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Homogenization of oblique boundary value problems

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Abstract: We consider a nonlinear Neumann problem, with periodic oscillation in the elliptic operator and on the boundary condition. Our focus is on problems posed in half-spaces, but with general normal directions that may not be parallel to the directions of periodicity. As the frequency of the oscillation grows, quantitative homogenization results are derived. When the homogenized operator is rotation-invariant, we prove the Hölder continuity of the homogenized boundary data. While we follow the outline of Choi and Kim (*Homogenization for nonlinear PDEs in general domains with oscillatory Neumann boundary data*, Journal de Mathématiques Pures et Appliquées **102** (2014), no. 2, 419–448), new challenges arise due to the presence of tangential derivatives on the boundary condition in our problem. In addition, we improve and optimize the rate of convergence within our approach. Our results appear to be new even for the linear oblique problem.

Keywords: homogenization, elliptic operator, oblique boundary problem

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1 Introduction

For given $\varepsilon > 0$, $\nu \in \mathcal{S}^{n-1}$ and $\tau \in \mathbb{R}^n$, let u_ε be a bounded solution of the following problem:

$$\begin{cases} F\left(D^2 u_\varepsilon, \dfrac{x}{\varepsilon}\right) = 0 & \text{in } \Pi := \{x \in \mathbb{R}^n : -1 < (x-\tau)\cdot \nu < 0\} \\ u_\varepsilon = h(x) & \text{on } H_{-1} := \{(x-\tau)\cdot \nu = -1\} \\ \partial_\nu u_\varepsilon = G\left(Du_\varepsilon, \dfrac{x}{\varepsilon}\right) & \text{on } H_0 := \{(x-\tau)\cdot \nu = 0\}. \end{cases} \qquad (P)_\varepsilon$$

Here, $F(M, y)$ and $G(p, y)$ are $\mathbb{Z}^n$-periodic in the y variable. We also assume the boundary condition to be *oblique* and F to be uniformly elliptic: see Section 1.1 for precise assumptions on F and G.

The examples of boundary conditions we consider include the linear oblique problem:

$$\overrightarrow{\gamma}\left(\frac{x}{\varepsilon}\right)\cdot Du + g\left(\frac{x}{\varepsilon}\right) = 0, \qquad (1)$$

where the vector field $\overrightarrow{\gamma}$ satisfies $c\left(\frac{x}{\varepsilon}, \nu\right) := \overrightarrow{\gamma}\left(\frac{x}{\varepsilon}\right)\cdot \nu > 0$. In this case, one can write

$$G(p, y) = (c(y))^{-1}[\overrightarrow{\gamma}(y)\cdot p_T + g(y)],$$

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where $p_T := p - (p \cdot \nu)$ is the tangential component of $p \in \mathbb{R}^n$ on H_0. A nonlinear example is *capillarity-type* conditions, for which G is given by

$$G(p, y) = \theta(y)\sqrt{1 + |p|^2}\,, \tag{2}$$

where $|\theta(x)| < 1$.

We are interested in the behavior of u_ε as ε tends to zero. Our objective is to extend the results of [7,8], to establish a general framework to understand nonlinear elliptic problems with oscillatory Neumann boundary data. In particular, we have tried to carefully detail the double-scale averaging argument given in Section 5, which has been central in understanding continuity properties of the homogenized boundary condition in both Neumann and Dirichlet boundary problems: see [7,8,12,13]. We focus on problems posed on half-spaces here. To deal with domains with general geometry, the approach taken in [7] or [13] uses fundamental solutions as barriers to bound the potential singularity generated at points with rational normals. For our problem, while our result is likely to hold in general domains, we suspect that these singular solutions may cause new challenges in dealing with perturbative arguments, due to their singularity in tangential derivatives.

Note that, as first pointed out by Bensoussan et al. [5], if ν is a multiple of a vector in $\mathbb{Z}^n$ (i.e., if ν is *rational*), then $\tau \cdot \nu$ must be zero for u^ε to converge, since otherwise the Neumann boundary condition changes drastically as ε changes, and thus, u_ε would not have a limit. When ν is irrational, we expect u_ε to average due to the ergodic property of its Neumann data. However, in this case, u^ε is no longer periodic, and thus, interesting challenges arise in dealing with the inherent lack of compactness. Compared to [7] where the linear Neumann problem was considered, there is an additional challenge in our setting given by the presence of tangential derivatives on the boundary condition. We will discuss some of the relevant literature on this issue.

Let us state a convergence result on $(P)_\varepsilon$ to begin the discussion. Let $\bar{F}$ be the homogenized operator of F obtained by Evans [11].

Theorem 1.1. *Let ν be irrational, or otherwise suppose that $\tau = 0$. Let us assume* (F1)–(F3) *and* (G1)–(G3) (*see Section* 1.1). *In addition, suppose that $F(\cdot, x)$ is convex when $G(\cdot, x)$ is nonlinear. Then there exists $\mu(\eta, q) : \mathcal{S}^{n-1} \times \mathbb{R}^{n-1} \to \mathbb{R}$, where μ is independent of τ, such that u_ε converges uniformly to the unique bounded solution $\bar{u}$ of the oblique boundary problem*:

$$\begin{cases} \bar{F}(D^2\bar{u}) = 0 & \text{in} \quad \Pi \\ \bar{u} = h(x) & \text{on} \quad H_{-1} \\ \partial_\nu \bar{u} = \mu(\nu, D_T\bar{u}) & \text{on} \quad H_0. \end{cases} \tag{$\bar{P}$}$$

(*here, $D_T u$ denotes the tangential derivative of u along the direction $\nu^\perp$.*) *Moreover, μ is Lipschitz continuous with respect to q. Finally, if $\bar{F}(M)$ is rotation-invariant, then μ is also Hölder continuous over irrational directions ν with exponent $\alpha = \frac{1}{5n}$.*

The proof of Theorem 1.1 will be given later in this section, based on our main result (Theorem 1.2), which establishes rates of convergence for (approximate) cell problem solutions. Our work extends the previous results in [8] on linear Neumann problems, where $G(p, y) = G(y)$. For general, $G(p, y)$ additional challenges arise due to the presence of tangential derivatives on the boundary condition, which necessitates Lipschitz regularity estimates for the solutions. As noted in [13], the continuity property of $\mu(\nu, q)$ fails when $\bar{F}$ is not rotation-invariant, even when it is convex. When the continuity result holds for μ we expect to be able to address domains of general geometry, building on our result and proceeding as in [7].

It is unknown whether the form of the boundary condition such as (1) or (2) is preserved in the limit $\varepsilon \to 0$. With the exception of linear problems, the interaction between the operator F and the boundary condition remains to be better understood to yield further characterizations of the homogenized problems.

Literature. Before proceeding further, let us briefly describe some of the relevant literature. In the classical article in [5], the following problem was considered:

$$-\nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon \right) = 0 \text{ in } \Omega, \quad \nu \cdot \left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon \right)(x) = g\left(\frac{x}{\varepsilon}\right) \text{ on } \partial\Omega. \tag{3}$$

For this co-normal boundary value problem, explicit integral formulas have been derived for the limiting operator as well as for the limiting boundary data, under the assumption that $\partial\Omega$ does not contain any flat piece with a rational normal.

For linear elliptic systems with either Dirichlet or Neumann problem with co-normal derivatives, there has been a recent surge of development in quantitative homogenization relying on the integral representation of solutions: we refer to [2,15,20] and the references therein.

For nonlinear problems, or even for linear problems with non co-normal boundary data, until recently, the focus has been on half-space type domains with rational normal, with the origin on the boundary. In [21], Tanaka considered some model problems in half-space whose boundary is parallel to the axes of the periodicity by purely probabilistic methods. In [1], Arisawa studied specific problems in oscillatory domains near half spaces going through the origin. Generalizing the results of Arisawa [1] for nonlinear boundary conditions, Barles et al. [4] studied the problem for operators with oscillating coefficients, in half-space type domains whose boundary is parallel to the axes of periodicity. We also refer to [14], which adopts an integro-differential approach to study linear scalar problems with the specific Neumann problem $G(p, y) = g(y)$.

For the linear Neumann problem $G(p, y) = g(y)$ in $(P)_\varepsilon$, corresponding results to Theorems 1.1 and 1.2 have been recently shown in [8]. General domains has been considered in [7] based on the cell problem analysis in [8]. Corresponding results for the Dirichlet boundary data have been obtained in [12]. Finally, for general operator F, [13] discusses the generic nature of discontinuity for the homogenized boundary data, for either linear Neumann or Dirichlet problem.

Cell problem. By the formal expansion $u_\varepsilon = \bar{u}(x) + \varepsilon v\left(x, \frac{x}{\varepsilon}\right) + O(\varepsilon^2)$, the cell problem for v was derived in [4] for a rational ν and $\tau = 0$. There they find a unique constant $\mu = \mu(\nu, q)$ for $q \in \langle \nu \rangle^\perp$ such that the boundary value problem

$$\begin{cases} F(D^2 v, y) = 0 & \text{in } \{y \cdot \nu \geq 0\}, \\ \mu = G(Dv + p, y) & \text{on } H_0, \end{cases} \tag{C}$$

with $p = \mu\nu + q$, has a bounded periodic solution v in $\{y \cdot \nu \geq 0\}$. The existence of bounded v leads to the uniform convergence of u_ε to $\bar{u}$ in the limit $\varepsilon \to 0$ with $p = D\bar{u}$ on H_0.

For general ν and τ, an approximate cell problem needs to be derived, since v is no longer expected to be periodic and thus compactness is lost. In the context of (C), our result shows that for irrational ν, there exists a unique constant $\mu = \mu(\nu, q)$ for $q \in \langle \nu \rangle^\perp$ such that the problem

$$\begin{cases} F(D^2 v, y + \tau) = 0 & \text{in } \{y \cdot \nu \geq 0\}, \\ \mu = G(Dv + p, y + \tau) & \text{on } H_0 \end{cases} \tag{$\tilde{C}$}$$

has a solution with sublinear growth at infinity, for any $\tau \in \mathbb{R}^n$. To show this, we use the ergodicity of Neumann data in a scale depending on ν, and the stability of solutions under perturbation of boundary conditions. When the homogenized operator $\bar{F}$ is rotation-invariant, we show that ν is stable as the normal direction of the domain ν varies. A quantitative version of this stability property yields the mode of continuity for μ as ν varies.

A discussion on assumptions on F and G. Our assumptions on F and G are mainly to obtain Lipschitz estimates for the solutions of $(\tilde{C})$. The Lipschitz estimates ensure that the solution of the cell problem has the ergodic structure with respect to translations along the Neumann boundary (see Lemma 3.5), when ε changes in $(P)_\varepsilon$ and when τ is not the origin. In particular to guarantee the Lipschitz bound, available literature restricts $F(M, x)$ to be convex with respect to M when G is a nonlinear function of Du. We refer to [3] for a detailed description of the regularity theory on nonlinear Neumann boundary problems. For the continuity properties of μ, we further need $C^{1,\alpha}$ estimates for solutions of $(\tilde{C})$; however, this does not further restrict the class of problems we can address.

1.1 Assumptions and main results

Let $\mathbb{T}$ be the 1-periodic torus in $\mathbb{R}^n$, and let $\mathcal{M}^n$ be the space of real $n \times n$ symmetric matrices. Consider the functions $F(M, y) : \mathcal{M}^n \times \mathbb{T} \to \mathbb{R}$ and $G(p, y) : \mathbb{R}^n \times \mathbb{T}$ satisfying the following properties:

(F1) (Uniform Ellipticity) There exist constants $0 < \lambda < \Lambda$ such that

$$\lambda \mathrm{Tr}(N) \leq F(M, y) - F(M + N, y) \leq \Lambda \mathrm{Tr}(N)$$

for all $y \in \mathbb{T}$ and $M, N \in \mathcal{M}^n$ with $N \geq 0$.

(F2) (1-Homogeneity) $F(tM, y) = tF(M, y)$ for all $y \in \mathbb{T}$, $t > 0$ and $M \in \mathcal{M}^n$.

(F3) (Lipschitz continuity) There exists $C > 0$ such that for all $y_1, y_2 \in \mathbb{T}$ and $M, N \in \mathcal{M}^n$,

$$|F(M, y_1) - F(N, y_2)| \leq C(|y_1 - y_2|(1 + \|M\| + \|N\|) + \|M - N\|).$$

(G1) (At most linear growth) $|G(p, x)| \leq \mu_0(1 + |p|)$.

(G2) (Lipschitz continuity) $(1 + |p|)|G_p|, |G_y| \leq m(1 + |p|)$ for some $m > 0$.

(G3) (Oblicity) $|G_p \cdot \nu| \leq c < 1$.

A typical example of an operator F satisfying (F1)–(F3) is the linear elliptic operator

$$F(D^2u, x) = -\Sigma_{i,j} a_{ij}(x) \partial_{x_i x_j} u, \tag{4}$$

where $a_{ij} : \mathbb{R}^n \to \mathbb{R}$ is periodic and Lipschitz continuous. A nonlinear example is the Bellman-Isaacs operator arising from stochastic optimal control and differential games

$$F(D^2u, x) = \inf_{\beta \in B} \sup_{\alpha \in A} \{\mathcal{L}^{\alpha,\beta} u\}, \tag{5}$$

where $\mathcal{L}^{\alpha,\beta}$ is a family of uniformly elliptic operators of the form (4). In fact, all operators satisfying (F1)–(F3) can be written as (5). As for G, the ones given in (1) and (2) with Lipschitz coefficients $c^{-1}\vec{\gamma}$, $c^{-1}g$ and θ satisfy (G1)–(G3).

For $\tau \in \mathbb{R}^n$ and $\nu \in \mathcal{S}^{n-1}$, let us define a strip domain

$$\Pi(\tau, \nu) := \{x \in \mathbb{R}^n : -1 \leq (x - \tau) \cdot \nu \leq 0\}$$

and a hyperplane

$$H_s(\tau, \nu) := \{(x - \tau) \cdot \nu = s\}.$$

We will denote $H_s(\tau, \nu)$ by H_s throughout the article when it is unambiguous. For a given $q \in \langle \nu \rangle^\perp$, let u_ε solve the following approximate cell problem:

$$\begin{cases} F\left(D^2 u_\varepsilon, \dfrac{x}{\varepsilon}\right) = 0 & \text{in } \Pi(\tau, \nu) \\ \partial_\nu u_\varepsilon = G(Du_\varepsilon, \dfrac{x}{\varepsilon}) & \text{on } H_0 \\ u_\varepsilon(x) = q \cdot x & \text{on } H_{-1} \end{cases} \qquad (P)_{\varepsilon,\nu,\tau,q}.$$

Now we are ready to state the main result.

Theorem 1.2. *Let u_ε solve $(P)_{\varepsilon,\nu,\tau,q}$. Suppose that either ν is irrational or $\tau = 0$. Then the following holds:*

(*a*) *There exists $\mu = \mu(\nu, q)$ such that u_ε converges uniformly to the linear profile*

$$u(x) := \mu((x - \tau) \cdot \nu + 1) + q \cdot x.$$

Here, $\mu(\nu, q)$ is independent of τ and Lipschitz continuous with respect to q. Moreover, we have

$$|u_\varepsilon - u| \leq C\Lambda(\varepsilon, \nu) \quad \text{in } \Pi(\tau, \nu), \tag{6}$$

where $\Lambda(\varepsilon, \nu)$ (as given in (23)) is an increasing function of ε such that $\lim_{\varepsilon \to 0} \Lambda(\varepsilon, \nu) = 0$.

