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Interface dynamics in a two-phase tumor growth model

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We study a tumor growth model in two space dimensions, where proliferation of the tumor cells leads to expansion of the tumor domain and migration of surrounding normal tissues into the exterior vacuum. The model features two moving interfaces separating the tumor, the normal tissue, and the exterior vacuum. We prove local-in-time existence and uniqueness of strong solutions for their evolution starting from a nearly radial initial configuration. It is assumed that the tumor has lower mobility than the normal tissue, which is in line with the well-known Saffman–Taylor condition in viscous fingering.

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1. Introduction

In this paper, we study free boundary dynamics arising in a model of avascular tumor growth which is adapted from [37].

1.1 *A two-species model of tumor growth*

Consider two species of cells in $\mathbb{R}^2$, one being actively growing tumor cell and the other being inactive normal cell. Spatial densities of tumor and normal cells, each denoted by m and n, satisfy

$$\partial_t m - \operatorname{div}(\mu m \nabla p) = mG(p), \tag{1.1}$$

$$\partial_t n - \operatorname{div}(\nu n \nabla p) = 0, \tag{1.2}$$

$$m + n \leqslant 1. \tag{1.3}$$

Here $\mu, \nu > 0$ denote mobilities of the tumor and normal cells. p is the pressure generated by the cells, serving as a Lagrange multiplier for the constraint $m + n \leqslant 1$. It satisfies

$$-\operatorname{div}\big((\mu m + \nu n)\nabla p\big) = mG(p) \quad \text{if } m + n = 1, \tag{1.4}$$

$$p = 0 \quad \text{if } m + n < 1. \tag{1.5}$$

See, e.g., [6, 40, 42]. In (1.1) and (1.4), $G(p)$ represents pressure-dependent proliferation rate of the tumor cell. In the spirit of [37], we assume that

1. $G \in C^1[0, +\infty)$.
2. $G(\cdot)$ is decreasing.
3. $G(0) > 0$ and $G(p_M) = 0$ for some $p_M > 0$.

In short, (1.1)–(1.5) models the scenario where the tumor keeps growing and where two species of cells migrate with different mobilities, according to the Darcy's law [7], under the pressure they generate together.

Mathematical analysis of strongly-coupled competitive systems such as (1.1)–(1.5) can be challenging [3, 4, 6, 11, 29, 33]. To the best of our knowledge, existing analyses of such problems are carried out either in one space dimension or with equal mobility of the two species. In contrast, it is suggested in [37] that the cells moving with different mobilities is an important feature of the model (1.1)–(1.5). Indeed, the numerical results in [37] show that when $\mu < \nu$, certain radially symmetric solution is stable, while when $\mu > \nu$ a Saffman–Taylor type instability [44] can occur.

1.2 *A free boundary problem*

In this paper, we study (1.1)–(1.5) with the restriction that m and n are segregated and fully saturated in their regions. Namely, we assume that $m = \chi_\Omega$ and $n = \chi_{\tilde{\Omega}\backslash\Omega}$, where $\Omega \subset\subset \tilde{\Omega}$ are two time-varying bounded domains. This gives rise to a free boundary problem that concerns dynamics of both $\gamma := \partial\Omega$ and $\tilde{\gamma} := \partial\tilde{\Omega}$, which are interfaces separating the tumor, the normal tissue, and the exterior vacuum.

First, the equation for p reduces to

$$-\mathrm{div}\left((\mu\chi_\Omega + \nu\chi_{\tilde{\Omega}\backslash\Omega})\nabla p\right) = \chi_\Omega G(p) \quad \text{in } \tilde{\Omega}, \quad p|_{\partial\tilde{\Omega}} = 0, \tag{1.6}$$

$$p = 0 \quad \text{in } \tilde{\Omega}^c. \tag{1.7}$$

Then the motion law of the free boundaries are given as follows. From (1.1), we may derive the normal velocity for γ:

$$V_{n,\gamma} = -\mu \frac{\partial p}{\partial \sigma_\Omega}. \tag{1.8}$$

Here σ_Ω denotes the unit outward normal vector of γ with respect to Ω. This is true heuristically because in (1.1), m migrates with the velocity field $-\mu\nabla p$. Formally, it can be derived by following the classic approach of studying singularity propagation in conservation laws (see, e.g., [25, § 3.4.1]). See also [34, 40, 42]. Similarly, the normal speed of $\tilde{\gamma}$ can be derived from (1.2):

$$V_{n,\tilde{\gamma}} = -\nu \frac{\partial p}{\partial \sigma_{\tilde{\Omega}}}, \tag{1.9}$$

where $\sigma_{\tilde{\Omega}}$ denotes the unit outward normal vector of $\tilde{\gamma}$ with respect to $\tilde{\Omega}$.

Our main result is the local-in-time well-posedness of the free boundary problem (1.6)–(1.9). Inspired by the numerical results in [37], we assume $\mu < \nu$ for the well-posedness. Interestingly, we will illustrate later that even with this assumption, instabilities may still occur along γ without further geometric assumptions on Ω and $\tilde{\Omega}$ (see Remark 2.5). We thus need to restrict ourselves to the case where the initial configuration is nearly radial (see Figure 1). More precise statement of our main results can be found in Theorem 2.1 and Theorem 2.3 in Section 2.3.

1.3 *Related works and our approach*

The evolution of the inner interface γ is similar to the 2-D Muskat problem [2, 41] with viscosity jump [39, 45], which is concerned with a close-to-flat interface between two fluids driven by the Darcy's law. In the case when the more viscous fluid is pushed towards the less viscous one, [45] establishes global well-posedness for small initial data; in the opposite case, ill-posedness is shown. With generalized Rayleigh–Taylor condition [23], [39] formulates similar result on the well-posedness in a more general setup allowing density-viscosity jumps. Note that these rigorous results agree very well with [44] and the aforementioned numerical results in [37]. They are obtained by exploring the inherent parabolicity in the interface motion with complex analysis [45] and functional analytic [39] approaches. However, it is not clear if these approaches can be directly applied here as our model involves a geometry-dependent source term, whose support touches γ.

Notably there is a lot more literature concerning the Muskat problem with density jump [1, 16–19] or density-viscosity jumps [13–15, 26, 39]. In both of these cases, the smoothing mechanism is essentially provided by the fact that a heavier fluid sits below a lighter one, where the gravity naturally damps the oscillation of the interface. In contrast, the smoothing mechanism is much less explicit when there is only jump in the viscosity across the interface [39, 45].

Motion of the outer interface $\tilde{\gamma}$ is reminiscent of the free boundary arising in the one-phase Hele–Shaw problem [43], where a blob of fluid is injected into a Hele–Shaw cell or a porous medium and expands according to the Darcy's law. Despite its similarity with the Muskat problem in some aspects, it admits a few other treatments. We direct the readers to [12, 20, 22, 24, 30, 31] and the references therein. Once again, in our problem, the presence of the source term depending nonlocally on $\tilde{\gamma}$ and γ may hinder direct applications of these approaches.

In this paper, we study the dynamics of both interfaces γ and $\tilde{\gamma}$ in a unified framework, adapted from the study of contour equations in the Muskat problem [15, 26]. We first reduce (1.6)–(1.9), which involves an elliptic equation for p in a time-varying domain, partially into contour equations for the interface configurations and quantities along them; see (2.16), (2.17), (2.33) and (2.34). A key step in this reduction is to represent the transporting velocity over $\tilde{\Omega}$ as a sum of three parts, which arise from the discontinuity of the cell mobilities across γ, the zero Dirichlet boundary condition of p along $\tilde{\gamma}$, and the source term in Ω, respectively; see (2.12) and also (2.3). Then by linearizing these contour equations around radially symmetric configurations, we show their parabolic nature under suitable conditions (cf. Section 2.4). In particular, the interfaces can smooth themselves according to a fractional-heat-type equation with source terms. After deriving good estimates for these source terms, we prove well-posedness of the interface motion by a fixed-point argument. Smallness of the geometric deviation of γ and $\tilde{\gamma}$ from radially symmetric configurations helps close the estimates needed in this argument. See Section 2 for more details.

1.4 *Difficulties arising from the source term*

This problem features a geometry-dependent source term $\chi_\Omega G(p)$ in (1.6) that is supported up to the inner interface. It may be tempting to think of it as an innocuous regular term, but in fact, it changes the dynamics in a crucial way compared to the related problems discussed above.

Firstly, on the technical level, the source term seems to prevent the complex analysis approach in [45] from being applied here. Secondly, the parabolicity of γ relies on the fact that the cell with lower mobility is displacing the other species, i.e., $(\mu - \nu)\frac{\partial p}{\partial \sigma_\Omega}|_\gamma > 0$ (cf. (1.8) and Remark 2.4), in line with the classic Saffman–Taylor condition [44]. Since $\chi_\Omega G(p)$ depends on the domain geometry, one can manufacture such Ω and $\tilde{\Omega}$, so that the tumor is pushed by the normal tissue along some part of γ under the assumption $\mu < \nu$. This is possible even when both γ and $\tilde{\gamma}$ are required to be graphs of functions over $\mathbb{T}$ in the polar coordinate; see Remark 2.5. In this sense, simply assuming $\mu < \nu$ is not enough for proving well-posedness, and it is reasonable to additionally require that γ and $\tilde{\gamma}$ are close to concentric circles (see Figure 1). Then characterization of the parabolicity of γ is based on a good understanding of p. In Section 3, we apply elliptic regularity theory to justify that given the domain geometry close to a radially symmetric one, the corresponding p should not be far away from a radially solution. That would be sufficient to guarantee parabolicity in the motion of γ as $\mu < \nu$. Furthermore, these elliptic estimates together with the results in Section 4 and Section 5 will help justify that such parabolicity can be characterized by a fractional heat operator with exponent $\frac{1}{2}$, which plays a central role in our analysis. See Section 2.4 and Section 8.

The source term also poses new difficulty in studying global well-posedness and stability properties near the radially symmetric solutions. Indeed, as the tumor grows larger, the pressure becomes more sensitive to the interface geometry. We demonstrate this by a scaling argument. Suppose at given time $T > 0$, Ω and $\tilde{\Omega}$ are close to two concentric discs, and $\tilde{\Omega}$ has radius of order $R \gg 1$. Define $p_R(x,t) := p(Rx, R(t-T))$ and let $\tilde{\Omega}_R$ and Ω_R denote the corresponding dilated version of $\tilde{\Omega}$ and Ω according to the scaling. Then (1.6) becomes

$$-\operatorname{div}\left((\mu\chi_{\Omega_R} + \nu\chi_{\tilde{\Omega}_R\setminus\Omega_R})\nabla p_R\right) = \chi_{\Omega_R} R^2 G(p_R), \tag{1.10}$$

with zero boundary data on $\partial\tilde{\Omega}_R$, while the boundary motion laws (1.8) and (1.9) remain the same. In this new problem, the proliferation rate $R^2 G(\cdot)$ can have a large magnitude where p_R is small and it is sensitive to the pressure. This results in concentration of the source term near the inner interface and a steep growth of p_R there. On the other hand, the total mass of the normal tissue is preserved due to (1.2), and thus $\tilde{\Omega}_R\setminus\Omega_R$ is extremely thin as $R \gg 1$. So in the rescaled problem the source term is close to both the inner and outer interfaces. It is then conceivable that p_R will be highly sensitive to the domain geometry in the sense that even when the domain is pretty close to being radial, p_R may be highly oscillatory and far from being radially symmetric. Therefore, because of the source term, nonlinear stability of the interface configurations around radially symmetric ones becomes a much more subtle issue when it comes to long time asymptotics.

2. Interface motion in an almost radially symmetric geometry

In this section, we will derive equations for the moving interfaces γ and $\tilde{\gamma}$ in the case when they are close to concentric circles. Our main result will be established in terms of these equations. Parabolicity of these equations will be revealed, which plays a key role in proving the well-posedness.

2.1 *Problem reformulation*

Define a potential φ to be

$$\varphi := \mu p \text{ in } \Omega, \quad \varphi := \nu p \text{ in } \Omega^c. \tag{2.1}$$

So φ solves

$$-\Delta\varphi = G(p)\chi_\Omega \quad \text{in } \tilde{\Omega}\setminus\gamma, \quad \varphi|_{\tilde{\gamma}} = 0,$$

and $\varphi \equiv 0$ on $\tilde{\Omega}^c$. When $\mu \neq \nu$, φ has discontinuity across γ, denoted by

$$[\varphi]_\gamma(x) := \varphi|_{\gamma,\Omega}(x) - \varphi|_{\gamma,\Omega^c}(x), \quad x \in \gamma.$$

(1.8) and (1.9) yield that each cell phase is transported by the velocity field $u = -\nabla\varphi$. It has discontinuity across γ in the tangential component, but not in its normal component.

Let Γ denote the fundamental solution of the Laplace equation in $\mathbb{R}^2$,

$$\Gamma(x) := -\frac{1}{2\pi}\ln|x|.$$

Let $\mathcal{D}_\gamma$ denote the double layer potential operator associated with γ. Namely, with a boundary potential ψ defined on γ, we define $\mathcal{D}_\gamma\psi$ on $\mathbb{R}^2$ to be

$$\mathcal{D}_\gamma\psi(x) := \int_\gamma \sigma_y \cdot \nabla_y\big(\Gamma(x-y)\big)\psi(y)\,dy. \tag{2.2}$$

Note that here the gradient is taken with respect to y. It is well-known that for γ and ψ sufficiently smooth, say $C^{1,\alpha}(\mathbb{T})$, $[\mathcal{D}_\gamma\psi]_\gamma = -\psi$ and $\mathcal{D}_\gamma\psi$ is harmonic in $\mathbb{R}^2\backslash\gamma$. Then φ admits the following representation:

$$\varphi = -\mathcal{D}_\gamma[\varphi] - \mathcal{D}_{\tilde{\gamma}}\phi + \Gamma * g \quad \text{in } \tilde{\Omega}\backslash\gamma, \tag{2.3}$$

where ϕ is some boundary potential defined along $\tilde{\gamma}$ to be determined in order for the boundary condition $\varphi|_{\tilde{\gamma}} = 0$, and where

$$g = G(p)\chi_\Omega = G(\mu^{-1}\varphi)\chi_\Omega \geqslant 0. \tag{2.4}$$

Assume $C^{1,\alpha}(\mathbb{T})$-regularity of γ and $[\varphi]$. Then the representation (2.3) along γ takes the average of φ on two sides of γ, i.e.,

$$(-\mathcal{D}_\gamma[\varphi] - \mathcal{D}_{\tilde{\gamma}}\phi + \Gamma * g)\big|_\gamma = \frac{1}{2}(\varphi|_{\gamma,\Omega} + \varphi|_{\gamma,\Omega^c}) = \frac{\mu+\nu}{2}p = \frac{\mu+\nu}{2(\mu-\nu)}[\varphi].$$

This implies

$$[\varphi] = 2A(-\mathcal{D}_\gamma[\varphi] - \mathcal{D}_{\tilde{\gamma}}\phi + \Gamma * g)|_\gamma, \tag{2.5}$$

where

$$A = \frac{\mu-\nu}{\mu+\nu}.$$

On the other hand, the zero Dirichlet boundary condition of φ along $\tilde{\gamma}$ requires that

$$\lim_{\substack{x\to\tilde{\gamma}(\theta)\\ x\in\tilde{\Omega}}} (-\mathcal{D}_{\tilde{\gamma}}\phi)(x) = (\mathcal{D}_\gamma[\varphi] - \Gamma * g)|_{\tilde{\gamma}(\theta)}.$$

Assuming $C^{1,\alpha}(\mathbb{T})$-regularity of $\tilde{\gamma}$ and ϕ, by the property of the double layer potential, ϕ should solve

$$-(\mathcal{D}_{\tilde{\gamma}}\phi)|_{\tilde{\gamma}} + \frac{1}{2}\phi = (\mathcal{D}_\gamma[\varphi] - \Gamma * g)|_{\tilde{\gamma}}$$

along $\tilde{\gamma}$, i.e.,

$$\phi = 2(\mathcal{D}_{\tilde{\gamma}}\phi + \mathcal{D}_\gamma[\varphi] - \Gamma * g)|_{\tilde{\gamma}}. \tag{2.6}$$

Finally, (1.8) and (1.9) become

$$V_{n,\gamma} = -\frac{\partial\varphi}{\partial\sigma_\Omega}, \quad V_{n,\tilde{\gamma}} = -\frac{\partial\varphi}{\partial\sigma_{\tilde{\Omega}}}. \tag{2.7}$$

(2.3)–(2.7) form a closed system.

2.2 *Derivation of contour equations*

We consider the case when γ and $\tilde{\gamma}$ are close to two concentric circles centered at the origin, with some radii $r < R$, respectively. See Figure 1. We parameterize γ and $\tilde{\gamma}$ using the polar coordinate,

$$\gamma(\theta,t) = f(\theta,t)(\cos\theta,\sin\theta), \tag{2.8}$$

$$\tilde{\gamma}(\theta,t) = F(\theta,t)(\cos\theta,\sin\theta), \tag{2.9}$$

where $\theta \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}) = [-\pi,\pi)$. Then $[\varphi]$ and ϕ can be naturally understood as functions of $\theta \in \mathbb{T}$. Next we shall derive equations for γ and $\tilde{\gamma}$ (or equivalently, for f and F). Since the derivation will be carried out at a fixed time t, for brevity, we will omit the time dependence of functions in most places.

Note that $\sigma_\Omega(\theta) = -\gamma'(\theta)^\perp/|\gamma'(\theta)|$, where $v^\perp$ denote a vector $v \in \mathbb{R}^2$ rotated counter-clockwise by $\pi/2$. By (2.2), all $x \in \tilde{\Omega}\backslash\gamma$,

$$\mathcal{D}_\gamma[\varphi](x) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\big(x-\gamma(\theta')\big)\cdot\big(-\gamma'(\theta')\big)^\perp}{|x-\gamma(\theta')|^2}[\varphi](\theta')\,d\theta'. \tag{2.10}$$

By assuming $[\varphi] \in C^1(\mathbb{T})$,

$$\begin{aligned}\nabla\mathcal{D}_\gamma[\varphi](x) &= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\partial}{\partial\theta'}\left(-\frac{\big(x-\gamma(\theta')\big)^\perp}{|x-\gamma(\theta')|^2}\right)[\varphi](\theta')\,d\theta' \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\big(x-\gamma(\theta')\big)^\perp}{|x-\gamma(\theta')|^2}[\varphi]'(\theta')\,d\theta'.\end{aligned} \tag{2.11}$$

which is a Birkhoff–Rott-type integral [38]. Hence,

$$\begin{aligned}u(x) &= -\nabla\varphi(x) \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\big(x-\gamma(\theta')\big)^\perp}{|x-\gamma(\theta')|^2}[\varphi]'(\theta')\,d\theta' + \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\big(x-\tilde{\gamma}(\theta')\big)^\perp}{|x-\tilde{\gamma}(\theta')|^2}\phi'(\theta')\,d\theta' - \nabla(\Gamma * g)(x).\end{aligned} \tag{2.12}$$

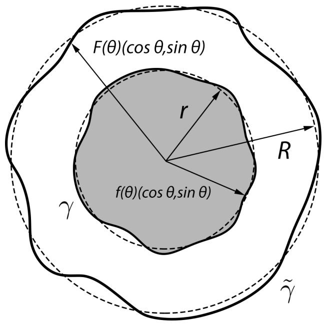

FIG. 1. An illustration of the geometry. The grey region represents the domain of the tumor cells, while the white region surrounding it is occupied by the normal cells. The solid curves γ and $\tilde{\gamma}$ are moving boundaries of the domains. The dashed circles indicate that γ and $\tilde{\gamma}$ are close to two concentric circles with radii r and R, respectively. γ and $\tilde{\gamma}$ are parameterized in the polar coordinate as functions of $\theta \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$.

On the other hand, by (2.7) and (2.8),

$$\partial_t f(\theta) = u\big(\gamma(\theta)\big) \cdot \sigma_\Omega(\theta) \cdot \frac{|\gamma'(\theta)|}{f(\theta)} = -\frac{1}{f} \cdot u\big(\gamma(\theta)\big) \cdot \gamma'(\theta)^\perp. \tag{2.13}$$

Although $u(\gamma(\theta))$ here should be understood as the limit of (2.12) when letting $x \to \gamma(\theta)$ from the inside of γ, it is safe to simply take $x = \gamma(\theta)$ since the normal component of u does not have discontinuity across γ. Define

$$\mathcal{K}_\gamma \psi(\theta) := \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{T}} \frac{\gamma(\theta) - \gamma(\theta')}{|\gamma(\theta) - \gamma(\theta')|^2} \cdot \psi(\theta')\, d\theta', \tag{2.14}$$

$$\mathcal{K}_{\gamma,\tilde{\gamma}} \psi(\theta) := \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\gamma(\theta) - \tilde{\gamma}(\theta')}{|\gamma(\theta) - \tilde{\gamma}(\theta')|^2} \cdot \psi(\theta')\, d\theta'. \tag{2.15}$$

Let $\mathcal{K}_{\tilde{\gamma},\gamma}\psi(\theta)$ be defined symmetrically by interchanging γ and $\tilde{\gamma}$ in (2.15). Thanks to (2.11) and (2.12), (2.13) can be rewritten as

$$\partial_t f = -\frac{1}{f}\gamma'(\theta) \cdot \mathcal{K}_\gamma[\varphi]' - \frac{1}{f}\gamma'(\theta) \cdot \mathcal{K}_{\gamma,\tilde{\gamma}}\phi' + \frac{1}{f}\nabla(\Gamma * g)|_{\gamma}(\theta) \cdot \gamma'(\theta)^\perp. \tag{2.16}$$

Similarly,

$$\partial_t F = -\frac{1}{F}\tilde{\gamma}'(\theta) \cdot \mathcal{K}_{\tilde{\gamma}}\phi' - \frac{1}{F}\tilde{\gamma}'(\theta) \cdot \mathcal{K}_{\tilde{\gamma},\gamma}[\varphi]' + \frac{1}{F}\nabla(\Gamma * g)|_{\tilde{\gamma}}(\theta) \cdot \tilde{\gamma}'(\theta)^\perp. \tag{2.17}$$

These equations are coupled with initial conditions

$$f(t=0) = f_0(\theta), \quad F(t=0) = F_0(\theta). \tag{2.18}$$

For future use, we introduce

$$h(\theta,t) = \frac{f(\theta,t)}{r} - 1, \quad H(\theta,t) = \frac{F(\theta,t)}{R} - 1. \tag{2.19}$$

They are relative deviations of γ and $\tilde{\gamma}$ from radially symmetric configurations.

2.3 *Main results*

We first introduce some norms that will be used in the rest of the paper.

For this moment, let $f = f(\theta)$ denote a general function defined on $\mathbb{T}$. With $\alpha \in (0,1)$, denote

$$\|f\|_{\dot{C}^\alpha(\mathbb{T})} := \sup_{\theta_1,\theta_2 \in \mathbb{T}, \theta_1 \neq \theta_2} \frac{|f(\theta_1) - f(\theta_2)|}{|\theta_1 - \theta_2|^\alpha}.$$

For $k \in \mathbb{N}$, let $f^{(k)}$ denote the k-th derivative of f with respect to θ. Then define

$$\|f\|_{\dot{C}^{k,\alpha}(\mathbb{T})} := \|f^{(k)}\|_{\dot{C}^\alpha(\mathbb{T})}, \quad \text{and} \quad \|f\|_{C^{k,\alpha}(\mathbb{T})} := \sum_{j=0}^{k} \|f^{(j)}\|_{C(\mathbb{T})} + \|f\|_{\dot{C}^{k,\alpha}(\mathbb{T})}.$$

We say that $f \in C^{k,\alpha}(\mathbb{T})$ if and only if $\|f\|_{C^{k,\alpha}(\mathbb{T})} < +\infty$.

For $k \in \mathbb{Z}_+$ and $p \in (1, \infty)$, we define

$$\|f\|_{\dot{W}^{k,p}(\mathbb{T})} := \|f^{(k)}\|_{L^p(\mathbb{T})}, \quad \text{and} \quad \|f\|_{W^{k,p}(\mathbb{T})} := \sum_{j=0}^{k} \|f^{(j)}\|_{L^p(\mathbb{T})}.$$

We say that $f \in W^{k,p}(\mathbb{T})$ if and only if $\|f\|_{W^{k,p}(\mathbb{T})} < +\infty$.

We also define $W^{k-\frac{1}{p},p}(\mathbb{T})$-space for $k \in \mathbb{Z}_+$ and $p \in (1,\infty)$ [47, §2.12.2]. Let $\{e^{-t(-\Delta)^{1/2}}\}_{t\geqslant 0}$ denote the Poisson semi-group on $\mathbb{T}$ with generator $-(-\Delta)^{1/2}$. For $f \in L^p(\mathbb{T})$, let

$$\|f\|_{\dot{W}^{k-\frac{1}{p},p}(\mathbb{T})} := \left\| e^{-t(-\Delta)^{1/2}} f \right\|_{L^p_{[0,\infty)} \dot{W}^{k,p}(\mathbb{T})}. \tag{2.20}$$

We say that $f \in W^{k-\frac{1}{p},p}(\mathbb{T})$ if and only if $f \in L^p(\mathbb{T})$ such that $\|f\|_{\dot{W}^{k-\frac{1}{p},p}(\mathbb{T})} < +\infty$.

Our main results are as follows.

Theorem 2.1 *Suppose $0 < \mu < \nu$. Let G satisfy the assumptions in Section 1. Suppose $f_0, F_0 \in W^{2-\frac{1}{p},p}(\mathbb{T})$ for some $p \in (2,\infty)$. Let*

$$r = \frac{1}{2\pi}\int_{\mathbb{T}} f_0(\theta)\,d\theta, \quad R = \frac{1}{2\pi}\int_{\mathbb{T}} F_0(\theta)\,d\theta. \tag{2.21}$$

With p_* be defined by (3.8), let c_* and $\tilde{c}_*$ be negative constants

$$c_* = -\frac{1}{2\pi r}\int_{B_r} G\big(p_*(X)\big)\,dX, \quad \tilde{c}_* = \frac{r}{R}c_*, \tag{2.22}$$

which are negative speeds of interfaces when they turn out to be concentric circles with radii r and R respectively (see, e.g., (3.13)). Take δ such that

$$\frac{R-r}{100R} \leqslant \delta \leqslant \frac{R-r}{10R}. \tag{2.23}$$

Define h_0 and H_0 as in (2.19).

Suppose h_0 and H_0 satisfy that, with $\alpha = 1 - \frac{2}{p}$ and for some $\varepsilon > 0$,

$$M := \delta^{-1}(\|h_0\|_{L^\infty(\mathbb{T})} + \|H_0\|_{L^\infty(\mathbb{T})}) + \delta^{\alpha-\varepsilon}\left(\|h_0\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})} + \|H_0\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})}\right) \leqslant M_*, \tag{2.24}$$

where M_* is a small constant depending on p, ε, μ, ν, $R/|\tilde{c}_*|$, G and δR^2, but not directly on δ. Then there exists $T > 0$ depending on the above quantities and additionally on δ, such that the system (2.16)–(2.18) admits a strong solution

$$f, F \in C_{[0,T]}C^{1,\alpha}(\mathbb{T}) \cap L^p_{[0,T]}W^{2,p}(\mathbb{T}), \tag{2.25}$$

with $\partial_t f, \partial_t F \in C_{[0,T]}C^{\alpha''}(\mathbb{T})$ for any $\alpha'' < \min\{\frac{1}{4}, \alpha\}$. The solution satisfies that, with h and H defined in (2.19),

$$\delta^{-1}(\|h\|_{C_{[0,T]}L^\infty} + \|H\|_{C_{[0,T]}L^\infty}) + \delta^{\alpha-\varepsilon}\left(\|h\|_{C_{[0,T]}\dot{C}^{1,\alpha}} + \|H\|_{C_{[0,T]}\dot{C}^{1,\alpha}}\right) \leqslant C(p,G)M, \tag{2.26}$$

$$\delta^{\alpha-\varepsilon}\left(\|\partial_t h\|_{L^p_{[0,T]}\dot{W}^{1,p}}+\|\partial_t H\|_{L^p_{[0,T]}\dot{W}^{1,p}}\right)\leqslant C(p,\mu,\nu,G)M, \tag{2.27}$$

and

$$\delta^{\alpha-\varepsilon}\left(\|h\|_{L^p_{[0,T]}\dot{W}^{2,p}}+\|H\|_{L^p_{[0,T]}\dot{W}^{2,p}}\right)\leqslant C(p,\mu,\nu,R/|\tilde{c}_*|,G)M. \tag{2.28}$$

REMARK 2.1 In the claim $\partial_t f, \partial_t F \in C_{[0,T]}C^{\alpha''}(\mathbb{T})$ ($\alpha'' < \min\{\frac{1}{4},\alpha\}$), we did not pursue the optimal range of the Hölder exponent α''.

REMARK 2.2 We use δ to characterize the relative thinness of the gap between γ and $\tilde{\gamma}$. The smallness condition (2.24) simply means that h_0 and H_0 need to be small in certain sense compared with δ. Note that requiring $\delta^{-1}(\|h_0\|_{L^\infty}+\|H_0\|_{L^\infty})\ll 1$ in (2.24) seems very natural, as otherwise the two interfaces may touch or cross each other. It is worthwhile to remark that the right hand side of (2.24) does not deteriorate as δ becomes smaller, in the sense that if all the model parameters and R are fixed and we let $r\to R$ (so that $\delta\to 0$), then the right hand side does not decrease to 0. Though δ also shows up on the right hand side in the form of δR^2, it will be clear later (see (8.43) in the proof of Theorem 2.1) that M_* increases as δR^2 decreases.

In contrast, the smallness of T has to depend on δ directly: when $\delta\ll 1$, we may need $T\ll 1$.

REMARK 2.3 In the 2-D Muskat problem, $\dot{W}^{1,\infty}$ and $\dot{H}^{3/2}$ are considered to be critical and scaling-invariant semi-norms [26]. Although our problem does not admit any scaling law, considering its similarity with the Muskat problem, the best thing one can do seems to be proving well-posedness with initial data being small in $W^{1,\infty}(\mathbb{T})$- or $H^{3/2}(\mathbb{T})$-norms. We note that in Theorem 2.1, the condition (2.24) on the initial data is proposed in the way that, by interpolation, $C^{1,\beta'}$-semi-norms of h_0 and H_0 are small for some $\beta'>0$ depending on p and ε (see (8.25) and (8.31)). In other words, although we are not able to prove well-posedness of our problem with smallness in the "critical" spaces, partly because of the source term, we manage to do that in all the "sub-critical" cases, which can be arbitrarily close to the "critical" one – note that $p>2$ and $\varepsilon>0$ are arbitrary.

Thanks to the estimates for the local solution, one can apply Theorem 2.1 iteratively and show that local solutions exist for an arbitrary time period $\tilde{T}>0$ as long as h_0 and H_0 are correspondingly sufficiently small.

COROLLARY 2.2 Under the assumptions of Theorem 2.1, for any $\tilde{T}>0$, if $h_0, H_0\in W^{2-\frac{1}{p},p}(\mathbb{T})$ satisfy $M\ll 1$, where the smallness depends on $p,\varepsilon,\mu,\nu,G,r,R$ and $\tilde{T}$, the local strong solution exists up to time $\tilde{T}$.

Uniqueness of local solutions can be shown if G is more regular.

Theorem 2.3 *Under the assumptions of Theorem 2.1, if in addition, $G\in C^{1,1}[0,+\infty)$, then the solution is unique.*

2.4 *Parabolic nature of the interface motion and scheme of the proof*

To elucidate the hidden parabolicity of (2.16)–(2.18), we linearize the system around the radially symmetric configurations.

It is convenient to first derive equations for $[\varphi]'$ and ϕ' by taking derivative in (2.5) and (2.6). Assuming $\gamma,[\varphi]\in C^1(\mathbb{T})$, we have

$$\frac{d}{d\theta}(\mathcal{D}_\gamma[\varphi])|_\gamma(\theta)=-\gamma'(\theta)^\perp\cdot\mathcal{K}_\gamma[\varphi]'. \tag{2.29}$$

Indeed, by integration by parts,

$$\begin{aligned}(\mathcal{D}_\gamma[\varphi])|_\gamma(\theta) &= -\frac{1}{2\pi}\text{p.v.}\int_{\mathbb{T}} \partial_{\theta'}\big[\arg\big((\gamma(\theta)-\gamma(\theta'))_1 + i(\gamma(\theta)-\gamma(\theta'))_2\big)\big]\cdot[\varphi](\theta')\,d\theta' \\ &= -\frac{1}{2\pi}\cdot\pi[\varphi](\theta) + \frac{1}{2\pi}\int_{\mathbb{T}}\arg\big((\gamma(\theta)-\gamma(\theta'))_1 + i(\gamma(\theta)-\gamma(\theta'))_2\big)\cdot[\varphi]'(\theta')\,d\theta'.\end{aligned} \tag{2.30}$$

Here the argument is defined such that its values at $\theta = \pm\pi$ coincide. In the last equality, we need the assumption $\gamma \in C^1(\mathbb{T})$. Hence, using the fact that $[\varphi] \in C^1(\mathbb{T})$,

$$\begin{aligned}\frac{d}{d\theta}(\mathcal{D}_\gamma[\varphi])|_\gamma &= -\frac{1}{2}[\varphi]' + \frac{1}{2\pi}\frac{d}{d\theta}\int_{\mathbb{T}}\arg\Big((\gamma(\theta)-\gamma(\theta'))_1 + i(\gamma(\theta)-\gamma(\theta'))_2\Big)\cdot[\varphi]'(\theta')\,d\theta' \\ &= \frac{1}{2\pi}\text{p.v.}\int_{\mathbb{T}}\frac{d}{d\theta}\left(\arg\Big((\gamma(\theta)-\gamma(\theta'))_1 + i(\gamma(\theta)-\gamma(\theta'))_2\Big)\right)\cdot[\varphi]'(\theta')\,d\theta'.\end{aligned} \tag{2.31}$$

This justifies (2.29). Next let

$$e_r := (\cos\theta, \sin\theta), \quad e_\theta := (-\sin\theta, \cos\theta). \tag{2.32}$$

Then $[\varphi]'$ and ϕ' satisfy

$$[\varphi]' = 2A\left((f'(\theta)e_r + f(\theta)e_\theta)\cdot\nabla(\Gamma * g)|_\gamma + \gamma'(\theta)^\perp\cdot\mathcal{K}_\gamma[\varphi]' + \gamma'(\theta)^\perp\cdot\mathcal{K}_{\gamma,\tilde\gamma}\phi'\right), \tag{2.33}$$

$$\phi' = -2\left((F'(\theta)e_r + F(\theta)e_\theta)\cdot\nabla(\Gamma * g)|_{\tilde\gamma} + \tilde\gamma'(\theta)^\perp\cdot\mathcal{K}_{\tilde\gamma}\phi' + \tilde\gamma'(\theta)^\perp\cdot\mathcal{K}_{\tilde\gamma,\gamma}[\varphi]'\right). \tag{2.34}$$

Now we shall linearize the equations (2.16), (2.17), (2.33) and (2.34) around the radially symmetric configurations, i.e., $f \equiv r$, $F \equiv R$, and $[\varphi]' = \phi' \equiv 0$. The following discussion is only formal and gives an overview of the analysis carried out in the rest of the paper. Let us begin by collecting a few facts that will be justified in later sections.

- It will be clear in Section 4 and Section 7 that

$$e_r\cdot\nabla(\Gamma * g)|_\gamma \approx c_* \quad \text{and} \quad e_r\cdot\nabla(\Gamma * g)|_{\tilde\gamma} \approx \tilde c_* := \frac{c_* r}{R}. \tag{2.35}$$

  Here c_* and $\tilde c_*$ are constants defined in (2.22).
- Let $\mathcal{H}$ be the Hilbert transform on $\mathbb{T}$ [28], i.e.,

$$\mathcal{H}f(\theta) := \frac{1}{2\pi}\text{p.v.}\int_{\mathbb{T}}\cot\left(\frac{\theta-\theta'}{2}\right)f(\theta')\,d\theta'. \tag{2.36}$$

  Then in Section 5 we shall show

$$\gamma'\cdot\mathcal{K}_\gamma \approx \frac{1}{2}\mathcal{H} \quad \text{and} \quad \tilde\gamma'\cdot\mathcal{K}_{\tilde\gamma} \approx \frac{1}{2}\mathcal{H}. \tag{2.37}$$

- Define $\mathcal{S}$ to be a smoothing operator on $\mathbb{T}$ with a Poisson kernel,

$$\mathcal{S}\psi(\theta) = \frac{1}{2\pi}P_{\frac{r}{R}} * \psi(\theta) = \frac{1}{2\pi}\int_{\mathbb{T}}\frac{1-\left(\frac{r}{R}\right)^2}{1+\left(\frac{r}{R}\right)^2 - 2\left(\frac{r}{R}\right)\cos\xi}\psi(\theta-\xi)\,d\xi. \tag{2.38}$$

The notation $P_{\frac{r}{R}}$ will be introduced in Section 6. Then in Section 6 we shall see

$$\gamma' \cdot \mathcal{K}_{\gamma,\tilde{\gamma}}\phi' \approx \frac{1}{2}\mathcal{H}\mathcal{S}\phi', \qquad \gamma'^{\perp} \cdot \mathcal{K}_{\gamma,\tilde{\gamma}}\phi' \approx \frac{1}{2}\mathcal{S}\phi', \tag{2.39}$$

$$\tilde{\gamma}' \cdot \mathcal{K}_{\tilde{\gamma},\gamma}[\varphi]' \approx \frac{1}{2}\mathcal{H}\mathcal{S}[\varphi]', \qquad \tilde{\gamma}'^{\perp} \cdot \mathcal{K}_{\tilde{\gamma},\gamma}[\varphi]' \approx -\frac{1}{2}\mathcal{S}[\varphi]'. \tag{2.40}$$

- The remaining terms in (2.16), (2.17), (2.33) and (2.34) and the error made above are considered to be smaller or more regular, which will be omitted for this moment.

Putting these facts together, the linearized system can be written as

$$\partial_t f + c_* = -\frac{1}{2r}\mathcal{H}([\varphi]' + \mathcal{S}\phi'), \tag{2.41}$$

$$\partial_t F + \frac{c_* r}{R} = -\frac{1}{2R}\mathcal{H}(\phi' + \mathcal{S}[\varphi]'). \tag{2.42}$$

$$[\varphi]' = 2Ac_* f' + A\mathcal{S}\phi', \tag{2.43}$$

$$\phi' = -\frac{2c_* r}{R}F' + \mathcal{S}[\varphi]'. \tag{2.44}$$

See Section 7 and Section 8 for the complete equations.

Combining (2.43) and (2.41), we obtain

$$\partial_t f + c_* = -\frac{Ac_*}{r}(-\Delta)^{1/2} f - \frac{1+A}{2r}\mathcal{H}\mathcal{S}\phi'. \tag{2.45}$$

(2.45) is a fractional heat equation only when $Ac_* > 0$. Note that the last term in (2.45) and all those omitted ones are supposed to be small or regular source terms. Since $c_* < 0$, it is natural to believe that the motion of γ can be well-posed only when $A < 0$, i.e., $\mu < \nu$.

Similarly, by combining (2.42) with (2.44),

$$\partial_t F + \frac{c_* r}{R} = \frac{c_* r}{R^2}(-\Delta)^{1/2} F - \frac{1}{R}\mathcal{H}\mathcal{S}[\varphi]'. \tag{2.46}$$

Note that it shows the smoothing of the outer interface not to depend on A, but only on the fact that $\frac{c_* r^2}{R} < 0$.

REMARK 2.4 The above formal derivation may be localized as long as the interfaces are locally graphs and sufficiently smooth. By doing so we may be able to show that the local parabolicity condition for the motion of γ is $(\mu - \nu)\frac{\partial p}{\partial \sigma_\Omega}|_\gamma > 0$, while it is $\frac{\partial p}{\partial \sigma_{\tilde{\Omega}}}|_{\tilde{\gamma}} < 0$ for the motion of $\tilde{\gamma}$. The former condition implies that when the less mobile cells are locally pushing the other one, we expect well-posedness in the motion of that local segment of γ. This is in the same spirit as the Saffman–Taylor condition [44] (see also the condition for well-posedness in [45]), and it is formulated in a more general setting in [39]. The parabolicity condition $\frac{\partial p}{\partial \sigma_{\tilde{\Omega}}}|_{\tilde{\gamma}} < 0$ indicates that $\tilde{\gamma}$ may stay regular when it is pushed towards the vacuum, but otherwise it may lose regularity. This fact echoes with many well-posedness and ill-posedness results on a variety of free boundary problems arising in, for instance, one-phase Hele–Shaw problems [12, 20, 22, 24, 30, 31] and porous medium equations [8–10, 32, 48].

In our problem, under the assumption of the almost radial symmetry, the parabolicity condition $(\mu - \nu)c_* > 0$ derived for (2.45) is an approximation of $(\mu - \nu)\frac{\partial p}{\partial \sigma_\Omega}|_\gamma > 0$, while the condition $\frac{c_* r^2}{R} < 0$ corresponding to (2.46) is an approximation of $\frac{\partial p}{\partial \sigma_{\tilde{\Omega}}}|_{\tilde{\gamma}} < 0$.

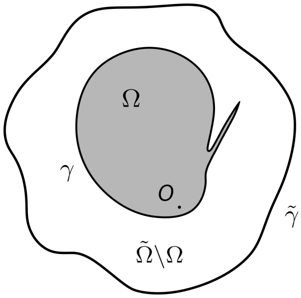

FIG. 2. A possible example exhibiting ill-posedness of motion of γ when $\mu < \nu$. Here Ω consists of a big chuck and a thin branch; the latter is expected to move towards right. Along a part of γ, more mobile normal cells are pushing less mobile tumor cells, i.e., $(\mu - \nu)\frac{\partial p}{\partial \sigma_\Omega}|_\gamma < 0$, making the local evolution of γ ill-posed. Note that both Ω and $\tilde{\Omega}$ are star-shaped with respect to the origin, denoted by O, so γ and $\tilde{\gamma}$ are still graphs of functions of θ in the polar coordinate at this moment.

REMARK 2.5 From the above discussion, we can tell that $\mu < \nu$ is not sufficient for the parabolicity of the motion of γ, since the domain geometry determines how γ moves in a nontrivial way. Even if both Ω and $\tilde{\Omega}$ are assumed to be star-shaped with respect to the same point, which means γ and $\tilde{\gamma}$ can be realized as graphs of functions of θ in the polar coordinate, we can still manufacture such domains so that the parabolicity fails along some portion of γ. A possible example is shown in Figure 2, where both Ω and $\tilde{\Omega}$ are star-shaped with respect to the origin, denoted by O in the figure. The tumor domain Ω consists of a big chunk and a thin branch, where the branch is so thin that it does not significantly affect p. Then it is conceivable that the thin branch will be pushed towards right under the expansion of the big chunk. So along the part of γ where the thin branch faces the main body of Ω, the more mobile normal cells are pushing the less mobile tumor cells (since $\mu < \nu$), which potentially gives rise to ill-posedness of the motion of γ locally.

Given this, in order to guarantee well-posedness of the motion of γ, it is then reasonable to assume γ and $\tilde{\gamma}$ are close to concentric circles, in which case the tumor cells should be always pushing the normal ones.

The parabolicity of (2.45) and (2.46) is sufficient to prove existence of local solutions in Section 8, and then uniqueness in Section 9. The proof of the local existence uses two layers of fixed-point arguments. We sketch it as follows.

1. Fix a pair of interface dynamics f and F.
2. First we need to solve for $[\varphi]'$ and ϕ' associated with the domain defined by f and F. To do that, in Section 7, we apply a fixed-point argument to static equations (7.1) and (7.2) (or equivalently, (2.33) and (2.34)) with the variable $([\varphi]', \phi')$. In this argument, we need estimates for the remainder terms that are omitted in (2.43) and (2.44), which turn out to be small.
3. Once $[\varphi]'$ and ϕ' are well-defined and their estimates are derived, we use them to bound $\mathcal{HS}\phi'$ and $\mathcal{HS}[\varphi]'$ in (2.45) and (2.46) as well as all the remainder terms omitted there (see (8.1) and (8.2) for the complete equations). They altogether will be put as the source terms in some fractional heat equations similar to (2.45) and (2.46) in order to construct a new pair of interface dynamics, $\tilde{f}$ and $\tilde{F}$. See (8.32)–(8.34). We then show in Section 8 that the map $(f, F) \mapsto (\tilde{f}, \tilde{F})$ has a fixed-point, which is a local solution.
4. In this process, bounds for all the remainder terms will rely on estimates derived in Sections 3–6. See Section 2.5 for what are exactly covered in them.

The proof of the uniqueness boils down to showing that $[\varphi]'$, ϕ' and all the remainder terms above depend in a Lipschitz manner on the interface configurations. Indeed, what we prove is a stability-type estimate for f and F based on that of the fractional heat equation. We carry out this idea in Section 9 with a twist in order to slightly reduce complexity of the proof.

2.5 *Organization of the paper*

In Section 3, we first study the pressure p in an almost radially symmetric geometry by elliptic regularity theory. In Section 4, we derive estimates concerning gradients of the growth potential $\Gamma * g$ (cf. (2.3) and (2.4)) restricted to inner and outer interfaces. Section 5 is devoted to proving estimates for singular integral operators $\mathcal{K}_\gamma$ and $\mathcal{K}_{\tilde{\gamma}}$, while Section 6 establishes estimates for integral operators $\mathcal{K}_{\gamma,\tilde{\gamma}}$ and $\mathcal{K}_{\tilde{\gamma},\gamma}$. Section 7 shows well-definedness of $[\varphi]$ and ϕ as well as their estimates. Finally, we prove existence of the local solution in Section 8, and uniqueness in Section 9. Some auxiliary estimates and non-essential lengthy proofs are collected in Appendices.

3. Pressure in an almost radially symmetric geometry

In this section, we focus on the elliptic equation (1.6) and (1.7) for the pressure p in $\tilde{\Omega}$. The goal is to quantify the fact that if Ω and $\tilde{\Omega}$ are close to two concentric discs then p should be almost radially symmetric.

3.1 *Geometric preliminaries*

First we introduce a diffeomorphism to transform the physical domain into a reference domain that is perfectly radially symmetric. Given δ satisfying (2.23), define a cut-off function $\eta_\delta \in C_0^\infty([0,+\infty))$, such that $\eta_\delta \in [0,1]$ is only supported on $[1-2\delta, 1+2\delta]$, $\eta_\delta = 1$ on $[1-\delta, 1+\delta]$, and for some universal constant C,

$$\delta|\eta_\delta'| + \delta^2|\eta_\delta''| \leqslant C. \tag{3.1}$$

Let $X = (\rho\cos\omega, \rho\sin\omega) \in \mathbb{R}^2$ be a point in the reference coordinate, with $\rho = |X|$. Define

$$x(X) = \left[1 + h(\omega)\eta_\delta\left(\frac{\rho}{r}\right) + H(\omega)\eta_\delta\left(\frac{\rho}{R}\right)\right] X =: \zeta(X)X, \tag{3.2}$$

where h and H are given in (2.19). In other words, x deforms the reference domain in the radial direction only in annuli around ∂B_r and ∂B_R. It depends only on γ in the annulus $B_{r(1+2\delta)}\backslash B_{r(1-2\delta)}$, and only on $\tilde{\gamma}$ in $B_R\backslash B_{R(1-2\delta)}$; $x(X) = X$ elsewhere. We may also write $\zeta(X)$ as $\zeta(\rho,\omega)$. We know that $x(X)$ is a diffeomorphism from $\mathbb{R}^2$ to itself provided that $\zeta(\rho,\omega)\rho$ is strictly increasing in ρ for all $\omega \in \mathbb{T}$. This is true if oscillations of γ and $\tilde{\gamma}$ in the radial direction are small with respect to the gap between them, i.e.,

$$\delta^{-1}\big(\|h\|_{L^\infty(\mathbb{T})} + \|H\|_{L^\infty(\mathbb{T})}\big) \ll 1. \tag{3.3}$$

Under this assumption, it is clear that $x(X)$ maps B_r, B_R, ∂B_r and ∂B_R to Ω, $\tilde{\Omega}$, γ and $\tilde{\gamma}$, respectively. We denote its inverse to be $X(x)$.

3.2 *Pressure in the reference coordinate*

Define

$$\tilde{p}(X) := p\big(x(X)\big). \tag{3.4}$$

By (1.6), $\tilde{p}$ in the X-coordinate satisfies

$$-\frac{\partial X_k}{\partial x_i}\nabla_{X_k}\left(a\frac{\partial X_j}{\partial x_i}\nabla_{X_j}\tilde{p}\right) = G(\tilde{p})\chi_{B_r} \quad \text{in } B_R, \quad \tilde{p}|_{\partial B_R} = 0. \tag{3.5}$$

Here the summation convention applies to repeated indices. We also used the notations

$$a(X) = \mu\chi_{B_r}(X) + \nu\chi_{B_R\setminus B_r}(X) \tag{3.6}$$

and

$$\frac{\partial X_k}{\partial x_i} = \left(\frac{\partial X}{\partial x}\right)_{ki} = \left[\left(\frac{\partial x}{\partial X}\right)^{-1}\right]_{ki}, \tag{3.7}$$

which are both functions in X. We may write $a = a(\rho)$.

In order to show $\tilde{p}$ is almost radially symmetric, we shall compare it with a radially symmetric solution p_* defined as follows.

Lemma 3.1 *Let p_* be the H^1-weak solution of*

$$-\nabla_{X_i}\big(a\nabla_{X_i}p_*\big) = G(p_*)\chi_{B_r} \quad \textit{in } B_R, \quad p_*|_{\partial B_R} = 0. \tag{3.8}$$

Then

1. *p_* is radially symmetric, i.e., $p_* = p_*(\rho)$, and $p_* \in W^{1,\infty}(B_R)$.*
2. *$p_* \in [0, p_M]$ and p_* is decreasing in ρ.*
3. *In $\overline{B}_R\setminus B_r$,*

$$p_*(\rho) = -\ln\left(\frac{\rho}{R}\right)\cdot\frac{1}{2\pi\nu}\int_{B_r} G(p_*)\,dx. \tag{3.9}$$

4. *For $\rho \in [0, r]$,*

$$\int_{B_\rho} G(p_*)\,dx \leqslant C\rho^2 \min\{1, \mu^{1/2}r^{-1}\}, \tag{3.10}$$

   where C only depends on G.
5. *For $\rho \in [0, r^-]$,*

$$|\nabla p_*|(\rho) \leqslant C \min\left\{\mu^{-1}\rho, \mu^{-1/2}\right\}. \tag{3.11}$$

   For $\rho \in [r^+, R]$,

$$|\nabla p_*|(\rho) \leqslant C\rho^{-1}\min\left\{\nu^{-1}r^2, \mu^{1/2}\nu^{-1}r\right\}. \tag{3.12}$$

   Here the constants C only depend on G. Note that ∇p_* has discontinuity across ∂B_r, so we use $|\nabla p_*|(r^\pm)$ to distinguish the gradients taken from two sides of ∂B_r.

Proof. The radial symmetry of p_* can be justified by a symmetrization argument in the variational formulation of (3.8). $W^{1,\infty}$-regularity of p_* follows from [36]. The fact that $p_* \in [0, p_M]$ and monotonicity of p_* follows from the maximum principle. (3.9) is obvious since p_* is harmonic in $\overline{B}_R \backslash B_r$.

The first bounds in (3.10)–(3.12) follow from the trivial fact $|G| \leqslant C$ and

$$|\nabla p_*|(\rho) = |\partial_\rho p_*(\rho)| = \frac{1}{2\pi a(\rho)\rho} \int_{B_\rho \cap B_r} G(p_*)\, dx. \tag{3.13}$$

To show the second bounds in (3.11) and (3.12), define $\mathcal{G}$ to be the anti-derivative of G with $\mathcal{G}(0) = 0$. Obviously, $\mathcal{G} \geqslant 0$ on $[0, p_M]$, attaining its maximum at p_M. Since in the polar coordinate, p_* solves $-\mu\partial_\rho(\rho\partial_\rho p_*) = \rho G(p_*)$ on $[0, r)$, by multiplying with $\rho^{-1}\partial_\rho p_*$,

$$\mu\rho^{-1}|\partial_\rho p_*|^2 + \mu\partial_\rho p_* \partial_\rho^2 p_* + G(p_*)\partial_\rho p_* = 0. \tag{3.14}$$

Taking integral in ρ from 0 to $\tau \in [0, r^-]$ yields

$$\mu \int_0^\tau \rho^{-1}|\partial_\rho p_*|^2\, d\rho + \frac{\mu}{2}|\partial_\rho p_*(\tau)|^2 + \mathcal{G}\big(p_*(\tau)\big) = \mathcal{G}\big(p_*(0)\big). \tag{3.15}$$

Hence,

$$\|\partial_\rho p_*\|^2_{L^\infty(B_r)} \leqslant 2\mu^{-1}\mathcal{G}(p_M). \tag{3.16}$$

By the nature of discontinuity of $\partial_\rho p_*$ across ∂B_r, $a(\rho)\partial_\rho p_*$ is continuous at $\rho = r$. Hence, for $\rho \in [r^+, R]$, $\partial_\rho p_*(\rho) = \frac{\mu r}{\nu\rho}\partial_\rho p_*|_{\rho=r_-}$. This gives the second bound in (3.12). Finally, the second bound in (3.10) follow from (3.12), (3.13) and the fact that $G(p_*(\rho))$ is increasing in ρ. □

In order to derive a bound for $(\tilde{p} - p_*)$, we need estimates concerning $x(X)$ and its inverse. Denote

$$m_0 := \delta^{-1}\|h\|_{L^\infty(\mathbb{T})} + \|h'\|_{L^\infty(\mathbb{T})}, \tag{3.17}$$

$$M_0 := \delta^{-1}\|H\|_{L^\infty(\mathbb{T})} + \|H'\|_{L^\infty(\mathbb{T})}. \tag{3.18}$$

Lemma 3.2 *Suppose $h, H \in W^{1,\infty}(\mathbb{T})$ satisfy that $m_0 + M_0 \ll 1$. Then*

$$\left\| \frac{\partial X}{\partial x} - Id \right\|_{L^\infty(B_{r(1+2\delta)} \backslash B_{r(1-2\delta)})} \leqslant Cm_0, \tag{3.19}$$

$$\left\| \frac{\partial X}{\partial x} - Id \right\|_{L^\infty(B_R \backslash B_{R(1-2\delta)})} \leqslant CM_0, \tag{3.20}$$

and

$$\left\| \nabla_{X_k} \frac{\partial X_k}{\partial x_i} \right\|_{L^\infty(B_{r(1+2\delta)} \backslash B_{r(1-2\delta)})} \leqslant C(\delta r)^{-1} m_0, \tag{3.21}$$

$$\left\| \nabla_{X_k} \frac{\partial X_k}{\partial x_i} \right\|_{L^\infty(B_R \backslash B_{R(1-2\delta)})} \leqslant C(\delta R)^{-1} M_0. \tag{3.22}$$

The constants C are all universal.

Proof. The proof is a straightforward calculation. By (3.2),

$$\frac{\partial x}{\partial X} = \zeta \cdot Id + X \otimes \nabla\zeta. \tag{3.23}$$

Its inverse is given by

$$\begin{aligned}\frac{\partial X}{\partial x} &= (\zeta^2 + \zeta\rho\partial_\rho\zeta)^{-1}\big((\zeta + \rho\partial_\rho\zeta)Id - X \otimes \nabla\zeta\big)\\ &= \zeta^{-1}Id - (\zeta^2 + \zeta\rho\partial_\rho\zeta)^{-1}X \otimes \nabla\zeta.\end{aligned} \tag{3.24}$$

On the other hand, since $\nabla_{X_k}(\frac{\partial X_k}{\partial x_i} \cdot \frac{\partial x_i}{\partial X_j}) = \nabla_{X_k}\delta_{kj} = 0$, we deduce that

$$\begin{aligned}\nabla_{X_k}\left(\frac{\partial X_k}{\partial x_l}\right) &= -\frac{\partial X_j}{\partial x_l}\frac{\partial X_k}{\partial x_i} \cdot \nabla_{X_k}\left(\frac{\partial x_i}{\partial X_j}\right)\\ &= -\frac{\partial X_j}{\partial x_l} \cdot (\zeta^2 + \zeta\rho\partial_\rho\zeta)^{-1}\big((\zeta + \rho\partial_\rho\zeta)\delta_{ki} - X_k(\nabla\zeta)_i\big) \cdot \nabla_{X_k}(\zeta\delta_{ij} + X_i\nabla_{X_j}\zeta)\\ &= -\frac{\partial X_j}{\partial x_l}(\zeta^2 + \zeta\rho\partial_\rho\zeta)^{-1}\nabla_{X_j}(\zeta^2 + \zeta\rho\partial_\rho\zeta).\end{aligned} \tag{3.25}$$

By (3.2),

$$\zeta - 1 = h\eta_\delta\left(\frac{\rho}{r}\right) + H\eta_\delta\left(\frac{\rho}{R}\right), \tag{3.26}$$

$$\rho\partial_\rho\zeta = h(\omega) \cdot \frac{\rho}{r}\eta_\delta'\left(\frac{\rho}{r}\right) + H(\omega) \cdot \frac{\rho}{R}\eta_\delta'\left(\frac{\rho}{R}\right). \tag{3.27}$$

Thanks to the smallness of m_0 and M_0,

$$|\zeta - 1| + |(\zeta^2 + \zeta\rho\partial_\rho\zeta) - 1| \ll 1. \tag{3.28}$$

Hence, by the last line in (3.2),

$$\left|\frac{\partial X}{\partial x} - Id\right| \leqslant C(|1 - \zeta| + \rho|\nabla\zeta|). \tag{3.29}$$

We calculate

$$\nabla\zeta = \left[h(\omega) \cdot \frac{1}{r}\eta_\delta'\left(\frac{\rho}{r}\right) + H(\omega) \cdot \frac{1}{R}\eta_\delta'\left(\frac{\rho}{R}\right)\right]e_r + \left[h'(\omega) \cdot \eta_\delta\left(\frac{\rho}{r}\right) + H'(\omega) \cdot \eta_\delta\left(\frac{\rho}{R}\right)\right]\rho^{-1}e_\theta, \tag{3.30}$$

where e_r and e_θ are defined in (2.32). Then (3.19) and (3.20) follow easily.

Similarly, (3.25) implies that

$$\left|\nabla_{X_k}\left(\frac{\partial X_k}{\partial x_l}\right)\right| \leqslant C|\nabla(\zeta^2 + \zeta\rho\partial_\rho\zeta)| \leqslant C\big(|\nabla\zeta| + |\nabla(\rho\partial_\rho\zeta)|\big). \tag{3.31}$$

Then (3.21) and (3.22) follow from (3.30) and the calculation

$$\begin{aligned}\nabla(\rho\partial_\rho\zeta) = &\left[h(\omega) \cdot \frac{\rho}{r^2}\eta_\delta''\left(\frac{\rho}{r}\right) + H(\omega) \cdot \frac{\rho}{R^2}\eta_\delta''\left(\frac{\rho}{R}\right)\right] \cdot e_r\\ &+ \left[h(\omega) \cdot \frac{1}{r}\eta_\delta'\left(\frac{\rho}{r}\right) + H(\omega) \cdot \frac{1}{R}\eta_\delta'\left(\frac{\rho}{R}\right)\right] \cdot e_r\\ &+ \left[h'(\omega) \cdot \frac{\rho}{r}\eta_\delta'\left(\frac{\rho}{r}\right) + H'(\omega) \cdot \frac{\rho}{R}\eta_\delta'\left(\frac{\rho}{R}\right)\right] \cdot \rho^{-1}e_\theta.\end{aligned} \tag{3.32}$$

This proves the lemma. □

By (3.5) and (3.8), $(\tilde{p}-p_*)$ solves

$$
\begin{aligned}
&-\nabla_{X_k}\left(a\frac{\partial X_k}{\partial x_i}\frac{\partial X_j}{\partial x_i}\nabla_{X_j}(\tilde{p}-p_*)\right)+c(\tilde{p}-p_*)\\
&\qquad=\nabla_{X_k}\left[a\left(\frac{\partial X_k}{\partial x_i}\frac{\partial X_j}{\partial x_i}-\delta_{kj}\right)\nabla_{X_j}p_*\right]-\nabla_{X_k}\frac{\partial X_k}{\partial x_i}\cdot a\frac{\partial X_j}{\partial x_i}\nabla_{X_j}\tilde{p} \qquad (3.33)
\end{aligned}
$$

in the reference coordinate with boundary condition $(\tilde{p}-p_*)|_{\partial B_R}=0$. Here

$$
c(X):=-\frac{G(\tilde{p})-G(p_*)}{\tilde{p}-p_*}\chi_{B_r}\geqslant 0 \qquad (3.34)
$$

due to the assumptions on G. Then we can prove stability of the pressure with respect to the domain geometry around the radially symmetric case.

Lemma 3.3 *Under the assumptions of Lemma 3.2,*

$$
\|\nabla(\tilde{p}-p_*)\|_{L^2(B_R)}\leqslant C(m_0+M_0)(\delta R^2)^{1/2}, \qquad (3.35)
$$

where $C=C(\mu,\nu,G)$.

Proof. We take inner product of $(\tilde{p}-p_*)$ and (3.33) and integrate by parts,

$$
\begin{aligned}
&\int_{B_R}a\left|\frac{\partial X_j}{\partial x_i}\nabla_{X_j}(\tilde{p}-p_*)\right|^2dX+\int_{B_r}c|\tilde{p}-p_*|^2\,dX\\
&\qquad=-\int_{B_R}\nabla_{X_k}(\tilde{p}-p_*)\cdot a\left(\frac{\partial X_k}{\partial x_i}\frac{\partial X_j}{\partial x_i}-\delta_{kj}\right)\nabla_{X_j}p_*\,dX\\
&\qquad\quad-\int_{B_R}(\tilde{p}-p_*)\nabla_{X_k}\frac{\partial X_k}{\partial x_i}\cdot a\frac{\partial X_j}{\partial x_i}[\nabla_{X_j}(\tilde{p}-p_*)+\nabla_{X_j}p_*]\,dX. \qquad (3.36)
\end{aligned}
$$

By the definition of a in (3.6), the assumptions on G, Lemma 3.2 and Hölder's inequality,

$$
\begin{aligned}
\|\nabla(\tilde{p}-p_*)\|_{L^2(B_R)}^2&\leqslant C[m_0(\delta r^2)^{1/2}+M_0(\delta R^2)^{1/2}]\|\nabla(\tilde{p}-p_*)\|_{L^2(B_R)}\|\nabla p_*\|_{L^\infty(B_R)}\\
&\quad+C(\delta r)^{-1}m_0\cdot\|\tilde{p}-p_*\|_{L^2(B_{r(1+2\delta)}\backslash B_{r(1-2\delta)})}\|\nabla(\tilde{p}-p_*)\|_{L^2(B_R)}\\
&\quad+C(\delta R)^{-1}M_0\cdot\|\tilde{p}-p_*\|_{L^2(B_R\backslash B_{R(1-2\delta)})}\|\nabla(\tilde{p}-p_*)\|_{L^2(B_R)}\\
&\quad+C(\delta r)^{-1}m_0\cdot\|\tilde{p}-p_*\|_{L^2(B_{r(1+2\delta)}\backslash B_{r(1-2\delta)})}\cdot(\delta r^2)^{1/2}\|\nabla p_*\|_{L^\infty(B_R)}\\
&\quad+C(\delta R)^{-1}M_0\cdot\|\tilde{p}-p_*\|_{L^2(B_R\backslash B_{R(1-2\delta)})}\cdot(\delta R^2)^{1/2}\|\nabla p_*\|_{L^\infty(B_R)},
\end{aligned}
\qquad (3.37)
$$

where $C=C(\mu,\nu,G)$. We proceed in two different cases.

CASE 1 If $R/2\leqslant r<R$, by (2.23) and Poincaré inequality on thin domains,

$$
\begin{aligned}
\|\tilde{p}-p_*\|_{L^2(B_R\backslash B_{r(1-2\delta)})}&\leqslant C\big(R-r(1-2\delta)\big)\|\nabla(\tilde{p}-p_*)\|_{L^2(B_R)}\\
&\leqslant C(\delta r)\|\nabla(\tilde{p}-p_*)\|_{L^2(B_R)}. \qquad (3.38)
\end{aligned}
$$

Combining this with (3.37) yields

$$\|\nabla(\tilde{p}-p_*)\|^2_{L^2(B_R)} \leqslant C(m_0+M_0)(\delta R^2)^{1/2}\|\nabla(\tilde{p}-p_*)\|_{L^2(B_R)}\|\nabla p_*\|_{L^\infty(B_R)} + C(m_0+M_0)\|\nabla(\tilde{p}-p_*)\|^2_{L^2(B_R)}. \quad (3.39)$$

By Young's inequality, smallness of m_0 and M_0 and Lemma 3.1, the desired estimate follows.

