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Porous Medium Equation with a Drift: Free Boundary Regularity

Inwon Kim & Yuming Paul Zhang

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Abstract

We study regularity properties of the free boundary for solutions of the porous medium equation with the presence of drift. We show the $C^{1,\alpha}$ regularity of the free boundary when the solution is directionally monotone in space variable in a local neighborhood. The main challenge lies in establishing a local non-degeneracy estimate (Theorem 1.3 and Proposition 1.5), which appears new even for the zero drift case.

1. Introduction

Let us consider the drift-diffusion equation

$$\varrho_t = \Delta \varrho^m + \nabla \cdot (\varrho \vec{b}) \quad \text{in } Q := \mathbb{R}^d \times (0, \infty), \tag{1.1}$$

with a smooth vector field $\vec{b}$: $Q \to \mathbb{R}^d$, a non-negative initial data $\varrho(\cdot, 0) = \varrho_0$ and $m > 1$. The nonlinear diffusion term in (1.1) represents an anti-congestion effect [5,7,14,24].

Our interest is in the regularity of the *free boundary* $\partial\{\varrho > 0\}$, which is present at all times if starting with a compactly supported initial data. We are motivated by the intriguing fact that the free boundary regularity is open even for the travelling wave solutions in two space dimensions, with a smooth and laminar drift $\vec{b}(x_1, x_2) = (\sin x_2, 0)$ (see [21]). Our analysis provides a starting point of the discussion in a general framework, but the full answer to this question remains open (see Theorem 1.6 and the discussion below). The presence of the drift generates several significant challenges that are new to the problem, as we will discuss below.

To illustrate the regularizing mechanism of the interface, let us write (1.1) in the form of continuity equation

$$\varrho_t - \nabla \cdot ((\nabla u + \vec{b})\varrho) = 0,$$

where

$$u = \frac{m}{m-1}\varrho^{m-1}. \tag{1.2}$$

Hence formally the normal velocity for the free boundary can be written as

$$V = -(\nabla u + \vec{b}) \cdot \vec{n} = |\nabla u| - \vec{b} \cdot \vec{n} \quad \text{on } (x,t) \in \Gamma := \partial\{u > 0\}, \tag{1.3}$$

where $\vec{n} = \vec{n}_{x,t}$ is the outward normal vector at given boundary points. Given that ϱ solves a diffusion equation, it would be natural to expect that the free boundary is regularized by the pressure gradient $|\nabla u|$ if $\vec{b}$ is smooth, as long as u stays non-degenerate near the free boundary and topological singularities are ruled out. In general neither can be guaranteed even with zero drift. Below we discuss our main results and new challenges in the context of the literature. We will always assume that

$$\vec{b} \in C^{3,1}_{x,t}(Q) \quad \text{and} \quad \varrho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \tag{1.4}$$

Literature Let us first discuss the case $\vec{b} = 0$, and then our problem (1.1) corresponds to the well-known *Porous Medium Equation* (PME). In this case a vast amount of literature is available: we refer to the book [23]. What follows is a brief discussion of several prominent results that are relevant to our results. ARONSON and BENILAN [2] showed the semi-convexity estimate $\Delta u > -\infty$ for $t > 0$ which played a fundamental role in the regularity theory of (PME). In general there can be a *waiting time* for degenerate initial data, where the free boundary does not move and regularization is delayed. When the initial data $u_0 = u(\cdot, 0)$ has super-quadratic growth at the free boundary, CAFFARELLI and FRIEDMAN [9] showed that there is no waiting time and the support of solution strictly expands in time. There an expansion rate of the support was obtained, by showing that its free boundary can be represented as $t = S(x)$ where S is Hölder continuous. To discuss further regularity results, it is natural to require some geometric properties of the solution to rule out topological singularities such as merging of two fingers. The $C^{1,\alpha}$ regularity of the free boundary is established by CAFFARELLI and WOLANSKI [10], under the assumption of non-degeneracy and Lipschitz continuity of solutions. Their assumptions are shown to hold after a finite time $T_0 > 0$ by CAFFARELLI et al. [11], where T_0 is the first time the support of solution expands to contain its initial convex hull. More recently, Kienzler explored the stability of solutions that are close to the flat traveling wave fronts to (PME) [16]. Later KIENZLER et al. [17] improved this result and showed that solutions that are locally close to the traveling waves are smooth; see further discussion on their result in comparison to ours below Theorem 1.3.

When $\vec{b} \neq 0$, few results are available on the free boundary regularity of (1.1). With the exception of the particular choice $\vec{b} = x$, in general there appears to be no change of coordinates that eliminates the drift dependence in (1.1). Numerical experiments in [22] present the interesting possibility that an initially planar solution with smooth drift could develop corners without topological changes. However the non-degeneracy of pressure or the free boundary regularity is unknown even for traveling wave solutions in $\mathbb{R}^2$ (see [21]). By comparison, well-posedness and regularity theory for the solutions of (1.1) has been much better understood. Existence and uniqueness results are shown in [4,6] for weak solutions and in [18] for viscosity solutions. Asymptotic convergence to equilibrium of (1.1) is shown in [12] using energy dissipation when $\vec{b}$ is the gradient of a convex potential. Recently [15, 19] proved Hölder continuity of solutions for uniformly bounded, but possibly non-smooth drifts.

Discussion of Main Results and Difficulties For our analysis, we will consider the pressure variable (1.2) and the equation it satisfies, which is

$$u_t = (m-1)u\,\Delta u + |\nabla u|^2 + \nabla u \cdot \vec{b} + (m-1)u\,\nabla \cdot \vec{b} \tag{1.5}$$

in $Q = \mathbb{R}^d \times (0, \infty)$.

We first show the semi-convexity (Aronsson–Benilan) estimate through a simple but novel barrier argument on Δu. This is where we use the C^3_x norm of $\vec{b}$.

Theorem 1.1. (Theorem 3.1) *Let ρ solve (1.1) in Q with (1.4), and let u be the corresponding pressure variable given by (1.2). Then for some $\sigma > 0$, $\Delta u > -\frac{\sigma}{t} - \sigma$ in the sense of distribution for all $t > 0$.*

Next we discuss a weak non-degeneracy property in the event of zero initial waiting time. With zero drift this corresponds to the strict expansion property of the positive set, see section 14 [23]. In our case this property needs to be understood in terms of the *streamlines*, defined as

$$X(t) := X(x_0, t_0; t) \text{ is the unique solution of the ODE}$$
$$\begin{cases} \partial_t X(t) = -\vec{b}(X(t), t_0 + t), \quad t \in \mathbb{R}, \\ X(0) = x_0. \end{cases} \tag{1.6}$$

While the streamlines are a natural coordinate for us to measure the strict expansion of the positive set over time, it does not cope well with the diffusion term in the equation. The most delicate scenario occurs with degenerate pressure, where the time range we need to observe is much larger than the space range. To deal with such a case we need to carefully localize $\vec{b}$.

Theorem 1.2. (Theorem 4.4) *Let u be as given in Theorem 1.1, and fix $(x_0, t_0) \in \Gamma := \partial\{u > 0\} \cap \{t > 0\}$. Write*

$$X(-s) := X(x_0, t_0; -s), \quad \Omega_t := \{u(\cdot, t) > 0\} \quad \textit{and} \quad \Gamma_t := \partial\Omega_t.$$

Then either of the following holds:

(Type one) $X(-s) \in \Gamma_{t_0 - s}$ for $s \in (0, t_0)$;

(Type two) there exist $C_*, \beta > 1$ and $h > 0$ such that for $s \in (0, h)$

$$u(x, t_0 - s) = 0 \quad \textit{if } |x - X(-s)| \leqq C_* s^\beta,$$
$$u(x, t_0 + s) > 0 \quad \textit{if } |x - X(s)| \leqq C_* s^\beta.$$

Moreover, if

$$\begin{aligned} &\Omega_0 \text{ is a bounded domain with Lipschitz boundary, and} \\ &u_0(x) \geqq \gamma(d(x, \Omega_0^C))^{2-\varsigma} \text{ for some } \gamma, \varsigma > 0, \end{aligned} \tag{1.7}$$

then any point on Γ *is of type two.*

The growth condition in (1.7) is optimal, since there is a stationary solution to (1.1) with a corner on its free boundary and with quadratic growth (see Theorem 7.3).

Next we proceed to show the non-degeneracy property of u, as it is essential for the regularity of its free boundary. This step presents the most challenging and novel part of our analysis. To illustrate the difficulties, let us briefly go over the main components of the celebrated arguments in [11], which provides non-degeneracy of solutions for (PME) for times $t > T_0$. One key ingredient in their analysis was the scale invariance of the equation under the transformation

$$u_{\varepsilon,A}(x,t) := \frac{1 + A\varepsilon}{(1+\varepsilon)^2} u((1+\varepsilon)x, (1+A\varepsilon)t + B) \quad \text{for any } A, B, \varepsilon > 0.$$

In [11] $u_{\varepsilon,A}$ was compared to u to obtain the space-time directional monotonicity

$$x \cdot \nabla u + (At + B)u_t \geqq 0 \quad \text{on } \Gamma. \tag{1.8}$$

Applying (1.3) with $\vec{b} = 0$, we then have

$$|\nabla u| = V = \frac{u_t}{|\nabla u|} \geqq \frac{1}{(At+B)} \nu \cdot (\frac{x}{|x|}) \quad \text{on } \Gamma,$$

where the first equality is from (1.3), the second equality is due to the level set formulation of the normal velocity, and the last inequality is due to (1.8) and the fact that ∇u is parallel to the negative normal $-\nu$ on the free boundary. Thus the non-degeneracy follows if we know that the free boundary is a Lipschitz graph with respect to the radial direction. This was shown in [11] for $t > T_0$ by the celebrated moving planes arguments, and thus we can conclude.

For nonzero drift, neither scaling invariance nor the moving planes method is available due to the inhomogeneity in $\vec{b}$. In fact it is not reasonable to expect consistent free boundary behavior for large times, except possibly when $\vec{b}$ is a potential vector field. Still, it is reasonable to expect that, without topological singularites and waiting time, the diffusive nature of the Eq. (1.5) regularizes the free boundary. With this in mind we show a local non-degeneracy result under the assumption of directional monotonicity and zero waiting time.

Let us define the spatial cone of directions

$$W_{\theta,\mu} := \left\{ y \in \mathbb{R}^d : \left| \frac{y}{|y|} - \mu \right| \leqq 2 \sin\frac{\theta}{2} \right\} \quad \text{with axis } \mu \in \mathcal{S}^{d-1} \text{ and } \theta \in (0, \pi/2]. \tag{1.9}$$

We say that u is *monotone* with respect to $W_{\theta,\mu}$ if $u(\cdot, t)$ is non-decreasing along directions in $W_{\theta,\mu}$. Using the notation $Q_r := \{|x| \leqq r\} \times (-r, r)$, we say that Γ is of type two in Q_r if all points on $\Gamma \cap Q_r$ are of type two.

Theorem 1.3. (Local Non-degeneracy, Corollary 5.7) *Let ϱ be a weak solution to* (1.1) *in Q_2, where Γ is of type two, and let u be the pressure. Suppose in Q_2, $\Delta u > -\infty$ and u is monotone with respect to $W_{\theta,\mu}$ for some θ and μ. Then there exists $\kappa_* > 0$ such that*

$$\liminf_{\varepsilon \to 0^+} \frac{u(x+\varepsilon\mu, t)}{\varepsilon} \geqq \kappa_* \quad \textit{for } (x,t) \in \Gamma \cap Q_1.$$

For the proof we adopt a local barrier argument introduced in [13] in the context of the Hele–Shaw flow. Heuristically speaking the barrier argument illustrates the fact that the nondegeneracy property of positive level sets propagates to the free boundary as the positive set expands out in diffusive free boundary problems.

As mentioned above, in the zero drift case [17] considered solutions that are locally close to a planar traveling wave solution. Their assumption in particular endows a discrete small-scale flatness and non-degeneracy. It was shown there that over time the flatness improves in its scale to yield the smoothness of the solutions. It was conjectured there whether a cone monotonicity assumption could replace proximity to the planar travelling waves. While we do not pursue improvement of flatness in scale, our result yields a positive partial answer to this question.

Building on the above non-degeneracy result, we proceed to study the free boundary regularity. To prevent sudden changes in the evolution caused by changes in the far-away region, we assume that, in the weak sense,

$$u_t \leqq A\,(\mu \cdot \nabla u + u + 1) \quad \text{in } Q_1 \text{ for some } A > 0. \tag{1.10}$$

Theorem 1.4. (Theorem 6.1) *Let u be given as in Theorem* 1.3*. If in addition* (1.10) *holds, then u is Lipschitz continuous and Γ is $C^{1,\alpha}$ in $Q_{\frac{1}{2}}$.*

The proof of the above theorem is given in Section 6. The novel ingredient in this section is the following result, which propagates the non-degeneracy of the solution at the free boundary to nearby positive level sets.

Proposition 1.5. (Propagation of non-degeneracy, Proposition 6.3) *Under the assumption of Theorem* 1.4*, there exist $\delta < \frac{1}{2}$ and $c_1 > 0$ such that*

$$\nabla_\mu u(x,t) \geqq c_1 \quad \textit{in } \{u > 0\} \cap Q_\delta.$$

From here, the proof of Theorem 1.4 largely follows the iterative argument given in [10], which compares in different scales the solution with its shifted version. For nonzero drifts (1.5) changes under coordinate shifts, and thus a notable modification is necessary in the iteration procedure. See Remark 6.9.

Now we address the traveling wave solutions discussed earlier in the introduction.