(*b*) *When $\bar{F}$ is rotation-invariant, there exists a continuous extension $\bar{\mu}(\nu, q) : \mathcal{S}^{n-1} \times \mathbb{R}^n \to \mathbb{R}$ of $\mu(\nu, q)$ over irrational directions $\nu \in \mathcal{S}^{n-1} - \mathbb{R}\mathbb{Z}^n$. Moreover, $\bar{\mu}$ is Lipschitz in q and C^α in ν, with $\alpha = \frac{1}{5n}$.*

The proof is given in Theorems 4.1, 5.2, and 5.1.

A discussion on the rate of convergence $\Lambda(\varepsilon, \nu)$. Here, we briefly describe the geometric process used in Section 4 to obtain an upper bound for the rate function Λ in (6). Given $\delta > 0$, we are interested in finding $\varepsilon_0 = \varepsilon_0(\nu, \delta)$ such that $|u_\varepsilon - u| \le C\delta \quad$ for $\varepsilon \le \varepsilon_0$.

If ν is rational and $\tau = 0$, F and G are periodic along ν-direction with period T_ν. Hence, we expect that ε_0 needs to be smaller than $1/T_\nu$ for a fixed δ. In fact, Theorem 4.1 (d) yields that

$$\Lambda(\varepsilon, \nu) \le \delta \quad \text{for } \varepsilon \le \varepsilon_0 = \delta^2/T_\nu$$

and thus yields a uniform bound

$$\Lambda(\varepsilon, \nu) \le C(\nu)\varepsilon^{1/2}. \tag{7}$$

If ν is irrational, for each δ, we choose a reference rational direction P as follows: choose a point $P = P(\nu, \delta) \in \mathbb{Z}^n$ such that

$$|T\nu - P| \le \delta \quad \text{for some} \quad T = T(\nu, \delta) > 0. \tag{8}$$

Then F and G are periodic along P-direction with period $T + O(\delta)$. If we let $\theta = \theta(\nu, \delta)$ be the angle between ν and P, then (8) can be written as $\theta < \delta/T$. If $R < 1/\theta$, then due to the proximity of ν to P direction, $G(p, \cdot)$ takes only limited values of G on $H_0 \cap B_R(\tau)$, even though ν is irrational. In other words, $G(p, \cdot)$ exhibits ergodicity on H_0 only in a neighborhood of size $R > 1/\theta$. For this reason, u_ε homogenizes only when $\varepsilon \le O(\theta)$. Indeed Theorem 4.1 (c) yields that

$$\Lambda(\varepsilon, \nu) \le \delta \quad \text{for } \varepsilon \le \varepsilon_0 = \delta^2\theta.$$

Since θ depends on not only ν but also δ, we are not able to separate the dependence of the rate function on ε and ν, without further estimate of θ or T as δ varies. Such estimate would require better understanding of the *discrepancy* function discussed in in [7], [8] and [12].

Proof of Theorem 1.1. Once Theorem 1.2 (a) is obtained, one can derive our main theorem by the *perturbed test function* arguments introduced by Evans [10].

Let u_ε solve $(P)_\varepsilon$ and define u^* and u_* as follows:

$$u^* = \text{limsup}^* u_\varepsilon := \lim_{r\to 0} \sup_{(y,\varepsilon)\in S_r^x} u_\varepsilon(y); \qquad u_* = \text{liminf}_* u_\varepsilon := \lim_{r\to 0} \inf_{(y,\varepsilon)\in S_r^x} u_\varepsilon(y),$$

where $S_r^x = \{(y, \varepsilon) : y \in \Pi, |x - y| < r, 0 < \varepsilon < r\}$. First, observe that, by using a barrier of the form

$$\varphi_M(x) := M((x - \tau)\cdot \nu + 1) + f(x),$$

where f is a C^2-approximation of h that is larger than h, one can conclude that $u_\varepsilon \le \varphi_M$ in Π for any large M, and thus, $u^* \le h$ on H_{-1}. Similar arguments yield that $u_* \ge h$ on H_{-1}.

We claim that u^* and u_* are, respectively, a viscosity subsolution and supersolution of (P). If the claim is true, then Corollary 3.4 applies to yield that $u^* \le u_*$. Since the opposite inequality is true from the definition, we conclude that $u^* = u_*$, which means that u_ε uniformly converges in $\bar{\Omega}$.

Below we will only show that u^* is a subsolution of (P), since the proof for u_* can be shown by parallel arguments. To this end, suppose that $u^* - \phi$ has a local max in $B_r(y_0) \cap \bar{\Pi}$ with a smooth test function ϕ. If y_0 is in the interior of Π, then $\bar{F}(D^2\phi)(y_0) \le 0$ due to standard interior homogenization (see, for instance, [10]). Hence, it remains to show that if y_0 is on the Neumann boundary, then ϕ satisfies

$$\partial_\nu \phi \le \mu(\nu, q := D_T\phi) \quad \text{at} \quad x = y_0. \tag{9}$$

First, suppose that ν is rational and $y_0 \cdot \nu = 0$. We may assume for simplicity that $u(y_0) = \phi(y_0) = 0$ and define $P(x) := D\phi(y_0)\cdot(x - y_0)$. Since $\Pi \subset \{x : x \cdot \nu < 0\}$, for any $\delta > 0$, we may choose r sufficiently small that $l_\delta(x) := P(x) - \delta(x\cdot\nu)$ is strictly larger than u^* on $B_r(0)\cap\Pi$. Then for sufficiently small choice of ε, we have

$$l_\delta > u_\varepsilon \quad \text{on } B_r(0)\cap H_{-r\delta}, \quad \text{where} \quad H_{-r\delta} = \{x\cdot\nu = -r\delta\}. \tag{10}$$

Let $\bar\varepsilon := (r\delta)^{-1}\varepsilon$ and consider the re-scaled function $v_\varepsilon(x) := (r\delta)^{-1}u_\varepsilon(r\delta x) - l_\delta(x)$. Then v_ε is a subsolution of $(P)_{\bar\varepsilon,\nu,0,q}$, in the local domain $\Pi\cap B_{\delta^{-1}}(0)$. Note that the corresponding Neumann boundary for v_ε remains to be H_0 since $y_0\cdot\nu = 0$: in general, it will be $\{(x-\tau)\cdot\nu = 0\}$ with

$$\tau = (\bar\varepsilon)^{-1}y_0, \tag{11}$$

and thus, the choice of τ must change as we vary $\bar\varepsilon$. We will compare v_ε with $w_{\bar\varepsilon}$, the unique bounded solution of $(P)_{\bar\varepsilon,\nu,0,q}$ in Π obtained in Lemma 3.3. Due to the localization lemma (Lemma 3.2), we have

$$v_\varepsilon \le w_{\bar\varepsilon} + M\delta \quad \text{in } \Pi\cap B_1(0). \tag{12}$$

Due to Theorem 1.2, we have

$$w_{\bar\varepsilon} \le \mu(\nu,q)(x\cdot\nu+1) + q\cdot x + \Lambda(\bar\varepsilon,\nu) \quad \text{in } \Pi.$$

Since $\Lambda(\varepsilon,\nu)\to 0$ as $\varepsilon\to 0$, (10) and (12) yield that

$$\limsup_{\varepsilon\to 0}\frac{u_\varepsilon(r\delta x)}{r\delta} \le \limsup_{\varepsilon\to 0} v_\varepsilon(x) + l_\delta(-\nu) \le \mu(\nu,q)(x\cdot\nu+1) + q\cdot x + l_\delta(-\nu) + M\delta \quad \text{in } \Pi\cap B_1(0). \tag{13}$$

Now suppose that (9) is false, then there exists $\delta > 0$ such that

$$\partial_\nu\phi(0) = \delta - l_\delta(-\nu) > \mu(\nu,q) + (M+1)\delta. \tag{14}$$

This means that the right-hand side of (13) is strictly negative at $x = 0$, which contradicts the assumption that $u^*(0) = 0$.

Next suppose that ν is irrational, we need to choose τ depending on $\bar\varepsilon$ so that (11) holds. Then we argue as earlier with a solution of $(P)_{\bar\varepsilon,\nu,\tau,q}$ in Π. Here, we must use the fact that ν is irrational, and thus, Theorem 1.2 ensures the uniform convergence of $w_{\bar\varepsilon}$ to the linear profile is regardless of the choice of τ. □

2 Preliminaries

We adopt the following definition of viscosity solutions, which is equivalent to the one given in [9]. Let Ω be domain in $\mathbb{R}^n$ with $\partial\Omega$ as a disjoint union of Γ_0 and Γ_1. Let F satisfy (F1)–(F3) in the previous section, and let G satisfy (G3) with $G(p,x)$ being uniformly continuous in p independent of the choice of x. For $f\in C(\Gamma_0)$, consider the following problem:

$$\begin{cases} F(D^2u,x) = 0 & \text{in } \Omega \\ u = f(x) & \text{on } \Gamma_0 \\ \dfrac{\partial}{\partial\nu}u = G(Du,x) & \text{on } \Gamma_1, \end{cases} \tag{P}$$

where $\nu = \nu(x)$ is the outward unit normal at $x\in\Gamma_1$. Here, we replace (G3) with

(G3)′ (Oblicity) $|G_p\cdot\nu| \le c < 1$ on $\partial\Omega$, where $\nu = \nu_x$ is the outward normal at $x\in\partial\Omega$.

Definition 2.1.

(a) An upper semi-continuous function $u:\bar\Omega\to\mathbb{R}$ is a viscosity subsolution of (P) if u cannot cross from below any C^2 function ϕ, which satisfies

$$\begin{cases} F(D^2\phi,x) > 0 \quad \text{in } \Omega, \quad \phi > f \quad \text{on } \Gamma_0, \\ \nu\cdot D\phi > G(D\phi,x) \quad \text{on } \Gamma_1. \end{cases}$$

(b) A lower semi-continuous function $u : \bar{\Omega} \to \mathbb{R}$ is a viscosity supersolution of (P) if u cannot cross from above any C^2 function φ, which satisfies

$$\begin{cases} F(D^2\phi, x) < 0 \quad \text{in } \Omega, \quad \phi < f \quad \text{on } \Gamma_0, \\ \nu \cdot D\phi < G(D\phi, x) \quad \text{on } \Gamma_1. \end{cases}$$

(c) u is a viscosity solution of (P) if its upper semi-continuous envelope u^* is a viscosity subsolution and its lower semi-continuous envelope u_* is a viscosity supersolution of (P).

Existence and uniqueness of viscosity solutions of (P) are based on the comparison principle we state later. We refer to [9,16] for details on the proof of the following theorem as well as the well-posedness of the problem (P).

Theorem 2.2. *Let G and F satisfy the conditions* (G1) *and* (G3)′ *and* (F1)–(F3) *in the previous section, with G being uniformly continuous in p independent of the choice of x. Let u and v be, respectively, bounded viscosity subsolution and supersolution of* (P) *in a bounded domain* Ω. *Then* $u \le v$ *in* Ω.

For a symmetric $n \times n$ matrix M, we decompose $M = M_+ - M_-$ with $M_\pm \ge 0$ and $M_+M_- = 0$. We define the Pucci operators as follows:

$$\mathcal{P}^+(M) = -\Lambda \mathrm{tr}(M_+) + \lambda \mathrm{tr}(M_-)$$

and

$$\mathcal{P}^-(M) = -\lambda \mathrm{tr}(M_+) + \Lambda \mathrm{tr}(M_-)$$

where $0 < \lambda < \Lambda$. Later this article, we will utilize the fact that the difference of two solutions of $F(D^2u, x) = 0$ is both a subsolution of $\mathcal{P}^+(D^2u) \le 0$ and a supersolution of $\mathcal{P}^-(D^2(u)) \ge 0$ (see [6]).

Next we state some regularity results that will be used throughout this article.

Theorem 2.3. [Chapter 8, [6], modified for our setting] *Let u be a viscosity solution of* $F(D^2u, x) = 0$ *in a domain* Ω. *Then for any compact subset* Ω' *of* Ω, *we have*

$$\|Du\|_{L^\infty(\Omega')} \le Cd^{-1}\|u\|_{L^\infty(\Omega)},$$

where $d = d(\Omega', \partial\Omega)$ *and* $C > 0$ *depends on* n, λ, *and* Λ.

As mentioned in Section 1, regularity results for nonlinear Neumann problems are rather limited. $C^{0,\alpha}$ estimates have been obtained by Barles and Da Lio in the general framework [3]. While a priori results for the gradient bounds are available for general F and G in [19], their results are based on linearization and thus require existence of classical solutions. For $G(p, x)$ that is linear in p, regularity estimates on Du were recently obtained by Li and Zhang [18].

Theorem 2.4. [18,19] *Let u be a viscosity solution of* (P) *with* $|u| \le M$.

$$B_r^+ := \{|x| < r\} \cap \{x \cdot e_n \ge 0\} \quad \text{and} \quad \Gamma := \{x \cdot e_n = 0\} \cap B_1.$$

Let u be a viscosity solution of

$$\begin{cases} F(D^2u, x) = 0 & \text{in } B_1^+ \\ \nu \cdot Du = G(Du, x) & \text{on } \Gamma. \end{cases}$$

For F and G satisfying (F1)–(F3) *and* (G1)–(G3), *suppose that either* (A) $F(M, x)$ *is convex with respect to M, or (B)* $G(p, x)$ *is linear with respect to p. Then for any* $0 < \alpha < 1$, *we have*

$$\|u\|_{C^{0,\alpha}(B_{1/2}^+)}, \|Du\|_{C^{0,\alpha}(B_{1/2}^+)} \le C, \tag{15}$$

where C depends on α *and M as well as the constants given in* (F1)–(F3) *and* (G1)–(G3).

Our proof extends in general to the cases where estimate (15) holds for some $\alpha > 0$.

Below we mention interior homogenization result from [7], which is a modified version of homogenization results such as in [11].

Theorem 2.5. (Theorem 2.14, [7]) *Let K be a positive constant and let $f : \mathbb{R}^n \to \mathbb{R}$ be bounded and Hölder continuous. Given $\nu \in \mathcal{S}^{n-1}$, let $u_N : \{-K \le x \cdot \nu \le 0\} \to \mathbb{R}$ be the unique bounded viscosity solution of*

$$\begin{cases} F(D^2u_N, Nx) = 0 \quad \text{in } \{-K \le x \cdot \nu \le 0\}; \\ \nu \cdot Du_N = f(x) \quad \text{on} \quad \{x \cdot \nu = 0\}, \quad u = 1 \quad \text{on} \quad \{x \cdot \nu = -K\}. \end{cases} \tag{P_N}$$

Then for any $\delta > 0$, there exists N_0 depending only on K, the bound of u_N, and the Hölder exponent of f, such that

$$|u_N - \bar{u}| \le \delta \quad \text{in } \{|x| \le K\} \quad \text{for } N \ge N_0, \tag{16}$$

where $\bar{u}$ is the unique bounded viscosity solution of

$$\begin{cases} \bar{F}(D^2\bar{u}) = 0 \quad \text{in } \{-K \le x \cdot \nu \le 0\}; \\ \nu \cdot D\bar{u} = f(x) \quad \text{on} \quad \{x \cdot \nu = 0\}, \quad u = 1 \quad \text{on} \quad \{x \cdot \nu = -K\}. \end{cases}$$

Next we state some consequences of ergodic property of irrational numbers in $\mathbb{R}$ mod $\mathbb{Z}$. First, we state a version of Dirichlet's approximation theorem, whose proof is based on the pigeon-hole principle.