CASE 2 If otherwise $r < R/2$, by (2.23), $\delta \geqslant C$ for some universal constant $C > 0$. We shall first derive a bound for $\|\tilde{p}-p_*\|_{L^\infty(B_R)}$.

Recall that p solves (1.6) and (1.7). Taking inner product of (1.6) and p, we find that

$$\|\nabla p\|^2_{L^2(\tilde{\Omega})} \leqslant C\int_\Omega G(p)p\,dx \leqslant C|\Omega| \leqslant Cr^2, \quad (3.40)$$

where $C = C(\mu,\nu,G)$. Hence, by Lemma 3.2, in the reference coordinate,

$$\left\|\frac{\partial X_j}{\partial x_i}\nabla_{X_j}\tilde{p}\right\|_{L^2(B_R)} \leqslant Cr. \quad (3.41)$$

Now consider (3.33). By boundedness of weak solutions [27, Theorem 8.16],

$$\|\tilde{p}-p_*\|_{L^\infty(B_R)} \leqslant C\left(R^{1/2}\left\|a\left(\frac{\partial X_k}{\partial x_i}\frac{\partial X_j}{\partial x_i}-\delta_{kj}\right)\nabla_{X_j}p_*\right\|_{L^4(B_R)} + R\left\|\nabla_{X_k}\frac{\partial X_k}{\partial x_i}\cdot a\frac{\partial X_j}{\partial x_i}\nabla_{X_j}\tilde{p}\right\|_{L^2(B_R)}\right). \quad (3.42)$$

Applying Lemma 3.1, Lemma 3.2, (3.41) and the fact $\delta \geqslant C$,

$$\begin{aligned}\|\tilde{p}-p_*\|_{L^\infty(B_R)} &\leqslant CR^{1/2}\big(m_0(\delta r^2)^{1/4}+M_0(\delta R^2)^{1/4}\big)\|\nabla p_*\|_{L^\infty(B_R)} + CR\big(m_0(\delta r)^{-1}+M_0(\delta R)^{-1}\big)\cdot r\\ &\leqslant C(m_0+M_0)(\delta R^2)^{1/2}, \end{aligned} \quad (3.43)$$

where $C = C(\mu,\nu,G)$.

With this estimate and Lemma 3.1, (3.37) becomes

$$\begin{aligned}\|\nabla(\tilde{p}-p_*)\|^2_{L^2(B_R)} &\leqslant C(m_0+M_0)(\delta R^2)^{1/2}\|\nabla(\tilde{p}-p_*)\|_{L^2(B_R)}\|\nabla p_*\|_{L^\infty(B_R)}\\ &\quad + C(m_0+M_0)\|\tilde{p}-p_*\|_{L^\infty(B_R)}\|\nabla(\tilde{p}-p_*)\|_{L^2(B_R)}\\ &\quad + C(m_0+M_0)\|\tilde{p}-p_*\|_{L^\infty(B_R)}\cdot(\delta R^2)^{1/2}\|\nabla p_*\|_{L^\infty(B_R)}\\ &\leqslant C(m_0+M_0)(\delta R^2)^{1/2}\|\nabla(\tilde{p}-p_*)\|_{L^2(B_R)}\\ &\quad + C(m_0+M_0)^2\delta R^2. \end{aligned} \quad (3.44)$$

Then the desired estimate follows from Young's inequality. □

REMARK 3.1 The above estimate involves δR^2. If $\delta^{-1}(\|h\|_{L^\infty}+\|H\|_{L^\infty}) \ll 1$, by (2.23), there exist universal constants $0 < c_1 < c_2$, such that

$$c_1|\tilde{\Omega}_0\backslash\Omega_0| \leqslant \delta R^2 \leqslant c_2|\tilde{\Omega}_0\backslash\Omega_0|.$$

It is noteworthy that $|\tilde{\Omega}_t\backslash\Omega_t|$ is constant in time provided that γ and $\tilde{\gamma}$ have sufficient regularity. This is because the transporting velocity field $-\nabla\varphi$ in $\tilde{\Omega}_t\backslash\Omega_t$ is divergence-free.

3.3 *More stability results*

For later use, further stability results are presented here for the interface velocities and the pressure, with respect to the interface configurations.

Fix $0 < r < R$ and take δ as in (2.23). Given two pairs of interface configurations $(\gamma_1, \tilde{\gamma}_1)$ and $(\gamma_2, \tilde{\gamma}_2)$, let (h_1, H_1), (h_2, H_2) be defined as in (2.8)–(2.19). As in (3.17) and (3.18), we define $m_{0,i}$ and $M_{0,i}$ that correspond to h_i and H_i $(i = 1, 2)$. We additionally introduce for some $\alpha \in (0, 1)$,

$$m_{\alpha,i} := \delta^{-1}\|h_i\|_{L^\infty} + \delta^\alpha \|h_i'\|_{\dot{C}^\alpha}, \tag{3.45}$$

$$M_{\alpha,i} := \delta^{-1}\|H_i\|_{L^\infty} + \delta^\alpha \|H_i'\|_{\dot{C}^\alpha}. \tag{3.46}$$

Also denote

$$\Delta m_0 := \delta^{-1}\|h_1 - h_2\|_{L^\infty(\mathbb{T})} + \|h_1' - h_2'\|_{L^\infty(\mathbb{T})}, \tag{3.47}$$

$$\Delta M_0 := \delta^{-1}\|H_1 - H_2\|_{L^\infty(\mathbb{T})} + \|H_1' - H_2'\|_{L^\infty(\mathbb{T})}, \tag{3.48}$$

$$\Delta m_\alpha := \delta^{-1}\|h_1 - h_2\|_{L^\infty(\mathbb{T})} + \delta^\alpha\|h_1' - h_2'\|_{\dot{C}^\alpha(\mathbb{T})}, \tag{3.49}$$

$$\Delta M_\alpha := \delta^{-1}\|H_1 - H_2\|_{L^\infty(\mathbb{T})} + \delta^\alpha\|H_1' - H_2'\|_{\dot{C}^\alpha(\mathbb{T})}. \tag{3.50}$$

Then we can show

Lemma 3.4 *Suppose $(h_1, H_1), (h_2, H_2) \in C^{1,\alpha}(\mathbb{T}) \times C^{1,\alpha}(\mathbb{T})$ for some $\alpha \in (0, \frac{1}{4})$, satisfying that for $i = 1, 2$, $m_{\alpha,i} + M_{\alpha,i} \ll 1$. Then*

$$\|\partial_t h_1 - \partial_t h_2\|_{C^\alpha(\mathbb{T})} + \|\partial_t H_1 - \partial_t H_2\|_{C^\alpha(\mathbb{T})} \leqslant C_*(\Delta m_\alpha + \Delta M_\alpha), \tag{3.51}$$

where $C_* = C_*(\alpha, \mu, \nu, r, R, G)$. Here $\partial_t h_i$ and $\partial_t H_i$ are the interface velocities in the radial direction, normalized by r and R respectively (see (2.13).)

Let p_i $(i = 1, 2)$ denote the pressure solving (1.6) and (1.7) on the physical domain that is determined by γ_i and $\tilde{\gamma}_i$, while $\tilde{p}_i$ denotes its pull back into the reference coordinate as in (3.4). An important intermediate result in proving Lemma 3.4 is the following lemma on $C^{1,\alpha}$-bound for $(\tilde{p}_1 - \tilde{p}_2)$, which will be also used when proving uniqueness of the local solution in Section 9.

Lemma 3.5 *Under the assumption of Lemma 3.4,*

$$\|\tilde{p}_1 - \tilde{p}_2\|_{L^\infty(B_R)} \leqslant C_*(\Delta m_0 + \Delta M_0), \tag{3.52}$$

and

$$\|\tilde{p}_1 - \tilde{p}_2\|_{C^{1,\alpha}(\overline{B_r})} + \|\tilde{p}_1 - \tilde{p}_2\|_{C^{1,\alpha}(\overline{B_R \setminus B_r})} \leqslant C_*(\Delta m_\alpha + \Delta M_\alpha), \tag{3.53}$$

where $C_* = C_*(\alpha, \mu, \nu, r, R, G)$.

Their proofs involve lengthy calculation, while they are relatively independent from the rest of the paper. So we leave them to Appendix B.

4. Gradient estimates for $\Gamma * g$ along interfaces

In this section, we shall derive estimates concerning $e_r \cdot \nabla(\Gamma * g)$ and $e_\theta \cdot \nabla(\Gamma * g)$ along γ and $\tilde{\gamma}$, where $e_r = (\cos\theta, \sin\theta)$ and $e_\theta = (-\sin\theta, \cos\theta)$. Aiming at greater generality, instead of working with g defined in (2.4), here we shall assume $g := g_0(X(x))$ for some g_0 defined in the reference coordinate and supported on $\overline{B_{(1+4\delta)r}}$, where $X(x)$ is the inverse of $x(X)$ defined by (3.2). We remark that the support is a slightly larger than the one corresponding to (2.4) ($\overline{B_r}$ in that case). The motivation for this will be clear in Section 9. Also note that $\overline{B_{(1+4\delta)r}} \subset B_{(1-2\delta)R}$.

4.1 *Preliminaries*

We introduce Poisson kernel P on the 2-D unit disc and its conjugate Q:

$$P(s,\xi) = \frac{1-s^2}{1+s^2-2s\cos\xi}, \tag{4.1}$$

$$Q(s,\xi) = \frac{2s\sin\xi}{1+s^2-2s\cos\xi}. \tag{4.2}$$

Elementary estimates for them as well as their derivatives are collected in Lemma A.1. Define

$$K(s,\xi) := \frac{2s^2\sin\xi}{1+s^2-2s\cos\xi} = sQ(s,\xi), \tag{4.3}$$

$$J(s,\xi) := \frac{2(s\cos\xi-1)s}{1+s^2-2s\cos\xi} = -s\big(1+P(s,\xi)\big). \tag{4.4}$$

See (4.43) and (4.44) for the motivation of defining these kernels. They have the following properties.

Lemma 4.1 *Let $z_i \in [0,2]$ ($i = 1,2,3,4$). Suppose for some $w \in [0,2]$ and $\xi \in \mathbb{T}$, $|z_i - w| \leqslant c(|\xi|+|1-w|)$. Here c is some universal small constant, whose smallness will be clear in the proof. Then*

$$|K(z_i,\xi)| \leqslant \frac{C|z_i|}{(1+w^2-2w\cos\xi)^{1/2}}, \tag{4.5}$$

$$\left|\frac{\partial K}{\partial s}(z_i,\xi)\right| + \left|\frac{\partial K}{\partial \xi}(z_i,\xi)\right| \leqslant \frac{C}{1+w^2-2w\cos\xi}, \tag{4.6}$$

$$|K(z_1,\xi)-K(z_2,\xi)| \leqslant \frac{C|z_1-z_2|}{1+w^2-2w\cos\xi}, \tag{4.7}$$

$$\left|\frac{\partial K}{\partial s}(z_1,\xi) - \frac{\partial K}{\partial s}(z_2,\xi)\right| + \left|\frac{\partial K}{\partial \xi}(z_1,\xi) - \frac{\partial K}{\partial \xi}(z_2,\xi)\right| \leqslant \frac{C|z_1-z_2|}{(1+w^2-2w\cos\xi)^{3/2}}, \tag{4.8}$$

and

$$\begin{aligned}&\left|\frac{\partial K}{\partial s}(z_1,\xi) - \frac{\partial K}{\partial s}(z_2,\xi) - \frac{\partial K}{\partial s}(z_3,\xi) + \frac{\partial K}{\partial s}(z_4,\xi)\right| \\ &\quad\leqslant \frac{C|z_1-z_2-z_3+z_4|}{(1+w^2-2w\cos\xi)^{3/2}} + \frac{C(|z_1-z_2|+|z_3-z_4|)(|z_1-z_3|+|z_2-z_4|)}{(1+w^2-2w\cos\xi)^{2}}.\end{aligned} \tag{4.9}$$

Here C are all universal constants. These estimates also hold if K is replaced by J.

Proof. We derive that

$$\left|\frac{1+z_i^2-2z_i\cos\xi}{1+w^2-2w\cos\xi}-1\right| \leqslant \frac{|z_i-w|+2|w-\cos\xi|}{1+w^2-2w\cos\xi}|z_i-w|. \tag{4.10}$$

When c is suitably small, the right hand side is bounded by $\frac{1}{2}$. This implies that $(1+z_i^2-2z_i\cos\xi)$ are comparable with $(1+w^2-2w\cos\xi)$, and thus they are comparable with each other. Then (4.5) and (4.6) follow from Lemma A.1 and the assumption $z_i\in[0,2]$. Using the same facts, we can also derive that

$$\begin{aligned}|K(z_1,\xi)-K(z_2,\xi)| &= \left|\frac{2\sin\xi(z_1-z_2)[z_1(1-z_2\cos\xi)+z_2(1-z_1\cos\xi)]}{(1+z_1^2-2z_1\cos\xi)(1+z_2^2-2z_2\cos\xi)}\right|\\ &\leqslant \frac{C|z_1-z_2|}{1+w^2-2w\cos\xi}.\end{aligned} \tag{4.11}$$

Moreover, by Lemma A.1,

$$\frac{\partial K}{\partial s} = Q+s\partial_s Q = Q+\frac{2s\sin\xi(1-s^2)}{(1+s^2-2s\cos\xi)^2} = Q(1+P), \tag{4.12}$$

$$\frac{\partial K}{\partial \xi} = s\partial_\xi Q = s^2\partial_s P = \frac{K}{\tan\xi}-QK. \tag{4.13}$$

Then (4.8) and (4.9) follow from

$$P(z_1,\xi)-P(z_2,\xi) = 2(z_1-z_2)\cdot\frac{(1-z_1)(1-z_2)-(1-\cos\xi)(1+z_1z_2)}{(1+z_1^2-2z_1\cos\xi)(1+z_2^2-2z_2\cos\xi)}, \tag{4.14}$$

$$Q(z_1,\xi)-Q(z_2,\xi) = 2(z_1-z_2)\cdot\frac{\sin\xi\big((1-z_1)+z_1(1-z_2)\big)}{(1+z_1^2-2z_1\cos\xi)(1+z_2^2-2z_2\cos\xi)}, \tag{4.15}$$

and Lemma A.1 by a direct calculation as in (4.11).

The estimates for J can be justified similarly. Indeed,

$$J(z_1,\xi)-J(z_2,\xi) = 2(z_1-z_2)\cdot\frac{z_1z_2\sin^2\xi-(1-z_1\cos\xi)(1-z_2\cos\xi)}{(1+z_1^2-2z_1\cos\xi)(1+z_2^2-2z_2\cos\xi)}, \tag{4.16}$$

and

$$\frac{\partial J}{\partial s} = -(1+P)-s\partial_s P = -1-P-\frac{Q}{\tan\xi}+Q^2, \tag{4.17}$$

$$\frac{\partial J}{\partial \xi} = -s\partial_\xi P = s^2\partial_s Q = PK. \tag{4.18}$$

□

Suppose the inner interface γ and the outer interface $\tilde{\gamma}$ are defined by h and H through (2.8)–(2.19), respectively. Let η_δ be defined as in the beginning of Section 3. With $\rho = rw$, let

$$\tilde{b}(w,\theta,\xi) := \frac{w\big(1+h(\theta+\xi)\eta_\delta(w)\big)}{1+h(\theta)}, \tag{4.19}$$

$$\tilde{B}(w,\theta,\xi) := \frac{r}{R}\cdot\frac{w\big(1+h(\theta+\xi)\eta_\delta(w)\big)}{1+H(\theta)}. \tag{4.20}$$

Additionally, we define

$$b(w,\theta) := \tilde{b}(w,\theta,0) = \frac{w\big(1+h(\theta)\eta_\delta(w)\big)}{1+h(\theta)}, \tag{4.21}$$

$$B(w,\theta) := \tilde{B}(w,\theta,0) = \frac{r}{R}\cdot\frac{w\big(1+h(\theta)\eta_\delta(w)\big)}{1+H(\theta)}. \tag{4.22}$$

The motivation of introducing these quantities will be clear later in (4.43) and (4.44). In what follows, we will work with several different configurations of interfaces, determined by h_i and H_i $(i=1,2)$, respectively. We define the corresponding quantities $\tilde{b}_i$, $\tilde{B}_i$, b_i and B_i as above, with h and H replaced by h_i and H_i.

Recall that $m_{0,i}$ and $M_{0,i}$ are defined in (3.17) and (3.18), while Δm_0 and ΔM_0 are defined in (3.47) and (3.48). It is straightforward to show that:

Lemma 4.2 *Suppose $h_i, H_i \in W^{1,\infty}(\mathbb{T})$ $(i=1,2)$, with $m_{0,i}+M_{0,i} \ll 1$. Then with C being universal constants, for all $w\in[0,1+4\delta]$ and $\xi\in\mathbb{T}$,*

$$|\tilde{b}_i - b_i| \leqslant C|\eta_\delta||\xi|\|h_i'\|_{L^\infty}, \tag{4.23}$$

$$|\tilde{b}_1 - \tilde{b}_2| \leqslant C(|\eta_\delta||\xi| + \delta|1-\eta_\delta|)\Delta m_0 \leqslant C(|\xi|+|1-w|)\Delta m_0, \tag{4.24}$$

$$|b_1 - b_2| \leqslant C|1-\eta_\delta|\|h_1-h_2\|_{L^\infty} \leqslant C|1-w|\Delta m_0, \tag{4.25}$$

$$|\tilde{b}_1 - b_1 - \tilde{b}_2 + b_2| \leqslant C|\eta_\delta||\xi|\Delta m_0, \tag{4.26}$$

$$|\tilde{B}_i - B_i| \leqslant \frac{Cr}{R}|\eta_\delta||\xi|\|h_i'\|_{L^\infty}, \tag{4.27}$$

$$|\tilde{B}_1 - \tilde{B}_2| + |B_1 - B_2| \leqslant \frac{Cr}{R}(|\eta_\delta|\|h_1-h_2\|_{L^\infty} + \|H_1-H_2\|_{L^\infty}) \leqslant \frac{Cr\delta}{R}(\Delta m_0 + \Delta M_0), \tag{4.28}$$

$$|\tilde{B}_1 - B_1 - \tilde{B}_2 + B_2| \leqslant \frac{Cr}{R}|\eta_\delta||\xi|(\|h_1'-h_2'\|_{L^\infty} + \|h_2'\|_{L^\infty}\|H_1-H_2\|_{L^\infty}), \tag{4.29}$$

$$\left|\frac{\partial\tilde{B}_i}{\partial\theta} - \frac{\partial B_i}{\partial\theta}\right| \leqslant \frac{Cr}{R}|\eta_\delta|\big(\|h_i'\|_{L^\infty} + \|H_i'\|_{L^\infty}\|h_i'\|_{L^\infty}|\xi|\big) \leqslant \frac{Cr}{R}|\eta_\delta|\|h_i'\|_{L^\infty}, \tag{4.30}$$

$$\left|\frac{\partial B_1}{\partial\theta} - \frac{\partial B_2}{\partial\theta}\right| \leqslant \frac{Cr}{R}(\Delta m_0 + \Delta M_0), \tag{4.31}$$

and

$$\left|\frac{\partial\tilde{B}_1}{\partial\theta} - \frac{\partial B_1}{\partial\theta} - \frac{\partial\tilde{B}_2}{\partial\theta} + \frac{\partial B_2}{\partial\theta}\right| \leqslant \frac{Cr}{R}|\eta_\delta|(\|h_1'-h_2'\|_{L^\infty} + \|h_2'\|_{L^\infty}\|H_1-H_2\|_{L^\infty} + \|H_1'-H_2'\|_{L^\infty}\|h_1'\|_{L^\infty}|\xi|). \tag{4.32}$$

If in addition, $h_i \in C^{1,\beta}(\mathbb{T})$ for some $\beta \in (0,1)$, then

$$\left|\frac{\partial \tilde{b}_i}{\partial \theta} - \frac{\partial b_i}{\partial \theta}\right| \leqslant C|\eta_\delta|(\|h_i'\|_{\dot{C}^\beta}|\xi|^\beta + \|h_i'\|_{L^\infty}^2|\xi|) \leqslant C|\eta_\delta|\|h_i'\|_{\dot{C}^\beta}|\xi|^\beta, \tag{4.33}$$

$$\left|\frac{\partial b_1}{\partial \theta} - \frac{\partial b_2}{\partial \theta}\right| \leqslant C|1-\eta_\delta|(\|h_1'-h_2'\|_{L^\infty} + \|h_2'\|_{L^\infty}\|h_1-h_2\|_{L^\infty}) \leqslant C|1-\eta_\delta|\Delta m_0, \tag{4.34}$$

and

$$\left|\frac{\partial \tilde{b}_1}{\partial \theta} - \frac{\partial b_1}{\partial \theta} - \frac{\partial \tilde{b}_2}{\partial \theta} + \frac{\partial b_2}{\partial \theta}\right| \leqslant C|\eta_\delta||\xi|^\beta(\|h_1'-h_2'\|_{\dot{C}^\beta} + \|h_2'\|_{\dot{C}^\beta}\|h_1-h_2\|_{L^\infty}). \tag{4.35}$$

Here all the constants C are universal.

Proof. These estimates follow directly from (4.19)–(4.22) and

$$\frac{\partial \tilde{b}_i}{\partial \theta} = \frac{w h_i'(\theta+\xi)\eta_\delta(w)\big(1+h_i(\theta)\big) - h_i'(\theta)w\big(1+h_i(\theta+\xi)\eta_\delta(w)\big)}{\big(1+h_i(\theta)\big)^2}, \tag{4.36}$$

$$\frac{\partial b_i}{\partial \theta} = \frac{w h_i'(\theta)(\eta_\delta(w)-1)}{\big(1+h_i(\theta)\big)^2}, \tag{4.37}$$

$$\frac{\partial \tilde{B}_i}{\partial \theta} = \frac{r}{R}\cdot\frac{w h_i'(\theta+\xi)\eta_\delta(w)(1+H_i(\theta)) - H_i'(\theta)w\big(1+h_i(\theta+\xi)\eta_\delta(w)\big)}{\big(1+H_i(\theta)\big)^2}, \tag{4.38}$$

$$\frac{\partial B_i}{\partial \theta} = \frac{r}{R}\cdot\frac{w h_i'(\theta)\eta_\delta(w)\big(1+H_i(\theta)\big) - H_i'(\theta)w\big(1+h_i(\theta)\eta_\delta(w)\big)}{\big(1+H_i(\theta)\big)^2}. \tag{4.39}$$

We omit the details. □

REMARK 4.1 Taking $h_1 = H_1 = 0$ (or $h_2 = H_2 = 0$), we find by (4.24), (4.25) and (4.28) that

$$|\tilde{b}_i - w| + |b_i - w| \leqslant C(|\xi| + |1-w|)m_{0,i}, \tag{4.40}$$

$$\left|\tilde{B}_i - \frac{rw}{R}\right| + \left|B_i - \frac{rw}{R}\right| \leqslant \frac{Cr\delta}{R}(m_{0,i}+M_{0,i}) \leqslant C\left(|\xi| + \left|1-\frac{rw}{R}\right|\right)(m_{0,i}+M_{0,i}). \tag{4.41}$$

Here we used the fact that $|1-\frac{rw}{R}| \geqslant C\delta$ for all $w \in [0, 1+4\delta]$ (c.f. (2.23)). If $m_{0,i} + M_{0,i}$ is assumed to be suitably small, $\tilde{b}_i(w,\theta,\xi)$ and $b_i(w,\theta)$ satisfy the assumption of Lemma 4.1, while $\tilde{B}_i(w,\theta,\xi)$ and $B_i(w,\theta)$ satisfy the assumption of Lemma 4.1 with w there replaced by $\frac{rw}{R}$.

4.2 *Estimates along γ*

Let $x = f(\theta)(\cos\theta, \sin\theta) \in \gamma$. With abuse of notations, let $y = x\big((\rho\cos(\theta+\xi), \rho\sin(\theta+\xi))\big)$ be an arbitrary point in $\mathbb{R}^2$, where the map x is defined in (3.2). Then

$$
\begin{aligned}
e_\theta \cdot \nabla(\Gamma * g) &= \frac{1}{2\pi}\int_{\tilde{\Omega}} \frac{(y-x)\cdot e_\theta}{|x-y|^2} g_0\big(X(y)\big)\,dy \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} d\xi \int_0^{r(1+4\delta)} \frac{|y|\sin\xi \cdot g_0(\rho,\theta+\xi)}{f(\theta)^2+|y|^2-2|y|f(\theta)\cos\xi}\cdot\frac{\partial|y|}{\partial\rho}|y|\,d\rho \\
&= \frac{1}{4\pi}\int_{\mathbb{T}} d\xi \int_0^{r(1+4\delta)} \frac{2\left(\frac{|y|}{f(\theta)}\right)^2\sin\xi}{1+\left(\frac{|y|}{f(\theta)}\right)^2-2\frac{|y|}{f(\theta)}\cos\xi}\cdot\frac{\partial|y|}{\partial\rho} g_0(\rho,\theta+\xi)\,d\rho. \quad (4.42)
\end{aligned}
$$

For $w \in [0, 1+4\delta]$, $|y| = |y(\rho,\theta+\xi)| = rw[1+h(\theta+\xi)\eta_\delta(w)]$. Note that the third term in (3.2) does not show up since $\rho = rw \leqslant R(1-2\delta)$. Then (4.42) becomes

$$
\big(e_\theta\cdot\nabla(\Gamma * g)\big)_{\gamma(\theta)} = \frac{r}{4\pi}\int_{\mathbb{T}} d\xi\int_0^{1+4\delta} K(\tilde{b},\xi)\cdot\frac{\partial|y|}{\partial\rho} g_0(rw,\theta+\xi)\,dw. \quad (4.43)
$$

Similarly,

$$
\big(e_r\cdot\nabla(\Gamma * g)\big)_{\gamma(\theta)} = \frac{r}{4\pi}\int_{\mathbb{T}} d\xi\int_0^{1+4\delta} J(\tilde{b},\xi)\cdot\frac{\partial|y|}{\partial\rho} g_0(rw,\theta+\xi)\,dw. \quad (4.44)
$$

We first show:

Lemma 4.3 *Suppose for $i = 1, 2$, $h_i \in W^{1,\infty}(\mathbb{T})$ such that $m_{0,i} \ll 1$. Let Δm_0 be defined in (3.47). Let $x_i(X)$ be the map (3.2) determined by h_i (H is irrelevant in this context, and one may take $H = 0$ in (3.2) without loss of generality.) Let $X_i(x)$ be its inverse. Define $g_i = g_0(X_i(x))$. Then*

$$
\left\|\big(e_\theta\cdot\nabla(\Gamma * g_1)\big)_{\gamma_1(\theta)} - \big(e_\theta\cdot\nabla(\Gamma * g_2)\big)_{\gamma_2(\theta)}\right\|_{L^\infty(\mathbb{T})} \leqslant Cr\delta|\ln\delta|\Delta m_0\|g_0\|_{L^\infty}, \quad (4.45)
$$

where C is a universal constant.

In addition, $\|(e_r\cdot\nabla(\Gamma * g_1))_{\gamma_1(\theta)} - (e_r\cdot\nabla(\Gamma * g_2))_{\gamma_2(\theta)}\|_{L^\infty(\mathbb{T})}$ satisfies an identical estimate.

Proof. Let $y_i = x_i(\rho,\theta+\xi)$, with $|y_i| = \rho[1+h_i(\theta+\xi)\eta_\delta(\rho/r)]$. We calculate

$$
\frac{\partial|y_i|}{\partial\rho}(\rho,\theta+\xi) - 1 = h_i(\theta+\xi)\big(\eta_\delta(w)+w\eta_\delta'(w)\big). \quad (4.46)
$$

By Lemma 4.1, Lemma 4.2 and Remark 4.1,

$$
\begin{aligned}
&\left|\int_{\mathbb{T}} d\xi\int_0^{1+4\delta}\big(K(\tilde{b}_1,\xi)-K(\tilde{b}_2,\xi)\big)\cdot\frac{\partial|y_1|}{\partial\rho}g_0(rw,\theta+\xi)\,dw\right| \\
&\qquad\leqslant C\Delta m_0\|g_0\|_{L^\infty}\int_{\mathbb{T}} d\xi\int_0^{1+4\delta}\frac{|\eta_\delta||\xi|+|1-\eta_\delta|\delta}{1+w^2-2w\cos\xi}\,dw \\
&\qquad\leqslant C\delta|\ln\delta|\Delta m_0\|g_0\|_{L^\infty}. \quad (4.47)
\end{aligned}
$$

On the other hand, by (4.46),

$$\begin{aligned}\left|\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} K(\tilde{b}_2,\xi)\cdot\left(\frac{\partial|y_1|}{\partial\rho}-\frac{\partial|y_2|}{\partial\rho}\right) g_0(rw,\theta+\xi)\,dw\right| \\ \leqslant C\int_{\mathbb{T}} d\xi\int_{1-2\delta}^{1+2\delta}\frac{1}{|1-w|+|\xi|}\cdot\|h_1-h_2\|_{L^\infty}\delta^{-1}\|g_0\|_{L^\infty}\,dw \\ \leqslant C\delta|\ln\delta|\Delta m_0\|g_0\|_{L^\infty}. \qquad (4.48)\end{aligned}$$

Combining these estimates with (4.43) yields (4.45). The estimate concerning $(e_r\cdot\nabla(\Gamma * g_i))_{\gamma_i(\theta)}$ can be justified in the same way. □

Lemma 4.4 *Let $h\in W^{1,\infty}(\mathbb{T})$ such that $m_0 \ll 1$, which defines the map x in (3.2) and $g = g_0(X(x))$. Then*

$$\begin{aligned}\left\|\left(e_\theta\cdot\nabla(\Gamma * g)\right)_{\gamma(\theta)}\right\|_{L^\infty(\mathbb{T})}+\left\|\left(e_r\cdot\nabla(\Gamma * g)\right)_{\gamma(\theta)}-c_{g_0}\right\|_{L^\infty(\mathbb{T})} \\ \leqslant Cr\left(m_0\delta|\ln\delta|\|g_0\|_{L^\infty(B_{(1+4\delta)r})}+\|e_\theta\cdot\nabla g_0\|_{L^2(B_{r(1+4\delta)})}\right), \qquad (4.49)\end{aligned}$$

where C is a universal constant and

$$c_{g_0} := -\frac{1}{2\pi r}\int_{B_r} g_0(X)\,dX. \qquad (4.50)$$

Proof. We first derive an L^∞-estimate of

$$\left(e_\theta\cdot\nabla(\Gamma * g_0)\right)_{\partial B_r} = \frac{r}{4\pi}\int_{\mathbb{T}} d\xi\int_0^{1+4\delta} K(w,\xi)g_0(rw,\theta+\xi)\,dw, \qquad (4.51)$$

which corresponds to the case $h=0$. Define $\bar{g}_0(rw)=(2\pi)^{-1}\int_{\mathbb{T}} g_0(rw,\xi)\,d\xi$. Since $K(w,\cdot)$ is an odd kernel, by Hölder's inequality and Sobolev embedding,

$$\begin{aligned}|\left(e_\theta\cdot\nabla(\Gamma * g_0)\right)_{\partial B_r}| &= \frac{r}{4\pi}\left|\int_{\mathbb{T}} d\xi\int_0^{1+4\delta} K(w,\xi)\left(g_0(rw,\theta+\xi)-\bar{g}_0(rw)\right)dw\right| \\ &\leqslant Cr\int_0^{1+4\delta}\left\|\frac{1}{|1-w|+|\xi|}\right\|_{L^1_\xi(\mathbb{T})}\|g_0(rw,\cdot)-\bar{g}_0(rw)\|_{L^\infty_\xi(\mathbb{T})}\,dw \\ &\leqslant Cr\int_0^{1+4\delta}(1+|\ln|1-w||)\|\partial_\theta g_0(rw,\cdot)\|_{L^2(\mathbb{T})}\,dw \\ &\leqslant Cr\|1+|\ln|1-w|\|_{L^2([0,1+4\delta])}\left(\int_0^{1+4\delta} r\|e_\theta\cdot\nabla g_0\|^2_{L^2(\partial B_{rw})}\,dw\right)^{1/2} \\ &\leqslant Cr\|e_\theta\cdot\nabla g_0\|_{L^2(B_{r(1+4\delta)})}. \qquad (4.52)\end{aligned}$$

Now we take in Lemma 4.3 that $h_1=h$ and $h_2=0$, and derive

$$\begin{aligned}\left\|\left(e_\theta\cdot\nabla(\Gamma * g)\right)_{\gamma(\theta)}\right\|_{L^\infty(\mathbb{T})} &\leqslant \left\|\left(e_\theta\cdot\nabla(\Gamma * g)\right)_{\gamma(\theta)}-\left(e_\theta\cdot\nabla(\Gamma * g_0)\right)_{\partial B_r}\right\|_{L^\infty(\mathbb{T})} \\ &\quad+\left\|\left(e_\theta\cdot\nabla(\Gamma * g_0)\right)_{\partial B_r}\right\|_{L^\infty(\mathbb{T})} \\ &\leqslant Cr\delta|\ln\delta|m_0\|g_0\|_{L^\infty}+Cr\|e_\theta\cdot\nabla g_0\|_{L^2(B_{r(1+4\delta)})}. \qquad (4.53)\end{aligned}$$

Next we study

$$\big(e_r\cdot\nabla(\Gamma*g_0)\big)_{\partial B_r}=\frac{r}{4\pi}\int_{\mathbb{T}}d\xi\int_0^{1+4\delta}J(w,\xi)\big(g_0(rw,\theta+\xi)-\bar g_0(rw)\big)\,dw$$
$$+\frac{r}{4\pi}\int_0^{1+4\delta}\int_{\mathbb{T}}d\xi\,J(w,\xi)\bar g_0(rw)\,dw. \quad (4.54)$$

The first term can be bounded exactly as in (4.52). We use the definition of J in (4.4) to simplify the second term as

$$\frac{r}{4\pi}\int_0^{1+4\delta}\int_{\mathbb{T}}d\xi\,J(w,\xi)\bar g_0(rw)\,dw=-r\int_0^1 w\bar g_0(rw)\,dw=-\frac{1}{2\pi r}\int_{B_r}g_0(X)\,dX. \quad (4.55)$$

Then the desired estimate follows. □

Next we derive $W^{1,p}$-estimates for $\big(e_\theta\cdot\nabla(\Gamma*g)\big)_{\gamma(\theta)}$ and $\big(e_r\cdot\nabla(\Gamma*g)\big)_{\gamma(\theta)}$.

Lemma 4.5 *Assume $h_1,h_2\in C^{1,\beta}(\mathbb{T})$ for some $\beta\in(0,1)$, such that $m_{0,i}\ll 1$. Let Δm_0 be defined in (3.47), and let $g_i(x)=g_0(X_i(x))$. Then for all $p\in[2,\infty)$,*

$$\big\|\big(e_\theta\cdot\nabla(\Gamma*g_1)\big)_{\gamma_1(\theta)}-\big(e_\theta\cdot\nabla(\Gamma*g_2)\big)_{\gamma_2(\theta)}\big\|_{\dot W^{1,p}(\mathbb{T})}$$
$$\leqslant Cr\|g_0\|_{L^\infty(B_{(1+4\delta)r})}\Big[\big(1+\delta^\beta(\|h_1'\|_{\dot C^\beta}+\|h_2'\|_{\dot C^\beta})\big)\Delta m_0+\delta^\beta\|h_1'-h_2'\|_{\dot C^\beta}\Big]$$
$$+Cr\Delta m_0\|e_\theta\cdot\nabla g_0\|_{L^2(B_{(1+4\delta)r})}, \quad (4.56)$$

where $C=C(p,\beta)$.

Proof. Let $y_i=x_i(\rho,\theta+\xi)$. We take θ-derivative in (4.43).

$$\frac{d}{d\theta}\big(e_\theta\cdot\nabla(\Gamma*g_i)\big)_{\gamma_i(\theta)}$$
$$=\frac{r}{4\pi}\int_{\mathbb{T}}d\xi\int_0^{1+4\delta}dw\,\frac{\partial}{\partial\theta}\left[K(\tilde b_i,\xi)\cdot\frac{\partial|y_i|}{\partial\rho}g_0(rw,\xi+\theta)\right]$$
$$=\frac{r}{4\pi}\int_{\mathbb{T}}d\xi\int_0^{1+4\delta}dw\left[\frac{\partial K}{\partial s}(\tilde b_i,\xi)\frac{\partial\tilde b_i}{\partial\theta}-\frac{\partial K}{\partial s}(b_i,\xi)\frac{\partial b_i}{\partial\theta}\right]\left[\frac{\partial|y_i|}{\partial\rho}g_0\right]_{(rw,\xi+\theta)}$$
$$+\frac{r}{4\pi}\int_{\mathbb{T}}d\xi\int_0^{1+4\delta}dw\,\frac{\partial K}{\partial s}(b_i,\xi)\frac{\partial b_i}{\partial\theta}\left(\left[\frac{\partial|y_i|}{\partial\rho}g_0\right]_{(rw,\xi+\theta)}-\left[\frac{\partial|y_i|}{\partial\rho}g_0\right]_{(rw,\theta)}\right)$$
$$-\frac{r}{4\pi}\int_{\mathbb{T}}d\xi\int_0^{1+4\delta}dw\left[\frac{\partial K}{\partial s}(\tilde b_i,\xi)\frac{\partial\tilde b_i}{\partial\xi}+\frac{\partial K}{\partial\xi}(\tilde b_i,\xi)\right]$$
$$\left(\left[\frac{\partial|y_i|}{\partial\rho}g_0\right]_{(rw,\xi+\theta)}-\left[\frac{\partial|y_i|}{\partial\rho}g_0\right]_{(rw,\theta)}\right)$$
$$=:J_{\theta,1}^{(i)}+J_{\theta,2}^{(i)}+J_{\theta,3}^{(i)}. \quad (4.57)$$

Here we exchanged the integral with the θ-derivative, which can be justified rigorously by a limiting argument. In $J_{\theta,2}^{(i)}$, an extra term is inserted without changing its value, since $\partial_s K(b_i,\xi)$ is odd in ξ. When deriving $J_{\theta,3}^{(i)}$, we used the fact that

$$\frac{\partial}{\partial\theta}\left[\frac{\partial|y_i|}{\partial\rho}g_0\right]_{(rw,\xi+\theta)} = \frac{\partial}{\partial\xi}\left[\frac{\partial|y_i|}{\partial\rho}g_0\right]_{(rw,\xi+\theta)} \tag{4.58}$$

and then integrated by parts. Note that it is not clear a priori whether these integrands are integrable at $(w,\xi)=(1,0)$, so we need to write them as principal value integrals in the w-variable in the first place. Yet, it will be clear in the following that all these integrands are absolutely integrable. For this reason, we omitted the notations for the principal value integral.

We start with bounding $J_{\theta,1}^{(1)} - J_{\theta,1}^{(2)}$.

$$\begin{aligned}
&J_{\theta,1}^{(1)} - J_{\theta,1}^{(2)}\\
&= \frac{r}{4\pi}\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw\,\left(\frac{\partial K}{\partial s}(\tilde b_1,\xi) - \frac{\partial K}{\partial s}(\tilde b_2,\xi)\right)\left(\frac{\partial \tilde b_1}{\partial\theta} - \frac{\partial b_1}{\partial\theta}\right)\left[\frac{\partial|y_1|}{\partial\rho}g_0\right]_{(rw,\xi+\theta)}\\
&\quad + \frac{r}{4\pi}\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw\,\frac{\partial K}{\partial s}(\tilde b_2,\xi)\left(\frac{\partial \tilde b_1}{\partial\theta} - \frac{\partial b_1}{\partial\theta} - \frac{\partial \tilde b_2}{\partial\theta} + \frac{\partial b_2}{\partial\theta}\right)\left[\frac{\partial|y_1|}{\partial\rho}g_0\right]_{(rw,\xi+\theta)}\\
&\quad + \frac{r}{4\pi}\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw\,\left(\frac{\partial K}{\partial s}(\tilde b_1,\xi) - \frac{\partial K}{\partial s}(b_1,\xi) - \frac{\partial K}{\partial s}(\tilde b_2,\xi) + \frac{\partial K}{\partial s}(b_2,\xi)\right)\frac{\partial b_1}{\partial\theta}\\
&\qquad\qquad \left[\frac{\partial|y_1|}{\partial\rho}g_0\right]_{(rw,\xi+\theta)}\\
&\quad + \frac{r}{4\pi}\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw\,\left(\frac{\partial K}{\partial s}(\tilde b_2,\xi) - \frac{\partial K}{\partial s}(b_2,\xi)\right)\left(\frac{\partial b_1}{\partial\theta} - \frac{\partial b_2}{\partial\theta}\right)\left[\frac{\partial|y_1|}{\partial\rho}g_0\right]_{(rw,\xi+\theta)}\\
&\quad + \frac{r}{4\pi}\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw\,\frac{\partial K}{\partial s}(\tilde b_2,\xi)\left(\frac{\partial \tilde b_2}{\partial\theta} - \frac{\partial b_2}{\partial\theta}\right)\left[\left(\frac{\partial|y_1|}{\partial\rho} - \frac{\partial|y_2|}{\partial\rho}\right)g_0\right]_{(rw,\xi+\theta)}\\
&\quad + \frac{r}{4\pi}\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw\,\left(\frac{\partial K}{\partial s}(\tilde b_2,\xi) - \frac{\partial K}{\partial s}(b_2,\xi)\right)\frac{\partial b_2}{\partial\theta}\left[\left(\frac{\partial|y_1|}{\partial\rho} - \frac{\partial|y_2|}{\partial\rho}\right)g_0\right]_{(rw,\xi+\theta)}.
\end{aligned}\tag{4.59}$$

By Lemma 4.1, Lemma 4.2, Lemma A.1 and (4.46),

$$\begin{aligned}
&|J_{\theta,1}^{(1)} - J_{\theta,1}^{(2)}|\\
&\leq Cr\|g_0\|_{L^\infty}\int_{\mathbb{T}} d\xi\int_0^{1+4\delta} dw\,\frac{|\tilde b_1 - \tilde b_2|}{(|1-w|+|\xi|)^3}\cdot|\eta_\delta|\|h_1'\|_{\dot C^\beta}|\xi|^\beta\\
&\quad + Cr\|g_0\|_{L^\infty}\int_{\mathbb{T}} d\xi\int_0^{1+4\delta} dw\,\frac{|\eta_\delta||\xi|^\beta(\|h_1'-h_2'\|_{\dot C^\beta} + \|h_2'\|_{\dot C^\beta}\|h_1-h_2\|_{L^\infty})}{(|1-w|+|\xi|)^2}\\
&\quad + Cr\|g_0\|_{L^\infty}\int_{\mathbb{T}} d\xi\int_0^{1+4\delta} dw\left[\frac{|\tilde b_1 - b_1 - \tilde b_2 + b_2|}{(|1-w|+|\xi|)^3}\right.\\
&\qquad\qquad \left. + \frac{(|\tilde b_1 - b_1| + |\tilde b_2 - b_2|)(|\tilde b_1 - \tilde b_2| + |b_1 - b_2|)}{(|1-w|+|\xi|)^4}\right]\|h_1'\|_{L^\infty}|1-\eta_\delta|
\end{aligned}$$

$$
\begin{aligned}
&+ Cr\|g_0\|_{L^\infty}\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw\, \frac{|\tilde{b}_2 - b_2|}{(|1-w|+|\xi|)^3}\cdot|1-\eta_\delta|\Delta m_0\\
&+ Cr\|g_0\|_{L^\infty}\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw\, \frac{1}{(|1-w|+|\xi|)^2}\cdot|\eta_\delta|\|h_2'\|_{\dot{C}^\beta}|\xi|^\beta\cdot|\eta_\delta + w\eta_\delta'|\|h_1 - h_2\|_{L^\infty}\\
&+ Cr\|g_0\|_{L^\infty}\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw\, \frac{|\tilde{b}_2 - b_2|}{(|1-w|+|\xi|)^3}\cdot\|h_2'\|_{L^\infty}|1-\eta_\delta|\cdot|\eta_\delta + w\eta_\delta'|\|h_1 - h_2\|_{L^\infty}\\
&\leqslant Cr\|g_0\|_{L^\infty}\Big[\delta^\beta(\|h_1'\|_{\dot{C}^\beta} + \|h_2'\|_{\dot{C}^\beta})\Delta m_0 + \delta^\beta\|h_1' - h_2'\|_{\dot{C}^\beta} + (\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty})\Delta m_0\Big].
\end{aligned}
\tag{4.60}
$$

In the last inequality, when calculating the integrals, we used the facts that η_δ is supported on $[1-2\delta, 1+2\delta]$ and that $\eta_\delta(1-\eta_\delta)$ is supported on $[1-2\delta, 1-\delta]\cup[1+\delta, 1+2\delta]$.

For $J_{\theta,2}^{(i)}$ and $J_{\theta,3}^{(i)}$, by (4.57),

$$
\begin{aligned}
&(J_{\theta,2}^{(1)} + J_{\theta,3}^{(1)}) - (J_{\theta,3}^{(2)} + J_{\theta,2}^{(2)})\\
&\quad = \frac{r}{4\pi}\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw\, \left[\frac{\partial K}{\partial s}(b_1,\xi)\frac{\partial b_1}{\partial \theta} - \frac{\partial K}{\partial s}(\tilde{b}_1,\xi)\frac{\partial \tilde{b}_1}{\partial \xi} - \frac{\partial K}{\partial \xi}(\tilde{b}_1,\xi)\right]\\
&\qquad\qquad \cdot\left(\left[\left(\frac{\partial|y_1|}{\partial\rho} - \frac{\partial|y_2|}{\partial\rho}\right)g_0\right]_{(rw,\xi+\theta)} - \left[\left(\frac{\partial|y_1|}{\partial\rho} - \frac{\partial|y_2|}{\partial\rho}\right)g_0\right]_{(rw,\theta)}\right)\\
&\qquad + \frac{r}{4\pi}\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw\, \left(\frac{\partial K}{\partial s}(b_1,\xi) - \frac{\partial K}{\partial s}(b_2,\xi)\right)\frac{\partial b_1}{\partial\theta}\\
&\qquad\qquad \cdot\left(\left[\frac{\partial|y_2|}{\partial\rho}g_0\right]_{(rw,\xi+\theta)} - \left[\frac{\partial|y_2|}{\partial\rho}g_0\right]_{(rw,\theta)}\right)\\
&\qquad - \frac{r}{4\pi}\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw\, \left[\left(\frac{\partial K}{\partial s}(\tilde{b}_1,\xi) - \frac{\partial K}{\partial s}(\tilde{b}_2,\xi)\right)\frac{\partial\tilde{b}_1}{\partial\xi} + \left(\frac{\partial K}{\partial \xi}(\tilde{b}_1,\xi) - \frac{\partial K}{\partial \xi}(\tilde{b}_2,\xi)\right)\right]\\
&\qquad\qquad \cdot\left(\left[\frac{\partial|y_2|}{\partial\rho}g_0\right]_{(rw,\xi+\theta)} - \left[\frac{\partial|y_2|}{\partial\rho}g_0\right]_{(rw,\theta)}\right)\\
&\qquad + \frac{r}{4\pi}\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw\, \left[\frac{\partial K}{\partial s}(b_2,\xi)\frac{\partial(b_1 - b_2)}{\partial\theta} - \frac{\partial K}{\partial s}(\tilde{b}_2,\xi)\frac{\partial(\tilde{b}_1 - \tilde{b}_2)}{\partial\xi}\right]\\
&\qquad\qquad \cdot\left(\left[\frac{\partial|y_2|}{\partial\rho}g_0\right]_{(rw,\xi+\theta)} - \left[\frac{\partial|y_2|}{\partial\rho}g_0\right]_{(rw,\theta)}\right).
\end{aligned}
\tag{4.61}
$$

We derive in a similar manner.

$$
\begin{aligned}
&\left|(J_{\theta,2}^{(1)} + J_{\theta,3}^{(1)}) - (J_{\theta,2}^{(2)} + J_{\theta,3}^{(2)})\right|\\
&\leqslant Cr\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} \frac{dw}{(|1-w|+|\xi|)^2}\cdot|\eta_\delta + w\eta_\delta'|\\
&\qquad \cdot\big(\|h_1 - h_2\|_{L^\infty}|g_0(rw,\xi+\theta) - g_0(rw,\theta)| + \|h_1 - h_2\|_{\dot{C}^\beta}|\xi|^\beta\|g_0\|_{L^\infty}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ Cr\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw\, \frac{|b_1-b_2|}{(|1-w|+|\xi|)^3}\cdot|1-\eta_\delta|\|h_1'\|_{L^\infty}\\
&\qquad\cdot\big(|g_0(rw,\xi+\theta)-g_0(rw,\theta)|+|\eta_\delta+w\eta_\delta'||\xi|^\beta\|h_2\|_{\dot C^\beta}\|g_0\|_{L^\infty}\big)\\
&+ Cr\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw\, \frac{|\tilde b_1-\tilde b_2|}{(|1-w|+|\xi|)^3}\\
&\qquad\cdot\big(|g_0(rw,\xi+\theta)-g_0(rw,\theta)|+|\eta_\delta+w\eta_\delta'||\xi|^\beta\|h_2\|_{\dot C^\beta}\|g_0\|_{L^\infty}\big)\\
&+ Cr\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} \frac{dw}{(|1-w|+|\xi|)^2}\cdot\Delta m_0\\
&\qquad\cdot\big(|g_0(rw,\xi+\theta)-g_0(rw,\theta)|+|\eta_\delta+w\eta_\delta'||\xi|^\beta\|h_2\|_{\dot C^\beta}\|g_0\|_{L^\infty}\big)\\
&\leqslant Cr\|g_0\|_{L^\infty}\delta^{\beta-1}\big(\|h_1-h_2\|_{\dot C^\beta}+\Delta m_0\|h_2\|_{\dot C^\beta}\big)\\
&+ Cr\Delta m_0\int_{\mathbb{T}} d\xi\int_0^{1+4\delta} dw\,\frac{|g_0(rw,\xi+\theta)-g_0(rw,\theta)|}{(|1-w|+|\xi|)^2}.
\end{aligned}
\tag{4.62}
$$

By Minkowski inequality and Hölder's inequality, with arbitrary $s\in\left(\frac12,\frac12+\frac1p\right)$ (for definiteness, take $s=\frac12+\frac{1}{2p}$),

$$
\begin{aligned}
&\left\|(J_{\theta,2}^{(1)}+J_{\theta,3}^{(1)})-(J_{\theta,2}^{(2)}+J_{\theta,3}^{(2)})\right\|_{L^p(\mathbb{T})}\\
&\leqslant Cr\|g_0\|_{L^\infty}\Delta m_0\\
&\quad+ Cr\Delta m_0\int_0^{1+4\delta} dw\left[\int_{\mathbb{T}} d\xi\,\frac{\|g_0(rw,\xi+\cdot)-g_0(rw,\cdot)\|^2_{L^p_\theta(\mathbb{T})}}{|\xi|^{1+2s}}\right]^{1/2}\left[\int_{\mathbb{T}}\frac{|\xi|^{1+2s}\,d\xi}{(|1-w|+|\xi|)^4}\right]^{1/2}\\
&\leqslant Cr\|g_0\|_{L^\infty}\Delta m_0 + Cr\Delta m_0\int_0^{1+4\delta} dw\,\frac{\|g_0(rw,\cdot)\|_{\dot B^s_{p,2}(\mathbb{T})}}{|1-w|^{1-s}}\\
&\leqslant Cr\|g_0\|_{L^\infty}\Delta m_0 + Cr\Delta m_0\int_0^{1+4\delta} dw\,\frac{\|\partial_\theta g_0(rw,\cdot)\|_{L^2(\mathbb{T})}}{|1-w|^{1-s}}\\
&\leqslant Cr\Delta m_0(\|g_0\|_{L^\infty}+\|e_\theta\cdot\nabla g_0\|_{L^2(B_{(1+4\delta)r})}).
\end{aligned}
\tag{4.63}
$$

See, e.g., [47, §2.5.12 and §2.7.1] for the definition of $B^s_{p,2}(\mathbb{T})$-space and the embedding of $H^1(\mathbb{T})$ into it. Combining this with (4.57) and (4.60), we conclude with (4.56). □

Lemma 4.6 *Under the assumptions of Lemma 4.5,*

$$
\begin{aligned}
&\|(e_r\cdot\nabla(\Gamma*g_1))_{\gamma_1(\theta)}-(e_r\cdot\nabla(\Gamma*g_2))_{\gamma_2(\theta)}\|_{\dot W^{1,p}(\mathbb{T})}\\
&\qquad\leqslant Cr\|g_0\|_{L^\infty(B_{(1+4\delta)r})}\Big[\big(1+\delta^\beta(\|h_1'\|_{\dot C^\beta}+\|h_2'\|_{\dot C^\beta})\big)\Delta m_0+\delta^\beta\|h_1'-h_2'\|_{\dot C^\beta}\Big]\\
&\qquad\qquad+ Cr\Delta m_0\|e_\theta\cdot\nabla g_0\|_{L^2(B_{(1+4\delta)r})},
\end{aligned}
\tag{4.64}
$$

where $C=C(p,\beta)$.

Proof. We proceed as the proof of Lemma 4.5. By (4.44) and integration by parts,

$$\begin{aligned}
&\frac{d}{d\theta}\big(e_r\cdot\nabla(\Gamma*g_i)\big)_{\gamma_i(\theta)}\\
&=\frac{r}{4\pi}\int_{\mathbb{T}}d\xi\int_0^{1+4\delta}\left[\frac{\partial J}{\partial s}(\tilde b_i,\xi)\frac{\partial\tilde b_i}{\partial\theta}-\frac{\partial J}{\partial s}(b_i,\xi)\frac{\partial b_i}{\partial\theta}\right]\left[\frac{\partial|y_i|}{\partial\rho}g_0\right]_{(rw,\theta+\xi)}dw\\
&+\frac{r}{4\pi}\int_{\mathbb{T}}d\xi\int_0^{1+4\delta}\frac{\partial J}{\partial s}(b_i,\xi)\frac{\partial b_i}{\partial\theta}\left(\left[\frac{\partial|y_i|}{\partial\rho}g_0\right]_{(rw,\theta+\xi)}-\left[\frac{\partial|y_i|}{\partial\rho}g_0\right]_{(rw,\theta)}\right)dw\\
&-\frac{r}{4\pi}\int_{\mathbb{T}}d\xi\int_0^{1+4\delta}\left[\frac{\partial J}{\partial s}(\tilde b_i,\xi)\frac{\partial\tilde b_i}{\partial\xi}+\frac{\partial J}{\partial\xi}(\tilde b_i,\xi)\right]\left(\left[\frac{\partial|y_i|}{\partial\rho}g_0\right]_{(rw,\theta+\xi)}-\left[\frac{\partial|y_i|}{\partial\rho}g_0\right]_{(rw,\theta)}\right)dw\\
&+\frac{r}{4\pi}\int_{\mathbb{T}}d\xi\int_0^{1+4\delta}\frac{\partial J}{\partial s}(b_i,\xi)\frac{\partial b_i}{\partial\theta}\left[\frac{\partial|y_i|}{\partial\rho}g_0\right]_{(rw,\theta)}dw\\
&=:J_{r,1}^{(i)}+J_{r,2}^{(i)}+J_{r,3}^{(i)}+J_{r,4}^{(i)}.
\end{aligned}\tag{4.65}$$

Estimates concerning $J_{r,1}^{(i)}+J_{r,2}^{(i)}+J_{r,3}^{(i)}$ can be derived exactly as in Lemma 4.5. It remains to bound $J_{r,4}^{(1)}-J_{r,4}^{(2)}$. By Lemma A.1,

$$\int_{\mathbb{T}}\frac{\partial J}{\partial s}(s,\xi)\,d\xi=\begin{cases}-4\pi, & \text{if } s\in[0,1),\\ 0, & \text{if } s>1.\end{cases}\tag{4.66}$$

Hence, thanks to Lemma 4.2 and (4.46),

$$\begin{aligned}
\big|J_{r,4}^{(1)}-J_{r,4}^{(2)}\big|&=r\left|\int_0^1\frac{\partial b_1}{\partial\theta}\left[\frac{\partial|y_1|}{\partial\rho}g_0\right]_{(rw,\theta)}-\frac{\partial b_2}{\partial\theta}\left[\frac{\partial|y_2|}{\partial\rho}g_0\right]_{(rw,\theta)}dw\right|\\
&\leqslant Cr\|g_0\|_{L^\infty}\int_0^1\left|\frac{\partial b_1}{\partial\theta}-\frac{\partial b_2}{\partial\theta}\right|\left|\frac{\partial|y_1|}{\partial\rho}\right|+\left|\frac{\partial b_2}{\partial\theta}\right|\left|\frac{\partial|y_1|}{\partial\rho}-\frac{\partial|y_2|}{\partial\rho}\right|dw\\
&\leqslant Cr\|g_0\|_{L^\infty}\Delta m_0.
\end{aligned}\tag{4.67}$$

This completes the proof. □

Lemma 4.7 *Assume $h\in C^{1,\beta}(\mathbb{T})$ for some $\beta\in(0,1)$, such that $m_0\ll 1$. Define $g(x)=g_0(X(x))$. Then for all $p\in[2,\infty)$,*

$$\begin{aligned}
&\big\|\big(e_\theta\cdot\nabla(\Gamma*g)\big)_{\gamma(\theta)}\big\|_{\dot W^{1,p}(\mathbb{T})}+\big\|\big(e_r\cdot\nabla(\Gamma*g)\big)_{\gamma(\theta)}\big\|_{\dot W^{1,p}(\mathbb{T})}\\
&\qquad\leqslant Cr\big(\|g_0\|_{L^\infty(B_{(1+4\delta)r})}m_\beta+\|e_\theta\cdot\nabla g_0\|_{L^2(B_{(1+4\delta)r})}\big),
\end{aligned}\tag{4.68}$$

where $C=C(p,\beta)$. Here m_β is defined as in (3.45).

Proof. As in Lemma 4.4, we first study the case with $h=0$. By (4.57),

$$\frac{d}{d\theta}\big(e_\theta\cdot\nabla(\Gamma*g_0)\big)_{\partial B_r}-\frac{r}{4\pi}\int_{\mathbb{T}}d\xi\int_0^{1+4\delta}dw\,\frac{\partial K}{\partial\xi}(w,\xi)\big(g_0(rw,\xi+\theta)-g_0(rw,\theta)\big).\tag{4.69}$$

Hence, arguing as in (4.63),

$$\begin{aligned}\left\|\big(e_\theta\cdot\nabla(\Gamma*g_0)\big)_{\partial B_r}\right\|_{\dot W^{1,p}} &\leqslant Cr\int_{\mathbb{T}}d\xi\int_0^{1+4\delta}dw\,\frac{\|g_0(rw,\xi+\cdot)-g_0(rw,\cdot)\|_{L^p_\theta(\mathbb{T})}}{(|1-w|+|\xi|)^2}\\ &\leqslant Cr\|e_\theta\cdot\nabla g_0\|_{L^2(B_{(1+4\delta)r})}. \end{aligned}\tag{4.70}$$

Now taking $h_1=h$ and $h_2=0$ in Lemma 4.5, we find that

$$\begin{aligned}&\left\|\big(e_\theta\cdot\nabla(\Gamma*g)\big)_{\gamma(\theta)}\right\|_{\dot W^{1,p}(\mathbb{T})}\\ &\quad\leqslant\left\|\big(e_\theta\cdot\nabla(\Gamma*g)\big)_{\gamma(\theta)}-\big(e_\theta\cdot\nabla(\Gamma*g_0)\big)_{\partial B_r}\right\|_{\dot W^{1,p}(\mathbb{T})}+\left\|\big(e_\theta\cdot\nabla(\Gamma*g_0)\big)_{\partial B_r}\right\|_{\dot W^{1,p}(\mathbb{T})}\\ &\quad\leqslant Cr\|g_0\|_{L^\infty(B_{(1+4\delta)r})}(m_0+\delta^\beta\|h'\|_{\dot C^\beta})+Cr\|e_\theta\cdot\nabla g_0\|_{L^2(B_{(1+4\delta)r})}. \end{aligned}\tag{4.71}$$

The estimate for $(e_r\cdot\nabla(\Gamma*g))_{\gamma(\theta)}$ can be derived in exactly the same way. □

4.3 *Estimates along $\tilde\gamma$*

Next, we derive estimates for $e_r\cdot\nabla(\Gamma*g)$ and $e_\theta\cdot\nabla(\Gamma*g)$ along $\tilde\gamma$, with $g(x)=g_0(X(x))$. We calculate as in (4.42) that

$$\begin{aligned}\big(e_\theta\cdot\nabla(\Gamma*g)\big)_{\tilde\gamma(\theta)}&=\frac{1}{2\pi}\int_{\mathbb{T}}d\xi\int_0^{r(1+4\delta)}\frac{|y|\sin\xi\cdot g_0(\rho,\theta+\xi)}{F(\theta)^2+|y|^2-2|y|F(\theta)\cos\xi}\cdot\frac{\partial|y|}{\partial\rho}|y|\,d\rho\\ &=\frac{r}{4\pi}\int_{\mathbb{T}}d\xi\int_0^{1+4\delta}K(\tilde B,\xi)\cdot\frac{\partial|y|}{\partial\rho}g_0(rw,\theta+\xi)\,dw,\end{aligned}\tag{4.72}$$

and

$$\big(e_r\cdot\nabla(\Gamma*g)\big)_{\tilde\gamma(\theta)}\frac{r}{4\pi}\int_{\mathbb{T}}d\xi\int_0^{1+4\delta}J(\tilde B,\xi)\cdot\frac{\partial|y|}{\partial\rho}g_0(rw,\theta+\xi)\,dw.\tag{4.73}$$

Arguing as in Lemma 4.3, we can show:

Lemma 4.8 *Under the assumptions of Lemma 4.3,*

$$\left\|\big(e_\theta\cdot\nabla(\Gamma*g_1)\big)_{\tilde\gamma_1(\theta)}-\big(e_\theta\cdot\nabla(\Gamma*g_2)\big)_{\tilde\gamma_2(\theta)}\right\|_{L^\infty(\mathbb{T})}\leqslant\frac{Cr^2}{R}\delta|\ln\delta|(\Delta m_0+\Delta M_0)\|g_0\|_{L^\infty},\tag{4.74}$$

where C is universal. Moreover, $\|(e_r\cdot\nabla(\Gamma*g_1))_{\tilde\gamma_1(\theta)}-(e_r\cdot\nabla(\Gamma*g_2))_{\tilde\gamma_2(\theta)}\|_{L^\infty(\mathbb{T})}$ satisfies the same estimate.

We omit its proof here, but only note that $|\tilde B_i|\leqslant\frac{Cr}{R}$ and $|\ln(1-\frac{(1+4\delta)r}{R})|\leqslant\frac{Cr}{R}|\ln\delta|$.

Then we prove as in Lemma 4.4 that:

Lemma 4.9 *Let $h,H\in W^{1,\infty}(\mathbb{T})$ such that $m_0,M_0\ll 1$, which define the map x in (3.2) and $g=g_0(X(x))$. Then*

$$\begin{aligned}&\left\|\big(e_\theta\cdot\nabla(\Gamma*g)\big)_{\tilde\gamma(\theta)}\right\|_{L^\infty(\mathbb{T})}+\left\|\big(e_r\cdot\nabla(\Gamma*g)\big)_{\tilde\gamma(\theta)}-\tilde c_{g_0}\right\|_{L^\infty(\mathbb{T})}\\ &\qquad\leqslant\frac{Cr^2}{R}\Big((m_0+M_0)\delta|\ln\delta|\|g_0\|_{L^\infty(B_{(1+4\delta)r})}+\|e_\theta\cdot\nabla g_0\|_{L^2(B_{r(1+4\delta)})}\Big),\end{aligned}\tag{4.75}$$

where C is universal and

$$\tilde c_{g_0}:=-\frac{1}{2\pi R}\int_{B_{r(1+4\delta)}}g_0(X)\,dX.\tag{4.76}$$

Proof. Let $\bar{g}_0$ be as in Lemma 4.4. We proceed as in (4.52) by noticing that $K(\frac{rw}{R},\cdot)$ is an odd kernel.

$$\begin{aligned}\left|\left(e_\theta\cdot\nabla(\Gamma*g_0)\right)_{\partial B_R}\right| &= \frac{r}{4\pi}\left|\int_{\mathbb{T}}d\xi\int_0^{1+4\delta}K\left(\frac{rw}{R},\xi\right)\left(g_0(rw,\theta+\xi)-\bar{g}_0(rw)\right)dw\right|\\ &\leqslant Cr\int_0^{1+4\delta}\left\|\frac{\frac{r}{R}}{|1-\frac{rw}{R}|+|\xi|}\right\|_{L^1_\xi(\mathbb{T})}\|g_0(rw,\cdot)-\bar{g}_0(rw)\|_{L^\infty_\xi(\mathbb{T})}\,dw\\ &\leqslant \frac{Cr^2}{R}\|e_\theta\cdot\nabla g_0\|_{L^2(B_{r(1+4\delta)})}. \end{aligned}\tag{4.77}$$

Combining this and Lemma 4.8 with $h_1=h$, $H_1=H$ and $h_2=H_2=0$, we argue as in (4.53) to find that $\|(e_\theta\cdot\nabla(\Gamma*g))_{\tilde{\gamma}(\theta)}\|_{L^\infty(\mathbb{T})}$ satisfies the desired bound.