Theorem 1.6. *Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a smooth and bounded function. Let u solve* (1.5) *in $Q = \mathbb{R}^2 \times (0,\infty)$ with $\vec{b} = (\alpha(x_2), 0)$ and the initial data $u_0(x) = u(x,0) = (x_1)_+$, under linear growth condition at infinity. Then Γ is locally uniformly $C^{1,\alpha}$ in Q.*

In [21] the existence of traveling wave solutions are shown with the above choice of $\vec{b}$. We consider the initially planar solution that was used in [22] to approximate the traveling waves. Our argument yields an exponentially decaying lower bound on the nondegeracy of u. While it rules out the possibility of finite time singularity for the approximate solutions, the free boundary regularity of travelling wave solutions remains open.

Lastly we present some examples which illustrate new types of free boundary singularities generated by drifts.

Theorem 1.7. (Theorem 7.3 and 7.4). *There is $\vec{b} \in C^3_x(\mathbb{R}^d)$ such that (1.5) has a stationary profile with a corner on its free boundary. There is a continuous spatial vector field $\vec{b}$ such that an initially smooth solution to (1.5) develops singularity on the free boundary in finite time.*

2. Preliminaries

◦ **Notations**

- $B(x, r) := \{x \in \mathbb{R}^d : |x| \leqq r\}$, $B_r := B(0, r)$, $Q = \mathbb{R}^d \times (0, \infty)$ and $Q_r := B_r \times (-r, r)$.
- Throughout the paper we denote σ as various *universal constants*, by which we mean constants that only depend on $m, d, \|\vec{b}\|_{C^{3,1}_{x,t}}$, and $\|\varrho_0\|_{L^1(\mathbb{R}^d)} + \|\varrho_0\|_{L^\infty(\mathbb{R}^d)}$ if ϱ solves (1.1) in Q, and $\|\varrho\|_{L^\infty(Q_r)}$ if ϱ is only assumed to be a solution in Q_r. By saying "X only depends on σ", we mean that X only depends on the above universal constants.
- We use C to represent constants which might depend on universal constants and other constants that are given in the assumptions of corresponding theorems.
- For a continuous, non-negative function $u : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$, we denote

$$\Omega(u) := \{u > 0\}, \quad \Omega_t(u) := \{u(\cdot, t) > 0\}$$

  and

$$\Gamma_t(u) := \partial\Omega_t, \quad \Gamma(u) := \bigcup_{t\in(0,\infty)} (\Gamma_t \times \{t\}).$$

  When it is clear from the context we will omit the dependence on u.
- We write

$$\fint_{B(x,r)} f(y)\mathrm{d}y := \frac{1}{|B(x, r)|} \int_{B(x,r)} f(y)\mathrm{d}y,$$

  where $|B(x, r)|$ is the volume of $B(x, r)$.
- $\nabla := \nabla_x$, and $\hat{\nabla} := (\nabla, \partial_t)$. We also denote $f_i := \partial_{x_i} f$, $f_{ij} := \partial^2_{x_i x_j} f$.
- For $\nu, \mu \in \mathbb{R}^d\backslash\{0\}$, the angle between them are denoted by

$$\langle \nu, \mu\rangle := \arccos\left(\frac{\nu \cdot \mu}{|\nu||\mu|}\right) \in [0, \pi].$$

For $\mu \in \mathbb{R}^d$, $\nu \in \mathbb{R}^{d+1}$ and $\theta \in [0, \pi/2]$, we define the space and space-time cones by

$$W_{\theta,\mu} := \{p \in \mathbb{R}^d \colon \langle p, \mu \rangle \leqq \theta\}, \quad \widehat{W}_{\theta,\nu} := \{p \in \mathbb{R}^{d+1} \colon \langle p, \nu \rangle \leqq \theta\}. \tag{2.1}$$

∘ **Notions of Solutions and Their Smooth Approximations.** Next we recall the notion of weak solutions and their properties, including their smooth approximations that will be used in this paper.

Definition 2.1. Let ϱ_0 be a non-negative function in $L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, and let $T > 0$. We say that a non-negative and bounded function $\varrho \colon \mathbb{R}^d \times [0, T] \to [0, \infty)$ is a subsolution (resp. supersolution) to (1.1) with initial data ϱ_0 if

$$\varrho \in C([0, T], L^1(\mathbb{R}^d)),\ \varrho \vec{b} \in L^2([0, T] \times \mathbb{R}^d) \text{ and } \varrho^m \in L^2(0, T, \dot{H}^1(\mathbb{R}^d)) \tag{2.2}$$

and

$$\int_0^T \int_{\mathbb{R}^d} \varrho\, \phi_t \mathrm{d}x\mathrm{d}t \geqq (\text{resp. } \leqq) \int_{\mathbb{R}^d} \varrho_0(x)\phi(0, x)\mathrm{d}x + \int_0^T \int_{\mathbb{R}^d} (\nabla \varrho^m + \varrho\, \vec{b})\nabla\phi \, \mathrm{d}x\mathrm{d}t, \tag{2.3}$$

for all non-negative $\phi \in C_c^\infty(\mathbb{R}^d \times [0, T))$.

We say ϱ is a weak solution to (1.1) if it is both sub- and supersolution of (1.1). We also say that $u := \frac{m}{m-1}\varrho^{m-1}$ is a *solution* (resp. *super/sub solution*) to (1.5) if ϱ is a weak solution (resp. super/sub solution) to (1.1).

The well-posedness result of general degenerate parabolic type equations is established in [1,2,6,7]. Kim and Zhang [19] proved the uniform in time L^∞-estimate of solutions (though $\vec{b} = \vec{b}(x)$ in the paper, the same proof applies to $\vec{b} \in L^\infty(Q)$). [3,4] proved the Hölder regularity of solutions.

Theorem 2.1. (Theorem 1.7, [1], Theorem 1.1, [19]) *Let ϱ_0 be as given in Definition 2.1. When $\vec{b} \in L^\infty(Q)$, then there exists a weak solution ϱ to (1.1) with initial data ϱ_0. Moreover ϱ is uniformly bounded for all $t \geqq 0$ with a bound depending only on m, d, $\|\vec{b}\|_\infty$, and $\|\varrho_0\|_{L^1} + \|\varrho_0\|_{L^\infty}$.*

Theorem 2.2. (Theorem 1, [4]) *Suppose ϱ is a non-negative, bounded weak solution to (1.1) in Q_1. Then ϱ is Hölder continuous in $Q_{\frac{1}{2}}$.*

Theorem 2.3. (Theorem 2.2, [1]) *Suppose U is an open subset of $\mathbb{R}^d$ and $\vec{b} \in C^{1,0}_{x,t}$. Let $\bar{\varrho}$, $\underline{\varrho}$ be respectively a subsolution and a supersolution of (1.1) in $U \times \mathbb{R}^+$ such that $\bar{\varrho} \leqq \underline{\varrho}$ a.e. in the parabolic boundary of $U \times \mathbb{R}^+$. Then $\bar{\varrho} \leqq \underline{\varrho}$ in $U \times \mathbb{R}^+$.*

Remark 2.4. Following from Theorem 2.3, we have comparison principle for (1.5): suppose $\bar{u}$, $\underline{u}$ are respectively a subsolution and a supersolution of (1.5) in $U \times \mathbb{R}^+$ such that $\bar{u} \leqq \underline{u}$ a.e. on the parabolic boundary of $U \times \mathbb{R}^+$. Then $\bar{u} \leqq \underline{u}$ in $U \times \mathbb{R}^+$.

In our analysis it is often convenient to work with classical solutions of (1.1), which is made possible by the following result (we will rely on this approximation lemma in Theorem 3.1 and in Section 5):

Lemma 2.5. (Section 9.3 [23]) *Let U be either B_1 or $\mathbb{R}^d$, and consider $\varrho_0 \in L^1(U) \cap L^\infty(U) \cap C(U)$. Let ϱ be a weak solution of (1.1) in $U \times [0,1]$ that is in $C(\overline{U} \times [0,1])$ with initial data ϱ_0. Then there exists a sequence of strictly positive, classical solutions ϱ_k of (1.1) such that $\varrho_k \to \varrho$ locally uniformly in $U \times (0,1]$ as $k \to \infty$.*

Proof. Let us consider $U = B_1$. Consider $\varrho_{0,k} = \varrho_0 + \frac{1}{k}$ and let ϱ_k be the weak solution to (1.1) in U with initial data $\varrho_{0,k}$ and Dirichlet boundary condition $\varrho_k = \varrho + \frac{1}{k}$ on $\partial U \times (0,1]$. Note that

$$\psi(x,t) := \frac{1}{k} \exp(-\|\nabla \cdot \vec{b}\|_\infty t)$$

is a subsolution to (1.1) in $U \times (0,1]$ with $\psi \leqq \frac{1}{k}$ on the parabolic boundary. Thus from the comparison principle it follows that

$$\varrho_k(x,t) \geqq \psi(x,t) > 0.$$

Since ϱ_k is uniformly bounded away from zero in $U \times [0,1]$, (1.1) is uniformly parabolic. In view of the standard parabolic theory, it follows that ϱ_k is smooth in $U \times (0,1]$. The proof for locally uniform convergence of ϱ_k to ϱ is parallel to that of Lemma 9.5 in [23]. □

To end this section, we state the following technical lemma which is used for comparison:

Lemma 2.6. *Set $U := B_1$ or $\mathbb{R}^d$, and let $T > 0$. Let ψ be a non-negative continuous function defined in $U \times [0,T]$ such that*

(a) *ψ is smooth in its positive set and in the set we have $\psi_t - \Delta\psi^m - \nabla \cdot (\vec{b}\,\psi) \geqq 0$,*
(b) *ψ^α is Lipschitz continuous for some $\alpha \in (0,m)$,*
(c) *$\Gamma(\psi)$ has Hausdorff dimension d.*

Then

$$\psi_t - \Delta\psi^m - \nabla \cdot (\vec{b}\,\psi) \geqq 0 \textit{ in } U \times [0,T]$$

in the weak sense i.e. for all non-negative $\phi \in C_c^\infty(U \times [0,T))$

$$\int_0^T \int_{\mathbb{R}^d} \psi\, \phi_t \mathrm{d}x\mathrm{d}t \leqq \int_{\mathbb{R}^d} \psi(0,x)\phi(0,x)\mathrm{d}x + \int_0^T \int_{\mathbb{R}^d} (\nabla\psi^m + \psi\,\vec{b})\nabla\phi \;\mathrm{d}x\mathrm{d}t. \tag{2.4}$$

We postpone the proof to the "Appendix".

3. Regularity of the Pressure

In this section we establish two basic properties for the pressure variable u that we will frequently use in the rest of the paper. We begin by obtaining the fundamental estimate.

Theorem 3.1. *Let u be a solution of (1.5) in $\mathbb{R}^d \times [0, \infty)$ with non-negative initial data u_0 such that $u_0^{\frac{1}{m-1}} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then there exists a universal constant σ such that*

$$\Delta u(x,t) > -\frac{\sigma}{t} - \sigma \quad \textit{for every } (x,t) \in \mathbb{R}^d \times (0,\infty) \tag{3.1}$$

in the sense of distribution.

Proof. By Lemma 2.5, it is enough to consider positive smooth solutions with positive smooth initial data. If (3.1) holds for the approximated smooth solutions, from the locally uniform convergence of the approximation we can conclude.

Assume that u is positive and smooth, and consider $p := \Delta u$. By differentiating (1.5) twice, we get

$$\begin{aligned} p_t = (m-1)u\Delta p + 2m\nabla u \cdot \nabla p + (m-1)p^2 + 2\Sigma\, u_{ij}u_{ij} \\ + \nabla p \cdot \vec{b} + 2\Sigma\, u_{ij}b^i_j + \nabla u \cdot \Delta\vec{b} + (m-1) \\ \left(p\nabla \cdot \vec{b} + 2\nabla u \cdot \nabla(\nabla \cdot \vec{b}) + u\Delta(\nabla \cdot \vec{b})\right). \end{aligned}$$

By Young's inequality,

$$\begin{aligned} \left|(m-1)p\nabla \cdot \vec{b} + 2\Sigma\, u_{ij}b^i_j\right| &\leqq \frac{m-1}{2}p^2 + \Sigma|u_{ij}|^2 + \sigma m \\ &\leqq \left(\frac{m-1}{2} - \frac{1}{d}\right)p^2 + 2\Sigma\,|u_{ij}|^2 + \sigma m; \\ \left|\nabla u \cdot \Delta\vec{b} + 2(m-1)(\nabla u \cdot \nabla(\nabla \cdot \vec{b})\right| &\leqq m|\nabla u|^2 + \sigma m; \\ (m-1)\left(u\Delta(\nabla \cdot \vec{b})\right) &\leqq \sigma m. \end{aligned}$$

Thus we obtain

$$\begin{aligned} &p_t - (m-1)u\Delta p - 2m\nabla u \cdot \nabla p \\ &-\left(\frac{m-1}{2} + \frac{1}{d}\right)p^2 - \nabla p \cdot \vec{b} + m|\nabla u|^2 + \sigma m \geqq 0. \end{aligned}$$

Viewing u as a known function, we may write the above quasilinear parabolic operator of p as $\mathcal{L}_0(p)$ and so we have $\mathcal{L}_0(p) \geqq 0$. Below will construct a barrier for this operator to obtain a lower bound for p.

Since u is smooth, then there exists $\tau > 0$ such that $\Delta u(\cdot, 0) \geqq -\frac{1}{\tau}$. By Theorem 2.1, u is uniformly bounded by a universal constant and we denote it as σ_0. Let $w := -\frac{\sigma_1}{t+\tau} + u - \sigma_2$ for some $\sigma_1 \geqq 1, \sigma_2 \geqq \sigma_0$ to be determined later. Then $p \geqq w$ at $t = 0$.