Lemma 2.6. [Lemma 2.11 in [13]] *For $\alpha_1,\dots,\alpha_n \in \mathbb{R}$ and $N \in \mathbb{N}$, there are integers $p_1,\dots,p_n$, $q \in \mathbb{Z}$ with $1 \le q \le N$ such that*

$$|q\alpha_i - p_i| \le N^{-1/n}.$$

Finally, we present a lemma that states ergodic property of hyperplanes with irrational normals in $\mathbb{R}^n$ mod $\mathbb{Z}^n$.

Lemma 2.7. [Lemma 2.7 in [8], Lemma 2.3 in [12]]. *For $\nu \in S^{n-1}$ and $x_0 \in \mathbb{R}^n$, let $H(x_0) := \{x \in R^n : (x - x_0) \cdot \nu = 0\}$. Then the following holds*:

(*a*) *Suppose that ν is a rational direction. Then for any $x \in H(x_0)$, there is $y \in H(x_0)$, such that*

$$|x - y| \le T_\nu; \quad y = x_0 \bmod \mathbb{Z}^n,$$

where T_ν is the smallest positive number such that $T_\nu \nu \in \mathbb{Z}^n$.

(*b*) *Suppose that ν is an irrational direction, and let $\omega_\nu : \mathbb{N} \to \mathbb{R}^+$ be defined as in* (2.2) *of* [12]. *Then there exists a dimensional constant $C = C(n) > 0$ such that the following is true: for any $x \in H(x_0)$ and $N \in \mathbb{N}$, there is $y \in \mathbb{R}^n$ such that*

$$|x - y| \le C(n)N; \quad y = x_0 \bmod \varepsilon\mathbb{Z}^n$$

and

$$\operatorname{dist}(y, H_{-d}) < \omega_\nu(N).$$

We recall that $\omega_\nu(N)$ converges to 0 as $N \to \infty$.

(*c*) *If ν is an irrational direction, then for any $z \in \mathbb{R}^n$ and $\delta > 0$, there is $w \in H(x_0)$ such that*

$$|z - w| \le \delta \bmod \mathbb{Z}^n.$$

3 Localization lemmas

In this section, we prove several lemmas on perturbing and localizing the solutions, which will be used frequently throughout the article. Below we prove a localization lemma, and as a corollary, we prove existence and uniqueness of solution u_ε of $(P)_{\varepsilon,\nu,\tau,q}$ with $\Pi = \Pi(\nu, \tau)$ for $\tau \in \mathbb{R}^n$ and $\nu \in \mathcal{S}^{n-1}$. Denote $B_R(\tau) := \{|x - \tau| \le R\}$ and recall $H_s := \{(x - \tau)\cdot \nu = s\}$.

First, we state a basic lemma, which will be frequently used. The proof is a direct consequence of the oblicity assumption $(G3)$.

Lemma 3.1. *There exists $M = M(|q|, c)$, such that $q \cdot x \pm Mx \cdot \nu$ are, respectively, super and subsolution of $(P)_{\varepsilon,\nu,\tau,q}$.*

Lemma 3.2. *Let $f \in C(\mathbb{R}^n)$ be bounded. Suppose w_1 and w_2 solve, in the viscosity sense,*

(a) $F\left(D^2 w_i, \frac{x}{\varepsilon}\right) = 0$ *in* $\Sigma_R := \Pi \cap B_R(0)$ *for* $i = 1, 2$

(b) $\nu \cdot Dw_i = G\left(Dw_i, \frac{x}{\varepsilon}\right)$ *on* H_0 *for* $i = 1, 2$

(c) $w_1 = w_2$ *on* H_{-1}

(d) $0 \le w_2 - w_1 \le M$ *on* $\Pi \cap \partial B_R(0)$.

Let $L := \|G_p\|_\infty$ and $0 < c < 1$ is the constant given in (G3). *Then there exists a constant $C\left(\frac{\Lambda}{\lambda}, c, L\right) > 0$, such that*

$$w_1 \le w_2 \le w_1 + \frac{CM}{(1-c)R} \quad \text{in } \Pi \cap B_1(0).$$

Proof. Without loss of generality, let us set $\nu = e_n$ and $\tau = 0$. The first inequality, $w_1 \le w_2$, directly follows from Theorem 2.2. To show the second inequality, let

$$w := w_1 + M(h_1 + h_2) + C_1 h_3,$$

where

$$h_1(x) = \frac{|x|^2}{R^2}, \quad h_2(x) = \frac{C}{R^2}(1 - (x_n)^2) \quad \text{with} \quad C = \frac{n\Lambda}{\lambda}, \quad h_3(x) = \frac{1 + x_n}{R},$$

and $C_1 > 0$ is a large constant depending on n, Λ, λ, L, and c, which will be chosen below in the proof.

Note that in Σ_R,

$$F\left(D^2 w, \frac{x}{\varepsilon}\right) = F\left(D^2 w_1 + M(D^2 h_1 + D^2 h_2), \frac{x}{\varepsilon}\right) \ge F\left(D^2 w_1, \frac{x}{\varepsilon}\right) - \mathcal{P}^+(M(D^2 h_1 + D^2 h_2)) = F\left(D^2 w_1, \frac{x}{\varepsilon}\right) = 0.$$

Also $w_2 = w_1 \le w$ on H_{-1} and $w_2 \le w_1 + M \le w$ on $\partial B_R(0) \cap \Pi$.

Hence, to show that $w_2 \le w$, it is enough to show that $\partial_{x_n} w \ge G\left(Dw, \frac{x}{\varepsilon}\right)$ on H_0. We will verify that this is true when C_1 is sufficiently large. Observe that in Σ_R,

$$|D(h_1 + h_2)| \le \frac{C_0}{R} \quad \text{for } C_0 = C_0(n, \Lambda, \lambda). \tag{17}$$

Hence, on $H_0 \cap \Sigma_R$, we have

$$\begin{aligned}
\partial_{x_n} w &\ge \partial_{x_n} w_1 + \frac{C_1}{R} - \frac{C_0}{R} \\
&= G\left(Dw_1, \frac{x}{\varepsilon}\right) + \frac{(C_1 - C_0)}{R} \\
&\ge G\left(Dw, \frac{x}{\varepsilon}\right) - \frac{cC_1}{R} + \frac{C_0 L}{R} + \frac{(C_1 - C_0)}{R},
\end{aligned}$$

where the last inequality follows from the Lipschitz property of G with (17), if $C_1 = C_1(n, \Lambda, \lambda, c)$ is chosen sufficiently large. It follows from Theorem 2.2 that $w_2 \le w$ in Σ_R, and we obtain the lemma. □

As a corollary of Lemma 3.2, we prove existence and uniqueness of solutions in strip regions.

Lemma 3.3. *There exists a unique solution u_ε of $(P)_{\varepsilon,\nu,\tau,q}$ with the property $\|u_\varepsilon(x) - q\cdot x\|_{L^\infty(\Pi)} < \infty$, such that*

$$\|u_\varepsilon - q\cdot x\| \le M.$$

Proof.

1. Let Σ_R be as given in Lemma 3.2, and consider the viscosity solution $w_R(x)$ of $(P)_{\varepsilon,\nu,\tau,q}$ in Σ_R with the lateral boundary data $q\cdot x$ on $\partial B_R(\tau)\cap\Pi$. The existence and uniqueness of the viscosity solution w_R is shown, for example, in [9, 16].

   From Lemma 3.1, $q\cdot x \pm M(x - \tau + \nu)\cdot \nu$ is a sub- and supersolution of $(P)_{\varepsilon,\nu,\tau,q}$, and thus, by comparison principle, we obtain that

   $$|w_R(x) - q\cdot x| \le M \quad \text{for } x\in\Sigma_R.$$

   Due to Theorem 2.5 and the Arzela-Ascoli Theorem, w_R locally uniformly converges to a continuous function $u_\varepsilon(x)$. From the stability property of viscosity solutions, it follows that $u_\varepsilon(x)$ is a viscosity solution of $(P)_{\varepsilon,\nu,\tau,q}$.
2. To show uniqueness, suppose both u_1 and u_2 are viscosity solutions of $(P)_{\varepsilon,\nu,\tau,q}$ with $|u_1 - q\cdot x|$, $|u_2 - q\cdot x| \le M$. Then Lemma 3.2 yields that, for any point $s\in H_0$,

   $$|u_1 - u_2| \le O(1/R) \quad \text{in} \quad B_1(s)\cap\Pi.$$

   Hence, $u_1 = u_2$. □

The following is immediate from Theorem 2.2 and the construction of u_ε in the aforementioned lemma.

Corollary 3.4. *Suppose u,v are bounded and continuous functions in $\overline{\Pi} = \overline{\Pi(\tau,\nu)}$. In addition, suppose they satisfy, for F satisfying* (F1)–(F3) *and G satisfying* (G1)–(G2),

(a) $F\left(D^2u, \frac{x}{\varepsilon}\right) \le 0 \le F\left(D^2v, \frac{x}{\varepsilon}\right)$ *in* Π;
(b) $u \le v$ *on* H_{-1};
(c) $\nu\cdot Du \le G(Du, x/\varepsilon)$; $\nu\cdot Dv \ge G(Dv, x/\varepsilon)$ *on* H_0.
Then $u \le v$ *in* Π.

Lemma 3.5. *There exists $C > 0$ such that the following holds: let u_i for $i = 1, 2$ solve*

$$\begin{cases} F(D^2u_i) = 0 & \text{in} \quad \Pi\cap B_R(0)\\ \partial_\nu u_i = G_i(Du_i, x) & \text{on} \quad H_0\cap B_R(0)\\ u_i = q\cdot x & \text{on} \quad H_{-1}\cap B_R(0),\end{cases}$$

where $\Pi = \Pi(\nu, 0)$. Furthermore, suppose that G_i satisfies the assumption in Theorem 2.4 *and G_1 and G_2 satisfy*

$$|G_1(p,x) - G_2(p,x)| \le \delta(1+|p|) \quad \text{and} \quad |u_1 - u_2| \le M. \tag{18}$$

Let L denote the Lipschitz bound for u_i and G's. Then there exists $C = C(\Lambda,\lambda,n)$, such that

$$|u_1 - u_2| \le \delta(L+1) + CM/R \quad \text{in } \Pi\cap B_1(0).$$

Proof. By our assumption, $v := (u_1 - u_2)/M$ satisfies $|v| \le 1$ in $B_R(0)$ with

$$\begin{cases} \mathcal{P}^+(D^2v) \le 0 & \text{in } \Pi\cap B_R(0)\\ v = 0 & \text{on } H_{-1}\cap B_R(0).\end{cases}$$

After a change of coordinates, we may assume $\nu = e_n$ so that $\Pi = \{x : -1 \le x_n \le 0\}$, and we denote $x = (x', x_n)$. Define

$$w(x) := (c_0/M + c_1/R)(x_n + 1) + 2\left(|x'|^2 - \frac{\Lambda n}{\lambda}(|x_n|^2 - 1)\right)/R^2,$$

where c_0 and $c_1 > 8$ will be chosen later. Then w is a supersolution of the aforementioned problem with the Neumann boundary condition:

$$\partial_n w = (c_0/M + c_1/R) \ge (c_0/M + 4|x'|/R^2) = (c_0/M + |D_T w|) \quad \text{on } \{x_n = 0\} \cap B_R(0).$$

Now suppose $v - w$ has positive maximum in $\Pi \cap \overline{B_R}(0)$. Then the maximum would need to be achieved at a point $\tau \in H_0 \cap B_R(0)$. At this point, we should have $\partial_n(v - w) \ge 0$ and $D_T v = D_T w$. Therefore,

$$\partial_n v \ge \partial_n w \ge (c_0/M + |D_T w|) = (c_0/M + |D_T v|). \quad \text{at} \quad x = \tau, \tag{19}$$

On the other hand,

$$G_1(Du_1, x) - G_2(Du_2, x) = DG_1(p^*, x)\cdot D(Mv) + G_1(Du_2, x) - G_2(Du_2, x),$$

and since $|DG_1(p^*, x)\cdot e_n| \le c$, we have, from (18) and the Lipschitz bound for u_i given in Theorem 2.4,

$$(1 - c)\partial_n v \le L|D_T v| + \frac{1}{M}|G_1(Du_2, x) - G_2(Du_2, x)| \le L|D_T v| + \frac{\delta}{M}(L + 1) \quad \text{at} \quad x = \tau.$$

Then using the fact that $|D_T w| = 4|x'|/R^2 \le 4/R$ in $B_R(0)$, it follows that

$$(1 - c)|\partial_n v| \le \frac{4L}{R} + \frac{\delta(L + 1)}{M}. \tag{20}$$

Hence, from (19), we obtain a contradiction if $c_0/M + c_1/R$ is larger than the right-hand side of (31). This happens if we choose $c_1 > 4L$ and $c_0 = \delta(L + 1)$. Therefore, it follows that $v \le w$ in $\Pi \cap \overline{B_R}$. We can now conclude that

$$u_1 - u_2 = Mv \le c_0 + c_1 M/R + 2M\left(1 + \frac{\Lambda n}{\lambda}\right)/R^2 \quad \text{in } \Pi \cap B_1(0).$$

The lower bound can be obtained with the aforementioned argument applied to $u_2 - u_1$. □

4 Homogenization in a strip domain

Let u_ε solve $(P)_{\varepsilon,\nu,\tau,q}$ with linear boundary data $l(x)$ on H_{-1}. We let v_ε be the unique linear function on Π such that v_ε coincides with u_ε on H_{-1} and at a reference point $\tau - \nu/2$. More precisely,

$$v_\varepsilon(x) = \mu_\varepsilon((x - \tau)\cdot \nu + 1) + l(x), \tag{21}$$

where $\mu_\varepsilon = 2(u_\varepsilon(\tau - \nu/2) - u_\varepsilon(\tau - \nu))$. Then we define the average slope $\mu(u_\varepsilon)$ of u as follows:

$$\mu(u_\varepsilon) := \partial_\nu v_\varepsilon = \mu_\varepsilon. \tag{22}$$

Theorem 4.1. *The followings hold for u_ε solving $(P)_{\varepsilon,\nu,\tau,q}$:*

(*a*) *For irrational directions ν, there exists a unique constant $\mu = \mu(\nu, q)$, such that u_ε converges uniformly to the linear profile*

$$u(x) := \mu((x - \tau)\cdot \nu + 1) + l(x),$$

where $l(x) := q \cdot x$. The same holds for rational directions ν with $\tau = 0$.

(*b*) [*Error estimate*] *There exists a constant $C > 0$ depending on λ, Λ, n, and the slope of $l(x)$ such that the following holds: if ν is an irrational direction or ν is a rational direction with $\tau = 0$, then*

$$|\mu(u_\varepsilon) - \mu| \le C\Lambda(\varepsilon, \nu) \quad \text{in } \Pi,$$

where

$$\Lambda(\varepsilon,\nu) = \begin{cases} \inf\limits_{0<k<1} \{\varepsilon^k T_\nu + \varepsilon^{1-k}\} & \text{if } \nu \text{ is a rational direction} \\ \inf\limits_{0<k<1, N\in\mathbb{N}} \{\varepsilon^k N + \omega_\nu(N) + \varepsilon^{1-k}\} & \text{if } \nu \text{ is an irrational direction.} \end{cases} \tag{23}$$

In (23), T_ν *and* ω_ν *are as given in* Lemma 2.7. T_ν *is the period of* $G(P, y)$ *on the Neumann boundary* H_0 *and* $\omega_\nu(N) \to 0$ *as* $N \to \infty$.