Similarly,

$$\begin{aligned}\left(e_r\cdot\nabla(\Gamma*g_0)\right)_{\partial B_R} = &\frac{r}{4\pi}\int_{\mathbb{T}}d\xi\int_0^{1+4\delta}J\left(\frac{rw}{R},\xi\right)\left(g_0(rw,\theta+\xi)-\bar{g}_0(rw)\right)dw\\ &+\frac{r}{4\pi}\int_0^{1+4\delta}\int_{\mathbb{T}}d\xi\,J\left(\frac{rw}{R},\xi\right)\bar{g}_0(rw)\,dw. \end{aligned}\tag{4.78}$$

The first term can be bounded exactly as in (4.77). For the second term, we notice that $\frac{r(1+4\delta)}{R}\leqslant 1$. By (4.4),

$$\begin{aligned}&\frac{r}{4\pi}\int_0^{1+4\delta}\int_{\mathbb{T}}d\xi\,J\left(\frac{rw}{R},\xi\right)\bar{g}_0(rw)\,dw\\ &\qquad=-\frac{r^2}{R}\int_0^{1+4\delta}w\bar{g}_0(rw)\,dw=-\frac{1}{2\pi R}\int_{B_{r(1+4\delta)}}g_0(X)\,dX. \end{aligned}\tag{4.79}$$

Then the desired estimate follows. □

We shall follow Lemma 4.5 and Lemma 4.6 to prove $W^{1,p}$-estimates concerning $(e_\theta\cdot\nabla(\Gamma*g))_{\tilde{\gamma}(\theta)}$ and $(e_r\cdot\nabla(\Gamma*g))_{\tilde{\gamma}(\theta)}$.

Lemma 4.10 *Assume $h_i,H_i\in W^{1,\infty}(\mathbb{T})$ $(i=1,2)$ such that $m_{0,i}+M_{0,i}\ll 1$. Let Δm_0 and ΔM_0 be defined in (3.47) and (3.48), respectively. Define $g_i(x)=g_0(X_i(x))$ as before. Then for all $p\in[2,\infty)$,*

$$\begin{aligned}&\left\|\left(e_\theta\cdot\nabla(\Gamma*g_1)\right)_{\tilde{\gamma}_1(\theta)}-\left(e_\theta\cdot\nabla(\Gamma*g_2)\right)_{\tilde{\gamma}_2(\theta)}\right\|_{\dot{W}^{1,p}(\mathbb{T})}\\ &\qquad\leqslant\frac{Cr^2}{R}\|g_0\|_{L^\infty(B_{(1+4\delta)r})}\left(\Delta m_0+(m_{0,1}+m_{0,2})\Delta M_0\right)\\ &\qquad\qquad+\frac{Cr^2}{R}(\Delta m_0+\Delta M_0)\|e_\theta\cdot\nabla g_0\|_{L^2(B_{(1+4\delta)r})}, \end{aligned}\tag{4.80}$$

where $C=C(p)$.

Proof. Following (4.57) and (4.72),

$$\begin{aligned}
&\frac{d}{d\theta}\big(e_\theta \cdot \nabla(\Gamma * g_i)\big)_{\tilde{\gamma}_i(\theta)} \\
&\quad = \frac{r}{4\pi}\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw \left[\frac{\partial K}{\partial s}(\tilde{B}_i,\xi)\frac{\partial \tilde{B}_i}{\partial \theta} - \frac{\partial K}{\partial s}(B_i,\xi)\frac{\partial B_i}{\partial \theta}\right]\left[\frac{\partial |y_i|}{\partial \rho} g_0\right]_{(rw,\xi+\theta)} \\
&\qquad + \frac{r}{4\pi}\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw \,\frac{\partial K}{\partial s}(B_i,\xi)\frac{\partial B_i}{\partial \theta}\left(\left[\frac{\partial |y_i|}{\partial \rho} g_0\right]_{(rw,\xi+\theta)} - \left[\frac{\partial |y_i|}{\partial \rho} g_0\right]_{(rw,\theta)}\right) \\
&\qquad - \frac{r}{4\pi}\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw \left[\frac{\partial K}{\partial s}(\tilde{B}_i,\xi)\frac{\partial \tilde{B}_i}{\partial \xi} + \frac{\partial K}{\partial \xi}(\tilde{B}_i,\xi)\right] \\
&\qquad\qquad \left(\left[\frac{\partial |y_i|}{\partial \rho} g_0\right]_{(rw,\xi+\theta)} - \left[\frac{\partial |y_i|}{\partial \rho} g_0\right]_{(rw,\theta)}\right) \\
&\quad =: \tilde{J}^{(i)}_{\theta,1} + \tilde{J}^{(i)}_{\theta,2} + \tilde{J}^{(i)}_{\theta,3}.
\end{aligned} \tag{4.81}$$

Then we derive as in (4.59) and (4.60) to find that

$$\begin{aligned}
&|\tilde{J}^{(1)}_{\theta,1} - \tilde{J}^{(2)}_{\theta,1}| \\
&\quad \leqslant \frac{Cr^2}{R}\|g_0\|_{L^\infty}(\Delta m_0 + \Delta M_0)\big(\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty}\big) + \frac{Cr^2}{R}\|g_0\|_{L^\infty}\|h_1' - h_2'\|_{L^\infty}.
\end{aligned} \tag{4.82}$$

Here we used the fact that $|1 - \frac{rw}{R}| \geqslant C\delta$ for all $w \in [0, 1+4\delta]$. Moreover, as in (4.61) and (4.62),

$$\begin{aligned}
\left|(\tilde{J}^{(1)}_{\theta,2} + \tilde{J}^{(1)}_{\theta,3}) - (\tilde{J}^{(2)}_{\theta,2} + \tilde{J}^{(2)}_{\theta,3})\right| &\leqslant \frac{Cr^2}{R}\|g_0\|_{L^\infty}(\Delta m_0 + \delta^{\beta-1}\|h_2\|_{\dot{C}^\beta}\Delta M_0) \\
&\quad + \frac{Cr^2}{R}(\Delta m_0 + \Delta M_0)\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} dw \,\frac{|g_0(rw,\xi+\theta) - g_0(rw,\theta)|}{(|1-\frac{rw}{R}| + |\xi|)^2}.
\end{aligned} \tag{4.83}$$

We proceed as in (4.63) to obtain that

$$\begin{aligned}
\left\|(\tilde{J}^{(1)}_{\theta,2} + \tilde{J}^{(1)}_{\theta,3}) - (\tilde{J}^{(2)}_{\theta,2} + \tilde{J}^{(2)}_{\theta,3})\right\|_{L^p(\mathbb{T})} &\leqslant \frac{Cr^2}{R}\|g_0\|_{L^\infty}\big(\Delta m_0 + \delta^{\beta-1}\|h_2\|_{\dot{C}^\beta}\Delta M_0\big) \\
&\quad + \frac{Cr^2}{R}(\Delta m_0 + \Delta M_0)\|e_\theta \cdot \nabla g_0\|_{L^2(B_{(1+4\delta)r})}.
\end{aligned} \tag{4.84}$$

Combining this with (4.81) and (4.82), we prove (4.80). □

Lemma 4.11 *Under the assumptions of Lemma 4.10,*

$$\begin{aligned}
&\big\|\big(e_r \cdot \nabla(\Gamma * g_1)\big)_{\tilde{\gamma}_1(\theta)} - \big(e_r \cdot \nabla(\Gamma * g_2)\big)_{\tilde{\gamma}_2(\theta)}\big\|_{\dot{W}^{1,p}(\mathbb{T})} \\
&\qquad \leqslant \frac{Cr^2}{R}(\Delta m_0 + \Delta M_0)\big(\|g_0\|_{L^\infty(B_{(1+4\delta)r})} + \|e_\theta \cdot \nabla g_0\|_{L^2(B_{(1+4\delta)r})}\big),
\end{aligned} \tag{4.85}$$

where $C = C(p)$.

Proof. Following the proofs of Lemma 4.6 and Lemma 4.10, we know that it remains to bound $\tilde{J}_{r,4}^{(1)} - \tilde{J}_{r,4}^{(2)}$, where

$$\tilde{J}_{r,4}^{(i)} := \frac{r}{4\pi}\int_{\mathbb{T}} d\xi \int_0^{1+4\delta} \frac{\partial J}{\partial s}(B_i,\xi)\frac{\partial B_i}{\partial \theta}\left[\frac{\partial |y_i|}{\partial \rho} g_0\right]_{(rw,\theta)} dw. \tag{4.86}$$

Since for all $w \in [0, 1+4\delta]$ and $\xi \in \mathbb{T}$, $B_i \lesssim 1$. By Lemma 4.2, (4.46) and (4.66),

$$\begin{aligned}\left|\tilde{J}_{r,4}^{(1)} - \tilde{J}_{r,4}^{(2)}\right| &\leqslant Cr\|g_0\|_{L^\infty}\int_0^{1+4\delta}\left|\frac{\partial B_1}{\partial\theta} - \frac{\partial B_2}{\partial\theta}\right|\left|\frac{\partial|y_1|}{\partial\rho}\right| + \left|\frac{\partial B_2}{\partial\theta}\right|\left|\frac{\partial|y_1|}{\partial\rho} - \frac{\partial|y_2|}{\partial\rho}\right| dw\\ &\leqslant \frac{Cr^2}{R}\|g_0\|_{L^\infty}(\Delta m_0 + \Delta M_0).\end{aligned} \tag{4.87}$$

Then by Lemma 4.10, (4.85) follows. □

Lemma 4.12 *Assume $h, H \in W^{1,\infty}(\mathbb{T})$, such that $m_0 + M_0 \ll 1$. Define $g(x) = g_0(X(x))$. Then for all $p \in [2,\infty)$,*

$$\begin{aligned}&\left\|\big(e_\theta\cdot\nabla(\Gamma * g)\big)_{\tilde{\gamma}(\theta)}\right\|_{\dot{W}^{1,p}(\mathbb{T})} + \left\|\big(e_r\cdot\nabla(\Gamma * g)\big)_{\tilde{\gamma}(\theta)}\right\|_{\dot{W}^{1,p}(\mathbb{T})}\\ &\qquad\leqslant \frac{Cr^2}{R}\Big((m_0+M_0)\|g_0\|_{L^\infty(B_{(1+4\delta)r})} + \|e_\theta\cdot\nabla g_0\|_{L^2(B_{(1+4\delta)r})}\Big),\end{aligned} \tag{4.88}$$

where $C = C(p)$.

Proof. We first study the case with $h = H = 0$. By (4.81),

$$\frac{d}{d\theta}\big(e_\theta\cdot\nabla(\Gamma * g_0)\big)_{\partial B_R} = -\frac{r}{4\pi}\int_{\mathbb{T}} d\xi\int_0^{1+4\delta} dw\,\frac{\partial K}{\partial\xi}\left(\frac{rw}{R},\xi\right)\big(g_0(rw,\xi+\theta) - g_0(rw,\theta)\big). \tag{4.89}$$

Hence, arguing as in (4.63),

$$\begin{aligned}\left\|\big(e_\theta\cdot\nabla(\Gamma * g_0)\big)_{\partial B_R}\right\|_{\dot{W}^{1,p}} &\leqslant Cr\int_{\mathbb{T}} d\xi\int_0^{1+4\delta} dw\,\frac{r}{R}\cdot\frac{\|g_0(rw,\xi+\cdot) - g_0(rw,\cdot)\|_{L^p_\theta(\mathbb{T})}}{(|1-\frac{rw}{R}| + |\xi|)^2}\\ &\leqslant \frac{Cr^2}{R}\|e_\theta\cdot\nabla g_0\|_{L^2(B_{(1+4\delta)r})}.\end{aligned} \tag{4.90}$$

The rest of the proof is the same as that of Lemma 4.7. □

5. Estimates for singular integral operators $\mathcal{K}_\gamma$ and $\mathcal{K}_{\tilde{\gamma}}$

In this section, we shall derive estimates for singular integrals of type $\gamma'(\theta)^\perp\cdot\mathcal{K}_\gamma\psi$ and $\gamma'(\theta)\cdot\mathcal{K}_\gamma\psi$ (see the definition in (2.14).) Singular integrals involving $\mathcal{K}_{\tilde{\gamma}}$ then follow similar estimates.

For convenience, for $\xi \in \mathbb{T}\backslash\{0\}$, denote

$$\Delta f(\theta) := \frac{f(\theta+\xi) - f(\theta)}{2\sin\frac{\xi}{2}}, \tag{5.1}$$

and

$$l(\theta,\theta+\xi) := \frac{(\Delta f)^2}{f(\theta)f(\theta+\xi)} = \frac{(\Delta h)^2}{\big(1+h(\theta)\big)\big(1+h(\theta+\xi)\big)}. \tag{5.2}$$

We first derive a Hölder estimate for $\gamma'^{\perp}\cdot\mathcal{K}_\gamma\psi$ for future use.

Lemma 5.1 *Fix $\beta\in(0,1)$. Assume $h\in C^{1,\beta}(\mathbb{T})$, such that $m_0\ll 1$. Then*

$$\|\gamma'(\theta)^{\perp}\cdot\mathcal{K}_\gamma\psi\|_{\dot C^\beta} \leqslant C\|h'\|_{\dot C^\beta}(\|\psi\|_{C^\beta}+\|\psi\|_{L^\infty}\|h'\|_{\dot C^\beta}\|h'\|_{L^\infty}), \tag{5.3}$$

where $C=C(\beta)$.

Proof. Using $\gamma(\theta)=f(\theta)(\cos\theta,\sin\theta)$,

$$2\pi\gamma'(\theta)^{\perp}\cdot\mathcal{K}_\gamma\psi = \text{p.v.}\int_{\mathbb{T}}\frac{-f(\theta)^2+f(\theta)f(\theta+\xi)\cos\xi-f'(\theta)f(\theta+\xi)\sin\xi}{f(\theta)^2+f(\theta+\xi)^2-2f(\theta)f(\theta+\xi)\cos\xi}\psi(\theta+\xi)\,d\xi. \tag{5.4}$$

With $f(\theta)=r(1+h(\theta))$, it can be rewritten as

$$\begin{aligned}
&2\pi\gamma'(\theta)^{\perp}\cdot\mathcal{K}_\gamma\psi\\
&= -\frac{1}{2}\int_{\mathbb{T}}\psi\,d\xi-\frac{1}{2}\int_{\mathbb{T}}\frac{\big(f(\theta+\xi)-f(\theta)\big)^2}{\big(f(\theta)-f(\theta+\xi)\big)^2+f(\theta)f(\theta+\xi)\cdot 4\sin^2\frac{\xi}{2}}\psi(\theta+\xi)\,d\xi\\
&\quad+\text{p.v.}\int_{\mathbb{T}}\frac{\big(f(\theta+\xi)-f(\theta)\big)f(\theta+\xi)-f'(\theta)f(\theta+\xi)\sin\xi}{\big(f(\theta)-f(\theta+\xi)\big)^2+f(\theta)f(\theta+\xi)\cdot 4\sin^2\frac{\xi}{2}}\psi(\theta+\xi)\,d\xi\\
&= -\frac{1}{2}\int_{\mathbb{T}}\psi\,d\xi-\frac{1}{2}\int_{\mathbb{T}}\frac{l(\theta,\theta+\xi)}{1+l(\theta,\theta+\xi)}\psi(\theta+\xi)\,d\xi\\
&\quad+\frac{1}{1+h(\theta)}\text{p.v.}\int_{\mathbb{T}}\frac{\Delta h}{2\sin\frac{\xi}{2}}\cdot\frac{\psi(\theta+\xi)}{1+l(\theta,\theta+\xi)}\,d\xi\\
&\quad+\frac{1}{1+h(\theta)}\text{p.v.}\int_{\mathbb{T}}-\frac{h'(\theta)}{2\tan\frac{\xi}{2}}\cdot\frac{\psi(\theta+\xi)}{1+l(\theta,\theta+\xi)}\,d\xi\\
&=: L_0+L_1(\theta)+L_2(\theta)+L_3(\theta).
\end{aligned} \tag{5.5}$$

Since $\|fg\|_{\dot C^\beta}\leqslant\|f\|_{\dot C^\beta}\|g\|_{L^\infty}+\|f\|_{L^\infty}\|g\|_{\dot C^\beta}$,

$$\begin{aligned}
\|L_1\|_{\dot C^\beta} &\leqslant C\sup_{\xi\in\mathbb{T}}\left\|\frac{l}{1+l}\psi(\theta+\xi)\right\|_{\dot C^\beta_\theta}\\
&\leqslant C\sup_{\xi\in\mathbb{T}}\left\|\frac{l}{1+l}\right\|_{\dot C^\beta_\theta}\|\psi\|_{L^\infty}+C\sup_{\xi\in\mathbb{T}}\left\|\frac{l}{1+l}\right\|_{L^\infty_\theta}\|\psi\|_{\dot C^\beta}.
\end{aligned} \tag{5.6}$$

By the Lipschitz continuity of $\frac{x}{1+x}$ on $[0,+\infty)$ and the smallness of h,

$$\begin{aligned}
\|L_1\|_{\dot C^\beta} &\leqslant C\sup_{\xi\in\mathbb{T}}\left\|\frac{(\Delta h)^2}{\big(1+h(\theta)\big)\big(1+h(\theta+\xi)\big)}\right\|_{\dot C^\beta_\theta}\|\psi\|_{L^\infty}+C\|h'\|^2_{L^\infty}\|\psi\|_{\dot C^\beta}\\
&\leqslant C\big(\|h'\|_{\dot C^\beta}\|h'\|_{L^\infty}\|\psi\|_{L^\infty}+\|h'\|^2_{L^\infty}\|\psi\|_{\dot C^\beta}\big).
\end{aligned} \tag{5.7}$$

Here we used

$$\|\Delta h\|_{\dot{C}^\beta_\theta} = \frac{\|h(\theta+\xi)-h(\theta)\|_{\dot{C}^\beta_\theta}}{\left|2\sin\frac{\xi}{2}\right|} \leqslant \left|\frac{1}{2\sin\frac{\xi}{2}}\int_0^\xi \|h'(\theta+\eta)\|_{\dot{C}^\beta_\theta}\,d\eta\right| \leqslant C\|h'\|_{\dot{C}^\beta}. \tag{5.8}$$

Take $\varepsilon \in \mathbb{T}$ and $\varepsilon \geqslant 0$ without loss of generality. Write

$$\begin{aligned}
&(L_2+L_3)(\theta+\varepsilon)-(L_2+L_3)(\theta)\\
&= \left(\frac{1}{1+h(\theta+\varepsilon)} - \frac{1}{1+h(\theta)}\right)\int_{\mathbb{T}} \frac{\Delta h(\theta+\varepsilon)-\cos\frac{\xi}{2}h'(\theta+\varepsilon)}{2\sin\frac{\xi}{2}} \cdot \frac{\psi(\theta+\varepsilon+\xi)}{1+l(\theta+\varepsilon,\theta+\varepsilon+\xi)}\,d\xi\\
&\quad + \frac{1}{1+h(\theta)}\int_{\mathbb{T}} \frac{\Delta h(\theta+\varepsilon)-\cos\frac{\xi}{2}\cdot h'(\theta+\varepsilon)}{2\sin\frac{\xi}{2}}\\
&\qquad \cdot\left(\frac{\psi(\theta+\varepsilon+\xi)}{1+l(\theta+\varepsilon,\theta+\varepsilon+\xi)} - \frac{\psi(\theta+\xi)}{1+l(\theta,\theta+\xi)}\right)d\xi\\
&\quad + \frac{1}{1+h(\theta)}\int_{\mathbb{T}} \frac{\Delta h(\theta+\varepsilon)-\Delta h(\theta)-\cos\frac{\xi}{2}(h'(\theta+\varepsilon)-h'(\theta))}{2\sin\frac{\xi}{2}} \frac{\psi(\theta)}{1+\frac{h'(\theta)^2}{(1+h(\theta))^2}}\,d\xi\\
&\quad + \frac{1}{1+h(\theta)}\int_{\mathbb{T}} \frac{\Delta h(\theta+\varepsilon)-\Delta h(\theta)-\cos\frac{\xi}{2}(h'(\theta+\varepsilon)-h'(\theta))}{2\sin\frac{\xi}{2}}\\
&\qquad \cdot\left(\frac{\psi(\theta+\xi)}{1+l(\theta,\theta+\xi)} - \frac{\psi(\theta)}{1+\frac{h'(\theta)^2}{(1+h(\theta))^2}}\right)d\xi.
\end{aligned} \tag{5.9}$$

We derive that

$$\begin{aligned}
\left|\Delta h(\theta+\varepsilon)-\cos\frac{\xi}{2}\cdot h'(\theta+\varepsilon)\right| &\leqslant \left|\frac{\int_0^\xi h'(\theta+\varepsilon+\eta)-h'(\theta+\varepsilon)\,d\eta}{2\sin\frac{\xi}{2}}\right| + \left|\frac{\xi-\sin\xi}{2\sin\frac{\xi}{2}}h'(\theta+\varepsilon)\right|\\
&\leqslant C|\xi|^\beta\|h'\|_{\dot{C}^\beta},
\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}
&\left|\Delta h(\theta+\varepsilon)-\Delta h(\theta)-\cos\frac{\xi}{2}\big(h'(\theta+\varepsilon)-h'(\theta)\big)\right|\\
&\qquad \leqslant \left|\frac{1}{2\sin\frac{\xi}{2}}\int_0^\xi h'(\theta+\varepsilon+\eta)-h'(\theta+\eta)\,d\eta\right| + |h'(\theta+\varepsilon)-h'(\theta)|\\
&\qquad \leqslant C\varepsilon^\beta\|h'\|_{\dot{C}^\beta}.
\end{aligned} \tag{5.11}$$

Thanks to (5.7) and (5.8),

$$
\begin{aligned}
&\left|\frac{\psi(\theta+\varepsilon+\xi)}{1+l(\theta+\varepsilon,\theta+\varepsilon+\xi)}-\frac{\psi(\theta+\xi)}{1+l(\theta,\theta+\xi)}\right| \\
&\leqslant C\varepsilon^\beta\|\psi\|_{\dot C^\beta}+C\|\psi\|_{L^\infty}\left|\frac{\big(\Delta h(\theta+\varepsilon)\big)^2}{\big(1+h(\theta+\varepsilon)\big)\big(1+h(\theta+\varepsilon+\xi)\big)}-\frac{(\Delta h)^2}{\big(1+h(\theta)\big)\big(1+h(\theta+\xi)\big)}\right| \\
&\leqslant C\varepsilon^\beta\big(\|\psi\|_{\dot C^\beta}+\|\psi\|_{L^\infty}\|h'\|_{\dot C^\beta}\|h'\|_{L^\infty}\big),
\end{aligned}
\tag{5.12}
$$

and similarly,

$$
\left|\frac{\psi(\theta+\xi)}{1+l(\theta,\theta+\xi)}-\frac{\psi(\theta)}{1+\frac{h'(\theta)^2}{(1+h(\theta))^2}}\right|\leqslant C|\xi|^\beta\big(\|\psi\|_{\dot C^\beta}+\|\psi\|_{L^\infty}\|h'\|_{\dot C^\beta}\|h'\|_{L^\infty}\big). \tag{5.13}
$$

Lastly,

$$
\begin{aligned}
&\left|\int_{\mathbb T}\frac{\Delta h(\theta+\varepsilon)-\Delta h(\theta)-\cos\frac{\xi}{2}\big(h'(\theta+\varepsilon)-h'(\theta)\big)}{2\sin\frac{\xi}{2}}\,d\xi\right| \\
&\qquad=\left|\mathrm{p.v.}\int_{\mathbb T}\frac{h(\theta+\varepsilon+\xi)-h(\theta+\varepsilon)-h(\theta+\xi)+h(\theta)}{4\sin^2\frac{\xi}{2}}\,d\xi\right| \\
&\qquad=C|\mathcal H h'(\theta+\varepsilon)-\mathcal H h'(\theta)| \\
&\qquad\leqslant C\varepsilon^\beta\|h'\|_{\dot C^\beta}.
\end{aligned}
\tag{5.14}
$$

Note that Hilbert transform is bounded in $C^\beta(\mathbb T)$.

Combining these estimates with (5.9), we obtain that

$$
|(L_2+L_3)(\theta+\varepsilon)-(L_2+L_3)(\theta)|\leqslant C\varepsilon^\beta\|h'\|_{\dot C^\beta}\big(\|\psi\|_{C^\beta}+\|\psi\|_{L^\infty}\|h'\|_{\dot C^\beta}\|h'\|_{L^\infty}\big). \tag{5.15}
$$

Then (5.3) follows from (5.5), (5.7) and (5.15). □

Now we turn to a $\dot W^{1,p}$-estimate of $\gamma'^\perp\cdot\mathcal K_\gamma\psi$.

Lemma 5.2 *Fix $p\in[2,\infty)$. Assume $h\in C^{1,\beta}(\mathbb T)$ for some $\beta\in(0,1)$, such that $m_0\ll 1$ with the needed smallness depending on p. Then*

$$
\begin{aligned}
\|\gamma'(\theta)^\perp\cdot\mathcal K_\gamma\psi\|_{\dot W^{1,p}}\leqslant{}& C\|h''\|_{L^p}\|\psi\|_{L^\infty}(1+\|h'\|_{\dot C^\beta}) \\
&+C\big(\|h''\|_{L^p}\|\psi\|_{\dot C^\beta}+\|h'\|_{L^\infty}\|\psi'\|_{L^p}\big),
\end{aligned}
\tag{5.16}
$$

where $C=C(p,\beta)$.

Proof. Let C_* and $C_\dagger$ be the constants introduced in Lemma A.2 and Lemma A.4, respectively, both of which only depend on p. Without loss of generality, we may assume $C_\dagger\geqslant C_*\geqslant 1$. We also recall that l is defined in (5.2).

Using the notation in (5.5), we take θ-derivative of L_1 to derive that

$$\|L_1\|_{\dot W^{1,p}} \leqslant C\left\|\int_{\mathbb{T}}\|h'\|_{L^\infty}^2|\psi'(\theta+\xi)|\,d\xi\right\|_{L^p} + C\left\|\int_{\mathbb{T}}(\|h'\|_{L^\infty}|\Delta h'| + \|h'\|_{L^\infty}^3)\|\psi\|_{L^\infty}\,d\xi\right\|_{L^p}$$
$$\leqslant C\|h'\|_{L^\infty}^2\|\psi'\|_{L^p} + C\|h'\|_{L^\infty}\|h''\|_{L^p}\|\psi\|_{L^\infty}. \tag{5.17}$$

Thanks to the smallness of h, we may assume $|l| < 1$. Hence, by Taylor expanding $(1+l)^{-1}$, we may rewrite L_2 in (5.5) as

$$L_2 = \sum_{j=0}^{\infty}(-1)^j\big(1+h(\theta)\big)^{-(j+1)}\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j+1}\big(1+h(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi =: \sum_{j=0}^{\infty}L_{2,j}. \tag{5.18}$$

By virtue of Lemma A.2,

$$\left\|\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j+1}\big(1+h(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\right\|_{L^p} \leqslant C_*^{2j+3}\|h'\|_{L^\infty}^{2j+1}\|(1+h)^{-j}\psi\|_{L^p}$$
$$\leqslant C\big(C_*^2C_2\|h'\|_{L^\infty}^2\big)^j\|h'\|_{L^\infty}\|\psi\|_{L^p}. \tag{5.19}$$

Here C_2 is a universal constant such that $\|(1+h)^{-1}\|_{L^\infty} \leqslant C_2$. Similarly, by Lemma A.4,

$$\left\|\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j+1}\big(1+h(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\right\|_{\dot W^{1,p}}$$
$$\leqslant (2j+2)C_\dagger^{2j+2}\|h'\|_{L^\infty}^{2j}\Big(\big\|\big((1+h)^{-j}\psi\big)'\big\|_{L^p}\|h'\|_{L^\infty} + \|(1+h)^{-j}\psi\|_{L^\infty}\|h''\|_{L^p}\Big)$$
$$\leqslant C(j+1)(C_\dagger^2C_2\|h'\|_{L^\infty}^2)^j\big(j\|h'\|_{L^\infty}^2\|\psi\|_{L^p} + \|h'\|_{L^\infty}\|\psi'\|_{L^p} + \|\psi\|_{L^\infty}\|h''\|_{L^p}\big). \tag{5.20}$$

Hence, with the assumption $C_\dagger \geqslant C_*$,

$$\|L_{2,j}\|_{\dot W^{1,p}}$$
$$\leqslant \|(1+h)^{-(j+1)}\|_{\dot W^{1,\infty}}\left\|\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j+1}\big(1+h(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\right\|_{L^p}$$
$$+ \|(1+h)^{-(j+1)}\|_{L^\infty}\left\|\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j+1}\big(1+h(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\right\|_{\dot W^{1,p}}$$
$$\leqslant C(j+1)(C_\dagger C_2\|h'\|_{L^\infty})^{2j}\big((j+1)\|h'\|_{L^\infty}^2\|\psi\|_{L^p} + \|h'\|_{L^\infty}\|\psi'\|_{L^p} + \|\psi\|_{L^\infty}\|h''\|_{L^p}\big). \tag{5.21}$$

To this end, by assuming $\|h'\|_{L^\infty} \ll 1$, where the smallness depends on p, we derive from (5.18) that

$$\|L_2\|_{\dot W^{1,p}} \leqslant C\big(\|h'\|_{L^\infty}\|\psi'\|_{L^p} + \|\psi\|_{L^\infty}\|h''\|_{L^p}\big). \tag{5.22}$$

Similarly, we write

$$L_3 = \sum_{j=0}^{\infty}h'(\theta)\big(-1-h(\theta)\big)^{-(j+1)}\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j}\big(1+h(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi =: \sum_{j=0}^{\infty}L_{3,j}. \tag{5.23}$$

In order to bound $\dot{W}^{1,p}$-semi-norm of $L_{3,j}$, we need an L^∞-bound of the integral above. This is possible thanks to the Hölder regularity of h' and ψ. Indeed, by the mean value theorem,

$$\begin{aligned}
&\left|\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j}\big(1+h(\theta+\xi)\big)^{-j}\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi\right| \\
&\quad=\left|\int_{\mathbb{T}}[(\Delta h)^{2j}\big(1+h(\theta+\xi)\big)^{-j}\psi(\theta+\xi)-h'(\theta)^{2j}\big(1+h(\theta)\big)^{-j}\psi(\theta)]\frac{1}{2\tan\frac{\xi}{2}}\,d\xi\right| \\
&\quad\leqslant C\int_{\mathbb{T}}2j(C_1\|h'\|_{L^\infty})^{2j-1}|\Delta h-h'(\theta)|\cdot C_2^j\|\psi\|_{L^\infty}|\xi|^{-1}\,d\xi \\
&\qquad+C\int_{\mathbb{T}}\|h'\|_{L^\infty}^{2j}\cdot jC_2^{j+1}|h(\theta+\xi)-h(\theta)|\cdot\|\psi\|_{L^\infty}|\xi|^{-1}\,d\xi \\
&\qquad+C\int_{\mathbb{T}}\|h'\|_{L^\infty}^{2j}\cdot C_2^j|\psi(\theta+\xi)-\psi(\theta)||\xi|^{-1}\,d\xi \\
&\quad\leqslant C\big(2jC_1^{2j}C_2^j\|h'\|_{L^\infty}^{2j-1}\|h'\|_{\dot{C}^\beta}\|\psi\|_{L^\infty}+C_2^j\|h'\|_{L^\infty}^{2j}\|\psi\|_{\dot{C}^\beta}\big).
\end{aligned}\tag{5.24}$$

Here $C_1=\frac{\pi}{2}$ introduced in the proof of Lemma A.2; note that $|\Delta h|\leqslant C_1\|h'\|_{L^\infty}$. Arguing as in (5.19)–(5.21),

$$\left\|\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j}\big(1+h(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi\right\|_{L^p}\leqslant C\big(C_*^2C_2\|h'\|_{L^\infty}^2\big)^j\|\psi\|_{L^p},\tag{5.25}$$

$$\begin{aligned}
&\left\|\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j}\big(1+h(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi\right\|_{\dot{W}^{1,p}} \\
&\leqslant C(2j+1)\big(C_\dagger^2C_2\|h'\|_{L^\infty}^2\big)^j\big(j\|h'\|_{L^\infty}\|\psi\|_{L^p}+\|\psi'\|_{L^p}+\mathbb{1}_{\{j>0\}}\|h'\|_{L^\infty}^{-1}\|h''\|_{L^p}\|\psi\|_{L^\infty}\big),
\end{aligned}\tag{5.26}$$

and hence,

$$\begin{aligned}
&\|L_{3,j}\|_{\dot{W}^{1,p}} \\
&\quad\leqslant\|h''\|_{L^p}\|\big(1+h(\theta)\big)^{-(j+1)}\|_{L^\infty}\left\|\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j}\big(1+h(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi\right\|_{L^\infty} \\
&\qquad+\|h'\|_{L^\infty}\|\big(1+h(\theta)\big)^{-(j+1)}\|_{\dot{W}^{1,\infty}}\left\|\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j}\big(1+h(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi\right\|_{L^p} \\
&\qquad+\|h'\|_{L^\infty}\|\big(1+h(\theta)\big)^{-(j+1)}\|_{L^\infty}\left\|\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j}\big(1+h(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi\right\|_{\dot{W}^{1,p}} \\
&\quad\leqslant C\cdot(C_2\|h'\|_{L^\infty})^{2j-1}\cdot(jC_1^{2j}\|h'\|_{\dot{C}^\beta}\|\psi\|_{L^\infty}+\|h'\|_{L^\infty}\|\psi\|_{\dot{C}^\beta})\|h''\|_{L^p} \\
&\qquad+C\cdot(j+1)(C_\dagger C_2\|h'\|_{L^\infty})^{2j} \\
&\qquad\quad\cdot\big((j+1)\|h'\|_{L^\infty}^2\|\psi\|_{L^p}+\|h'\|_{L^\infty}\|\psi'\|_{L^p}+\mathbb{1}_{\{j>0\}}\|h''\|_{L^p}\|\psi\|_{L^\infty}\big).
\end{aligned}\tag{5.27}$$

By (5.23), provided that $\|h'\|_{L^\infty} \ll 1$,

$$\|L_3\|_{\dot{W}^{1,p}} \leqslant C\big(\|h''\|_{L^p}\|h'\|_{\dot{C}^\beta}\|\psi\|_{L^\infty} + \|h''\|_{L^p}\|\psi\|_{\dot{C}^\beta} + \|h'\|_{L^\infty}\|\psi'\|_{L^p}\big). \tag{5.28}$$

Combining (5.17), (5.22) and (5.28), we prove the desired estimate. □

We also prove a $\dot{W}^{1,p}$-estimate for $\gamma' \cdot \mathcal{K}_\gamma\psi - \frac{1}{2}\mathcal{H}\psi$.

Lemma 5.3 *Under the assumptions of Lemma 5.2,*

$$\begin{aligned}\left\|\gamma'(\theta)\cdot\mathcal{K}_\gamma\psi - \frac{1}{2}\mathcal{H}\psi\right\|_{\dot{W}^{1,p}} &\leqslant C\|h''\|_{L^p}\|\psi\|_{L^\infty}(1+\|h'\|_{\dot{C}^\beta})\\ &\quad + C\big(\|h'\|_{L^\infty}\|h''\|_{L^p}\|\psi\|_{\dot{C}^\beta} + \|h'\|_{L^\infty}^2\|\psi'\|_{L^p}\big),\end{aligned} \tag{5.29}$$

where $C = C(p,\beta)$.

Proof. Using $\gamma(\theta) = f(\theta)(\cos\theta, \sin\theta)$, by definition,

$$2\pi\gamma'(\theta)\cdot\mathcal{K}_\gamma\psi = \text{p.v.}\int_{\mathbb{T}} \frac{f'(\theta)f(\theta) - f'(\theta)f(\theta+\xi)\cos\xi - f(\theta)f(\theta+\xi)\sin\xi}{f(\theta)^2 + f(\theta+\xi)^2 - 2f(\theta)f(\theta+\xi)\cos\xi}\psi(\theta+\xi)\,d\xi. \tag{5.30}$$

With $f(\theta) = r(1+h(\theta))$ and $l(\theta,\theta+\xi)$ defined in (5.2), it can be rewritten as

$$\begin{aligned}2\pi\gamma'(\theta)\cdot\mathcal{K}_\gamma\psi &= f'(\theta)\int_{\mathbb{T}} \frac{f(\theta+\xi)\cdot 2\sin^2\frac{\xi}{2}}{\big(f(\theta+\xi)-f(\theta)\big)^2 + f(\theta)f(\theta+\xi)\cdot 4\sin^2\frac{\xi}{2}}\psi(\theta+\xi)\,d\xi\\ &\quad - f'(\theta)\text{p.v.}\int_{\mathbb{T}} \frac{f(\theta+\xi)-f(\theta)}{\big(f(\theta+\xi)-f(\theta)\big)^2 + f(\theta)f(\theta+\xi)\cdot 4\sin^2\frac{\xi}{2}}\psi(\theta+\xi)\,d\xi\\ &\quad - \text{p.v.}\int_{\mathbb{T}} \frac{f(\theta)f(\theta+\xi)\sin\xi}{\big(f(\theta+\xi)-f(\theta)\big)^2 + f(\theta)f(\theta+\xi)\cdot 4\sin^2\frac{\xi}{2}}\psi(\theta+\xi)\,d\xi\\ &= \frac{h'(\theta)}{2(1+h(\theta))}\left(\int_{\mathbb{T}}\psi\,d\xi - \int_{\mathbb{T}}\frac{l(\theta,\theta+\xi)}{1+l(\theta,\theta+\xi)}\psi(\theta+\xi)\,d\xi\right)\\ &\quad - \frac{h'(\theta)}{1+h(\theta)}\text{p.v.}\int_{\mathbb{T}}\frac{\frac{\Delta h}{2\sin\frac{\xi}{2}}}{1+l(\theta,\theta+\xi)}\frac{\psi(\theta+\xi)}{1+h(\theta+\xi)}\,d\xi\\ &\quad + \text{p.v.}\int_{\mathbb{T}}\frac{l(\theta,\theta+\xi)}{1+l(\theta,\theta+\xi)}\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi + \pi\mathcal{H}\psi\\ &=: \tilde{L}_1(\theta) + \tilde{L}_2(\theta) + \tilde{L}_3(\theta) + \pi\mathcal{H}\psi.\end{aligned} \tag{5.31}$$

Since

$$\tilde{L}_1 = \frac{h'(\theta)}{1+h(\theta)}\left(\frac{1}{2}\int_{\mathbb{T}}\psi\,d\xi + L_1\right), \tag{5.32}$$

we derive by (5.17) that

$$\begin{aligned}\|\tilde{L}_1\|_{\dot{W}^{1,p}} &\leqslant C\left\|\frac{h'}{1+h}\right\|_{\dot{W}^{1,p}}\|\psi\|_{L^\infty} + C\|h'\|_{L^\infty}\|L_1\|_{\dot{W}^{1,p}}\\ &\leqslant C\big(\|h''\|_{L^p}\|\psi\|_{L^\infty} + \|h'\|_{L^\infty}^3\|\psi'\|_{L^p}\big).\end{aligned} \tag{5.33}$$

For $\tilde{L}_2$,

$$\tilde{L}_2 = \sum_{j=0}^{\infty} h'(\theta)\big(-1-h(\theta)\big)^{-(j+1)}\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j+1}\big(1+h(\theta+\xi)\big)^{-(j+1)}\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}. \tag{5.34}$$

Arguing as in (5.24),

$$\left\|\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j+1}\big(1+h(\theta+\xi)\big)^{-(j+1)}\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\right\|_{L^\infty} \leqslant C\big(C_2\|h'\|_{L^\infty}^2\big)^j\big((2j+1)C_1^{2j}\|h'\|_{\dot{C}^\beta}\|\psi\|_{L^\infty}+\|h'\|_{L^\infty}\|\psi\|_{\dot{C}^\beta}\big). \tag{5.35}$$

Moreover, by Lemma A.2 and Lemma A.4,

$$\left\|\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j+1}\big(1+h(\theta+\xi)\big)^{-(j+1)}\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\right\|_{L^p} \leqslant C\big(C_*^2C_2\|h'\|_{L^\infty}^2\big)^j\|h'\|_{L^\infty}\|\psi\|_{L^\infty}, \tag{5.36}$$

and

$$\left\|\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j+1}\big(1+h(\theta+\xi)\big)^{-(j+1)}\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\right\|_{\dot{W}^{1,p}} \leqslant C(2j+2)\big(C_\dagger^2C_2\|h'\|_{L^\infty}^2\big)^j\big((j+1)\|h'\|_{L^\infty}^2\|\psi\|_{L^\infty}+\|h''\|_{L^p}\|\psi\|_{L^\infty}+\|h'\|_{L^\infty}\|\psi'\|_{L^p}\big). \tag{5.37}$$

Hence,

$$\|\tilde{L}_2\|_{\dot{W}^{1,p}} \leqslant C(\|h''\|_{L^p}\|h'\|_{\dot{C}^\beta}\|\psi\|_{L^\infty}+\|h'\|_{L^\infty}\|h''\|_{L^p}\|\psi\|_{\dot{C}^\beta}+\|h'\|_{L^\infty}^2\|\psi'\|_{L^p}). \tag{5.38}$$

For $\tilde{L}_3$,

$$\tilde{L}_3 = \sum_{j=0}^{\infty}(-1)^j\big(1+h(\theta)\big)^{-(j+1)}\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j+2}\big(1+h(\theta+\xi)\big)^{-(j+1)}\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi. \tag{5.39}$$

Since

$$\left\|\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j+2}\big(1+h(\theta+\xi)\big)^{-(j+1)}\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi\right\|_{L^p} \leqslant C(C_*^2C_2\|h'\|_{L^\infty}^2)^j\|h'\|_{L^\infty}^2\|\psi\|_{L^\infty}, \tag{5.40}$$

and

$$\left\|\text{p.v.}\int_{\mathbb{T}}(\Delta h)^{2j+2}\big(1+h(\theta+\xi)\big)^{-(j+1)}\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi\right\|_{\dot{W}^{1,p}} \leqslant C(2j+3)\big(C_\dagger^2C_2\|h'\|_{L^\infty}^2\big)^j\|h'\|_{L^\infty} \cdot\big((j+1)\|h'\|_{L^\infty}^2\|\psi\|_{L^\infty}+\|h'\|_{L^\infty}\|\psi'\|_{L^p}+\|h''\|_{L^p}\|\psi\|_{L^\infty}\big), \tag{5.41}$$

we find that

$$\|\tilde{L}_3\|_{\dot{W}^{1,p}} \leqslant C\big(\|h'\|_{L^\infty}\|h''\|_{L^p}\|\psi\|_{L^\infty} + \|h'\|_{L^\infty}^2\|\psi'\|_{L^p}\big). \tag{5.42}$$

Combining (5.31), (5.33), (5.38) and (5.42), we obtain (5.29). □

In order to show uniqueness of the solution in Section 9, we need the following three lemmas, which are generalizations of Lemmas 5.1–5.3, respectively.

Lemma 5.4 *Fix $\beta \in (0,1)$. Assume $h_1, h_2 \in C^{1,\beta}(\mathbb{T})$, such that $m_{0,1}, m_{0,2} \ll 1$. Here $m_{0,i}$ are defined for $i = 1,2$ as in (3.17). Then*

$$\|\gamma_1'(\theta)^\perp \cdot \mathcal{K}_{\gamma_1}\psi - \gamma_2'(\theta)^\perp \cdot \mathcal{K}_{\gamma_2}\psi\|_{\dot{C}^\beta} \leqslant C\|h_1 - h_2\|_{C^{1,\beta}}(1 + \|h_1'\|_{\dot{C}^\beta} + \|h_2'\|_{\dot{C}^\beta})^2\|\psi\|_{C^\beta}, \tag{5.43}$$

where $C = C(\beta)$.

Lemma 5.5 *Fix $p \in [2,\infty)$ and $\beta \in (0,1)$. Assume $h_i \in C^{1,\beta} \cap W^{2,p}(\mathbb{T})$ $(i = 1,2)$, such that $m_{0,i} \ll 1$ with the needed smallness depending only on p. Then*

$$\begin{aligned}
&\big\|\gamma_1'(\theta)^\perp \cdot \mathcal{K}_{\gamma_1}\psi - \gamma_2'(\theta)^\perp \cdot \mathcal{K}_{\gamma_2}\psi\big\|_{\dot{W}^{1,p}} \\
&\quad\leqslant C\|h_1'' - h_2''\|_{L^p}\big(1 + \|h_1'\|_{\dot{C}^\beta} + \|h_2'\|_{\dot{C}^\beta}\big)\|\psi\|_{C^\beta} \\
&\qquad + C\big(\|h_1''\|_{L^p} + \|h_2''\|_{L^p}\big)\|h_1 - h_2\|_{C^{1,\beta}}\big(1 + \|h_1'\|_{\dot{C}^\beta} + \|h_2'\|_{\dot{C}^\beta}\big)\|\psi\|_{C^\beta} \\
&\qquad + C\|h_1 - h_2\|_{W^{1,\infty}}\|\psi'\|_{L^p},
\end{aligned} \tag{5.44}$$

where $C = C(p,\beta)$.

Lemma 5.6 *Under the assumptions of Lemma 5.5,*

$$\begin{aligned}
&\big\|\gamma_1'(\theta) \cdot \mathcal{K}_{\gamma_1}\psi - \gamma_2'(\theta) \cdot \mathcal{K}_{\gamma_2}\psi\big\|_{\dot{W}^{1,p}} \\
&\quad\leqslant C\|h_1'' - h_2''\|_{L^p}\left|\int_{\mathbb{T}} \psi\, d\xi\right| \\
&\qquad + C\|h_1'' - h_2''\|_{L^p}\|\psi\|_{C^\beta}\big(\|h_1\|_{C^{1,\beta}} + \|h_2\|_{C^{1,\beta}}\big)\big(1 + \|h_1\|_{C^{1,\beta}} + \|h_2\|_{C^{1,\beta}}\big)^2 \\
&\qquad + C\big(\|h_1''\|_{L^p} + \|h_2''\|_{L^p}\big)\|\psi\|_{C^\beta}\|h_1 - h_2\|_{C^{1,\beta}}\big(1 + \|h_1\|_{C^{1,\beta}} + \|h_2\|_{C^{1,\beta}}\big)^3 \\
&\qquad + C\|h_1 - h_2\|_{W^{1,\infty}}\|\psi'\|_{L^p}(\|h_1\|_{W^{1,\infty}} + \|h_2\|_{W^{1,\infty}}),
\end{aligned} \tag{5.45}$$

where $C = C(p,\beta)$.

These estimates can be justified by following similar arguments as those in Lemmas 5.1–5.3. However, since their proofs turn out to be extremely lengthy and somewhat tedious, we shall leave them to Appendix C.

6. Estimates for integral operators $\mathcal{K}_{\gamma,\tilde{\gamma}}$ and $\mathcal{K}_{\tilde{\gamma},\gamma}$

Recall that the integral operators $\mathcal{K}_{\gamma,\tilde{\gamma}}$ and $\mathcal{K}_{\tilde{\gamma},\gamma}$ are defined in (2.15), while the Poisson kernel P on the 2-D unit disc and its conjugate Q are defined in (4.1) and (4.2). For convenience, we denote

$$P_{\frac{r}{R}} := P\left(\frac{r}{R},\cdot\right) \quad \text{and} \quad Q_{\frac{r}{R}} := Q\left(\frac{r}{R},\cdot\right). \tag{6.1}$$

Lemma 6.1 *Assume* $h, H \in W^{1,\infty}(\mathbb{T})$, *such that* $\delta^{-1}(\|h\|_{L^\infty} + \|H\|_{L^\infty}) \ll 1$. *Denote* $\bar{\psi} = (2\pi)^{-1}\int_{\mathbb{T}} \psi(\theta)\,d\theta$. *Then*

$$\left\| fe_r(\theta)\cdot \mathcal{K}_{\gamma,\tilde{\gamma}}\psi + \frac{1}{4\pi} P_{\frac{r}{R}} * (\psi - \bar{\psi}) \right\|_{L^\infty(\mathbb{T})} + \left\| fe_\theta(\theta)\cdot \mathcal{K}_{\gamma,\tilde{\gamma}}\psi - \frac{1}{4\pi} Q_{\frac{r}{R}} * (\psi - \bar{\psi}) \right\|_{L^\infty(\mathbb{T})} \leqslant \frac{Cr}{R}\delta^{-1}\big(\|h\|_{L^\infty} + \|H\|_{L^\infty}\big)\|\psi\|_{L^\infty}, \quad (6.2)$$

where C *is a universal constant.*

Proof. With $\theta' = \theta + \xi$ and $D(\theta, \theta+\xi) := f(\theta)/F(\theta+\xi)$, we calculate that

$$\begin{aligned}
&2\pi e_r(\theta)\cdot \mathcal{K}_{\gamma,\tilde{\gamma}}\psi \\
&= \int_{\mathbb{T}} \frac{e_r(\theta)\cdot\big(\gamma(\theta) - \tilde{\gamma}(\theta')\big)}{|\gamma(\theta) - \tilde{\gamma}(\theta')|^2}\psi(\theta')\,d\theta' \\
&= f(\theta)^{-1}\int_{\mathbb{T}} \left[\frac{1}{2} - \frac{1}{2}\cdot\frac{1 - D(\theta,\theta+\xi)^2}{1 + D(\theta,\theta+\xi)^2 - 2D(\theta,\theta+\xi)\cos\xi}\right]\psi(\theta+\xi)\,d\xi \\
&=: f(\theta)^{-1}(I_{r,1} + I_{r,2}),
\end{aligned} \quad (6.3)$$

where

$$I_{r,1} = -\frac{1}{2}\int_{\mathbb{T}} P\left(\frac{r}{R},\xi\right)\big(\psi(\theta+\xi) - \bar{\psi}\big)\,d\xi = -\frac{1}{2}P_{\frac{r}{R}} * (\psi - \bar{\psi}), \quad (6.4)$$

$$I_{r,2} = \frac{1}{2}\int_{\mathbb{T}} \left[P\left(\frac{r}{R},\xi\right) - P(D,\xi)\right]\psi(\theta+\xi)\,d\xi. \quad (6.5)$$

Here we used the fact that $P_{\frac{r}{R}}$ is an even function and has integral 2π on $\mathbb{T}$. $I_{r,1}$ is already in the desired shape. For $I_{r,2}$, since

$$\left|\frac{r}{R} - D(\theta,\theta+\xi)\right| = \frac{r}{R}\left|1 - \frac{1 + h(\theta)}{1 + H(\theta+\xi)}\right| \leqslant \frac{Cr}{R}(\|h\|_{L^\infty} + \|H\|_{L^\infty}), \quad (6.6)$$

we may assume that $D \in [0, 1 - C\delta]$ for some universal $C > 0$. Hence, by the mean value theorem and Lemma A.1,

$$\begin{aligned}
\|I_{r,2}\|_{L^\infty} &\leqslant \frac{Cr}{R}\big(\|h\|_{L^\infty} + \|H\|_{L^\infty}\big)\|\psi\|_{L^\infty}\int_{\mathbb{T}}(\delta^2 + \xi^2)^{-1}\,d\xi \\
&\leqslant \frac{Cr}{R}\delta^{-1}\big(\|h\|_{L^\infty} + \|H\|_{L^\infty}\big)\|\psi\|_{L^\infty}.
\end{aligned} \quad (6.7)$$

The estimate for $fe_r\cdot\mathcal{K}_{\gamma,\tilde{\gamma}}\psi$ in (6.2) follows.

Similarly, since $Q_{\frac{r}{R}}$ is an odd kernel,

$$\begin{aligned}
2\pi e_\theta(\theta)\cdot\mathcal{K}_{\gamma,\tilde{\gamma}}\psi &= -f(\theta)^{-1}\int_{\mathbb{T}}\frac{D(\theta,\theta+\xi)\cdot\sin\xi}{1 + D(\theta,\theta+\xi)^2 - 2D(\theta,\theta+\xi)\cos\xi}\psi(\theta+\xi)\,d\xi. \\
&=: f(\theta)^{-1}(I_{\theta,1} + I_{\theta,2}),
\end{aligned} \quad (6.8)$$

where

$$I_{\theta,1} = -\frac{1}{2}\int_{\mathbb{T}} Q\left(\frac{r}{R},\xi\right)\left(\psi(\theta+\xi)-\bar{\psi}\right)d\xi = \frac{1}{2}Q_{\frac{r}{R}} * (\psi-\bar{\psi}), \tag{6.9}$$

$$I_{\theta,2} = \frac{1}{2}\int_{\mathbb{T}}\left[Q\left(\frac{r}{R},\xi\right)-Q(D,\xi)\right]\psi(\theta+\xi)\,d\xi. \tag{6.10}$$

Then the estimate for $fe_\theta \cdot \mathcal{K}_{\gamma,\tilde{\gamma}}\psi$ in (6.2) can be derived as before. □

Lemma 6.2 *Assume $h, H \in C^{1,\alpha}(\mathbb{T})$ for some $\alpha \in (0,1)$, such that $m_0 + M_0 \ll 1$. Then for $\beta \in (0, \frac{\alpha}{1+\alpha})$,*

$$\left\| fe_r(\theta)\cdot\mathcal{K}_{\gamma,\tilde{\gamma}}\psi + \frac{1}{4\pi}P_{\frac{r}{R}} * (\psi-\bar{\psi})\right\|_{\dot{C}^\beta(\mathbb{T})} + \left\| fe_\theta(\theta)\cdot\mathcal{K}_{\gamma,\tilde{\gamma}}\psi - \frac{1}{4\pi}Q_{\frac{r}{R}} * (\psi-\bar{\psi})\right\|_{\dot{C}^\beta(\mathbb{T})}$$
$$\leqslant \frac{Cr}{R}(m_0+M_0)\|\psi\|_{\dot{C}^\beta} + \frac{Cr}{R}\|\psi\|_{L^\infty}\left(\delta^{-1}(\|h\|_{L^\infty}+\|H\|_{L^\infty}) + \|h'\|_{\dot{C}^\alpha} + \|H'\|_{\dot{C}^\alpha}\right), \tag{6.11}$$

where $C = C(\alpha,\beta)$.

Proof. Let $I_{r,1}$, $I_{r,2}$, $I_{\theta,1}$ and $I_{\theta,2}$ be defined as in the proof of Lemma 6.1.

Consider $I_{r,2}$. For $\theta_1, \theta_2 \in \mathbb{T}$,

$$\begin{aligned} I_{r,2}(\theta_1) - I_{r,2}(\theta_2) &= \frac{1}{2}\int_{\mathbb{T}}\left[P(D(\theta_2,\theta_2+\xi),\xi) - P\left(D(\theta_1,\theta_1+\xi),\xi\right)\right]\left(\psi(\theta_1+\xi)-\psi(\theta_1)\right)d\xi \\ &\quad + \frac{1}{2}\psi(\theta_1)\int_{\mathbb{T}} P\left(D(\theta_2,\theta_2+\xi),\xi\right) - P(D(\theta_1,\theta_1+\xi),\xi)\,d\xi \\ &\quad - \frac{1}{2}\int_{\mathbb{T}}\left[P\left(\frac{r}{R},\xi\right) - P(D(\theta_2,\theta_2+\xi),\xi)\right]\left(\psi(\theta_2+\xi)-\psi(\theta_1+\xi)\right)d\xi \\ &=: I_{r,2,1} + I_{r,2,2} + I_{r,2,3}. \end{aligned} \tag{6.12}$$

Following the argument of (6.6) and (6.7),

$$\begin{aligned} |I_{r,2,1}| &\leqslant C\int_{\mathbb{T}}\frac{1}{\delta^2+|\xi|^2}\cdot\left|\frac{f(\theta_1)}{F(\theta_1+\xi)} - \frac{f(\theta_2)}{F(\theta_2+\xi)}\right|\cdot|\xi|^\beta\|\psi\|_{\dot{C}^\beta}\,d\xi \\ &\leqslant C\|\psi\|_{\dot{C}^\beta}\int_{\mathbb{T}}\frac{|\xi|^\beta}{\delta^2+|\xi|^2}\cdot\frac{r}{R}|\theta_1-\theta_2|^\beta\left(\|h\|_{\dot{C}^\beta}+\|H\|_{\dot{C}^\beta}\right)d\xi \\ &\leqslant C|\theta_1-\theta_2|^\beta\cdot\frac{r}{R}(m_0+M_0)\|\psi\|_{\dot{C}^\beta}, \end{aligned} \tag{6.13}$$

and similarly,

$$|I_{r,2,3}| \leqslant C|\theta_1-\theta_2|^\beta\cdot\frac{r}{R}(m_0+M_0)\|\psi\|_{\dot{C}^\beta}. \tag{6.14}$$

To handle $I_{r,2,2}$, we first note that

$$\begin{aligned}
&\int_{\mathbb{T}} P\big(D(\theta_2,\theta_2+\xi),\xi\big) - P\big(D(\theta_1,\theta_1+\xi),\xi\big)\,d\xi \\
&= \int_{\mathbb{T}} P\big(D(\theta_2,\theta_2+\xi),\xi\big) - P\big(D(\theta_2,\theta_2),\xi\big) - P\big(D(\theta_1,\theta_1+\xi),\xi\big) + P\big(D(\theta_1,\theta_1),\xi\big)\,d\xi.
\end{aligned} \tag{6.15}$$

We may bound the integrands in (6.15) as follows. By the mean value theorem and Lemma A.1,

$$\begin{aligned}
&\big|P\big(D(\theta_2,\theta_2+\xi),\xi\big) - P\big(D(\theta_2,\theta_2),\xi\big) - P(D(\theta_1,\theta_1+\xi),\xi) + P\big(D(\theta_1,\theta_1),\xi\big)\big| \\
&\quad\leqslant \frac{C}{\delta^2+\xi^2}\big(\big|D(\theta_2,\theta_2+\xi) - D(\theta_2,\theta_2)\big| + \big|D(\theta_1,\theta_1+\xi) - D(\theta_1,\theta_1)\big|\big) \\
&\quad\leqslant \frac{C|\xi|^{\beta'}}{\delta^2+\xi^2}\cdot\frac{r}{R}\|H\|_{\dot C^{\beta'}},
\end{aligned} \tag{6.16}$$

where $\beta' \in (0,1)$ is to be determined. Here we used the bound $|\partial_s P| \leqslant C(\delta^2+\xi^2)^{-1}$ since $D \leqslant 1 - C\delta$ (see the proof of Lemma 6.1). Alternatively,

$$\begin{aligned}
&\big|P\big(D(\theta_2,\theta_2+\xi),\xi\big) - P\big(D(\theta_1,\theta_1+\xi),\xi\big) - P\big(D(\theta_2,\theta_2),\xi\big) + P\big(D(\theta_1,\theta_1),\xi\big)\big| \\
&\quad\leqslant \frac{C}{\delta^2+\xi^2}\big(|D(\theta_2,\theta_2+\xi) - D(\theta_1,\theta_1+\xi)| + |D(\theta_2,\theta_2) - D(\theta_1,\theta_1)|\big) \\
&\quad\leqslant \frac{C}{\delta^2+\xi^2}\cdot\frac{r}{R}|\theta_1-\theta_2|(\|h'\|_{L^\infty} + \|H'\|_{L^\infty}).
\end{aligned} \tag{6.17}$$

If $|\theta_1-\theta_2| \geqslant \delta$, by (6.15) and (6.16),

$$\left|\int_{\mathbb{T}} P\big(D(\theta_2,\theta_2+\xi),\xi\big) - P\big(D(\theta_1,\theta_1+\xi),\xi\big)\,d\xi\right| \leqslant \frac{Cr}{R}\|H\|_{\dot C^{\beta'}}\delta^{\beta'-\beta-1}|\theta_1-\theta_2|^{\beta}. \tag{6.18}$$

Otherwise, if $|\theta_1-\theta_2| \leqslant \delta$, we deduce by (6.15) and (6.17) that

$$\left|\int_{\mathbb{T}} P\big(D(\theta_2,\theta_2+\xi),\xi\big) - P\big(D(\theta_1,\theta_1+\xi),\xi\big)\,d\xi\right| \leqslant \frac{Cr}{R}|\theta_1-\theta_2|^{\beta}\delta^{-\beta}\big(\|h'\|_{L^\infty} + \|H'\|_{L^\infty}\big). \tag{6.19}$$

Recall that $\beta < \frac{\alpha}{1+\alpha}$, so we take $\beta' = \frac{\beta(1+\alpha)}{\alpha}$. Combining these estimates with the definition of $I_{r,2,2}$ in (6.12), by interpolation inequality,

$$|I_{r,2,2}| \leqslant \frac{Cr}{R}|\theta_1-\theta_2|^{\beta}\|\psi\|_{L^\infty}\big(\delta^{-1}(\|h\|_{L^\infty} + \|H\|_{L^\infty}) + \|h'\|_{\dot C^\alpha} + \|H'\|_{\dot C^\alpha}\big). \tag{6.20}$$

Combining this with (6.12)–(6.14), we obtain that

$$\|I_{r,2}\|_{\dot C^\beta} \leqslant \frac{Cr}{R}(m_0+M_0)\|\psi\|_{\dot C^\beta} + \frac{Cr}{R}\|\psi\|_{L^\infty}\big(\delta^{-1}(\|h\|_{L^\infty} + \|H\|_{L^\infty}) + \|h'\|_{\dot C^\alpha} + \|H'\|_{\dot C^\alpha}\big). \tag{6.21}$$

The estimate for $I_{\theta,2}$ can be derived in the same manner. □

Lemma 6.3 *Assume $h \in W^{1,\infty}(\mathbb{T})$ and $H \in W^{2,p}(\mathbb{T})$ for some $p \in (1,\infty)$, satisfying that $m_0 + M_0 \ll 1$. Then*

$$\left\| fe_r(\theta) \cdot \mathcal{K}_{\gamma,\tilde{\gamma}}\psi + \frac{1}{4\pi} P_{\frac{r}{R}} * (\psi - \bar{\psi}) \right\|_{\dot{W}^{1,p}(\mathbb{T})} + \left\| fe_\theta(\theta) \cdot \mathcal{K}_{\gamma,\tilde{\gamma}}\psi - \frac{1}{4\pi} Q_{\frac{r}{R}} * (\psi - \bar{\psi}) \right\|_{\dot{W}^{1,p}(\mathbb{T})} \leqslant \frac{Cr}{R}\big(\|H''\|_{L^p}\|\psi\|_{L^\infty} + (m_0 + M_0)\|\psi'\|_{L^p}\big), \tag{6.22}$$

where $C = C(p)$.

Proof. Let $I_{r,1}$, $I_{r,2}$, $I_{\theta,1}$ and $I_{\theta,2}$ be defined as in the proof of Lemma 6.1.