Direct computation yields

$$\mathcal{L}_0(w) = \frac{\sigma_1}{(t+\tau)^2} + u_t - (m-1)u\Delta u - 2m|\nabla u|^2 - \left(\frac{m-1}{2} + \frac{1}{d}\right) \left(-\frac{\sigma_1}{t+\tau} + u - \sigma_2\right)^2 - \nabla u \cdot \vec{b} + m|\nabla u|^2 + \sigma m.$$

Now we use the Eq. (1.5), and $\sigma_2 \geqq \sigma_0$ to obtain

$$\begin{aligned}
\mathcal{L}_0(w) &\leqq \frac{\sigma_1}{(t+\tau)^2} - (m-1)\,|\nabla u|^2 - \left(\frac{m-1}{2} + \frac{1}{d}\right)\left(-\frac{\sigma_1}{t+\tau} + u - \sigma_2\right)^2 + \sigma m \\
&\leqq \frac{\sigma_1}{(t+\tau)^2} - \left(\frac{m-1}{2} + \frac{1}{d}\right)\frac{\sigma_1^2}{(t+\tau)^2} - \left(\frac{m-1}{2} + \frac{1}{d}\right)(\sigma_2 - u)^2 + \sigma m \\
&\leqq 0,
\end{aligned}$$

where the last inequality holds if we choose $\sigma_1 := d$ and $\sigma_2 := \sigma_0 + (2d\sigma)^{1/2}$. Hence $\mathcal{L}_0(w) \leqq 0 \leqq \mathcal{L}_0(p)$, and from the comparison principle for $\mathcal{L}_0$ we conclude that

$$\Delta u = p \geqq w \geqq -\frac{\sigma_1}{t+\tau} - \sigma_2.$$

After taking $\tau \to 0$, we obtain that (3.1) holds for smooth solutions. We can conclude by Lemma 2.5. $\square$

Remark 3.2. Using the same barrier in the proof of the lemma, it can be seen that if $\Delta u_0 \geqq -C_0$ in the sense of distribution, then $\Delta u \geqq -\frac{\sigma_1}{t+(1/C_0)} - \sigma_2$ in the distribution sense for all time.

Next we prove a useful property: the consistency of positivity set of a solution along streamlines over time. The proof is parallel to the proof of Lemma 3.5 [20] where they used a barrier argument. Recall that we denote $\Omega_t = \{u(\cdot, t) > 0\}$.

Lemma 3.3. *Let u solve (1.5) with $\Delta u > -\infty$ in Q_2. Then for $X(x,t;s)$ given in (1.6) and for $c_0 := \frac{1}{2(1+\|\vec{b}\|_\infty)}$ the following is true:*

$$(X(\Omega_t, t; s) \cap B_1) \subseteq \Omega_{t+s} \text{ for all } t \in (-1, 1-c_0) \text{ and } s \in (0, c_0].$$

If u solves (1.5) in $\mathbb{R}^d \times [0,\infty)$ with initial data u_0 given as in Theorem 3.1, then

$$X(\Omega_t, t; s) \subseteq \Omega_{t+s} \quad \text{for all } s, t > 0.$$

Proof. In view of Theorem 3.1, the second statement follows easily from the first one. To prove the first statement, it is suffices to show that for all $x \in \Omega_t$ and $s \in (0, c_0]$, if $X(x,t;s) \in B_1$ then $u(X(x,t;s)) > 0$.

If $x \in B^c_{\frac{3}{2}}$, by the choice of c_0,

$$|X(x,t;s)| \geqq |x| - \|\vec{b}\|_\infty s > 1 \quad \text{for all } s \in (0, c_0].$$

Thus we take $x \in \Omega_t \cap B_{\frac{3}{2}}$ and then $X(x, t; s)$ is inside the domain B_2 for all $s \in (0, c_0]$. By Theorem 2.2, u is continuous in Q_2. Then we can suppose for contradiction that there exists $s_0 \in (0, c_0]$ such that

$$u(X(x, t; s), t + s) > 0 \text{ for all } s \in (0, s_0) \quad \text{and} \quad u(X(x, t; s_0), t + s_0) = 0.$$

Suppose $\Delta u \geqq -C_0$ in Q_2. Note that (1.5) is uniformly parabolic in any compact subset of $\{u > 0\}$, due to the continuity of u. Therefore by the standard parabolic theory, u is smooth in $\Omega \cap Q_2$. It follows from (1.5) that for all $s \in (0, s_0)$,

$$\begin{aligned}\partial_s u(X(x, t; s), t + s) &= (u_t + \nabla u \cdot \vec{b})(X(x, t; s), t + s) \\ &\geqq (-C_0(m-1)u + |\nabla u|^2 + (m-1)u\nabla \cdot \vec{b})(X(x, t; s), t + s) \\ &\geqq -Cu(X(x, t; s), t + s)\end{aligned}$$

where $C := (m-1)(C_0 + \|\nabla \cdot \vec{b}\|_\infty)$. This yields

$$u(X(x, t; s), t + s) \geqq e^{-Cs} u(x, t) > 0, \tag{3.2}$$

which, after taking $s \to s_0 < 1$, contradicts the assumption that $u(X(x, t; s_0), t + s_0) = 0$. $\square$

4. Regularity of the Free Boundary

In this section we study finer properties on expansion of the positive set $\{u > 0\}$ along the streamlines associated with the drift $\vec{b}$. We largely follow the ideas in [9] applied to the zero drift case, and obtain corresponding statements (Lemma 4.1 and 4.2) for our problem.

Lemma 4.1. *Let u be given as in Theorem 3.1, and let $\eta_0 > 0$. For any $t_0 \geqq \eta_0$ there exist τ_0, c_0 depending only on η_0 and universal constants such that the following holds: for any $R > 0$ and $\tau \in (0, \tau_0)$, if*

$$u(\cdot, t_0) = 0 \textit{ in } B(x_0, R) \quad \textit{and} \quad \fint_{B(X(x_0,t_0;\tau),R)} u(x, t_0 + \tau)\mathrm{d}x \leqq \frac{c_0 R^2}{\tau}, \tag{4.1}$$

then

$$u(x, t_0 + \tau) = 0 \quad \textit{for } x \in B(X(x_0, t_0; \tau), R/6). \tag{4.2}$$

Proof. For simplicity, suppose $x_0 = 0$, $t_0 = 0$, and consider the rescaled function

$$\tilde{u}(x, t) := \frac{\tau}{R^2} u(Rx, \tau t) \quad \text{with } \vec{b}'(x, t) := \frac{\tau}{R}\vec{b}(Rx, \tau t), \ \tilde{X}(t) := \frac{1}{R} X(0, 0; \tau t). \tag{4.3}$$

Then $\tilde{u}$ satisfies

$$\tilde{u}_t = (m-1)\tilde{u}\Delta\tilde{u} + |\nabla\tilde{u}|^2 + \nabla\tilde{u} \cdot \vec{b}' + (m-1)\tilde{u}\, \nabla\vec{b}'.$$

Theorem 3.1 yields

$$\Delta u \geqq -C_0 = -C_0(\eta_0) \text{ for } t \geqq \eta_0. \tag{4.4}$$

Set $\varepsilon := C_0\tau_0$ so that $\Delta\tilde{u} = \tau\Delta u \geqq -\varepsilon$. From our assumption, it follows that

$$\fint_{B(\tilde{X}(1),1)} \tilde{u}(x,1)\mathrm{d}x \leqq c_0.$$

Using this and that $\tilde{u} + \varepsilon|x|^2/(2d)$ is subharmonic, we find for $x \in B(\tilde{X}(1), \frac{1}{2})$,

$$\begin{aligned}\tilde{u}(x,1) &\leqq -\frac{\varepsilon|x|^2}{2d} + \fint_{B(\tilde{X}(1),\frac{1}{2})} \tilde{u}(y,1) + \frac{\varepsilon|y|^2}{2d}\mathrm{d}y \\ &\leqq 2^d \fint_{B_1} \tilde{u}(y,1)\mathrm{d}y + \sigma\varepsilon \leqq 2^d c_0 + \sigma\varepsilon.\end{aligned} \tag{4.5}$$

Now consider

$$v(x,t) := \tilde{u}(x + \tilde{X}(t), t).$$

Then $\Delta v \geqq -\varepsilon$. Moreover, observe that v is the weak solution of

$$\begin{aligned}\mathcal{L}_1(v) &:= v_t - (m-1)v\Delta v - |\nabla v|^2 - \nabla v\cdot(\vec{b}'(x+\tilde{X},t) - \vec{b}'(\tilde{X},t)) \\ &\quad -(m-1)v\nabla\cdot\vec{b}'(x+\tilde{X},t) = 0.\end{aligned}$$

We used Definition 2.1 as the notion of weak solutions, where $\vec{b}$ is replaced by $\vec{b}'(x+\tilde{X},t) - \vec{b}'(\tilde{X},t)$. Since the operator $\mathcal{L}_1$ is locally uniformly parabolic in its positive set, v is smooth in the set due to the standard parabolic theory. From the above equation, v satisfies the following in the classical sense in its positive set:

$$\begin{aligned}v_t(x,t) &\geqq -\varepsilon(m-1)v + |\nabla v|^2 - \sigma\tau|\nabla v||x| - \sigma\tau v \\ &\geqq -\varepsilon(m-1)v - \sigma\tau v - \sigma\tau^2|x|^2.\end{aligned}$$

Here the first inequality is due to the fact that $|\nabla\vec{b}'| \leqq \tau\sigma$ and the second inequality follows from Young's inequality. Because v is continuous and non-negative, the above estimate also holds weakly in the whole domain.

Since $\varepsilon = C_0\tau \geqq \tau$, we obtain

$$v_t(x,t) \geqq -\sigma\varepsilon\, v(x,t) - \sigma\varepsilon^2|x|^2, \tag{4.6}$$

and thus by Gronwall

$$\begin{aligned}v(x,1) &\geqq e^{\sigma\varepsilon(t-1)}v(x,t) - \sigma(1 - e^{\sigma\varepsilon(t-1)})\varepsilon\,|x|^2 \\ &\geqq e^{-\sigma\varepsilon}v(x,t) - \sigma\varepsilon \text{ in } B_{\frac{1}{2}} \times (0,1).\end{aligned}$$

Using (4.5), we conclude that for all $(x,t) \in B_{\frac{1}{2}} \times (0,1)$ and some $\sigma \geqq 1$,

$$\begin{aligned}v(x,t) &\leqq e^{\sigma\varepsilon}v(x,1) + e^{\sigma\varepsilon}\sigma\varepsilon = e^{\sigma\varepsilon}\tilde{u}(x+\tilde{X}(t),1) + e^{\sigma\varepsilon}\sigma\varepsilon \\ &\leqq e^{\sigma\varepsilon}(2^d c_0 + 2\sigma\varepsilon) \leqq \sigma(c_0 + \varepsilon).\end{aligned} \tag{4.7}$$

if ε is sufficiently small.

To conclude we proceed with a barrier argument applied to the operator $\mathcal{L}_1$. Define

$$\varphi(x,t) := \lambda\left(\frac{t}{36} + \frac{(|x|-1/3)}{6}\right)_+$$

and we aim at showing $\mathcal{L}_1(\varphi) \geqq 0$ weakly. Using Lemma 2.6 to $(\frac{m-1}{m}\varphi)^{\frac{1}{m-1}}$, the corresponding density variable of φ, and the Lipschitz continuity of φ, we find that to show φ is a supersolution of $\mathcal{L}_1$, it suffices to prove $\mathcal{L}_1(\varphi) \geqq 0$ in the positive set of φ.

Notice

$$\nabla\varphi\cdot(\vec{b}'(x+\tilde{X},t)+\vec{b}'(\tilde{X},t)) - (m-1)\varphi\nabla\cdot\vec{b}'(x+\tilde{X},t) \leqq \sigma\varepsilon|\nabla\varphi|\,|x| + \sigma\varepsilon\varphi,$$

so direct computations yield that if

$$\frac{1}{\lambda} \geqq \left(\frac{t}{6}+|x|-\frac{1}{3}\right)\left((m-1)(d-1)|x|^{-1}+\frac{\sigma\varepsilon}{\lambda}\right)+1+\frac{\sigma\varepsilon}{\lambda^2}, \tag{4.8}$$

then $\mathcal{L}_1(\varphi) \geqq 0$ for $\frac{1}{3}-\frac{t}{6} < |x| < \frac{1}{2}$ in the classical sense. The inequality (4.8) is valid for $t \in (0,1)$ provided that we take $0 < \varepsilon \ll \lambda \ll 1$. With this choice of ε, λ, we get $\mathcal{L}_1(\varphi) \geqq 0$ in $|x| < \frac{1}{2}$ weakly. By the assumption $v(x,0) = 0$ in $B_{\frac{1}{2}}$ and thus $v \leqq \varphi$ on $|x| \leqq \frac{1}{2}$, $t = 0$. On the lateral boundary $|x| = \frac{1}{2}$, $t \in (0,1)$, by (4.7) if c_0, ε are small enough depending on universal constants we have

$$v \leqq \sigma(c_0+\varepsilon) \leqq \frac{\lambda}{36} \leqq \varphi.$$

Hence by comparison principle for the operator $\mathcal{L}_1$ (see Remark 2.4) in $B_{\frac{1}{2}} \times (0,1)$ we have $v \leqq \varphi$. In particular,

$$\tilde{u}(x+\tilde{X}(1),1) = v(x,1) \leqq \varphi(x,1) = 0$$

for $|x| < \frac{1}{6}$, and we have proved the lemma. □

Remark 4.2. One can check that the conclusion of the lemma also holds in a local setting: If u solves (1.5) with $\Delta u \geqq -C_0$ in Q_1 for some C_0, then there exist τ_0, c_0, σ such that (4.1) implies (4.2) for any $R \in (0,\sigma)$ and $\tau \in (0,\tau_0)$. Here τ_0, c_0 depend only on C_0 and universal constants, and σ is universal. This local version of the lemma will be used in Lemma 6.2.