(*c*) *Let* ν *be an irrational direction. For any* $\delta > 0$, *there exist* $T > 0$ *and* $P \in \mathbb{Z}^n$ *such that*

$$|T\nu - P| \le \delta.$$

Let $\theta = \theta(\delta, \nu)$ *be the angle between* ν *and* P, *then*

$$\Lambda(\varepsilon, \nu) \le 3\delta \quad \text{for } \varepsilon < \delta^2\theta.$$

(*d*) *Let* ν *be a rational direction, and let* $\delta > 0$. *Then*

$$\Lambda(\varepsilon, \nu) \le 2\delta \quad \text{for } \varepsilon < \frac{\delta^2}{T_\nu}.$$

To prove Theorem 4.1 we begin with a preliminary lemma. The following lemma states that u_ε looks like a linear profile (almost flat) on each hyperplane normal to ν.

Lemma 4.2. *Away from the Neumann boundary* H_0 *and* $u_\varepsilon - l(x)$ *is almost a constant on hyperplanes parallel to* H_0. *More precisely, for* $x_0 \in \Pi$, *we denote*

$$d := \operatorname{dist}(x_0, H_0) > 0$$

and $H_{-d} := \{(x - \tau)\cdot \nu = -d\} = \{(x - x_0)\cdot \nu = 0\}$. *Then the following holds*:

(*a*) *If* ν *is a rational direction, there exists a constant* $C > 0$ *depending on* α, λ, Λ, n, *and the slope of* l, *such that for any* $x \in H_{-d}$,

$$|(u_\varepsilon(x) - l(x)) - (u_\varepsilon(x_0) - l(x_0))| \le C(d^{-1} + 1)(T_\nu\varepsilon), \tag{24}$$

where T_ν *is a constant depending on* ν, *given as in* (a) *of* Lemma 2.7.

(*b*) *If* ν *is an irrational direction, there exists a constant* $C > 0$ *depending on* α, λ, Λ, n, *and the slope of* l, *such that for any* $x \in H_{-d}$,

$$|(u_\varepsilon(x) - l(x)) - (u_\varepsilon(x_0) - l(x_0))| \le C(d^{-1}\varepsilon\omega_\nu(N) + \omega_\nu(N)), \tag{25}$$

for any $N \in \mathbb{N}$ *and* $\varepsilon > 0$ *with* $\varepsilon\omega_\nu(N) < 1$, *where* $\omega_\nu(N)$ *is given as in* Lemma 2.7.

Proof. First, we consider a rational direction ν. By (a) of Lemma 2.7, for any $x \in H_{-d}$, there is $y \in H_{-d}$ such that $|x - y| \le T_\nu\varepsilon$ and $y = x_0 \mod \varepsilon\mathbb{Z}^n$. Then by comparison,

$$u_\varepsilon(x) = u_\varepsilon(x + (y - x_0)) - l(y) + l(x_0). \tag{26}$$

Hence, $u_\varepsilon(x_0) = u_\varepsilon(y) - l(y) + l(x_0)$, and we obtain

$$\begin{aligned} |(u_\varepsilon(x) - l(x)) - (u_\varepsilon(x_0) - l(x_0))| &\le |u_\varepsilon(x) - u_\varepsilon(y)| + |l(y) - l(x)| \\ &\le |u_\varepsilon(x) - u_\varepsilon(y)| + CT_\nu\varepsilon \\ &\le Cd^{-1}T_\nu\varepsilon + CT_\nu\varepsilon, \end{aligned}$$

where the third inequality follows from Theorem 2.3.

Next, we consider an irrational direction ν and let $x \in H_{-d}$. By (b) of Lemma 2.7, for any $N \in \mathbb{N}$, there exists $y \in \mathbb{R}^n$ such that $|x - y| \le \varepsilon\omega_\nu(N)$, $y = x_0 \mod \varepsilon\mathbb{Z}^n$ and

$$\operatorname{dist}(y, H_{-d}) < \varepsilon\omega_\nu(N). \tag{27}$$

Observe that

$$|(u_\varepsilon(x) - l(x)) - (u_\varepsilon(x_0) - l(x_0))| \le |u_\varepsilon(x) - u_\varepsilon(y)| + |(u_\varepsilon(y) - l(y)) - (u_\varepsilon(x_0) - l(x_0))| + |l(y) - l(x)|,$$

where, from Theorem 2.3,

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le Cd^{-1}\varepsilon\omega_\nu(N).$$

Next we project y to $x_1 \in H_{-d}$ and use Lemma 3.5 for $G_1 = G$ and $G_2(p, x) = G(p, x + (x_0 - x_1)) = G(p, x + (y - x_1))$ with $\delta = \omega_\nu(N)$ to conclude that

$$|(u_\varepsilon(x_0) - l(x_0)) - (u_\varepsilon(x_1) - l(x_1))| \le C\omega_\nu(N).$$

Then by using Theorem 2.3 with (27) once again, we compare $u(y)$ with $u(x_1)$ and conclude that

$$|(u_\varepsilon(y) - l(y)) - (u_\varepsilon(x_0) - l(x_0))| \le C(\omega_\nu(N) + \varepsilon).$$

Finally,

$$|l(y) - l(x)| \le C|y - x| \le C\varepsilon\omega_\nu(N) \le Cd^{-1}\varepsilon\omega_\nu(N),$$

where the last inequality follows since $|y - x| \le \varepsilon\omega_\nu(N)$ and $d \le 1$. □

Since u_ε is flat on each hyperplanes located, a constant d-away from the Neumann boundary, u_ε can be approximated well by a linear solution as in the following corollary. The proof of Corollary 4.3 follows from the comparison principle (Theorem 2.2) and Lemma 4.2 with $d = \varepsilon^{1-k}$.

Corollary 4.3. *For a solution u_ε of $(P)_{\varepsilon,\nu,\tau,q}$, let v_ε be the unique linear function given as in* (21). *Then there exists a constant C depending on λ, Λ, n, and the slope of l such that for any $N \in \mathbb{N}$ and* $0 < k < 1$,

$$|u_\varepsilon(x) - v_\varepsilon(x)| \le \begin{cases} C(\varepsilon^k T_\nu + \varepsilon^{1-k}) & \text{if } \nu \text{ is a rational direction} \\ C(\varepsilon^k N + \omega_\nu(N) + \varepsilon^{1-k}) & \text{if } \nu \text{ is an irrational direction,} \end{cases}$$

and hence,

$$|u_\varepsilon(x) - v_\varepsilon(x)| \le C\Lambda(\varepsilon, \nu).$$

Due to the uniform interior regularity of $\{u_\varepsilon\}$ (Theorem 2.3), along a subsequence, they locally uniformly converges to u in Π. Let us choose one of the convergent subsequence u_{ε_j} and denote it by u_j, i.e., $u_j = u_{\varepsilon_j}$. Let $v_j = v_{\varepsilon_j}$ and $\mu_j = \mu(u_{\varepsilon_j})$, both as given in (21) and (22). Corollary 4.3 implies that for any $\nu \in \mathcal{S}^{n-1}$, $\lim u_j$ is linear. More precisely, the slope μ_j converges as $j \to \infty$ (see Lemma 4.1 of [8]), and hence, by Corollary 4.3,

$$\lim u_j = \lim v_j = \mu((x - \tau)\cdot \nu + 1) + l(x) = u$$

for $\mu := \lim \mu_j$.

Next, we prove that the subsequential limit is unique, i.e., μ does not depend on the subsequence $\{\varepsilon_j\}$, when ν is irrational or ν is rational with $\tau = 0$. We will also obtain a mode of convergence of μ_ε.

Proof of Theorem 4.1(**a**) **and** (**b**) **for irrational directions:** Let ν be an irrational direction and let u be a subsequential limit of u_ε. We claim that

$$\partial u / \partial \nu = \mu(\nu, q)$$

for a constant $\mu(\nu, q)$, which depends on ν and q, not on τ or the subsequence $\{\varepsilon_j\}$. More precisely,

$$|\mu(u_\eta) - \mu(u_\varepsilon)| \le C(\Lambda(\varepsilon, \nu) + \eta). \tag{28}$$

where we let $0 < \eta < \varepsilon$ be sufficiently small.

For the proof of (28), let

$$w_\varepsilon(x) = \frac{u_\varepsilon(\varepsilon x)}{\varepsilon}, \quad w_\eta(x) = \frac{u_\eta(\eta x)}{\eta}$$

and denote by H^1 and H^2, the Neumann boundaries of w_ε and w_η, respectively. By (c) of Lemma 2.7, for $\tau \in \mathbb{R}^n$, there exist $s_1 \in H^1$ and $s_2 \in H^2$, such that

$$|\tau - s_1| \le \eta \mod \mathbb{Z}^n, \quad \text{and} \quad |\tau - s_2| \le \eta \mod \mathbb{Z}^n.$$

Hence, after translations by $\tau - s_1$ and $\tau - s_2$, we may suppose that $w_\varepsilon(x)$ and $w_\eta(x)$ are defined on the extended strips

$$\Omega_\varepsilon := \left\{x : -\frac{1}{\varepsilon} \le (x-\tau)\cdot \nu \le 0\right\} \quad \text{and} \quad \Omega_\eta := \left\{x : -\frac{1}{\eta} \le (x-\tau)\cdot \nu \le 0\right\},$$

respectively, with

$$w_\varepsilon = l_\varepsilon(x) \quad \text{on } \left\{(x-\tau)\cdot \nu = -\frac{1}{\varepsilon}\right\}$$

and

$$w_\eta = l_\eta(x) \quad \text{on } \left\{(x-\tau)\cdot \nu = -\frac{1}{\eta}\right\},$$

where l_ε and l_η are linear functions with the same slope as $l(x)$. Moreover on H_0, we have

$$\partial w_\varepsilon / \partial \nu = G(Dw_\varepsilon, x - z_1) \quad \text{and} \quad \partial w_\eta / \partial \nu = G(Dw_\eta, x - z_2)$$

for some $|z_1|, |z_2| \le \eta$. Observe that by Lipschitz continuity of G, i.e., by (G2),

$$|G(p, x - z_1) - G(p, x - z_2)| < m(1 + |p|)\eta. \tag{29}$$

Let v_ε be given in (21). Then by Corollary 4.3 (after a translation),

$$|w_\varepsilon(x) - \frac{v_\varepsilon(\varepsilon x)}{\varepsilon}| \le \frac{C\Lambda(\varepsilon,\nu)}{\varepsilon}. \tag{30}$$

Note that

$$\frac{v_\varepsilon(\varepsilon x)}{\varepsilon} = \mu_\varepsilon\left((x-\tau)\cdot \nu + \frac{1}{\varepsilon}\right) + l_\varepsilon(x).$$

From (30) and the comparison principle, it follows that

$$(\mu_\varepsilon - C\Lambda(\varepsilon,\nu))\left((x-\tau)\cdot\nu + \frac{1}{\varepsilon}\right) \le w_\varepsilon(x) - l_\varepsilon(x) \le (\mu_\varepsilon + C\Lambda(\varepsilon,\nu))\left((x-\tau)\cdot\nu + \frac{1}{\varepsilon}\right). \tag{31}$$

Here, we denote by l_1 and l_2, the following linear profiles

$$l_1(x) = a_1(x-\tau)\cdot\nu + b_1 \quad \text{and} \quad l_2(x) = a_2(x-\tau)\cdot\nu + b_2,$$

whose respective slopes are $a_1 = \mu_\varepsilon + C\Lambda(\varepsilon,\nu)$ and $a_2 = \mu_\varepsilon - C\Lambda(\varepsilon,\nu)$. b_1 and b_2 are chosen, so that

$$l_1(x) = l_2(x) = w_\eta(x) - l_\eta(x) = 0 \quad \text{on } \left\{x : (x-\tau)\cdot\nu = -\frac{1}{\eta}\right\}. \tag{32}$$

Now we define

$$\overline{w}(x) := l_\eta(x) + \begin{cases} l_1(x) & \text{in } \{-1/\eta \le (x-\tau)\cdot\nu \le -1/\varepsilon\} \\ w_\varepsilon(x) - l_\varepsilon(x) + c_1 & \text{in } \{-1/\varepsilon \le (x-\tau)\cdot\nu \le 0\} \end{cases}$$

and

$$\underline{w}(x) := l_\eta(x) + \begin{cases} l_2(x) & \text{in } \{-1/\eta \le (x-\tau)\cdot \nu \le -1/\varepsilon\} \\ w_\varepsilon(x) - l_\varepsilon(x) + c_2 & \text{in } \{-1/\varepsilon \le (x-\tau)\cdot \nu \le 0\}, \end{cases}$$

where c_1 and c_2 are constants satisfying

$$l_1 = w_\varepsilon - l_\varepsilon + c_1 = c_1 \quad \text{and} \quad l_2 = w_\varepsilon - l_\varepsilon + c_2 = c_2$$

on $\{(x-\tau)\cdot \nu = -1/\varepsilon\}$. Note that by (32),

$$\underline{w} = \overline{w} = w_\eta \quad \text{on} \quad \left\{x : (x-\tau)\cdot \nu = -\frac{1}{\eta}\right\},$$

and also due to (31),

$$\overline{w}(x) = l_\eta(x) + \min(l_1(x), w_\varepsilon(x) - l_\varepsilon(x) + c_1)$$

and

$$\underline{w}(x) = l_\eta(x) + \max(l_2(x), w_\varepsilon(x) - l_\varepsilon(x) + c_2)$$

in $\left\{-\frac{1}{\varepsilon} \le (x-\tau)\cdot \nu \le 0\right\}$. Thus, it follows that $\overline{w}$ and $\underline{w}$ are, respectively, viscosity super- and subsolution of (P). Hence, we obtain

$$\underline{w} \le \tilde{w}_\eta \le \overline{w}, \tag{33}$$

where $\tilde{w}_\eta$ is a solution of (P) in Ω_η with $\tilde{w}_\eta = w_\eta = l_\eta(x)$ on $\{(x-\tau)\cdot \nu = -1/\eta\}$, and $\partial\tilde{w}_\eta/\partial\nu = G(D\tilde{w}_\eta, x - z_1)$ on H_0. Then by (33) and Lemma 3.5 with (29),

$$|\mu_\eta - \mu_\varepsilon| \le |\mu_\eta - \mu(\tilde{w}_\eta)| + |\mu(\tilde{w}_\eta) - \mu_\varepsilon| \le C(\Lambda(\varepsilon, \nu) + \eta),$$

where $\mu(\tilde{w}_\eta)$ is the slope of the linear approximation of $\tilde{w}_\varepsilon$. The aforementioned inequality implies that the slope μ of a subsequential limit of u_ε depends on neither the subsequence $\{\varepsilon_j\}$ nor τ. Also sending $\eta \to 0$, we obtain an error estimate (d) when ν is irrational.