We calculate that

$$I'_{r,2}(\theta) = \frac{1}{2}\int_{\mathbb{T}} \left[P\left(\frac{r}{R},\xi\right) - P(D,\xi)\right]\psi'(\theta+\xi)\,d\xi - \frac{1}{2}\int_{\mathbb{T}} \partial_s P(D,\xi)\frac{\partial D}{\partial\theta}\psi(\theta+\xi)\,d\xi. \tag{6.23}$$

Arguing as in (6.6) and (6.7),

$$\left\|\int_{\mathbb{T}} \left[P\left(\frac{r}{R},\xi\right) - P(D,\xi)\right]\psi'(\theta+\xi)\,d\xi\right\|_{L^p} \leqslant \frac{Cr}{R}\delta^{-1}(\|h\|_{L^\infty} + \|H\|_{L^\infty})\|\psi'\|_{L^p}. \tag{6.24}$$

For the second term in $I'_{r,2}$, we derive by Lemma A.1 that

$$\begin{aligned}
&\int_{\mathbb{T}} \partial_s P(D,\xi)\frac{\partial D}{\partial\theta}\psi(\theta+\xi)\,d\xi \\
&\quad= \frac{f'(\theta)}{f(\theta)}\int_{\mathbb{T}} D\partial_s P(D,\xi)\psi(\theta+\xi)\,d\xi + \int_{\mathbb{T}} \partial_s P(D,\xi)\frac{\partial D}{\partial\xi}\psi(\theta+\xi)\,d\xi \\
&\quad= \frac{f'(\theta)}{f(\theta)}\int_{\mathbb{T}} \frac{\partial Q(D,\xi)}{\partial\xi}\psi(\theta+\xi)\,d\xi + \int_{\mathbb{T}} \partial_s P(D,\xi)\frac{\partial D}{\partial\xi}\psi(\theta+\xi)\,d\xi \\
&\qquad - \frac{f'(\theta)}{f(\theta)}\int_{\mathbb{T}} \partial_s Q(D,\xi)\frac{\partial D}{\partial\xi}\psi(\theta+\xi)\,d\xi \\
&\quad=: I_{r,2,a} + I_{r,2,b} + I_{r,2,c}.
\end{aligned} \tag{6.25}$$

Here $\frac{\partial Q(D,\xi)}{\partial\xi}$ denotes total derivative of $Q(D(\theta,\theta+\xi),\xi)$ with respect to ξ.

We integrate by parts in $I_{2,r,a}$. Arguing as in (6.24),

$$\begin{aligned}
\|I_{r,2,a}\|_{L^p} &\leqslant C\|h'\|_{L^\infty}\left(\left\|\int_{\mathbb{T}} \left[Q(D,\xi) - Q\left(\frac{r}{R},\xi\right)\right]\psi'(\theta+\xi)\,d\xi\right\|_{L^p} + \|Q_{\frac{r}{R}} * \psi'\|_{L^p}\right) \\
&\leqslant \frac{Cr}{R}\|h'\|_{L^\infty}\delta^{-1}(\|h\|_{L^\infty} + \|H\|_{L^\infty})\|\psi'\|_{L^p} + C\|h'\|_{L^\infty}\|P_{\frac{r}{R}} * \mathcal{H}\psi'\|_{L^p}.
\end{aligned} \tag{6.26}$$

Using the fact that $\mathcal{H}\psi'$ has mean zero on $\mathbb{T}$, we derive that

$$P_{\frac{r}{R}} * \mathcal{H}\psi' = \int_{\mathbb{T}} \left(P_{\frac{r}{R}}(\xi) - P_{\frac{r}{R}}(\pi)\right)\mathcal{H}\psi'(\theta-\xi)\,d\xi. \tag{6.27}$$

By Young's inequality,

$$\|P_{\frac{r}{R}} * \mathcal{H}\psi'\|_{L^p} \leqslant \int_{\mathbb{T}} \left|P_{\frac{r}{R}}(\xi) - P_{\frac{r}{R}}(\pi)\right| \, d\xi \cdot \|\mathcal{H}\psi'\|_{L^p} \leqslant \frac{Cr}{R}\|\psi'\|_{L^p}. \tag{6.28}$$

Therefore,

$$\|I_{r,2,a}\|_{L^p} \leqslant \frac{Cr}{R}\|h'\|_{L^\infty}\|\psi'\|_{L^p}. \tag{6.29}$$

Next we deal with $I_{r,2,b}$. Since

$$\frac{\partial D}{\partial \xi} = -D\frac{F'(\theta+\xi)}{F(\theta+\xi)}, \tag{6.30}$$

we find by Lemma A.1 that

$$\begin{aligned}
I_{r,2,b} &= -\int_{\mathbb{T}} D\partial_s P(D,\xi)\frac{F'(\theta+\xi)}{F(\theta+\xi)}\psi(\xi+\theta)\, d\xi \\
&= -\int_{\mathbb{T}} \left[\frac{\partial Q(D,\xi)}{\partial \xi} - \partial_s Q(D,\xi)\frac{\partial D}{\partial \xi}\right] \cdot \frac{F'\psi}{F}(\xi+\theta)\, d\xi \\
&= -\int_{\mathbb{T}} \frac{\partial Q(D,\xi)}{\partial \xi} \cdot \frac{F'\psi}{F}(\xi+\theta)\, d\xi - \int_{\mathbb{T}} D\partial_s Q(D,\xi) \cdot \frac{F'^2\psi}{F^2}(\xi+\theta)\, d\xi \\
&= -\int_{\mathbb{T}} \frac{\partial Q(D,\xi)}{\partial \xi} \cdot \frac{F'\psi}{F}(\xi+\theta)\, d\xi + \int_{\mathbb{T}} \frac{\partial P(D,\xi)}{\partial \xi} \cdot \frac{F'^2\psi}{F^2}(\xi+\theta)\, d\xi \\
&\qquad - \int_{\mathbb{T}} \partial_s P(D,\xi)\frac{\partial D}{\partial \xi} \cdot \frac{F'^2\psi}{F^2}(\xi+\theta)\, d\xi.
\end{aligned} \tag{6.31}$$

Arguing as in (6.26)–(6.29),

$$\begin{aligned}
&\left\|\int_{\mathbb{T}} \frac{\partial Q(D,\xi)}{\partial \xi} \cdot \frac{F'\psi}{F}(\xi+\theta)\, d\xi\right\|_{L^p} + \left\|\int_{\mathbb{T}} \frac{\partial P(D,\xi)}{\partial \xi} \cdot \frac{F'^2\psi}{F^2}(\xi+\theta)\, d\xi\right\|_{L^p} \\
&\qquad\qquad \leqslant \frac{Cr}{R}\left\|\frac{F'\psi}{F}\right\|_{\dot{W}^{1,p}} + \frac{Cr}{R}\left\|\frac{F'^2\psi}{F^2}\right\|_{\dot{W}^{1,p}} \\
&\qquad\qquad \leqslant \frac{Cr}{R}\big(\|H''\|_{L^p}\|\psi\|_{L^\infty} + \|H'\|_{L^\infty}\|\psi'\|_{L^p}\big).
\end{aligned} \tag{6.32}$$

We notice that the last term in (6.31), which has not been bounded, is in a similar form as the original $I_{r,2,b}$. Following (6.31) and (6.32), it is not difficult to argue by induction that for all $k \in \mathbb{N}$,

$$\begin{aligned}
&\|I_{r,2,b}\|_{L^p} \\
&\leqslant \frac{Cr}{R}\big(\|H''\|_{L^p}\|\psi\|_{L^\infty} + \|H'\|_{L^\infty}\|\psi'\|_{L^p}\big) + \left\|\int_{\mathbb{T}} \partial_s P(D,\xi)\frac{\partial D}{\partial \xi}\frac{F'^{2k}\psi}{F^{2k}}(\xi+\theta)\, d\xi\right\|_{L^p} \\
&\leqslant \frac{Cr}{R}\big(\|H''\|_{L^p}\|\psi\|_{L^\infty} + \|H'\|_{L^\infty}\|\psi'\|_{L^p}\big) + \frac{Cr}{R}\int_{\mathbb{T}} \frac{d\xi}{\delta^2+\xi^2}\left(\frac{\|H'\|_{L^\infty}}{1-\|H\|_{L^\infty}}\right)^{2k+1}\|\psi\|_{L^p}.
\end{aligned} \tag{6.33}$$

Here the constants C are uniformly bounded in k provided the smallness of H. Since $M_0 \ll 1$, we take $k \to \infty$ and obtain

$$\|I_{r,2,b}\|_{L^p} \leqslant \frac{Cr}{R}\big(\|H''\|_{L^p}\|\psi\|_{L^\infty} + \|H'\|_{L^\infty}\|\psi'\|_{L^p}\big). \tag{6.34}$$

$\|I_{r,2,c}\|_{L^p}$ can be estimated in a similar manner, so is $\|I'_{\theta,2}\|_{L^p}$. □

Estimates for the operator $\mathcal{K}_{\tilde{\gamma},\gamma}$ can be derived in a similar manner.

Lemma 6.4 1. *Under the assumptions of Lemma 6.1,*

$$\left\|Fe_r(\theta)\cdot\mathcal{K}_{\tilde{\gamma},\gamma}\psi - \bar{\psi} - \frac{1}{4\pi}P_{\frac{r}{R}} * (\psi-\bar{\psi})\right\|_{L^\infty(\mathbb{T})} + \left\|Fe_\theta(\theta)\cdot\mathcal{K}_{\tilde{\gamma},\gamma}\psi - \frac{1}{4\pi}Q_{\frac{r}{R}} * (\psi-\bar{\psi})\right\|_{L^\infty(\mathbb{T})} \leqslant \frac{Cr}{R}\delta^{-1}\big(\|h\|_{L^\infty} + \|H\|_{L^\infty}\big)\|\psi\|_{L^\infty}, \tag{6.35}$$

where C is a universal constant.

2. *Under the assumptions of Lemma 6.2,*

$$\left\|Fe_r(\theta)\cdot\mathcal{K}_{\tilde{\gamma},\gamma}\psi - \frac{1}{4\pi}P_{\frac{r}{R}} * (\psi-\bar{\psi})\right\|_{\dot{C}^\beta(\mathbb{T})} + \left\|Fe_\theta(\theta)\cdot\mathcal{K}_{\tilde{\gamma},\gamma}\psi - \frac{1}{4\pi}Q_{\frac{r}{R}} * (\psi-\bar{\psi})\right\|_{\dot{C}^\beta(\mathbb{T})} \leqslant \frac{Cr}{R}(m_0+M_0)\|\psi\|_{\dot{C}^\beta} + \frac{Cr}{R}\|\psi\|_{L^\infty}\big(\delta^{-1}(\|h\|_{L^\infty}+\|H\|_{L^\infty}) + \|h'\|_{\dot{C}^\alpha} + \|H'\|_{\dot{C}^\alpha}\big), \tag{6.36}$$

where $C = C(\alpha,\beta)$.

3. *Assume $h \in W^{2,p}(\mathbb{T})$ for some $p \in (1,\infty)$ and $H \in W^{1,\infty}(\mathbb{T})$, satisfying that $m_0 + M_0 \ll 1$. Then*

$$\left\|Fe_r(\theta)\cdot\mathcal{K}_{\tilde{\gamma},\gamma}\psi - \frac{1}{4\pi}P_{\frac{r}{R}} * (\psi-\bar{\psi})\right\|_{\dot{W}^{1,p}(\mathbb{T})} + \left\|Fe_\theta(\theta)\cdot\mathcal{K}_{\tilde{\gamma},\gamma}\psi - \frac{1}{4\pi}Q_{\frac{r}{R}} * (\psi-\bar{\psi})\right\|_{\dot{W}^{1,p}(\mathbb{T})} \leqslant \frac{Cr}{R}\big(\|h''\|_{L^p}\|\psi\|_{L^\infty} + (m_0+M_0)\|\psi'\|_{L^p}\big), \tag{6.37}$$

where $C = C(p)$.

Proof. We derive as in Lemma 6.1.

$$2\pi F(\theta)e_r(\theta)\cdot\mathcal{K}_{\tilde{\gamma},\gamma}\psi - 2\pi\bar{\psi} = \frac{1}{2}\int_{\mathbb{T}} P\left(\frac{r}{R},\xi\right)\left(\psi(\theta+\xi)-\bar{\psi}\right)d\xi + \frac{1}{2}\int_{\mathbb{T}}\left[P\left(\frac{f(\theta+\xi)}{F(\theta)},\xi\right) - P\left(\frac{r}{R},\xi\right)\right]\psi(\theta+\xi)\,d\xi, \tag{6.38}$$

and

$$\begin{aligned}&2\pi F(\theta)e_\theta(\theta)\cdot\mathcal{K}_{\tilde{\gamma},\gamma}\psi\\&=-\frac{1}{2}\int_{\mathbb{T}}Q\left(\frac{r}{R},\xi\right)\left(\psi(\theta+\xi)-\bar{\psi}\right)d\xi+\frac{1}{2}\int_{\mathbb{T}}\left[Q\left(\frac{r}{R},\xi\right)-Q\left(\frac{f(\theta+\xi)}{F(\theta)},\xi\right)\right]\psi(\theta+\xi)\,d\xi.\end{aligned}\tag{6.39}$$

Then the desired estimate can be proved by arguing as in Lemmas 6.1–6.3. □

Lastly, for those convolution terms on the left hand sides of the estimates in Lemmas 6.1–6.4, we have that

Lemma 6.5 *For $\beta\in(0,1)$, we have*

$$\|P_{\frac{r}{R}}*(\psi-\bar{\psi})\|_{L^\infty}\leqslant\frac{4\pi r}{R+r}\|\psi-\bar{\psi}\|_{L^\infty},\tag{6.40}$$

$$\|Q_{\frac{r}{R}}*(\psi-\bar{\psi})\|_{L^\infty}\leqslant C\|\psi\|_{\dot{C}^\beta},\tag{6.41}$$

and

$$\|P_{\frac{r}{R}}*(\psi-\bar{\psi})\|_{\dot{C}^\beta}\leqslant\frac{4\pi r}{R+r}\|\psi\|_{\dot{C}^\beta},\tag{6.42}$$

$$\|Q_{\frac{r}{R}}*(\psi-\bar{\psi})\|_{\dot{C}^\beta}\leqslant C\|\psi\|_{\dot{C}^\beta},\tag{6.43}$$

where these two constants C depend on β. Moreover, for $p\in(1,\infty)$,

$$\|P_{\frac{r}{R}}*(\psi-\bar{\psi})\|_{\dot{W}^{1,p}}\leqslant\frac{4\pi r}{R+r}\|\psi'\|_{L^p},\tag{6.44}$$

$$\|Q_{\frac{r}{R}}*(\psi-\bar{\psi})\|_{\dot{W}^{1,p}}\leqslant C\|\psi'\|_{L^p},\tag{6.45}$$

where C depends on p.

Proof. Since

$$P_{\frac{r}{R}}*(\psi-\bar{\psi})=\int_{\mathbb{T}}\big(P_{\frac{r}{R}}(\xi)-P_{\frac{r}{R}}(\pi)\big)(\psi(\theta-\xi)-\bar{\psi})\,d\xi,\tag{6.46}$$

and $P_{\frac{r}{R}}(\xi)\geqslant P_{\frac{r}{R}}(\pi)$, we have that

$$\|P_{\frac{r}{R}}*(\psi-\bar{\psi})\|_{L^\infty}\leqslant\|\psi-\bar{\psi}\|_{L^\infty}\int_{\mathbb{T}}P_{\frac{r}{R}}(\xi)-P_{\frac{r}{R}}(\pi)\,d\xi=\frac{4\pi r}{R+r}\|\psi-\bar{\psi}\|_{L^\infty}.\tag{6.47}$$

Since $Q_{\frac{r}{R}}$ has integral zero over $\mathbb{T}$, by Lemma A.1,

$$|Q_{\frac{r}{R}}*(\psi-\bar{\psi})|=\left|\int_{\mathbb{T}}Q_{\frac{r}{R}}(\xi)\big(\psi(\theta-\xi)-\psi(\theta)\big)\,d\xi\right|\leqslant C\|\psi\|_{\dot{C}^\beta}\int_{\mathbb{T}}\frac{|\xi|^\beta}{\delta+|\xi|}\,d\xi\leqslant C\|\psi\|_{\dot{C}^\beta}.\tag{6.48}$$

It is straightforward to derive that for $\theta_1, \theta_2 \in \mathbb{T}$,

$$
\begin{aligned}
&|P_{\frac{r}{R}} * (\psi - \bar{\psi})(\theta_1) - P_{\frac{r}{R}} * (\psi - \bar{\psi})(\theta_2)| \\
&\qquad = \left| \int_{\mathbb{T}} \big(P_{\frac{r}{R}}(\xi) - P_{\frac{r}{R}}(\pi)\big)\big(\psi(\theta_1 - \xi) - \psi(\theta_2 - \xi)\big)\, d\xi \right| \\
&\qquad\qquad \leqslant \frac{4\pi r}{R + r} \|\psi\|_{\dot{C}^{\beta}} |\theta_1 - \theta_2|^{\beta}. \qquad (6.49)
\end{aligned}
$$

Moreover, by Young's inequality,

$$
\|P_{\frac{r}{R}} * (\psi - \bar{\psi})\|_{\dot{W}^{1,p}} = \big\|\big(P_{\frac{r}{R}} - P_{\frac{r}{R}}(\pi)\big) * \psi'\big\|_{L^p} \leqslant \frac{4\pi r}{R + r} \|\psi'\|_{L^p}. \qquad (6.50)
$$

The estimates involving $Q_{\frac{r}{R}}$ follows from the fact $Q_{\frac{r}{R}} = \mathcal{H} P_{\frac{r}{R}}$. Note that the boundedness of Hilbert transform on $C^{\beta}(\mathbb{T})$ can be justified by that of its counterpart on $C^{\beta}(\mathbb{R})$ with some adaptation. □

7. Existence, uniqueness and estimates for $[\varphi]$ and ϕ

This section aims at establishing well-definedness, regularity and estimates for $[\varphi]_\gamma$ and ϕ. The main approach is to apply a fixed-point argument to the static equations (2.33) and (2.34), by using many estimates in Sections 3–6.

With the domain determined by r, R, h and H, let $\tilde{p}$ be defined by (3.4) and (3.5), and let the radially symmetric solution p_* be defined as in (3.8). Recall that c_* and $\tilde{c}_*$ are defined in (2.22). In fact, $c_* = -\mu|\nabla p_*(r^-)|$ and $\tilde{c}_* = -\nu|\nabla p_*(R)|$, so their estimates can be found in Lemma 3.1. Also recall that $\mathcal{S}\psi := \frac{1}{2\pi} P_{\frac{r}{R}} * \psi$ defined in (2.38). Then $\mathcal{H}\mathcal{S}\psi = \frac{1}{2\pi} Q_{\frac{r}{R}} * \psi$ thanks to Lemma A.1.

In the spirit of the linearized equations (2.43) and (2.44), we rewrite (2.33) and (2.34) as

$$
[\varphi]' - 2Ac_* f' - A\mathcal{S}\phi' = \mathcal{R}_{[\varphi]'}, \qquad (7.1)
$$

$$
\phi' + 2\tilde{c}_* F' - \mathcal{S}[\varphi]' = \mathcal{R}_{\phi'}, \qquad (7.2)
$$

where

$$
\begin{aligned}
\mathcal{R}_{[\varphi]'} :=\ & 2Af'(\theta)\big(e_r \cdot \nabla(\Gamma * g)|_\gamma - c_*\big) + 2Af(\theta) e_\theta \cdot \nabla(\Gamma * g)|_\gamma \\
& + 2A\gamma'(\theta)^{\perp} \cdot \mathcal{K}_\gamma [\varphi]' + 2A\left(\gamma'(\theta)^{\perp} \cdot \mathcal{K}_{\gamma,\tilde{\gamma}}\phi' - \frac{1}{2}\mathcal{S}\phi'\right), \qquad (7.3)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_{\phi'} :=\ & -2F'\big(e_r \cdot \nabla(\Gamma * g)|_{\tilde{\gamma}} - \tilde{c}_*\big) - 2F e_\theta \cdot \nabla(\Gamma * g)|_{\tilde{\gamma}} \\
& - 2\tilde{\gamma}'(\theta)^{\perp} \cdot \mathcal{K}_{\tilde{\gamma}}\phi' - 2\left(\tilde{\gamma}'(\theta)^{\perp} \cdot \mathcal{K}_{\tilde{\gamma},\gamma}[\varphi]' + \frac{1}{2}\mathcal{S}[\varphi]'\right). \qquad (7.4)
\end{aligned}
$$

In what follows, we will need to apply the lemmas in Section 4 with $g_0 = G(\tilde{p}(X))\chi_{B_r}(X)$. For that purpose, according to (4.50) and (4.76), we define

$$
c = -\frac{1}{2\pi r}\int_{B_r} G\big(\tilde{p}(X)\big)\, dX, \quad \tilde{c} = \frac{r}{R}c. \qquad (7.5)
$$

We can show the following relation between c and c_*.

Lemma 7.1 *Let c_* and c be defined in (2.22) and (7.5), respectively. Then under the assumption $m_0 + M_0 \ll 1$,*

$$|c - c_*| \leqslant Cr(m_0 + M_0)(\delta R^2)^{1/2}, \tag{7.6}$$

where $C = C(\mu, \nu, G)$.

Proof. Thanks to the C^1-smoothness of G,

$$|c - c_*| \leqslant Cr^{-1} \int_{B_r} |\tilde{p} - p_*| \, dX. \tag{7.7}$$

If $r \geqslant R/2$, by Lemma 3.3, Hölder's inequality and Poincaré inequality,

$$|c - c_*| \leqslant C\|\tilde{p} - p_*\|_{L^2(B_R)} \leqslant CR\|\nabla(\tilde{p} - p_*)\|_{L^2(B_R)} \leqslant CR(m_0 + M_0)(\delta R^2)^{1/2}. \tag{7.8}$$

Since r and R are comparable, the desired estimate follows.
Otherwise, the estimate follows from (3.43). □

Then we turn to prove that the static equations (2.33) and (2.34) have solutions $[\varphi]'$ and ϕ'.

Proposition 7.2 *Let $\beta' \in (0, 1)$ and $\beta \in (0, \frac{\beta'}{1+\beta'})$. Suppose $h, H \in C^{1,\beta'}(\mathbb{T})$, such that*

$$m_0 + M_0 + \|h'\|_{\dot{C}^{\beta'}} + \|H'\|_{\dot{C}^{\beta'}} \ll 1, \tag{7.9}$$

where the smallness depends on μ, ν, β and β'. Then there exist unique $[\varphi]', \phi' \in C^\beta(\mathbb{T})$ solving (2.33) and (2.34), or equivalently (7.1)–(7.4). They satisfy that

$$\begin{aligned}\|[\varphi]'\|_{\dot{C}^\beta} + \|\phi'\|_{\dot{C}^\beta} &\leqslant C|c_*|r(\|h'\|_{\dot{C}^\beta} + \|H'\|_{\dot{C}^\beta}) + Cr^2\big(\delta^\beta\|h'\|_{\dot{C}^\beta} + (m_0 + M_0)(1 + \delta R^2)^{1/2}\big)\\ &=: N_{1,\beta},\end{aligned} \tag{7.10}$$

where $C = C(\mu, \nu, G, \beta, \beta')$.

Proof. We will first derive a priori estimates for $[\varphi]'$ and ϕ', and then briefly discuss the proof of their existence and uniqueness at the end.

By Lemmas 3.3, 4.4 and 4.7 (with $p = (1 - \beta)^{-1}$), the C^1-smoothness of G and the smallness of h,

$$\begin{aligned}&\|f'(e_r \cdot \nabla(\Gamma * g)|_\gamma - c)\|_{\dot{C}^\beta} + \|fe_\theta \cdot \nabla(\Gamma * g)|_\gamma\|_{\dot{C}^\beta}\\
&\quad\leqslant \|f'\|_{\dot{C}^\beta}\|e_r \cdot \nabla(\Gamma * g)|_\gamma - c\|_{L^\infty} + \|f'\|_{L^\infty}\|e_r \cdot \nabla(\Gamma * g)|_\gamma\|_{\dot{W}^{1,p}}\\
&\qquad + \|f\|_{\dot{C}^\beta}\|e_\theta \cdot \nabla(\Gamma * g)|_\gamma\|_{L^\infty} + \|f\|_{L^\infty}\|e_\theta \cdot \nabla(\Gamma * g)|_\gamma\|_{\dot{W}^{1,p}}\\
&\quad\leqslant Cr^2\|h'\|_{\dot{C}^\beta}\big(m_0\delta|\ln\delta| + \|\nabla(\tilde{p} - p_*)\|_{L^2(B_r)}\big)\\
&\qquad + Cr^2\big(m_\beta + \|\nabla(\tilde{p} - p_*)\|_{L^2(B_r)}\big)\\
&\quad\leqslant Cr^2\big(m_\beta + (m_0 + M_0)(\delta R^2)^{1/2}\big).\end{aligned} \tag{7.11}$$

On the other hand, for $\beta \in (0, \frac{\beta'}{1+\beta'})$, by Lemmas 5.1, 6.1, 6.2 and 6.5,

$$\begin{aligned} &\|\gamma'(\theta)^{\perp}\cdot\mathcal{K}_\gamma[\varphi]'\|_{\dot{C}^\beta} + \left\|\gamma'(\theta)^{\perp}\cdot\mathcal{K}_{\gamma,\tilde{\gamma}}\phi' - \frac{1}{2}\mathcal{S}\phi'\right\|_{\dot{C}^\beta} \\ &\quad\leqslant C\|h'\|_{\dot{C}^\beta}(\|[\varphi]'\|_{C^\beta} + \|[\varphi]'\|_{L^\infty}\|h'\|_{\dot{C}^\beta}\|h'\|_{L^\infty}) \\ &\qquad + \|f'/f\|_{\dot{C}^\beta}\|fe_\theta\cdot\mathcal{K}_{\gamma,\tilde{\gamma}}\phi'\|_{L^\infty} + \|f'/f\|_{L^\infty}\|fe_\theta\cdot\mathcal{K}_{\gamma,\tilde{\gamma}}\phi'\|_{\dot{C}^\beta} \\ &\qquad + \left\|fe_r\cdot\mathcal{K}_{\gamma,\tilde{\gamma}}\phi' + \frac{1}{2}\mathcal{S}\phi'\right\|_{\dot{C}^\beta} \\ &\quad\leqslant C\|h'\|_{\dot{C}^\beta}\|[\varphi]'\|_{\dot{C}^\beta} + C(m_0 + M_0 + \|h'\|_{\dot{C}^{\beta'}} + \|H'\|_{\dot{C}^{\beta'}})\|\phi'\|_{\dot{C}^\beta}, \end{aligned} \tag{7.12}$$

where $C = C(\beta, \beta')$. Hence, by (7.3), Lemma 7.1 and the fact that $|A| \leqslant 1$,

$$\begin{aligned} \|\mathcal{R}_{[\varphi]'}\|_{\dot{C}^\beta} &\leqslant |A|C(\beta,\beta')(m_0 + M_0 + \|h'\|_{\dot{C}^{\beta'}} + \|H'\|_{\dot{C}^{\beta'}})\|\phi'\|_{\dot{C}^\beta} \\ &\quad + C(\beta,\beta')\|h'\|_{\dot{C}^\beta}\|[\varphi]'\|_{\dot{C}^\beta} + Cr^2\big(m_\beta + (m_0 + M_0)(\delta R^2)^{1/2}\big), \end{aligned} \tag{7.13}$$

and thus by (7.1),

$$\begin{aligned} \|[\varphi]'\|_{\dot{C}^\beta} &\leqslant |A|\left(\frac{2r}{R+r} + C(\beta,\beta')(m_0 + M_0 + \|h'\|_{\dot{C}^{\beta'}} + \|H'\|_{\dot{C}^{\beta'}})\right)\|\phi'\|_{\dot{C}^\beta} \\ &\quad + C(\beta,\beta')\|h'\|_{\dot{C}^\beta}\|[\varphi]'\|_{\dot{C}^\beta} + C|c_*|r\|h'\|_{\dot{C}^\beta} + Cr^2\big(m_\beta + (m_0 + M_0)(\delta R^2)^{1/2}\big), \end{aligned} \tag{7.14}$$

where $C = C(\mu, \nu, G, \beta, \beta')$ unless otherwise stated.

Similarly, by Lemmas 3.3, 4.9 and 4.12,

$$\begin{aligned} &\|F'(e_r\cdot\nabla(\Gamma * g) - \tilde{c})|_{\tilde{\gamma}}\|_{\dot{C}^\beta} + \|Fe_\theta\cdot\nabla(\Gamma * g)|_{\tilde{\gamma}}\|_{\dot{C}^\beta} \\ &\quad\leqslant \|F'\|_{\dot{C}^\beta}\|e_r\cdot\nabla(\Gamma * g)|_{\tilde{\gamma}} - \tilde{c}\|_{L^\infty} + \|F'\|_{L^\infty}\|e_r\cdot\nabla(\Gamma * g)|_{\tilde{\gamma}}\|_{\dot{W}^{1,p}} \\ &\qquad + \|F\|_{\dot{C}^\beta}\|e_\theta\cdot\nabla(\Gamma * g)|_{\tilde{\gamma}}\|_{L^\infty} + \|F\|_{L^\infty}\|e_\theta\cdot\nabla(\Gamma * g)|_{\tilde{\gamma}}\|_{\dot{W}^{1,p}} \\ &\quad\leqslant Cr^2(m_0 + M_0)(1 + \delta R^2)^{1/2}. \end{aligned} \tag{7.15}$$

By Lemmas 5.1, 6.4 and 6.5,

$$\begin{aligned} &\|\tilde{\gamma}'(\theta)^{\perp}\cdot\mathcal{K}_{\tilde{\gamma}}\phi'\|_{\dot{C}^\beta} + \left\|\tilde{\gamma}'(\theta)^{\perp}\cdot\mathcal{K}_{\tilde{\gamma},\gamma}[\varphi]' + \frac{1}{2}\mathcal{S}[\varphi]\right\|_{\dot{C}^\beta} \\ &\quad\leqslant C\|H'\|_{\dot{C}^\beta}\big(\|\phi'\|_{C^\beta} + \|\phi'\|_{L^\infty}\|H'\|_{\dot{C}^\beta}\|H'\|_{L^\infty}\big) \\ &\qquad + \|F'/F\|_{\dot{C}^\beta}\|Fe_\theta\cdot\mathcal{K}_{\tilde{\gamma},\gamma}[\varphi]'\|_{L^\infty} + \|F'/F\|_{L^\infty}\|Fe_\theta\cdot\mathcal{K}_{\tilde{\gamma},\gamma}[\varphi]'\|_{\dot{C}^\beta} \\ &\qquad + \left\|Fe_r\cdot\mathcal{K}_{\tilde{\gamma},\gamma}[\varphi]' - \frac{1}{2}\mathcal{S}[\varphi]'\right\|_{\dot{C}^\beta} \\ &\quad\leqslant C\|H'\|_{\dot{C}^\beta}\|\phi'\|_{\dot{C}^\beta} + C\big(m_0 + M_0 + \|h'\|_{\dot{C}^{\beta'}} + \|H'\|_{\dot{C}^{\beta'}}\big)\|[\varphi]'\|_{\dot{C}^\beta}, \end{aligned} \tag{7.16}$$

where $C = C(\beta, \beta')$. Combining them with (7.2), (7.4) and Lemma 7.1, we obtain that

$$\begin{aligned} \|\mathcal{R}_{\phi'}\|_{\dot{C}^\beta} &\leqslant C(\beta,\beta')\big(m_0 + M_0 + \|h'\|_{\dot{C}^{\beta'}} + \|H'\|_{\dot{C}^{\beta'}}\big)\|[\varphi]'\|_{\dot{C}^\beta} \\ &\quad + C(\beta,\beta')\|H'\|_{\dot{C}^\beta}\|\phi'\|_{\dot{C}^\beta} + Cr^2(m_0 + M_0)(1 + \delta R^2)^{1/2}, \end{aligned} \tag{7.17}$$

and

$$\|\phi'\|_{\dot{C}^\beta} \leqslant \left(\frac{2r}{R+r} + C(\beta,\beta')(m_0+M_0+\|h'\|_{\dot{C}^{\beta'}} + \|H'\|_{\dot{C}^{\beta'}})\right)\|[\varphi]'\|_{\dot{C}^\beta} + C(\beta,\beta')\|H'\|_{\dot{C}^\beta}\|\phi'\|_{\dot{C}^\beta} + C|\tilde{c}_*|R\|H'\|_{\dot{C}^\beta} + Cr^2(m_0+M_0)(1+\delta R^2)^{1/2}, \tag{7.18}$$

where $C = C(\mu,\nu,G,\beta,\beta')$.

Since $|A|<1$ and $\tilde{c}_* = \frac{r}{R}c_*$, by the smallness assumption (7.9), we combine (7.14) and (7.18) to obtain (7.10).

Let us briefly explain the proof of existence and uniqueness of $[\varphi]'$ and ϕ'. Let V denote the space of $C^\beta(\mathbb{T})$-functions with mean zero. Take h and H satisfying the assumptions. According to (7.1) and (7.2), define a map from $V\times V$ to itself by

$$([\varphi]',\phi') \mapsto \left(2Ac_*f' + AS\phi' + \mathcal{R}_{[\varphi]'}, -2\tilde{c}_*F' + \mathcal{S}[\varphi]' + \mathcal{R}_{\phi'}\right). \tag{7.19}$$

Thanks to the estimates above, one can easily show that the map is well-defined and it is a contraction mapping provided the smallness of h and H. Then the existence and uniqueness of $([\varphi]',\phi')$ follow. □

Proposition 7.3 *Let $\beta'\in(0,1)$, $\beta\in(0,\frac{\beta'}{1+\beta'})$ and $p\in[2,\infty)$. Suppose $h,H\in C^{1,\beta'}\cap W^{2,p}(\mathbb{T})$, such that*

$$m_0+M_0+\|h'\|_{\dot{C}^{\beta'}}+\|H'\|_{\dot{C}^{\beta'}} \ll 1, \tag{7.20}$$

where the smallness depends on μ, ν, p, β and β'. Then $[\varphi]'$ and ϕ' obtained in Proposition 7.2 also belong to $W^{1,p}(\mathbb{T})$. They satisfy

$$\begin{aligned}\|[\varphi]''\|_{L^p} + \|\phi''\|_{L^p} &\leqslant C|c_*|r\left(\|h''\|_{L^p}+\|H''\|_{L^p}\right) \\ &\quad + Cr^2(1+\|h''\|_{L^p}+\|H''\|_{L^p})\left(\delta^\beta\|h'\|_{\dot{C}^\beta}+(m_0+M_0)(1+\delta R^2)^{1/2}\right) \\ &=: N_{2,p},\end{aligned} \tag{7.21}$$

where $C = C(\mu,\nu,p,G,\beta,\beta')$.

Proof. The proof is similar to that of Proposition 7.2.

Let c and $\tilde{c}$ be defined as in (7.5). We proceed as before.

$$\begin{aligned}&\|f'(e_r\cdot\nabla(\Gamma*g)|_\gamma - c)\|_{\dot{W}^{1,p}} + \|fe_\theta\cdot\nabla(\Gamma*g)|_\gamma\|_{\dot{W}^{1,p}} \\ &\leqslant \|f'\|_{\dot{W}^{1,p}}\|e_r\cdot\nabla(\Gamma*g)|_\gamma - c\|_{L^\infty} + \|f'\|_{L^\infty}\|e_r\cdot\nabla(\Gamma*g)|_\gamma\|_{\dot{W}^{1,p}} \\ &\qquad + \|f'\|_{L^p}\|e_\theta\cdot\nabla(\Gamma*g)|_\gamma\|_{L^\infty} + \|f\|_{L^\infty}\|e_\theta\cdot\nabla(\Gamma*g)|_\gamma\|_{\dot{W}^{1,p}} \\ &\leqslant Cr^2\|h''\|_{L^p}\left(m_0\delta|\ln\delta| + \|\nabla(\tilde{p}-p_*)\|_{L^2(B_r)}\right) \\ &\qquad + Cr^2\left(m_\beta + \|\nabla(\tilde{p}-p_*)\|_{L^2(B_r)}\right) \\ &\leqslant Cr^2\delta^\beta\|h'\|_{\dot{C}^\beta} + Cr^2(1+\|h''\|_{L^p})(m_0+M_0)(1+\delta R^2)^{1/2},\end{aligned} \tag{7.22}$$

where $C = C(\mu, \nu, p, G, \beta)$, and by Lemma 5.2 and Lemma 6.3,

$$\begin{aligned}
&\|\gamma'(\theta)^{\perp}\cdot\mathcal{K}_{\gamma}[\varphi]'\|_{\dot{W}^{1,p}} + \left\|\gamma'(\theta)^{\perp}\cdot\mathcal{K}_{\gamma,\tilde{\gamma}}\phi' - \frac{1}{2}\mathcal{S}\phi'\right\|_{\dot{W}^{1,p}} \\
&\quad\leqslant C\|h''\|_{L^p}\|[\varphi]'\|_{L^\infty}(1+\|h'\|_{\dot{C}^\beta}) + C(\|h''\|_{L^p}\|[\varphi]'\|_{\dot{C}^\beta} + \|h'\|_{L^\infty}\|[\varphi]''\|_{L^p}) \\
&\qquad + \|f'/f\|_{\dot{W}^{1,p}}\|fe_\theta\cdot\mathcal{K}_{\gamma,\tilde{\gamma}}\phi'\|_{L^\infty} + \|f'/f\|_{L^\infty}\|fe_\theta\cdot\mathcal{K}_{\gamma,\tilde{\gamma}}\phi'\|_{\dot{W}^{1,p}} \\
&\qquad + \left\|fe_r\cdot\mathcal{K}_{\gamma,\tilde{\gamma}}\phi' + \frac{1}{2}\mathcal{S}\phi'\right\|_{\dot{W}^{1,p}} \\
&\quad\leqslant C\big(\|h''\|_{L^p}\|[\varphi]'\|_{\dot{C}^\beta} + \|h'\|_{L^\infty}\|[\varphi]''\|_{L^p}\big) \\
&\qquad + C\|h''\|_{L^p}\|\phi'\|_{\dot{C}^\beta} + C\|h'\|_{L^\infty}\|\phi''\|_{L^p} \\
&\qquad + C\big(\|H''\|_{L^p}\|\phi'\|_{L^\infty} + (m_0+M_0)\|\phi''\|_{L^p}\big) \\
&\quad\leqslant C(m_0+M_0)\|\phi''\|_{L^p} + C\|h'\|_{L^\infty}\|[\varphi]''\|_{L^p} \\
&\qquad + C(\|h''\|_{L^p} + \|H''\|_{L^p})\big(\|[\varphi]'\|_{\dot{C}^\beta} + \|\phi'\|_{\dot{C}^\beta}\big),
\end{aligned} \tag{7.23}$$

where $C = C(p, \beta)$. Combining them with (7.1) and (7.3), by Lemma 6.5 and Lemma 7.1

$$\begin{aligned}
\|\mathcal{R}'_{[\varphi]'}\|_{L^p} \leqslant{}& C(p,\beta)(m_0+M_0)\|\phi''\|_{L^p} + C(p,\beta)\|h'\|_{L^\infty}\|[\varphi]''\|_{L^p} \\
&+ C(\|h''\|_{L^p} + \|H''\|_{L^p})(\|[\varphi]'\|_{\dot{C}^\beta} + \|\phi'\|_{\dot{C}^\beta}) \\
&+ Cr^2\delta^\beta\|h'\|_{\dot{C}^\beta} + Cr^2(1+\|h''\|_{L^p})(m_0+M_0)(1+\delta R^2)^{1/2},
\end{aligned} \tag{7.24}$$

and thus

$$\begin{aligned}
\|[\varphi]''\|_{L^p} \leqslant{}& \left(\frac{2|A|r}{R+r} + C(p,\beta)(m_0+M_0)\right)\|\phi''\|_{L^p} + C(p,\beta)\|h'\|_{L^\infty}\|[\varphi]''\|_{L^p} \\
&+ C(\|h''\|_{L^p} + \|H''\|_{L^p})\big(\|[\varphi]'\|_{\dot{C}^\beta} + \|\phi'\|_{\dot{C}^\beta}\big) \\
&+ C|c_*|r\|h''\|_{L^p} + Cr^2\delta^\beta\|h'\|_{\dot{C}^\beta} + Cr^2(1+\|h''\|_{L^p})(m_0+M_0)(1+\delta R^2)^{1/2},
\end{aligned} \tag{7.25}$$

where $C = C(\mu, \nu, p, G, \beta)$ unless otherwise stated.

Moreover,

$$\begin{aligned}
&\|F'(e_r\cdot\nabla(\Gamma * g) - \tilde{c})|_{\tilde{\gamma}}\|_{\dot{W}^{1,p}} + \|Fe_\theta\cdot\nabla(\Gamma * g)|_{\tilde{\gamma}}\|_{\dot{W}^{1,p}} \\
&\quad\leqslant \|F'\|_{\dot{W}^{1,p}}\|e_r\cdot\nabla(\Gamma * g)|_{\tilde{\gamma}} - \tilde{c}\|_{L^\infty} + \|F'\|_{L^\infty}\|e_r\cdot\nabla(\Gamma * g)|_{\tilde{\gamma}}\|_{\dot{W}^{1,p}} \\
&\qquad + \|F\|_{\dot{W}^{1,p}}\|e_\theta\cdot\nabla(\Gamma * g)|_{\tilde{\gamma}}\|_{L^\infty} + \|F\|_{L^\infty}\|e_\theta\cdot\nabla(\Gamma * g)|_{\tilde{\gamma}}\|_{\dot{W}^{1,p}} \\
&\quad\leqslant Cr^2(1+\|H''\|_{L^p})(m_0+M_0)(1+\delta R^2)^{1/2},
\end{aligned} \tag{7.26}$$

where $C = C(p, G, \beta)$, and

$$
\begin{aligned}
&\|\tilde{\gamma}'(\theta)^\perp \cdot \mathcal{K}_{\tilde{\gamma}}\phi'\|_{\dot{W}^{1,p}} + \left\|\tilde{\gamma}'(\theta)^\perp \cdot \mathcal{K}_{\tilde{\gamma},\gamma}[\varphi]' + \frac{1}{2}\mathcal{S}[\varphi]'\right\|_{\dot{W}^{1,p}} \\
&\quad\leqslant C\|H''\|_{L^p}\|\phi'\|_{L^\infty}(1 + \|H'\|_{\dot{C}^\beta}) + C\big(\|H''\|_{L^p}\|\phi'\|_{\dot{C}^\beta} + \|H'\|_{L^\infty}\|\phi''\|_{L^p}\big) \\
&\qquad + \|F'/F\|_{\dot{W}^{1,p}}\|Fe_\theta \cdot \mathcal{K}_{\tilde{\gamma},\gamma}[\varphi]'\|_{L^\infty} + \|F'/F\|_{L^\infty}\|Fe_\theta \cdot \mathcal{K}_{\tilde{\gamma},\gamma}[\varphi]'\|_{\dot{W}^{1,p}} \\
&\qquad + \left\|Fe_r \cdot \mathcal{K}_{\tilde{\gamma},\gamma}[\varphi]' - \frac{1}{2}\mathcal{S}[\varphi]'\right\|_{\dot{W}^{1,p}} \\
&\quad\leqslant C(m_0 + M_0)\|[\varphi]''\|_{L^p} + C\|H'\|_{L^\infty}\|\phi''\|_{L^p} \\
&\qquad + C(\|h''\|_{L^p} + \|H''\|_{L^p})\big(\|\phi'\|_{\dot{C}^\beta} + \|[\varphi]'\|_{\dot{C}^\beta}\big),
\end{aligned} \tag{7.27}
$$

where $C = C(p, \beta)$. Hence, by (7.2), (7.4), Lemma 6.5 and Lemma 7.1, with $C = C(p, G, \beta)$,

$$
\begin{aligned}
\|\mathcal{R}'_{\phi'}\|_{L^p} &\leqslant C(p, \beta)(m_0 + M_0)\|[\varphi]''\|_{L^p} + C(p, \beta)\|H'\|_{L^\infty}\|\phi''\|_{L^p} \\
&\quad + C(\|h''\|_{L^p} + \|H''\|_{L^p})\big(\|\phi'\|_{\dot{C}^\beta} + \|[\varphi]'\|_{\dot{C}^\beta}\big) \\
&\quad + Cr^2(1 + \|H''\|_{L^p})(m_0 + M_0)(1 + \delta R^2)^{1/2},
\end{aligned} \tag{7.28}
$$

and

$$
\begin{aligned}
\|\phi''\|_{L^p} &\leqslant \left(\frac{2r}{R + r} + C(p, \beta)(m_0 + M_0)\right)\|[\varphi]''\|_{L^p} + C(p, \beta)\|H'\|_{L^\infty}\|\phi''\|_{L^p} \\
&\quad + C(\|h''\|_{L^p} + \|H''\|_{L^p})\big(\|\phi'\|_{\dot{C}^\beta} + \|[\varphi]'\|_{\dot{C}^\beta}\big) \\
&\quad + C|\tilde{c}_*|R\|H''\|_{L^p} + Cr^2(1 + \|H''\|_{L^p})(m_0 + M_0)(1 + \delta R^2)^{1/2}.
\end{aligned} \tag{7.29}
$$

Since $|A| < 1$ and $m_0 + M_0 \ll 1$, we combine (7.25) and (7.29) to obtain that

$$
\begin{aligned}
&\|[\varphi]''\|_{L^p} + \|\phi''\|_{L^p} \\
&\quad\leqslant C(\|h''\|_{L^p} + \|H''\|_{L^p})(\|[\varphi]'\|_{\dot{C}^\beta} + \|\phi'\|_{\dot{C}^\beta}) + C|c_*|r\big(\|h''\|_{L^p} + \|H''\|_{L^p}\big) \\
&\qquad + Cr^2\delta^\beta\|h'\|_{\dot{C}^\beta} + Cr^2(1 + \|h''\|_{L^p} + \|H''\|_{L^p})(m_0 + M_0)(1 + \delta R^2)^{1/2},
\end{aligned} \tag{7.30}
$$

where $C = C(\mu, \nu, p, G, \beta)$. Applying Proposition 7.2 yields the desired estimate.

To prove $[\varphi]', \phi' \in W^{1,p}(\mathbb{T})$, we simply define $\tilde{V}$ to be the space of mean-zero $C^\beta \cap W^{1,p}(\mathbb{T})$-functions. One can show that the map in (7.19) is well-defined from $\tilde{V} \times \tilde{V}$ to itself and it is a contraction mapping, provided smallness of h and H. □

Lemma 7.4 *Under the assumptions of Proposition 7.3,*

$$
\begin{aligned}
\|\mathcal{R}_{[\varphi]'}\|_{\dot{C}^\beta} + \|\mathcal{R}_{\phi'}\|_{\dot{C}^\beta} &\leqslant C|c_*|r\big(\|h'\|_{\dot{C}^{\beta'}} + \|H'\|_{\dot{C}^{\beta'}}\big)^2 \\
&\quad + Cr^2\big(\delta^\beta\|h'\|_{\dot{C}^\beta} + (m_0 + M_0)(1 + \delta R^2)^{1/2}\big),
\end{aligned} \tag{7.31}
$$

and

$$
\begin{aligned}
&\|\mathcal{R}_{[\varphi]'}\|_{\dot{W}^{1,p}} + \|\mathcal{R}_{\phi'}\|_{\dot{W}^{1,p}} \\
&\quad\leqslant C|c_*|r(\|h''\|_{L^p} + \|H''\|_{L^p})\big(\|h'\|_{\dot{C}^\beta} + \|H'\|_{\dot{C}^\beta}\big) \\
&\qquad + Cr^2(1 + \|h''\|_{L^p} + \|H''\|_{L^p})\big(\delta^\beta\|h'\|_{\dot{C}^\beta} + (m_0 + M_0)(1 + \delta R^2)^{1/2}\big) \\
&\quad =: \tilde{N}_{2,p},
\end{aligned} \tag{7.32}
$$

where $C = C(\mu, \nu, p, G, \beta, \beta')$.

Proof. The estimates immediately follow by combining (7.13), (7.17), (7.24) and (7.28) with Proposition 7.2, Proposition 7.3 and Lemma 3.1. ☐

8. Local existence

In this section, we prove existence of local solutions of (2.16)–(2.18).

8.1 *Preliminaries*

Inspired by (2.45) and (2.46), we may rewrite (2.16) and (2.17) as

$$\partial_t h + \frac{c_*}{r} = -\frac{Ac_*}{r}(-\Delta)^{1/2}h - \frac{1+A}{2r^2}\mathcal{H}\mathcal{S}\phi' + \frac{1}{r}\mathcal{R}_h, \tag{8.1}$$

$$\partial_t H + \frac{\tilde{c}_*}{R} = \frac{\tilde{c}_*}{R}(-\Delta)^{1/2}H - \frac{1}{R^2}\mathcal{H}\mathcal{S}[\varphi]' + \frac{1}{R}\mathcal{R}_H, \tag{8.2}$$

where

$$\begin{aligned}\mathcal{R}_h := &-\frac{1}{f}\gamma'(\theta)\cdot\mathcal{K}_\gamma\mathcal{R}_{[\varphi]'}\\ &-\left(\frac{1}{f}\gamma'(\theta)\cdot\mathcal{K}_\gamma(2Ac_*f' + A\mathcal{S}\phi') - \frac{1}{2r}\mathcal{H}(2Ac_*f' + A\mathcal{S}\phi')\right)\\ &+\left(\frac{1}{f}\nabla(\Gamma * g)|_\gamma\cdot\gamma'(\theta)^\perp + c_*\right) - \left(\frac{1}{f}\gamma'(\theta)\cdot\mathcal{K}_{\gamma,\tilde{\gamma}}\phi' - \frac{1}{2r}\mathcal{H}\mathcal{S}\phi'\right),\end{aligned} \tag{8.3}$$

and

$$\begin{aligned}\mathcal{R}_H := &-\frac{1}{F}\tilde{\gamma}'(\theta)\cdot\mathcal{K}_{\tilde{\gamma}}\mathcal{R}_{\phi'}\\ &-\left(\frac{1}{F}\tilde{\gamma}'(\theta)\cdot\mathcal{K}_{\tilde{\gamma}}(-2\tilde{c}_*F' + \mathcal{S}[\varphi]') - \frac{1}{2R}\mathcal{H}(-2\tilde{c}_*F' + \mathcal{S}[\varphi]')\right)\\ &+\left(\frac{1}{F}\nabla(\Gamma * g)|_{\tilde{\gamma}}\cdot\tilde{\gamma}'(\theta)^\perp + \tilde{c}_*\right) - \left(\frac{1}{F}\tilde{\gamma}'(\theta)\cdot\mathcal{K}_{\tilde{\gamma},\gamma}[\varphi]' - \frac{1}{2R}\mathcal{H}\mathcal{S}[\varphi]'\right).\end{aligned} \tag{8.4}$$

For future use, we also denote

$$\tilde{\mathcal{R}}_h := -\frac{1+A}{2r}\mathcal{H}\mathcal{S}\phi' + \mathcal{R}_h, \quad \tilde{\mathcal{R}}_H := -\frac{1}{R}\mathcal{H}\mathcal{S}[\varphi]' + \mathcal{R}_H. \tag{8.5}$$

We need estimates for $\mathcal{R}_h$ and $\mathcal{R}_H$.

Lemma 8.1 *Under the assumptions of Proposition 7.3,*

$$r\|\mathcal{R}_h\|_{\dot{W}^{1,p}} + R\|\mathcal{R}_H\|_{\dot{W}^{1,p}} \leqslant C\tilde{N}_{2,p}, \tag{8.6}$$

where $C = C(\mu, \nu, p, G, \beta, \beta')$.

Proof. By (7.1), $\mathcal{R}_{[\varphi]'}$ has zero integral on $\mathbb{T}$. By Lemma 5.3 and Lemma 7.4,

$$\|\gamma'(\theta)\cdot\mathcal{K}_\gamma\mathcal{R}_{[\varphi]'}\|_{\dot W^{1,p}} \leqslant C\|\mathcal{R}_{[\varphi]'}\|_{\dot W^{1,p}} + C\|h''\|_{L^p}\|\mathcal{R}_{[\varphi]'}\|_{\dot C^\beta} \leqslant C\tilde N_{2,p}. \tag{8.7}$$

When γ' and ψ are Hölder continuous on $\mathbb{T}$ and h satisfies the smallness assumption, one can rigorously show that

$$\gamma'\cdot\mathcal{K}_\gamma\psi = \frac{d}{d\theta}\left[\frac{1}{2\pi}\int_{\mathbb{T}}\ln|\gamma(\theta)-\gamma(\theta')|\psi(\theta')\,d\theta'\right], \tag{8.8}$$

and thus it has mean zero on $\mathbb{T}$. Hence, by Poincaré inequality and (8.7),

$$\|f^{-1}\gamma'(\theta)\cdot\mathcal{K}_\gamma\mathcal{R}_{[\varphi]'}\|_{\dot W^{1,p}} \leqslant Cr^{-1}\tilde N_{2,p}. \tag{8.9}$$

Similarly,

$$\begin{aligned}
&\left\|\frac{1}{f}\gamma'(\theta)\cdot\mathcal{K}_\gamma(2Ac_*f'+A\mathcal{S}\phi') - \frac{1}{2r}\mathcal{H}(2Ac_*f'+A\mathcal{S}\phi')\right\|_{\dot W^{1,p}}\\
&\quad\leqslant \left\|\frac{1}{f}\left(\gamma'(\theta)\cdot\mathcal{K}_\gamma(2Ac_*f'+A\mathcal{S}\phi') - \frac{1}{2}\mathcal{H}(2Ac_*f'+A\mathcal{S}\phi')\right)\right\|_{\dot W^{1,p}}\\
&\qquad\quad + \left\|\left(\frac{1}{2f}-\frac{1}{2r}\right)\mathcal{H}(2Ac_*f'+A\mathcal{S}\phi')\right\|_{\dot W^{1,p}}\\
&\quad\leqslant Cr^{-1}\|h''\|_{L^p}\|2Ac_*f'+A\mathcal{S}\phi'\|_{\dot C^\beta} + Cr^{-1}m_0\|2Ac_*f'+A\mathcal{S}\phi'\|_{\dot W^{1,p}}\\
&\quad\leqslant Cr^{-1}\tilde N_{2,p}.
\end{aligned} \tag{8.10}$$

By Lemmas 3.3, 4.4 and 4.7,

$$\begin{aligned}
&\|f^{-1}\nabla(\Gamma*g)|_\gamma\cdot\gamma'(\theta)^\perp + c_*\|_{\dot W^{1,p}}\\
&\quad\leqslant C\|f'/f\|_{\dot W^{1,p}}\|e_\theta\cdot\nabla(\Gamma*g)|_\gamma\|_{L^\infty} + C\|f'/f\|_{L^\infty}\|e_\theta\cdot\nabla(\Gamma*g)|_\gamma\|_{\dot W^{1,p}}\\
&\qquad\quad + \|e_r\cdot\nabla(\Gamma*g)|_\gamma\|_{\dot W^{1,p}}\\
&\quad\leqslant Cr\|h''\|_{L^p}\big(m_0\delta|\ln\delta| + \|\nabla(\tilde p-p_*)\|_{L^2(B_r)}\big)\\
&\qquad\quad + Cr\big(m_\beta + \|\nabla(\tilde p-p_*)\|_{L^2(B_r)}\big)\\
&\quad\leqslant Cr^{-1}\tilde N_{2,p}.
\end{aligned} \tag{8.11}$$

Finally, by Lemmas 6.1, 6.3 and 6.5,

$$\begin{aligned}
&\left\|\frac{1}{f}\gamma'(\theta)\cdot\mathcal{K}_{\gamma,\tilde\gamma}\phi' - \frac{1}{2r}\mathcal{H}\mathcal{S}\phi'\right\|_{\dot W^{1,p}}\\
&\quad\leqslant Cr^{-1}\left\|fe_\theta\cdot\mathcal{K}_{\gamma,\tilde\gamma}\phi' - \frac{1}{2}\mathcal{H}\mathcal{S}\phi'\right\|_{\dot W^{1,p}} + Cr^{-1}\|f'e_r\cdot\mathcal{K}_{\gamma,\tilde\gamma}\phi'\|_{\dot W^{1,p}}\\
&\qquad\quad + Cr^{-1}\|h\|_{W^{1,\infty}}\|\mathcal{H}\mathcal{S}\phi'\|_{\dot W^{1,p}}\\
&\quad\leqslant Cr^{-1}\big(\|H''\|_{L^p}\|\phi'\|_{L^\infty} + (m_0+M_0)\|\phi''\|_{L^p}\big) + Cr^{-1}\|h''\|_{L^p}\|\phi'\|_{L^\infty}\\
&\quad\leqslant Cr^{-1}\tilde N_{2,p}.
\end{aligned} \tag{8.12}$$

Combining these estimates with (8.3), we obtain the estimate for $\mathcal{R}_h$ in (8.6).

The estimate for $\mathcal{R}_H$ can be derived in a similar manner. □

We shall also need bounds for integrals of $\mathcal{R}_h$ and $\mathcal{R}_H$ on $\mathbb{T}$.

Lemma 8.2 *Under the assumptions of Proposition 7.3,*

$$r\left|\int_{\mathbb{T}} \mathcal{R}_h \, d\theta\right| + R\left|\int_{\mathbb{T}} \mathcal{R}_H \, d\theta\right| \leqslant C(\|h\|_{L^\infty} + \|H\|_{L^\infty})N_{2,p} + Cr^2(m_0+M_0)(1+\delta R^2)^{1/2}, \tag{8.13}$$

where $C = C(\mu, \nu, p, G, \beta, \beta')$.

Proof. We shall again use the fact that, provided γ', $\tilde{\gamma}'$ and ψ to be Hölder continuous on $\mathbb{T}$,

$$(\gamma' \cdot \mathcal{K}_\gamma \psi),\ (\tilde{\gamma}' \cdot \mathcal{K}_{\tilde{\gamma}} \psi),\ (\gamma' \cdot \mathcal{K}_{\gamma,\tilde{\gamma}} \psi),\ (\tilde{\gamma}' \cdot \mathcal{K}_{\tilde{\gamma},\gamma} \psi) \text{ have integrals 0 on } \mathbb{T}. \tag{8.14}$$

This is because they all can be represented as θ-derivatives of certain quantities as in (8.8).

Applying this fact to (8.3),

$$\begin{aligned}\int_{\mathbb{T}} \mathcal{R}_h \, d\theta = & \int_{\mathbb{T}} \left(\frac{1}{r} - \frac{1}{f}\right)\left(\gamma'(\theta) \cdot \mathcal{K}_\gamma(\mathcal{R}_{[\varphi]'} + 2Ac_* f' + A\mathcal{S}\phi')\right) d\theta \\ & + \int_{\mathbb{T}} (-e_r \cdot \nabla(\Gamma * g)|_\gamma + c_*) \, d\theta + \int_{\mathbb{T}} \frac{f'}{f} e_\theta \cdot \nabla(\Gamma * g)|_\gamma \, d\theta \\ & + \int_{\mathbb{T}} \left(\frac{1}{r} - \frac{1}{f}\right) \gamma'(\theta) \cdot \mathcal{K}_{\gamma,\tilde{\gamma}} \phi' \, d\theta. \end{aligned} \tag{8.15}$$

By (7.1), Poincaré inequality, Lemmas 3.3, 4.4, 5.3, 6.1, 6.5 and 7.1, as well as Propositions 7.2 and 7.3, we derive that

$$\begin{aligned}\left|\int_{\mathbb{T}} \mathcal{R}_h \, d\theta\right| \leqslant & Cr^{-1}\|h\|_{L^\infty}\|\gamma'(\theta) \cdot \mathcal{K}_\gamma[\varphi]'\|_{\dot{W}^{1,p}} \\ & + C\|e_r \cdot \nabla(\Gamma * g)|_\gamma - c_*\|_{L^\infty} + C\|h'\|_{L^\infty}\|e_\theta \cdot \nabla(\Gamma * g)|_\gamma\|_{L^\infty} \\ & + Cr^{-1}\|h\|_{L^\infty}\left(\|h'\|_{L^\infty}\|fe_r \cdot \mathcal{K}_{\gamma,\tilde{\gamma}}\phi'\|_{L^\infty} + \|fe_\theta \cdot \mathcal{K}_{\gamma,\tilde{\gamma}}\phi'\|_{L^\infty}\right) \\ \leqslant & Cr^{-1}\|h\|_{L^\infty}\left(\|h''\|_{L^p}\|[\varphi]'\|_{\dot{C}^\beta} + \|[\varphi]''\|_{L^p}\right) \\ & + Cr\left(m_0\delta|\ln\delta| + (m_0+M_0)(\delta R^2)^{1/2}\right) \\ & + Cr^{-1}\|h\|_{L^\infty}\left(\delta^{-1}(\|h\|_{L^\infty} + \|H\|_{L^\infty})\|\phi'\|_{L^\infty} + \|\phi'\|_{\dot{C}^\beta}\right) \\ \leqslant & Cr^{-1}\|h\|_{L^\infty}N_{2,p} + Cr(m_0+M_0)(1+\delta R^2)^{1/2}, \end{aligned} \tag{8.16}$$

where $C = C(\mu, \nu, p, G, \beta, \beta')$.

The estimate for the $\int_{\mathbb{T}} \mathcal{R}_H$ can be derived similarly. □

8.2 *Proof of existence of local solutions*

Now we are ready to show existence of local solutions.

Proof of Theorem 2.1. The proof is an application of the Schauder fixed-point theorem.

STEP 1 (Setup) Let δ be chosen according to (2.23). Also recall that $\alpha = 1 - \frac{2}{p}$, and $\varepsilon > 0$ and M are given in (2.24). We assume $M \leqslant 1$. The exact smallness of M will be specified later.

With $0 < T \leqslant \min\{1, \delta M\}$ to be determined, we define

$$\begin{aligned} X_{M,T} := \Big\{ & v \in L^p_{[0,T]} W^{2,p} \cap C_{[0,T]} C^{1,\alpha}(\mathbb{T}) : \ v_t \in L^p_{[0,T]} W^{1,p}(\mathbb{T}), \\ & v|_{t=0} = 0, \ \|v\|_{C_{[0,T]} L^\infty(\mathbb{T})} \leqslant \delta M, \\ & \|v\|_{L^p_{[0,T]} \dot{W}^{2,p}(\mathbb{T})} + \|v\|_{C_{[0,T]} \dot{C}^{1,\alpha}(\mathbb{T})} + \|v_t\|_{L^p_{[0,T]} \dot{W}^{1,p}(\mathbb{T})} \leqslant \delta^{-\alpha+\varepsilon} M \Big\}. \end{aligned} \tag{8.17}$$

$X_{M,T}$ is a non-empty, convex, closed subset of $\{v \in C_{[0,T]} C^{1,\alpha}(\mathbb{T}) : \ v_t \in L^p_{[0,T]} W^{1,p}(\mathbb{T})\}$. Take $\alpha' \in (0, \alpha)$ to be determined. Denote

$$Z := L^\infty_{[0,T]} C^{1,\alpha'}(\mathbb{T}). \tag{8.18}$$

By Aubin–Lions Lemma [46], the embedding

$$\big\{v \in C_{[0,T]} C^{1,\alpha}(\mathbb{T}) : \ v_t \in L^p_{[0,T]} W^{1,p}(\mathbb{T})\big\} \hookrightarrow Z \tag{8.19}$$

is compact, so $X_{M,T}$ is compact in Z. In what follows, we shall apply Schauder fixed-point theorem on

$$Y_{M,T} := \left(e^{-\frac{Ac_*}{r} t(-\Delta)^{1/2}} h_0 - \frac{c_* t}{r} + X_{M,T} \right) \times \left(e^{\frac{\tilde{c}_*}{R} t(-\Delta)^{1/2}} H_0 - \frac{\tilde{c}_* t}{R} + X_{M,T} \right), \tag{8.20}$$

which is a non-empty, convex, compact subset of $Z \times Z$.