Lemma 4.3. *Let u be as in Theorem 3.1, and let $\eta_0 > 0$. For any $t_0 \geqq \eta_0$ and any $c_1 > 0$, there exist $\lambda, c_2, \tau_0 > 0$ depending on c_1, η_0 and universal constants such that the following holds: for any $R > 0$ and $0 < \tau \leqq \tau_0$, if*

$$\fint_{B(x_0,R)} u(x,t_0)\mathrm{d}x \geqq c_1\frac{R^2}{\tau}, \tag{4.9}$$

then

$$u(X(x_0,t_0;\lambda\tau), t_0+\lambda\tau) \geqq c_2\frac{R^2}{\tau}. \tag{4.10}$$

Proof. Let C_0 be as in (4.4), and set $(x_0, t_0) = (0, 0)$ by shifting coordinates. We consider the corresponding density variable $\varrho(x,t) := (\frac{m-1}{m}u(x,t))^{\frac{1}{m-1}}$ and its rescaled version

$$\tilde{\varrho}(x,t) := \left(\frac{\tau}{R^2}\right)^{\frac{1}{m-1}} \varrho(Rx, \tau t).$$

Let $\vec{b}'$, $\tilde{X}$ be as in (4.3) and let $\varepsilon = C_0\tau$ as in the proof of Lemma 4.1. Then $\tilde{\varrho}$ solves the re-scaled density equation

$$\tilde{\varrho}_t = \Delta\tilde{\varrho}^m + \nabla\cdot(\tilde{\varrho}\,\vec{b}').$$

The fundamental estimate on u implies that $\Delta\tilde{\varrho}^m \geqq -\varepsilon\tilde{\varrho}$ in the sense of distribution.

Let us define $\xi(x,t) := \tilde{\varrho}(x + \tilde{X}, t)$ and $Y(t) := \int_{B_1} \xi^m(x,t)\mathrm{d}x$. Below we study properties on the growth rate of Y using properties of $\tilde{\varrho}$, namely we derive (4.12) and (4.13). We then use these estimates to argue by a contradiction to prove our main statement.

First let us show that $Y(\lambda)$ stays sufficiently positive if $\varepsilon\lambda$ is small. Since $\tilde{X}(0) = 0$, our assumption yields that

$$\begin{aligned} Y(0) &= \fint_{B_1} \xi^m(x,0)\mathrm{d}x = \sigma\left(\frac{\tau}{R^2}\right)^{\frac{m}{m-1}} \fint_{B(0,R)} \varrho^m(x + \tilde{X}(0), 0)\mathrm{d}x \\ &= \sigma \fint_{B(0,R)} \left(\frac{\tau}{R^2}u\right)^{\frac{m}{m-1}}(x,0)\mathrm{d}x \\ &\geqq \sigma\left(\frac{\tau}{R^2}\fint_{B(0,R)} u(x,0)\mathrm{d}x\right)^{\frac{m}{m-1}} \geqq \sigma c_1^{\frac{m}{m-1}} =: c_1'. \end{aligned}$$

Due to (4.6) and $v(x,t) = \frac{m}{m-1}\xi^{m-1}(x,t)$, for ε small enough

$$(\xi^m)_t \geqq -\sigma\varepsilon\xi^m - \sigma\varepsilon^2|x|^2\xi \geqq -\sigma\varepsilon\xi^m - \sigma\varepsilon \text{ for } x \in B_1 \cap \{\xi > 0\}. \tag{4.11}$$

Consequently,

$$Y(t) \geqq e^{-\sigma\varepsilon t}Y(0) - \sigma\varepsilon t \geqq e^{-\sigma\varepsilon\lambda}c_1' - \sigma\varepsilon\lambda > \frac{c_1'}{2} \sim c_1^{\frac{m}{m-1}} \tag{4.12}$$

for $t \in (0, \lambda]$ if $\varepsilon\lambda \ll_\sigma 1$.

Next we obtain an upper bound for the growth of Y over time. □

Claim. For some universal constants σ_1, σ_2 and γ,

$$e^{-\sigma_1\varepsilon t}\int_0^t Y(s)\mathrm{d}s \leqq \sigma_2\left(\int_0^t \xi^m(0,s)\mathrm{d}s + \varepsilon^\gamma + Y^{\frac{1}{m}}\right). \tag{4.13}$$

Proof of the Claim. As in [9], we introduce the Green's function in a unit ball so that G solves

$$\Delta G = -\sigma_d\delta(x) + \sigma_d I_{B_1} \quad \text{and} \quad G = |\nabla G| = 0 \text{ on } \partial B_1. \tag{4.14}$$

Let us only discuss the dimension $d \geqq 3$, where G is defined as

$$G(x) = |x|^{2-d} - 1 - \frac{d-2}{2}(1 - |x|^2). \tag{4.15}$$

We want to differentiate $\int_{B(\tilde{X},1)} G(x - \tilde{X})\tilde{\varrho}(x,t)\mathrm{d}x$ with respect to t. Since $G(x - \tilde{X}) = 0$ on $\partial B(\tilde{X}, 1)$,

$$\begin{aligned}\left(\int_{B(\tilde{X},1)} G(x - \tilde{X})\tilde{\varrho}(x,t)\mathrm{d}x\right)' &= \int_{B(\tilde{X},1)} \nabla G(x - \tilde{X}) \cdot \vec{b}'(\tilde{X})\tilde{\varrho}\,\mathrm{d}x \\ &\quad + \int_{B(\tilde{X},1)} G(x - \tilde{X})\,\tilde{\varrho}_t\,\mathrm{d}x \\ &= \int_{B(\tilde{X},1)} \nabla G(x - \tilde{X}) \cdot (\vec{b}'(\tilde{X}) - \vec{b}'(x))\tilde{\varrho}\,\mathrm{d}x \\ &\quad + \int_{B(\tilde{X},1)} \Delta G(x - \tilde{X})\,\tilde{\varrho}^m\,\mathrm{d}x =: A_1 + A_2.\end{aligned} \tag{4.16}$$

Since $\nabla\vec{b}' \geqq -\sigma\varepsilon I_d$,

$$\begin{aligned}A_1 &= -\int_{B(\tilde{X},1)} (d-2)(|x - \tilde{X}|^{-d} - 1)(x - \tilde{X}) \cdot (\vec{b}'(\tilde{X}) - \vec{b}'(x))\tilde{\varrho}\,\mathrm{d}x \\ &\geqq -\sigma\varepsilon \int_{B(\tilde{X},1)} (d-2)(|x - \tilde{X}|^{-d} - 1)|x - \tilde{X}|^2\tilde{\varrho}\,\mathrm{d}x \\ &\geqq -\sigma\varepsilon \int_{B(\tilde{X},1)} G(x - \tilde{X})\tilde{\varrho}\,\mathrm{d}x.\end{aligned} \tag{4.17}$$

As for A_2, applying (4.14), we obtain

$$A_2 = -\sigma_d\,\tilde{\varrho}^m(\tilde{X}, t) + \sigma \int_{B(\tilde{X},1)} \tilde{\varrho}^m(x,t)\,\mathrm{d}x. \tag{4.18}$$

Using (4.17), (4.18), we find, for some universal $\sigma > 0$

$$\begin{aligned}\left(\int_{B(\tilde{X},t)} G(x - \tilde{X})\tilde{\varrho}(x,t)\mathrm{d}x\right)' &\geqq -\sigma_d\,\tilde{\varrho}^m(\tilde{X}, t) + \sigma \int_{B(\tilde{X},t)} \tilde{\varrho}^m(x,t)\,\mathrm{d}x \\ &\quad - \sigma\varepsilon \int_{B(\tilde{X},t)} G(x - \tilde{X})\,\tilde{\varrho}(x,t)\mathrm{d}x.\end{aligned}$$

Hence we derive

$$\begin{aligned}e^{\sigma\varepsilon t} \int_{B_1} G(|x|)\xi(x,t)\mathrm{d}x &\geqq -\sigma_d \int_0^t e^{\sigma\varepsilon s}\xi^m(0,s)\mathrm{d}s \\ &\quad + \sigma \int_0^t \int_{B_1} e^{\sigma\varepsilon s}\xi^m(x,s)\,\mathrm{d}x\mathrm{d}s,\end{aligned}$$

which simplifies to

$$\int_0^t e^{-\sigma\varepsilon t} Y(s)\mathrm{d}s \leqq \sigma \int_{B_1} G(|x|)\xi(x,t)\mathrm{d}x + \sigma \int_0^t \xi^m(0,s)\mathrm{d}s. \tag{4.19}$$

Now following the proof of Lemma 2.3 [9], using (4.19) and the integrability property of G, we can obtain the upper bound $\int_{B_1} G\xi \,\mathrm{d}x$ to conclude. We omit the computation since it is parallel to [9]. □

Going back to the proof of Lemma 4.3, let us suppose that our statement is false, which means $u(X(\lambda\tau), \lambda\tau) < c_2 \frac{R^2}{\tau}$ for any choice of λ, c_2, τ_0, where $X(t) := X(0, 0; t)$. Later we will pick the constants satisfying

$$\lambda \gg 1, \quad c_2^{\frac{m}{m-1}} \lambda \ll 1, \quad \varepsilon\lambda \ll 1.$$

In terms of $\xi = \tilde{\varrho}(\cdot + \tilde{X}, \cdot)$, we have

$$\xi^m(0, \lambda) \leqq \sigma(m)\, c_2^{\frac{m}{m-1}}.$$

Since $\varepsilon\lambda \ll 1$, by (4.11) again, we obtain

$$\xi^m(0, t) \leqq \sigma e^{\sigma\varepsilon\lambda} c_2^{\frac{m}{m-1}} + \sigma\varepsilon\lambda \text{ for } t \in (0, \lambda].$$

If follows from (4.13) that for all $t \in (0, \lambda]$ and some $\sigma = \sigma(\sigma_2)$,

$$e^{-\sigma_1 \varepsilon t} \int_0^t Y(s)\mathrm{d}s \leqq \sigma(e^{\sigma\varepsilon\lambda} c_2^{\frac{m}{m-1}} \lambda + \varepsilon\lambda^2 + \varepsilon^\gamma + Y^{\frac{1}{m}}).$$

Recall (4.12), and we have

$$\sigma Y^{\frac{1}{m}} \geqq \sigma c_1^{\frac{1}{m-1}} \geqq \sigma(e^{\sigma\varepsilon\lambda} c_2^{\frac{m}{m-1}} \lambda + \varepsilon\lambda^2 + \varepsilon^\gamma). \tag{4.20}$$

Hence we get for $t \in (0, \lambda]$ and some universal $\sigma > 0$,

$$\sigma Y^{\frac{1}{m}} \geqq e^{-\sigma_1 \varepsilon t} \int_0^t Y \mathrm{d}s.$$

Writing $Z(t) := \int_0^t Y(s)\mathrm{d}s$, in view of (4.12) we obtain $Z(\frac{\lambda}{2}) \geqq c_3 \lambda$ with

$$c_3 := \frac{1}{2}(e^{-\sigma\varepsilon\lambda} c_1' - \varepsilon\lambda) \geqq \sigma c_1^{\frac{m}{m-1}} > 0.$$

Solving the ODE problem

$$\sigma Z' \geqq e^{-\sigma\varepsilon t} Z^m, \quad \text{with } Z\left(\frac{\lambda}{2}\right) \geqq c_3\lambda$$

shows that

$$Z\left(t + \frac{\lambda}{2}\right) \geqq \left((c_3\lambda)^{1-m} - f(t)\right)^{\frac{1}{1-m}}, \quad \text{for } t \in (0, \frac{\lambda}{2}] \tag{4.21}$$

where

$$f(t) := \int_{\lambda/2}^{t+\lambda/2} \sigma e^{-\sigma\varepsilon s} \mathrm{d}s = \sigma e^{-\sigma\lambda\varepsilon/2} \frac{(e^{\sigma\varepsilon t} - 1)}{\sigma\varepsilon}.$$

Since $\sigma\varepsilon \ll 1$,

$$f(t) \geqq \sigma t - \sigma\varepsilon t^2.$$

It is obvious that f is monotone increasing in t. Notice the right-hand side of (4.21) goes to $+\infty$ as

$$t \to f^{-1}((c_3\lambda)^{1-m})$$

which is impossible provided that $f^{-1}((c_3\lambda)^{1-m}) \leqq \frac{\lambda}{2}$. However if $\lambda \geqq C(c_3, \sigma)$ and $\varepsilon\lambda \ll 1$, we indeed have

$$f\left(\frac{\lambda}{2}\right) \geqq \sigma\frac{\lambda}{2} - \sigma\varepsilon\frac{\lambda^2}{4} \geqq (c_3\lambda)^{1-m}, \tag{4.22}$$

which leads to a contradiction.