Proof of Theorem 4.1(**a**) **and** (**b**) **for rational directions:** Let ν be a rational direction with $\tau = 0$. We claim that $\partial u/\partial\nu = \mu(\nu, q)$ for a constant $\mu(\nu, q)$, which depends on ν and q, not on the subsequence $\{\varepsilon_j\}$. More precisely, if $\eta \le \varepsilon$, then

$$|\mu(u_\eta) - \mu(u_\varepsilon)| \le C\Lambda(\varepsilon, \nu). \tag{34}$$

The proof of (34) is parallel to that of (28). Let w_ε and w_η be as given in the proof of (28). Note that since Ω_ε and Ω_η have their Neumann boundaries passing through the origin, $\partial w_\varepsilon/\partial\nu = G(x) = \partial w_\eta/\partial\nu$ without translation of the x variable, and thus, we do not need to use the properties of hyperplanes with an irrational normal (Lemma 2.7 (b)) to estimate the error between the shifted Neumann boundary datas. In other words, there exist $q_1 \in H^1$ and $q_2 \in H^2$ such that $p = q_1 = q_2 \mod \mathbb{Z}^n$, and hence, $G(\cdot, x - z_1) = G(\cdot, x - z_2)$ in the proof of (28). Following the proof of (28), we obtain an upper bound $\Lambda(\varepsilon, k)$ of $|\mu_\eta - \mu_\varepsilon|$. Note that we do not have the term η in (34) since $G(\cdot, x - z_1) = G(\cdot, x - z_2)$. By sending $\eta \to 0$ in (34), we obtain the error estimate (b) for rational directions with $\tau = 0$.

Proof of Theorem 4.1(**c**) **and** (**d**)**:** Let $\delta > 0$ and let ν be an irrational direction. Lemma 2.6 implies that there is a positive number $T_\nu(\delta) \le \delta^{-(n-1)}$ such that $|T_\nu(\delta)\nu| \le \delta \mod \mathbb{Z}^n$. Then, for some $P \in \mathbb{Z}^n$ and $T = T_\nu(\delta) + O(\delta)$,

$$|T\nu - P| \le \delta$$

and $T\nu \in P + \langle\vec{P}\rangle^\perp$. Let $\theta = \theta(\delta, \nu) > 0$ be the angle between ν and $\vec{P}$, then

$$|T\nu - P| = T\theta \le \delta. \tag{35}$$

If we define $q := T\nu - P \in \langle\vec{P}\rangle^\perp$, then $|q| \le \delta$ by (35). Then for $0 \le m \le \left[\frac{1}{T\theta}\right]$, $mT\nu = mP + mq$ with $1 - \delta \le \left[\frac{1}{T\theta}\right]|q| \le 1$. Hence, we obtain

$$\omega_\nu(N) \le T\theta \quad \text{when} \quad N = \left[\frac{1}{\theta}\right]. \tag{36}$$

Let $\varepsilon(\delta, \nu)$ be a constant depending on δ and the direction ν such that

$$\varepsilon(\delta, \nu) = \delta^2\theta = \delta^2\theta(\delta, \nu). \tag{37}$$

Then for $0 < \varepsilon < \varepsilon(\delta, \nu)$,

$$\Lambda(\varepsilon, \nu) = \inf_{0<k<1, N\in\mathbb{N}} \{\varepsilon^k N + \omega_\nu(N) + \varepsilon^{1-k}\} \le \inf_{0<k<1} \{\varepsilon^k/\theta + T\theta + \varepsilon^{1-k}\} \le \inf_{0<k<1} \{\varepsilon^k/\theta + \varepsilon^{1-k}\} + \delta,$$

where the first and last inequalities follow from (36) and (35), respectively. Then by (37),

$$\inf_{0<k<1} \{\varepsilon^k/\theta + \varepsilon^{1-k}\} \le \inf_{0<k<1} \{(\delta^2\theta)^k/\theta + (\delta^2\theta)^{1-k}\}.$$

The infimum is taken when $0 < k = \ln(\theta\delta)/\ln(\theta\delta^2) < 1$ and

$$\inf_{0<k<1} \{(\delta^2\theta)^k/\theta + (\delta^2\theta)^{1-k}\} = 2\delta.$$

Hence, we can conclude $\Lambda(\varepsilon, \nu) \le 3\delta$ for $\varepsilon < \varepsilon(\delta, \nu) = \delta^2\theta$.

Next, we consider a rational direction ν. For $\delta > 0$, let $\varepsilon < \delta^2/T_\nu$. Then we can check

$$\Lambda(\varepsilon, \nu) = \inf_{0<k<1} \{\varepsilon^k T_\nu + \varepsilon^{1-k}\} \le \inf_{0<k<1} \{\delta^{2k} T_\nu^{1-k} + \delta^{2(1-k)} T_\nu^{k-1}\} = 2\delta.$$

The following lemma will be used in the next section.

Lemma 4.4. *Let $\nu = e_n$, $\tau = 0$, and let w solve*

$$\begin{cases} F(D^2 w, x/\varepsilon) = 0 & \text{in} \quad \{-N\varepsilon \le x_n \le 0\}; \\ \partial w/\partial x_n = G(Dw, x/\varepsilon) & \text{on} \quad H_0; \\ w = A & \text{on} \quad H_{-N\varepsilon}, \end{cases}$$

where N and A are constants. Then there is a constant $C = C(\lambda, \Lambda, n)$ such that

$$|w(x) - w(x_0)| \le C\varepsilon \quad \text{for } x, x_0 \in H_s, \quad -N\varepsilon \le s \le -\frac{N\varepsilon}{2}.$$

Proof. For $x_0, x \in H_s$ with $s \in [-N\varepsilon, -\frac{N\varepsilon}{2}]$, choose $y \in H_s$ such that $|x - y| \le \varepsilon$ and $y = x_0 \bmod \varepsilon\mathbb{Z}^n$. Observe that $w(y) = w(x_0)$, since G is 1-periodic on H_0. Therefore,

$$|w(x) - w(x_0)| = |w(x) - w(y)| \le C\|w - A\|_{L^\infty} \left|\frac{x - y}{N\varepsilon}\right| \le C\varepsilon,$$

where the second inequality is from the interior Lipschitz regularity (Theorem 2.3) applied to $w(N\varepsilon x) - A$. ☐

5 Continuity over normal directions

In the previous section, we have shown that for an irrational direction $\nu \in \mathcal{S}^{n-1} - \mathbb{R}\mathbb{Z}^n$, there is a unique homogenized slope $\mu(\nu, q)$ for any solution u_ε^ν of $(P)_{\varepsilon,\nu,\tau,q}$ in $\Pi(\nu, \tau)$. In this section, we investigate the continuity properties of μ with respect to ν and q, as well as the mode of convergence for u_ε^ν as the normal direction ν of the domain varies.

We first show that μ is Lipschitz with respect to q, which directly follows from the 1-homogeneity of G.

Theorem 5.1. *For $\nu \in \mathcal{S}^{n-1} - \mathbb{R}\mathbb{Z}^n$, $\mu(\nu, q)$ is uniformly Lipschitz in $q \in \langle\nu\rangle^\perp$, independent of ν.*

Proof. For $q_1, q_2 \in \langle \nu \rangle^\perp$, let u_ε^i be the unique bounded solution of $(P)_{\varepsilon,\nu,\tau,q_i}$ for $i = 1, 2$. Let m be the Lipschitz constant for G given in (G1) and c be as given in (G3). Then it follows that

$$w_\pm(x) := u_\varepsilon^1(x) + (q_2 - q_1)\cdot x \pm \frac{m}{1-c}|q_1 - q_2|(x \cdot \nu)$$

is, respectively, a super and subsolution of $(P)_{\varepsilon,\nu,\tau,q_2}$. Hence, by Corollary 3.4, we have

$$w_- \le u_\varepsilon^2 \le w_+ \quad \text{in } \Pi.$$

From here and Theorem 4.1, it follows that

$$|\mu(\nu, q_1) - \mu(\nu, q_2)| \le \frac{m}{1-c}|q_1 - q_2|. \qquad \square$$

The dependence of μ on ν is a much more subtle matter due to the change of the domain and the resulting changes in boundary conditions on the Neumann boundary. From now on, we work with a fixed choice of q and denote $\mu = \mu(\nu)$.

For $s \ge 0$, let $T_\nu(s)$ be the smallest positive number ≥ 1 such that

$$|T_\nu(s)\nu| \le s \mod \mathbb{Z}^n.$$

Note $T_\nu(0)$ is larger than all $T_\nu(s)$. In general, Lemma 2.6 yields

$$T_\nu(s) \le \sqrt{n} \cdot s^{-(n-1)}. \tag{38}$$

Theorem 5.2. *With fixed q, let us denote $\mu = \mu(\cdot, q) : (\mathcal{S}^{n-1} - \mathbb{R}\mathbb{Z}^n) \to \mathbb{R}$ be as given in* Theorem 4.1. *Then μ has a continuous extension $\bar{\mu}(\nu) : \mathcal{S}^{n-1} \to \mathbb{R}$. More precisely, let us fix a direction $\nu \in \mathcal{S}^{n-1}$ and a constant $\delta > 0$. If ν_1 and ν_2 are irrational directions such that*

$$\tan\theta_i < \frac{\delta^{5/2}}{T_\nu(\delta^{5/2})} \quad \textit{for} \quad \theta_i := |\nu_i - \nu| \quad \textit{and} \quad i = 1, 2, \tag{39}$$

then we have
(*a*) $|\mu(\nu_1) - \mu(\nu_2)| < C\delta^{1/2}$ *for $C = C(\nu)$.*
(*b*) $\bar{\mu}(\nu)$ *is Hölder continuous on $\mathcal{S}^{n-1}$ with a Hölder exponent of $\frac{1}{5n}$.*

Remark 5.3. In the proof, we indeed show that, for any directions ν_1 and ν_2 satisfying (39), the range of $\{\mu(u_\varepsilon^{\nu_i})\}_{\varepsilon,i}$ fluctuates only by δ, if ε is sufficiently small. The fact that ν_i's are irrational is only used to guarantee that there is only one subsequential limit for $\mu(u_\varepsilon^{\nu_i})$.

Remark 5.4. For notational simplicity and clarity in the proof, we will assume that $n = 2$ and $\nu = e_2$. We explain in Remark 5.6 how to modify the notations and proof for $\nu \ne e_2$. For general dimension n, we refer to Remark 5.7.

For the rest of the article, we prove (a) of Theorem 5.2. Theorem 5.2 (b) follows from (38), (39), and Theorem 5.2 (a).

5.1 Basic settings and Sketch of the proof

We denote

$$\Pi := \Pi(e_2, 0) \quad \text{and} \quad \Pi^{\nu_i} := \Pi(\nu_i, 0), \quad \text{for } i = 1, 2.$$

We also denote

$$H_0 = H_0(e_2), \quad H_0^{\nu_i} := H_0(\nu^i) \quad \text{for } i = 1, 2.$$

For given

$$m \in \mathbb{N} \quad \text{and} \quad \delta := 1/m > 0,$$

we divide the unit strip $\mathbb{R} \times [0, 1]$ by m numbers of small horizontal strips of width δ and define a family of functions $\{G_k\}_k$ so that the value of G_k at (x_1, x_2) is same as the value of G at $(x_1, \tilde{x}_2)$, where $(x_1, \tilde{x}_2)$ is the projection of (x_1, x_2) onto the bottom of the k-th strip. More precisely, we define

$$G_k(x_1, x_2) := G(x_1, \delta(k-1)) \quad \text{for } k = 1, \dots, m. \tag{40}$$

Then G_k is a 1-periodic function with respect to x_1.

Next we introduce the parameters

$$\theta_1 := |\nu_1 - e_2|, \quad \theta_2 := |\nu_2 - e_2| \tag{41}$$

and

$$N := \left[\frac{\delta}{\tan\theta_1}\right], \quad M := \left[\frac{\delta}{\tan\theta_2}\right]. \tag{42}$$

Without loss of generality, assume $\theta_2 \le \theta_1$, and thus, $N \le M$.

If θ_i's are sufficiently small, then we will be able to approximate G on both of the Neumann boundary $H_0^{\nu_1}$ and $H_0^{\nu_2}$ using the universal boundary data G_k's, which depends only on δ, but not on the direction ν_1 nor ν_2. In particular, in meso-scopic scale G can be approximated by many repeating pieces of G_k's on $H_0^{\nu_i}$ (approximately, N number of pieces of G_k for ν_1 and M for ν_2). Thus, the problem already experiences averaging phenomena: we call this as *the first* or *near-boundary* homogenization. Note that in this step, the only difference in the averaging phenomena between the two directions ν_1 and ν_2, besides the errors in terms of G and G_k on $H_0^{\nu_i}$, is the number of repeating data G_k for each k. This explains the proximity of $\mu(\nu_1)$ and $\mu(\nu_2)$.

On the other hand, since $\nu's$ are irrational directions, the distribution of G_k approximates the given G on $H_0^{\nu_i}$ in large scale. Since ν_1 and ν_2 are close to the rational direction e_2, the averaging behavior of a solution $u_\varepsilon^{\nu_i}$ in Π^{ν_i} would appear in a very large scale, and in other words, only after ε obtains very small. We call this as the *secondary* homogenization.

The two-scale homogenization procedure has been introduced in [7,8]. It allows studying continuity properties of the homogenized boundary data as we approach the rational direction, which might be singular points as described in Section 1. This point of view was also employed in [12,13] to study homogenization for general operators, by studying the singularity of homogenized operator at rational directions. Let us also point out near the boundary the small-scale oscillation of the operator interacts with that of boundary data to create a meso-scale averaging phenomena. Due to this interaction, characterizing the homogenized boundary condition remains a challenging and interesting open problem. After the first homogenization, the boundary data change to periodic data in a meso-scale (which will be $N\varepsilon$ below), and hence, the operator is well approximated by the homogenized operator $\bar{F}$ in the second homogenization in large scale.

Below we begin the analysis of the two-step homogenization as described earlier. We will work with small $\varepsilon > 0$ satisfying

$$\varepsilon \le \frac{\delta \tan\theta_i}{T_\nu(\delta^{5/2})} \quad \text{for } i = 1, 2, \tag{43}$$

which can be stated as follows:

$$0 < \varepsilon \le \delta \tan\theta_i \quad \text{for } i = 1, 2 \tag{44}$$

since $T_\nu(s) \equiv 1$ when $\nu = e_2$. It follows that

$$mN\varepsilon \le mM\varepsilon \le \delta. \tag{45}$$

After the near-boundary homogenization, $u_\varepsilon^{\nu_1}$ will be approximated by a solution, which has periodic boundary data with period $mN\varepsilon$. With (45), it follows that $u_\varepsilon^{\nu_1}$ fluctuates in order of δ in the interior of the strip domain.

On the other hand, (39) of Theorem 5.2 can be stated as follows:

$$0 < \tan\theta_1 \le \delta^{5/2}. \tag{46}$$

It follows then that

$$1/N \le \delta^{3/2} \tag{47}$$

which ensures $u_\varepsilon^{\nu_i}$ to homogenize $N\varepsilon$-close to the Neumann boundary.

