta}-(1-\theta)(1-x)^{3+\beta}}{\Gamma(4+\beta)}-12\tfrac{\theta x^{2+\beta}-(1-\theta)(1-x)^{2+\beta}}{\Gamma(3+\beta)}+2\tfrac{\theta x^{\beta+1}-(1-\theta)(1-x)^{\beta+1}}{\Gamma(2+\beta)}\Big\}e^t\\
&\quad+bx^2(1-x)^2e^t,\\
f(x,t)&=-\Big\{24\tfrac{\theta x^{\beta+2}+(1-\theta)(1-x)^{\beta+2}}{\Gamma(\beta+3)}-12\tfrac{\theta x^{\beta+1}+(1-\theta)(1-x)^{\beta+1}}{\Gamma(\beta+2)}+2\tfrac{\theta x^{\beta}+(1-\theta)(1-x)^{\beta}}{\Gamma(\beta+1)}\Big\}e^t\\
&\quad+b(2x-6x^2+4x^3)e^t+x^2(1-x)^2e^t.
\end{aligned}$$

In this example, we use the lowest Raviart-Thomas space $W_h\times\mathbf{V}_h$ with the space index $k=0$, that is, $\mathbf{V}_h$ is the piecewise linear polynomial space as well as W_h is the piecewise constant space. $\boldsymbol{\Lambda}_h$ is taken to be the piecewise constant space. We partition $[0,T]$ uniformly with time step $\tau=h$ and use the backward Euler scheme to discrete the temporal derivative. The spacial errors and convergence rates for $\beta=\frac{1}{3},\frac{2}{3}$ at $t=T=1$ are presented in Table 5.5. The numerical results in Table 5.5 show that the convergence rates for the unknown u, the spacial derivative $\boldsymbol{\lambda}$ and the fractional flux $\boldsymbol{\sigma}$ at $t=T=1$ is 1, which are in agreement with the theoretical findings predicted by Theorem 4.7.

Table 5.5: Spatial errors and convergence rates at $t = T = 1$.

β	h	$\|u - u_h\|$	rate	$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|$	rate	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H^1(\Omega)}$	rate
$\frac{1}{3}$	2^{-4}	6.6255E-03		2.3632E-02		1.7993E-01	
	2^{-5}	3.2268E-03	1.038	1.1523E-02	1.036	9.2358E-02	0.962
	2^{-6}	1.6001E-03	1.012	5.6926E-03	1.017	4.7994E-02	0.944
	2^{-7}	7.9816E-04	1.003	2.8294E-03	1.009	2.5285E-02	0.925
	2^{-8}	3.9881E-04	1.001	1.4107E-03	1.004	1.3492E-02	0.906
	2^{-9}	1.9937E-04	1.000	7.0440E-04	1.002	7.2783E-03	0.890
$\frac{2}{3}$	2^{-4}	6.4674E-03		2.3478E-02		1.2406E-01	
	2^{-5}	3.1863E-03	1.021	1.1704E-02	1.004	6.2922E-02	0.979
	2^{-6}	1.5865E-03	1.006	5.8584E-03	0.998	3.1710E-02	0.988
	2^{-7}	7.9237E-04	1.002	2.9343E-03	0.997	1.5937E-02	0.993
	2^{-8}	3.9607E-04	1.000	1.4695E-03	0.998	7.9982E-03	0.995
	2^{-9}	1.9802E-04	1.000	7.3575E-04	0.998	4.0105E-03	0.996

6. Concluding Remarks

We propose the new finite element space $\boldsymbol{\Lambda}_h$ so that it, as coupled with the Raviart-Thomas spaces, can preserve the same approximate capability as the traditional mixed methods do as well as ensure the solvability of the EMFEM for linear elliptic problems of second-order. The strong requirement $\mathbf{V}_h \subset \boldsymbol{\Lambda}_h$ in [9] is weakened and the new $\boldsymbol{\Lambda}_h$ possesses minimum degrees of freedom. The redesigned $\boldsymbol{\Lambda}_h$ combined with the Raviart-Thomas mixed space satisfies the coerciveness condition and the inf-sup condition. We also prove the existence, uniqueness and the same approximate capability of the EMFEM solution as the traditional mixed methods [9,10], by an application of the coerciveness and the inf-sup condition for rectangular partitions of $\Omega \subset R^d, d = 2,3$ and triangular partitions of $\Omega \subset R^2$. In this way, the EMFEM in [9] is generalized to those finite element spaces $\boldsymbol{\Lambda}_h$ with lower indices, such as the commonly used piecewise constant space, to reduce computation cost and improve the stability of the EMFEM obviously.

This approach can be extended to other mixed finite element spaces, for examples, the BDM elements and the BDDF elements [2,3].

Although the solvability for triangular partitions, even for general partitions in $\mathbb{R}^d, d = 2,3$, can be proved by a direct proof, the proof based on the inf-sup condition for triangular partitions in $\mathbb{R}^3$ is still left open due to the lack of knowledge on the orthogonal decomposition of the space $\mathbf{V}_h$.

Acknowledgments. The work of Z. Zhou was supported by NSF of China grant 11971276; H. Chen was supported by NSF of China grants 12171287, 10971254 and 11471196; H. Wang was supported by the ARO MURI Grant W911NF-15-1-0562, and by the National Science Foundation under Grant DMS-2012291.

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