We proved that $\xi^m(0,\lambda) \leqq \sigma(m)\, c_2^{\frac{m}{m-1}}$. Since c_3 only depends on c_1, σ, the choices of $\lambda, c_2, \varepsilon$ only depend on c_1, σ. We conclude the lemma with $\tau_0 = \varepsilon/C_0$, λ satisfying (4.22), and c_2, ε satisfying (4.20) and $\varepsilon\lambda \ll_C 1$. □

For any $(x_0, t_0) \in \Gamma$, we use the notation

$$\Upsilon(x_0, t_0) := \{(X(x_0, t_0; -s), t_0 - s), \quad s \in (0, t_0)\}.$$

Theorem 4.4. *For a given point $(x_0, t_0) \in \Gamma$ with $t_0 \geqq \eta_0 > 0$, the following is true:*

(1) *Either (a) $\Upsilon(x_0, t_0) \subset \Gamma$ or (b) $\Upsilon(x_0, t_0) \cap \Gamma = \emptyset$.*
(2) *If (b) holds, then there exist positive constants C_*, β, h such that for all $s \in (0, h)$*

$$\varrho(x, t_0 - s) = 0 \quad \textit{if} \quad |x - X(x_0, t_0; -s)| \leqq C_* s^\beta;$$
$$\varrho(x, t_0 + s) > 0 \quad \textit{if} \quad |x - X(x_0, t_0; s)| \leqq C_* s^\beta.$$

Here β only depends on η_0 and universal constants. If (b) holds for $(x_0, t_0) \in \Gamma$, we say (x_0, t_0) is "of the second type" free boundary point.

Sketch of the proof The proof is parallel to those for Theorems 3.1–3.2 [9], based on the Lemmas 4.1 and 4.3. Let us only sketch the proof for part (1) below.

If the assertion of (1) is not true, then we can find $t_0 > t_1 > t_2 > 0$ such that $t_0 - t_1 \gg t_1 - t_2$ and

$$x_0 \in \Gamma_{t_0}, \quad x_1 := X(x_0, t_0; t_1 - t_0) \in \Gamma_{t_1}, \quad x_2 := X(x_0, t_0; t_2 - t_0) \notin \Gamma_{t_2}.$$

Consequently $u(\cdot, t_2) = 0$ in $B(x_2, R)$ for some $R > 0$. Since $x_1 = X(x_2, t_2; t_1 - t_2)$, by Lemma 4.1,

$$\fint_{B(x_1, R)} u(x, t_1)\mathrm{d}x \geqq \frac{c_0 R^2}{t_1 - t_2}.$$

Since $t_0 - t_1 \gg (t_1 - t_2)$, Lemma 4.3 yields $u(x_0, t_0) = u(X(x_1, t_1; t_0 - t_1), t_0) > 0$, which is a contradiction. □

When the initial data grows faster than quadratically near its free boundary and the boundary is Lipschitz, it is possible to characterize the constants C_*, h in above theorem in terms of time variable. Note that using both assumptions in (1.7), Lemma 3.3, and Lemma 4.3 yields that $X(\overline{\Omega_0}, 0; t) \subseteq \Omega_t$ for all $t > 0$. Thus by a compactness argument, iteratively using Theorem 4.4 and arguing as in the remark on Theorem 3.2 in [9], we have the following theorem:

Theorem 4.5. *Suppose* (1.7). *Then any point* $x_0 \in \Gamma_{t_0}$ *with* $t_0 > 0$ *is of the second type and the constants* C_*, h *in Theorem* 4.4 (2) *only depend on* t_0*,* (1.7)*, and universal constants.*

5. Monotonicity Implies Non-degeneracy

In this section we discuss non-degeneracy property of solutions in local settings. We start with the following theorem:

Theorem 5.1. *Let* u *solve* (1.5) *in* Q_2 *with* $\Delta u \geqq -C_0$. *Suppose that* Γ *is of type two in* Q_2*, and that*

$$u \text{ is monotone with respect to } W_{\theta,\mu} \text{ in } Q_2 \text{ for some } \theta \in (0, \pi/2) \text{ and } \mu \in \mathcal{S}^{d-1}. \tag{5.1}$$

Then there exist constants $C, \varepsilon_0 > 0$ *such that we have*

$$u(X(x, t; C\varepsilon) - \varepsilon\mu, t + C\varepsilon) > 0 \text{ for } (x, t) \in \Gamma \cap Q_1 \text{ and for } \varepsilon < \varepsilon_0.$$

Remark 5.2. The constants C, ε_0 in Theorem 5.1 only depend on

$$C_0, \theta, C_*, h, \beta, \text{ and universal constants,} \tag{5.2}$$

where C_*, h, β are constants given in Theorem 4.4. In the global setting, an estimate of C_0 can be found in Theorem 3.1.

Let us also mention that Theorem 2.2 allows us to consider continuous local solutions.

The central ingredient of the proof is a barrier argument motivated from [13] in the context of Hele–Shaw flow. The barrier argument illustrates the fact that in diffusive free boundary problems the nice regularity properties of u propagate from positive level sets to the free boundary as the positive set expands out. This argument in our setting corresponds to the proof of (5.35). Compared to the Hele–Shaw flow which is driven by a harmonic function, our solutions features a nonlinear diffusion that degenerates near the free boundary and thus it requires more careful arguments. On the other hand, we will benefit from the weak formulation of the problem using the density formula (see $\mathcal{G}$ below.)

For u as given above we consider

$$v(x, t) := u(x + X(t), t), \quad \text{where } X(t) := X(0, 0; t) \text{ is given in (1.6).} \tag{5.3}$$

Then v is a weak solution of $\mathcal{L}_2(\cdot) = 0$, where the operator $\mathcal{L}_2$ is given by

$$\begin{aligned}\mathcal{L}_2(f) := \partial_t f - (m-1) f \Delta f - |\nabla f|^2 - \nabla f \cdot (\vec{b}(x + X(t), t) \\ - \vec{b}(X(t), t)) - (m-1) f \nabla \cdot \vec{b}(x + X(t), t).\end{aligned} \tag{5.4}$$

Since the operator $\mathcal{L}_2$ is the same as in (1.5) with $\vec{b}$ replaced by $\vec{b}(x + X(t), t) - \vec{b}(X(t), t))$, the notion of sub- and supersolution is given in Definition 2.1.

Below we construct a supersolution for the operator $\mathcal{L}_2$ for the aforementioned barrier argument, using a inf-convolution construction introduced first by [8]. Since the supersolution to be constructed is a rescaled inf-convolution of v [see (5.8)], comparison of the two functions gives a space-time monotonicity of v, yielding the theorem. To this end, we will use both smooth approximations of u and the density version of the equation $\mathcal{L}_2$.

We begin with some basic properties of the inf-convolution of smooth functions. Let $\psi, h \in C^\infty(B_2)$ with $0 < \psi < \frac{1}{2}$ and $h \geqq 0$. Define

$$f(x) := \inf_{B(x,\psi(x))} h(y), \tag{5.5}$$

which is Lipschitz continuous. The proofs of the next two lemmas are in the appendix.

Lemma 5.3. *Let h and f be as given in (5.5). Furthermore, suppose $\Delta h \geqq -C$ for some $C \in \mathbb{R}$ and $\|\nabla \psi\|_\infty \leqq 1$. Then there are dimensional constants $\sigma_1 > 0$ and $\sigma_2 \geqq 3$ such that if ψ satisfies*

$$\Delta \psi \geqq \frac{\sigma_1 |\nabla \psi|^2}{|\psi|} \quad \textit{in } B_2,$$

we have

$$\begin{aligned}&\Delta f(\cdot) - (1 + \sigma_2 \|\nabla \psi\|_\infty) \Delta h(y(\cdot)) \\ &\leqq \sigma_2 \|\nabla \psi\|_\infty C \quad \textit{in } B_1 \textit{ in the sense of distribution},\end{aligned}$$

where $y(\cdot)$ satisfies that $f(\cdot) = h(y(\cdot))$ a.e. in B_1.

Lemma 5.4. *Let h, f be as given in (5.5). Then for a.e. $x \in B_1$ we have*

$$|\nabla f(x) - \nabla h(y)| = |\nabla h(y)||\nabla \psi(x)| \quad \textit{if } f(x) = h(y) \textit{ and } y \in B(x, \psi(x)).$$

Now for a weak solution u to (1.5) in Q_2, let $\{u_k\}_k$ be its smooth approximations as given in Lemma 2.5. In particular u_k is positive in Q_2 for each k. Set $v_k(x,t) := u_k(x + X(t), t)$ and introduce the corresponding density variable of v_k as

$$\xi_k(x,t) := \left(\frac{m-1}{m} v_k(x,t)\right)^{\frac{1}{m-1}} = \left(\frac{m-1}{m} u_k(x + X(t), t)\right)^{\frac{1}{m-1}}. \tag{5.6}$$

We define the density version of the operator $\mathcal{L}_2$ as $\mathcal{G}(\xi) := \mathcal{L}_2(v)$ where $\xi = (\frac{m-1}{m} v)^{\frac{1}{m-1}}$ i.e.

$$\mathcal{G}(f) := \partial_t f - \Delta f - \nabla \cdot (f(\vec{b}(x,t) - f(x + X(t), t))),$$

and thus $\mathcal{G}(\xi_k) = 0$.

Let $\varphi\colon \mathbb{R}^d \to (0, \infty)$ be a smooth function and σ_1, σ_2 be from Lemma 5.3. For some constants α, A_0, $M_0 \geqq 1$ to be determined, we define

$$w_k(x,t) := e^{A_0\varepsilon t} \inf_{y\in B(x,R_\varepsilon(x,t))} v_k(y + r\varepsilon\mu, p_\varepsilon(t)), \tag{5.7}$$

$$w(x,t) := e^{A_0\varepsilon t} \inf_{y\in B(x,R_\varepsilon(x,t))} v(y + r\varepsilon\mu, p_\varepsilon(t)), \tag{5.8}$$

and

$$\eta_k(x,t) := e^{A_1\varepsilon t} \inf_{y\in B(x,R_\varepsilon(x,t))} \xi_k(y + r\varepsilon\mu, p_\varepsilon(t)) \quad \text{with } A_1 := \frac{A_0}{m-1}, \tag{5.9}$$

where

$$R_\varepsilon(x,t) := \varepsilon\varphi(x)(1-\alpha t) \tag{5.10}$$

$$p_\varepsilon(t) := (1 + \sigma_2 M_0\varepsilon)\left(\frac{e^{A_0\varepsilon t}-1}{A_0\varepsilon}\right). \tag{5.11}$$

Then w_k is Lipschitz continuous, and

$$\eta_k(x,t) = \left(\frac{m-1}{m} w_k(x,t)\right)^{\frac{1}{m-1}}.$$

Thus to show that w_k is a supersolution for $\mathcal{L}_2$, it suffices to show that η_k is a supersolution for $\mathcal{G}$.

We will apply Lemmas 5.3, 5.4 with

$$h = \xi_k^m(\cdot + r\varepsilon\mu, p_\varepsilon) \quad \text{and} \quad \psi = R_\varepsilon(\cdot, t).$$

Based on these lemmas we estimate the density equation $\mathcal{G}(\eta_k)$ in the weak sense, to go around the potential lack of smoothness for inf-convolutions, to conclude.

We will choose the constants $A_0 = A_0(M_0)$ and $\alpha = \alpha(M_0)$ in Proposition 5.5, the constants M_0, r and the function φ in the proof of Theorem 5.1.

Proposition 5.5. *Let u_k, w_k be defined from above, and suppose that u_k satisfies $\Delta u_k \geqq -C_0$ in Q_2. Fix any $M_0 \geqq 1$ and consider $\varphi\colon B_2 \to \mathbb{R}$ such that*

$$\begin{cases} \Delta\varphi = \frac{\sigma_1|\nabla\varphi|^2}{|\varphi|}, \\ \frac{r}{M_0} \leqq \varphi(\cdot) \leqq rM_0, \quad \|\nabla\varphi\|_\infty \leqq M_0 \quad \textit{for some } r \in (0,1). \end{cases} \tag{5.12}$$

Then there exist positive constants A_0, α, τ depending only on M_0 and universal constants such that for all $\varepsilon < \frac{1}{M_0}$ the function w_k given in (5.7) is a weak supersolution of

$$\mathcal{L}_2(w_k) \geqq 0 \quad \textit{in } B_r \times (0,\tau).$$

Proof. Let ξ_k, η_k be from (5.6), (5.9) respectively. As discussed before to prove the statement, it suffices to show that $\mathcal{G}(\eta_k) \geqq 0$ weakly in $B_r \times (0, \tau)$.

Below we estimate each term in $\mathcal{G}(\eta_k)$ in $B_r \times (0, \tau)$ using ξ_k. We begin with some preliminary estimates on η_k.

Since u_k is smooth and positive, ξ_k is also smooth and positive. From the definition of the inf-convolution, it follows that η_k is Lipschitz continuous. Since $\Delta u_k \geqq -C_0$, direct computation yields that

$$\Delta(\xi_k^m) \geqq -\sigma C_0 \xi_k \text{ for some } \sigma = \sigma(m) > 0. \tag{5.13}$$

Let us set the constants

$$A_0 := \sigma_3 M_0 (1 + C_0), \quad \alpha := \sigma_3 M_0^2 \tag{5.14}$$

for some $\sigma_3 \geqq \sigma_2$ to be determined, and

$$\tau := \min\left\{\frac{1}{2A_0}, \frac{1}{2A_1}, \frac{1}{\sigma_2 M_0}, \frac{1}{5\alpha}\right\}. \tag{5.15}$$

By definition of η_k, there is $z(x)$ satisfying

$$|z(x) - x| \leqq |R_\varepsilon| + r\varepsilon \leqq 2M_0 r\varepsilon, \tag{5.16}$$

such that

$$\eta_k(x, t) = g(t)\, \xi_k(z(x), p_\varepsilon(t)),$$

where we use the notation $g(t) := e^{A_1 \varepsilon t}$.