Next, we define vertical strips $\bar{I}_k$'s so that in each $\bar{I}_k$, the Neumann boundary $H_0^{\nu_1}$ is contained in the horizontal strip (parallel to H_0)of width approximately $\delta\varepsilon$. Let $N_0 = 0$ and

$$N_k := \max\left\{ N \in \mathbb{N} \mid \left(\sum_{j=0}^{k-1} N_j + N\right)\varepsilon\tan\theta_1 < k\delta\varepsilon \right\} \quad \text{for } k \in \mathbb{N}.$$

We define

$$\bar{I}_k = \begin{cases} \left[\sum_{j=0}^{k-1} N_j\varepsilon, \sum_{j=0}^{k} N_j\varepsilon\right] \times \mathbb{R} & \text{for } k \in \mathbb{N} \\ \left[-\sum_{j=0}^{k+1} N_j\varepsilon, -\sum_{j=0}^{k} N_j\varepsilon\right] \times \mathbb{R} & \text{for } k \in -\mathbb{N} \cup \{0\}. \end{cases}$$

Then we can observe

$$\frac{\delta}{\tan\theta_1} - 1 \le N_k \le \frac{\delta}{\tan\theta_1} + 1 \tag{48}$$

since the definition of N_k implies

$$(N_k - 1)\varepsilon\tan\theta_1 \le \delta\varepsilon \le (N_k + 1)\varepsilon\tan\theta_1.$$

On the other hand, by the definitions of N_k and $\bar{I}_k$, $H_0^{\nu_1} \cap \bar{I}_k$ is located within $\delta\varepsilon$-distance from $H_0 + \delta\varepsilon(k-1)e_2$, mod $\varepsilon\mathbb{Z}^n$, for each $k \in \mathbb{Z}$. Thus, G is approximated well by G_k on $H_0^{\nu_1} \cap \bar{I}_k$, for $1 \le k \le m$. Indeed, if we extend the definition of G_k over $k \in \mathbb{Z}$ by letting $G_k = G_{\bar{k}}$ for $k = \bar{k}$ (mod m), then we have

$$\left| G\left(p, \frac{x}{\varepsilon}\right) - G_k\left(p, \frac{x}{\varepsilon}\right) \right| < C(1 + |p|)\delta \quad \text{on } H_0^{\nu_1} \cap \bar{I}_k \quad \text{for } k \in \mathbb{Z}. \tag{49}$$

Similarly for ν_2, we define M_k for $k \in \mathbb{N} \cup \{0\}$ and the vertical strips $\bar{J}_k$ for $k \in \mathbb{Z}$.

Remark 5.5. Observe that (48) implies N_k and M_k are comparable, respectively, with $N = \left[\frac{\delta}{\tan\theta_1}\right]$ and $M = \left[\frac{\delta}{\tan\theta_2}\right]$ with $|N_k - N|$, $|M_k - M| \le 1$. Thus, for simplicity of our proof, we assume

$$N_k = N; \quad M_k = M \quad \text{for } k \in \mathbb{N}$$

and

$$I_k = [(k-1)N\varepsilon, kN\varepsilon] \times \mathbb{R}; \; J_k = [(k-1)M\varepsilon, kM\varepsilon] \times \mathbb{R} \quad \text{for } k \in \mathbb{Z}. \tag{50}$$

Our simplification of N_k does not affect our analysis in Section 5.2: For the first homogenization near the boundary, the estimate in Lemma 5.9 does not change since N_k and N differ at most by 1, and the analysis is done in a local ball in the proof of Lemma 5.9. More precisely, Lemma 5.9 holds with $\mu^N(G_k)$ replaced by $\mu^{N_k}(G_k)$, where $|\mu^N(G_k) - \mu^{N_k}(G_k)|$ is small enough by parallel arguments that show (75). For the second homogenization in the middle region, we can construct a periodic function $\Lambda(x)$ with period $\varepsilon/\sin\theta_1$ similarly as in step 2 of Section 5.2, since there we view each m union of $\bar{I}_k$ as a "block," and

$$\frac{\varepsilon(1-\tan\theta_1)}{\sin\theta_1} \leq \left|\bigcup_{k=1}^{m} \bar{I}_k \cap H_0^{\nu_1}\right| \leq \frac{\varepsilon(1+\tan\theta_1)}{\sin\theta_1}$$

and also

$$\frac{\varepsilon(L-\tan\theta_1)}{\sin\theta_1} \leq \left|\bigcup_{k=1}^{Lm} \bar{I}_k \cap H_0^{\nu_1}\right| \leq \frac{\varepsilon(L+\tan\theta_1)}{\sin\theta_1}$$

for any $L \in \mathbb{N}$. This shows that the required period of Λ is $\varepsilon/\sin\theta_1$, approximating the average period of $\bigcup_{k=1}^{Lm}\bar{I}_k$ with the error $\frac{\varepsilon\tan\theta_1}{L\sin\theta_1} \sim \varepsilon/L$.

Remark 5.6. For $\nu \neq e_2$ in $\mathbb{R}^2$, there exists a rational direction $\tilde{\nu}$ such that for $T = T_\nu(\delta^{5/2})$,

$$T\tilde{\nu} = 0 \pmod{\mathbb{Z}^2}; \quad |\nu - \tilde{\nu}| \leq \delta^{5/2}/T.$$

Observe that if Theorem 5.2 holds for the rational direction $\tilde{\nu}$, it also holds for ν. For the proof of the theorem for $\tilde{\nu}$, let $x' = x - (x\cdot\tilde{\nu})\tilde{\nu}$ and define

$$G_k = G_k(x', x - x') = G(x', \delta(k-1)\tilde{\nu}) \quad \text{for } 1 \leq k \leq m.$$

Then G_k is a periodic function on $\{x\cdot\tilde{\nu} = 0\}$ with a period of T. The only difference between the case of $\tilde{\nu}$ and e_2 is in the periodicity of the function G_k, and it does not make any essential difference in the proof. we point out that instead of the conditions (46), (47), and (45), we will need

$$\frac{1}{TN} \leq \delta^{3/2}; \quad T\tan\theta_1 \leq \delta^{5/2}; \quad mTM\varepsilon \leq \delta$$

since G_k has a period of T. These conditions will be ensured if θ_i and ε satisfy the assumptions as in Theorem 5.2.

Remark 5.7. For the dimension $n > 2$ and $\nu = e_n$, for a fixed $m \in \mathbb{N}$ and $\delta = \frac{1}{m}$, let us define

$$G_i(x_1, \ldots, x_{n-1}, x_n) := G(x_1, \ldots, x_{n-1}, \delta(i-1)) \quad \text{for } i = 0, \ldots, m$$

and

$$I_{k_1,k_2,\ldots,k_{n-1}} := [(k_1-1)N\varepsilon, k_1N\varepsilon]\times\cdots\times[(k_{n-1}-1)N\varepsilon, k_{n-1}N\varepsilon] \times \mathbb{R}.$$

Then parallel arguments as in steps 1–9 in the next section would apply to yield the results in $\mathbb{R}^n$.

5.2 Proof of Theorem 5.2

In the first three steps, we follow the aforementioned heuristics and replace the Neumann condition with the locally projected boundary data G_k. Then we go through the two-step homogenization procedures to obtain the first slope $\mu^N(G_k)$ on each I_k near the boundary, and then the global slope $\mu(\nu_1)$. While the actual first homogenization takes place in Π^{ν_1}, it turns out that its value has a small difference from $\mu^N(G_k)$ taken in Π (see Lemma 5.9). This fact is important in establishing a universal domain for both directions ν_1 and ν_2. In fact, we rotate the middle and inner regions to compare the slopes in Π^{ν_1} and Π^{ν_2}. For this, we use the rotational invariance of the homogenized operator $\bar{F}$. (See Lemmas 5.10 and 5.11.) The rest of steps are to verify that indeed $\mu(\nu_1)$ is the correct averaged slope for the problem $(P)_{\varepsilon,\nu_1,\tau,q}$.

Step 1. First homogenization near Boundary ($N\varepsilon$-**away from** $H_0^{\nu_1}$)

We proceed to discuss the first homogenization. Denote $x = (x_1, x_2)$ throughout this section. For a given linear function $l(x) = l(x_1)$ and $k \in \mathbb{Z}$, let $u = u^{N,\varepsilon}$ and $v_k = v_k^{N,\varepsilon}$ solve the following problem with $u = l(x)$ on $H_{-N\varepsilon}^{\nu_1}$ and $v_k = l(x)$ on $H_{-N\varepsilon}$:

$$\begin{cases} F(D^2u, x/\varepsilon) = 0 & \text{in} \quad \{-N\varepsilon \le x \cdot \nu_1 \le 0\}; \\ \dfrac{\partial u}{\partial \nu_1}(x) = G(Du, x/\varepsilon) & \text{on } H_0^{\nu_1} \end{cases} \tag{51}$$

and

$$\begin{cases} F(D^2v_k, x/\varepsilon) = 0 & \text{in } \{-N\varepsilon \le x_2 \le 0\}; \\ \dfrac{\partial v_k}{\partial x_2}(x) = G_k(Dv_k, x/\varepsilon) & \text{on } H_0. \end{cases} \tag{52}$$

Definition 5.8. For a given function $u : \{-N\varepsilon \le x \cdot \nu \le 0\} \to \mathbb{R}$ and I_k given as in (50), let a_k and b_k be the middle points of $I_k \cap H_{-N\varepsilon/2}^{\nu}$ and $I_k \cap H_{-N\varepsilon}^{\nu}$, respectively, and consider the unique linear function h given by $h = u$ at $x = a_k, b_k$ and $D_T h(b_k) = D_T u(b_k)$. (Here, $D_T h$ denotes the tangential derivative of h along the direction $\nu^\perp$.) Then $\mu_k(u)$ is defined by

$$\mu_k(u) := \partial h/\partial \nu.$$

Note that the Neumann boundary data of v_k are G_k on each boundary piece $H_0 \cap I_i$ ($i \in \mathbb{Z}$), and hence, $\mu_i(v_k) = \mu(v_k)$. (Here, $\mu(v_k)$ is the average slope of v_k given as in (22) with $\tau = (N\varepsilon/2)e_1$.) For N as given in (42), we denote

$$\mu^N(G_k) := \mu(v_k). \tag{53}$$

Lemma 5.9. *For* $k \in \mathbb{Z}$ *and* $\mu_k(u)$ *as given in* Definition 5.8,

$$|\mu_k(u) - \mu^N(G_k)| < C\delta^{1/2}. \tag{54}$$

Proof. We will prove the lemma for $k = 1$, i.e., we will compare $\mu_1(u)$ with $\mu(v_1)$. Let $\tilde{u}$ and $\tilde{v}_1$ solve the following problem with $\tilde{u} = l$ on $H_{-\varepsilon/\delta}^{\nu_1}$ and $\tilde{v}_1 = l$ on $H_{-\varepsilon/\delta}$:

$$\begin{cases} F(D^2\tilde{u}, x/\varepsilon) = 0 & \text{in } \{-\varepsilon/\delta \le x \cdot \nu_1 \le 0\}; \\ \dfrac{\partial \tilde{u}}{\partial \nu_1}(x) = G(D\tilde{u}, x/\varepsilon) & \text{on } H_0^{\nu_1} \end{cases}$$

and

$$\begin{cases} F(D^2\tilde{v}_1, x/\varepsilon) = 0 & \text{in } \{-\varepsilon/\delta \le x_2 \le 0\} \\ \dfrac{\partial \tilde{v}_1}{\partial x_2}(x) = G_1(D\tilde{v}_1, x/\varepsilon) & \text{on } H_0. \end{cases}$$

We will compare both of $\tilde{u}(x)$ and $\tilde{v}_1(x)$ to $w_1(x)$ in the ball $|x| \le \delta^{-1-\alpha_0}\varepsilon$, where $\alpha_0 = 1/2$. For computational convenience, we will call this number as α_0. Let $w_1(x)$ solve $w_1 = l$ on $H_{-\varepsilon/\delta}^{\nu_1}$ with

$$\begin{cases} F(D^2w_1, x/\varepsilon) = 0 & \text{in } \{-\varepsilon/\delta \le x \cdot \nu_1 \le 0\}; \\ \dfrac{\partial w_1}{\partial \nu_1}(x) = G_1(Dw_1, x/\varepsilon) & \text{on } H_0^{\nu_1}. \end{cases} \tag{55}$$

Here, observe that in the ball $|x| \le \delta^{-1-\alpha_0}\varepsilon$, the hyperplanes $H_0^{\nu_1}$ and H_0 only differ by $\tan\theta_1\delta^{-1-\alpha_0}\varepsilon$.

Below we derive some properties of w_1. Consider

$$\bar{w}(x) := \varepsilon^{-1}w_1(\varepsilon x).$$

Then by Theorem 2.4, $\bar{w}$ is $C^{1,1}$ regular up to the Neumann boundary in a unit ball, if $\bar{w}$ has a bounded oscillation in the ball $|x| \le 1/\delta$. Observe that $(\varepsilon/\delta)^{-1}w_1(\varepsilon x/\delta)$ is defined in the strip $\{-1 \le x \cdot \nu_1 \le 0\}$, and it has a periodic Neumann data $G_1(\cdot,\cdot,x/\delta)$ with period δ. Since it has a periodic boundary data, it corresponds to the case of rational direction with Neumann boundary passing through the origin. Hence, we can use the error estimate Theorem 4.1 (b) for the rational direction passing through the origin, with $T_\nu = 1$. Then we obtain

$$\left| \left(\frac{\varepsilon}{\delta}\right)^{-1} w_1\left(\frac{\varepsilon x}{\delta}\right) - h(x) \right| \le \inf_{0<k<1} C(\delta^k + \delta^{1-k}) = C\delta^{1/2}, \tag{56}$$

where h is a linear solution approximating $(\varepsilon/\delta)^{-1}w_1(\varepsilon x/\delta)$. Then by (56),

$$\left| w_1\left(\frac{\varepsilon x}{\delta}\right) - \frac{\varepsilon}{\delta} h(x) \right| \le C\delta^{-1/2}\varepsilon, \tag{57}$$

and hence, the oscillation of $\bar{w}$ becomes less than $C\delta^{-1/2}$ in the ball $|x| \le 1/\delta$. Later in the proof, we will use $C^{1,1}$ regularity of $\bar{w}$ as well as the linear approximation (57) of w_1.

First, we compare $\tilde{u}$ to w_1 in $B_{\delta^{-1-\alpha_0}\varepsilon}(0)$. For this, we compare the boundary data of $\tilde{u}$, that is G, to G_1. Observe that if $x \in H_0^{\nu_1} \cap B_{\delta^{-1-\alpha_0}\varepsilon}(0)$, then $x \in I_k$ for some $|k| \le \delta^{-1-\alpha_0}/N = \delta^{-2-\alpha_0}\tan\theta_1$. Hence, for $x \in H_0^{\nu_1} \cap B_{\delta^{-1-\alpha_0}\varepsilon}(0)$ (i.e., for $x \in H_0^{\nu_1} \cap I_k$ with $|k| \le \delta^{-2-\alpha_0}\tan\theta_1$),

$$\begin{aligned} |G(p,x/\varepsilon) - G_1(p,x/\varepsilon)| &\le |G(p,x/\varepsilon) - G_k(p,x/\varepsilon)| + |G_k(p,x/\varepsilon) - G_1(p,x/\varepsilon)| \\ &\le C[(1+|p|)\delta + (1+|p|)|k-1|\delta] \\ &\le C(1+|p|)\left(\delta + \left(\tan\theta_1 \delta^{(-1-\alpha_0)}\right)\right) \\ &\le C(1+|p|)\left(\delta + \delta^{(3/2-\alpha_0)}\right) \\ &\le C(1+|p|)\delta, \end{aligned} \tag{58}$$

where the second inequality follows from (49) and the construction of G_k, third inequality follows from $|k| \le \delta^{-2-\alpha_0}\tan\theta_1$, the fourth inequality follows from (46), and the last inequality follows since $\alpha_0 \le 1/2$.