STEP 2 (Estimates for elements in $Y_{M,T}$) Take $(h, H) \in Y_{M,T}$. By the definition of $X_{M,T}$ and Lemma 3.1,

$$\|h\|_{C_{[0,T]} L^\infty(\mathbb{T})} \leqslant \left\| e^{-\frac{Ac_*}{r} t(-\Delta)^{1/2}} h_0 \right\|_{L^\infty(\mathbb{T})} + \frac{|c_*| T}{r} + \delta M \leqslant C(G) \delta M. \tag{8.21}$$

By the definition of the $\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})$-seminorm in (2.20),

$$\|h\|_{L^p_{[0,T]} \dot{W}^{2,p}(\mathbb{T})} \leqslant \left(\frac{r}{|Ac_*|} \right)^{\frac{1}{p}} \|h_0\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})} + \delta^{-\alpha+\varepsilon} M \leqslant C(p, \mu, \nu, r/|c_*|) \delta^{-\alpha+\varepsilon} M. \tag{8.22}$$

Moreover, $W^{2-\frac{1}{p},p}(\mathbb{T}) \hookrightarrow h^{1,\alpha}(\mathbb{T})$ [47, § 2.7], where $h^{1,\alpha}(\mathbb{T})$ is the closure of $C^\infty(\mathbb{T})$ in the $C^{1,\alpha}$-topology. So $e^{-\frac{Ac_*}{r} t(-\Delta)^{1/2}} h_0$ is continuous in t valued in $C^{1,\alpha}(\mathbb{T})$ and hence

$$\|h\|_{C_{[0,T]} \dot{C}^{1,\alpha}(\mathbb{T})} \leqslant \|h_0\|_{\dot{C}^{1,\alpha}(\mathbb{T})} + \delta^{-\alpha+\varepsilon} M \leqslant C(p) \delta^{-\alpha+\varepsilon} M. \tag{8.23}$$

Applying interpolation to (8.21) and (8.23) yields

$$\|h\|_{C_{[0,T]} \dot{C}^{1,\beta'}(\mathbb{T})} \leqslant C(G, p) \delta^{1 - \frac{1+\alpha-\varepsilon}{1+\alpha}(1+\beta')} M. \tag{8.24}$$

Hence, taking

$$\beta' = \frac{\varepsilon}{1 + \alpha - \varepsilon}, \tag{8.25}$$

we find that

$$\|h\|_{C_{[0,T]}\dot{C}^{1,\beta'}(\mathbb{T})} \leqslant C(G,p)M. \tag{8.26}$$

Similarly,

$$\|H\|_{C_{[0,T]}L^\infty(\mathbb{T})} \leqslant C(G)\delta M, \tag{8.27}$$

$$\|H\|_{L^p_{[0,T]}\dot{W}^{2,p}(\mathbb{T})} \leqslant C(p,R/|\tilde{c}_*|)\delta^{-\alpha+\varepsilon}M, \tag{8.28}$$

$$\|H\|_{C_{[0,T]}\dot{C}^{1,\alpha}(\mathbb{T})} \leqslant C(p)\delta^{-\alpha+\varepsilon}M, \tag{8.29}$$

and, with the same β' as above,

$$\|H\|_{C_{[0,T]}\dot{C}^{1,\beta'}(\mathbb{T})} \leqslant C(G,p)M. \tag{8.30}$$

In what follows, we shall assume M to be suitably small, which depends on p and G, so that (8.21), (8.23), (8.26), (8.27), (8.29) and (8.30) implies that for $(h,H)\in Y_{M,T}$,

$$\sup_{t\in[0,T]}\big(m_{1,\alpha}+M_{1,\alpha}+\|h'\|_{\dot{C}^{\beta'}}+\|H'\|_{\dot{C}^{\beta'}}\big) \leqslant C(G,p)M \ll 1. \tag{8.31}$$

STEP 3 (Construction of a map on $Y_{M,T}$) Inspired by (8.1) and (8.2), for given $(h,H)\in Y_{M,T}$, we let $(h_\dagger,H_\dagger)$ solve

$$\partial_t h_\dagger = -\frac{Ac_*}{r}(-\Delta)^{1/2}h_\dagger + \frac{1}{r}\tilde{\mathcal{R}}_h, \quad h_\dagger|_{t=0}=0, \tag{8.32}$$

$$\partial_t H_\dagger = \frac{\tilde{c}_*}{R}(-\Delta)^{1/2}H_\dagger + \frac{1}{R}\tilde{\mathcal{R}}_H, \quad H_\dagger|_{t=0}=0. \tag{8.33}$$

Recall that $\tilde{\mathcal{R}}_h$ and $\tilde{\mathcal{R}}_H$ are defined in (8.5), which are uniquely determined by (h,H) via (2.33) (cf. Proposition 7.3), (2.34), (8.3) and (8.4). Then let

$$(\tilde{h},\tilde{H}) = \left(e^{-\frac{Ac_*}{r}t(-\Delta)^{1/2}}h_0 - \frac{c_*t}{r} + h_\dagger,\ e^{\frac{\tilde{c}_*}{R}t(-\Delta)^{1/2}}H_0 - \frac{\tilde{c}_*t}{R} + H_\dagger\right). \tag{8.34}$$

A fixed-point of the map $\mathcal{T}:(h,H)\mapsto(\tilde{h},\tilde{H})$ is then a solution of (8.1) and (8.2).

We shall show that $\mathcal{T}$ is continuous from $Y_{M,T}$ to itself in the topology of $Z\times Z$ and then apply Schauder fixed-point theorem. It suffices to prove that:

- the map $\mathcal{T}':(h,H)\mapsto(h_\dagger,H_\dagger)$ is well-defined as a continuous function on $Y_{M,T}$ in the topology of $Z\times Z$, and
- $(h_\dagger,H_\dagger)\in X_{M,T}\times X_{M,T}$ for properly chosen M and T.

STEP 4 (Continuity of $\mathcal{T}'$) We choose $\alpha'<\alpha''<\min\{\frac{1}{4},\alpha\}$. By (8.1) and (8.2),

$$(\tilde{\mathcal{R}}_h,\tilde{\mathcal{R}}_H) = \big(r\partial_t h + c_* + Ac_*(-\Delta)^{1/2}h,\ R\partial_t H + \tilde{c}_* - \tilde{c}_*(-\Delta)^{1/2}H\big). \tag{8.35}$$

By (8.31) and Lemma 3.4, provided that $M \ll 1$ which depends on p, G and α'', for any pair $(h_1,H_1),(h_2,H_2)\in Y_{M,T}$,

$$\|\tilde{\mathcal{R}}_{h_1}-\tilde{\mathcal{R}}_{h_2}\|_{L^\infty_{[0,T]}C^{\alpha''}(\mathbb{T})} + \|\tilde{\mathcal{R}}_{H_1}-\tilde{\mathcal{R}}_{H_2}\|_{L^\infty_{[0,T]}C^{\alpha''}(\mathbb{T})} \leqslant C(\alpha'',\mu,\nu,r,R,G)\cdot d_{\alpha''}\big((h_1,H_1),(h_2,H_2)\big), \tag{8.36}$$

where

$$d_{\alpha''}\big((h_1,H_1),(h_2,H_2)\big) := \|h_1-h_2\|_{L^\infty_{[0,T]}C^{1,\alpha''}(\mathbb{T})} + \|H_1-H_2\|_{L^\infty_{[0,T]}C^{1,\alpha''}(\mathbb{T})}. \tag{8.37}$$

We abbreviate it as $d_{\alpha''}$ if it incurs no confusion. By taking $h_2 = H_2 = 0$ in (8.36) which corresponds to $\tilde{\mathcal{R}}_{h_2} = \tilde{\mathcal{R}}_{H_2} = 0$, we show that $\tilde{\mathcal{R}}_{h_1}, \tilde{\mathcal{R}}_{H_1} \in L^\infty_{[0,T]}C^{\alpha''}(\mathbb{T})$; so are $\tilde{\mathcal{R}}_{h_2}$ and $\tilde{\mathcal{R}}_{H_2}$. Following a similar argument, we may apply Lemma 3.4 to different time slices of h_i and H_i, and use the time continuity $h_i, H_i \in C_{[0,T]}C^{1,\alpha''}(\mathbb{T})$ to prove $\tilde{\mathcal{R}}_{h_i}, \tilde{\mathcal{R}}_{H_i} \in C_{[0,T]}C^{\alpha''}(\mathbb{T})$.

Let $(h_{i,\dagger}, H_{i,\dagger})$ $(i=1,2)$ be the unique solution of (8.32) and (8.33) in $Z\times Z$ corresponding to $(h_i,H_i)\in Y_{M,T}$. By Lemma A.7 and (8.36),

$$\|h_{1,\dagger}-h_{2,\dagger}\|_{C_{[0,T]}\dot{C}^{1,\alpha'}(\mathbb{T})} \leqslant C(\alpha',\alpha'',\mu,\nu,r,R,G)\cdot d_{\alpha''}. \tag{8.38}$$

On the other hand, let $\bar{h}_{i,\dagger} = \frac{1}{2\pi}\int_{\mathbb{T}} h_{i,\dagger}\,d\theta$. By (8.32) and (8.36),

$$\|\bar{h}_{1,\dagger}-\bar{h}_{2,\dagger}\|_{C_{[0,T]}L^\infty(\mathbb{T})} \leqslant Cr^{-1}\|\tilde{\mathcal{R}}_{h_1}-\tilde{\mathcal{R}}_{h_2}\|_{C_{[0,T]}C^{\alpha''}(\mathbb{T})} \leqslant C(\alpha'',\mu,\nu,r,R,G)\cdot d_{\alpha''}. \tag{8.39}$$

Combining this with (8.38), we use interpolation as well as (8.21), (8.23), (8.27) and (8.29) to derive that

$$\begin{aligned}\|h_{1,\dagger}-h_{2,\dagger}\|_{C_{[0,T]}C^{1,\alpha'}(\mathbb{T})} &\leqslant C(\alpha',\alpha'',\mu,\nu,r,R,G)\cdot d_{\alpha'}^{\theta}d_{\alpha}^{1-\theta}\\ &\leqslant C(\alpha',\alpha'',p,\mu,\nu,r,R,G)\cdot d_{\alpha'}^{\theta},\end{aligned} \tag{8.40}$$

where $\theta = \frac{\alpha-\alpha''}{\alpha-\alpha'}$. Similarly, $\|H_{1,\dagger}-H_{2,\dagger}\|_{C_{[0,T]}C^{1,\alpha'}(\mathbb{T})}$ enjoys the same bound. This proves (Hölder) continuity of $\mathcal{T}'$ in $Y_{M,T}$ in the topology of $Z\times Z$.

In fact, if one improves Lemma A.7, it can be shown that $\mathcal{T}'$ is log-Lipschitz continuous in $Y_{M,T}$ in the topology of $Z\times Z$. We omit the details although it may be of independent interest.

STEP 5 (Justification of $(h_\dagger, H_\dagger)\in X_{M,T}\times X_{M,T}$) Let β' be taken as before, and let $\beta = \frac{\beta'}{4} < \frac{\beta'}{1+\beta'}$. It is not difficult to show that

$$\|\mathcal{H}\mathcal{S}\psi'\|_{\dot{W}^{1,p}} \leqslant C\|\mathcal{S}\psi'\|_{\dot{W}^{1,p}} \leqslant C\delta^{\beta-1+\frac{1}{p}}\|\psi'\|_{\dot{C}^\beta}. \tag{8.41}$$

Combining with Lemma 8.1,

$$\|\tilde{\mathcal{R}}_h\|_{\dot{W}^{1,p}} + \|\tilde{\mathcal{R}}_H\|_{\dot{W}^{1,p}} \leqslant Cr^{-1}\big(\tilde{N}_{2,p} + \delta^{\beta-1+\frac{1}{p}}N_{1,\beta}\big), \tag{8.42}$$

Then we derive by Lemma 3.1, Proposition 7.2, Lemma 7.4, (8.22), (8.28) and (8.31) that

$$\begin{aligned}&\|r^{-1}\tilde{\mathcal{R}}_h\|_{L^p_{[0,T]}\dot{W}^{1,p}} + \|R^{-1}\tilde{\mathcal{R}}_H\|_{L^p_{[0,T]}\dot{W}^{1,p}}\\ &\leqslant C|c_*|r^{-1}(\|h''\|_{L^p_{[0,T]}L^p} + \|H''\|_{L^p_{[0,T]}L^p})\sup_{t\in[0,T]}\big(\|h'\|_{\dot{C}^\beta}+\|H'\|_{\dot{C}^\beta}\big)\\ &\quad + C(T^{1/p} + \|h''\|_{L^p_{[0,T]}L^p} + \|H''\|_{L^p_{[0,T]}L^p})\sup_{t\in[0,T]}\big(\delta^\beta\|h'\|_{\dot{C}^\beta} + (m_0+M_0)(1+\delta R^2)^{1/2}\big)\\ &\quad + C\delta^{\beta-1+\frac{1}{p}}T^{1/p}|c_*|r^{-1}\sup_{t\in[0,T]}\big(\|h'\|_{\dot{C}^\beta}+\|H'\|_{\dot{C}^\beta}\big)\\ &\quad + C\delta^{\beta-1+\frac{1}{p}}T^{1/p}\sup_{t\in[0,T]}\big(\delta^\beta\|h'\|_{\dot{C}^\beta} + (m_0+M_0)(1+\delta R^2)^{1/2}\big)\\ &\leqslant C\delta^{-\alpha+\varepsilon}M^2(1+\delta R^2)^{1/2} + C\delta^{\beta-1+\frac{1}{p}}T^{1/p}M(1+\delta R^2)^{1/2},\end{aligned} \tag{8.43}$$

where $C = C(p, \varepsilon, \mu, \nu, R/|\tilde{c}_*|, G)$. Here we rewrote the β- and β'-dependence into dependence on p and ε, and used the fact that $r/|c_*| \leqslant R/|\tilde{c}_*|$. In particular, C does not deteriorate as δ becomes smaller. Hence,

$$\|r^{-1}\tilde{\mathcal{R}}_h\|_{L^p_{[0,T]}\dot{W}^{1,p}} + \|R^{-1}\tilde{\mathcal{R}}_H\|_{L^p_{[0,T]}\dot{W}^{1,p}} \leqslant C\big(\delta^{\beta-1+\frac{1}{p}}T^{1/p} + \delta^{-\alpha+\varepsilon}M\big)M, \tag{8.44}$$

where $C = C(p, \varepsilon, \mu, \nu, R/|\tilde{c}_*|, G, \delta R^2)$.

To this end, applying Lemma A.5 and Lemma A.6 to (8.32) and (8.33), we obtain that

$$\begin{aligned}\|h_\dagger\|_{L^p_{[0,T]}\dot{W}^{2,p}} + \|\partial_t h_\dagger\|_{L^p_{[0,T]}\dot{W}^{1,p}} + \|h_\dagger\|_{C_{[0,T]}\dot{C}^{1,\alpha}}&\\ + \|H_\dagger\|_{L^p_{[0,T]}\dot{W}^{2,p}} + \|\partial_t H_\dagger\|_{L^p_{[0,T]}\dot{W}^{1,p}} + \|H_\dagger\|_{C_{[0,T]}\dot{C}^{1,\alpha}}&\\ \leqslant C\big(\delta^{\beta-1+\frac{1}{p}}T^{1/p} + \delta^{-\alpha+\varepsilon}M\big)M.&\end{aligned} \tag{8.45}$$

Here the universal constant C has the same dependence as above. Now we take

$$M \leqslant M_*(p, \varepsilon, \mu, \nu, R/|\tilde{c}_*|, G, \delta R^2) \ll 1, \tag{8.46}$$

$$T \leqslant T_*(\delta, p, \varepsilon, \mu, \nu, R/|\tilde{c}_*|, G, \delta R^2) \ll 1, \tag{8.47}$$

so that (8.45) becomes

$$\begin{aligned}\|h_\dagger\|_{L^p_{[0,T]}\dot{W}^{2,p}} + \|\partial_t h_\dagger\|_{L^p_{[0,T]}\dot{W}^{1,p}} + \|h_\dagger\|_{C_{[0,T]}\dot{C}^{1,\alpha}}&\\ + \|H_\dagger\|_{L^p_{[0,T]}\dot{W}^{2,p}} + \|\partial_t H_\dagger\|_{L^p_{[0,T]}\dot{W}^{1,p}} + \|H_\dagger\|_{C_{[0,T]}\dot{C}^{1,\alpha}}&\\ \leqslant \delta^{-\alpha+\varepsilon}M.&\end{aligned} \tag{8.48}$$

Note that the smallness needed for M will not be more stringent as δ becomes smaller.

Finally, we show $(h_\dagger, H_\dagger)$ satisfies the $C_{[0,T]}L^\infty(\mathbb{T})$-bound in the definition (8.17) of $X_{M,T}$. By Lemma 8.2, Sobolev inequality and (8.42),

$$\begin{aligned}\|r^{-1}\tilde{\mathcal{R}}_h\|_{L^\infty} + \|R^{-1}\tilde{\mathcal{R}}_H\|_{L^\infty} &\leqslant Cr^{-2}(\|h\|_{L^\infty} + \|H\|_{L^\infty})N_{2,p} + C(m_0 + M_0)(1 + \delta R^2)^{1/2}\\ &\quad + Cr^{-2}(\tilde{N}_{2,p} + \delta^{\beta-1+\frac{1}{p}}N_{1,\beta})\\ &\leqslant Cr^{-2}(\tilde{N}_{2,p} + \delta^{\beta-1+\frac{1}{p}}N_{1,\beta}).\end{aligned} \tag{8.49}$$

Following (8.43) and (8.44),

$$\|r^{-1}\tilde{\mathcal{R}}_h\|_{L^1_{[0,T]}L^\infty} + \|R^{-1}\tilde{\mathcal{R}}_H\|_{L^1_{[0,T]}L^\infty} \leqslant CT^{1-\frac{1}{p}}\big(\delta^{\beta-1+\frac{1}{p}}T^{1/p} + \delta^{-\alpha+\varepsilon}M\big)M. \tag{8.50}$$

Combining this with (8.32) and (8.33), we use the fact $\|e^{-t(-\Delta)^{1/2}}\|_{L^\infty\to L^\infty} \leqslant 1$ to obtain that

$$\|h_\dagger\|_{C_{[0,T]}L^\infty(\mathbb{T})} + \|H_\dagger\|_{C_{[0,T]}L^\infty(\mathbb{T})} \leqslant CT^{1-\frac{1}{p}}\big(\delta^{\beta-1+\frac{1}{p}}T^{1/p} + \delta^{-\alpha+\varepsilon}M\big)M, \tag{8.51}$$

where $C = C(p, \varepsilon, \mu, \nu, R/|\tilde{c}_*|, G, \delta R^2)$. Take T_* in (8.47) even smaller if necessary, so that the required $C_{[0,T]}L^\infty(\mathbb{T})$-bound for $(h_\dagger, H_\dagger)$ in (8.17) is achieved.

This shows that $\mathcal{T}'$ has its image $(h_\dagger, H_\dagger)$ in $X_{M,T} \times X_{M,T}$.

STEP 6 (Existence and estimates) By Schauder fixed-point theorem, the map $\mathcal{T}$ has a fixed-point $(h, H) \in Y_{M,T}$, which is a mild solution of (8.1) and (8.2). Moreover, the pointwise well-definedness of $\partial_t h$ and $\partial_t H$ has been readily shown in Step 4, as they are at least in $C_{[0,T]}C^{\alpha''}(\mathbb{T})$, where $\alpha'' < \min\{\frac{1}{4}, \alpha\}$ is arbitrary. Therefore, (h, H) is a strong solution of (8.1) and (8.2).

Estimates for h and H follow from (8.21)–(8.23) and (8.27)–(8.29). For $\partial_t h$ and $\partial_t H$, we derive by (8.34), (8.48) and the definition of $W^{2-\frac{1}{p},p}(\mathbb{T})$-space (2.20),

$$\begin{aligned}\|\partial_t h\|_{L^p_{[0,T]}\dot{W}^{1,p}} &\leqslant \|\partial_t e^{-\frac{Ac*}{r}t(-\Delta)^{1/2}} h_0\|_{L^p_{[0,T]}\dot{W}^{1,p}} + \|\partial_t h_\dagger\|_{L^p_{[0,T]}\dot{W}^{1,p}} \\ &\leqslant C(\mu,\nu,p,G)\|h_0\|_{\dot{W}^{2-\frac{1}{p},p}} + \delta^{-\alpha+\varepsilon}M,\end{aligned} \tag{8.52}$$

and similarly,

$$\|\partial_t H\|_{L^p_{[0,T]}\dot{W}^{1,p}} \leqslant C(\mu,\nu,p,G)\|H_0\|_{\dot{W}^{2-\frac{1}{p},p}} + \delta^{-\alpha+\varepsilon}M. \tag{8.53}$$

□

8.3 *Continuation of the local solutions*

A local solution can be extended to longer time intervals as long as $f(T)$ and $F(T)$ still satisfy the smallness assumption (2.24) on the initial data. We start with the following lemma that links estimates for $f(T)$ and $F(T)$ when they are treated as new initial datum, with the estimates for f_0 and F_0.

Lemma 8.3 *Under the assumptions of Theorem 2.1 with M_* suitably small, let f and F be a local solution over $[0,T]$. Define $f_1(\theta) = f(\theta,T)$ and $F_1(\theta) = F(\theta,T)$. Let*

$$r_1 := \frac{1}{2\pi}\int_{\mathbb{T}} f_1(\theta)\,d\theta, \quad R_1 := \frac{1}{2\pi}\int_{\mathbb{T}} F_1(\theta)\,d\theta, \tag{8.54}$$

and according to (2.19),

$$h_1(\theta) := \frac{f_1}{r_1} - 1, \quad H_1(\theta) := \frac{F_1}{R_1} - 1. \tag{8.55}$$

Let

$$\delta_1 = \frac{1-\frac{r_1}{R_1}}{1-\frac{r}{R}}\cdot\delta. \tag{8.56}$$

Then r_1, R_1 and δ_1 satisfy (2.23). Moreover, with some universal constant $\tilde{C} = \tilde{C}(p,\varepsilon,G)$,

$$\delta_1^{-1}(\|h_1\|_{L^\infty(\mathbb{T})} + \|H_1\|_{L^\infty(\mathbb{T})}) + \delta_1^{\alpha-\varepsilon}\left(\|h_1\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})} + \|H_1\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})}\right) \leqslant \tilde{C}(p,\varepsilon,G)M, \tag{8.57}$$

where M is defined in (2.24) with h_0, H_0 and δ.

Proof. That r_1, R_1 and δ_1 satisfy (2.23) is obvious since r, R and δ satisfy (2.23).

To show (8.57), we first study $h(T)$ and $H(T)$. Note that (2.26) readily provides

$$\delta^{-1}\big(\|h(T)\|_{L^\infty} + \|H(T)\|_{L^\infty}\big) \leqslant C(p,G)M. \tag{8.58}$$

A bound for $W^{2-\frac{1}{p},p}$-seminorm of $h(T)$ and $H(T)$ may be derived as follows. Denote $h_* = h - e^{-\frac{Ac_*}{r}t(-\Delta)^{1/2}}h_0$ and $H_* = H - e^{\frac{\tilde{c}_*}{R}t(-\Delta)^{1/2}}H_0$. By (8.34) and (8.48), they satisfy

$$\partial_x h_*, \partial_x H_* \in W^{1,p}([0,T]\times\mathbb{T}) \tag{8.59}$$

and

$$h_*|_{t=0} = H_*|_{t=0} = \partial_x h_*|_{t=0} = \partial_x H_*|_{t=0} = 0. \tag{8.60}$$

We make zero extension of h_* and H_* to the region $t < 0$ while still denote the extension to be h_* and H_*. Then the above properties imply that $\partial_x h_*, \partial_x H_* \in W^{1,p}((-\infty,T]\times\mathbb{T})$. By trace theorem (see, e.g., [47, §2.7.2]) and (8.48),

$$\begin{aligned}
&\|\partial_x h_*(T)\|_{\dot{W}^{1-\frac{1}{p},p}(\mathbb{T})} + \|\partial_x H_*(T)\|_{\dot{W}^{1-\frac{1}{p},p}(\mathbb{T})} \\
&\qquad \leqslant C\left(\|\partial_x h_*\|_{\dot{W}^{1,p}((-\infty,T]\times\mathbb{T})} + \|\partial_x H_*\|_{\dot{W}^{1,p}((-\infty,T]\times\mathbb{T})}\right) \\
&\qquad \leqslant C\left(\|h_*\|_{L^p_{[0,T]}\dot{W}^{2,p}(\mathbb{T})} + \|H_*\|_{L^p_{[0,T]}\dot{W}^{2,p}(\mathbb{T})}\right) \\
&\qquad\qquad + C\left(\|\partial_t h_*\|_{L^p_{[0,T]}\dot{W}^{1,p}(\mathbb{T})} + \|\partial_t H_*\|_{L^p_{[0,T]}\dot{W}^{1,p}(\mathbb{T})}\right) \\
&\qquad \leqslant C\delta^{-\alpha+\varepsilon}M.
\end{aligned} \tag{8.61}$$

It is noteworthy that the constants C may only depend on p but not on T. On the other hand, by the definition (2.20) of the $W^{2-\frac{1}{p},p}(\mathbb{T})$-seminorm,

$$\begin{aligned}
&\|e^{-\frac{Ac_*}{r}t(-\Delta)^{1/2}}h_0(T)\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})} + \|e^{-\frac{\tilde{c}_*}{R}t(-\Delta)^{1/2}}H_0(T)\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})} \\
&\qquad \leqslant \|h_0\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})} + \|H_0\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})}.
\end{aligned} \tag{8.62}$$

Combining this with (2.24) and (8.61), we conclude that

$$\|h(T)\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})} + \|H(T)\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})} \leqslant C(p)\delta^{-\alpha+\varepsilon}M. \tag{8.63}$$

Thanks to (8.58) and the way r_1 and R_1 are defined

$$\left|\frac{r_1}{r} - 1\right| + \left|\frac{R_1}{R} - 1\right| \leqslant C(p,G)\delta M. \tag{8.64}$$

Assume M_* is already small enough, depending on p and G, to guarantee that the right hand side of the above inequality is sufficiently small and that

$$c_1\delta \leqslant \delta_1 \leqslant c_2\delta \tag{8.65}$$

for some universal $0 < c_1 < 1 < c_2$. Hence,

$$\delta_1^{-1}\big(\|h_1\|_{L^\infty(\mathbb{T})} + \|H_1\|_{L^\infty(\mathbb{T})}\big) \leqslant C\delta^{-1}\big(\|h(T)\|_{L^\infty(\mathbb{T})} + \|H(T)\|_{L^\infty(\mathbb{T})}\big) + C(G)M, \tag{8.66}$$

and

$$\begin{aligned}
&\delta_1^{\alpha-\varepsilon}\Big(\|h_1\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})} + \|H_1\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})}\Big) \\
&\qquad \leqslant C(p,\varepsilon)\delta^{\alpha-\varepsilon}\Big(\|h(T)\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})} + \|H(T)\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})}\Big).
\end{aligned} \tag{8.67}$$

They combined with (8.58) and (8.63) imply (8.57). □

Proof of Corollary 2.2. We would like to construct a local solution over $[0, \tilde{T}]$ by making successive continuations.

STEP 1 (Setup) We can always start with f_0 and F_0 satisfying the smallness condition of Theorem 2.1. To make the notations more systematic, we rewrite r, R and δ in Theorem 2.1 as r_0, R_0 and δ_0, respectively. Let h_0 and H_0 be defined as in (2.19). Since

$$M^0 := \delta_0^{-1}\big(\|h_0\|_{L^\infty(\mathbb{T})} + \|H_0\|_{L^\infty(\mathbb{T})}\big) + \delta_0^{\alpha-\varepsilon}\big(\|h_0\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})} + \|H_0\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})}\big) \leqslant M_{*,0}, \tag{8.68}$$

where according to (8.46),

$$M_{*,0} := M_*\big(p, \varepsilon, \mu, \nu, R_0/|\tilde{c}_*(r_0, R_0)|, G, \delta_0 R_0^2\big), \tag{8.69}$$

by Theorem 2.1, there exists a solution (f^0, F^0) on $[0, t_0]$, where by (8.47),

$$t_0 \leqslant T_*\big(\delta_0, p, \varepsilon, \mu, \nu, R_0/|\tilde{c}_*(r_0, R_0)|, G, \delta_0 R_0^2\big). \tag{8.70}$$

Define $T_0 = t_0$.

Suppose we have obtained a solution on $[0, T_{k-1}]$ for some $k \in \mathbb{Z}_+$. We define

$$f_k = f(T_{k-1}), \quad F_k(t = 0) = F(T_{k-1}), \tag{8.71}$$

$$r_k = \frac{1}{2\pi}\int_{\mathbb{T}} f_k(\theta)\, d\theta, \quad R_k = \frac{1}{2\pi}\int_{\mathbb{T}} F_k(\theta)\, d\theta. \tag{8.72}$$

Also let

$$\delta_k = \frac{1 - \frac{r_k}{R_k}}{1 - \frac{r_{k-1}}{R_{k-1}}} \cdot \delta_{k-1}. \tag{8.73}$$

With this choice, r_k, R_k and δ_k satisfy (2.23). Let h_k and H_k be defined by f_k, F_k, r_k and R_k as in (2.19). Then if

$$M^k := \delta_k^{-1}\big(\|h_k\|_{L^\infty(\mathbb{T})} + \|H_k\|_{L^\infty(\mathbb{T})}\big) + \delta_k^{\alpha-\varepsilon}\big(\|h_k\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})} + \|H_k\|_{\dot{W}^{2-\frac{1}{p},p}(\mathbb{T})}\big) \leqslant M_{*,k}, \tag{8.74}$$

where

$$M_{*,k} := M_*\big(p, \varepsilon, \mu, \nu, R_k/|\tilde{c}_*(r_k, R_k)|, G, \delta_k R_k^2\big), \tag{8.75}$$

Theorem 2.1 claims that there exists a solution (f^k, F^k) on $[0, t_k]$, where by (8.47),

$$t_k \leqslant T_*\big(\delta_k, p, \varepsilon, \mu, \nu, R_k/|\tilde{c}_*(r_k, R_k)|, G, \delta_k R_k^2\big). \tag{8.76}$$

To this end, we let $T_k = T_{k-1} + t_k$, and define $f(t) = f^k(t - T_{k-1})$ and $F(t) = F^k(t - T_{k-1})$ for $t \in [T_{k-1}, T_k]$. Then it is easy to verify that (f, F) is a local strong solution on $[0, T_k]$.

Starting from the initial data, if we are able to make such continuation until $T_K \geqslant \tilde{T}$ for some finite K, then we prove the existence of a strong solution on $[0, \tilde{T}]$. Otherwise,

1. either (8.74) is first violated for some finite K_* (depending on the initial data) with $T_{K_*} < \tilde{T}$;
2. or we are able to make continuation for infinitely many times but still can not reach $\tilde{T}$. This implies that for all $k \in \mathbb{N}$, $T_k < \tilde{T}$ and (8.74) holds, while

$$\lim_{k\to\infty} T_*\big(\delta_k, p, \varepsilon, \mu, \nu, R_k/|\tilde{c}_*(r_k, R_k)|, G, \delta_k R_k^2\big) = 0. \tag{8.77}$$

We are going to show that both of them would not occur if we take initial datum h_0 and H_0 to be sufficiently small.

STEP 2 (A priori estimates for configurations staying almost circular) Consider an arbitrary k such that $T_k < \tilde{T}$ and (8.74) holds for all numbers from 0 to k. We shall first derive upper and lower bounds for r_k and R_k.

Since (8.74) holds, in which M_* is sufficiently small, the inner and outer interfaces at times $T_{-1}, \cdots, T_{k-1}$ are all sufficiently close to circles (we use the convention $T_{-1} = 0$). In this case, we must have $r_k < R_k$ as the interfaces can not cross by the proof of Theorem 2.1. Moreover, with some universal constants c and C,

$$c|\Omega_{T_{k-1}}|^{1/2} \leqslant r_k < R_k \leqslant C|\tilde{\Omega}_{T_{k-1}}|^{1/2}. \tag{8.78}$$

The increment of $|\tilde{\Omega}|$ is due to the growth of the tumor, which provides a naive bound for $|\tilde{\Omega}|$

$$\frac{d}{dt}|\tilde{\Omega}| \leqslant G(0)|\tilde{\Omega}|. \tag{8.79}$$

Therefore, for all such k, r_k and R_k admit an upper bound that only depends on G, $|\tilde{\Omega}_0|$ and $\tilde{T}$. Since the initial data is assumed to satisfy the smallness condition (8.68), $|\tilde{\Omega}_0|$ is comparable with R_0^2 up to universal constants. Hence, the $|\tilde{\Omega}_0|$-dependence can be rewritten as R_0-dependence. We note that Lemma 3.1 may provide a better upper bound that depends linearly on T, but the naive bound here is enough for this qualitative discussion. On the other hand, because of the growth of the tumor, $|\Omega_{T_{k-1}}| \geqslant |\Omega_0|$. This gives a positive lower bound for r_k and R_k that only depends on $|\Omega_0|$, and thus only on r_0 by the same reasoning as above.

To this end, we note that $R/|\tilde{c}_*(r, R)|$ is a continuous function in $r, R \in \mathbb{R}_+$. The continuity can be justified using Lemma 3.4 with $h_1 = H_1 = 0$ and h_2 and H_2 being small constants. Indeed, $|\tilde{c}_*(r, R)|$ is the speed of the outer interface when the interfaces are concentric circles with radii r and R, respectively. Therefore, for all such k, $R_k/|\tilde{c}_*(r_k, R_k)|$ admits positive lower and upper bounds depending only on μ, ν, G, r_0, R_0 and $\tilde{T}$.

By Remark 3.1, $\delta_k R_k^2$ has lower and upper bounds that only depend on $|\tilde{\Omega}_0 \backslash \Omega_0|$. This together with the bound for R_k implies that δ_k has positive lower and upper bounds only depending on G, r_0, R_0 and $\tilde{T}$.

By the proof of Theorem 2.1 (cf. (8.47) and (8.51)), T_* has continuous dependence on δ, $R/|\tilde{c}_*|$ and δR^2. Combining all the facts above, there is a universal $T_{**} = T_{**}(\mu, \nu, G, r_0, R_0, \tilde{T}) > 0$, such that for all such k,

$$T_*\big(\delta_k, p, \varepsilon, \mu, \nu, R_k/|\tilde{c}_*(r_k, R_k)|, G, \delta_k R_k^2\big) \geqslant T_{**}. \tag{8.80}$$

This contradicts with (8.77), so case (2) above is ruled out.

Similarly, there exists a universal $M_{**} = M_{**}(\mu, \nu, G, r_0, R_0, \tilde{T}) > 0$ such that for all such k,

$$M_*\big(p, \varepsilon, \mu, \nu, R_k/|\tilde{c}_*(r_k, R_k)|, G, \delta_k R_k^2\big) \geqslant M_{**}. \tag{8.81}$$

STEP 3 (Estimates for total number of continuations) It suffices to consider the case (1) above.

Thanks to (8.80), if (8.74) always holds, we only need to make continuation for finitely many times to cover the time interval $[0, \tilde{T}]$. To be more precise, by choosing the longest possible lifespan of the local solution in each stage of continuation, we can have $T_N \geqslant \tilde{T}$ for some N that admits an upper bound

$$N \leqslant N_{**}(\mu, \nu, G, r_0, R_0, \tilde{T}), \tag{8.82}$$

provided that (8.74) is not violated along the way. In order to make (8.74) hold for N_{**} times, we take M sufficiently small (recall that M is defined by h_0, H_0 and δ_0 in (2.24)), such that

$$\tilde{C}(p, \varepsilon, G)^{N_{**}} \cdot M \leqslant M_{**}, \tag{8.83}$$

where $\tilde{C}$ is given in Lemma 8.3 and M_{**} is introduced in (8.81). Note that the required smallness for M only depends on μ, ν, G, r_0, R_0 and $\tilde{T}$. With (8.83), it is easy to justify by Lemma 8.3 that (8.74) will always be satisfied before the solution is extended beyond $\tilde{T}$.

This completes the proof. □

9. Uniqueness

In this section, we prove uniqueness of the local solution under the additional assumption $G \in C^{1,1}$.

9.1 *Basic setup*

We start with basic setups that will be used throughout this section. Let $p \in (2, \infty)$ and $\varepsilon > 0$ as in Theorem 2.1, and $\alpha = 1 - \frac{2}{p}$. Let β' be defined in (8.25) and $\beta = \beta'/4$ as in the proof of local existence (see step 5). In particular, $\beta < \frac{\beta'}{1+\beta'}$ and $\beta < \frac{1}{4}$.

Suppose there are two solutions f_i and F_i $(i = 1, 2)$ of (2.16)–(2.18) with regularity and estimates given in Theorem 2.1. We define h_i and H_i $(i = 1, 2)$ as in (2.19). Let $m_{0,i}$, $M_{0,i}$, $m_{\alpha,i}$ and $M_{\alpha,i}$ be defined as in (3.17), (3.18), (3.45) and (3.46), respectively, and let Δm_0, ΔM_0, Δm_α and ΔM_α be defined in (3.47)–(3.50). By virtue of (2.26), by imposing sufficient smallness in (2.24) that depends on G, p and ε, we may assume that for all $t \in [0, T]$, $\gamma_i(t) \subset B_{r(1+\delta)}$ and $\tilde{\gamma}_i(t) \subset B^c_{r(1+5\delta)}$, and

$$m_{0,i} + M_{0,i} + \|h_i\|_{\dot{C}^{\beta'}} + \|H_i\|_{\dot{C}^{\beta'}} \ll 1. \tag{9.1}$$

Later we shall see the smallness needs to depend on p and ε.

Let p_i solve (1.6) and (1.7) in the (time-varying) physical domain that is determined by f_i and F_i. Let $x_i(X)$ be the diffeomorphism between the physical and the (time-invariant) reference domains, determined by h_i and H_i via (3.2), and let $X_i(x)$ be its inverse. Define $\tilde{p}_i(X) := p_i(x_i(X))$ as the pull-back of p_i to the reference domain. Let φ_i be the potential defined in (2.1) corresponding to p_i. Let c_i and $\tilde{c}_i$ be defined as in (7.5).

The idea of proving uniqueness is to first derive bounds for $\tilde{\mathcal{R}}_{h_1} - \tilde{\mathcal{R}}_{h_2}$ and $\tilde{\mathcal{R}}_{H_1} - \tilde{\mathcal{R}}_{H_2}$ (see (8.5)) in terms of $h_1 - h_2$ and $H_1 - H_2$ by following the arguments in previous sections, and then use regularity theory of (8.1) and (8.2) to conclude that $h_1 - h_2$ and $H_1 - H_2$ can only be zero if they initially are. Such a process would be extremely involved if carried out naively, requiring more estimates than we currently have. To slightly reduce the complexity, we shall segregate inner and outer interfaces by a cut-off function in space, which decouples their dynamics in some sense.

With abuse of notation, let $\eta(x)$ be a time-independent, radially symmetric, smooth cut-off function on the physical domain, such that $\eta \in [0, 1]$ in $\mathbb{R}^2$, $\eta \equiv 1$ on $B_{r(1+3\delta)}$, and $\eta \equiv 0$ outside $B_{r(1+4\delta)}$. Moreover, we need $|\nabla \eta| \leqslant C(r\delta)^{-1}$ and $|\nabla^2 \eta| \leqslant C(r\delta)^{-2}$ for some universal C. For $i = 1, 2$, define

$$\psi_i = \eta \varphi_i, \quad \Psi_i = (1 - \eta)\varphi_i.$$

The equation satisfied by ψ_i can be derived from (1.6), (1.7) and (2.1). Proceeding as in Section 2,

$$\begin{aligned}\psi_i &= -\mathcal{D}_{\gamma_i}[\varphi_i] + \Gamma * (G(p_i)\chi_{\Omega_i} - 2\nabla\varphi_i\nabla\eta - \varphi_i\Delta\eta)\\ &= -\mathcal{D}_{\gamma_i}[\varphi_i] + \Gamma * (g_{\psi,i}(X_i)) \quad \text{in } \mathbb{R}^2\backslash\gamma_i,\end{aligned} \tag{9.2}$$

where we define in the reference coordinate

$$g_{\psi,i}(X) = G\big(\tilde{p}_i(X)\big)\chi_{B_r}(X) - 2\nu\nabla\tilde{p}_i(X)\nabla\eta(X) - \nu\tilde{p}_i(X)\Delta\eta(X). \tag{9.3}$$

Note that the last two terms above are only supported on $\overline{B}_{r(1+4\delta)}\backslash B_{r(1+3\delta)}$, where the diffeomorphism is identity. Comparing (2.3) and (9.2), we find

$$-\mathcal{D}_{\tilde{\gamma}_i}\phi_i + \Gamma * (2\nabla\varphi_i\nabla\eta + \varphi_i\Delta\eta) = 0 \quad \text{in } B_{r(1+3\delta)}. \tag{9.4}$$

Hence, we claim that

$$\Psi_i = -\mathcal{D}_{\tilde{\gamma}_i}\phi_i + \Gamma * (2\nabla\varphi_i\nabla\eta + \varphi_i\Delta\eta) \quad \text{in } \tilde{\Omega}_i. \tag{9.5}$$

Indeed, we may first assume $\Psi_i = -\mathcal{D}_{\tilde{\gamma}_i}\Phi_i + \Gamma * (2\nabla\varphi_i\nabla\eta + \varphi_i\Delta\eta)$ for some boundary potential Φ_i to be determined along $\tilde{\gamma}_i$. Then we observe $\mathcal{D}_{\tilde{\gamma}_i}\Phi_i$ and $\mathcal{D}_{\tilde{\gamma}_i}\phi_i$ have to coincide in $B_{r(1+3\delta)}$ because of (9.4) and the fact $\Psi_i = 0$ there. Since $\mathcal{D}_{\tilde{\gamma}_i}\Phi_i$ and $\mathcal{D}_{\tilde{\gamma}_i}\phi_i$ are harmonic inside $\tilde{\Omega}_i$, this proves $\Phi_i = \phi_i$. For convenience, we also introduce

$$g_{\Psi,i}(X) = 2\nu\nabla\tilde{p}_i(X)\nabla\eta(X) + \nu\tilde{p}_i(X)\Delta\eta(X). \tag{9.6}$$

Then (9.5) becomes

$$\Psi_i = -\mathcal{D}_{\tilde{\gamma}_i}\phi_i + \Gamma * g_{\Psi,i}\big(X_i(x)\big) \quad \text{in } \tilde{\Omega}_i. \tag{9.7}$$

This also implies

$$\Gamma * g_{\Psi,i}\big(X_i(x)\big) = \Gamma * \big(G(p_i)\chi_{\Omega_i}\big) - \mathcal{D}_{\gamma_i}[\varphi_i] \quad \text{in } B^c_{r(1+4\delta)}. \tag{9.8}$$

Recall that $[\varphi_i]$ and ϕ_i satisfy (7.1)–(7.4). They can be rewritten as (see (2.33) and (2.34))

$$[\varphi_i]' - 2Ac_* f_i' = \tilde{\mathcal{R}}_{[\varphi_i]'}, \tag{9.9}$$

$$\phi_i' + 2\tilde{c}_* F_i' = \tilde{\mathcal{R}}_{\phi_i'}, \tag{9.10}$$

where

$$\begin{aligned}\tilde{\mathcal{R}}_{[\varphi_i]'} := 2Af_i'(\theta)\Big(e_r\cdot\nabla\big(\Gamma * g_{\psi,i}(X_i)\big)\big|_{\gamma_i} - c_*\Big)\\ + 2Af_i(\theta)e_\theta\cdot\nabla\big(\Gamma * g_{\psi,i}(X_i)\big)\big|_{\gamma_i} + 2A\gamma_i'^{\perp}\cdot\mathcal{K}_{\gamma_i}[\varphi_i]',\end{aligned} \tag{9.11}$$

and

$$\begin{aligned}\tilde{\mathcal{R}}_{\phi_i'} := -2F_i'(\theta)\Big(e_r\cdot\nabla\big(\Gamma * g_{\psi,i}(X_i)\big)\big|_{\tilde{\gamma}_i} - \tilde{c}_*\Big)\\ - 2F_i(\theta)e_\theta\cdot\nabla\big(\Gamma * g_{\psi,i}(X_i)\big)\big|_{\tilde{\gamma}_i} - 2\tilde{\gamma}_i'^{\perp}\cdot\mathcal{K}_{\tilde{\gamma}_i}\phi_i'.\end{aligned} \tag{9.12}$$

On the other hand, following the derivation of (2.16) and (2.17), (8.1)–(8.5) admit the following new representations,

$$\partial_t h_i + \frac{c_*}{r} = -\frac{Ac_*}{r}(-\Delta)^{1/2} h_i + \frac{1}{r}\tilde{\mathcal{R}}_{h_i}, \tag{9.13}$$

$$\partial_t H_i + \frac{\tilde{c}_*}{R} = \frac{\tilde{c}_*}{R}(-\Delta)^{1/2} H_i + \frac{1}{R}\tilde{\mathcal{R}}_{H_i}, \tag{9.14}$$

where

$$\tilde{\mathcal{R}}_{h_i} = -\frac{1}{f_i}\gamma_i' \cdot \mathcal{K}_{\gamma_i}\tilde{\mathcal{R}}_{[\varphi_i]'} - 2Ac_*\left(\frac{1}{f_i}\gamma_i' \cdot \mathcal{K}_{\gamma_i} f_i' - \frac{1}{2r}\mathcal{H} f_i'\right) + \left(\frac{f_i'}{f_i} e_\theta \cdot \nabla\big(\Gamma * g_{\psi,i}(X_i)\big)\big|_{\gamma_i} - e_r \cdot \nabla\big(\Gamma * g_{\psi,i}(X_i)\big)\big|_{\gamma_i} + c_*\right), \tag{9.15}$$

and

$$\tilde{\mathcal{R}}_{H_i} = -\frac{1}{F_i}\tilde{\gamma}_i' \cdot \mathcal{K}_{\tilde{\gamma}_i}\tilde{\mathcal{R}}_{\phi_i'} + 2\tilde{c}_*\left(\frac{1}{F_i}\tilde{\gamma}_i' \cdot \mathcal{K}_{\tilde{\gamma}_i} F_i' - \frac{1}{2R}\mathcal{H} F_i'\right) + \left(\frac{F_i'}{F_i} e_\theta \cdot \nabla\big(\Gamma * g_{\Psi,i}(X_i)\big)\big|_{\tilde{\gamma}_i} - e_r \cdot \nabla\big(\Gamma * g_{\Psi,i}(X_i)\big)\big|_{\tilde{\gamma}_i} + \tilde{c}_*\right). \tag{9.16}$$

(9.13) and (9.14) are coupled with initial data $h_i(t=0) = h_0$ and $H_i(t=0) = H_0$.

9.2 *Estimates for differences of two solutions*

Next we shall bound $\tilde{R}_{h_1} - \tilde{R}_{h_2}$ and $\tilde{R}_{H_1} - \tilde{R}_{H_2}$.

Lemma 9.1 *$g_{\psi,i}$ and $g_{\Psi,i}$ are supported in $B_{r(1+4\delta)}$, satisfying that*

$$\|g_{\psi,i}\|_{L^\infty} + \|g_{\Psi,i}\|_{L^\infty} \leqslant C(\nu, r, R, G), \tag{9.17}$$

$$\|g_{\psi,1} - g_{\psi,2}\|_{L^\infty} + \|g_{\Psi,1} - g_{\Psi,2}\|_{L^\infty} \leqslant C(\beta, \mu, \nu, r, R, G)(\Delta m_0 + \Delta M_0), \tag{9.18}$$

and

$$\|e_\theta \cdot \nabla g_{\psi,i}\|_{L^2} + \|e_\theta \cdot \nabla g_{\Psi,i}\|_{L^2} \leqslant C(\mu, \nu, r, R, G)(m_{0,i} + M_{0,i}), \tag{9.19}$$

$$\|e_\theta \cdot \nabla(g_{\psi,1} - g_{\psi,2})\|_{L^2} + \|e_\theta \cdot \nabla(g_{\Psi,1} - g_{\Psi,2})\|_{L^2} \leqslant C(\beta, \mu, \nu, r, R, G)(\Delta m_\beta + \Delta M_\beta). \tag{9.20}$$

Proof. Note that $\tilde{p}_i$ and p_* are harmonic in a neighborhood (whose size depends on r and R) of the support of $\nabla\eta$, so gradient estimates apply. Then the desired estimates follow from Lemma 3.3 and Lemma 3.5. The assumption $G \in C^{1,1}$ is used when proving the last inequality. □

Proposition 9.2 *Assume (9.1) with the smallness depending on p and β (and thus on p and ε.)*

$$\|[\varphi_1]' - [\varphi_2]'\|_{\dot{C}^\beta} \leqslant Cr^2\big(\|h_1' - h_2'\|_{\dot{C}^\beta} + \Delta m_0 + \Delta M_\beta\big), \tag{9.21}$$

and

$$\|[\varphi_1]'' - [\varphi_2]''\|_{L^p} \leqslant C(p, \varepsilon, \mu, \nu, G) r^2 \|h_1'' - h_2''\|_{L^p}(1 + \delta R^2)^{1/2} + Cr^2\big(\|h_1' - h_2'\|_{\dot{C}^\beta} + \Delta m_0 + \Delta M_\beta\big)\big(1 + \|h_1''\|_{L^p} + \|h_2''\|_{L^p} + \|H_1''\|_{L^p}\big), \tag{9.22}$$

where $C = C(p, \varepsilon, \mu, \nu, r, R, G)$ unless otherwise stated.

Proof. We proceed as in Proposition 7.2 and Proposition 7.3. By Lemmas 3.5, 4.3–4.7, 7.1 and 9.1,

$$\begin{aligned}
&\big\| f_1'(\theta)(e_r\cdot\nabla\big(\Gamma * g_{\psi,1}(X_1)\big)\big|_{\gamma_1} - c_*) - f_2'(\theta)(e_r\cdot\nabla\big(\Gamma * g_{\psi,2}(X_2)\big)\big|_{\gamma_2} - c_*)\big\|_{\dot C^\beta} \\
&\quad\leqslant \big\|(f_1'-f_2')(e_r\cdot\nabla\big(\Gamma * g_{\psi,1}(X_1)\big)\big|_{\gamma_1} - c_*)\big\|_{\dot C^\beta} \\
&\qquad + \big\| f_2'(\theta)(e_r\cdot\nabla\big(\Gamma * g_{\psi,1}(X_1)\big)\big|_{\gamma_1} - e_r\cdot\nabla\big(\Gamma * g_{\psi,1}(X_2)\big)\big|_{\gamma_2})\big\|_{\dot C^\beta} \\
&\qquad + \big\| f_2'(\theta)(e_r\cdot\nabla\big(\Gamma * (g_{\psi,1}-g_{\psi,2})(X_2)\big)\big|_{\gamma_2}\big\|_{\dot C^\beta} \\
&\quad\leqslant Cr^2\|h_1'-h_2'\|_{\dot C^\beta}(m_{0,1}\delta|\ln\delta|\|g_{\psi,1}\|_{L^\infty(B_{(1+4\delta)r})} \\
&\qquad + \|e_\theta\cdot\nabla g_{\psi,1}\|_{L^2(B_{r(1+4\delta)})} + (m_{0,1}+M_{0,1})(\delta R^2)^{1/2}) \\
&\qquad + Cr^2\|h_1'-h_2'\|_{L^\infty}(\|g_{\psi,1}\|_{L^\infty(B_{(1+4\delta)r})}m_{\beta,1} + \|e_\theta\cdot\nabla g_{\psi,1}\|_{L^2(B_{(1+4\delta)r})}) \\
&\qquad + Cr^2\|h_2'\|_{\dot C^\beta}\cdot\delta|\ln\delta|\Delta m_0\|g_{\psi,1}\|_{L^\infty(B_{r(1+4\delta)})} \\
&\qquad + Cr^2\|h_2'\|_{L^\infty}(\|g_{\psi,1}\|_{L^\infty(B_{(1+4\delta)r})}\Delta m_\beta + \Delta m_0\|e_\theta\cdot\nabla g_{\psi,1}\|_{L^2(B_{(1+4\delta)r})}) \\
&\qquad + Cr^2\|h_2'\|_{\dot C^\beta}(m_{0,2}\delta|\ln\delta|\|g_{\psi,1}-g_{\psi,2}\|_{L^\infty(B_{(1+4\delta)r})} \\
&\qquad + \|e_\theta\cdot\nabla(g_{\psi,1}-g_{\psi,2})\|_{L^2(B_{r(1+4\delta)})} + |c_1-c_2|) \\
&\qquad + Cr^2\|h_2'\|_{L^\infty}(\|g_{\psi,1}-g_{\psi,2}\|_{L^\infty(B_{(1+4\delta)r})}m_{\beta,2} + \|e_\theta\cdot\nabla(g_{\psi,1}-g_{\psi,2})\|_{L^2(B_{(1+4\delta)r})}) \\
&\quad\leqslant Cr^2\|h_1'-h_2'\|_{\dot C^\beta}(m_{\beta,1}+M_{0,1}) + Cr^2\|h_2'\|_{\dot C^\beta}(\Delta m_\beta + \Delta M_\beta),
\end{aligned}\tag{9.23}$$

where $C = C(\beta,\mu,\nu,r,R,G)$. Here we used the estimate by (7.5) and Lemma 3.5 that

$$|c_1-c_2| \leqslant \frac{C}{r}\int_{B_r}|G(\tilde p_1)-G(\tilde p_2)|\,dX \leqslant Cr\|\tilde p_1-\tilde p_2\|_{L^\infty(B_r)} \leqslant Cr(\Delta m_0+\Delta M_0), \tag{9.24}$$

where $C = C(\beta,\mu,\nu,r,R,G)$. Similarly,

$$\begin{aligned}
&\big\| f_1(\theta)e_\theta\cdot\nabla\big(\Gamma * g_{\psi,1}(X_1)\big)\big|_{\gamma_1} - f_2(\theta)e_\theta\cdot\nabla\big(\Gamma * g_{\psi,2}(X_2)\big)\big|_{\gamma_2}\big\|_{\dot C^\beta} \\
&\qquad\leqslant Cr^2(\Delta m_\beta+\Delta M_\beta).
\end{aligned}\tag{9.25}$$

On the other hand, by (9.1), Lemma 5.1 and Lemma 5.4,

$$\begin{aligned}
&\|\gamma_1'^{\perp}\cdot\mathcal{K}_{\gamma_1}[\varphi_1]' - \gamma_2'^{\perp}\cdot\mathcal{K}_{\gamma_2}[\varphi_2]'\|_{\dot C^\beta} \\
&\qquad\leqslant C(\beta)\big(\|h_1-h_2\|_{C^{1,\beta}}\|[\varphi_1]'\|_{\dot C^\beta} + \|h_2'\|_{\dot C^\beta}\|[\varphi_1]'-[\varphi_2]'\|_{\dot C^\beta}\big).
\end{aligned}\tag{9.26}$$

Combining these estimates with (9.1), (9.9), (9.11) and Proposition 7.2 yields

$$\begin{aligned}
&\|\tilde{\mathcal{R}}_{[\varphi_1]'} - \tilde{\mathcal{R}}_{[\varphi_2]'}\|_{\dot C^\beta} \\
&\quad\leqslant Cr^2\|h_1'-h_2'\|_{\dot C^\beta}(m_{0,1}+M_{0,1}+\|h_1'\|_{\dot C^\beta}+\|H_1'\|_{\dot C^\beta}) + Cr^2(\Delta m_\beta+\Delta M_\beta) \\
&\qquad + C(\beta)\|h_2'\|_{\dot C^\beta}\|[\varphi_1]'-[\varphi_2]'\|_{\dot C^\beta},
\end{aligned}\tag{9.27}$$

and

$$\|[\varphi_1]'-[\varphi_2]'\|_{\dot C^\beta} \leqslant Cr^2\|h_1'-h_2'\|_{\dot C^\beta} + Cr^2(\Delta m_\beta+\Delta M_\beta), \tag{9.28}$$

where $C = C(\beta,\beta',\mu,\nu,r,R,G)$. Note that β and β' essentially depend on p and ε.

To show (9.22), we derive as in (7.22) that

$$\begin{aligned}
&\big\| f_1'(\theta)(e_r\cdot\nabla\big(\Gamma * g_{\psi,1}(X_1)\big)|_{\gamma_1}-c_*) - f_2'(\theta)(e_r\cdot\nabla\big(\Gamma * g_{\psi,2}(X_2)\big)\big|_{\gamma_2}-c_*)\big\|_{\dot W^{1,p}}\\
&\quad\leqslant \|f_1'-f_2'\|_{\dot W^{1,p}}\big\|e_r\cdot\nabla\big(\Gamma * g_{\psi,1}(X_1)\big)\big|_{\gamma_1}-c_*\big\|_{L^\infty}\\
&\qquad\quad + \|f_1'-f_2'\|_{L^\infty}\big\|e_r\cdot\nabla\big(\Gamma * g_{\psi,1}(X_1)\big)\big|_{\gamma_1}\big\|_{\dot W^{1,p}}\\
&\qquad\quad + \Big\|f_2'(\theta)\Big(e_r\cdot\nabla\big(\Gamma * g_{\psi,1}(X_1)\big)\big|_{\gamma_1}-e_r\cdot\nabla\big(\Gamma * g_{\psi,1}(X_2)\big)\big|_{\gamma_2}\Big)\Big\|_{\dot W^{1,p}}\\
&\qquad\quad + \Big\|f_2'(\theta)e_r\cdot\nabla\big(\Gamma * (g_{\psi,1}-g_{\psi,2})(X_2)\big)\big|_{\gamma_2}\Big\|_{\dot W^{1,p}}\\
&\quad\leqslant \|f_1'-f_2'\|_{\dot W^{1,p}}\big\|e_r\cdot\nabla\big(\Gamma * g_{\psi,1}(X_1)\big)\big|_{\gamma_1}-c_*\big\|_{L^\infty}\\
&\qquad\quad + Cr^2\|h_1'-h_2'\|_{L^\infty}(m_{\beta,1}+M_{0,1}) + Cr^2\|h_2''\|_{L^p}(\Delta m_\beta+\Delta M_\beta).
\end{aligned}\tag{9.29}$$

We shall need an estimate for $\|e_r\cdot\nabla(\Gamma * g_{\psi,1}(X_1))|_{\gamma_1}-c_*\|_{L^\infty}$ with explicit r- and R-dependence. By (9.4), and then (2.11), Lemmas 3.3, 4.4, 6.1, 6.5, 7.1 and Proposition 7.2,

$$\begin{aligned}
&\big\|e_r\cdot\nabla\big(\Gamma * g_{\psi,1}(X_1)\big)|_{\gamma_1}-c_*\big\|_{L^\infty}\\
&\quad\leqslant \big\|e_r\cdot\nabla\big(\Gamma * (G(p_1)\chi_{\Omega_1})\big)|_{\gamma_1}-c_*\big\|_{L^\infty} + \|e_r\cdot\nabla(\mathcal{D}_{\tilde\gamma_1}\phi_1)|_{\gamma_1}\|_{L^\infty}\\
&\quad\leqslant \big\|e_r\cdot\nabla\big(\Gamma * (G(p_1)\chi_{\Omega_1})\big)|_{\gamma_1}-c_1\big\|_{L^\infty} + |c_1-c_*| + \|e_\theta\cdot\mathcal{K}_{\gamma_1,\tilde\gamma_1}\phi_1'\|_{L^\infty}\\
&\quad\leqslant C(\mu,\nu,G,\beta,\beta')r\big(\|h_1'\|_{\dot C^\beta}+\|H_1'\|_{\dot C^\beta}+(m_{0,1}+M_{0,1})(1+\delta R^2)^{1/2}\big).
\end{aligned}\tag{9.30}$$

Hence,

$$\begin{aligned}
&\big\| f_1'(\theta)\big(e_r\cdot\nabla\big(\Gamma * g_{\psi,1}(X_1)\big)\big|_{\gamma_1}-c_*\big) - f_2'(\theta)(e_r\cdot\nabla\big(\Gamma * g_{\psi,2}(X_2)\big)\big|_{\gamma_2}-c_*)\big\|_{\dot W^{1,p}}\\
&\quad\leqslant C(\mu,\nu,G,\beta,\beta')r^2\|h_1''-h_2''\|_{L^p}\big(\|h_1'\|_{\dot C^\beta}+\|H_1'\|_{\dot C^\beta}+(m_{0,1}+M_{0,1})(1+\delta R^2)^{1/2}\big)\\
&\qquad\qquad + Cr^2\|h_1'-h_2'\|_{L^\infty}(m_{\beta,1}+M_{0,1}) + Cr^2\|h_2''\|_{L^p}(\Delta m_\beta+\Delta M_\beta).
\end{aligned}\tag{9.31}$$

On the other hand,

$$\begin{aligned}
&\big\| f_1(\theta)e_\theta\cdot\nabla\big(\Gamma * g_{\psi,1}(X_1)\big)\big|_{\gamma_1} - f_2(\theta)e_\theta\cdot\nabla\big(\Gamma * g_{\psi,2}(X_2)\big)\big|_{\gamma_2}\big\|_{\dot W^{1,p}}\\
&\qquad\qquad\qquad\qquad \leqslant Cr^2(\Delta m_\beta+\Delta M_\beta),
\end{aligned}\tag{9.32}$$

and by Lemma 5.2 and Lemma 5.5,

$$\begin{aligned}
&\|\gamma_1'^{\perp}\cdot\mathcal{K}_{\gamma_1}[\varphi_1]' - \gamma_2'^{\perp}\cdot\mathcal{K}_{\gamma_2}[\varphi_2]'\|_{\dot W^{1,p}}\\
&\quad\leqslant C\big(\Delta m_0+\|h_1'-h_2'\|_{\dot C^\beta}\big)\big(\|h_1''\|_{L^p}+\|h_2''\|_{L^p}\big)\|[\varphi_1]'\|_{\dot C^\beta}\\
&\qquad + C\|h_1''-h_2''\|_{L^p}\|[\varphi_1]'\|_{\dot C^\beta} + C\Delta m_0\|[\varphi_1]''\|_{L^p}\\
&\qquad + C\big(\|h_2''\|_{L^p}\|[\varphi_1]'-[\varphi_2]'\|_{\dot C^\beta}+\|h_2'\|_{L^\infty}\|[\varphi_1]''-[\varphi_2]''\|_{L^p}\big),
\end{aligned}\tag{9.33}$$

where $C = C(p,\beta)$. Combining these estimates with (9.1), (9.9), (9.11), (9.21) as well as Propositions 7.2 and 7.3, we can show

$$\begin{aligned}
&\|\tilde{\mathcal{R}}_{[\varphi_1]'}-\tilde{\mathcal{R}}_{[\varphi_2]'}\|_{\dot W^{1,p}}\\
&\quad\leqslant C(p,\varepsilon,\mu,\nu,G)r^2\|h_1''-h_2''\|_{L^p}\big(\|h_1'\|_{\dot C^\beta}+\|H_1'\|_{\dot C^\beta}+(m_{0,1}+M_{0,1})(1+\delta R^2)^{1/2}\big)\\
&\qquad + Cr^2\big(\|h_1'-h_2'\|_{\dot C^\beta}+\Delta m_0+\Delta M_\beta\big)\big(1+\|h_1''\|_{L^p}+\|h_2''\|_{L^p}+\|H_1''\|_{L^p}\big)\\
&\qquad + C(p,\beta)\|h_2'\|_{L^\infty}\|[\varphi_1]''-[\varphi_2]''\|_{L^p},
\end{aligned}\tag{9.34}$$

and thus (9.22). □

Proposition 9.3 *Under the assumption of Proposition 9.2,*

$$\|\phi_1' - \phi_2'\|_{\dot{C}^\beta} \leqslant Cr^2\big(\|H_1' - H_2'\|_{\dot{C}^\beta} + \Delta m_\beta + \Delta M_0\big), \tag{9.35}$$

and

$$\begin{aligned}\|\phi_1'' - \phi_2''\|_{L^p} &\leqslant C(p,\varepsilon,\mu,\nu,G)r^2\|H_1'' - H_2''\|_{L^p}(1+\delta R^2)^{1/2} \\ &\quad + Cr^2\big(\Delta m_\beta + \Delta M_0 + \|H_1' - H_2'\|_{\dot{C}^\beta}\big)\big(1 + \|h_1''\|_{L^p} + \|H_1''\|_{L^p} + \|H_2''\|_{L^p}\big),\end{aligned} \tag{9.36}$$

where $C = C(p,\varepsilon,\mu,\nu,r,R,G)$ unless otherwise stated.