It follows from the definition of $p_\varepsilon(t)$ in (5.11) that

$$p'_\varepsilon(t) = (1 + \sigma_2 M_0 \varepsilon) g(t)^{m-1} \tag{5.17}$$

and

$$0 \leqq p_\varepsilon(t) - t \leqq \sigma M_0 t\varepsilon \leqq \sigma\varepsilon \quad \text{for } 0 < t < \tau. \tag{5.18}$$

We now proceed to estimating each terms in $\mathcal{G}(\eta_k)$, starting with $\partial_t \eta_k$. All estimates in the domain $B_r(0) \times (0, \tau)$. In the rest of the proof, for simplicity, $X(t) := X(0, 0, ; t)$, p_ε, η_k denotes the values of them at (x, t), and $\xi_k, \partial_t \xi_k, \nabla \xi_k, \Delta \xi_k$ denotes the values of them evaluated at point $(z(x), p_\varepsilon(t))$.

In [20], $\partial_t \eta_k$ is computed in the viscosity sense. Since our η_k is Lipschitz continuous, the same computation carries out almost everywhere in $B_r \times (0, \tau)$. We have

$$\partial_t \eta_k \geqq A_1 \varepsilon\, \eta_k - \partial_t R_\varepsilon\, |\nabla \eta_k| + (p'_\varepsilon) g\, \partial_t \xi_k. \tag{5.19}$$

Applying (5.10), (5.17) and the assumption that $\varphi \geqq \frac{r}{M_0}$, (5.19) implies

$$\partial_t \eta_k \geqq A_1 \varepsilon\, \eta_k + \frac{\alpha r \varepsilon}{M_0} |\nabla \eta_k| + (1 + \sigma_2 M_0 \varepsilon) g^m \partial_t \xi_k. \tag{5.20}$$

From the assumptions on φ, $\|R_\varepsilon\|_\infty \leqq rM_0\varepsilon$, $\|\nabla R_\varepsilon\|_\infty \leqq M_0\varepsilon$. We now apply Lemma 5.3 with $h = \xi_k^m(\cdot + r\varepsilon\mu, p_\varepsilon)$ and $\psi = R_\varepsilon(\cdot, t)$. From (5.12) and (5.13), the following holds in the sense of distribution:

$$\begin{aligned} -\Delta\eta_k^m &\geqq -(1+\sigma_2\|R_\varepsilon\|_\infty)g^m\Delta\xi_k^m - \sigma_2\|\nabla R_\varepsilon\|_\infty C_0\xi_k \\ &\geqq -(1+\sigma_2 M_0\varepsilon)g^m\Delta\xi_k^m - \sigma M_0 C_0\varepsilon\,\eta_k. \end{aligned} \tag{5.21}$$

Next we consider the terms coming from the drift. Due to Lemma 5.4,

$$|\nabla\eta_k - g\nabla\xi_k| = |\nabla R_\varepsilon||g\nabla\xi_k| \leqq M_0\varepsilon g|\nabla\xi_k|,$$

since $\varepsilon < \frac{1}{M_0}$, we have $|\nabla\eta_k - g\nabla\xi_k| \leqq \sigma M_0\varepsilon|\nabla\eta_k|$. This implies that for $t \leqq \tau$,

$$\left|\nabla\eta_k - (1+\sigma_2 M_0\varepsilon)g^m\nabla\xi_k\right| \leqq \sigma M_0\varepsilon|\nabla\eta_k|. \tag{5.22}$$

Next, using the regularity of $\vec{b}$ and $|x| \leqq r$, we have

$$\left|\vec{b}(x+X(p_\varepsilon), p_\varepsilon) - \vec{b}(X(p_\varepsilon), p_\varepsilon)\right| \leqq \|D\vec{b}\|_\infty r \leqq \sigma r, \tag{5.23}$$

and, by (5.16),

$$\left|\vec{b}(x+X(p_\varepsilon), p_\varepsilon) - \vec{b}(z+X(p_\varepsilon), p_\varepsilon)\right| \leqq \sigma M_0 r\varepsilon. \tag{5.24}$$

Then (5.22)–(5.24) imply

$$\begin{aligned} &-\nabla\eta_k\cdot\left(\vec{b}(x+X(p_\varepsilon), p_\varepsilon) - \vec{b}(X(p_\varepsilon), p_\varepsilon)\right) \\ &\quad\geqq -(1+\sigma_2 M_0\varepsilon)g^m\nabla\xi_k\cdot\left(\vec{b}(x+X(p_\varepsilon), p_\varepsilon) - \vec{b}(X(p_\varepsilon), p_\varepsilon)\right) - \sigma M_0 r\varepsilon|\nabla\eta_k| \\ &\quad\geqq -(1+\sigma_2 M_0\varepsilon)g^m\nabla\xi_k\cdot\left(\vec{b}(z+X(p_\varepsilon), p_\varepsilon) - \vec{b}(X(p_\varepsilon), p_\varepsilon)\right) - \sigma M_0 r\varepsilon|\nabla\eta_k|. \end{aligned} \tag{5.25}$$

Parallel computations yield

$$\begin{aligned} &-\eta_k\nabla\cdot\vec{b}(x+X(p_\varepsilon)) \\ &\quad\geqq -(1+\sigma_2 M_0\varepsilon)g^m\xi_k\nabla\cdot\vec{b}(x+X(p_\varepsilon)) - \sigma\,\eta_k\left|g-(1+\sigma_2 M_0\varepsilon)g^m\right|\|D\vec{b}\|_\infty \\ &\quad\geqq -(1+\sigma_2 M_0\varepsilon)g^m\xi_k\nabla\cdot\vec{b}(z+X(p_\varepsilon)) - \sigma M_0\varepsilon\,\eta_k - \sigma\,\eta_k\|D^2\vec{b}\|_\infty M_0 r\varepsilon \\ &\quad\geqq -(1+\sigma_2 M_0\varepsilon)g^m\xi_k\nabla\cdot\vec{b}(z+X(p_\varepsilon)) - \sigma M_0\varepsilon\,\eta_k. \end{aligned} \tag{5.26}$$

Combining the estimates (5.20), (5.21), (5.25) and (5.26), we have

$$\begin{aligned} \tilde{\mathcal{G}}(\eta_k) &:= \partial_t\eta_k - \Delta\eta_k^m - \nabla\left(\eta_k\cdot\left(\vec{b}(x+X(p_\varepsilon), p_\varepsilon) - \vec{b}(X(p_\varepsilon), p_\varepsilon)\right)\right) \\ &\geqq A_1\varepsilon\,\eta_k + \frac{\alpha r\varepsilon}{M_0}|\nabla\eta_k| + (1+\sigma_2 M_0\varepsilon)g^m(\partial_t\xi_k - \Delta\xi_k^m) \\ &\quad - (1+\sigma_2 M_0\varepsilon)g^m\,\nabla\left(\xi_k\cdot\left(\vec{b}(z+X(p_\varepsilon), p_\varepsilon) - \vec{b}(X(p_\varepsilon), p_\varepsilon)\right)\right) \\ &\quad - \sigma M_0(1+C_0)\varepsilon\,\eta_k - \sigma M_0 r\varepsilon|\nabla\eta_k|. \end{aligned}$$

Since $\mathcal{G}(\xi_k) = 0$ we obtain

$$\begin{aligned}\widetilde{\mathcal{G}}(\eta_k) &\geqq A_1\varepsilon\,\eta_k + \frac{\alpha r\varepsilon}{M_0}|\nabla\xi_k| + (1+\sigma_2 M_0\varepsilon)g^m\mathcal{G}(\xi_k(\cdot,\cdot))(z,p_\varepsilon)\\ &\quad - \sigma M_0(1+C_0)\varepsilon\,\eta_k - \sigma M_0 r\varepsilon|\nabla\eta_k|\\ &= C_1\varepsilon\,\eta_k + C_2 r\varepsilon|\nabla\eta_k|,\end{aligned}$$

where

$$C_1 := A_1 - \sigma M_0(1+C_0), \quad C_2 := \frac{\alpha}{M_0} - \sigma M_0. \tag{5.27}$$

Finally we proceed from $\widetilde{\mathcal{G}}(\eta_k)$ to $\mathcal{G}(\eta_k)$ to get

$$\begin{aligned}\mathcal{G}(\eta_k) &\geqq \mathcal{G}(\eta_k) - \widetilde{\mathcal{G}}(\eta_k) + C_1\varepsilon\,\eta_k + C_2 r|\nabla\eta_k|\\ &\geqq C_1\varepsilon\,\eta_k + C_2 r\varepsilon|\nabla\eta_k| - \eta_k\left|\nabla\cdot\vec{b}(x+X(p_\varepsilon),p_\varepsilon) - \nabla\cdot\vec{b}(x+X(t),t)\right|\\ &\quad - |\nabla\eta_k|\underbrace{\left|\vec{b}(x+X(p_\varepsilon),p_\varepsilon) - \vec{b}(X(p_\varepsilon),p_\varepsilon) - (\vec{b}(x+X(t),t) - \vec{b}(X(t),t))\right|}_{V_0:=}.\end{aligned} \tag{5.28}$$

Let us estimate V_0 as follows:

$$\begin{aligned}V_0 &= \left|\int_t^{p_\varepsilon} \partial_s\vec{b}(x+X(s),s) - \partial_s\vec{b}(X(s),s)\mathrm{d}s\right|\\ &\leqq \int_t^{p_\varepsilon} \left|\left((D\vec{b})(x+X(s),s) - (D\vec{b})(X(s),s)\right)\vec{b}(X(s))\right|\\ &\quad + \left|(\partial_t\vec{b})(x+X(s),s) - (\partial_t\vec{b})(X(s),s)\right|\mathrm{d}s\\ &\leqq \sigma|x|\int_t^{p_\varepsilon} \|D^2\vec{b}\|_\infty\|\vec{b}\|_\infty + \|D\partial_t\vec{b}\|_\infty\mathrm{d}s \leqq \sigma r\varepsilon.\end{aligned}$$

Similarly,

$$\begin{aligned}&\left|\nabla\cdot\vec{b}(x+X(p_\varepsilon),p_\varepsilon) - \nabla\cdot\vec{b}(x+X(t),t)\right|\\ &\leqq \int_t^{p_\varepsilon} \left|(D\nabla\cdot\vec{b})(x+X(s),s)\vec{b}(X(s))\right| + \left|(\partial_t\nabla\cdot\vec{b})(x+X(s),s)\right|\mathrm{d}s\\ &\leqq \sigma\left(\|D^2\vec{b}\|_\infty\|\vec{b}\|_\infty + \|D\partial_t\vec{b}\|_\infty\right)\varepsilon.\end{aligned}$$

Thus it follows from (5.28) that, if $C_1 \geqq \sigma$, $C_2 \geqq \sigma$,

$$\mathcal{G}(\eta_k) \geqq (C_1 - \sigma r)\varepsilon\,\eta_k + (C_2 - \sigma)r\varepsilon|\nabla\eta_k| \geqq 0 \text{ in } B_R\times(0,\tau). \tag{5.29}$$

In view of (5.27), $C_1, C_2 \geqq \sigma$ if σ_3 in (5.14) is chosen to be large enough depending only on universal constants. Hence with this choice of σ_3 we have proved that $\mathcal{G}(\eta_k) \geqq 0$ in the sense of distribution in $B_r\times(0,\tau)$. From the Lipschitz continuity of η_k we conclude that $\mathcal{G}(\eta_k) \geqq 0$ weakly in $B_r\times(0,\tau)$.

Lastly, it is not hard to see that the choices of A_0, α, τ are independent of r and k. □

Corollary 5.6. *Let u be from Theorem 5.1, and let v and w be given by (5.3) and (5.8) respectively. Suppose that the assumptions in Proposition 5.5 are satisfied. Then for any open set $U \subseteq B_r$, if $w \geqq v$ on the parabolic boundary of $U \times (0, \tau)$, then*

$$w \geqq v \quad \text{in } U \times (0, \tau).$$

Proof. Let $\{u_k\}_k$ be the smooth approximations of u and $u_k \geqq u$. Let $v_k(x,t) = u_k(x+X(t),t)$ and w_k be from (5.7). It follows from the proposition that $\mathcal{L}_2(w_k) \geqq 0$ weakly in $B_r \times (0,\tau)$. We have $w_k \geqq w$ due to the fact that $u_k \geqq u$. Then by the assumption, $w_k \geqq v$ on the parabolic boundary of $U \times (0,\tau)$. By comparison principle for $\mathcal{L}_2$, we get $w_k \geqq v$ in $U \times (0,\tau)$. Due to Lemma 2.5, u_k converges locally uniformly to u, and so w_k converges locally uniformly to w. We conclude by sending $k \to \infty$. □

Now we are able to prove Theorem 5.1.

Proof of Theorem 5.1. Let σ_1 be given in Lemma 5.3, and let Φ be the unique solution of

$$\begin{cases} \Delta(\Phi^{-\sigma_1+1}) = 0 & \text{in } B_{\frac{1}{2}} \backslash B_{\sin\theta/10} \\ \Phi = A_{d,\theta} & \text{on } \partial B_{\sin\theta/10} \\ \Phi = \dfrac{1}{2}\sin\theta & \text{on } \partial B_{\frac{1}{2}}. \end{cases}$$

Here $A_{d,\theta}$ is chosen sufficiently large so that

$$\Phi\left(y + \frac{\mu}{5}\right) \geqq 3 \quad \text{for all } y \in B_{\sin\theta/10}. \tag{5.30}$$

Then Φ satisfies $\Delta\Phi \leq \frac{\sigma_1|\nabla\Phi|^2}{\Phi}$ and for some $M_0(\theta, d) \geqq 1$

$$\frac{1}{M_0} \leqq \Phi \leqq M_0, \quad \|\nabla\Phi\|_\infty \leqq M_0 \quad \text{in } B_{\frac{1}{2}}.$$

With this M_0, let A_0, α, τ be as given in Proposition 5.5.