Note that $|\tilde{u} - w_1| \le C\frac{\varepsilon}{\delta}$ in $|x| \le 2\delta^{-1-\alpha_0}\varepsilon$. This implies that, by Lemma 3.5, $|\tilde{u} - w_1| \le \delta(L+1) + C\delta^{\alpha_0}\frac{\varepsilon}{\delta}$ in $|x| \le \delta^{-1-\alpha_0}\varepsilon$. Now we can compare $\tilde{u} - w_1$ with linear profiles in the strip to obtain

$$|\tilde{u}(x) - w_1(x)| \le C\left(\delta + \delta^{\alpha_0}\right)\left(x \cdot \nu_1 + \frac{\varepsilon}{\delta}\right) \le C\delta^{\alpha_0}\left(x \cdot \nu_1 + \frac{\varepsilon}{\delta}\right) \quad \text{in } |x| \le \delta^{-1-\alpha_0}\varepsilon. \tag{59}$$

Observe that (57) and (59) yield

$$|\tilde{u}(x) - L_1(x)| \le C\left(\delta^{\alpha_0} + \delta^{1/2}\right)\left(x \cdot \nu_1 + \frac{\varepsilon}{\delta}\right) \le C\delta^{\alpha_0}\left(x \cdot \nu_1 + \frac{\varepsilon}{\delta}\right) \quad \text{in } |x| \le \delta^{-1-\alpha_0}\varepsilon,$$

where $L_1(x) = l(x) + \mu(w_1)\left(x \cdot \nu_1 + \frac{\varepsilon}{\delta}\right)$, and $\mu(w_1)$ is the average slope of w_1. In other words, we obtain

$$|\mu_1(\tilde{u}) - \mu(w_1)| \le C\delta^{\alpha_0}. \tag{60}$$

Next, we compare $\tilde{v}_1$ and w_1 and prove that

$$|\mu(\tilde{v}_1) - \mu(w_1)| \le C\delta^{\alpha_0}.$$

Recall that the oscillation of $\bar{w}$ is less than $C\delta^{-1/2}$ in the ball $|x| \le 1/\delta$ (see (57)). If we consider $\tilde{w} = \delta^{1/2}\bar{w}$, then this function solves the boundary condition:

$$\partial\tilde{w}/\partial\nu = \tilde{G}(D\tilde{w}, x) = \delta^{1/2}G_1(\delta^{-1/2}D\tilde{w}, x),$$

which satisfies the assumptions for the $C^{1,1}$ regularity theory, Theorem 2.4. Thus, we have

$$\|\bar{w}\|_{C^{1,1}(B_1)} \le O(\delta^{-1/2}).$$

For x in the $\sigma\varepsilon$-neighborhood of $H_0^{\nu_1}$, choose $\tilde{x}$ to be the closest point to x on H_0. Then by (G1) and (G2) with the $C^{1,1}$ regularity of $\bar{w}$ given earlier, w_1 satisfies on H_0,

$$\left| G\left(Dw_1(x), \frac{x}{\varepsilon}\right) - G\left(Dw_1(\tilde{x}), \frac{\tilde{x}}{\varepsilon}\right) \right| \le O(\delta^{-1/2}\sigma)(1 + |Dw_1(x)|).$$

Recall that the Neumann boundaries of w_1 and v_1 ($H_0^{\nu_1}$ and H_0) only differ in the ball $|x| \le \delta^{-1-\alpha_0}\varepsilon$, by $\tan\theta_1\delta^{-1-\alpha_0}\varepsilon \le \delta^{3/2-\alpha_0}\varepsilon$ (see (46)). So putting $\sigma = \delta^{3/2-\alpha_0}$,

$$\left| G\left(Dw_1(x), \frac{x}{\varepsilon}\right) - G\left(Dw_1(\tilde{x}), \frac{\tilde{x}}{\varepsilon}\right) \right| \le O\left(\delta^{1-\alpha_0}\right)(1 + |Dw_1(x)|) \quad \text{on } H_0,$$

and Lemma 3.5 yields that in $|x| \le \delta^{-1-\alpha_0}\varepsilon$,

$$|(\tilde{v}_1 - w_1)(x)| \le C\left(\delta^{1-\alpha_0} + \delta^{\alpha_0}\right)\left(x_n + \frac{\varepsilon}{\delta}\right) \le C\delta^{\alpha_0}\left(x_n + \frac{\varepsilon}{\delta}\right).$$

This and (57) yield that in $|x| \le \delta^{-1-\alpha_0}\varepsilon$,

$$|\tilde{v}_1(x) - L(x)| \le C\left(\delta^{\alpha_0} + \delta^{1/2}\right)\left(x_n + \frac{\varepsilon}{\delta}\right) \le C\delta^{\alpha_0}\left(x_n + \frac{\varepsilon}{\delta}\right),$$

where $L(x) = l(x) + \mu(w_1)\left(x_2 + \frac{\varepsilon}{\delta}\right)$. In other words, we obtain

$$|\mu(w_1) - \mu(\tilde{v}_1)| \le C\delta^{\alpha_0}. \tag{61}$$

Recalling $\alpha_0 = 1/2$, we conclude from (60) and (61) that

$$|\mu_1(\tilde{u}) - \mu(\tilde{v}_1)| \le C\delta^{1/2}. \tag{62}$$

In the rest of proof, we will show

$$|\mu(v_1) - \mu(\tilde{v}_1)|, \quad |\mu_1(u) - \mu_1(\tilde{u})| \le C\delta^{1/2}.$$

Then the aforementioned inequalities and (62) would imply

$$|\mu_1(u) - \mu(v_1)| \le |\mu_1(u) - \mu_1(\tilde{u})| + |\mu_1(\tilde{u}) - \mu(\tilde{v}_1)| + |\mu(\tilde{v}_1) - \mu(v_1)| \le C\delta^{1/2}.$$

First, observe that v_1 and $\tilde{v}_1$ have periodic Neumann data G_1 on H_0. Hence, by similar arguments as in the proof of (28),

$$|\mu(v_1) - \mu(\tilde{v}_1)| \le C(\Lambda(\delta, e_2) + N^{-1}) \le C(\delta^{1/2} + N^{-1}) \le C\delta^{1/2}, \tag{63}$$

where the last inequality follows from (47).

Next, recall that

$$|\mu_1(\tilde{u}) - \mu(w_1)| \le C\delta^{1/2}$$

for a solution w_1 of (55). (See (60).) Similarly, one can prove that

$$|\mu_1(u) - \mu(\tilde{w}_1)| \le CN^{-1/2} \le C\delta^{1/2},$$

where $\tilde{w}_1$ solves similar equations as in (55) in the domain $\{-N\varepsilon \le x \cdot \nu_1 \le 0\}$, and the last inequality follows from (47). Then since w_1 and $\tilde{w}_1$ have periodic Neumann data G_1 on $H_0^{\nu_1}$, it corresponds to the case of $\nu = e_2$. Hence, by similar arguments as in (63),

$$|\mu(w_1) - \mu(\tilde{w}_1)| \le C(\Lambda(\delta, e_2) + N^{-1}) \le C(\delta^{1/2} + N^{-1}) \le C\delta^{1/2},$$

and we can conclude

$$|\mu_1(u) - \mu_1(\tilde{u})| \le |\mu_1(u) - \mu(\tilde{w}_1)| + |\mu(\tilde{w}_1) - \mu(w_1)| + |\mu(w_1) - \mu_1(\tilde{u})| \le C\delta^{1/2}.$$ □

Step 2. Constructing middle region barrier ω_ε (between $H_{-N\varepsilon/2}$ and $H_{-KmN\varepsilon}$)

In step 1, we showed that $N\varepsilon$ away from the boundary $H_0^{\nu_1}$, $u_\varepsilon^{\nu_1}$ is homogenized with average slope approximated by $\mu^N(G_k)$ in each vertical strip I_k. Now more than $N\varepsilon$ away from $H_0^{\nu_1}$, we obtain the second homogenization of $u_\varepsilon^{\nu_1}$, whose slope is determined by $\mu^N(G_k)$, $k = 1, \ldots, m$. Since the width of $I_k = N\varepsilon$, the homogenized slopes $\mu^N(G_1), \ldots, \mu^N(G_m)$ are repeated K times in a vertical strip of width $KmN\varepsilon$, $N\varepsilon$-away from $H_0^{\nu_1}$. We will specify

$$K := 1/\delta,$$

but for computational clarity, we will keep the symbol K.

We will construct middle region barrier ω_ε in the region $\{-KmN\varepsilon \le x_2 \le -N\varepsilon/2\}$. To ensure that ω_ε is regular near its Neumann boundary, we introduce a regularization of the original Neumann boundary data $\mu^N(G_k)$ as follows:

Consider a ball $B_{\delta^{-\alpha_0/2}N\varepsilon}(0)$. If $I_k \cap H_0$, $I_j \cap H_0 \subset B_{\delta^{-\alpha_0/2}N\varepsilon}(0)$, then $|k - j| \le \delta^{-\alpha_0/2}$ and

$$|G_k(p, x/\varepsilon) - G_j(p, x/\varepsilon)| \le C(1 + |p|)(|k - j|\delta) \le C(1 + |p|)\delta^{(1-\alpha_0/2)}. \tag{64}$$

By using this fact with Lemma 3.5, we can construct a C^1 function $\Lambda(x)$ on $H_{-N\varepsilon/2}$, such that

(a) $\Lambda \in C^1(H_{-N\varepsilon/2})$ with $\|\Lambda\|_{C^1} \le \delta(N\varepsilon)^{-1}$;

(b) $\mu^N(G_k) + \delta^{\alpha_0} \le \Lambda(x) \le \mu^N(G_k) + \delta^{\alpha_0} + \delta$ on each I_k;

(c) $\Lambda(x)$ is periodic with period $mN\varepsilon$.

Note that when we patch the middle region barrier ω_ε with the near-boundary barrier f_ε in step 6, we will need that the average slope of ω_ε is “sufficiently” larger than that of f_ε. For this, we will make the average slope of ω_ε to be $\mu^N(G_k) + O(\delta^{\alpha_0})$, i.e., (b) is to ensure that $\mu_k(\omega_\varepsilon)$ is sufficiently larger than $\mu_k(f_\varepsilon)$. Also when we show the flatness of barriers in steps 4 and 5, we will localize them in a “large” ball of size $\delta^{-\alpha_0/2}N\varepsilon$.

Let $\Sigma := \{-KmN\varepsilon \le x_2 \le -N\varepsilon/2\}$ and ω_ε solve the following Neumann boundary problem:

$$\begin{cases} F(D^2\omega_\varepsilon, x/\varepsilon) = 0 & \text{in } \Sigma \\ \dfrac{\partial \omega_\varepsilon}{\partial x_2} = \Lambda(x) & \text{on } H_{-N\varepsilon/2} \\ \omega_\varepsilon = l(x) & \text{on } H_{-KmN\varepsilon}. \end{cases} \tag{65}$$

Step 3. Homogenization of the operator in the middle region

Next we show, similar to Lemma 5.9, that the second homogenization does not change too much if the domain Π is replaced by Π^{ν_1}. More precisely, we will show that ω_ε is close to $\tilde{\omega}_\varepsilon$ solving

$$\begin{cases} \bar{F}(D^2\tilde{\omega}_\varepsilon) = 0 & \text{in } \{-KmN\varepsilon \le x \cdot \nu_1 \le -N\varepsilon/2\} \\ \dfrac{\partial \tilde{\omega}_\varepsilon}{\partial \nu_1} = \Lambda(x) & \text{on } H^{\nu_1}_{-N\varepsilon/2} \\ \tilde{\omega}_\varepsilon = l(x) & \text{on } H^{\nu_1}_{-KmN\varepsilon}. \end{cases}$$

Here, $\Lambda(x)$ is a C^1 function constructed as in step 2, which approximates $\mu^N(G_k)$ on each I_k, which is extended to $\mathbb{R}^2$ so that $\Lambda(x) = \Lambda(p(x))$ for a projection $p(x)$ onto $H_{-N\varepsilon/2}$.

To this end, we will first compare ω_ε with $\bar{\omega}_\varepsilon$, with the same Dirichlet data l on $H_{-kmN\varepsilon}$ and solving

$$\begin{cases} \bar{F}(D^2\bar{\omega}_\varepsilon) = 0 & \text{in } \Sigma \\ \dfrac{\partial \bar{\omega}_\varepsilon}{\partial x_2} = \Lambda(x) & \text{on } H_{-N\varepsilon/2}. \end{cases} \tag{66}$$

Lemma 5.10. *For any $\sigma > 0$, there exists N_0 such that for $N_0 > N$, we have*

$$|\omega_\varepsilon(x) - \bar{\omega}_\varepsilon(x)| \le \sigma\delta N\varepsilon \quad \text{in } \Sigma.$$

Proof. The proof follows from Theorem 2.5 applied to $(\delta N\varepsilon)^{-1}\omega_\varepsilon(N\varepsilon x)$. ☐

Next we compare $\bar{\omega}_\varepsilon$ to $\tilde{\omega}_\varepsilon$ to conclude. Here, we will use the rotational invariance of $\bar{F}$.

Lemma 5.11. *Let O be the rotation matrix that maps e_2 to v_1. Then*

$$|\tilde{\omega}_\varepsilon(Ox) - \bar{\omega}_\varepsilon(x)| \le \delta^{1/2}(KmN\varepsilon)$$

in $\Sigma \cap \{|x| \le \delta^{-1/2}(N\varepsilon)\}$.

Proof. Observe that $v(x) := \tilde{\omega}(Ox)$ solves $\bar{F}(D^2 v) = 0$ in Σ with Neumann boundary data $\Lambda(Ox)$ on $H_{-N\varepsilon/2}$ and Dirichlet data $l(Ox)$ on $H_{-KmN\varepsilon}$. Note that due to (46) and the C^1 bound of Λ, we have

$$|\Lambda(KmN\varepsilon Ox) - \Lambda(KmN\varepsilon x)| \le \tan\theta_1 |KmN\varepsilon x| \sup|D\Lambda| \le \delta|x|.$$

and $|l(KmN\varepsilon Ox) - l(KmN\varepsilon x)| \le KmN\varepsilon \tan\theta_1 |x| \le \delta|x|$.