Proof. We justify as before. By Lemmas 3.5, 4.8–4.12, 7.1 and 9.1,

$$\begin{aligned}&\big\|F_1'(\theta)\big(e_r\cdot\nabla\big(\Gamma * g_{\Psi,1}(X_1)\big)\big|_{\tilde{\gamma}_1} - \tilde{c}_*\big) - F_2'(\theta)\big(e_r\cdot\nabla\big(\Gamma * g_{\Psi,2}(X_2)\big)\big|_{\tilde{\gamma}_2} - \tilde{c}_*\big)\big\|_{\dot{C}^\beta} \\ &\qquad \leqslant Cr^2\|H_1' - H_2'\|_{\dot{C}^\beta}(m_{0,1} + M_{0,1}) + Cr^2\|H_2'\|_{\dot{C}^\beta}(\Delta m_\beta + \Delta M_\beta),\end{aligned} \tag{9.37}$$

where $C = C(\beta,\mu,\nu,r,R,G)$. Here we used the fact that, by (4.76), (7.5) and (9.6), with σ being the unit outer normal vector of $\partial B_{r(1+3\delta)}$,

$$\tilde{c}_{g_{\Psi,i}} = -\frac{1}{2\pi R}\int_{B_{r(1+4\delta)}\setminus B_{r(1+3\delta)}} \nu\Delta(\eta\tilde{p}_i)\,dX = \frac{\nu}{2\pi R}\int_{\partial B_{r(1+3\delta)}} \frac{\partial\tilde{p}_i}{\partial\sigma}\,dy = \tilde{c}_i,$$

which yields by (9.24) that

$$|\tilde{c}_1 - \tilde{c}_2| \leqslant \frac{Cr^2}{R}(\Delta m_0 + \Delta M_0). \tag{9.38}$$

Similarly,

$$\begin{aligned}&\big\|F_1(\theta)e_\theta\cdot\nabla\big(\Gamma * g_{\Psi,1}(X_1)\big)\big|_{\tilde{\gamma}_1} - F_2(\theta)e_\theta\cdot\nabla\big(\Gamma * g_{\Psi,2}(X_2)\big)\big|_{\tilde{\gamma}_2}\big\|_{\dot{C}^\beta} \\ &\qquad \leqslant Cr^2(\Delta m_\beta + \Delta M_\beta).\end{aligned} \tag{9.39}$$

Again by (9.1), Lemma 5.1 and Lemma 5.4,

$$\begin{aligned}&\|\tilde{\gamma}_1'^{\perp}\cdot\mathcal{K}_{\tilde{\gamma}_1}\phi_1' - \tilde{\gamma}_2'^{\perp}\cdot\mathcal{K}_{\tilde{\gamma}_2}\phi_2'\|_{\dot{C}^\beta} \\ &\qquad \leqslant C(\beta)\big(\|H_1 - H_2\|_{C^{1,\beta}}\|\phi_1'\|_{\dot{C}^\beta} + \|H_2'\|_{\dot{C}^\beta}\|\phi_1' - \phi_2'\|_{\dot{C}^\beta}\big).\end{aligned} \tag{9.40}$$

By (9.12), (9.37), (9.39), (9.40) and Proposition 7.2,

$$\begin{aligned}&\|\tilde{\mathcal{R}}_{\phi_1'} - \tilde{\mathcal{R}}_{\phi_2'}\|_{\dot{C}^\beta} \\ &\quad \leqslant Cr^2\|H_1' - H_2'\|_{\dot{C}^\beta}\big(\|h_1'\|_{\dot{C}^\beta} + \|H_1'\|_{\dot{C}^\beta} + m_{0,1} + M_{0,1}\big) + Cr^2(\Delta m_\beta + \Delta M_\beta) \\ &\qquad + C(\beta)\|H_2'\|_{\dot{C}^\beta}\|\phi_1' - \phi_2'\|_{\dot{C}^\beta},\end{aligned} \tag{9.41}$$

where $C = C(p,\beta,\mu,\nu,r,R,G)$. Combining this with (9.10) yields (9.35).

In addition, thanks to (9.8),

$$
\begin{aligned}
&\big\|F_1'(\theta)(e_r\cdot\nabla\big(\Gamma * g_{\Psi,1}(X_1)\big)\big|_{\tilde{\gamma}_1}-\tilde{c}_*)-F_2'(\theta)(e_r\cdot\nabla\big(\Gamma * g_{\Psi,2}(X_2)\big)\big|_{\tilde{\gamma}_2}-\tilde{c}_*)\big\|_{\dot{W}^{1,p}}\\
&\quad\leqslant R\|H_1''-H_2''\|_{L^p}\|e_r\cdot\nabla\big(\Gamma * g_{\Psi,1}(X_1)\big)\big|_{\tilde{\gamma}_1}-\tilde{c}_*\|_{L^\infty}\\
&\qquad\qquad+Cr^2\|H_1'-H_2'\|_{L^\infty}(m_{0,1}+M_{0,1})+Cr^2\|H_2''\|_{L^p}(\Delta m_\beta+\Delta M_\beta)\\
&\quad\leqslant C(\mu,\nu,G,\beta,\beta')r^2\|H_1''-H_2''\|_{L^p}\big(\|h_1'\|_{\dot{C}^\beta}+\|H_1'\|_{\dot{C}^\beta}+(m_{0,1}+M_{0,1})(1+\delta R^2)^{1/2}\big)\\
&\qquad\qquad+Cr^2\|H_1'-H_2'\|_{L^\infty}(m_{0,1}+M_{0,1})+Cr^2\|H_2''\|_{L^p}(\Delta m_\beta+\Delta M_\beta),
\end{aligned}\tag{9.42}
$$

and

$$
\begin{aligned}
&\big\|F_1(\theta)e_\theta\cdot\nabla\big(\Gamma * g_{\Psi,1}(X_1)\big)\big|_{\tilde{\gamma}_1}-F_2(\theta)e_\theta\cdot\nabla\big(\Gamma * g_{\Psi,2}(X_2)\big)\big|_{\tilde{\gamma}_2}\big\|_{\dot{W}^{1,p}}\\
&\qquad\qquad\leqslant Cr^2(\Delta m_\beta+\Delta M_\beta).
\end{aligned}\tag{9.43}
$$

By Lemma 5.2 and Lemma 5.5,

$$
\begin{aligned}
&\|\tilde{\gamma}_1'^{\perp}\cdot\mathcal{K}_{\tilde{\gamma}_1}\phi_1'-\tilde{\gamma}_2'^{\perp}\cdot\mathcal{K}_{\tilde{\gamma}_2}\phi_2'\|_{\dot{W}^{1,p}}\\
&\qquad\leqslant C(\Delta M_0+\|H_1'-H_2'\|_{\dot{C}^\beta})(\|H_1''\|_{L^p}+\|H_2''\|_{L^p})\|\phi_1'\|_{\dot{C}^\beta}\\
&\qquad\qquad+C(p,\beta)\|H_1''-H_2''\|_{L^p}\|\phi_1'\|_{\dot{C}^\beta}+C\Delta M_0\|\phi_1''\|_{L^p}\\
&\qquad\qquad+C(p,\beta)\big(\|H_2''\|_{L^p}\|\phi_1'-\phi_2'\|_{\dot{C}^\beta}+\|H_2'\|_{L^\infty}\|\phi_1''-\phi_2''\|_{L^p}\big).
\end{aligned}\tag{9.44}
$$

Combining (9.42)–(9.44) with (9.35) and Propositions 7.2 and 7.3, we find

$$
\begin{aligned}
&\|\tilde{\mathcal{R}}_{\phi_1'}-\tilde{\mathcal{R}}_{\phi_2'}\|_{\dot{W}^{1,p}}\\
&\quad\leqslant C(p,\varepsilon,\mu,\nu,G)r^2\|H_1''-H_2''\|_{L^p}\big(\|h_1'\|_{\dot{C}^\beta}+\|H_1'\|_{\dot{C}^\beta}+(m_{0,1}+M_{0,1})(1+\delta R^2)^{1/2}\big)\\
&\qquad+Cr^2\big(\|H_1'-H_2'\|_{\dot{C}^\beta}+\Delta m_\beta+\Delta M_0\big)\big(1+\|h_1''\|_{L^p}+\|H_1''\|_{L^p}+\|H_2''\|_{L^p}\big)\\
&\qquad+C(p,\beta)\|H_2'\|_{L^\infty}\|\phi_1''-\phi_2''\|_{L^p}.
\end{aligned}\tag{9.45}
$$

Then (9.36) follows from (9.10) and (9.45). □

Lemma 9.4 *Under the assumption of Proposition 9.2,*

$$
\begin{aligned}
\|\tilde{\mathcal{R}}_{[\varphi_1]'}-\tilde{\mathcal{R}}_{[\varphi_2]'}\|_{\dot{C}^\beta}&\leqslant Cr^2(\Delta m_\beta+\Delta M_\beta)\\
&\quad+Cr^2\|h_1-h_2\|_{\dot{C}^{1,\beta}}\big(\|h_1'\|_{\dot{C}^\beta}+\|h_2'\|_{\dot{C}^\beta}+\|H_1'\|_{\dot{C}^\beta}+m_{0,1}+M_{0,1}\big),
\end{aligned}\tag{9.46}
$$

$$
\begin{aligned}
\|\tilde{\mathcal{R}}_{[\varphi_1]'}-\tilde{\mathcal{R}}_{[\varphi_2]'}\|_{\dot{W}^{1,p}}&\leqslant C(p,\varepsilon,\mu,\nu,G)r^2\|h_1''-h_2''\|_{L^p}\\
&\qquad\cdot\big(\|h_1'\|_{\dot{C}^\beta}+\|H_1'\|_{\dot{C}^\beta}+(m_{0,1}+m_{0,2}+M_{0,1})(1+\delta R^2)^{1/2}\big)\\
&\quad+Cr^2\big(\|h_1'-h_2'\|_{\dot{C}^\beta}+\Delta m_0+\Delta M_\beta\big)\big(1+\|h_1''\|_{L^p}+\|h_2''\|_{L^p}+\|H_1''\|_{L^p}\big),
\end{aligned}\tag{9.47}
$$

$$
\begin{aligned}
\|\tilde{\mathcal{R}}_{\phi_1'}-\tilde{\mathcal{R}}_{\phi_2'}\|_{\dot{C}^\beta}&\leqslant Cr^2(\Delta m_\beta+\Delta M_\beta)\\
&\quad+Cr^2\|H_1-H_2\|_{\dot{C}^{1,\beta}}\big(\|h_1'\|_{\dot{C}^\beta}+\|H_1'\|_{\dot{C}^\beta}+\|H_2'\|_{\dot{C}^\beta}+m_{0,1}+M_{0,1}\big),
\end{aligned}\tag{9.48}
$$

and

$$\|\tilde{\mathcal{R}}_{\phi_1'} - \tilde{\mathcal{R}}_{\phi_2'}\|_{\dot{W}^{1,p}} \leqslant C(p,\varepsilon,\mu,\nu,G)r^2\|H_1''-H_2''\|_{L^p} \cdot \big(\|h_1'\|_{\dot{C}^\beta} + \|H_1'\|_{\dot{C}^\beta} + (m_{0,1}+M_{0,1}+M_{0,2})(1+\delta R^2)^{1/2}\big) + Cr^2\big(\|H_1'-H_2'\|_{\dot{C}^\beta} + \Delta m_\beta + \Delta M_0\big)\big(1+\|h_1''\|_{L^p}+\|H_1''\|_{L^p}+\|H_2''\|_{L^p}\big), \quad (9.49)$$

where $C = C(p,\varepsilon,\mu,\nu,r,R,G)$ *unless otherwise stated.*

Proof. It suffices to apply Proposition 9.2 and Proposition 9.3 to (9.27), (9.34), (9.41) and (9.45). □

Lemma 9.5 *Under the assumption of Proposition 9.2,*

$$\|\tilde{\mathcal{R}}_{h_1} - \tilde{\mathcal{R}}_{h_2}\|_{W^{1,p}} \leqslant C(p,\varepsilon,\mu,\nu,G)r\|h_1''-h_2''\|_{L^p} \cdot \big(\|h_1'\|_{\dot{C}^\beta} + \|h_2'\|_{\dot{C}^\beta} + \|H_1'\|_{\dot{C}^\beta} + (m_{0,1}+m_{0,2}+M_{0,1})(1+\delta R^2)^{1/2}\big) + Cr\big(\|h_1'-h_2'\|_{\dot{C}^\beta} + \Delta m_0 + \Delta M_\beta\big)\big(1+\|h_1''\|_{L^p}+\|h_2''\|_{L^p}+\|H_1''\|_{L^p}\big), \quad (9.50)$$

and

$$\|\tilde{\mathcal{R}}_{H_1} - \tilde{\mathcal{R}}_{H_2}\|_{W^{1,p}} \leqslant C(p,\varepsilon,\mu,\nu,G)R^{-1}r^2\|H_1''-H_2''\|_{L^p} \cdot \big(\|h_1'\|_{\dot{C}^\beta} + \|H_1'\|_{\dot{C}^\beta} + \|H_2'\|_{\dot{C}^\beta} + (m_{0,1}+M_{0,1}+M_{0,2})(1+\delta R^2)^{1/2}\big) + CR^{-1}r^2\big(\|H_1'-H_2'\|_{\dot{C}^\beta} + \Delta m_\beta + \Delta M_0\big)\big(1+\|h_1''\|_{L^p}+\|H_1''\|_{L^p}+\|H_2''\|_{L^p}\big), \quad (9.51)$$

where $C = C(p,\varepsilon,\mu,\nu,r,R,G)$ *unless otherwise stated.*

Proof. We argue as in Lemma 8.1. Note that $\tilde{\mathcal{R}}_{[\varphi_i]'}$ has mean zero on $\mathbb{T}$. By Poincaré inequality and Lemmas 5.3 and 5.6,

$$\begin{aligned}
&\left\|\frac{1}{f_1}\gamma_1'\cdot\mathcal{K}_{\gamma_1}\tilde{\mathcal{R}}_{[\varphi_1]'} - \frac{1}{f_2}\gamma_2'\cdot\mathcal{K}_{\gamma_2}\tilde{\mathcal{R}}_{[\varphi_2]'}\right\|_{W^{1,p}}\\
&\leqslant C\left\|\frac{1}{f_1}-\frac{1}{f_2}\right\|_{W^{1,\infty}}\|\gamma_1'\cdot\mathcal{K}_{\gamma_1}\tilde{\mathcal{R}}_{[\varphi_1]'}\|_{\dot{W}^{1,p}}\\
&\quad + C\|f_2^{-1}\|_{W^{1,\infty}}\|\gamma_1'\cdot\mathcal{K}_{\gamma_1}\tilde{\mathcal{R}}_{[\varphi_1]'} - \gamma_2'\cdot\mathcal{K}_{\gamma_2}\tilde{\mathcal{R}}_{[\varphi_1]'}\|_{\dot{W}^{1,p}}\\
&\quad + C\|f_2^{-1}\|_{W^{1,\infty}}\|\gamma_2'\cdot\mathcal{K}_{\gamma_2}(\tilde{\mathcal{R}}_{[\varphi_1]'} - \tilde{\mathcal{R}}_{[\varphi_2]'})\|_{\dot{W}^{1,p}}\\
&\leqslant Cr^{-1}\Delta m_0\|\tilde{\mathcal{R}}_{[\varphi_1]'}\|_{\dot{W}^{1,p}}\\
&\quad + C(p,\beta)r^{-1}\|\tilde{\mathcal{R}}_{[\varphi_1]'}\|_{\dot{C}^\beta}\big(\|h_1''-h_2''\|_{L^p} + (\|h_1''\|_{L^p}+\|h_2''\|_{L^p})(\|h_1'-h_2'\|_{\dot{C}^\beta}+\Delta m_0)\big)\\
&\quad + C(p,\beta)r^{-1}\big(\|\tilde{\mathcal{R}}_{[\varphi_1]'} - \tilde{\mathcal{R}}_{[\varphi_2]'}\|_{\dot{W}^{1,p}} + \|\tilde{\mathcal{R}}_{[\varphi_1]'} - \tilde{\mathcal{R}}_{[\varphi_2]'}\|_{\dot{C}^\beta}\|h_2''\|_{L^p}\big).
\end{aligned} \quad (9.52)$$

By (9.9) and (9.10),

$$\|\tilde{\mathcal{R}}_{[\varphi_i]'}\|_{\dot{C}^\beta} + \|\tilde{\mathcal{R}}_{\phi_i'}\|_{\dot{C}^\beta} \leqslant \|[\varphi_i]'\|_{\dot{C}^\beta} + \|\phi_i'\|_{\dot{C}^\beta} + C(\mu,\nu,G)r^2\big(\|h_i'\|_{\dot{C}^\beta} + \|H_i'\|_{\dot{C}^\beta}\big), \quad (9.53)$$

$$\|\tilde{\mathcal{R}}_{[\varphi_i]'}\|_{\dot{W}^{1,p}} + \|\tilde{\mathcal{R}}_{\phi_i'}\|_{\dot{W}^{1,p}} \leqslant \|[\varphi_i]'\|_{\dot{W}^{1,p}} + \|\phi_i'\|_{\dot{W}^{1,p}} + C(\mu,\nu,G)r^2\big(\|h_i'\|_{\dot{W}^{1,p}} + \|H_i'\|_{\dot{W}^{1,p}}\big). \quad (9.54)$$

So by Proposition 7.2, Proposition 7.3 and Lemma 9.4,

$$\begin{aligned}
&\left\| \frac{1}{f_1}\gamma_1' \cdot \mathcal{K}_{\gamma_1}\tilde{\mathcal{R}}_{[\varphi_1]'} - \frac{1}{f_2}\gamma_2' \cdot \mathcal{K}_{\gamma_2}\tilde{\mathcal{R}}_{[\varphi_2]'} \right\|_{W^{1,p}} \\
&\leqslant Cr^{-1}\Delta m_0\big(\|[\varphi_1]''\|_{L^p} + \|\phi_1''\|_{L^p} + r^2(\|h_1''\|_{L^p} + \|H_1''\|_{L^p})\big) \\
&\quad + C(p,\varepsilon,\mu,\nu,G)r^{-1}\big(\|[\varphi_1]'\|_{\dot{C}^\beta} + \|\phi_1'\|_{\dot{C}^\beta} + r^2(\|h_1'\|_{\dot{C}^\beta} + \|H_1'\|_{\dot{C}^\beta})\big) \\
&\qquad\quad \cdot \big(\|h_1'' - h_2''\|_{L^p} + (\|h_1''\|_{L^p} + \|h_2''\|_{L^p})(\|h_1' - h_2'\|_{\dot{C}^\beta} + \Delta m_0)\big) \\
&\quad + C(p,\varepsilon,\mu,\nu,G)r\|h_1'' - h_2''\|_{L^p}\big(\|h_1'\|_{\dot{C}^\beta} + \|H_1'\|_{\dot{C}^\beta} + (m_{0,1} + m_{0,2} + M_{0,1})(1+\delta R^2)^{1/2}\big) \\
&\quad + Cr\big(\|h_1' - h_2'\|_{\dot{C}^\beta} + \Delta m_0 + \Delta M_\beta\big)\big(1 + \|h_1''\|_{L^p} + \|h_2''\|_{L^p} + \|H_1''\|_{L^p}\big) \\
&\quad + Cr(\Delta m_\beta + \Delta M_\beta)\|h_2''\|_{L^p} \\
&\quad + Cr\|h_1 - h_2\|_{\dot{C}^{1,\beta}}(\|h_1'\|_{\dot{C}^\beta} + \|h_2'\|_{\dot{C}^\beta} + \|H_1'\|_{\dot{C}^\beta} + m_{0,1} + M_{0,1})\|h_2''\|_{L^p}, \\
&\leqslant C(p,\varepsilon,\mu,\nu,G)r\|h_1'' - h_2''\|_{L^p}(\|h_1'\|_{\dot{C}^\beta} + \|H_1'\|_{\dot{C}^\beta} + (m_{0,1} + m_{0,2} + M_{0,1})(1+\delta R^2)^{1/2}) \\
&\quad + Cr\big(\|h_1' - h_2'\|_{\dot{C}^\beta} + \Delta m_0 + \Delta M_\beta\big)\big(1 + \|h_1''\|_{L^p} + \|h_2''\|_{L^p} + \|H_1''\|_{L^p}\big),
\end{aligned} \tag{9.55}$$

where $C = C(p,\varepsilon,\mu,\nu,r,R,G)$ unless otherwise stated. Similarly,

$$\begin{aligned}
&\left\| \left(\frac{1}{f_1}\gamma_1' \cdot \mathcal{K}_{\gamma_1} f_1' - \frac{1}{2r}\mathcal{H} f_1'\right) - \left(\frac{1}{f_2}\gamma_2' \cdot \mathcal{K}_{\gamma_2} f_2' - \frac{1}{2r}\mathcal{H} f_2'\right) \right\|_{W^{1,p}} \\
&\leqslant \left\| \frac{1}{f_1} - \frac{1}{f_2} \right\|_{W^{1,\infty}} \|\gamma_1' \cdot \mathcal{K}_{\gamma_1} f_1'\|_{\dot{W}^{1,p}} + \left\| \frac{1}{f_2} \right\|_{W^{1,\infty}} \|\gamma_1' \cdot \mathcal{K}_{\gamma_1} f_1' - \gamma_2' \cdot \mathcal{K}_{\gamma_2} f_1'\|_{\dot{W}^{1,p}} \\
&\quad + \left\| \frac{1}{f_2} - \frac{1}{r} \right\|_{W^{1,\infty}} \|\gamma_2' \cdot \mathcal{K}_{\gamma_2}(f_1 - f_2)'\|_{\dot{W}^{1,p}} + \frac{1}{r}\left\| \gamma_2' \cdot \mathcal{K}_{\gamma_2}(f_1 - f_2)' - \frac{1}{2}\mathcal{H}(f_1 - f_2)' \right\|_{\dot{W}^{1,p}} \\
&\leqslant C(p,\beta)\|h_1'' - h_2''\|_{L^p}\big(m_{0,1} + m_{0,2} + \|h_1'\|_{\dot{C}^\beta} + \|h_2'\|_{\dot{C}^\beta}\big) \\
&\quad + C\big(\|h_1''\|_{L^p} + \|h_2''\|_{L^p}\big)(\|h_1' - h_2'\|_{\dot{C}^\beta} + \Delta m_0).
\end{aligned} \tag{9.56}$$

By (9.24) and Lemmas 4.3–4.7 and 9.1,

$$\begin{aligned}
&\big\|e_r \cdot \nabla\big(\Gamma * g_{\psi,1}(X_1)\big)\big|_{\gamma_1} - e_r \cdot \nabla\big(\Gamma * g_{\psi,2}(X_2)\big)\big|_{\gamma_2}\big\|_{L^\infty} \\
&\qquad \leqslant \big\|e_r \cdot \nabla\big(\Gamma * g_{\psi,1}(X_1)\big)\big|_{\gamma_1} - e_r \cdot \nabla\big(\Gamma * g_{\psi,1}(X_2)\big)\big|_{\gamma_2}\big\|_{L^\infty} \\
&\qquad\qquad + \big\|e_r \cdot \nabla\big(\Gamma * (g_{\psi,1} - g_{\psi,2})(X_2)\big)\big|_{\gamma_2}\big\|_{L^\infty} \\
&\qquad \leqslant Cr(\Delta m_\beta + \Delta M_\beta) + |c_1 - c_2| \\
&\qquad \leqslant Cr(\Delta m_\beta + \Delta M_\beta),
\end{aligned} \tag{9.57}$$

and

$$\big\|e_r \cdot \nabla\big(\Gamma * g_{\psi,1}(X_1)\big)\big|_{\gamma_1} - e_r \cdot \nabla\big(\Gamma * g_{\psi,2}(X_2)\big)\big|_{\gamma_2}\big\|_{\dot{W}^{1,p}} \leqslant Cr(\Delta m_\beta + \Delta M_\beta). \tag{9.58}$$

Note that this term is not of mean zero on $\mathbb{T}$, so we have to bound its L^∞-norm and $\dot{W}^{1,p}$-seminorm in order to prove (9.50). Finally,

$$\begin{aligned}&\left\|\frac{f_1'}{f_1}e_\theta\cdot\nabla\big(\varGamma*g_{\psi,1}(X_1)\big)\big|_{\gamma_1}-\frac{f_2'}{f_2}e_\theta\cdot\nabla\big(\varGamma*g_{\psi,2}(X_2)\big)\big|_{\gamma_2}\right\|_{L^\infty}\\&\quad\leqslant C\left\|\frac{f_1'}{f_1}-\frac{f_2'}{f_2}\right\|_{L^\infty}\big\|e_\theta\cdot\nabla\big(\varGamma*g_{\psi,1}(X_1)\big)\big|_{\gamma_1}\big\|_{L^\infty}\\&\qquad+C\left\|\frac{f_2'}{f_2}\right\|_{L^\infty}\big\|(e_\theta\cdot\nabla(\varGamma*g_{\psi,1}(X_1))|_{\gamma_1}-e_\theta\cdot\nabla\big(\varGamma*g_{\psi,2}(X_2)\big)\big|_{\gamma_2})\big\|_{L^\infty}\\&\quad\leqslant Cr(m_{0,1}+m_{0,2}+M_{0,1})(\varDelta m_\beta+\varDelta M_\beta),\end{aligned}\tag{9.59}$$

and by proceeding as in (9.29)–(9.31),

$$\begin{aligned}&\left\|\frac{f_1'}{f_1}e_\theta\cdot\nabla\big(\varGamma*g_{\psi,1}(X_1)\big)\big|_{\gamma_1}-\frac{f_2'}{f_2}e_\theta\cdot\nabla\big(\varGamma*g_{\psi,2}(X_2)\big)\big|_{\gamma_2}\right\|_{\dot W^{1,p}}\\&\quad\leqslant C(p,\varepsilon,\mu,\nu,G)r\|h_1''-h_2''\|_{L^p}\big(\|h_1'\|_{\dot C^\beta}+\|H_1'\|_{\dot C^\beta}+(m_{0,1}+M_{0,1})(1+\delta R^2)^{1/2}\big)\\&\qquad+Cr\big(m_{\beta,1}+M_{0,1}+\|h_2''\|_{L^p}\big)(\varDelta m_\beta+\varDelta M_\beta).\end{aligned}\tag{9.60}$$

Combining these estimates with (9.15), we use the fact $|c_*|\leqslant Cr$ by Lemma 3.1 to prove (9.50).

To show (9.51), we derive as before.

$$\begin{aligned}&\left\|\frac{1}{F_1}\tilde\gamma_1'\cdot\mathcal{K}_{\tilde\gamma_1}\tilde{\mathcal{R}}_{\phi_1'}-\frac{1}{F_2}\tilde\gamma_2'\cdot\mathcal{K}_{\tilde\gamma_2}\tilde{\mathcal{R}}_{\phi_2'}\right\|_{W^{1,p}}\\&\leqslant C(p,\varepsilon,\mu,\nu,G)R^{-1}r^2\|H_1''-H_2''\|_{L^p}\\&\quad\cdot\big(\|h_1'\|_{\dot C^\beta}+\|H_1'\|_{\dot C^\beta}+(m_{0,1}+M_{0,1}+M_{0,2})(1+\delta R^2)^{1/2}\big)\\&\quad+CR^{-1}r^2\big(\|H_1'-H_2'\|_{\dot C^\beta}+\varDelta m_\beta+\varDelta M_0\big)\big(1+\|h_1''\|_{L^p}+\|H_1''\|_{L^p}+\|H_2''\|_{L^p}\big)\end{aligned}\tag{9.61}$$

and

$$\begin{aligned}&\left\|\left(\frac{1}{F_1}\tilde\gamma_1'\cdot\mathcal{K}_{\tilde\gamma_1}F_1'-\frac{1}{2R}\mathcal{H}F_1'\right)-\left(\frac{1}{F_2}\tilde\gamma_2'\cdot\mathcal{K}_{\tilde\gamma_2}F_2'-\frac{1}{2R}\mathcal{H}F_2'\right)\right\|_{W^{1,p}}\\&\quad\leqslant C(p,\beta)\|H_1''-H_2''\|_{L^p}\big(M_{0,1}+M_{0,2}+\|H_1'\|_{\dot C^\beta}+\|H_2'\|_{\dot C^\beta}\big)\\&\qquad+C\big(\|H_1''\|_{L^p}+\|H_2''\|_{L^p}\big)\big(\|H_1'-H_2'\|_{\dot C^\beta}+\varDelta M_0\big).\end{aligned}\tag{9.62}$$

By (9.38) and Lemmas 4.8–4.12 and 9.1,

$$\begin{aligned}&\big\|e_r\cdot\nabla\big(\varGamma*g_{\Psi,1}(X_1)\big)\big|_{\tilde\gamma_1}-e_r\cdot\nabla\big(\varGamma*g_{\Psi,2}(X_2)\big)\big|_{\tilde\gamma_2}\big\|_{L^\infty}\\&\quad+\big\|e_r\cdot\nabla\big(\varGamma*g_{\Psi,1}(X_1)\big)\big|_{\tilde\gamma_1}-e_r\cdot\nabla\big(\varGamma*g_{\Psi,2}(X_2)\big)\big|_{\tilde\gamma_2}\big\|_{\dot W^{1,p}}\\&\qquad\leqslant CR^{-1}r^2(\varDelta m_\beta+\varDelta M_\beta),\end{aligned}\tag{9.63}$$

$$\begin{aligned}&\left\|\frac{F_1'}{F_1}e_\theta\cdot\nabla\big(\varGamma*g_{\Psi,1}(X_1)\big)\big|_{\tilde\gamma_1}-\frac{F_2'}{F_2}e_\theta\cdot\nabla\big(\varGamma*g_{\Psi,2}(X_2)\big)\big|_{\tilde\gamma_2}\right\|_{L^\infty}\\&\qquad\leqslant CR^{-1}r^2(m_{0,1}+M_{0,1}+M_{0,2})(\varDelta m_\beta+\varDelta M_\beta),\end{aligned}\tag{9.64}$$

and

$$\begin{aligned}&\Big\| \frac{F_1'}{F_1} e_\theta \cdot \nabla\big(\Gamma * g_{\Psi,1}(X_1)\big)\big|_{\tilde{\gamma}_1} - \frac{F_2'}{F_2} e_\theta \cdot \nabla\big(\Gamma * g_{\Psi,2}(X_2)\big)\big|_{\tilde{\gamma}_2} \Big\|_{\dot{W}^{1,p}} \\ &\leqslant C(p,\varepsilon,\mu,\nu,G) R^{-1} r^2 \|H_1'' - H_2''\|_{L^p} \\ &\qquad \cdot \big(\|h_1'\|_{\dot{C}^\beta} + \|H_1'\|_{\dot{C}^\beta} + (m_{0,1} + M_{0,1})(1+\delta R^2)^{1/2}\big) \\ &\qquad + CR^{-1} r^2 (m_{0,1} + M_{0,1} + \|H_2''\|_{L^p})(\Delta m_\beta + \Delta M_\beta). \quad (9.65)\end{aligned}$$

Combining these estimates and the fact $\tilde{c}_* \leqslant C(\mu,\nu,G)R^{-1}r^2$ with (9.16) yields (9.51). □

9.3 *Proof of the uniqueness*

Now we are ready to prove uniqueness.

Proof of Theorem 2.3. In this proof, we always assume that the constant C has the dependence $C = C(p,\varepsilon,\mu,\nu,r,R,G)$ unless otherwise stated.

As stated at the beginning of this section, suppose there are two solutions f_i and F_i $(i = 1,2)$ of (2.16)–(2.18) with regularity and estimates given in Theorem 2.1. By (9.13) and (9.14), $(h_1 - h_2)$ and $(H_1 - H_2)$ solve

$$\partial_t (h_1 - h_2) = -\frac{Ac_*}{r}(-\Delta)^{1/2}(h_1 - h_2) + \frac{1}{r}(\tilde{\mathcal{R}}_{h_1} - \tilde{\mathcal{R}}_{h_2}), \quad (9.66)$$

$$\partial_t (H_1 - H_2) = \frac{\tilde{c}_*}{R}(-\Delta)^{1/2}(H_1 - H_2) + \frac{1}{R}(\tilde{\mathcal{R}}_{H_1} - \tilde{\mathcal{R}}_{H_2}), \quad (9.67)$$

with initial condition $(h_1 - h_2)|_{t=0} = (H_1 - H_2)|_{t=0} = 0$.

Let $T_0 \in (0,T)$, $T_0 < 1$ to be chosen. By virtue of Lemma A.5 and Lemma A.6, with $\alpha = 1 - \frac{2}{p}$,

$$\begin{aligned}&\|h_1'' - h_2''\|_{L^p_{[0,T_0]}L^p(\mathbb{T})} + \|H_1'' - H_2''\|_{L^p_{[0,T_0]}L^p(\mathbb{T})} \\ &\qquad + \|h_1' - h_2'\|_{C_{[0,T_0]}\dot{C}^\alpha(\mathbb{T})} + \|H_1' - H_2'\|_{C_{[0,T_0]}\dot{C}^\alpha(\mathbb{T})} \\ &\leqslant C(p,\mu,\nu,G) \\ &\quad \left(\frac{r}{|Ac_*|} \cdot \frac{1}{r} \|\tilde{\mathcal{R}}_{h_1} - \tilde{\mathcal{R}}_{h_2}\|_{L^p_{[0,T_0]}\dot{W}^{1,p}(\mathbb{T})} + \frac{R}{|\tilde{c}_*|} \cdot \frac{1}{R} \|\tilde{\mathcal{R}}_{H_1} - \tilde{\mathcal{R}}_{H_2}\|_{L^p_{[0,T_0]}\dot{W}^{1,p}(\mathbb{T})} \right). \quad (9.68)\end{aligned}$$

Here we first applied change of time variables to normalize the coefficients of fractional Laplacians in (9.66) and (9.67), and then applied Lemma A.5 and Lemma A.6 to obtain these estimates. To fulfill the condition of Lemma A.6, we need

$$T_0 \leqslant \min\left\{ \frac{r}{Ac_*}, \frac{R}{|\tilde{c}_*|} \right\}. \quad (9.69)$$

Note that by Lemma 3.1, the right hand side is bounded from below by some constant depending only on μ, ν and G.

On the other hand, by Sobolev embedding (in space) and Hölder's inequality (in time)

$$\|h_1 - h_2\|_{C_{[0,T_0]}L^\infty(\mathbb{T})} + \|H_1 - H_2\|_{C_{[0,T_0]}L^\infty(\mathbb{T})} \leqslant C(p)T_0^{1-\frac{1}{p}} \left(\frac{1}{r} \|\tilde{\mathcal{R}}_{h_1} - \tilde{\mathcal{R}}_{h_2}\|_{L^p_{[0,T_0]}W^{1,p}(\mathbb{T})} + \frac{1}{R} \|\tilde{\mathcal{R}}_{H_1} - \tilde{\mathcal{R}}_{H_2}\|_{L^p_{[0,T_0]}W^{1,p}(\mathbb{T})} \right). \quad (9.70)$$

Denote

$$\begin{aligned} \mathcal{N}(T_0) := {}& \|h_1'' - h_2''\|_{L^p_{[0,T_0]}L^p(\mathbb{T})} + \|H_1'' - H_2''\|_{L^p_{[0,T_0]}L^p(\mathbb{T})} \\ & + \|h_1' - h_2'\|_{C_{[0,T_0]}\dot{C}^\beta(\mathbb{T})} + \|H_1' - H_2'\|_{C_{[0,T_0]}\dot{C}^\beta(\mathbb{T})} \\ & + \delta^{-1}\|h_1 - h_2\|_{C_{[0,T_0]}L^\infty(\mathbb{T})} + \delta^{-1}\|H_1 - H_2\|_{C_{[0,T_0]}L^\infty(\mathbb{T})}. \end{aligned} \quad (9.71)$$

By interpolation and Lemma 9.5, with $\theta = (1 - \frac{1}{p}) \cdot \frac{\alpha - \beta}{1+\alpha}$,

$$\begin{aligned} &\mathcal{N}(T_0) \\ &\leqslant C(p,\mu,\nu,G) \left(\frac{r}{|Ac_*|} \cdot \frac{1}{r} \|\tilde{\mathcal{R}}_{h_1} - \tilde{\mathcal{R}}_{h_2}\|_{L^p_{[0,T_0]}\dot{W}^{1,p}(\mathbb{T})} + \frac{R}{|\tilde{c}_*|} \cdot \frac{1}{R} \|\tilde{\mathcal{R}}_{H_1} - \tilde{\mathcal{R}}_{H_2}\|_{L^p_{[0,T_0]}\dot{W}^{1,p}(\mathbb{T})} \right) \\ &\qquad + CT_0^\theta \left(\frac{1}{r} \|\tilde{\mathcal{R}}_{h_1} - \tilde{\mathcal{R}}_{h_2}\|_{L^p_{[0,T_0]}\dot{W}^{1,p}(\mathbb{T})} + \frac{1}{R} \|\tilde{\mathcal{R}}_{H_1} - \tilde{\mathcal{R}}_{H_2}\|_{L^p_{[0,T_0]}\dot{W}^{1,p}(\mathbb{T})} \right) \\ &\leqslant \left[C(p,\varepsilon,\mu,\nu,G) \cdot \frac{r}{|c_*|} + C_1 T_0^\theta \right] \mathcal{N}(T_0) \\ &\qquad \cdot \sup_{t\in[0,T_0]} \big(\|h_1'\|_{\dot{C}^\beta} + \|h_2'\|_{\dot{C}^\beta} + \|H_1'\|_{\dot{C}^\beta} + \|H_2'\|_{\dot{C}^\beta} \\ &\qquad\quad + (m_{0,1} + m_{0,2} + M_{0,1} + M_{0,2})(1 + \delta R^2)^{1/2} \big) \\ &\quad + C_2 \mathcal{N}(T_0) \big(T_0^{1/p} + \|h_1''\|_{L^p_{[0,T_0]}L^p} + \|h_2''\|_{L^p_{[0,T_0]}L^p} + \|H_1''\|_{L^p_{[0,T_0]}L^p} + \|H_2''\|_{L^p_{[0,T_0]}L^p} \big). \end{aligned} \quad (9.72)$$

Here the constants C_1 and C_2 have the same dependence as C introduced above.

Now we take T_0 such that $C_1 T_0^\theta \leqslant \frac{1}{2}$ and

$$C_2 \Big(T_0^{1/p} + \|h_1''\|_{L^p_{[0,T_0]}L^p} + \|h_2''\|_{L^p_{[0,T_0]}L^p} + \|H_1''\|_{L^p_{[0,T_0]}L^p} + \|H_2''\|_{L^p_{[0,T_0]}L^p} \Big) \leqslant \frac{1}{2}. \quad (9.73)$$

Such T_0 relies on p, ε, μ, ν, r, R, G as well as the fixed solutions h_i and H_i. Then (9.72) becomes

$$\begin{aligned} \mathcal{N}(T_0) &\leqslant \left[C(p,\varepsilon,\mu,\nu,G) \cdot \frac{r}{|c_*|} + 1 \right] \mathcal{N}(T_0) \\ &\qquad \cdot \sup_{t\in[0,T_0]} \big(\|h_1'\|_{\dot{C}^\beta} + \|h_2'\|_{\dot{C}^\beta} + \|H_1'\|_{\dot{C}^\beta} + \|H_2'\|_{\dot{C}^\beta} \\ &\qquad\quad + (m_{0,1} + m_{0,2} + M_{0,1} + M_{0,2})(1 + \delta R^2)^{1/2} \big) \\ &\leqslant C(p,\varepsilon,\mu,\nu,G,r/|c_*|,\delta R^2) M \cdot \mathcal{N}(T_0). \end{aligned} \quad (9.74)$$

In the last inequality, we used the estimate (2.26). If we assume M to be suitably small, depending only on p, ε, μ, ν, G, $r/|c_*|$ and δR^2, we obtain that $\mathcal{N}(T_0) = 0$. Note that here the smallness of M has no additional dependence on other parameters compared to that in the proof of existence of local solutions.

We can continue this process starting from $t = T_0$ and find a second time interval $[T_0, T_0 + T_1]$ on which uniqueness holds. By repeating this argument for finitely many times (see (2.28) and the way we chose T_0 above), we can prove the uniqueness of local solution on $[0, T]$. □

A. Some auxiliary estimates

A.1 *Estimates for the Poisson kernel and its conjugate*

Lemma A.1 *Let Poisson kernel P on the 2-D unit disc and its conjugate Q be defined as in* (4.1) *and* (4.2)*, respectively.*

1. *Let $\mathcal{H}_\xi$ denote the Hilbert transform on $\mathbb{T}$ with respect to ξ. Then for $s \neq 1$,*

$$Q(s, \xi) = \operatorname{sgn}(1 - s)\mathcal{H}_\xi P(s, \xi). \tag{A1}$$

2. *For all $\xi \in \mathbb{T}$ and all $s \in [0, 2]$,*

$$|P(s, \xi)| + |Q(s, \xi)| \leqslant C(|1 - s|^2 + \xi^2)^{-1/2}. \tag{A2}$$

3. *For derivatives of P and Q, we have*

$$\left|\frac{\partial P}{\partial s}(s, \xi)\right| + \left|\frac{\partial Q}{\partial \xi}(s, \xi)\right| \leqslant C\left((1 - s)^2 + \xi^2\right)^{-1}, \tag{A3}$$

$$\left|\frac{\partial P}{\partial \xi}(s, \xi)\right| + \left|\frac{\partial Q}{\partial s}(s, \xi)\right| \leqslant C|\sin \xi|\left((1 - s)^2 + \xi^2\right)^{-3/2}, \tag{A4}$$

and

$$\left|\frac{\partial^2 P}{\partial s^2}(s, \xi)\right| + \left|\frac{\partial^2 P}{\partial \xi \partial s}(s, \xi)\right| + \left|\frac{\partial^2 Q}{\partial s^2}(s, \xi)\right| \leqslant C\left((1 - s)^2 + \xi^2\right)^{-3/2}. \tag{A5}$$

Moreover,

$$\frac{\partial P}{\partial \xi}(s, \xi) = -s\frac{\partial Q}{\partial s}, \quad \frac{\partial Q}{\partial \xi}(s, \xi) = s\frac{\partial P}{\partial s}. \tag{A6}$$

Proof. (A1) can be proved by calculating Fourier transforms of $P(s, \cdot)$ and $Q(s, \cdot)$.

For any $s \geqslant 0$,

$$1 + s^2 - 2s \cos \xi = (1 - s \cos \xi)^2 + (s \sin \xi)^2 = (s - \cos \xi)^2 + \sin^2 \xi \geqslant 0. \tag{A7}$$

If $\cos \xi \geqslant \frac{1}{2}$,

$$\begin{aligned} 1 + s^2 - 2s \cos \xi &= (1 + s^2)(1 - \cos \xi) + \cos \xi (1 - s)^2 \\ &\geqslant C\left(|\xi|^2 + |1 - s|^2\right). \end{aligned} \tag{A8}$$

Otherwise,

$$1 + s^2 - 2s \cos \xi \geqslant C(1 + s^2) \geqslant C\left(|\xi|^2 + |1 - s|^2\right). \tag{A9}$$

Then (A2) follows easily.

Finally, we calculate that

$$\frac{\partial P}{\partial s}(s,\xi) = \frac{2(1+s^2)\cos\xi - 4s}{(1+s^2-2s\cos\xi)^2} = \frac{2\cos\xi}{1+s^2-2s\cos\xi} - \frac{4s\sin^2\xi}{(1+s^2-2s\cos\xi)^2}, \tag{A10}$$

$$\frac{\partial Q}{\partial s}(s,\xi) = \frac{2(1-s^2)\sin\xi}{(1+s^2-2s\cos\xi)^2}, \tag{A11}$$

$$\frac{\partial^2 P}{\partial\xi\partial s}(s,\xi) = -\frac{2(1+s^2)\sin\xi}{(1+s^2-2s\cos\xi)^2} - \frac{\partial P}{\partial s}\cdot\frac{4s\sin\xi}{1+s^2-2s\cos\xi}, \tag{A12}$$

$$\frac{\partial^2 P}{\partial s^2}(s,\xi) = \frac{4s\cos\xi - 4}{(1+s^2-2s\cos\xi)^2} - \frac{8(s-\cos\xi)\big((1+s^2)\cos\xi - 2s\big)}{(1+s^2-2s\cos\xi)^3}, \tag{A13}$$

and

$$\frac{\partial P}{\partial\xi}(s,\xi) = -s\frac{\partial Q}{\partial s}, \tag{A14}$$

$$\frac{\partial Q}{\partial\xi}(s,\xi) = s\frac{\partial P}{\partial s}, \tag{A15}$$

$$\frac{\partial^2 Q}{\partial s^2}(s,\xi) = -\frac{1}{s}\left(\frac{\partial^2 P}{\partial\xi\partial s} + \frac{\partial Q}{\partial s}\right). \tag{A16}$$

Then (A3)–(A6) follow. □

A.2 *Some Calderón-commutator-type estimates*

In this part we shall establish some Calderón-commutator-type estimates used in Section 5. Recall that

$$\Delta f(\theta) := \frac{f(\theta+\xi) - f(\xi)}{2\sin\frac{\xi}{2}}. \tag{A17}$$

Lemma A.2 *Let* $\mathbf{k} = (k_1,\cdots,k_n)$ *be a multi-index of length* $n \in \mathbb{Z}_+$. *Assume* $h_1,\cdots,h_n \in W^{1,\infty}(\mathbb{T})$ *and* $\psi \in L^p(\mathbb{T})$ *for some* $p \in [2,\infty)$. *Define*

$$M_{\mathbf{k},\psi}(\theta) = \text{p.v.}\int_{\mathbb{T}}\prod_{i=1}^{n}(\Delta h_i)^{k_i}\cdot\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi, \tag{A18}$$

$$N_{\mathbf{k},\psi}(\theta) = \text{p.v.}\int_{\mathbb{T}}\prod_{i=1}^{n}(\Delta h_i)^{k_i}\cdot\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi. \tag{A19}$$

Then

$$\|M_{\mathbf{k},\psi}\|_{L^p} + \|N_{\mathbf{k},\psi}\|_{L^p} \leqslant C_*^{|\mathbf{k}|+2}\|\psi\|_{L^p}\prod_{i=1}^{n}\|h_i'\|_{L^\infty}^{k_i}, \tag{A20}$$

where C_* *is a universal constant depending only on* p. *Here* $|\mathbf{k}| := \sum_{i=1}^{n} k_i$.

Proof. The proof essentially follows the classic argument of L^p-boundedness of the Calderón commutator [21, § 9.3]. For completeness, we elaborate it as follows.

First we notice that $\sin(\xi/2)$ is not continuous on $\mathbb{T}$ at $\pm\pi$. For this technical reason, with abuse of notations, we introduce an even cut-off function $\eta \in C_0^\infty([-2,2])$, such that $\eta \equiv 1$ on $[-1,1]$, $\eta \in [0,1]$ on $[-2,2]$, and $|\eta'| \leqslant C$. Write (A18) as

$$M_{\mathbf{k},\psi} = \text{p.v.} \int_{\mathbb{T}} \prod_{i=1}^n (\Delta h_i)^{k_i} \cdot \frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}} \big[\eta(\xi) + \big(1-\eta(\xi)\big)\big]\, d\xi =: M_{\mathbf{k},\psi}^{(1)} + M_{\mathbf{k},\psi}^{(2)}. \tag{A21}$$

It is straightforward to bound $M_{\mathbf{k},\psi}^{(2)}$ as it involves no singularity,

$$\|M_{\mathbf{k},\psi}^{(2)}\|_{L^p} \leqslant C C_1^{|\mathbf{k}|} \|\psi\|_{L^p} \prod_{i=1}^n \|h_i'\|_{L^\infty}^{k_i}. \tag{A22}$$

Here $C_1 = \pi/2$ comes from the fact that

$$\left|2\sin\frac{\xi}{2}\right|^{-1} \leqslant C_1 |\xi|^{-1} \quad \text{on } \mathbb{T}. \tag{A23}$$

To derive an L^p-bound for $M_{\mathbf{k},\psi}^{(1)}$, we first show that $M_{\mathbf{k},1}^{(1)} \in BMO$ by mathematical induction.

STEP 1 For $\mathbf{k} = \mathbf{0}$, $M_{\mathbf{0},1}^{(1)} = -\pi\mathcal{H}\eta(0) = 0$ since η is even.

STEP 2 Suppose for some $N \geqslant 1$ and any multi-index $\mathbf{k}$ such that $|\mathbf{k}| \leqslant N-1$, we have shown that $M_{\mathbf{k},1}^{(1)} \in BMO$ and, with some constant C_* that will be specified later,

$$\|M_{\mathbf{k},1}^{(1)}\|_{BMO} \leqslant C_*^{|\mathbf{k}|+1} \prod_{i=1}^n \|h_i'\|_{L^\infty}^{k_i}. \tag{A24}$$

It is known that the map $\psi \mapsto M_{\mathbf{k},\psi}^{(1)}$ is associated with the kernel

$$\prod_{i=1}^n \left(\frac{h_i(x)-h_i(y)}{2\sin\frac{x-y}{2}}\right)^{k_i} \cdot \frac{\eta(x-y)}{2\tan\frac{x-y}{2}}, \tag{A25}$$

which is a standard anti-symmetric kernel, vanishing whenever $|x-y| > 2$. It can be naturally understand as a kernel on $\mathbb{R}$ with a bound similar to (A24). Hence, by the $T1$ Theorem, it is $(2,2)$-bounded. Its operator norm depends linearly [21, § 9.3] on the constant in (A24) and the kernel constant of (A25), which is bounded by

$$C C_1^{|\mathbf{k}|+1} \big(|\mathbf{k}|+1\big) \prod_{i=1}^n \|h_i'\|_{L^\infty}^{k_i}. \tag{A26}$$

This further implies that [21, Theorem 6.6] for all $\mathbf{k}$ satisfying $|\mathbf{k}| \leqslant N-1$, and $\psi \in L^\infty$,

$$\|M_{\mathbf{k},\psi}^{(1)}\|_{BMO} \leqslant C\big(C_1^{|\mathbf{k}|+1}(|\mathbf{k}|+1) + C_*^{|\mathbf{k}|+1}\big) \|\psi\|_{L^\infty} \prod_{i=1}^n \|h_i'\|_{L^\infty}^{k_i}. \tag{A27}$$

Now consider the case when $|\mathbf{k}| = N$. Observe that

$$\left(\frac{1}{2\sin\frac{\xi}{2}}\right)^{|\mathbf{k}|}\frac{1}{2\tan\frac{\xi}{2}} = -\frac{1}{|\mathbf{k}|}\cdot\frac{d}{d\xi}\left(\frac{1}{2\sin\frac{\xi}{2}}\right)^{|\mathbf{k}|}. \tag{A28}$$

We integrate by parts in $M^{(1)}_{\mathbf{k},1}$. For almost all $\theta \in \mathbb{T}$,

$$\begin{aligned}
M^{(1)}_{\mathbf{k},1}(\theta) &= \frac{1}{|\mathbf{k}|}\text{p.v.}\int_{[-2,2]}\left(\frac{1}{2\sin\frac{\xi}{2}}\right)^{|\mathbf{k}|} d\left[\prod(h_i(\theta+\xi)-h_i(\theta))^{k_i}\eta(\xi)\right]\\
&= \frac{1}{|\mathbf{k}|}\int_{[-2,2]}\prod_{i=1}^{n}(\Delta h_i)^{k_i}\cdot\eta'(\xi)\,d\xi\\
&\qquad + \sum_{i=1}^{n}\frac{k_i}{|\mathbf{k}|}\text{p.v.}\int_{[-2,2]}(\Delta h_1)^{k_1}\cdots(\Delta h_i)^{k_i-1}\cdots(\Delta h_n)^{k_n}\cdot\frac{\eta(\xi)h_i'(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\\
&=: M^{(1,0)}_{\mathbf{k},1} + \sum_{i=1}^{n}M^{(1,i)}_{\mathbf{k},1}.
\end{aligned} \tag{A29}$$

Indeed, this can be rigorously justified by the fact that h_i are differentiable almost everywhere. It is straightforward to derive that

$$\|M^{(1,0)}_{\mathbf{k},1}\|_{L^\infty} \leqslant CC_1^{|\mathbf{k}|}\prod_{i=1}^{n}\|h_i'\|_{L^\infty}^{k_i}. \tag{A30}$$

On the other hand, by (A27),

$$\begin{aligned}
&\|M^{(1,i)}_{\mathbf{k},1}\|_{BMO}\\
&\leqslant \frac{k_i}{|\mathbf{k}|}\|M^{(1)}_{(k_1,\cdots,k_i-1,\cdots,k_n),h_i'}\|_{BMO} + \frac{k_i}{|\mathbf{k}|}C_1^{|\mathbf{k}|-1}\prod_{j=1}^{n}\|h_j'\|_{L^\infty}^{k_j}\left\|\frac{1}{2\sin\frac{\xi}{2}} - \frac{1}{2\tan\frac{\xi}{2}}\right\|_{L^\infty([-2,2])}\\
&\leqslant \frac{Ck_i}{|\mathbf{k}|}(C_1^{|\mathbf{k}|}|\mathbf{k}| + C_*^{|\mathbf{k}|})\prod_{j=1}^{n}\|h_j'\|_{L^\infty}^{k_j}.
\end{aligned} \tag{A31}$$

Hence, with some universal constant C,

$$\|M^{(1)}_{\mathbf{k},1}\|_{BMO} \leqslant C\left(C_1^{|\mathbf{k}|}|\mathbf{k}| + C_*^{|\mathbf{k}|}\right)\prod_{i=1}^{n}\|h_i'\|_{L^\infty}^{k_i}. \tag{A32}$$

Now assuming C_* sufficiently large but still universal, such that

$$C\left[\left(\frac{C_1}{C_*}\right)^{|\mathbf{k}|}|\mathbf{k}| + 1\right] \leqslant C_*, \tag{A33}$$

we conclude with (A24) for $|\mathbf{k}| = N$. By induction, (A24) holds for all multi-indices $\mathbf{k}$.

To this end, we argue as in (A25)–(A27) to find that $\psi \mapsto M^{(1)}_{\mathbf{k},\psi}$ is bounded from L^2 to L^2, and also from L^∞ to BMO. By interpolation, it is (p,p)-bounded as well. In particular,

$$\|M^{(1)}_{\mathbf{k},\psi}\|_{L^p} \leqslant C_p\big(C_1^{|\mathbf{k}|+1}(|\mathbf{k}|+1) + C_*^{|\mathbf{k}|+1}\big)\|\psi\|_{L^p}\prod_{i=1}^n \|h_i'\|^{k_i}_{L^\infty}. \tag{A34}$$

Combining (A22) and (A34) yields a bound for $\|M_{\mathbf{k},\psi}\|_{L^p}$ that has the same form as in (A34). A bound for $\|N_{\mathbf{k},\psi}\|_{L^p}$ can be derived easily since $(M_{\mathbf{k},\psi} - N_{\mathbf{k},\psi})$ is an integral with no singularity.

Assuming C_* to be even larger if needed, we obtain the desired estimate from (A34). □

Lemma A.3 *Let $\mathbf{k} = (k_1, \cdots, k_n)$ be a multi-index of length $n \in \mathbb{Z}_+$. With $p \in [2,\infty)$, assume that $h_1, \cdots, h_n \in W^{2,p}(\mathbb{T})$, and $h_{n+1}, \psi \in W^{1,p}(\mathbb{T})$. Define*

$$\tilde{M}_{\mathbf{k},\psi}(\theta) = \text{p.v.}\int_{\mathbb{T}} \prod_{i=1}^n (\Delta h_i)^{k_i}\cdot \Delta h_{n+1}\cdot \frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi, \tag{A35}$$

$$\tilde{N}_{\mathbf{k},\psi}(\theta) = \text{p.v.}\int_{\mathbb{T}} \prod_{i=1}^n (\Delta h_i)^{k_i}\cdot \Delta h_{n+1}\cdot \frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi. \tag{A36}$$

Then

$$\begin{aligned}
&\|\tilde{M}_{\mathbf{k},\psi}\|_{L^p} + \|\tilde{N}_{\mathbf{k},\psi}\|_{L^p}\\
&\leqslant C_{**}^{|\mathbf{k}|+1}(\|h_{n+1}'\|_{L^p}\|\psi\|_{L^\infty} + \|h_{n+1}\|_{L^\infty}\|\psi'\|_{L^p})\prod_{i=1}^n\|h_i'\|^{k_i}_{L^\infty}\\
&\quad + C_{**}^{|\mathbf{k}|+1}\|h_{n+1}\|_{L^\infty}\|\psi\|_{L^\infty}\sum_{i=1}^n \|h_1'\|^{k_1}_{L^\infty}\cdots\|h_i'\|^{k_i-1}_{L^\infty}\cdots\|h_n'\|^{k_n}_{L^\infty}\cdot \mathbb{1}_{\{k_i>0\}}\|h_i''\|_{L^p},
\end{aligned} \tag{A37}$$

where C_{**} is a universal constant depending only on p.

Proof. We shall prove (A37) by induction. It suffices to prove it for h_{n+1} and ψ being smooth.

Step 1 Consider $\mathbf{k} = \mathbf{0}$. Note that even in this simple case, the estimate (A37) does not trivially follow from Lemma A.2.

By integration by parts as in (A29),

$$\begin{aligned}
&\tilde{M}_{\mathbf{0},\psi}(\theta)\\
&= \text{p.v.}\int_{\mathbb{T}} \frac{1}{2\sin\frac{\xi}{2}}\,d\big[(h_{n+1}(\theta+\xi) - h_{n+1}(\theta))\psi(\theta+\xi)\big]\\
&\qquad - \big[\big(h_{n+1}(\theta+\pi) - h_{n+1}(\theta)\big)\psi(\theta+\pi)\big]\\
&= \int_{\mathbb{T}} \frac{1-\cos\frac{\xi}{2}}{2\sin\frac{\xi}{2}}\big[h_{n+1}'(\theta+\xi)\psi(\theta+\xi) + \big(h_{n+1}(\theta+\xi) - h_{n+1}(\theta)\big)\psi'(\theta+\xi)\big]\,d\xi\\
&\qquad + \text{p.v.}\int_{\mathbb{T}} \frac{1}{2\tan\frac{\xi}{2}}\big[h_{n+1}'(\theta+\xi)\psi(\theta+\xi) + \big(h_{n+1}(\theta+\xi) - h_{n+1}(\theta)\big)\psi'(\theta+\xi)\big]\,d\xi\\
&\qquad - \big[\big(h_{n+1}(\theta+\pi) - h_{n+1}(\theta)\big)\psi(\theta+\pi)\big].
\end{aligned} \tag{A38}$$

By Sobolev embedding and L^p-boundedness of the Hilbert transform,

$$\|\tilde{M}_{\mathbf{0},\psi}\|_{L^p} \leqslant C\big(\|h'_{n+1}\|_{L^p}\|\psi\|_{L^\infty} + \|h_{n+1}\|_{L^\infty}\|\psi'\|_{L^p}\big). \tag{A39}$$

Since

$$|\tilde{N}_{\mathbf{0},\psi} - \tilde{M}_{\mathbf{0},\psi}| \leqslant C\int_{\mathbb{T}} |h_{n+1}(\theta+\xi) - h_{n+1}(\theta)||\psi(\theta+\xi)|\,d\xi, \tag{A40}$$

it is easy to show that $\tilde{N}_{\mathbf{0},\psi}$ satisfies the same estimate as (A39).

STEP 2 Suppose (A37) holds for all multi-indices $\mathbf{k}$ satisfying $|\mathbf{k}| \leqslant N-1$, where $C_{**} > 0$ is some constant to be chosen later. Then consider the case with $|\mathbf{k}| = N$. By integration by parts as in (A29), for almost all $\theta \in \mathbb{T}$,

$$\begin{aligned}
&\tilde{M}_{\mathbf{k},\psi}(\theta)\\
&= \sum_{i=1}^{n} \frac{k_i}{|\mathbf{k}|+1}\text{p.v.}\int_{\mathbb{T}} (\Delta h_1)^{k_1}\cdots(\Delta h_i)^{k_i-1}\cdots(\Delta h_n)^{k_n}\cdot\Delta h_{n+1}\frac{h'_i(\theta+\xi)\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\\
&\quad + \frac{1}{|\mathbf{k}|+1}\text{p.v.}\int_{\mathbb{T}}\prod_{i=1}^{n}(\Delta h_i)^{k_i}\frac{h'_{n+1}(\theta+\xi)\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\\
&\quad + \frac{1}{|\mathbf{k}|+1}\text{p.v.}\int_{\mathbb{T}}\prod_{i=1}^{n}(\Delta h_i)^{k_i}(h_{n+1}(\theta+\xi)-h_{n+1}(\theta))\cdot\frac{\psi'(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\\
&\quad - \frac{1}{|\mathbf{k}|+1}\cdot\frac{1-(-1)^{|\mathbf{k}|+1}}{2^{|\mathbf{k}|+1}}\prod_{i=1}^{n}\big(h_i(\theta+\pi)-h_i(\theta)\big)^{k_i}\big(h_{n+1}(\theta+\pi)-h_{n+1}(\theta)\big)\psi(\theta+\pi)\\
&= \sum_{i=1}^{n}\frac{k_i}{|\mathbf{k}|+1}\tilde{N}_{(k_1,\cdots,k_i-1,\cdots,k_n),h'_i\psi} + \frac{1}{|\mathbf{k}|+1}\big(N_{\mathbf{k},(h_{n+1}\psi)'} - h_{n+1}(\theta)N_{\mathbf{k},\psi'}\big)\\
&\quad - \frac{1}{|\mathbf{k}|+1}\cdot\frac{1-(-1)^{|\mathbf{k}|+1}}{2^{|\mathbf{k}|+1}}\prod_{i=1}^{n}\big(h_i(\theta+\pi)-h_i(\theta)\big)^{k_i}\big(h_{n+1}(\theta+\pi)-h_{n+1}(\theta)\big)\psi(\theta+\pi).
\end{aligned} \tag{A41}$$

By the induction hypothesis (A37),

$$\begin{aligned}
&\|k_i\tilde{N}_{(k_1,\cdots,k_i-1,\cdots,k_n),h'_i\psi}\|_{L^p}\\
&\leqslant k_iC_{**}^{|\mathbf{k}|}(\|h'_{n+1}\|_{L^p}\|h'_i\psi\|_{L^\infty} + \|h_{n+1}\|_{L^\infty}\|(h'_i\psi)'\|_{L^p})\cdot\|h'_1\|_{L^\infty}^{k_1}\cdots\|h'_i\|_{L^\infty}^{k_i-1}\cdots\|h'_n\|_{L^\infty}^{k_n}\\
&\quad + k_iC_{**}^{|\mathbf{k}|}\|h_{n+1}\|_{L^\infty}\|h'_i\psi\|_{L^\infty}\\
&\qquad \cdot\sum_{\substack{j=1,\cdots n\\ j\neq i}}\|h'_1\|_{L^\infty}^{k_1}\cdots\|h'_i\|_{L^\infty}^{k_i-1}\cdots\|h'_j\|_{L^\infty}^{k_j-1}\cdots\|h'_n\|_{L^\infty}^{k_n}\cdot\mathbb{1}_{\{k_j>0\}}\|h''_j\|_{L^p}\\
&\quad + k_iC_{**}^{|\mathbf{k}|}\|h_{n+1}\|_{L^\infty}\|h'_i\psi\|_{L^\infty}\cdot\|h'_1\|_{L^\infty}^{k_1}\cdots\|h'_i\|_{L^\infty}^{k_i-2}\cdots\|h'_n\|_{L^\infty}^{k_n}\cdot\mathbb{1}_{\{k_i>1\}}\|h''_i\|_{L^p}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant k_i C_{**}^{|\mathbf{k}|}(\|h_{n+1}'\|_{L^p}\|\psi\|_{L^\infty}+\|h_{n+1}\|_{L^\infty}\|\psi'\|_{L^p})\cdot\prod_{j=1}^{n}\|h_j'\|_{L^\infty}^{k_j}\\
&\quad+Ck_iC_{**}^{|\mathbf{k}|}\|h_{n+1}\|_{L^\infty}\|\psi\|_{L^\infty}\sum_{j=1}^{n}\|h_1'\|_{L^\infty}^{k_1}\cdots\|h_j'\|_{L^\infty}^{k_j-1}\cdots\|h_n'\|_{L^\infty}^{k_n}\cdot\mathbb{1}_{\{k_j>0\}}\|h_j''\|_{L^p}.
\end{aligned}
\tag{A42}
$$

By Lemma A.2,

$$
\begin{aligned}
&\|(N_{\mathbf{k},(h_{n+1}\psi)'}-h_{n+1}(\theta)N_{\mathbf{k},\psi'})\|_{L^p}\\
&\qquad\leqslant C_*^{|\mathbf{k}|+2}(\|h_{n+1}'\|_{L^p}\|\psi\|_{L^\infty}+\|h_{n+1}\|_{L^\infty}\|\psi'\|_{L^p})\prod_{i=1}^{n}\|h_i'\|_{L^\infty}^{k_i}.
\end{aligned}
\tag{A43}
$$

Combining these estimates with (A41), we obtain by Sobolev embedding that

$$
\begin{aligned}
&\|\tilde{M}_{\mathbf{k},\psi}\|_{L^p}\\
&\leqslant C_{**}^{|\mathbf{k}|}(\|h_{n+1}'\|_{L^p}\|\psi\|_{L^\infty}+\|h_{n+1}\|_{L^\infty}\|\psi'\|_{L^p})\prod_{i=1}^{n}\|h_i'\|_{L^\infty}^{k_i}\\
&\quad+CC_{**}^{|\mathbf{k}|}\|h_{n+1}\|_{L^\infty}\|\psi\|_{L^\infty}\sum_{i=1}^{n}\|h_1'\|_{L^\infty}^{k_1}\cdots\|h_i'\|_{L^\infty}^{k_i-1}\cdots\|h_n'\|_{L^\infty}^{k_n}\cdot\mathbb{1}_{\{k_i>0\}}\|h_i''\|_{L^p}\\
&\quad+C_*^{|\mathbf{k}|+2}(\|h_{n+1}'\|_{L^p}\|\psi\|_{L^\infty}+\|h_{n+1}\|_{L^\infty}\|\psi'\|_{L^p})\prod_{i=1}^{n}\|h_i'\|_{L^\infty}^{k_i}\\
&\quad+CC_1^{|\mathbf{k}|}\prod_{i=1}^{n}\|h_i'\|_{L^\infty}^{k_i}\cdot\|h_{n+1}'\|_{L^p}\|\psi\|_{L^\infty}\\
&\leqslant(C_{**}^{|\mathbf{k}|}+C_*^{|\mathbf{k}|+2}+CC_1^{|\mathbf{k}|})(\|h_{n+1}'\|_{L^p}\|\psi\|_{L^\infty}+\|h_{n+1}\|_{L^\infty}\|\psi'\|_{L^p})\prod_{i=1}^{n}\|h_i'\|_{L^\infty}^{k_i}\\
&\quad+CC_{**}^{|\mathbf{k}|}\|h_{n+1}\|_{L^\infty}\|\psi\|_{L^\infty}\sum_{i=1}^{n}\|h_1'\|_{L^\infty}^{k_1}\cdots\|h_i'\|_{L^\infty}^{k_i-1}\cdots\|h_n'\|_{L^\infty}^{k_n}\cdot\mathbb{1}_{\{k_i>0\}}\|h_i''\|_{L^p}.
\end{aligned}
\tag{A44}
$$

The estimate for $\tilde{N}_{\mathbf{k},\psi}$ can be derived easily, since

$$
|\tilde{N}_{\mathbf{k},\psi}-\tilde{M}_{\mathbf{k},\psi}|\leqslant CC_1^{|\mathbf{k}|}\prod_{i=1}^{n}\|h_i'\|_{L^\infty}^{k_i}\int_{\mathbb{T}}|h_{n+1}(\theta+\xi)-h_{n+1}(\theta)||\psi(\theta+\xi)|\,d\xi.
\tag{A45}
$$

Taking $C_{**}>0$ to be suitably large, we prove (A37) when $|\mathbf{k}|=N$.