Fix any $(\hat{x}, \hat{t}) \in Q_1 \cap \Gamma$ and let C^*, h, β be from Theorem 4.4 and τ be from (5.15). We will show that the support of the solution strictly expands relatively to the streamlines at $(\hat{x}, \hat{t})$.

Let $\delta = \delta(\theta, C_0) > 0$, which will be chosen as a constant satisfying (5.42) and (5.44). Define

$$t_\delta := \min\{\tau, h, \delta\}, \tag{5.31}$$

and

$$r_\delta := \min\left\{C_* t_\delta^\beta, \frac{1}{4}\right\} > 0. \tag{5.32}$$

Due to Theorem 4.4,

$$u(x, \hat{t} - t_\delta) = 0 \quad \text{for } x \in B(X(\hat{x}, \hat{t}; -t_\delta), r_\delta). \tag{5.33}$$

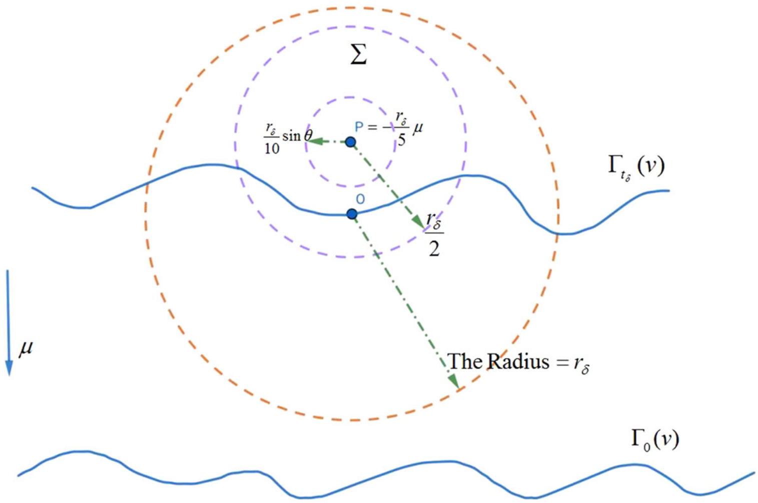

Fig. 1. The local domain

After translation, we assume $(X(\hat{x}, \hat{t}; -t_\delta), \hat{t} - t_\delta)$ to be the origin. Using the notation $X(t) = X(0, 0; t)$, we have

$$(X(t_\delta), t_\delta) = (X(X(\hat{x}, \hat{t}; -t_\delta)), \hat{t} - t_\delta; t_\delta), t_\delta) = (\hat{x}, \hat{t}) \in \Gamma(u).$$

Let v be as given in (5.3), and then $\mathcal{L}_2(v) = 0$ weakly in $Q_{\frac{1}{2}}$, where $\mathcal{L}_2$ is given in (5.4). It follows from (5.33) that

$$v(x, 0) = 0 \quad \text{in } B_{r_\delta}. \tag{5.34}$$

For $P := -\frac{r_\delta}{5}\mu$, set $\varphi(x) := r_\delta \Phi(\frac{x-P}{r_\delta})$.

Let w be defined as in (5.8) with the above φ and $r = r_\delta$:

$$\begin{aligned} w(x, t) &:= e^{A_0 \varepsilon t} \inf_{B(x, \varepsilon\varphi(x)(1-\alpha t))} u(y + r_\delta \varepsilon \mu + X(p_\varepsilon(t)), p_\varepsilon(t)) \\ &= e^{A_0 \varepsilon t} \inf_{B(x, \varepsilon\varphi(x)(1-\alpha t))} v(y + r_\delta \varepsilon \mu, p_\varepsilon(t)). \end{aligned}$$

Next denote the cylindrical domain (see Fig. 1)

$$\Sigma := (B(P, \frac{r_\delta}{2}) \backslash B(P, r_\theta)) \times [0, t_\delta],$$

where $r_\theta := \frac{r_\delta}{10} \sin\theta$. We claim that

$$w \geqq v \text{ in } \Sigma. \tag{5.35}$$

Roughly speaking, (5.35) states that the nondegeneracy property of u propagates from the positive set to the free boundary, as the positive set expands out relative to the streamlines.

The proof of (5.35) will be given below. We first discuss its consequences.

Using (5.15) and (5.30),

$$\varphi(x) = r_\delta \Phi\left(\frac{x}{r_\delta} + \frac{\mu}{5}\right) \geqq 3r_\delta \quad \text{for } x \in B_{\frac{r_\delta}{10}}(0).$$

From this, it follows that for all $|x| \leqq \frac{r_\delta \varepsilon}{5} \leqq \frac{r_\delta}{10}$,

$$-r_\delta \varepsilon \mu \in B\left(x, \frac{12}{5} r_\delta \varepsilon\right) + r_\delta \varepsilon \mu \subseteq B(x, \varepsilon\varphi(x)(1 - \alpha t_\delta)) + r_\delta \varepsilon \mu.$$

In the inclusion, we used that $\alpha t_\delta \leqq \frac{1}{5}$. Then using (5.35) and the definition of w, we get for $|x| \leqq \frac{r_\delta \varepsilon}{5}$,

$$\begin{aligned} e^{A_0 \varepsilon t_\delta} v(-r_\delta \varepsilon \mu, p_\varepsilon(t_\delta)) &\geqq e^{A_0 \varepsilon t_\delta} \inf_{B(x, \varepsilon\varphi(x)(1-\alpha t_\delta))} v(y + r_\delta \varepsilon \mu, p_\varepsilon(t_\delta)) \\ &\geqq w(x, t_\delta) \geqq v(x, t_\delta). \end{aligned}$$

From (5.11) it follows that $p_\varepsilon(t_\delta) = t_\delta + c\varepsilon$ for some $c = c(t_\delta, \sigma)$ which is independent of ε. Thus

$$u(-r_\delta \varepsilon \mu + X(t_\delta + c\varepsilon), t_\delta + c\varepsilon) \geqq e^{-A_0 \varepsilon t_\delta} \sup_{|x| \leqq r_\delta \varepsilon / 5} u(x + X(t_\delta), t_\delta).$$

Recall that $(X(t_\delta), t_\delta) = (\hat{x}, \hat{t}) \in \Gamma(u)$ and $X(t_\delta + c\varepsilon) = X(X(t_\delta), t_\delta; c\varepsilon)$. We proved

$$u(-r_\delta \varepsilon \mu + X(\hat{x}, \hat{t}; c\varepsilon), \hat{t} + c\varepsilon) > 0,$$

which implies

$$u(X(\cdot, \cdot; c\varepsilon) - r_\delta \varepsilon \mu, \cdot + c\varepsilon) > 0 \quad \text{on } \Gamma \cap Q_1.$$

Now we proceed to prove our claim. □

Proof of (5.35). Here we apply Corollary 5.6 with the choice of $U := B(P, \frac{r_\delta}{2}) \setminus B(P, r_\theta)$. To this end, it suffices to show that $w \geqq v$ on the parabolic boundary of Σ.

First observe that from (5.34),

$$w(x, 0) \geqq 0 = v(x, 0) \text{ in } B\left(P, \frac{r_\delta}{2}\right).$$

Since $v(0, t_\delta) = u(X(t_\delta), t_\delta) = 0$ and due to Lemma 3.3,

$$v(0, t) = u(X(t), t) = 0 \text{ for } t \in [0, t_\delta].$$

Due to the cone monotonicity assumption (5.1),

$$w \geqq v = 0 \text{ in } B(P, r_\theta) \subset B\left(P, \frac{r_\delta}{5} \sin\theta\right) \times [0, t_\delta].$$

Hence to show (5.35), it remains to show that $w \geqq v$ on $\partial B(P, \frac{r_\delta}{2}) \times [0, t_\delta]$.

By definition of φ, we have $\varphi(x) = \frac{r_\delta}{2}\sin\theta$ on $\partial B(P, r_\delta/2)$. From (5.14), we know $A_0 \geqq \sigma_2 M_0$. It follows that for $x \in \partial B(P, r_\delta/2)$,

$$\begin{aligned} w(x,t) &\geqq e^{\sigma_2 M_0 \varepsilon} \inf_{y\in B\left(x, r\varepsilon(1-\alpha t)\frac{\sin\theta}{2}\right)} v(y + r_\delta\varepsilon\mu, p_\varepsilon(t)) \\ &= e^{\sigma_2 M_0 \varepsilon} \inf_{y\in B\left(x, \frac{r_\delta\varepsilon}{2}\sin\theta\right)} u(y + r_\delta\varepsilon\mu + X(p_\varepsilon(t)), p_\varepsilon(t)) \\ &=: e^{\sigma_2 M_0 \varepsilon} V_1(x,t). \end{aligned} \tag{5.36}$$

In view of (5.1), we have

$$\inf_{B(x, r_\delta\varepsilon\sin\theta)} u(y + r_\delta\varepsilon\mu + X(t), t) \geqq v(x,t).$$

Thus it remains to show that

$$\begin{aligned} &e^{\sigma_2 M_0 \varepsilon} V_1(\cdot,\cdot) \geqq \inf_{B(\cdot, r_\delta\varepsilon\sin\theta)} u(y + r_\delta\varepsilon\mu + X(\cdot), \cdot) \\ &\quad \text{on } \partial B(P, \frac{r_\delta}{2}) \times [0, t_\delta]. \end{aligned} \tag{5.37}$$

Take any $(x,t) \in \partial B(P, \frac{r_\delta}{2}) \times [0, t_\delta]$, and denote

$$z := z(y,\varepsilon) = y + r_\delta\varepsilon\mu + X(t) \quad \text{for any } y \in B\left(x, \frac{r_\delta\varepsilon}{2}\sin\theta\right).$$

With this notation we can rewrite $V_1(x,t)$ as

$$\inf_{y\in B\left(x, \frac{r_\delta\varepsilon}{2}\sin\theta\right)} u(z - X(p_\varepsilon(t)) + X(t), p_\varepsilon(t)). \tag{5.38}$$

By (5.18) and (5.31), we know

$$s_\varepsilon(t) := p_\varepsilon(t) - t \leqq \sigma\delta\varepsilon. \tag{5.39}$$

Then

$$\begin{aligned} |X(z,t;s_\varepsilon(t)) - z - X(p_\varepsilon(t)) + X(t)| &= |X(z,t;s_\varepsilon(t)) - X(z,t;0) - X(X(t),t;s_\varepsilon(t)) \\ &\quad + X(X(t),t;0)| \\ &= \left| \int_0^{s_\varepsilon(t)} \vec{b}(X(z,t;h),h) - \vec{b}(X(X(t),t;h),h)\mathrm{d}h \right| \\ &\leqq \int_0^{s_\varepsilon(t)} \left(\|D\vec{b}\|_\infty |X(z,t;h) - X(X(t),t;h)| \right) \mathrm{d}h. \end{aligned} \tag{5.40}$$

Note that, for some universal σ,

$$\begin{aligned} |X(z,t;h) - X(X(t),t;h)| &\leqq |X(z,t;0) - X(X(t),t;0)| + \sigma h \\ &= |z - X(t)| + \sigma h \\ &\leqq \sigma r_\delta + \sigma h. \end{aligned}$$

Therefore, (5.40) and (5.39) imply that

$$
\begin{aligned}
|X(z,t;s_\varepsilon(t)) - z - X(p_\varepsilon(t)) + X(t)| &\leqq \sigma r_\delta s_\varepsilon(t) + \sigma s_\varepsilon(t)^2 \\
&\leqq \sigma(\delta r_\delta \varepsilon + \delta^2\varepsilon^2) \leqq \frac{r_\delta \varepsilon}{2}\sin\theta,
\end{aligned} \tag{5.41}
$$

where the last inequality holds if

$$
\delta \leqq \frac{\sin\theta}{4\sigma} \quad \text{and} \quad \varepsilon \leqq \frac{r_\delta \sin\theta}{4\sigma\delta^2}. \tag{5.42}
$$

Using (5.38), (5.39), and (5.41), it follows that

$$
V_1(x,t) \geqq \inf_{y\in B(x,r_\delta\varepsilon\sin\theta)} u(X(z,t;s_\varepsilon(t)), t+s_\varepsilon(t)).
$$

Due to (3.2), for $C := (m-1)(C_0 + \|\nabla\cdot\vec{b}\|_\infty)$,

$$
\begin{aligned}
&\inf_{y\in B(x,r_\delta\varepsilon\sin\theta)} u(X(z(y),t;s_\varepsilon(t)), t+s_\varepsilon(t)) \\
&\geqq e^{-Cs_\varepsilon(t)} \inf_{y\in B(x,r_\delta\varepsilon\sin\theta)} u(y + r_\delta\varepsilon\mu + X(t), t).
\end{aligned}
$$

In view of (5.36), we derive

$$
w(x,t) \geqq e^{\sigma_2 M_0\varepsilon} e^{-Cs_\varepsilon(t)} \inf_{B(x,r_\delta\varepsilon\sin\theta)} u(y + r_\delta\varepsilon\mu + X(t), t). \tag{5.43}
$$

Using (5.39) again shows

$$
e^{\sigma_2 M_0\varepsilon - Cs_\varepsilon(t)} \geqq e^{\sigma_2 M_0\varepsilon - C\sigma\delta\varepsilon} \geqq 1 \quad \text{if} \quad \delta \leqq \frac{\sigma}{1+C_0}. \tag{5.44}
$$

Now after fixing $\delta = \delta(\theta, C_0) > 0$ such that (5.42) and (5.44) hold, we can conclude with (5.37) and then the claim (5.35). □

In view of the velocity law (1.3), non-degeneracy follows once we know that the positive set of the solution is strictly expanding relatively to the streamlines. In the next theorem, we are going to show that indeed the solution u grows linearly near the free boundary.