Hence, one can apply Lemma 2.9 of [7] to $\tau^{-1}v(\tau x)$ and $\tau^{-1}\bar{w}(\tau x)$ in $\tau^{-1}\Sigma$, where $\tau = KmN\varepsilon$ and choose $R := \delta^{-1/2}$ and $\varepsilon = 2$ to conclude. □

Step 4. Flatness of ω_ε on $H_{-N\varepsilon}$, and the construction of near-boundary barrier f_ε

Lemma 5.12. [Flatness of ω_ε] *Let x_0 be any point on $H_{-N\varepsilon}$. Then for $x \in H_{-N\varepsilon} \cap B_{\delta^{-\alpha_0/2}N\varepsilon}(x_0)$,*

$$|\omega_\varepsilon(x) - \omega_\varepsilon(x_0) - \partial_1\omega_\varepsilon(x_0)(x - x_0)_1| \le C\delta^{1-\alpha_0}N\varepsilon.$$

Proof. Due to Lemma 5.10, it is enough to show aforementioned lemma for $\bar{\omega}_\varepsilon$. Let $\omega_1(x) := (KmN\varepsilon)^{-1}\bar{\omega}_\varepsilon((KmN\varepsilon)x)$, then it solves

$$\begin{cases} \bar{F}(D^2\omega_1) = 0 & \text{in } \left\{-1 \le x_2 \le -\dfrac{1}{2Km}\right\} \\ \dfrac{\partial\omega_1}{\partial x_2} = \Lambda(KmN\varepsilon x) & \text{on } H_{-\frac{1}{2Km}} \\ \omega_1(x) = l(x) + C & \text{on } H_{-1}. \end{cases}$$

We know that $\|\Lambda\|_{C^1} \le \delta(N\varepsilon)^{-1}$, so the aforementioned Neumann boundary data has C^1 norm of δKm. From Theorem 2.4, we have that

$$\|\omega_1\|_{C^{1,1}} \le C\delta Km.$$

Hence,

$$|\omega_1(x) - \omega_1(x_0) - \partial_1\omega_1(x_0)\cdot(x - x_0)| \le C\delta Km|x - x_0|^2, \tag{67}$$

which can be written in terms of $\bar{\omega}_\varepsilon$,

$$\begin{aligned} |\bar{\omega}_\varepsilon(x) - \bar{\omega}_\varepsilon(x_0) - \partial_1\bar{\omega}_\varepsilon(x_0)\cdot(x - x_0)| &\le C\delta(Km)^2(N\varepsilon)|(KmN\varepsilon)^{-1}(x - x_0)|^2 \\ &\le C\delta\left(\delta^{-\alpha_0/2}\right)^2(N\varepsilon) = C\delta^{1-\alpha_0}N\varepsilon \end{aligned}$$

in $\delta^{-\alpha_0/2}N\varepsilon$-neighborhood of x_0. □

Now we construct the near-boundary barrier f_ε using ω_ε. Let f_ε solve

$$\begin{cases} F(D^2 f_\varepsilon, x/\varepsilon) = 0 & \text{in } \{-N\varepsilon \le x_2 \le 0\}; \\ f_\varepsilon = \omega_\varepsilon + \delta^{1-\alpha_0}N\varepsilon & \text{on } H_{-N\varepsilon}; \\ \dfrac{\partial f_\varepsilon}{\partial x_2} = G\left(Df_\varepsilon, \dfrac{x}{\varepsilon}\right) & \text{on } H_0. \end{cases}$$

Step 5. Flatness of f_ε

In this step, we compare $\mu^N(G_k)$ given in (53) with $\mu_k(f_\varepsilon)$ given in Definition 5.8. For simplicity, we put $k = 1$. Note that Lemmas 3.2, 5.12, and 3.5 with (64) imply that

$$|\mu^N(G_1) - \mu_1(f_\varepsilon)| \le C\Big(\delta^{1-\alpha_0/2} + \delta + \delta^{1-\alpha_0}\Big) \le C\delta^{1-\alpha_0}. \tag{68}$$

Also from Lemma 5.12 and the definition of f_ε, it follows that f_ε is close to a linear function

$$|f_\varepsilon(x) - L_0(x)| \le C\delta^{1-\alpha_0}N\varepsilon \quad \text{on } H_{-N\varepsilon} \cap B_{\delta^{-\alpha_0/2}N\varepsilon}(0), \tag{69}$$

where $L_0(x) := f_\varepsilon(-N\varepsilon e_2) + \mu^N(G_1)(x_2 + N\varepsilon) + \partial_1 f_\varepsilon(-N\varepsilon e_2)x_1$. Then Lemma 4.4, (69), and Lemma 3.2 applied to the rescaled function $(N\varepsilon)^{-1}f_\varepsilon(N\varepsilon x)$ in the region $\{-1 \le x_2 \le -1/2\} \cap B_{\delta^{-\alpha_0/2}}$ yield that

$$|f_\varepsilon - L_0| \le C\Big(\delta^{1-\alpha_0} + \delta^{(1-\alpha_0/2)}\Big)N\varepsilon + C\varepsilon \le C\delta^{1-\alpha_0}N\varepsilon \tag{70}$$

in $\{-N\varepsilon \le x_2 \le -N\varepsilon/2\} \cap B_{\delta^{-\alpha_0/2}N\varepsilon}(0)$, where the last inequality follows from (47).

Before we proceed to the next step, observe that the C^1 regularity of Λ, Theorem 2.4, as well as Lemma 5.11 yield that

$$|\omega_\varepsilon(x_1, x_2) - \omega_\varepsilon(x_1, -N\varepsilon) - \Lambda(x)(x_2 + N\varepsilon)| \le C\delta^{1-\alpha_0}N\varepsilon \quad \text{on } \Big\{-N\varepsilon \le x_2 \le -\frac{N\varepsilon}{2}\Big\}. \tag{71}$$

Step 6. Patching up

Let $h(x) := l(x) + (\mu(\omega_\varepsilon) - C\delta^{1/2})(x_2 + KmN\varepsilon)$, where $C > 0$ is a constant given as in (b) of Theorem 4.1, and $l(x) = l(x_1)$ is a linear function chosen so that $h(x) = q \cdot x$ on H_{-1}. We define

$$\rho_\varepsilon := \begin{cases} h & \text{in } \{-1 \le x_2 \le -KmN\varepsilon\}, \\ \omega_\varepsilon & \text{in } \{-KmN\varepsilon \le x_2 \le -N\varepsilon/2\}. \end{cases}$$

Since Λ is $mN\varepsilon$-periodic, (b) of Theorem 4.1 implies that on $\{x_2 = -KmN\varepsilon\}$,

$$\partial_{x_2}\omega_\varepsilon \ge \mu(\omega_\varepsilon) - C\Lambda(1/K, e_2) = \mu(\omega_\varepsilon) - CK^{-1/2} = \mu(\omega_\varepsilon) - C\delta^{1/2} = \partial_{x_2}h.$$

Thus, it follows that $F(D^2\rho_\varepsilon, \frac{x}{\varepsilon}) \le 0$ in $\{-1 \le x_2 \le -N\varepsilon/2\}$.

Due to the flatness estimates (70) and (71), we can approximate f_ε and ρ_ε by linear functions, respectively, with normal derivatives of $\mu^N(G_k)$ and $\Lambda(x)$, with the error of $O(\delta^{1-\alpha_0}N\varepsilon)$. Here, recall that $\Lambda(x)$ was constructed so that $\Lambda(x) \ge \mu^N(G_k) + \delta^{\alpha_0}$, and α_0 is a constant satisfying $\alpha_0 \le 1/2$. Then since $f_\varepsilon = \rho_\varepsilon + \delta^{1-\alpha_0}N\varepsilon$ on $\{x_2 = -N\varepsilon\}$,

$$\rho_\varepsilon > f_\varepsilon \quad \text{on } \{x_2 = -N\varepsilon/2\} \quad \text{and} \quad f_\varepsilon > \rho_\varepsilon \quad \text{on } \{x_2 = -N\varepsilon\}. \tag{72}$$

Define $\underline{\rho}_\varepsilon$ as follows:

$$\underline{\rho}_\varepsilon := \begin{cases} \rho_\varepsilon & \text{in } \{-1 \le x_2 \le -N\varepsilon\}, \\ \min(\rho_\varepsilon, f_\varepsilon) & \text{in } \{-N\varepsilon \le x_2 \le -N\varepsilon/2\}, \\ f_\varepsilon & \text{in } \{-N\varepsilon/2 \le x_2 \le 0\}. \end{cases}$$

Then by (72), $\underline{\rho}_\varepsilon$ is a viscosity supersolution of $(P)_{\varepsilon,e_2,0,q}$ in $\{-1 \le x_2 \le 0\}$. Let us mention that, due to Lemmas 5.9, 5.10, and 5.11, a small perturbation of these barriers also yield a supersolution in $\{-1 \le x \cdot \nu_1 \le 0\}$. Similarly, one can construct a subsolution $\bar{\rho}_\varepsilon$ of $(P)_{\varepsilon,e_2,0,q}$ by replacing $\Lambda(x)$ given in the construction of ρ_ε by $\tilde{\Lambda}(x) \le \mu^N(G_k) - \delta^{\alpha_0}$. Then by Lemmas 5.9 and 5.11,

$$|\mu\big(u_\varepsilon^{\nu_1}\big) - \mu(\underline{\rho}_\varepsilon)| \le |\mu(\bar{\rho}_\varepsilon) - \mu(\underline{\rho}_\varepsilon)| + C\delta^{1/2} \le C\Big(\delta^{1/2} + \delta^{\alpha_0}\Big) \le C\delta^{\alpha_0} = C\delta^{1/2}, \tag{73}$$

where the last inequality follows by choosing $\alpha_0 = 1/2$.

We denote $\bar{\rho}_\varepsilon = \bar{\rho}_\varepsilon^{\nu_1}$ and $\underline{\rho}_\varepsilon = \underline{\rho}_\varepsilon^{\nu_1}$ indicating that they are obtained from the direction ν_1, i.e., with the scale $N\varepsilon$.

Step 7. Comparing the solutions $u_\varepsilon^{\nu_1}$ and $u_\varepsilon^{\nu_2}$: Proof of Theorem 5.2(a)

Parallel arguments as in the previous steps apply to the other direction ν_2. Recall that

$$\theta_2 = |\nu_2 - e_2| < \theta_1, \quad M = \left[\frac{\delta}{\tan\theta_2}\right] > N.$$

Then similarly as in the direction ν_1, we can construct barriers $\bar{\rho}_\varepsilon^{\nu_2}$ and $\underline{\rho}_\varepsilon^{\nu_2}$, such that

$$|\mu(u_\varepsilon^{\nu_2}) - \mu(\underline{\rho}_\varepsilon^{\nu_2})| \le |\mu(\bar{\rho}_\varepsilon^{\nu_2}) - \mu(\underline{\rho}_\varepsilon^{\nu_2})| + C\delta^{1/2} \le C\delta^{1/2}. \tag{74}$$

Here, their corresponding Neumann boundary conditions satisfy

$$\mu^M(G_k) - \delta^{\alpha_0} - \delta \le \frac{\partial}{\partial x_2}\bar{\rho}_\varepsilon^{\nu_2}; \quad \frac{\partial}{\partial x_2}\underline{\rho}_\varepsilon^{\nu_2} \le \mu^M(G_k) + \delta^{\alpha_0} + \delta \quad \text{on } H_{-M\varepsilon} \cap I_k,$$

where $\alpha_0 = 1/2$, and the respective derivatives of $\bar{\rho}_\varepsilon^{\nu_2}$ and $\underline{\rho}_\varepsilon^{\nu_2}$ are taken as a limit from the region $\{-1 \le x_2 < -M\varepsilon\}$.

Thus, to compare $\mu(u_\varepsilon^{\nu_1})$ and $\mu(u_\varepsilon^{\nu_2})$, we compare $\mu^N(G_k)$ and $\mu^M(G_k)$. Recall that we define $\mu^M(G_k)$ similarly as $\mu^N(G_k)$. More precisely, $\mu^M(G_k)$ is the slope of the linear approximation of $v_k^{M,\varepsilon}$, where $v_k^{M,\varepsilon}$ is defined similarly as in (52) in the region $\{-M\varepsilon \le x_2 \le 0\}$ with the boundary condition:

$$\partial_{x_2} v_k^{M,\varepsilon}(x) = G_k(Dv_k^{M,\varepsilon}, x/\varepsilon) \quad \text{on} \quad H_0$$

and $v_k^{M,\varepsilon} = l(x)$ on $H_{-M\varepsilon}$. Since G_k is periodic on the Neumann boundary, it corresponds to the case of Neumman boundary with rational normal, passing through the origin. Hence, by applying arguments as in the proof of (34),

$$|\mu^N(G_k) - \mu^M(G_k)| \le C\Lambda(1/N, e_2) = C \inf_{0<k<1}\{1/N^k + 1/N^{1-k}\} = C/N^{1/2}. \tag{75}$$

Now we prove the following lemma using the estimate (75).

Lemma 5.13. *For any ε satisfying* (44),

$$|\mu(u_\varepsilon^{\nu_1}) - \mu(u_\varepsilon^{\nu_2})| \le C\delta^{1/2}.$$

Proof. By the construction of the viscosity supersolution $\underline{\rho}_\varepsilon^{\nu_1}$ and Lemma 5.10,

$$|\mu(\underline{\rho}_\varepsilon^{\nu_1}) - \mu(\bar{\omega}_\varepsilon)| \le C\delta^{1/2}, \tag{76}$$

where $\bar{\omega}_\varepsilon$ is given as in (66). Similarly, we obtain

$$|\mu(\underline{\rho}_\varepsilon^{\nu_2}) - \mu(\bar{\omega}_\varepsilon^{\nu_2})| \le C\delta^{1/2}, \tag{77}$$

where $\bar{\omega}_\varepsilon^{\nu_2}$ solves

$$\begin{cases} \bar{F}(D^2\bar{\omega}_\varepsilon^{\nu_2}) = 0 & \text{in } \{-KmM\varepsilon \le x_2 \le -M\varepsilon/2\}; \\ \dfrac{\partial \bar{\omega}_\varepsilon^{\nu_2}}{\partial \nu} = \Lambda^{\nu_2}(x) & \text{on } H_{-M\varepsilon/2}; \\ \bar{\omega}_\varepsilon^{\nu_2} = l(x) & \text{on } H_{-KmM\varepsilon}. \end{cases}$$

Here, $\Lambda^{\nu_2}(x)$ is constructed similarly as $\Lambda(x)$ with N replaced by M, i.e., with $\mu^N(G_k)$ replaced by $\mu^M(G_k)$. Then by (73), (74), (76), and (77), it suffices to prove

$$|\mu(\bar{\omega}_\varepsilon) - \mu(\bar{\omega}_\varepsilon^{\nu_2})| \le C\delta^{1/2}.$$

Recall that $|\Lambda(x) - \mu^N(G_k)| \le \delta^{\alpha_0} + \delta$ on I_k, and similarly, $|\Lambda^{\nu_2}(x) - \mu^M(G_k)| \le \delta^{\alpha_0} + \delta$ on I_k, with $\alpha_0 = 1/2$. Hence,

$$|\mu(\bar{\omega}_\varepsilon) - \mu(h_1)|, \quad |\mu(\bar{\omega}_\varepsilon^{\nu_2}) - \mu(h_2)| \le C\delta^{1/2} \tag{78}$$

for solutions h_1 and h_2 of

$$\begin{cases} \bar{F}(D^2 h_1) = 0 & \text{in } \{-KmN\varepsilon \le x_2 \le -N\varepsilon/2\} \\ \dfrac{\partial h_1}{\partial \nu} = \mu^N(G_k) & \text{on } H_{-N\varepsilon/2} \cap I_k \\ h_1 = l(x) & \text{on } H_{-KmN\varepsilon} \end{cases}$$

and

$$\begin{cases} \bar{F}(D^2 h_2) = 0 & \text{in } \{-KmM\varepsilon \le x_2 \le -M\varepsilon/2\} \\ \dfrac{\partial h_2}{\partial \nu} = \mu^M(G_k) & \text{on } H_{-M\varepsilon/2} \cap I_k \\ h_2 = l(x) & \text{on } H_{-KmM\varepsilon}. \end{cases}$$

Note that h_1 has a periodic Neumann condition on $H_{-N\varepsilon/2}$ with period $mN\varepsilon$, and also h_2 has a periodic Neumann condition on $H_{-M\varepsilon/2}$ with period $mM\varepsilon$. Hence, they correspond to the case of periodic Neumann boundary data, i.e., the case of Neumann boundary with a normal direction e_2, and passing through the origin. Hence, by Theorem 4.1 with (75) and $K = 1/\delta$, we obtain

$$|\mu(h_1) - \mu(h_2)| \le \Lambda(\delta, e_2) + C/N^{1/2} \le C(\delta^{1/2} + (1/N)^{1/2}) \le C\delta^{1/2}, \tag{79}$$

where the last inequality follows from (47). Then we can conclude from (78) and (79). □

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