This completes the proof. □

Lemma A.4 *Under the hypotheses of Lemma A.2, we additionally assume $h_1, \cdots, h_n \in W^{2,p}(\mathbb{T})$ and $\psi \in W^{1,p}(\mathbb{T})$.*

$$\begin{aligned}
&\|M_{\mathbf{k},\psi}\|_{\dot W^{1,p}} + \|N_{\mathbf{k},\psi}\|_{\dot W^{1,p}} \\
&\leqslant (|\mathbf{k}|+1)C_\dagger^{|\mathbf{k}|+1}\|\psi'\|_{L^p}\prod_{i=1}^n \|h_i'\|_{L^\infty}^{k_i} \\
&\quad + (|\mathbf{k}|+1)C_\dagger^{|\mathbf{k}|+1}\|\psi\|_{L^\infty}\sum_{i=1}^n \|h_1'\|_{L^\infty}^{k_1}\cdots\|h_i'\|_{L^\infty}^{k_i-1}\cdots\|h_n'\|_{L^\infty}^{k_n}\cdot \mathbb{1}_{\{k_i>0\}}\|h_i''\|_{L^p}, \qquad \text{(A46)}
\end{aligned}$$

where $C_\dagger$ is a universal constant depending only on p.

Proof. Instead of studying weak derivatives of $M_{\mathbf{k},\psi}$ and $N_{\mathbf{k},\psi}$ directly, we turn to difference quotients first. Without loss of generality, let $\varepsilon > 0$ be arbitrary and sufficiently small. It suffices to prove uniform-in-ε L^p-bounds for $\varepsilon^{-1}(M_{\mathbf{k},\psi}(\theta+\varepsilon) - M_{\mathbf{k},\psi}(\theta))$ and $\varepsilon^{-1}(N_{\mathbf{k},\psi}(\theta+\varepsilon) - N_{\mathbf{k},\psi}(\theta))$. Write

$$\begin{aligned}
&\varepsilon^{-1}\big(M_{\mathbf{k},\psi}(\theta+\varepsilon) - M_{\mathbf{k},\psi}(\theta)\big) \\
&= \sum_{i=1}^n \int_{\mathbb{T}} \big(\Delta h_1(\theta)\big)^{k_1}\cdots\big(\Delta h_{i-1}(\theta)\big)^{k_{i-1}}\big(\Delta h_{i+1}(\theta+\varepsilon)\big)^{k_{i+1}}\cdot\big(\Delta h_n(\theta+\varepsilon)\big)^{k_n} \\
&\qquad \cdot \sum_{l=0}^{k_i-1}\big(\Delta h_i(\theta)\big)^l\big(\Delta h_i(\theta+\varepsilon)\big)^{k_i-1-l}\Delta\left(\frac{h_i(\theta+\varepsilon)-h_i(\theta)}{\varepsilon}\right)\frac{\psi(\theta+\varepsilon+\xi)}{2\tan\frac{\xi}{2}}\,d\xi \\
&\qquad\qquad + \int_{\mathbb{T}}\prod_{i=1}^n\big(\Delta h_i(\theta)\big)^{k_i}\cdot\frac{\varepsilon^{-1}(\psi(\theta+\varepsilon+\xi)-\psi(\theta+\xi))}{2\tan\frac{\xi}{2}}\,d\xi. \qquad \text{(A47)}
\end{aligned}$$

Applying Lemma A.2 and Lemma A.3,

$$\begin{aligned}
&\big\|\varepsilon^{-1}\big(M_{\mathbf{k},\psi}(\theta+\varepsilon) - M_{\mathbf{k},\psi}(\theta)\big)\big\|_{L^p} \\
&\leqslant \sum_{i=1}^n k_i C_{**}^{|\mathbf{k}|}\big(\|\varepsilon^{-1}\big(h_i(\theta+\varepsilon)-h_i(\theta)\big)'\|_{L^p}\|\psi\|_{L^\infty} + \|\varepsilon^{-1}\big(h_i(\theta+\varepsilon)-h_i(\theta)\big)\|_{L^\infty}\|\psi'\|_{L^p}\big) \\
&\qquad \cdot \|h_1'\|_{L^\infty}^{k_1}\cdots\|h_i'\|_{L^\infty}^{k_i-1}\cdots\|h_n'\|_{L^\infty}^{k_n} \\
&\quad + \sum_{i=1}^n C_{**}^{|\mathbf{k}|}\big\|\varepsilon^{-1}\big(h_i(\theta+\varepsilon)-h_i(\theta)\big)\big\|_{L^\infty}\|\psi\|_{L^\infty} \\
&\qquad \cdot k_i \sum_{\substack{j=1,\cdots,n\\ j\neq i}} \|h_1'\|_{L^\infty}^{k_1}\cdots\|h_i'\|_{L^\infty}^{k_i-1}\cdots\|h_j'\|_{L^\infty}^{k_j-1}\cdots\|h_n'\|_{L^\infty}^{k_n}\cdot\mathbb{1}_{\{k_j>0\}}\|h_j''\|_{L^p} \\
&\quad + \sum_{i=1}^n C_{**}^{|\mathbf{k}|}\big\|\varepsilon^{-1}\big(h_i(\theta+\varepsilon)-h_i(\theta)\big)\big\|_{L^\infty}\|\psi\|_{L^\infty} \\
&\qquad \cdot \|h_1'\|_{L^\infty}^{k_1}\cdots\|h_i'\|_{L^\infty}^{k_i-2}\cdots\|h_n'\|_{L^\infty}^{k_n}\cdot C(k_i-1)\mathbb{1}_{\{k_i>1\}}\|h_i''\|_{L^p} \\
&\quad + C_*^{|\mathbf{k}|+2}\prod_{i=1}^n\|h_i'\|_{L^\infty}^{k_i}\cdot\|\varepsilon^{-1}(\psi(\cdot+\varepsilon)-\psi(\cdot))\|_{L^p}
\end{aligned}$$

$$\leqslant C\big(|\mathbf{k}|C_{**}^{|\mathbf{k}|} + C_*^{|\mathbf{k}|+2}\big)\|\psi'\|_{L^p}\prod_{i=1}^{n}\|h_i'\|_{L^\infty}^{k_i}$$

$$+ C|\mathbf{k}|C_{**}^{|\mathbf{k}|}\|\psi\|_{L^\infty}\sum_{i=1}^{n}\|h_1'\|_{L^\infty}^{k_1}\cdots\|h_i'\|_{L^\infty}^{k_i-1}\cdots\|h_n'\|_{L^\infty}^{k_n}\cdot \mathbb{1}_{\{k_i>0\}}\|h_i''\|_{L^p}. \tag{A48}$$

Note that this bound is uniform in ε. Hence, $M_{\mathbf{k},\psi}(\theta)$ has weak derivative, with an identical L^p-bound as above. The estimate for $N_{\mathbf{k},\psi}$ can be derived similarly. Therefore, (A46) holds if $C_\dagger$ is taken to be suitably large. □

A.3 *Regularity theory of fractional heat equations*

We focus on the following Cauchy problem of fractional heat equation on $\mathbb{T}$ with special exponent $\frac{1}{2}$.

$$\partial_t v = -(-\Delta)^{1/2}v + f(t,\theta), \quad v(0,\theta) = 0. \tag{A49}$$

For our purpose, we have that:

Lemma A.5 *Suppose $f \in L^p_{[0,T]}L^p(\mathbb{T})$ for some $p \in [2,\infty)$. Then there exists $v \in L^p_{[0,T]}W^{1,p}(\mathbb{T})$ solving (A49), satisfying that*

$$\|v_t\|_{L^p_{[0,T]}L^p(\mathbb{T})} + \|(-\Delta)^{1/2}v\|_{L^p_{[0,T]}L^p(\mathbb{T})} \leqslant C\|f\|_{L^p_{[0,T]}L^p(\mathbb{T})}, \tag{A50}$$

where $C = C(p)$.

This immediately follows from [35, Theorem 1]; see also [5, Theorem 4.1].

Lemma A.6 *Suppose $T \leqslant 1$ and $p \in (2,\infty)$. Under the assumption of Lemma A.5, $v \in C_{[0,T]}C^\alpha(\mathbb{T})$ with $\alpha = 1 - \frac{2}{p}$, satisfying that*

$$\|v\|_{C_{[0,T]}\dot{C}^\alpha(\mathbb{T})} \leqslant C\|f\|_{L^p_{[0,T]}L^p(\mathbb{T})}, \tag{A51}$$

where $C = C(p)$.

Proof. Let $\mathcal{P}(t,\theta)$ be the Poisson kernel on $\mathbb{T}$, with t being the time variable, solving

$$\partial_t\mathcal{P} = -(-\Delta)^{1/2}\mathcal{P}, \quad \mathcal{P}(0,\theta) = \delta_0 \tag{A52}$$

in the sense of distribution. Here δ_0 is the delta measure at $0 \in \mathbb{T}$. Note that $\mathcal{P}(t,\theta)$ is related to $P(s,\xi)$, which is defined in Section 4, in the following sense

$$\mathcal{P}(t,\theta) = \frac{1}{2\pi}P(e^{-t},\theta). \tag{A53}$$

Then v can be represented by

$$v(t,\theta) = \int_0^t\int_{\mathbb{T}}\mathcal{P}(t-\tau,\theta-\xi)f(\tau,\xi)\,d\xi d\tau. \tag{A54}$$

Take arbitrary $\theta_1, \theta_2 \in \mathbb{T}$, such that $d_\theta := |\theta_1 - \theta_2| \leqslant 1$. Denote $\bar{\theta} = (\theta_1 + \theta_2)/2$. Then

$$
\begin{aligned}
&|v(t,\theta_1) - v(t,\theta_2)| \\
&\leqslant \int_{[0,t]\times\mathbb{T}\cap\{(\tau,\xi):\,|t-\tau|+|\bar{\theta}-\xi|\leqslant d_\theta\}} \big(|\mathcal{P}(t-\tau,\theta_1-\xi)| + |\mathcal{P}(t-\tau,\theta_2-\xi)|\big)|f(\tau,\xi)|\,d\xi d\tau \\
&\quad + \int_{[0,t]\times\mathbb{T}\cap\{(\tau,\xi):\,|t-\tau|+|\bar{\theta}-\xi|\geqslant d_\theta\}} |\mathcal{P}(t-\tau,\theta_1-\xi) - \mathcal{P}(t-\tau,\theta_2-\xi)||f(\tau,\xi)|\,d\xi d\tau. \qquad \text{(A55)}
\end{aligned}
$$

By the mean value theorem, Lemma A.1 and Hölder's inequality,

$$
\begin{aligned}
&|v(t,\theta_1) - v(t,\theta_2)| \\
&\quad\leqslant C\int_{[0,t]\times\mathbb{T}\cap\{(\tau,\xi):\,|t-\tau|+|\theta_1-\xi|\leqslant 2d_\theta\}} \frac{|f(\tau,\xi)|}{|t-\tau|+|\theta_1-\xi|}\,d\xi d\tau \\
&\qquad + C\int_{[0,t]\times\mathbb{T}\cap\{(\tau,\xi):\,|t-\tau|+|\theta_2-\xi|\leqslant 2d_\theta\}} \frac{|f(\tau,\xi)|}{|t-\tau|+|\theta_2-\xi|}\,d\xi d\tau \\
&\qquad + C|\theta_1-\theta_2|\int_{[0,t]\times\mathbb{T}\cap\{(\tau,\xi):\,|t-\tau|+|\bar{\theta}-\xi|\geqslant d_\theta\}} \frac{|f(\tau,\xi)|}{|t-\tau|^2+|\bar{\theta}-\xi|^2}\,d\xi d\tau \\
&\quad\leqslant C\|f\|_{L^p([0,T]\times\mathbb{T})}\left(\int_0^{2d_\theta}\rho^{1-p'}\,d\rho\right)^{1/p'} \\
&\qquad + C|\theta_1-\theta_2|\|f\|_{L^p([0,T]\times\mathbb{T})}\left(\int_{d_\theta/\sqrt{2}}^{\infty}\rho^{1-2p'}\,d\rho\right)^{1/p'}. \qquad \text{(A56)}
\end{aligned}
$$

Here $p' = (1-\frac{1}{p})^{-1} \in (1,2)$. Calculating the integral above yields

$$
|v(t,\theta_1) - v(t,\theta_2)| \leqslant C|\theta_1-\theta_2|^\alpha\|f\|_{L^p_{[0,T]}L^p(\mathbb{T})}. \qquad \text{(A57)}
$$

It is then straightforward to justify the case $|\theta_1 - \theta_2| > 1$.

The time-continuity of v in $C^{1,\alpha}$ follows from the absolute continuity of the Lebesgue integral with respect to translation. □

Lemma A.7 *Suppose $T \leqslant 1$ and $f \in L^\infty_{[0,T]}C^\alpha(\mathbb{T})$ for some $\alpha \in (0,1)$. Then for all $\beta \in (0,\alpha)$, there exists a unique $v \in C_{[0,T]}C^{1,\beta}(\mathbb{T})$ solving (A49), satisfying that*

$$
\|v\|_{C_{[0,T]}\dot{C}^{1,\beta}(\mathbb{T})} \leqslant C\|f\|_{L^\infty_{[0,T]}\dot{C}^\alpha(\mathbb{T})}, \qquad \text{(A58)}
$$

where $C = C(\alpha,\beta)$.

Proof. Once again, v can be represented by (A54). It then suffices to bound its $\dot{C}^{1,\beta}$-seminorm, which also implies the uniqueness.

For arbitrary $\theta_1, \theta_2 \in \mathbb{T}$,

$$
\begin{aligned}
&\partial_\theta v(t,\theta_1) - \partial_\theta v(t,\theta_2) \\
&\quad= \int_0^t\int_{\mathbb{T}} \partial_\theta\mathcal{P}(t-\tau,\xi)\big(f(\tau,\theta_1-\xi) - f(\tau,\theta_1) - f(\tau,\theta_2-\xi) + f(\tau,\theta_2)\big)\,d\xi d\tau \qquad \text{(A59)}
\end{aligned}
$$

Since

$$|f(\tau,\theta_1-\xi)-f(\tau,\theta_1)-f(\tau,\theta_2-\xi)+f(\tau,\theta_2)| \leqslant C\|f(\tau,\cdot)\|_{\dot{C}^\alpha}\min\{|\xi|^\alpha,|\theta_1-\theta_2|^\alpha\}, \quad \text{(A60)}$$

by (A53) and Lemma A.1, we have that

$$\begin{aligned}
|\partial_\theta v(t,\theta_1)-\partial_\theta v(t,\theta_2)| &\leqslant \int_0^t\int_{\mathbb{T}} |\partial_\theta \mathcal{P}(t-\tau,\xi)||\xi|^{\alpha-\beta}\,d\xi d\tau\cdot|\theta_1-\theta_2|^\beta\|f\|_{L^\infty_{[0,T]}\dot{C}^\alpha(\mathbb{T})}\\
&\leqslant C\int_0^t (1-e^{-(t-\tau)})^{\alpha-\beta-1}\,d\tau\cdot|\theta_1-\theta_2|^\beta\|f\|_{L^\infty_{[0,T]}\dot{C}^\alpha(\mathbb{T})}\\
&\leqslant C|\theta_1-\theta_2|^\beta\|f\|_{L^\infty_{[0,T]}\dot{C}^\alpha(\mathbb{T})}. \qquad \text{(A61)}
\end{aligned}$$

Finally, the time continuity of v can be justified by interpolating between the facts that $v \in C_{[0,T]}C^\alpha(\mathbb{T})$ and $v\in L^\infty_{[0,T]}C^{1,\beta'}(\mathbb{T})$ for some $\beta'\in(\beta,\alpha)$. □

B. Proofs of Lemma 3.4 and Lemma 3.5

We need several preparatory results.

Let h_i and H_i be given as in Section 3.3. Let $x_i(X)$ $(i=1,2)$ denote the diffeomorphism (3.2) defined by h_i and H_i,

$$x_i(X)=\zeta_i(X)X, \quad \zeta_i(X):=1+h_i(\omega)\eta_\delta\left(\frac{\rho}{r}\right)+H_i(\omega)\eta_\delta\left(\frac{\rho}{R}\right). \qquad \text{(B1)}$$

Let p_i denote the pressure on the physical domain that is determined by γ_i and $\tilde{\gamma}_i$, while $\tilde{p}_i$ denotes its pull back into the reference coordinate as in (3.4). By (3.5), $(\tilde{p}_1-\tilde{p}_2)$ solves

$$\begin{aligned}
&-\nabla_{X_k}\left(a\frac{\partial X_k}{\partial x_{1,i}}\frac{\partial X_j}{\partial x_{1,i}}\nabla_{X_j}(\tilde{p}_1-\tilde{p}_2)\right)\\
&=\nabla_{X_k}\left[a\left(\frac{\partial X_k}{\partial x_{1,i}}\frac{\partial X_j}{\partial x_{1,i}}-\frac{\partial X_k}{\partial x_{2,i}}\frac{\partial X_j}{\partial x_{2,i}}\right)\nabla_{X_j}\tilde{p}_2\right]\\
&\qquad+\big(G(\tilde{p}_1)-G(\tilde{p}_2)\big)\chi_{B_r}-\nabla_{X_k}\frac{\partial X_k}{\partial x_{1,i}}\cdot a\frac{\partial X_j}{\partial x_{1,i}}\nabla_{X_j}(\tilde{p}_1-\tilde{p}_2)\\
&\qquad-a\left[\nabla_{X_k}\frac{\partial X_k}{\partial x_{1,i}}\cdot\frac{\partial X_j}{\partial x_{1,i}}-\nabla_{X_k}\frac{\partial X_k}{\partial x_{2,i}}\cdot\frac{\partial X_j}{\partial x_{2,i}}\right]\nabla_{X_j}\tilde{p}_2 \qquad \text{(B2)}
\end{aligned}$$

in B_R, with $(\tilde{p}_1-\tilde{p}_2)|_{\partial B_R}=0$. Here $a=a(X)$ is given in (3.6), and $x_{1,i}$ and $x_{2,i}$ denote i-th components of x_1 and x_2, respectively.

We first derive estimates for several ingredients in (B2).

Lemma B.1 *Assume $h_i, H_i\in W^{1,\infty}(\mathbb{T})$ satisfy that $m_{0,i}+M_{0,i}\ll 1$. Then*

$$\left\|\frac{\partial X}{\partial x_1}-\frac{\partial X}{\partial x_2}\right\|_{L^\infty(B_R)}\leqslant C(\Delta m_0+\Delta M_0), \qquad \text{(B3)}$$

$$\left\|\frac{\partial X_k}{\partial x_{1,i}}\frac{\partial X_j}{\partial x_{1,i}}-\frac{\partial X_k}{\partial x_{2,i}}\frac{\partial X_j}{\partial x_{2,i}}\right\|_{L^\infty(B_R)}\leqslant C(\Delta m_0+\Delta M_0), \qquad \text{(B4)}$$

$$\left\|\nabla_{X_k}\frac{\partial X_k}{\partial x_{1,i}}-\nabla_{X_k}\frac{\partial X_k}{\partial x_{2,i}}\right\|_{L^\infty(B_R)}\leqslant C(\delta r)^{-1}(\Delta m_0+\Delta M_0), \qquad \text{(B5)}$$

where the constants C are all universal.

If in addition, $h_i, H_i \in C^{1,\alpha}(\mathbb{T})$ for some $\alpha \in (0,1)$, such that $m_{\alpha,i} + M_{\alpha,i} \ll 1$, then

$$\left\| \frac{\partial X}{\partial x_1} - \frac{\partial X}{\partial x_2} \right\|_{\dot{C}^\alpha(B_R)} \leqslant C(\delta r)^{-\alpha}(\Delta m_\alpha + \Delta M_\alpha), \tag{B6}$$

and

$$\left\| \frac{\partial X_k}{\partial x_{1,i}} \frac{\partial X_j}{\partial x_{1,i}} - \frac{\partial X_k}{\partial x_{2,i}} \frac{\partial X_j}{\partial x_{2,i}} \right\|_{\dot{C}^\alpha(B_R)} \leqslant C(\delta r)^{-\alpha}(\Delta m_\alpha + \Delta M_\alpha). \tag{B7}$$

Here C are universal constants only depending on α. All the quantities above are only supported on $\overline{B_{r(1+2\delta)}} \backslash B_{r(1-2\delta)}$ and $\overline{B_R} \backslash B_{R(1-2\delta)}$.

Proof. The proof is once again a straightforward calculation.

We derive by (3.2) that

$$\begin{aligned}\frac{\partial X}{\partial x_1} - \frac{\partial X}{\partial x_2} &= (\zeta_1^2 + \zeta_1 \rho \partial_\rho \zeta_1)^{-1}\Big((\zeta_1 - \zeta_2)\cdot Id + \big(\nabla(\zeta_1 - \zeta_2)\big)^\perp \otimes X^\perp\Big) \\ &\quad + \frac{(\zeta_2^2 + \zeta_2\rho\partial_\rho\zeta_2) - (\zeta_1^2 + \zeta_1\rho\partial_\rho\zeta_1)}{(\zeta_1^2 + \zeta_1\rho\partial_\rho\zeta_1)(\zeta_2^2 + \zeta_2\rho\partial_\rho\zeta_2)}(\zeta_2 \cdot Id + (\nabla\zeta_2)^\perp \otimes X^\perp). \end{aligned} \tag{B8}$$

By (3.27),

$$\begin{aligned}\big|(\zeta_2^2 + \zeta_2\rho\partial_\rho\zeta_2) - (\zeta_1^2 + \zeta_1\rho\partial_\rho\zeta_1)\big| &\leqslant |\zeta_1 - \zeta_2||\zeta_1 + \zeta_2 + \rho\partial_\rho\zeta_2| + |\zeta_1||\rho\partial_\rho(\zeta_1 - \zeta_2)| \\ &\leqslant C\delta^{-1}\big(\|h_1 - h_2\|_{L^\infty} + \|H_1 - H_2\|_{L^\infty}\big) \\ &\leqslant C(\Delta m_0 + \Delta M_0). \end{aligned} \tag{B9}$$

Combining (3.28), (3.30) and (B9) with (B8), we find that

$$\begin{aligned}\left|\frac{\partial X}{\partial x_1} - \frac{\partial X}{\partial x_2}\right| &\leqslant C\big(|\zeta_1 - \zeta_2| + \rho|\nabla(\zeta_1 - \zeta_2)|\big) \\ &\qquad + C\big|(\zeta_2^2 + \zeta_2\rho\partial_\rho\zeta_2) - (\zeta_1^2 + \zeta_1\rho\partial_\rho\zeta_1)\big|(|\zeta_2| + \rho|\nabla\zeta_2|) \\ &\leqslant C(\Delta m_0 + \Delta M_0), \end{aligned} \tag{B10}$$

which proves (B3). It is easy to derive (B4) from (B3) and Lemma 3.2.

To show (B5), we use (3.25) to derive that

$$\begin{aligned}\nabla_{X_k}\frac{\partial X_k}{\partial x_{1,i}} - \nabla_{X_k}\frac{\partial X_k}{\partial x_{2,i}} &= \left(\frac{\partial X_j}{\partial x_{2,i}} - \frac{\partial X_j}{\partial x_{1,i}}\right)(\zeta_1^2 + \zeta_1\rho\partial_\rho\zeta_1)^{-1}\nabla_{X_j}(\zeta_1^2 + \zeta_1\rho\cdot\partial_\rho\zeta_1) \\ &\quad + \frac{\partial X_j}{\partial x_{2,i}}\frac{(\zeta_1^2 + \zeta_1\rho\partial_\rho\zeta_1) - (\zeta_2^2 + \zeta_2\rho\partial_\rho\zeta_2)}{(\zeta_1^2 + \zeta_1\rho\partial_\rho\zeta_1)(\zeta_2^2 + \zeta_2\rho\partial_\rho\zeta_2)}\nabla_{X_j}(\zeta_1^2 + \zeta_1\rho\cdot\partial_\rho\zeta_1) \\ &\quad + \frac{\partial X_j}{\partial x_{2,i}}(\zeta_2^2 + \zeta_2\rho\partial_\rho\zeta_2)^{-1}\nabla_{X_j}[(\zeta_2^2 + \zeta_2\rho\cdot\partial_\rho\zeta_2) - (\zeta_1^2 + \zeta_1\rho\cdot\partial_\rho\zeta_1)]. \end{aligned} \tag{B11}$$

Then by (3.27), (3.28), (3.30), (3.32), (B9) and Lemma 3.2,

$$\begin{aligned}\left|\nabla_{X_k}\frac{\partial X_k}{\partial x_{1,i}}-\nabla_{X_k}\frac{\partial X_k}{\partial x_{2,i}}\right| &\leqslant C\left|\frac{\partial X}{\partial x_2}-\frac{\partial X}{\partial x_1}\right|\left|\nabla_{X_j}(\zeta_1^2+\zeta_1\rho\partial_\rho\zeta_1)\right| \\ &\quad + C\left|(\zeta_1^2+\zeta_1\rho\partial_\rho\zeta_1)-(\zeta_2^2+\zeta_2\rho\partial_\rho\zeta_2)\right|\left|\nabla_{X_j}(\zeta_1^2+\zeta_1\rho\partial_\rho\zeta_1)\right| \\ &\quad + C\left|\nabla_{X_j}[(\zeta_2^2+\zeta_2\rho\partial_\rho\zeta_2)-(\zeta_1^2+\zeta_1\rho\partial_\rho\zeta_1)]\right| \\ &\leqslant C(\delta r)^{-1}(\varDelta m_0+\varDelta M_0). \end{aligned} \tag{B12}$$

To prove (B6), we start with a Hölder estimate of $(\zeta_1^2+\zeta_1\rho\partial_\rho\zeta_1)-(\zeta_2^2+\zeta_2\rho\partial_\rho\zeta_2)$. Using the fact that $\|fg\|_{\dot C^\alpha}\leqslant\|f\|_{\dot C^\alpha}\|g\|_{L^\infty}+\|f\|_{L^\infty}\|g\|_{\dot C^\alpha}$,

$$\begin{aligned}&\left\|(\zeta_1^2+\zeta_1\rho\partial_\rho\zeta_1)-(\zeta_2^2+\zeta_2\rho\partial_\rho\zeta_2)\right\|_{\dot C^\alpha(B_R)} \\ &\quad\leqslant\|\zeta_1-\zeta_2\|_{\dot C^\alpha(B_R)}(\|\zeta_1+\zeta_2\|_{L^\infty}+\|\rho\partial_\rho\zeta_1\|_{L^\infty}) \\ &\qquad+\|\zeta_1-\zeta_2\|_{L^\infty}(\|\zeta_1+\zeta_2\|_{\dot C^\alpha(B_R)}+\|\rho\partial_\rho\zeta_1\|_{\dot C^\alpha(B_R)}) \\ &\qquad+\|\zeta_2\|_{\dot C^\alpha(B_R)}\|\rho\partial_\rho\zeta_1-\rho\partial_\rho\zeta_2\|_{L^\infty}+\|\zeta_2\|_{L^\infty}\|\rho\partial_\rho\zeta_1-\rho\partial_\rho\zeta_2\|_{\dot C^\alpha(B_R)}. \end{aligned} \tag{B13}$$

Note that the Hölder semi-norms are taken over B_R with respect to the Euclidean distance in X-coordinate instead of the (ρ,ω)-coordinate. Using

$$(\zeta_1-\zeta_2)(X)=(h_1-h_2)(\omega)\eta_\delta\left(\frac{\rho}{r}\right)+(H_1-H_2)(\omega)\eta_\delta\left(\frac{\rho}{R}\right), \tag{B14}$$

and the fact that $\eta_\delta(\frac{\rho}{r})$ and $\eta_\delta(\frac{\rho}{R})$ are supported near ∂B_r and ∂B_R, respectively, we find that

$$\begin{aligned}\|\zeta_1-\zeta_2\|_{\dot C^\alpha(B_R)}&\leqslant Cr^{-\alpha}\|h_1-h_2\|_{\dot C^\alpha(\mathbb{T})}+C\|h_1-h_2\|_{L^\infty}\left\|\eta_\delta\left(\frac{\rho}{r}\right)\right\|_{\dot C^\alpha(B_R)} \\ &\quad+CR^{-\alpha}\|H_1-H_2\|_{\dot C^\alpha(\mathbb{T})}+C\|H_1-H_2\|_{L^\infty}\left\|\eta_\delta\left(\frac{\rho}{R}\right)\right\|_{\dot C^\alpha(B_R)} \\ &\leqslant Cr^{-\alpha}\|h_1-h_2\|_{\dot C^\alpha(\mathbb{T})}+C\|h_1-h_2\|_{L^\infty}(\delta r)^{-\alpha} \\ &\quad+CR^{-\alpha}\|H_1-H_2\|_{\dot C^\alpha(\mathbb{T})}+C\|H_1-H_2\|_{L^\infty}(\delta R)^{-\alpha} \\ &\leqslant C\delta^{1-\alpha}r^{-\alpha}\varDelta m_0+C\delta^{1-\alpha}R^{-\alpha}\varDelta M_0. \end{aligned} \tag{B15}$$

In the last line, we applied interpolation inequalities. Setting $h_1=H_1=0$ (or $h_2=H_2=0$), we obtain estimates for $\|\zeta_i\|_{\dot C^\alpha(B_R)}$.

Similarly, since

$$\rho\partial_\rho(\zeta_1-\zeta_2)=(h_1-h_2)(\omega)\cdot\frac{\rho}{r}\eta_\delta'\left(\frac{\rho}{r}\right)+(H_1-H_2)(\omega)\cdot\frac{\rho}{R}\eta_\delta'\left(\frac{\rho}{R}\right), \tag{B16}$$

we deduce that

$$\|\rho\partial_\rho\zeta_1-\rho\partial_\rho\zeta_2\|_{\dot C^\alpha(B_R)}\leqslant C(\delta r)^{-\alpha}\varDelta m_0+C(\delta R)^{-\alpha}\varDelta M_0. \tag{B17}$$

Combining (B15) and (B17) with (B13) yields that

$$\begin{aligned}\|(\zeta_1^2+\zeta_1\rho\partial_\rho\zeta_1)-(\zeta_2^2+\zeta_2\rho\partial_\rho\zeta_2)\|_{\dot C^\alpha(B_R)}\\ &\leqslant C\|\zeta_1-\zeta_2\|_{\dot C^\alpha(B_R)}\\ &\quad + C\delta(\Delta m_0+\Delta M_0)\cdot\big(\|\zeta_1\|_{\dot C^\alpha(B_R)}+\|\zeta_2\|_{\dot C^\alpha(B_R)}+\|\rho\partial_\rho\zeta_1\|_{\dot C^\alpha(B_R)}\big)\\ &\quad + C\|\zeta_2\|_{\dot C^\alpha}(\Delta m_0+\Delta M_0)+C\|\rho\partial_\rho\zeta_1-\rho\partial_\rho\zeta_2\|_{\dot C^\alpha(B_R)}\\ &\leqslant C(\delta r)^{-\alpha}(\Delta m_0+\Delta M_0). \end{aligned}\tag{B18}$$

Setting $h_2=H_2=0$ gives

$$\|\zeta_1^2+\zeta_1\rho\partial_\rho\zeta_1\|_{\dot C^\alpha(B_R)}\leqslant C(\delta r)^{-\alpha}(m_{0,1}+M_{0,1}).\tag{B19}$$

Thanks to (3.28), it is not difficult to derive that $\|(\zeta_1^2+\zeta_1\rho\partial_\rho\zeta_1)^{-1}\|_{\dot C^\alpha(B_R)}$ has the same bound, with a different constant C.

In addition, by (3.30),

$$\begin{aligned}\nabla(\zeta_1-\zeta_2)&=\left[(h_1-h_2)(\omega)\cdot\frac{1}{r}\eta_\delta'\left(\frac{\rho}{r}\right)+(H_1-H_2)(\omega)\cdot\frac{1}{R}\eta_\delta'\left(\frac{\rho}{R}\right)\right]e_r\\ &\quad+\left[(h_1'-h_2')(\omega)\cdot\eta_\delta\left(\frac{\rho}{r}\right)+(H_1'-H_2')(\omega)\cdot\eta_\delta\left(\frac{\rho}{R}\right)\right]\rho^{-1}e_\theta.\end{aligned}\tag{B20}$$

So

$$\begin{aligned}&\|\nabla(\zeta_1-\zeta_2)\|_{\dot C^\alpha(B_R)}\\ &\leqslant Cr^{-\alpha}\|(h_1-h_2)e_r\|_{\dot C^\alpha(\mathbb{T})}(\delta r)^{-1}+C\|h_1-h_2\|_{L^\infty}\left\|\frac{1}{r}\eta_\delta'\left(\frac{\rho}{r}\right)\right\|_{\dot C^\alpha(B_R)}\\ &\quad+CR^{-\alpha}\|(H_1-H_2)e_r\|_{\dot C^\alpha(\mathbb{T})}(\delta R)^{-1}+C\|H_1-H_2\|_{L^\infty}\left\|\frac{1}{R}\eta_\delta'\left(\frac{\rho}{R}\right)\right\|_{\dot C^\alpha(B_R)}\\ &\quad+Cr^{-\alpha}\|(h_1'-h_2')e_\theta\|_{\dot C^\alpha(\mathbb{T})}r^{-1}+C\|h_1'-h_2'\|_{L^\infty}\left\|\frac{1}{\rho}\eta_\delta\left(\frac{\rho}{r}\right)\right\|_{\dot C^\alpha(B_R)}\\ &\quad+CR^{-\alpha}\|(H_1'-H_2')e_\theta\|_{\dot C^\alpha(\mathbb{T})}R^{-1}+C\|H_1'-H_2'\|_{L^\infty}\left\|\frac{1}{\rho}\eta_\delta\left(\frac{\rho}{R}\right)\right\|_{\dot C^\alpha(B_R)}\\ &\leqslant C\delta^{-\alpha}(r^{-1-\alpha}\Delta m_\alpha+R^{-1-\alpha}\Delta M_\alpha).\end{aligned}\tag{B21}$$

Here we used the fact that $\Delta m_0+\Delta M_0\leqslant C(\Delta m_\alpha+\Delta M_\alpha)$ by interpolation.

To this end, combining (3.28), (3.30), (B8), (B9), (B15), (B18) and (B21),

$$\begin{aligned}\left\|\frac{\partial X}{\partial x_1}-\frac{\partial X}{\partial x_2}\right\|_{\dot C^\alpha(B_R)}&\leqslant C\|(\zeta_1^2+\zeta_1\rho\partial_\rho\zeta_1)^{-1}\|_{\dot C^\alpha(B_R)}\big(\|\zeta_1-\zeta_2\|_{L^\infty}+\|\rho\nabla(\zeta_1-\zeta_2)\|_{L^\infty}\big)\\ &\quad+C\big(\|\zeta_1-\zeta_2\|_{\dot C^\alpha(B_R)}+\|\nabla(\zeta_1-\zeta_2)^\perp\otimes X^\perp\|_{\dot C^\alpha(B_R)}\big)\\ &\quad+C\big\|(\zeta_2^2+\zeta_2\rho\partial_\rho\zeta_2)-(\zeta_1^2+\zeta_1\rho\partial_\rho\zeta_1)\big\|_{\dot C^\alpha(B_R)}\\ &\quad+C\big\|(\zeta_2^2+\zeta_2\rho\partial_\rho\zeta_2)-(\zeta_1^2+\zeta_1\rho\partial_\rho\zeta_1)\big\|_{L^\infty}\end{aligned}$$

$$
\begin{aligned}
&\cdot\Big[\|(\zeta_1^2+\zeta_1\rho\partial_\rho\zeta_1)^{-1}\|_{\dot C^\alpha(B_R)}+\|(\zeta_2^2+\zeta_2\rho\partial_\rho\zeta_2)^{-1}\|_{\dot C^\alpha(B_R)}\\
&\qquad+\|\zeta_2\|_{\dot C^\alpha(B_R)}+\|(\nabla\zeta_2)^\perp\otimes X^\perp\|_{\dot C^\alpha(B_R)}\Big]\\
&\leqslant C(\delta r)^{-\alpha}(\Delta m_\alpha+\Delta M_\alpha).
\end{aligned}\tag{B22}
$$

In the last inequality, we needed the assumption $m_{\alpha,i}+M_{\alpha,i}\ll 1$.

Finally, (B7) follows from (B3), (B6) and Lemma 3.2. □

Lemma B.2 *Assume $h_2, H_2\in C^{1,\alpha}(\mathbb{T})$ with $\alpha<\frac14$, satisfying that $m_{\alpha,2}+M_{\alpha,2}\ll 1$. Then*

$$
\|\tilde p_2\|_{C^{1,\alpha}(\overline{B_r})}+\|\tilde p_2\|_{C^{1,\alpha}(\overline{B_R\backslash B_r})}\leqslant C(\alpha,\mu,\nu,r,R,G).\tag{B23}
$$

Proof. By (3.5), $\tilde p_2$ solves

$$
-\nabla_{X_k}\left(a\frac{\partial X_k}{\partial x_{2,i}}\frac{\partial X_j}{\partial x_{2,i}}\nabla_{X_j}\tilde p_2\right)=G(\tilde p_2)\chi_{B_r}-\nabla_{X_k}\frac{\partial X_k}{\partial x_{2,i}}\cdot a\frac{\partial X_j}{\partial x_{2,i}}\nabla_{X_j}\tilde p_2\tag{B24}
$$

in B_R, with $\tilde p_2|_{\partial B_R}=0$. By putting $h_1=H_1=0$ in (B4) and (B7), we obtain that

$$
\left\|\frac{\partial X_k}{\partial x_{2,i}}\frac{\partial X_j}{\partial x_{2,i}}-Id\right\|_{L^\infty(B_R)}\leqslant C(m_{0,2}+M_{0,2}),\tag{B25}
$$

$$
\left\|\frac{\partial X_k}{\partial x_{2,i}}\frac{\partial X_j}{\partial x_{2,i}}\right\|_{\dot C^\alpha(B_R)}\leqslant C(\delta r)^{-\alpha}(m_{\alpha,2}+M_{\alpha,2}).\tag{B26}
$$

By assuming $m_{\alpha,2}+M_{\alpha,2}$ to be suitably small (ans thus $m_{0,2}+M_{0,2}$ is small by interpolation), we may have the coefficient matrix satisfy

$$
\frac12\min\{\mu,\nu\}Id\leqslant a\frac{\partial X_k}{\partial x_{2,i}}\frac{\partial X_j}{\partial x_{2,i}}\leqslant 2\max\{\mu,\nu\}Id,\tag{B27}
$$

which is symmetric and piecewise C^α in B_R. Therefore, by [36, Corollary 1.3] and Lemma 3.2, for $\alpha<\frac14$,

$$
\begin{aligned}
&\|\tilde p_2\|_{C^{1,\alpha}(\overline{B_r})}+\|\tilde p_2\|_{C^{1,\alpha}(\overline{B_R\backslash B_r})}\\
&\quad\leqslant C\left(\alpha,\mu,\nu,r,R,\left\|\frac{\partial X_k}{\partial x_{2,i}}\frac{\partial X_j}{\partial x_{2,i}}\right\|_{C^\alpha(B_R)}\right)\left\|G(\tilde p_2)\chi_{B_r}-\nabla_{X_k}\frac{\partial X_k}{\partial x_{2,i}}\cdot a\frac{\partial X_j}{\partial x_{2,i}}\nabla_{X_j}\tilde p_2\right\|_{L^\infty}\\
&\quad\leqslant C(\alpha,\mu,\nu,r,R,G)\big(1+\|\nabla\tilde p_2\|_{L^\infty(B_R)}\big).
\end{aligned}\tag{B28}
$$

We omit the dependence of C on $m_{0,2}+M_{0,2}$ and $m_{\alpha,2}+M_{\alpha,2}$ since they can be bounded by universal constants. The δ-dependence of C is encoded in the (r,R)-dependence. By interpolation inequality, with $\epsilon>0$ to be chosen and C_ϵ depending on ϵ and α,

$$
\|\nabla\tilde p_2\|_{L^\infty(B_R)}\leqslant\epsilon\big(\|\tilde p_2\|_{C^{1,\alpha}(\overline{B_r})}+\|\tilde p_2\|_{C^{1,\alpha}(\overline{B_R\backslash B_r})}\big)+C_\epsilon\|\tilde p_2\|_{L^\infty(B_R)}.\tag{B29}
$$

Taking ϵ suitably small, we conclude from (B28) that

$$
\|\tilde p_2\|_{C^{1,\alpha}(\overline{B_r})}+\|\tilde p_2\|_{C^{1,\alpha}(\overline{B_R\backslash B_r})}\leqslant C(\alpha,\mu,\nu,r,R,G)(1+\|\tilde p_2\|_{L^\infty(B_R)}).\tag{B30}
$$

Then the desired estimate follows from the fact $p_2\in[0,p_M]$ (see Section 1). □

Now we are ready to prove Lemma 3.5.

Proof of Lemma 3.5. In this proof, we shall use C_* to denote universal constants with the dependence $C_* = C_*(\alpha, \mu, \nu, r, R, G)$. Its precise definition may vary from line to line.

STEP 1 (L^∞-bound) Rewrite (B2) as

$$\begin{aligned}\nabla_{X_k}&\left(a\frac{\partial X_k}{\partial x_{1,i}}\frac{\partial X_j}{\partial x_{1,i}}\nabla_{X_j}(\tilde p_1-\tilde p_2)\right)\\&\qquad+\frac{G(\tilde p_1)-G(\tilde p_2)}{\tilde p_1-\tilde p_2}\chi_{B_r}\cdot(\tilde p_1-\tilde p_2)-\nabla_{X_k}\frac{\partial X_k}{\partial x_{1,i}}\cdot a\frac{\partial X_j}{\partial x_{1,i}}\nabla_{X_j}(\tilde p_1-\tilde p_2)\\&=-\nabla_{X_k}\left[a\left(\frac{\partial X_k}{\partial x_{1,i}}\frac{\partial X_j}{\partial x_{1,i}}-\frac{\partial X_k}{\partial x_{2,i}}\frac{\partial X_j}{\partial x_{2,i}}\right)\nabla_{X_j}\tilde p_2\right]\\&\qquad+a\left[\nabla_{X_k}\frac{\partial X_k}{\partial x_{1,i}}\cdot\frac{\partial X_j}{\partial x_{1,i}}-\nabla_{X_k}\frac{\partial X_k}{\partial x_{2,i}}\cdot\frac{\partial X_j}{\partial x_{2,i}}\right]\nabla_{X_j}\tilde p_2.\end{aligned}\tag{B31}$$

Arguing as in the proof of Lemma B.2, we may assume the coefficient matrix satisfies

$$\frac{1}{2}\min\{\mu,\nu\}Id\leqslant a\frac{\partial X_k}{\partial x_{1,i}}\frac{\partial X_j}{\partial x_{1,i}}\leqslant 2\max\{\mu,\nu\}Id,\tag{B32}$$

and it is symmetric and piecewise C^α in B_R. Moreover,

$$\frac{G(\tilde p_1)-G(\tilde p_2)}{\tilde p_1-\tilde p_2}\chi_{B_r}\leqslant 0,\tag{B33}$$

and

$$\left|\frac{G(\tilde p_1)-G(\tilde p_2)}{\tilde p_1-\tilde p_2}\right|+\left|\nabla_{X_k}\frac{\partial X_k}{\partial x_{1,i}}\cdot a\frac{\partial X_j}{\partial x_{1,i}}\right|\leqslant C(\mu,\nu,r,R,G).\tag{B34}$$

Recall that $(\tilde p_1-\tilde p_2)|_{\partial B_R}=0$. By the L^∞-bound of the weak solution [27, Theorem 8.16], together with Lemma B.1 and Lemma B.2,

$$\begin{aligned}&\|\tilde p_1-\tilde p_2\|_{L^\infty(B_R)}\\&\leqslant C(\mu,\nu,r,R,G)\left\|a\left(\frac{\partial X_k}{\partial x_{1,i}}\frac{\partial X_j}{\partial x_{1,i}}-\frac{\partial X_k}{\partial x_{2,i}}\frac{\partial X_j}{\partial x_{2,i}}\right)\nabla_{X_j}\tilde p_2\right\|_{L^4(B_R)}\\&\qquad+C(\mu,\nu,r,R,G)\left\|a\left[\nabla_{X_k}\frac{\partial X_k}{\partial x_{1,i}}\cdot\frac{\partial X_j}{\partial x_{1,i}}-\nabla_{X_k}\frac{\partial X_k}{\partial x_{2,i}}\cdot\frac{\partial X_j}{\partial x_{2,i}}\right]\nabla_{X_j}\tilde p_2\right\|_{L^2(B_R)}\\&\leqslant C_*(\Delta m_0+\Delta M_0).\end{aligned}\tag{B35}$$

This proves (3.52).

STEP 2 ($C^{1,\alpha}$-bound) This part of the proof is similar to that of Lemma B.2.

In addition to (B32), we know that

$$a\left(\frac{\partial X_k}{\partial x_{1,i}}\frac{\partial X_j}{\partial x_{1,i}}-\frac{\partial X_k}{\partial x_{2,i}}\frac{\partial X_j}{\partial x_{2,i}}\right)\nabla_{X_j}\tilde p_2\tag{B36}$$

is piecewise C^α thanks to Lemma B.1 and Lemma B.2. Applying [36, Corollary 1.3] to (B2), for $\alpha < \frac{1}{4}$,

$$\begin{aligned}
&\|\tilde{p}_1 - \tilde{p}_2\|_{C^{1,\alpha}(\overline{B_r})} + \|\tilde{p}_1 - \tilde{p}_2\|_{C^{1,\alpha}(\overline{B_R \setminus B_r})} \\
&\leqslant C\|G(\tilde{p}_1) - G(\tilde{p}_2)\|_{L^\infty(B_r)} + C\left\|\nabla_{X_k} \frac{\partial X_k}{\partial x_{1,i}} \frac{\partial X_j}{\partial x_{1,i}}\right\|_{L^\infty(B_R)} \|\nabla(\tilde{p}_1 - \tilde{p}_2)\|_{L^\infty(B_R)} \\
&\quad + C\left\|\nabla_{X_k} \frac{\partial X_k}{\partial x_{1,i}} \cdot \frac{\partial X_j}{\partial x_{1,i}} - \nabla_{X_k} \frac{\partial X_k}{\partial x_{2,i}} \cdot \frac{\partial X_j}{\partial x_{2,i}}\right\|_{L^\infty(B_R)} \|\nabla \tilde{p}_2\|_{L^\infty(B_R)} \\
&\quad + C\left\|\left(\frac{\partial X_k}{\partial x_{1,i}} \frac{\partial X_j}{\partial x_{1,i}} - \frac{\partial X_k}{\partial x_{2,i}} \frac{\partial X_j}{\partial x_{2,i}}\right) \nabla_{X_j} \tilde{p}_2\right\|_{C^\alpha(\overline{B_r})} \\
&\quad + C\left\|\left(\frac{\partial X_k}{\partial x_{1,i}} \frac{\partial X_j}{\partial x_{1,i}} - \frac{\partial X_k}{\partial x_{2,i}} \frac{\partial X_j}{\partial x_{2,i}}\right) \nabla_{X_j} \tilde{p}_2\right\|_{C^\alpha(\overline{B_R \setminus B_r})}. \qquad \text{(B37)}
\end{aligned}$$

Here the constants

$$C = C\left(\alpha, \mu, \nu, r, R, \left\|\frac{\partial X_k}{\partial x_{1,i}} \frac{\partial X_j}{\partial x_{1,i}}\right\|_{C^\alpha(B_R)}\right). \qquad \text{(B38)}$$

By (3.52), Lemma B.1 and Lemma B.2, we simplify (B37) to be

$$\begin{aligned}
&\|\tilde{p}_1 - \tilde{p}_2\|_{C^{1,\alpha}(\overline{B_r})} + \|\tilde{p}_1 - \tilde{p}_2\|_{C^{1,\alpha}(\overline{B_R \setminus B_r})} \\
&\qquad \leqslant C_*\|\tilde{p}_1 - \tilde{p}_2\|_{L^\infty(B_r)} + C_*(\delta r)^{-1}(m_{0,1} + M_{0,1})\|\nabla(\tilde{p}_1 - \tilde{p}_2)\|_{L^\infty(B_R)} \\
&\qquad\quad + C_*(\delta r)^{-1}(\Delta m_0 + \Delta M_0) \\
&\qquad\quad + C_*(\Delta m_0 + \Delta M_0) + C_*(\delta r)^{-\alpha}(\Delta m_\alpha + \Delta M_\alpha) \\
&\qquad \leqslant C_*\|\nabla(\tilde{p}_1 - \tilde{p}_2)\|_{L^\infty(B_R)} + C_*(\Delta m_\alpha + \Delta M_\alpha). \qquad \text{(B39)}
\end{aligned}$$

By interpolation and arguing as in the proof of Lemma B.2,

$$\|\tilde{p}_1 - \tilde{p}_2\|_{C^{1,\alpha}(\overline{B_r})} + \|\tilde{p}_1 - \tilde{p}_2\|_{C^{1,\alpha}(\overline{B_R \setminus B_r})} \leqslant C_*\big(\Delta m_\alpha + \Delta M_\alpha + \|\tilde{p}_1 - \tilde{p}_2\|_{L^\infty(B_R)}\big). \quad \text{(B40)}$$

Now by the L^∞-bound (3.52), we conclude with (3.53). □

Lemma 3.4 follows from Lemma 3.5 immediately.

Proof of Lemma 3.4. Back in the physical coordinate, by (2.13),

$$\partial_t h_i = -\frac{1}{r f_i} \cdot u_i\big(\gamma(\theta)\big) \cdot \gamma_i'(\theta)^\perp = -\frac{\mu\big((1 + h_i)e_r - h_i' e_\theta\big)_j}{r(1 + h_i)} \cdot \left[\frac{\partial X_k}{\partial x_{i,j}} \cdot \nabla_{X_k} \tilde{p}_i\right]\bigg|_{\partial B_r}. \qquad \text{(B41)}$$

Here ∇_{X_k} is taken from the inside of ∂B_r. Similarly,

$$\partial_t H_i = -\frac{\nu\big((1 + H_i)e_r - H_i' e_\theta\big)_j}{R(1 + H_i)} \cdot \left[\frac{\partial X_k}{\partial x_{i,j}} \cdot \nabla_{X_k} \tilde{p}_i\right]\bigg|_{\partial B_R}. \qquad \text{(B42)}$$

By definition (B1), $\zeta_i = 1 + h_i(\theta)$ in a neighborhood of ∂B_r, while $\zeta_i = 1 + H_i(\theta)$ near ∂B_R. So (3.30) reduces to

$$\nabla \zeta_i = \begin{cases} h_i'(\theta) r^{-1} e_\theta & \text{on } \partial B_r, \\ H_i'(\theta) R^{-1} e_\theta & \text{on } \partial B_R. \end{cases} \qquad \text{(B43)}$$

Hence, (3.2) can be simplified as

$$\frac{\partial X_k}{\partial x_{i,j}} = \begin{cases} \big(1+h_i(\theta)\big)^{-2}\big[\big(1+h_i(\theta)\big)\delta_{kj} - h_i'(\theta)e_{r,k}\otimes e_{\theta,j}\big] & \text{on } \partial B_r, \\ \big(1+H_i(\theta)\big)^{-2}\big[\big(1+H_i(\theta)\big)\delta_{kj} - H_i'(\theta)e_{r,k}\otimes e_{\theta,j}\big] & \text{on } \partial B_R. \end{cases} \tag{B44}$$

Now we calculate by (B41) and (B42) that

$$\partial_t h_i = -\frac{\mu}{r}\left[\frac{(1+h_i)^2+(h_i')^2}{(1+h_i)^3}e_r - \frac{h_i'}{(1+h_i)^2}e_\theta\right]\nabla \tilde{p}_i|_{\partial B_r}, \tag{B45}$$

$$\partial_t H_i = -\frac{\nu}{R}\cdot\frac{(1+H_i)^2+(H_i')^2}{(1+H_i)^3}\cdot e_r\cdot\nabla\tilde{p}_i|_{\partial B_R}. \tag{B46}$$

In (B46), we used the fact that $\tilde{p}_i|_{\partial B_R} = 0$ and thus $\nabla\tilde{p}_i|_{\partial B_R}$ is in the e_r-direction.

To prove (3.51), we start with the trivial bound

$$\big\|\big(1+h_i(\theta)\big)^{-1}\big\|_{C^\alpha(\mathbb{T})} \leqslant C \tag{B47}$$

due to the smallness of $m_{0,i}$, where C is a universal constant. Then we simply use $\|fg\|_{C^\alpha(\mathbb{T})} \leqslant 3\|f\|_{C^\alpha(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}$ to derive that

$$\begin{aligned} &\left\|\frac{(1+h_1)^2+(h_1')^2}{(1+h_1)^3} - \frac{(1+h_2)^2+(h_2')^2}{(1+h_2)^3}\right\|_{C^\alpha(\mathbb{T})} \\ &\leqslant \left\|\frac{1}{1+h_1}-\frac{1}{1+h_2}\right\|_{C^\alpha} + \left\|\frac{(h_1')^2-(h_2')^2}{(1+h_1)^3}\right\|_{C^\alpha} + \left\|(h_2')^2\frac{(1+h_1)^3-(1+h_2)^3}{(1+h_1)^3(1+h_2)^3}\right\|_{C^\alpha} \\ &\leqslant C\|h_1-h_2\|_{C^\alpha} + C\|h_1'+h_2'\|_{C^\alpha}\|h_1'-h_2'\|_{C^\alpha} + C\|h_2'\|_{C^\alpha}^2\|h_1-h_2\|_{C^\alpha} \\ &\leqslant C(\alpha,\delta,m_{\alpha,1}+m_{\alpha,2})\Delta m_\alpha. \end{aligned} \tag{B48}$$

Similarly,

$$\left\|\frac{h_1'}{(1+h_1)^2} - \frac{h_2'}{(1+h_2)^2}\right\|_{C^\alpha(\mathbb{T})} \leqslant C(\alpha,\delta,m_{\alpha,1}+m_{\alpha,2})\Delta m_\alpha. \tag{B49}$$

Setting $h_1 = 0$ or $h_2 = 0$ above yields

$$\left\|\frac{(1+h_i)^2+(h_i')^2}{(1+h_i)^3}\right\|_{C^\alpha(\mathbb{T})} + \left\|\frac{h_i'}{(1+h_i)^2}\right\|_{C^\alpha(\mathbb{T})} \leqslant C(\alpha,\delta,m_{\alpha,i}). \tag{B50}$$

Then it is not difficult to derive from (B45) that

$$\begin{aligned} \|\partial_t h_1 - \partial_t h_2\|_{C^\alpha(\mathbb{T})} &\leqslant C(\mu,r)\cdot C(\alpha,\delta,m_{\alpha,1}+m_{\alpha,2})\Delta m_\alpha\cdot\|\nabla\tilde{p}_1\|_{C^\alpha(\mathbb{T})} \\ &\quad + C(\mu,r)\cdot C(\delta,m_{\alpha,2})\|\nabla(\tilde{p}_1-\tilde{p}_2)\|_{C^\alpha(\mathbb{T})} \\ &\leqslant C(\alpha,\mu,r,R,m_{\alpha,1}+m_{\alpha,2})\big(\Delta m_\alpha\|\nabla\tilde{p}_1\|_{C^\alpha(\overline{B_r})} + \|\nabla(\tilde{p}_1-\tilde{p}_2)\|_{C^\alpha(\overline{B_r})}\big). \end{aligned} \tag{B51}$$

By Lemma B.2 and Lemma 3.5,

$$\|\partial_t h_1 - \partial_t h_2\|_{C^\alpha(\mathbb{T})} \leqslant C_*(\Delta m_\alpha + \Delta M_\alpha), \tag{B52}$$

where $C_* = C_*(\alpha,\mu,\nu,r,R,G)$. Once again, the dependence of C_* on $m_{\alpha,i}+M_{\alpha,i}$ is omitted since it is assumed to be small.

Estimates for $(\partial_t H_1 - \partial_t H_2)$ can be derived from (B46) in a similar manner. □

C. Proofs of Lemmas 5.4–5.6

In this section, we prove Lemmas 5.4–5.6.

Proof of Lemma 5.4. Let l_i be defined as in (5.2) corresponding to h_i. By virtue of (5.5),

$$\begin{aligned}
&2\pi\big(\gamma_1'(\theta)^{\perp}\cdot\mathcal{K}_{\gamma_1}\psi-\gamma_2'(\theta)^{\perp}\cdot\mathcal{K}_{\gamma_2}\psi\big)\\
&\quad=\frac{1}{2}\int_{\mathbb{T}}\left(\frac{1}{1+l_1}-\frac{1}{1+l_2}\right)\psi(\theta+\xi)\,d\xi\\
&\qquad+\frac{h_2(\theta)-h_1(\theta)}{1+h_2(\theta)}\cdot\frac{1}{1+h_1(\theta)}\int_{\mathbb{T}}\frac{\Delta h_1(\theta)-\cos\frac{\xi}{2}\cdot h_1'(\theta)}{2\sin\frac{\xi}{2}}\,\frac{\psi(\theta+\xi)}{1+l_1}\,d\xi\\
&\qquad+\frac{1}{1+h_2(\theta)}\int_{\mathbb{T}}\frac{\Delta(h_1-h_2)(\theta)-\cos\frac{\xi}{2}\cdot(h_1-h_2)'(\theta)}{2\sin\frac{\xi}{2}}\,\frac{\psi(\theta+\xi)}{1+l_1}\,d\xi\\
&\qquad+\frac{1}{1+h_2(\theta)}\int_{\mathbb{T}}\frac{\Delta h_2(\theta)-\cos\frac{\xi}{2}\cdot h_2'(\theta)}{2\sin\frac{\xi}{2}}\left(\frac{\psi(\theta+\xi)}{1+l_1}-\frac{\psi(\theta+\xi)}{1+l_2}\right)d\xi\\
&\quad=:J_1+J_2+J_3+J_4.
\end{aligned}\tag{C1}$$

We start with the integrand of J_1.

$$\begin{aligned}
&\left\|\left(\frac{1}{1+l_1}-\frac{1}{1+l_2}\right)\psi(\theta+\xi)\right\|_{\dot{C}^{\beta}_{\theta}}\\
&\quad\leqslant\left\|\frac{1}{1+l_1}-\frac{1}{1+l_2}\right\|_{\dot{C}^{\beta}_{\theta}}\|\psi\|_{L^{\infty}}+\left\|\frac{1}{1+l_1}-\frac{1}{1+l_2}\right\|_{L^{\infty}_{\theta}}\|\psi\|_{\dot{C}^{\beta}}\\
&\quad\leqslant C\|l_1-l_2\|_{\dot{C}^{\beta}_{\theta}}\|\psi\|_{L^{\infty}}+C\|l_1-l_2\|_{L^{\infty}_{\theta}}(\|l_1\|_{\dot{C}^{\beta}_{\theta}}+\|l_2\|_{\dot{C}^{\beta}_{\theta}})\|\psi\|_{L^{\infty}}\\
&\qquad+C\|l_1-l_2\|_{L^{\infty}_{\theta}}\|\psi\|_{\dot{C}^{\beta}}.
\end{aligned}\tag{C2}$$

We derive that

$$\begin{aligned}
&\|l_1-l_2\|_{L^{\infty}_{\theta}}\\
&\quad\leqslant\left\|\frac{(\Delta h_1)^2-(\Delta h_2)^2}{\big(1+h_1(\theta)\big)\big(1+h_1(\theta+\xi)\big)}\right\|_{L^{\infty}_{\theta}}\\
&\qquad+\left\|(\Delta h_2)^2\left(\frac{1}{\big(1+h_1(\theta)\big)\big(1+h_1(\theta+\xi)\big)}-\frac{1}{\big(1+h_2(\theta)\big)\big(1+h_2(\theta+\xi)\big)}\right)\right\|_{L^{\infty}}\\
&\quad\leqslant C(\|h_1'\|_{L^{\infty}}+\|h_2'\|_{L^{\infty}})\|h_1-h_2\|_{W^{1,\infty}},
\end{aligned}\tag{C3}$$

and

$$\begin{aligned}
&\|l_1-l_2\|_{\dot{C}^{\beta}_{\theta}}\\
&\quad\leqslant\left\|\frac{(\Delta h_1)^2-(\Delta h_2)^2}{\big(1+h_1(\theta)\big)\big(1+h_1(\theta+\xi)\big)}\right\|_{\dot{C}^{\beta}_{\theta}}
\end{aligned}$$

$$+\left\|(\Delta h_2)^2\left(\frac{1}{\big(1+h_1(\theta)\big)\big(1+h_1(\theta+\xi)\big)}-\frac{1}{\big(1+h_2(\theta)\big)\big(1+h_2(\theta+\xi)\big)}\right)\right\|_{\dot C^\beta_\theta}$$
$$\begin{aligned}&\leqslant C\|h_1'+h_2'\|_{\dot C^\beta}\|h_1'-h_2'\|_{L^\infty}+C\|h_1'+h_2'\|_{L^\infty}\|h_1'-h_2'\|_{\dot C^\beta}\\&\quad+C\|h_1'+h_2'\|_{L^\infty}\|h_1'-h_2'\|_{L^\infty}\|h_1\|_{\dot C^\beta}\\&\quad+C\|h_2'\|_{\dot C^\beta}\|h_2'\|_{L^\infty}\|h_1-h_2\|_{L^\infty}+C\|h_2'\|_{L^\infty}^2\|h_1-h_2\|_{\dot C^\beta}\\&\quad+C\|h_2'\|_{L^\infty}^2\|h_1-h_2\|_{L^\infty}\big(\|h_1\|_{\dot C^\beta}+\|h_2\|_{\dot C^\beta}\big)\\&\leqslant C\big(\|h_1'\|_{\dot C^\beta}+\|h_2'\|_{\dot C^\beta}\big)\|h_1-h_2\|_{C^{1,\beta}}.\end{aligned}\tag{C4}$$

Taking $h_2=0$ in the second last step of (C4) yields that

$$\|l_i\|_{\dot C^\beta_\theta}\leqslant C\|h_i'\|_{\dot C^\beta}\|h_i'\|_{L^\infty}.\tag{C5}$$

Combining these estimates with (C2), we argue as in (5.6) that

$$\begin{aligned}\|J_1\|_{\dot C^\beta}&\leqslant C\sup_{\xi\in\mathbb{T}}\left\|\left(\frac{1}{1+l_1}-\frac{1}{1+l_2}\right)\psi(\theta+\xi)\right\|_{\dot C^\beta_\theta}\\&\leqslant C\|h_1-h_2\|_{C^{1,\beta}}(\|h_1'\|_{\dot C^\beta}+\|h_2'\|_{\dot C^\beta})\|\psi\|_{C^\beta}.\end{aligned}\tag{C6}$$

Next, by taking advantage of (5.10) and (5.15),

$$\begin{aligned}\|J_2\|_{\dot C^\beta}&\leqslant\left\|\frac{h_2-h_1}{1+h_2}\right\|_{\dot C^\beta}\left\|\frac{1}{1+h_1}\int_{\mathbb{T}}\frac{\Delta h_1(\theta)-\cos\frac{\xi}{2}\cdot h_1'(\theta)}{2\sin\frac{\xi}{2}}\frac{\psi(\theta+\xi)}{1+l_1}\,d\xi\right\|_{L^\infty}\\&\quad+\left\|\frac{h_2-h_1}{1+h_2}\right\|_{L^\infty}\left\|\frac{1}{1+h_1}\int_{\mathbb{T}}\frac{\Delta h_1(\theta)-\cos\frac{\xi}{2}\cdot h_1'(\theta)}{2\sin\frac{\xi}{2}}\frac{\psi(\theta+\xi)}{1+l_1}\,d\xi\right\|_{\dot C^\beta}\\&\leqslant C\big(\|h_2-h_1\|_{\dot C^\beta}+\|h_2-h_1\|_{L^\infty}\|h_2\|_{\dot C^\beta}\big)\cdot\|h_1'\|_{\dot C^\beta}\|\psi\|_{L^\infty}\\&\quad+C\|h_2-h_1\|_{L^\infty}\cdot\|h_1'\|_{\dot C^\beta}\big(\|\psi\|_{C^\beta}+\|\psi\|_{L^\infty}\|h_1'\|_{\dot C^\beta}\|h_1'\|_{L^\infty}\big).\end{aligned}\tag{C7}$$

Arguing as in (5.9)–(5.15),

$$\|J_3\|_{\dot C^\beta}\leqslant C\|(h_1-h_2)'\|_{\dot C^\beta}\big(\|\psi\|_{C^\beta}+\|\psi\|_{L^\infty}\|h_1'\|_{\dot C^\beta}\|h_1'\|_{L^\infty}\big).\tag{C8}$$

In order to apply the same argument to J_4, we need the following estimate.

$$\begin{aligned}&\left|\frac{\psi(\theta+\xi)}{1+l_1}-\frac{\psi(\theta+\xi)}{1+l_2}-\frac{\psi(\theta)}{1+\frac{h_1'(\theta)^2}{(1+h_1)^2}}+\frac{\psi(\theta)}{1+\frac{h_2'(\theta)^2}{(1+h_2)^2}}\right|\\&\leqslant C|\psi(\theta+\xi)-\psi(\theta)||l_1-l_2|\\&\quad+C|\psi(\theta)|\left|l_1-l_2-\frac{h_1'(\theta)^2}{(1+h_1)^2}+\frac{h_2'(\theta)^2}{(1+h_2)^2}\right|\\&\quad+C|\psi(\theta)|\left|\frac{h_1'(\theta)^2}{\big(1+h_1(\theta)\big)^2}-\frac{h_2'(\theta)^2}{\big(1+h_2(\theta)\big)^2}\right|\left(\left|l_1-\frac{h_1'(\theta)^2}{(1+h_1)^2}\right|+\left|l_2-\frac{h_2'(\theta)^2}{(1+h_2)^2}\right|\right).\end{aligned}\tag{C9}$$

Since

$$\begin{aligned}
&\left| l_1 - l_2 - \frac{h_1'(\theta)^2}{(1+h_1)^2} + \frac{h_2'(\theta)^2}{(1+h_2)^2} \right| \\
&\quad \leqslant \left| \frac{\big(\Delta h_1(\theta)\big)^2 - h_1'(\theta)^2}{(1+h_1)\big(1+h_1(\theta+\xi)\big)} - \frac{\big(\Delta h_2(\theta)\big)^2 - h_2'(\theta)^2}{(1+h_2)\big(1+h_2(\theta+\xi)\big)} \right| \\
&\qquad + \left| \frac{h_1'(\theta)^2\big(h_1(\theta+\xi)-h_1(\theta)\big)}{(1+h_1)^2\big(1+h_1(\theta+\xi)\big)} - \frac{h_2'(\theta)^2\big(h_2(\theta+\xi)-h_2(\theta)\big)}{(1+h_2)^2\big(1+h_2(\theta+\xi)\big)} \right| \\
&\quad \leqslant C|\xi|^\beta \|h_1 - h_2\|_{C^{1,\beta}} (\|h_1'\|_{\dot C^\beta} + \|h_2'\|_{\dot C^\beta}), \qquad \text{(C10)}
\end{aligned}$$

we apply this and (C3) to (C9) to conclude that

$$\begin{aligned}
&\left| \frac{\psi(\theta+\xi)}{1+l_1} - \frac{\psi(\theta+\xi)}{1+l_2} - \frac{\psi(\theta)}{1+\frac{h_1'(\theta)^2}{(1+h_1)^2}} + \frac{\psi(\theta)}{1+\frac{h_2'(\theta)^2}{(1+h_2)^2}} \right| \\
&\leqslant C|\xi|^\beta \|\psi\|_{\dot C^\beta} \cdot \big(\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty}\big) \|h_1 - h_2\|_{W^{1,\infty}} \\
&\quad + C\|\psi\|_{L^\infty} \cdot |\xi|^\beta \|h_1 - h_2\|_{C^{1,\beta}} \big(\|h_1'\|_{\dot C^\beta} + \|h_2'\|_{\dot C^\beta}\big) \\
&\quad + C\|\psi\|_{L^\infty} \big(\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty}\big) \|h_1 - h_2\|_{W^{1,\infty}} \cdot |\xi|^\beta \big(\|h_1'\|_{L^\infty}\|h_1'\|_{\dot C^\beta} + \|h_2'\|_{L^\infty}\|h_2'\|_{\dot C^\beta}\big) \\
&\leqslant C|\xi|^\beta \|\psi\|_{C^\beta} (\|h_1'\|_{\dot C^\beta} + \|h_2'\|_{\dot C^\beta}) \|h_1 - h_2\|_{C^{1,\beta}}. \qquad \text{(C11)}
\end{aligned}$$

Now we proceed as in (5.9)–(5.15).

$$\begin{aligned}
&|J_4(\theta+\varepsilon) - J_4(\theta)| \\
&\leqslant \left| \frac{1}{1+h_2(\theta+\varepsilon)} - \frac{1}{1+h_2(\theta)} \right| \int_{\mathbb{T}} \left| \frac{\Delta h_2(\theta+\varepsilon) - \cos\frac{\xi}{2} \cdot h_2'(\theta+\varepsilon)}{2\sin\frac{\xi}{2}} \right| \\
&\qquad \cdot \left| \frac{\psi(\theta+\varepsilon+\xi)}{1+l_1(\theta+\varepsilon,\theta+\varepsilon+\xi)} - \frac{\psi(\theta+\varepsilon+\xi)}{1+l_2(\theta+\varepsilon,\theta+\varepsilon+\xi)} \right| d\xi \\
&\quad + C\int_{\mathbb{T}} \left| \frac{\Delta h_2(\theta+\varepsilon) - \cos\frac{\xi}{2} \cdot h_2'(\theta+\varepsilon)}{2\sin\frac{\xi}{2}} \right| \\
&\qquad \cdot |\varepsilon|^\beta \sup_\xi \left\| \frac{\psi(\theta+\xi)}{1+l_1(\theta,\theta+\xi)} - \frac{\psi(\theta+\xi)}{1+l_2(\theta,\theta+\xi)} \right\|_{\dot C^\beta_\theta} d\xi \\
&\quad + C\int_{\mathbb{T}} \left| \frac{\Delta h_2(\theta+\varepsilon) - \Delta h_2(\theta) - \cos\frac{\xi}{2}\big(h_2'(\theta+\varepsilon) - h_2'(\theta)\big)}{2\sin\frac{\xi}{2}} \right| \\
&\qquad \cdot \left| \frac{\psi(\theta+\xi)}{1+l_1(\theta,\theta+\xi)} - \frac{\psi(\theta+\xi)}{1+l_2(\theta,\theta+\xi)} - \frac{\psi(\theta)}{1+\frac{h_1'(\theta)^2}{(1+h_1(\theta))^2}} + \frac{\psi(\theta)}{1+\frac{h_2'(\theta)^2}{(1+h_2(\theta))^2}} \right| d\xi \\
&\quad + C\left| \int_{\mathbb{T}} \frac{\Delta h_2(\theta+\varepsilon) - \Delta h_2(\theta) - \cos\frac{\xi}{2}\big(h_2'(\theta+\varepsilon) - h_2'(\theta)\big)}{2\sin\frac{\xi}{2}} \, d\xi \right|
\end{aligned}$$

$$\cdot\left|\frac{\psi(\theta)}{1+\frac{h_1'(\theta)^2}{(1+h_1(\theta))^2}}-\frac{\psi(\theta)}{1+\frac{h_2'(\theta)^2}{(1+h_2(\theta))^2}}\right|. \tag{C12}$$

By (5.11), (5.14), (C3)–(C6) and (C11),

$$|J_4(\theta+\varepsilon)-J_4(\theta)|\leqslant C\varepsilon^\beta\|\psi\|_{C^\beta}\|h_2'\|_{\dot C^\beta}(\|h_1'\|_{\dot C^\beta}+\|h_2'\|_{\dot C^\beta})\|h_1-h_2\|_{C^{1,\beta}}. \tag{C13}$$

Combining (C6)–(C8) and (C13) yield the desired estimate. □

Proof of Lemma 5.5. Let C_* and $C_\dagger$ be the constants in Lemma A.2 and Lemma A.4, respectively, both of which only depend on p. Without loss of generality, we may assume $C_\dagger \geqslant C_* \geqslant 1$. Following (5.5), we use $L_k^{(i)}$ ($k=0,1,2,3$) to denote the corresponding quantities defined by h_i ($i=1,2$). l_i are defined as in (5.2) by h_i. Thanks to the smallness of h_i, we may assume $|l_i|<1$, and that $C_2>0$ is a universal constant such that $\|(1+h_i)^{-1}\|_{L^\infty}\leqslant C_2$.

We start with bounding $L_1^{(1)}-L_1^{(2)}$. Taking their θ-derivatives, we use (C3) to derive that

$$\begin{aligned}
&\|L_1^{(1)}-L_1^{(2)}\|_{\dot W^{1,p}}\\
&\leqslant \frac12\Bigg\|\int_{\mathbb T}\Bigg[\frac{\frac{2\Delta h_1\Delta h_1'}{(1+h_1(\theta))(1+h_1(\theta+\xi))}-\frac{(\Delta h_1)^2(h_1'(\theta)+h_1'(\theta+\xi)+h_1'(\theta)h_1(\theta+\xi)+h_1(\theta)h_1'(\theta+\xi))}{(1+h_1(\theta))^2(1+h_1(\theta+\xi))^2}}{\big(1+l_1(\theta,\theta+\xi)\big)^2}\\
&\quad-\frac{\frac{2\Delta h_2\Delta h_2'}{(1+h_2(\theta))(1+h_2(\theta+\xi))}-\frac{(\Delta h_2)^2(h_2'(\theta)+h_2'(\theta+\xi)+h_2'(\theta)h_2(\theta+\xi)+h_2(\theta)h_2'(\theta+\xi))}{(1+h_2(\theta))^2(1+h_2(\theta+\xi))^2}}{\big(1+l_2(\theta,\theta+\xi)\big)^2}\Bigg]\psi(\theta+\xi)\,d\xi\Bigg\|_{L^p}\\
&\quad+\frac12\left\|\int_{\mathbb T}\left(\frac{1}{1+l_1}-\frac{1}{1+l_2}\right)\psi'(\theta+\xi)\,d\xi\right\|_{L^p}\\
&\leqslant C(\|h_1'-h_2'\|_{L^\infty}\|h_1''\|_{L^p}+\|h_2'\|_{L^\infty}\|h_1''-h_2''\|_{L^p}+\|h_2'\|_{L^\infty}\|h_2''\|_{L^p}\|h_1-h_2\|_{W^{1,\infty}})\|\psi\|_{L^\infty}\\
&\quad+C(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty})\|h_1-h_2\|_{W^{1,\infty}}\|\psi'\|_{L^p}.
\end{aligned}\tag{C14}$$

As in (5.18), we Taylor expand $(1+l_i)^{-1}$ and rewrite $L_2^{(i)}$ as

$$\begin{aligned}
L_2^{(i)}&=\sum_{j=0}^\infty(-1)^j\big(1+h_i(\theta)\big)^{-(j+1)}\mathrm{p.v.}\int_{\mathbb T}(\Delta h_i)^{2j+1}\big(1+h_i(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\\
&=:\sum_{j=0}^\infty L_{2,j}^{(i)}.
\end{aligned}\tag{C15}$$

We derive

$$\begin{aligned}
&(-1)^j(L_{2,j}^{(1)}-L_{2,j}^{(2)})\\
&\quad=\Big[\big(1+h_1(\theta)\big)^{-(j+1)}-\big(1+h_2(\theta)\big)^{-(j+1)}\Big]\\
&\qquad\cdot\mathrm{p.v.}\int_{\mathbb T}(\Delta h_1)^{2j+1}\big(1+h_1(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi
\end{aligned}$$

$$
\begin{aligned}
&+\big(1+h_2(\theta)\big)^{-(j+1)} \\
&\quad\cdot \mathrm{p.v.}\int_{\mathbb{T}} \Delta(h_1-h_2)\sum_{l=0}^{2j}(\Delta h_1)^l(\Delta h_2)^{2j-l}\cdot\big(1+h_1(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi \\
&+\big(1+h_2(\theta)\big)^{-(j+1)} \\
&\quad\cdot \mathrm{p.v.}\int_{\mathbb{T}}(\Delta h_2)^{2j+1}\left(\frac{1}{1+h_1(\theta+\xi)}-\frac{1}{1+h_2(\theta+\xi)}\right) \\
&\qquad\cdot\sum_{l=0}^{j-1}\big(1+h_1(\theta+\xi)\big)^{-l}\big(1+h_2(\theta+\xi)\big)^{-(j-1-l)}\cdot\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi. \qquad \text{(C16)}
\end{aligned}
$$

Note that here in this proof, with abuse of notations, we use l as a summation index, which has nothing to do with (5.2).

By Lemma A.2, for $0\leqslant l\leqslant k$,

$$
\begin{aligned}
&\left\|\mathrm{p.v.}\int_{\mathbb{T}}\Delta(h_1-h_2)(\Delta h_1)^l(\Delta h_2)^{k-l}\cdot\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\right\|_{L^p} \\
&\qquad\leqslant CC_*^k\big(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty}\big)^k\|h_1'-h_2'\|_{L^\infty}\|\psi\|_{L^p}. \qquad \text{(C17)}
\end{aligned}
$$

Letting $k=2j$ and replacing ψ by $(1+h_1)^{-j}\psi$,

$$
\begin{aligned}
&\left\|\mathrm{p.v.}\int_{\mathbb{T}}\Delta(h_1-h_2)(\Delta h_1)^l(\Delta h_2)^{2j-l}\big(1+h_1(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\right\|_{L^p} \\
&\qquad\leqslant C\big(C_*^2C_2(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty})^2\big)^j\|h_1'-h_2'\|_{L^\infty}\|\psi\|_{L^p}. \qquad \text{(C18)}
\end{aligned}
$$

Further taking $h_2=0$ and $l=2j$, we find

$$
\begin{aligned}
&\left\|\mathrm{p.v.}\int_{\mathbb{T}}(\Delta h_1)^{2j+1}\big(1+h_1(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\right\|_{L^p} \\
&\qquad\leqslant C\big(C_*^2C_2\|h_1'\|_{L^\infty}^2\big)^j\|h_1'\|_{L^\infty}\|\psi\|_{L^p}. \qquad \text{(C19)}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\Bigg\|\mathrm{p.v.}\int_{\mathbb{T}}(\Delta h_2)^{2j+1}\left(\frac{1}{1+h_1(\theta+\xi)}-\frac{1}{1+h_2(\theta+\xi)}\right) \\
&\qquad\cdot\big(1+h_1(\theta+\xi)\big)^{-l}\big(1+h_2(\theta+\xi)\big)^{-(j-1-l)}\cdot\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\Bigg\|_{L^p} \\
&\qquad\leqslant C\big(C_*^2C_2\|h_2'\|_{L^\infty}^2\big)^j\|h_2'\|_{L^\infty}\|h_1-h_2\|_{L^\infty}\|\psi\|_{L^p}. \qquad \text{(C20)}
\end{aligned}
$$

On the other hand, by Lemma A.4, for $0 \leqslant l \leqslant k$,

$$\begin{aligned}
&\left\| \mathrm{p.v.} \int_{\mathbb{T}} \Delta(h_1 - h_2)(\Delta h_1)^l (\Delta h_2)^{k-l} \cdot \frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\, d\xi \right\|_{\dot W^{1,p}} \\
&\leqslant (k+2) C_\dagger^{k+2} \|\psi'\|_{L^p} \|h_1' - h_2'\|_{L^\infty} \big(\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty}\big)^k \\
&\quad + (k+2) C_\dagger^{k+2} \|\psi\|_{L^\infty} \|h_1'' - h_2''\|_{L^p} \big(\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty}\big)^k \\
&\quad + (k+2) C_\dagger^{k+2} \|\psi\|_{L^\infty} \|h_1' - h_2'\|_{L^\infty} \big(\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty}\big)^{k-1} \cdot \mathbb{1}_{\{k>0\}} \big(\|h_1''\|_{L^p} + \|h_2''\|_{L^p}\big).
\end{aligned} \tag{C21}$$

Taking $k = 2j$ and replacing ψ by $(1+h_1)^{-j}\psi$,

$$\begin{aligned}
&\left\| \mathrm{p.v.} \int_{\mathbb{T}} \Delta(h_1 - h_2)(\Delta h_1)^l (\Delta h_2)^{2j-l} \big(1 + h_1(\theta+\xi)\big)^{-j} \cdot \frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\, d\xi \right\|_{\dot W^{1,p}} \\
&\leqslant (2j+2) C_\dagger^{2j+2} (j C_2^{j+1} \|h_1'\|_{L^\infty} \|\psi\|_{L^p} + C_2^j \|\psi'\|_{L^p}) \|h_1' - h_2'\|_{L^\infty} (\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty})^{2j} \\
&\quad + (2j+2) C_\dagger^{2j+2} C_2^j \|\psi\|_{L^\infty} \|h_1'' - h_2''\|_{L^p} \big(\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty}\big)^{2j} \\
&\quad + (2j+2) C_\dagger^{2j+2} C_2^j \|\psi\|_{L^\infty} \|h_1' - h_2'\|_{L^\infty} \\
&\qquad \cdot \mathbb{1}_{\{j>0\}} (\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty})^{2j-1} \big(\|h_1''\|_{L^p} + \|h_2''\|_{L^p}\big).
\end{aligned} \tag{C22}$$

Further taking $h_2 = 0$ and $l = 2j$,

$$\begin{aligned}
&\left\| \mathrm{p.v.} \int_{\mathbb{T}} (\Delta h_1)^{2j+1} \big(1 + h_1(\theta+\xi)\big)^{-j} \cdot \frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\, d\xi \right\|_{\dot W^{1,p}} \\
&\quad \leqslant C(j+1)\big(C_\dagger^2 C_2 \|h_1'\|_{L^\infty}^2\big)^j \big[(j\|h_1'\|_{L^\infty} \|\psi\|_{L^p} + \|\psi'\|_{L^p}) \|h_1'\|_{L^\infty} + \|\psi\|_{L^\infty} \|h_1''\|_{L^p}\big].
\end{aligned} \tag{C23}$$

Similarly,

$$\begin{aligned}
&\Bigg\| \mathrm{p.v.} \int_{\mathbb{T}} (\Delta h_2)^{2j+1} \left(\frac{1}{1 + h_1(\theta+\xi)} - \frac{1}{1 + h_2(\theta+\xi)} \right) \\
&\qquad \cdot \big(1 + h_1(\theta+\xi)\big)^{-l} \big(1 + h_2(\theta+\xi)\big)^{-(j-1-l)} \cdot \frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\, d\xi \Bigg\|_{\dot W^{1,p}} \\
&\quad \leqslant (2j+2) C_\dagger^{2j+2} \|h_2'\|_{L^\infty}^{2j+1} \\
&\qquad \cdot \left\| \left(\frac{1}{1+h_1} - \frac{1}{1+h_2} \right) (1+h_1)^{-l} (1+h_2)^{-(j-1-l)} \psi \right\|_{\dot W^{1,p}} \\
&\qquad + C(2j+2) C_\dagger^{2j+2} \|h_2'\|_{L^\infty}^{2j} \|h_2''\|_{L^p} \cdot C_2^{j+1} \|h_1 - h_2\|_{L^\infty} \|\psi\|_{L^\infty} \\
&\quad \leqslant C(j+1) (C_\dagger^2 C_2 \|h_2'\|_{L^\infty}^2)^j \|h_2'\|_{L^\infty} \cdot \big[\|h_1' - h_2'\|_{L^\infty} \|\psi\|_{L^p} \\
&\qquad\quad + j\|h_1 - h_2\|_{L^\infty} (\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty}) \|\psi\|_{L^p} + \|h_1 - h_2\|_{L^\infty} \|\psi'\|_{L^p}\big] \\
&\qquad + C(j+1) (C_\dagger^2 C_2 \|h_2'\|_{L^\infty}^2)^j \|h_2''\|_{L^p} \cdot \|h_1 - h_2\|_{L^\infty} \|\psi\|_{L^\infty}.
\end{aligned} \tag{C24}$$

Combining these estimates with (C16), we use the fact $\|fg\|_{\dot W^{1,p}} \leqslant \|f\|_{\dot W^{1,\infty}}\|g\|_{L^p} + \|f\|_{L^\infty}\|g\|_{\dot W^{1,p}}$ to derive that

$$
\begin{aligned}
&\|L_{2,j}^{(1)} - L_{2,j}^{(2)}\|_{\dot W^{1,p}} \\
&\leqslant C(j+1)C_2^j\big(\|h_1'-h_2'\|_{L^\infty} + (j+2)\|h_1-h_2\|_{L^\infty}\|h_2'\|_{L^\infty}\big) \\
&\qquad\cdot (C_*^2C_2\|h_1'\|_{L^\infty}^2)^j\|h_1'\|_{L^\infty}\|\psi\|_{L^p} \\
&\quad + C(j+1)C_2^j\|h_1-h_2\|_{L^\infty} \\
&\qquad\cdot (j+1)(C_\dagger^2C_2\|h_1'\|_{L^\infty}^2)^j\big[(j\|h_1'\|_{L^\infty}\|\psi\|_{L^p} + \|\psi'\|_{L^p})\|h_1'\|_{L^\infty} + \|\psi\|_{L^\infty}\|h_1''\|_{L^p}\big] \\
&\quad + C(j+1)C_2^j\|h_2'\|_{L^\infty}\sum_{l=0}^{2j}\big(C_*^2C_2(\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty})^2\big)^j\|h_1'-h_2'\|_{L^\infty}\|\psi\|_{L^p} \\
&\quad + CC_2^j\sum_{l=0}^{2j}(j+1)\big(C_\dagger^2C_2(\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty})^2\big)^j \\
&\qquad\cdot\big[(j\|h_1'\|_{L^\infty}\|\psi\|_{L^p} + \|\psi'\|_{L^p})\|h_1'-h_2'\|_{L^\infty} + \|\psi\|_{L^\infty}\|h_1''-h_2''\|_{L^p} \\
&\qquad\qquad + \mathbb{1}_{\{j>0\}}\|\psi\|_{L^\infty}\|h_1'-h_2'\|_{L^\infty}(\|h_1''\|_{L^p} + \|h_2''\|_{L^p})(\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty})^{-1}\big] \\
&\quad + C(j+1)C_2^j\|h_2'\|_{L^\infty}\sum_{l=0}^{j-1}(C_*^2C_2\|h_2'\|_{L^\infty}^2)^j\|h_2'\|_{L^\infty}\|h_1-h_2\|_{L^\infty}\|\psi\|_{L^p} \\
&\quad + CC_2^j\sum_{l=0}^{j-1}(j+1)(C_\dagger^2C_2\|h_2'\|_{L^\infty}^2)^j\|h_2'\|_{L^\infty}\cdot\big[\|h_1'-h_2'\|_{L^\infty}\|\psi\|_{L^p} \\
&\qquad + j\|h_1-h_2\|_{L^\infty}(\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty})\|\psi\|_{L^p} + \|h_1-h_2\|_{L^\infty}\|\psi'\|_{L^p}\big] \\
&\qquad + (j+1)(C_\dagger^2C_2\|h_2'\|_{L^\infty}^2)^j\|h_2''\|_{L^p}\cdot\|h_1-h_2\|_{L^\infty}\|\psi\|_{L^\infty}.
\end{aligned}
\tag{C25}
$$

Assuming $\|h_i'\|_{L^\infty} \ll 1$,

$$
\begin{aligned}
\|L_2^{(1)} - L_2^{(2)}\|_{\dot W^{1,p}} &\leqslant \sum_{j=0}^\infty \|L_{2,j}^{(1)} - L_{2,j}^{(2)}\|_{\dot W^{1,p}} \\
&\leqslant C\big(\|\psi'\|_{L^p}\|h_1-h_2\|_{W^{1,\infty}} + \|\psi\|_{L^\infty}\|h_1''-h_2''\|_{L^p}\big) \\
&\qquad + C\|\psi\|_{L^\infty}\|h_1-h_2\|_{W^{1,\infty}}\big(\|h_1''\|_{L^p} + \|h_2''\|_{L^p}\big).
\end{aligned}
\tag{C26}
$$

We similarly write

$$
\begin{aligned}
L_3^{(i)} &= \sum_{j=0}^\infty h_i'(\theta)\big(-1-h_i(\theta)\big)^{-(j+1)}\text{p.v.}\int_{\mathbb{T}}(\Delta h_i)^{2j}\big(1+h_i(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi \\
&=: \sum_{j=0}^\infty L_{3,j}^{(i)}
\end{aligned}
\tag{C27}
$$

and

$$
\begin{aligned}
&(-1)^{j+1}(L_{3,j}^{(1)} - L_{3,j}^{(2)}) \\
&\quad = \Big[h_1'(\theta)(1+h_1(\theta))^{-(j+1)} - h_2'(\theta)\big(1+h_2(\theta)\big)^{-(j+1)}\Big] \cdot \text{p.v.} \int_{\mathbb{T}} A_i^j \cdot \frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\, d\xi \\
&\qquad + h_2'(\theta)\big(1+h_2(\theta)\big)^{-(j+1)} \cdot \text{p.v.} \int_{\mathbb{T}} (A_1^j - A_2^j) \cdot \frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\, d\xi, \qquad \text{(C28)}
\end{aligned}
$$

where

$$
A_i := \frac{(\Delta h_i)^2}{1+h_i(\theta+\xi)} = \big(1+h_i(\theta)\big) \cdot l_i(\theta, \theta+\xi). \qquad \text{(C29)}
$$

To proceed as before, we need L^∞-bounds for the integrals in (C28). We additionally define

$$
B_i = \frac{h_i'(\theta)^2}{1+h_i(\theta)}. \qquad \text{(C30)}
$$

It is easy to show that $|A_i|, |B_i| \leqslant C_1^2 C_2 \|h_i'\|_{L^\infty}^2$, where $C_1 = \pi/2$ is introduced in the proof of Lemma A.2, and

$$
|A_i - B_i| \leqslant C\|h_i'\|_{L^\infty}\|h_i'\|_{\dot{C}^\beta}|\xi|^\beta. \qquad \text{(C31)}
$$

Hence, by the mean value theorem,

$$
\begin{aligned}
\left|\text{p.v.} \int_{\mathbb{T}} A_i^j \cdot \frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\, d\xi\right| &= \left|\int_{\mathbb{T}} \big(A_i^j \psi(\theta+\xi) - B_i^j \psi(\theta)\big) \frac{1}{2\tan\frac{\xi}{2}}\, d\xi\right| \\
&\leqslant C \int_{\mathbb{T}} j(C_1^2 C_2 \|h_i'\|_{L^\infty}^2)^{j-1} \cdot \|h_i'\|_{L^\infty}\|h_i'\|_{\dot{C}^\beta}|\xi|^\beta \cdot \|\psi\|_{L^\infty}|\xi|^{-1}\, d\xi \\
&\quad + C \int_{\mathbb{T}} (C_1^2 C_2 \|h_i'\|_{L^\infty}^2)^j \cdot |\psi(\theta+\xi) - \psi(\theta)||\xi|^{-1}\, d\xi \\
&\leqslant C(C_1^2 C_2)^j \big(j\|h_i'\|_{L^\infty}^{2j-1}\|h_i'\|_{\dot{C}^\beta}\|\psi\|_{L^\infty} + \|h_i'\|_{L^\infty}^{2j}\|\psi\|_{\dot{C}^\beta}\big). \qquad \text{(C32)}
\end{aligned}
$$

We also derive that

$$
\begin{aligned}
&\left|\text{p.v.} \int_{\mathbb{T}} (A_1^j - A_2^j) \frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\, d\xi\right| \\
&\leqslant \int_{\mathbb{T}} |A_1^j - A_2^j - B_1^j + B_2^j| \left|\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\right| d\xi + |B_1^j - B_2^j| \left|\int_{\mathbb{T}} \frac{\psi(\theta+\xi) - \psi(\theta)}{2\tan\frac{\xi}{2}}\, d\xi\right|. \qquad \text{(C33)}
\end{aligned}
$$

Write

$$
\begin{aligned}
&A_1^j - A_2^j - B_1^j + B_2^j \\
&\quad = (A_1 - A_2 - B_1 + B_2) \sum_{l=0}^{j-1} A_1^l A_2^{j-1-l} + (B_1 - B_2) \sum_{l=0}^{j-1} \big(A_1^l A_2^{j-1-l} - B_1^l B_2^{j-1-l}\big). \qquad \text{(C34)}
\end{aligned}
$$

Since

$$
\begin{aligned}
A_1 - A_2 - B_1 + B_2 &= \big(1+h_1(\theta)\big)\left(l_1 - \frac{h_1'(\theta)^2}{\big(1+h_1(\theta)\big)^2}\right) - \big(1+h_2(\theta)\big)\left(l_2 - \frac{h_2'(\theta)^2}{\big(1+h_2(\theta)\big)^2}\right)\\
&= \frac{h_1-h_2}{1+h_1}(A_1 - B_1)\\
&\quad + \big(1+h_2(\theta)\big)\left(l_1 - l_2 - \frac{h_1'(\theta)^2}{\big(1+h_1(\theta)\big)^2} + \frac{h_2'(\theta)^2}{\big(1+h_2(\theta)\big)^2}\right),
\end{aligned} \tag{C35}
$$

we use (C10) and (C31) to derive that

$$
|A_1 - A_2 - B_1 + B_2| \leqslant C|\xi|^\beta \|h_1 - h_2\|_{C^{1,\beta}} \big(\|h_1'\|_{\dot C^\beta} + \|h_2'\|_{\dot C^\beta}\big) \tag{C36}
$$

Combining this with (C31) and (C34) yields that

$$
\begin{aligned}
&|A_1^j - A_2^j - B_1^j + B_2^j|\\
&\leqslant |A_1 - A_2 - B_1 + B_2| \sum_{l=0}^{j-1} \big(C_1^2 C_2 \|h_1'\|_{L^\infty}^2\big)^l \big(C_1^2 C_2 \|h_2'\|_{L^\infty}^2\big)^{j-1-l}\\
&\quad + C|B_1 - B_2| \sum_{l=0}^{j-1} l\big(C_1^2 C_2 \|h_1'\|_{L^\infty}^2\big)^{l-1} \cdot \|h_1'\|_{L^\infty} \|h_1'\|_{\dot C^\beta} |\xi|^\beta \cdot \big(C_1^2 C_2 \|h_2'\|_{L^\infty}^2\big)^{j-1-l}\\
&\quad + C|B_1 - B_2| \sum_{l=0}^{j-1} \big(C_1^2 C_2 \|h_1'\|_{L^\infty}^2\big)^l \cdot (j-1-l)\big(C_1^2 C_2 \|h_2'\|_{L^\infty}^2\big)^{j-2-l} \cdot \|h_2'\|_{L^\infty} \|h_2'\|_{\dot C^\beta} |\xi|^\beta\\
&\leqslant C(C_1^2 C_2)^{j-1} |\xi|^\beta \cdot j\|h_1 - h_2\|_{C^{1,\beta}} \big(\|h_1'\|_{\dot C^\beta} + \|h_2'\|_{\dot C^\beta}\big)\big(\|h_1'\|_{L^\infty}^2 + \|h_2'\|_{L^\infty}^2\big)^{j-1}.
\end{aligned} \tag{C37}
$$

Applying this to (C33), we obtain that

$$
\begin{aligned}
&\left|\text{p.v.} \int_{\mathbb{T}} (A_1^j - A_2^j) \frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\, d\xi\right|\\
&\leqslant C(C_1^2 C_2)^{j-1} \cdot j\|h_1 - h_2\|_{C^{1,\beta}} \big(\|h_1'\|_{\dot C^\beta} + \|h_2'\|_{\dot C^\beta}\big)\big(\|h_1'\|_{L^\infty}^2 + \|h_2'\|_{L^\infty}^2\big)^{j-1} \|\psi\|_{C^\beta}.
\end{aligned} \tag{C38}
$$

Arguing as in (C17)–(C20), for $j \geqslant 1$ and $0 \leqslant l \leqslant 2j-1$,

$$
\begin{aligned}
&\left\|\text{p.v.} \int_{\mathbb{T}} \Delta(h_1 - h_2)(\Delta h_1)^l (\Delta h_2)^{2j-1-l} \big(1 + h_1(\theta+\xi)\big)^{-j} \cdot \frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\, d\xi\right\|_{L^p}\\
&\qquad \leqslant C C_*^{2j-1} C_2^j \big(\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty}\big)^{2j-1} \|h_1' - h_2'\|_{L^\infty} \|\psi\|_{L^p},
\end{aligned} \tag{C39}
$$

$$
\left\|\text{p.v.} \int_{\mathbb{T}} A_1^j \cdot \frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\, d\xi\right\|_{L^p} \leqslant C C_*^{2j-1} C_2^j \|h_1'\|_{L^\infty}^{2j} \|\psi\|_{L^p}, \tag{C40}
$$

and

$$\Big\|\text{p.v.}\int_{\mathbb{T}}(\Delta h_2)^{2j}\left(\frac{1}{1+h_1(\theta+\xi)}-\frac{1}{1+h_2(\theta+\xi)}\right)$$
$$\cdot\big(1+h_1(\theta+\xi)\big)^{-l}\big(1+h_2(\theta+\xi)\big)^{-(j-1-l)}\cdot\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\Big\|_{L^p}$$
$$\leqslant CC_*^{2j-1}C_2^j\|h_2'\|_{L^\infty}^{2j}\|h_1-h_2\|_{L^\infty}\|\psi\|_{L^p}. \tag{C41}$$

Hence,

$$\Big\|\text{p.v.}\int_{\mathbb{T}}(A_1^j-A_2^j)\cdot\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi\Big\|_{L^p}$$
$$\leqslant\sum_{l=0}^{2j-1}\Big\|\text{p.v.}\int_{\mathbb{T}}\Delta(h_1-h_2)(\Delta h_1)^l(\Delta h_2)^{2j-1-l}\big(1+h_1(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi\Big\|_{L^p}$$
$$+\sum_{l=0}^{j-1}\Big\|\text{p.v.}\int_{\mathbb{T}}(\Delta h_2)^{2j}\left(\frac{1}{1+h_1(\theta+\xi)}-\frac{1}{1+h_2(\theta+\xi)}\right)$$
$$\cdot\big(1+h_1(\theta+\xi)\big)^{-l}\big(1+h_1(\theta+\xi)\big)^{-(j-1-l)}\cdot\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi\Big\|_{L^p}$$
$$\leqslant CjC_*^{2j-1}C_2^j\big(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty}\big)^{2j-1}\|h_1-h_2\|_{W^{1,\infty}}\|\psi\|_{L^p}. \tag{C42}$$

Similar to (C21)–(C24), for $j\geqslant 1$,

$$\Big\|\text{p.v.}\int_{\mathbb{T}}\Delta(h_1-h_2)(\Delta h_1)^l(\Delta h_2)^{2j-1-l}\big(1+h_1(\theta+\xi)\big)^{-j}\cdot\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi\Big\|_{\dot W^{1,p}}$$
$$\leqslant(2j+1)C_\dagger^{2j+1}\big(jC_2^{j+1}\|h_1'\|_{L^\infty}\|\psi\|_{L^p}+C_2^j\|\psi'\|_{L^p}\big)\|h_1'-h_2'\|_{L^\infty}\big(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty}\big)^{2j-1}$$
$$+(2j+1)C_\dagger^{2j+1}C_2^j\|\psi\|_{L^\infty}\|h_1''-h_2''\|_{L^p}\big(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty}\big)^{2j-1}$$
$$+(2j+1)C_\dagger^{2j+1}C_2^j\|\psi\|_{L^\infty}\|h_1'-h_2'\|_{L^\infty}\big(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty}\big)^{2j-2}\cdot\big(\|h_1''\|_{L^p}+\|h_2''\|_{L^p}\big), \tag{C43}$$

$$\Big\|\text{p.v.}\int_{\mathbb{T}}A_1^j\cdot\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi\Big\|_{\dot W^{1,p}}$$
$$\leqslant(2j+1)C_\dagger^{2j+1}\big(jC_2^{j+1}\|h_1'\|_{L^\infty}\|\psi\|_{L^p}+C_2^j\|\psi'\|_{L^p}\big)\|h_1'\|_{L^\infty}^{2j}$$
$$+C(2j+1)C_\dagger^{2j+1}C_2^j\|\psi\|_{L^\infty}\|h_1''\|_{L^p}\|h_1'\|_{L^\infty}^{2j-1}, \tag{C44}$$

and

$$\begin{aligned}
&\Big\| \text{p.v.} \int_{\mathbb{T}} (\Delta h_2)^{2j} \Big(\frac{1}{1+h_1(\theta+\xi)} - \frac{1}{1+h_2(\theta+\xi)} \Big) \\
&\qquad\qquad \cdot \big(1+h_1(\theta+\xi)\big)^{-l} \big(1+h_2(\theta+\xi)\big)^{-(j-1-l)} \cdot \frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\, d\xi \Big\|_{\dot{W}^{1,p}} \\
&\quad \leqslant (2j+1)C_{\dagger}^{2j+1} \Big\| \Big(\frac{1}{1+h_1} - \frac{1}{1+h_2} \Big) (1+h_1)^{-l}(1+h_2)^{-(j-1-l)}\psi \Big\|_{\dot{W}^{1,p}} \|h_2'\|_{L^\infty}^{2j} \\
&\qquad + C(2j+1)C_{\dagger}^{2j+1} C_2^{j+1} \|h_1-h_2\|_{L^\infty} \|\psi\|_{L^\infty} \|h_2''\|_{L^p} \|h_2'\|_{L^\infty}^{2j-1} \\
&\quad \leqslant C(2j+1)C_{\dagger}^{2j+1} C_2^{j} \|h_2'\|_{L^\infty}^{2j} \cdot \big[\|h_1'-h_2'\|_{L^\infty} \|\psi\|_{L^p} \\
&\qquad + j\|h_1-h_2\|_{L^\infty} (\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty}) \|\psi\|_{L^p} + \|h_1-h_2\|_{L^\infty} \|\psi'\|_{L^p} \big] \\
&\qquad + C(2j+1)C_{\dagger}^{2j+1} C_2^{j+1} \|h_2'\|_{L^\infty}^{2j-1} \|h_1-h_2\|_{L^\infty} \|\psi\|_{L^\infty} \|h_2''\|_{L^p}.
\end{aligned} \tag{C45}$$

Hence,

$$\begin{aligned}
&\Big\| \text{p.v.} \int_{\mathbb{T}} (A_1^j - A_2^j) \cdot \frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\, d\xi \Big\|_{\dot{W}^{1,p}} \\
&\leqslant \sum_{l=0}^{2j-1} \Big\| \text{p.v.} \int_{\mathbb{T}} \Delta(h_1-h_2)(\Delta h_1)^l (\Delta h_2)^{2j-1-l} \big(1+h_1(\theta+\xi)\big)^{-j} \cdot \frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\, d\xi \Big\|_{\dot{W}^{1,p}} \\
&\quad + \sum_{l=0}^{j-1} \Big\| \text{p.v.} \int_{\mathbb{T}} (\Delta h_2)^{2j} \Big(\frac{1}{1+h_1(\theta+\xi)} - \frac{1}{1+h_2(\theta+\xi)} \Big) \\
&\qquad\qquad \cdot \big(1+h_1(\theta+\xi)\big)^{-l} \big(1+h_2(\theta+\xi)\big)^{-(j-1-l)} \cdot \frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\, d\xi \Big\|_{\dot{W}^{1,p}} \\
&\leqslant Cj^2(2j+1)C_{\dagger}^{2j+1} C_2^{j} \|\psi\|_{L^p} \|h_1-h_2\|_{W^{1,\infty}} \big(\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty}\big)^{2j} \\
&\quad + Cj(2j+1)C_{\dagger}^{2j+1} C_2^{j} \|\psi'\|_{L^p} \|h_1-h_2\|_{W^{1,\infty}} \big(\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty}\big)^{2j-1} \\
&\quad + Cj(2j+1)C_{\dagger}^{2j+1} C_2^{j} \|\psi\|_{L^\infty} \|h_1''-h_2''\|_{L^p} \big(\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty}\big)^{2j-1} \\
&\quad + Cj(2j+1)C_{\dagger}^{2j+1} C_2^{j} \|\psi\|_{L^\infty} \|h_1-h_2\|_{W^{1,\infty}} (\|h_1'\|_{L^\infty} + \|h_2'\|_{L^\infty})^{2j-2} \big(\|h_1''\|_{L^p} + \|h_2''\|_{L^p}\big).
\end{aligned} \tag{C46}$$

To this end, by (C28),

$$\begin{aligned}
&\|L_{3,j}^{(1)} - L_{3,j}^{(2)}\|_{\dot{W}^{1,p}} \\
&\quad \leqslant \|h_1''(1+h_1)^{-(j+1)} - h_2''(1+h_2)^{-(j+1)}\|_{L^p} \Big\| \text{p.v.} \int_{\mathbb{T}} A_1^j \frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\, d\xi \Big\|_{L^\infty} \\
&\qquad + (j+1)\|(h_1')^2(1+h_1)^{-(j+2)} - (h_2')^2(1+h_2)^{-(j+2)}\|_{L^\infty} \Big\| \text{p.v.} \int_{\mathbb{T}} A_1^j \frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\, d\xi \Big\|_{L^p}
\end{aligned}$$

$$+\|h_1'(1+h_1)^{-(j+1)}-h_2'(1+h_2)^{-(j+1)}\|_{L^\infty}\left\|\mathrm{p.v.}\int_{\mathbb{T}}A_1^j\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi\right\|_{\dot W^{1,p}}$$
$$+\|h_2''(1+h_2)^{-(j+1)}\|_{L^p}\left\|\mathrm{p.v.}\int_{\mathbb{T}}(A_1^j-A_2^j)\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi\right\|_{L^\infty}$$
$$+(j+1)\|(h_2')^2(1+h_2)^{-(j+2)}\|_{L^\infty}\left\|\mathrm{p.v.}\int_{\mathbb{T}}(A_1^j-A_2^j)\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi\right\|_{L^p}$$
$$+\|h_2'(1+h_2)^{-(j+1)}\|_{L^\infty}\left\|\mathrm{p.v.}\int_{\mathbb{T}}(A_1^j-A_2^j)\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi\right\|_{\dot W^{1,p}}. \tag{C47}$$

For $j=0$, this can be simplified as

$$\begin{aligned}\|L_{3,0}^{(1)}-L_{3,0}^{(2)}\|_{\dot W^{1,p}}\leqslant\ & C\big(\|h_1''-h_2''\|_{L^p}+\|h_2''\|_{L^p}\|h_1-h_2\|_{L^\infty}\big)\|\psi\|_{\dot C^\beta}\\ &+C\|h_1-h_2\|_{W^{1,\infty}}\big(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty}\big)\|\psi\|_{L^p}\\ &+C\|h_1-h_2\|_{W^{1,\infty}}\|\psi'\|_{L^p}.\end{aligned} \tag{C48}$$

For $j\geqslant 1$, by applying (C32), (C38), (C40), (C42), (C44) and (C46) to (C47), we derive that

$$\begin{aligned}&\|L_{3,j}^{(1)}-L_{3,j}^{(2)}\|_{\dot W^{1,p}}\\ &\leqslant C(C_2^{j+1}\|h_1''-h_2''\|_{L^p}+\|h_2''\|_{L^p}(j+1)C_2^{j+2}\|h_1-h_2\|_{L^\infty})\\ &\qquad\cdot(C_1^2C_2)^j\cdot j\|h_1'\|_{L^\infty}^{2j-1}\|h_1'\|_{\dot C^\beta}\|\psi\|_{C^\beta}\\ &\quad+C(j+1)(j+2)C_2^{j+2}\|h_1-h_2\|_{W^{1,\infty}}(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty})\\ &\qquad\cdot C_*^{2j-1}C_2^j\|h_1'\|_{L^\infty}^{2j}\|\psi\|_{L^p}\\ &\quad+C(j+1)C_2^{j+1}\|h_1-h_2\|_{W^{1,\infty}}\\ &\qquad\cdot\Big[(2j+1)C_\dagger^{2j+1}(jC_2^{j+1}\|h_1'\|_{L^\infty}\|\psi\|_{L^p}+C_2^j\|\psi'\|_{L^p})\|h_1'\|_{L^\infty}^{2j}\\ &\qquad\quad+(2j+1)C_\dagger^{2j+1}C_2^j\|\psi\|_{L^\infty}\|h_1''\|_{L^p}\|h_1'\|_{L^\infty}^{2j-1}\Big]\\ &\quad+CC_2^{j+1}\|h_2''\|_{L^p}\\ &\qquad\cdot(C_1^2C_2)^{j-1}\cdot j\|h_1-h_2\|_{C^{1,\beta}}(\|h_1'\|_{\dot C^\beta}+\|h_2'\|_{\dot C^\beta})(\|h_1'\|_{L^\infty}^2+\|h_2'\|_{L^\infty}^2)^{j-1}\|\psi\|_{C^\beta}\\ &\quad+CC_2^{j+2}(j+1)\|h_2'\|_{L^\infty}^2\\ &\qquad\cdot jC_*^{2j-1}C_2^j(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty})^{2j-1}\|h_1-h_2\|_{W^{1,\infty}}\|\psi\|_{L^p}\\ &\quad+CC_2^{j+1}\|h_2'\|_{L^\infty}\cdot j(2j+1)C_\dagger^{2j+1}C_2^j(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty})^{2j-2}\\ &\qquad\cdot\big[j\|\psi\|_{L^p}\|h_1-h_2\|_{W^{1,\infty}}(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty})^2\\ &\qquad\quad+\|\psi'\|_{L^p}\|h_1-h_2\|_{W^{1,\infty}}(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty})\\ &\qquad\quad+\|\psi\|_{L^\infty}\|h_1''-h_2''\|_{L^p}(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty})\\ &\qquad\quad+\|\psi\|_{L^\infty}\|h_1-h_2\|_{W^{1,\infty}}(\|h_1''\|_{L^p}+\|h_2''\|_{L^p})\big].\end{aligned} \tag{C49}$$

This together with (C48) and the smallness of h_i implies

$$\begin{aligned}\|L_3^{(1)}-L_3^{(2)}\|_{\dot W^{1,p}} &\leqslant \|L_{3,0}^{(1)}-L_{3,0}^{(2)}\|_{\dot W^{1,p}}+\sum_{j=1}^{\infty}\|L_{3,j}^{(1)}-L_{3,j}^{(2)}\|_{\dot W^{1,p}}\\ &\leqslant C\|h_1''-h_2''\|_{L^p}\big(1+\|h_1'\|_{\dot C^\beta}+\|h_2'\|_{\dot C^\beta}\big)\|\psi\|_{C^\beta}\\ &\quad + C\big(\|h_1''\|_{L^p}+\|h_2''\|_{L^p}\big)\|h_1-h_2\|_{C^{1,\beta}}(1+\|h_1'\|_{\dot C^\beta}+\|h_2'\|_{\dot C^\beta})\|\psi\|_{C^\beta}\\ &\quad + C\|h_1-h_2\|_{W^{1,\infty}}\|\psi'\|_{L^p}. \end{aligned}\tag{C50}$$

Then the desired estimate follows from (C14), (C26) and (C50). □

Proof of Lemma 5.6. Following (5.31), we use $\tilde L_k^{(i)}$ $(k=1,2,3)$ to denote the corresponding quantities defined by h_i $(i=1,2)$.

Using (5.32), we find that

$$\begin{aligned}&\|\tilde L_1^{(1)}-\tilde L_1^{(2)}\|_{\dot W^{1,p}}\\ &\leqslant \left\|\left(\frac{h_1''}{1+h_1}-\frac{(h_1')^2}{(1+h_1)^2}-\frac{h_2''}{1+h_2}+\frac{(h_2')^2}{(1+h_2)^2}\right)\left(\frac12\int_{\mathbb T}\psi\,d\xi+L_1^{(1)}\right)\right\|_{L^p}\\ &\quad+\left\|\left(\frac{h_2''}{1+h_2}-\frac{(h_2')^2}{(1+h_2)^2}\right)(L_1^{(1)}-L_1^{(2)})\right\|_{L^p}\\ &\quad+\left\|\left(\frac{h_1'}{1+h_1}-\frac{h_2'}{1+h_2}\right)(L_1^{(1)})'\right\|_{L^p}+\left\|\frac{h_2'}{1+h_2}(L_1^{(1)}-L_1^{(2)})'\right\|_{L^p}\\ &\leqslant C\big(\|h_1''-h_2''\|_{L^p}+\|h_2''\|_{L^p}\|h_1-h_2\|_{L^\infty}\big)\left(\left|\int_{\mathbb T}\psi\,d\xi\right|+\|L_1^{(1)}\|_{L^\infty}\right)\\ &\quad+C\|h_2''\|_{L^p}\|L_1^{(1)}-L_1^{(2)}\|_{L^\infty}\\ &\quad+C\|h_1-h_2\|_{W^{1,\infty}}\|L_1^{(1)}\|_{\dot W^{1,p}}+C\|h_2'\|_{L^\infty}\|L_1^{(1)}-L_1^{(2)}\|_{\dot W^{1,p}}. \end{aligned}\tag{C51}$$

It is not difficult to show by (C3) that

$$\begin{aligned}\|L_1^{(1)}-L_1^{(2)}\|_{L^\infty}&\leqslant C\|\psi\|_{L^\infty}\int_{\mathbb T}\|l_1-l_2\|_{L^\infty_\theta}\,d\xi\\ &\leqslant C\|\psi\|_{L^\infty}(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty})\|h_1-h_2\|_{W^{1,\infty}}. \end{aligned}\tag{C52}$$

Taking $h_2=0$ yields $\|L_1^{(1)}\|_{L^\infty}\leqslant C\|\psi\|_{L^\infty}\|h_1'\|_{L^\infty}$; here we used the fact $m_{0,i}\ll 1$. Substituting these estimates as well as (5.17) and (C14) into (C51), we obtain that

$$\begin{aligned}\|\tilde L_1^{(1)}-\tilde L_1^{(2)}\|_{\dot W^{1,p}}&\leqslant C(\|h_1''-h_2''\|_{L^p}+(\|h_1''\|_{L^p}+\|h_2''\|_{L^p})\|h_1-h_2\|_{W^{1,\infty}})\\ &\qquad\cdot\left(\left|\int_{\mathbb T}\psi\,d\xi\right|+\|\psi\|_{L^\infty}(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty})\right)\\ &\quad+C(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty})^2\|h_1-h_2\|_{W^{1,\infty}}\|\psi'\|_{L^p}. \end{aligned}\tag{C53}$$

To bound $\tilde L_2^{(1)}-\tilde L_2^{(2)}$, we are going to make use of the estimates for $L_2^{(1)}-L_2^{(2)}$ in Lemma 5.5, since $\tilde L_2^{(i)}$ coincides with $-h_i'(\theta)L_2^{(i)}$ if ψ in the definition of $L_2^{(i)}$ is replaced by $\psi/(1+h_i)$. For this purpose, an L^∞-estimate for $L_2^{(1)} - L_2^{(2)}$ is needed. We start with

$$\left|L_2^{(1)} - L_2^{(2)}\right| \leqslant \left|\text{p.v.} \int_{\mathbb{T}} \left(\frac{\frac{\Delta h_1 - h_1'(\theta)}{1+h_1(\theta)}}{1+l_1} - \frac{\frac{\Delta h_2 - h_2'(\theta)}{1+h_2(\theta)}}{1+l_2} \right) \frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\, d\xi \right| + \left|\text{p.v.} \int_{\mathbb{T}} \left(\frac{\frac{h_1'(\theta)}{1+h_1(\theta)}}{1+l_1} - \frac{\frac{h_2'(\theta)}{1+h_2(\theta)}}{1+l_2} \right) \frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\, d\xi \right|. \tag{C54}$$

It is straightforward to bound the first term.

$$\begin{aligned} &\left|\text{p.v.} \int_{\mathbb{T}} \left(\frac{\frac{\Delta h_1 - h_1'(\theta)}{1+h_1(\theta)}}{1+l_1} - \frac{\frac{\Delta h_2 - h_2'(\theta)}{1+h_2(\theta)}}{1+l_2} \right) \frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\, d\xi \right| \\ &\leqslant C \int_{\mathbb{T}} |\xi|^\beta \big(\|h_1' - h_2'\|_{\dot C^\beta} + \|h_2'\|_{\dot C^\beta} (\|h_1 - h_2\|_{L^\infty} + |l_1 - l_2|) \big) \|\psi\|_{L^\infty} |\xi|^{-1}\, d\xi \\ &\leqslant C \big(\|h_1' - h_2'\|_{\dot C^\beta} + \|h_2'\|_{\dot C^\beta} \|h_1 - h_2\|_{W^{1,\infty}} \big) \|\psi\|_{L^\infty}. \end{aligned} \tag{C55}$$

To bound the second term in (C54), we first note that (C32) and (C38) still hold if $2\tan\frac{\xi}{2}$ in their denominators are replaced by $2\sin\frac{\xi}{2}$. Hence, we argue as in the proof of Lemma 5.5 by Taylor expanding $(1+l_i)^{-1}$ that

$$\begin{aligned} &\left|\text{p.v.} \int_{\mathbb{T}} \left(\frac{1}{1+l_1} - \frac{1}{1+l_2} \right) \frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\, d\xi \right| \\ &\leqslant \sum_{j=1}^\infty \left|\text{p.v.} \int_{\mathbb{T}} \left(\frac{A_1^j}{(1+h_1(\theta))^j} - \frac{A_2^j}{(1+h_2(\theta))^j} \right) \frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\, d\xi \right| \\ &\leqslant \sum_{j=1}^\infty C_2^j \left|\text{p.v.} \int_{\mathbb{T}} (A_1^j - A_2^j) \frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\, d\xi \right| + \left| \frac{1}{(1+h_1)^j} - \frac{1}{(1+h_2)^j} \right| \left|\text{p.v.} \int_{\mathbb{T}} A_2^j \frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\, d\xi \right| \\ &\leqslant C \|\psi\|_{C^\beta} \|h_1 - h_2\|_{C^{1,\beta}} \big(\|h_1'\|_{\dot C^\beta} + \|h_2'\|_{\dot C^\beta} \big). \end{aligned} \tag{C56}$$

Taking $h_2 = 0$ here yields

$$\left|\text{p.v.} \int_{\mathbb{T}} \left(\frac{1}{1+l_1} - 1 \right) \frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\, d\xi \right| \leqslant C \|\psi\|_{C^\beta} \|h_1\|_{C^{1,\beta}} \|h_1'\|_{\dot C^\beta}, \tag{C57}$$

which further implies

$$\left|\text{p.v.} \int_{\mathbb{T}} \frac{1}{1+l_1} \frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\, d\xi \right| \leqslant C \|\psi\|_{C^\beta} \big(1 + \|h_1\|_{C^{1,\beta}}\big)^2. \tag{C58}$$

To this end, we may bound the second term in (C54) as follows

$$\begin{aligned}&\left|\text{p.v.}\int_{\mathbb{T}}\left(\frac{\frac{h_1'(\theta)}{1+h_1(\theta)}}{1+l_1}-\frac{\frac{h_2'(\theta)}{1+h_2(\theta)}}{1+l_2}\right)\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\right|\\
&\qquad\leqslant\left|\frac{h_1'}{1+h_1}-\frac{h_2'}{1+h_2}\right|\left|\text{p.v.}\int_{\mathbb{T}}\frac{1}{1+l_1}\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\right|\\
&\qquad\quad+\left|\frac{h_2'}{1+h_2}\right|\left|\text{p.v.}\int_{\mathbb{T}}\left(\frac{1}{1+l_1}-\frac{1}{1+l_2}\right)\frac{\psi(\theta+\xi)}{2\sin\frac{\xi}{2}}\,d\xi\right|\\
&\qquad\leqslant C\|\psi\|_{C^\beta}\|h_1-h_2\|_{C^{1,\beta}}\big(1+\|h_1\|_{C^{1,\beta}}+\|h_2\|_{C^{1,\beta}}\big)^2.\end{aligned}\tag{C59}$$

Combining this with (C54) and (C55),

$$\|L_2^{(1)}-L_2^{(2)}\|_{L^\infty}\leqslant C\|\psi\|_{C^\beta}\|h_1-h_2\|_{C^{1,\beta}}\big(1+\|h_1\|_{C^{1,\beta}}+\|h_2\|_{C^{1,\beta}}\big)^2.\tag{C60}$$

Setting $h_1=0$ (or $h_2=0$) provides

$$\|L_2^{(i)}\|_{L^\infty}\leqslant C\|\psi\|_{C^\beta}\|h_i\|_{C^{1,\beta}}\big(1+\|h_i\|_{C^{1,\beta}}\big)^2.\tag{C61}$$

To emphasize the ψ-dependence of $L_2^{(i)}$, we shall rewrite $L_2^{(i)}$ as $L_{2,\psi}^{(i)}$. Since $\tilde{L}_2^{(i)}=-h_i'(\theta)L^{(i)}_{2,\psi/(1+h_i)}$, we derive with (C26), (C60) and (C61) that

$$\begin{aligned}&\|\tilde{L}_2^{(1)}-\tilde{L}_2^{(2)}\|_{\dot{W}^{1,p}}\\
&\leqslant\|h_1''-h_2''\|_{L^p}\|L^{(1)}_{2,\psi/(1+h_1)}\|_{L^\infty}+\|h_2''\|_{L^p}\|L^{(1)}_{2,\psi/(1+h_1)}-L^{(2)}_{2,\psi/(1+h_1)}\|_{L^\infty}\\
&\quad+\|h_2''\|_{L^p}\|L^{(2)}_{2,\psi/(1+h_1)-\psi/(1+h_2)}\|_{L^\infty}\\
&\quad+\|h_1'-h_2'\|_{L^\infty}\|L^{(1)}_{2,\psi/(1+h_1)}\|_{\dot{W}^{1,p}}+\|h_2'\|_{L^\infty}\|L^{(1)}_{2,\psi/(1+h_1)}-L^{(2)}_{2,\psi/(1+h_1)}\|_{\dot{W}^{1,p}}\\
&\quad+\|h_2'\|_{L^\infty}\|L^{(2)}_{2,\psi/(1+h_1)-\psi/(1+h_2)}\|_{\dot{W}^{1,p}}\\
&\leqslant C\|h_1''-h_2''\|_{L^p}\left\|\frac{\psi}{1+h_1}\right\|_{C^\beta}\|h_1\|_{C^{1,\beta}}(1+\|h_1\|_{C^{1,\beta}})^2\\
&\quad+C\|h_2''\|_{L^p}\left\|\frac{\psi}{1+h_1}\right\|_{C^\beta}\|h_1-h_2\|_{C^{1,\beta}}(1+\|h_1\|_{C^{1,\beta}}+\|h_2\|_{C^{1,\beta}})^2\\
&\quad+C\|h_2''\|_{L^p}\left\|\frac{\psi}{1+h_1}-\frac{\psi}{1+h_2}\right\|_{C^\beta}\|h_2\|_{C^{1,\beta}}(1+\|h_2\|_{C^{1,\beta}})^2\\
&\quad+C\|h_1'-h_2'\|_{L^\infty}\left(\left\|\frac{\psi}{1+h_1}\right\|_{\dot{W}^{1,p}}\|h_1\|_{W^{1,\infty}}+\|\psi\|_{L^\infty}\|h_1''\|_{L^p}\right)\\
&\quad+C\|h_2'\|_{L^\infty}\left(\left\|\frac{\psi}{1+h_1}\right\|_{\dot{W}^{1,p}}\|h_1-h_2\|_{W^{1,\infty}}+\|\psi\|_{L^\infty}\|h_1''-h_2''\|_{L^p}\right.\\
&\qquad\left.+\|\psi\|_{L^\infty}\|h_1-h_2\|_{W^{1,\infty}}(\|h_1''\|_{L^p}+\|h_2''\|_{L^p})\right)\end{aligned}$$

$$+ C\|h_2'\|_{L^\infty}\left(\left\|\frac{\psi}{1+h_1}-\frac{\psi}{1+h_2}\right\|_{\dot W^{1,p}}\|h_2\|_{W^{1,\infty}}+\left\|\frac{\psi}{1+h_1}-\frac{\psi}{1+h_2}\right\|_{L^\infty}\|h_2''\|_{L^p}\right). \tag{C62}$$

This gives

$$\begin{aligned}\|\tilde L_2^{(1)}-\tilde L_2^{(2)}\|_{\dot W^{1,p}} &\leqslant C\|h_1''-h_2''\|_{L^p}\|\psi\|_{C^\beta}\big(\|h_1\|_{C^{1,\beta}}+\|h_2\|_{C^{1,\beta}}\big)\big(1+\|h_1\|_{C^{1,\beta}}+\|h_2\|_{C^{1,\beta}}\big)^2\\ &\quad+C(\|h_1''\|_{L^p}+\|h_2''\|_{L^p})\|\psi\|_{C^\beta}\|h_1-h_2\|_{C^{1,\beta}}\big(1+\|h_1\|_{C^{1,\beta}}+\|h_2\|_{C^{1,\beta}}\big)^3\\ &\quad+C\|h_1-h_2\|_{W^{1,\infty}}\|\psi'\|_{L^p}\big(\|h_1\|_{W^{1,\infty}}+\|h_2\|_{W^{1,\infty}}\big).\end{aligned} \tag{C63}$$

For $\tilde L_3^{(i)}$, we rewrite

$$\tilde L_3^{(i)}=\sum_{j=1}^{\infty}(-1)^{j+1}\big(1+h_i(\theta)\big)^{-j}\,\text{p.v.}\int_{\mathbb T}A_i^j\cdot\frac{\psi(\theta+\xi)}{2\tan\frac{\xi}{2}}\,d\xi. \tag{C64}$$

Thanks to (C40), (C42), (C44) and (C46), we derive as in the proof of Lemma 5.5 that

$$\begin{aligned}\big\|\tilde L_3^{(1)}-\tilde L_3^{(2)}\big\|_{\dot W^{1,p}} &\leqslant C\|h_1-h_2\|_{W^{1,\infty}}\big(\|\psi'\|_{L^p}(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty})+\|\psi\|_{L^\infty}(\|h_1''\|_{L^p}+\|h_2''\|_{L^p})\big)\\ &\quad+C\|h_1''-h_2''\|_{L^p}\|\psi\|_{L^\infty}\big(\|h_1'\|_{L^\infty}+\|h_2'\|_{L^\infty}\big).\end{aligned} \tag{C65}$$

Combining (C53), (C63) and (C65), we obtain (5.45). □

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