Corollary 5.7. *Under the conditions of Theorem 5.1, there exist $\varepsilon_0, \kappa_* > 0$ depending only on constants in (5.2) such that, for all $\varepsilon \in (0, \varepsilon_0)$,*

$$
u(x+\varepsilon\mu, t) \geqq \kappa_*\varepsilon \quad \textit{for all } (x,t) \in \Gamma \cap Q_1. \tag{5.45}
$$

Proof. Let c_0 be from Lemma 4.1 and C be from Theorem 5.1. Define $\kappa := \frac{c_0 \sin^2\theta}{4C}$. We first claim that for all $\varepsilon > 0$ sufficiently small

$$
\sup_{y\in B(x,\varepsilon)} u(y,t) \geqq \kappa\varepsilon \quad \text{for } (x,t) \in \Gamma \cap Q_1. \tag{5.46}
$$

We argue by contradiction. Suppose that the above claim is false. Then for any $\varepsilon_0 > 0$ there exist $\varepsilon \in (0, \varepsilon_0]$ and $(\hat{x}, \hat{t}) \in \Gamma \cap Q_1$ such that (5.46) fails.

Set $t_1 := \hat{t} - C\varepsilon$ and consider the map $X(\cdot, t_1; C\varepsilon)\colon \mathbb{R}^d \to \mathbb{R}^d$, which is an isomorphism when ε_0 is small enough. Since the positive set of u is strictly expanding relatively to the streamlines, we have

$$u(X(x, t_1; C\varepsilon), \hat{t}) > 0 \quad \text{for } x \in B_1 \cap \Gamma_{t_1}.$$

Using the cone monotonicity condition (5.1) and the fact that $u(\hat{x}, \hat{t}) = 0$, it follows that $(\hat{x} + \mathbb{R}^+\mu) \cap \Gamma_{t_1} \neq \emptyset$. Therefore there exists $(x_1, t_1) \in \Gamma$ such that

$$X(x_1, t_1; C\varepsilon) = \hat{x} + C_1\varepsilon\mu \quad \text{for some } C_1 > 0.$$

Due to (5.1) again, we have

$$d(x_1 - c\varepsilon\mu, \Gamma_{t_1}) \geqq c\varepsilon \sin\theta \quad \text{for all } c \geqq 0. \tag{5.47}$$

In view of Theorem 5.1, for all ε sufficiently small

$$u(X(x_1, t_1; C\varepsilon) - \varepsilon\mu, t_1 + C\varepsilon) > 0.$$

Therefore, combining with the fact that

$$u(X(x_1, t_1; C\varepsilon) - C_1\varepsilon u, t_1 + C\varepsilon) = u(\hat{x}, \hat{t}) = 0,$$

we obtain $C_1 \geqq 1$.

Next define

$$x_2 := X(\hat{x}, \hat{t}; -C\varepsilon), \quad f(t) := X(\hat{x} + C_1\varepsilon\mu, \hat{t}; t) - X(\hat{x}, \hat{t}; t).$$

Due to (1.6),

$$|f'(t)| \leqq \|D_x\vec{b}\|_\infty |f(t)| = \sigma|f(t)|, \quad f(0) = C_1\varepsilon\mu \text{ and } f(-C\varepsilon) = x_1 - x_2.$$

Thus

$$|x_1 - x_2 - C_1\varepsilon\mu| = |f(-C\varepsilon) - f(0)| \leqq \sigma C C_1 \varepsilon^2.$$

Using this, (5.47) and the fact that $C_1 \geqq 1$, if $\varepsilon \leqq \varepsilon_0$ is sufficiently small compared to C, it follows that

$$d(x_2, \Gamma_{t_1}) \geqq \frac{C_1\varepsilon\sin\theta}{2} \geqq \frac{\varepsilon\sin\theta}{2} =: R,$$

which yields

$$u(\cdot, t_1) = 0 \text{ in } B(x_2, R). \tag{5.48}$$

Note that $t_1 + C\varepsilon = \hat{t}$ and $X(x_2, t_1; C\varepsilon) = \hat{x}$ from definition. Therefore the failure of (5.46) implies that

$$\fint_{B(X(x_2,t_1;C\varepsilon),R)} u(x, \hat{t})\mathrm{d}x = \fint_{B(\hat{x},R)} u(x, \hat{t})\mathrm{d}x \leqq \kappa\varepsilon = \frac{c_0 R^2}{C\varepsilon}. \tag{5.49}$$

In the last equality, we used that $\kappa = \frac{c_0 \sin^2\theta}{4C}$.

With (5.48)–(5.49), we are able to apply Lemma 4.1 to get

$$u(x, \hat{t}) = 0 \quad \text{in } B(X(x_2, t_1; C\varepsilon), R/6) = B(\hat{x}, R/6),$$

which is in contradiction with the assumption that $(\hat{x}, \hat{t}) \in \Gamma$. We proved (5.46). It can be seen from the proof that ε_0 only depends on constants in (5.2).

Now we show (5.45). Let us take $\gamma \in (0, 1)$ to be small enough depending only on θ such that $B(\mu, \gamma) \subseteq W_{\theta,\mu}$, which implies that, for any $\varepsilon \in (0, 1)$,

$$\varepsilon\mu \in \bigcap_{z \in B(0,\gamma\varepsilon)} \{z + W_{\theta,\mu}\}. \tag{5.50}$$

Fix any $(x, t) \in \Gamma \cap Q_1$, and set $\kappa_* := \kappa\gamma$. By (5.46), there exists $\varepsilon_0 > 0$ such that

$$\sup_{y \in B(x,\gamma\varepsilon)} u(y, t) \geqq \kappa_*\varepsilon \quad \text{for any } \varepsilon \in (0, \varepsilon_0].$$

Therefore we can find $y \in B(x, \gamma\varepsilon)$ that $u(y, t) \geqq \kappa_*\varepsilon$. It follows from (5.50) that $x + \varepsilon\mu \in y + W_{\theta,\mu}$. Due to (5.1), we conclude with

$$u(x + \varepsilon\mu, t) \geqq \kappa_*\varepsilon \quad \text{for any } (x, t) \in \Gamma \cap Q_1 \text{ and } \varepsilon \in (0, \varepsilon_0].$$

□

6. Flatness Implies Smoothness

In this section we prove the following theorem:

Theorem 6.1. *Let u be as given in Theorem 5.1. If (1.10) holds in Q_1, then u is Lipschitz continuous and $\Gamma \cap Q_{\frac{1}{2}}$ is a d-dimensional $C^{1,\alpha}$ surface for some $\alpha \in (0, 1)$.*

The cone monotonicity and (1.10) provide sufficient monotonicity properties for the solution to rule out topological singularities and to localize the regularization phenomena driven by the diffusion in the interior of the domain. We follow the outline for the zero drift built on [10,11], while we elaborate on the differences. Most notable difference is in establishing Proposition 6.3.

Lemma 6.2. *Under the conditions of Theorem 6.1, u is Lipschitz continuous in Q_1, and $\Gamma \cap Q_{1/2}$ is a d-dimensional Lipschitz continuous surface.*

Proof. First let us prove that u is Lipschitz continuous in Q_1. Since u satisfies a parabolic equation locally uniformly in its positive set, u is smooth in $\{u > 0\}$. From the equation and $\Delta u \geqq -C_0$, we obtain

$$u_t \geqq |\nabla u|^2 - \sigma(C_0 + 1)u + \nabla u \cdot \vec{b} \text{ in } \{u > 0\}, \tag{6.1}$$

where σ is universal. Above estimate combined with condition (1.10) yields

$$(A+\sigma)|\nabla u| + C(C_0, A, \sigma)\, u + A \geqq |\nabla u|^2,$$

which turns into a bound on $|\nabla u|$ in $\{u > 0\}$. From (1.10), we also get a bound on $|u_t|$. Notice the bounds are independent of the ellipticity constants of the equation satisfied by u. Indeed we have,

$$|\nabla u| + |u_t| \leqq C \quad \text{in } Q_1 \cap \{u > 0\} \tag{6.2}$$

for some C only depending on A, C_0 and universal constants. Since u is continuous and nonnegative, it is not hard to see that the same estimate holds weakly in Q_1.

Next we turn to the Lipschitz continuity of Γ, using the cone monotonicity and Lipschitz continuity of u. The spatial cone monotonicity of u implies that for each $t \in (-1, 1)$, Γ_t is a Lipschitz continuous graph in $\mathbb{R}^d$. Thus it remains to show that for each $\tau \in (-1, 1)$, $\Gamma_{t+\tau} \cap B_{\frac{1}{2}}$ is in a $C\tau$ neighbourhood of $\Gamma_t \cap B_1$ for some $C > 0$. To this end, it is enough to show the following: for $(x, t) \in \Gamma \cap Q_{\frac{1}{2}}$ and for $\tau > 0$ sufficiently small, we have

$$d(x, \Omega_{t+\tau}) \quad \textit{and} \quad d(x, \{u(\cdot, t+\tau) = 0\}) \leqq C\tau. \tag{6.3}$$

To show (6.3) let us fix $(x, t) \in \Gamma \cap Q_{\frac{1}{2}}$. Observe that from Lemma 3.3 there exists $C > 0$ such that, if $\tau > 0$ is small,

$$d(x, \Omega_{t+\tau}(u)) \leqq C\tau.$$

Thus it remains to show the second inequality in (6.3). Let $C_1 > 0$ be a sufficiently large constant to be chosen later. From the cone monotonicity

$$u(\cdot, t) = 0 \text{ in } B(y, R),$$

where $y := x - C_1\tau\mu$ and $R := C_1 \sin\theta\, \tau$. By the Lipschitz continuity of u,

$$\begin{aligned}
\sup_{z\in B(X(y,t;\tau),R)} u(z, t+\tau) &\leqq u(X(y, t; \tau), t) + C(R+\tau) \\
&\leqq u(y, t) + \sigma C\tau + C(1 + C_1 \sin\theta)\tau \\
&\quad (\text{ since } |X(y, t; \tau) - y| \\
&\leqq \|\vec{b}\|_\infty \tau) \\
&\leqq C\, C_1\, \tau,
\end{aligned}$$

where C depends on $Lip(u)$ and $\|\vec{b}\|_\infty$. Thus, for c_0 given in Lemma 4.1,

$$\fint_{B(X(y,t;\tau),R)} u(z, t+\tau)\mathrm{d}z \leqq CC_1\tau \leqq (c_0\, C_1^2 \sin^2\theta)\, \tau = c_0 \frac{R^2}{\tau},$$

where the last inequality holds if C_1 is large enough compared to $1/c_0$, $1/\theta$, $Lip(u)$, $\|\vec{b}\|_\infty$. Remark 4.2 then yields for small τ,

$$u(x - C_1\tau\mu, t+\tau) = 0$$

and therefore (6.3) is proved. $\square$

Now we start proving the $C^{1,\alpha}$ regularity of the free boundary. By considering $\tilde{u}(x,t) := 2u(x_0+\frac{1}{2}x, t_0+\frac{1}{2}t)$ for any $(x_0,t_0) \in Q_{\frac{1}{2}} \cap \Gamma$, we can assume $(0,0) \in \Gamma$ and u is a solution in Q_1. Then to prove the rest of Theorem 6.1, it suffices to show that Γ is $C^{1,\alpha}$ at point $(0,0)$.

The following proposition propagates the free boundary non-degeneracy in Corollary 5.7 to the nearby level sets:

Proposition 6.3. *Assume the conditions of Theorem 6.1 and $(0,0) \in \Gamma$. Then there exist constants $0 < \delta_1 < \frac{1}{2}$ and $c_1 > 0$ such that*

$$\nabla_\mu u(x,t) \geqq c_1 \quad \text{a.e. in } Q_{\delta_1} \cap \Omega(u).$$

Proof. Fix a sufficiently small $\delta > 0$ to be determined and pick $(\hat{x},\hat{t}) \in \{u > 0\} \cap Q_\delta$. Let $h := d(\hat{x}, \Gamma_{\hat{t}}) < \delta$. From Lemma 6.1, $\Gamma(u)$ is space-time Lipschitz continuous, and actually it can be written as the graph of $x_\nu = F_u(x^\perp, t)$ where $x_\nu := x \cdot \nu$ and $x^\perp \in \{x \cdot \nu = 0\}$. Let us denote the space-time Lipschitz constant of F_u as C, and choose $C_2 := C + 1$. Then

$$d(\hat{x}, \Gamma_{\hat{t}-h}) \leqq (C_2 - 1)h.$$

Denote (y,s) such that $s = \hat{t}-h$, $y \in \Gamma_s$ and $d(\hat{x}, y) = d(\hat{x}, \Gamma_s) \leqq (C_2-1)h$. Thus $B(y,h) \subseteq B(\hat{x}, C_2 h)$. Also by Lipschitz continuity of Γ_s in space, $\partial B(y,h) \cap \{u > 0\}$ is of strictly positive measure Ch with C independent of h.

By the divergence theorem,

$$\fint_{B(y,h)\cap\{u>0\}} \nabla_\mu u(x,s)\mathrm{d}x \geqq \frac{\sigma}{h} \fint_{\partial B(y,h)\cap\{u>0\}} u(x,s)\mu \cdot n_x \mathrm{d}x$$

where n_x is the outward pointing unit normal on $\partial B(y,h)$. So in view of (5.45) and the assumption that u is monotone with respect to $W_{\theta,\mu}$, we have

$$\fint_{B(y,h)\cap\{u>0\}} \nabla_\mu u(x,s)\mathrm{d}x \geqq \kappa$$

for some $\kappa > 0$ only depending on κ_* and C_2.

Let us define

$$\Omega^r := \{(x,t) \in \Omega, d((x,t), \partial\Omega) > r\}. \tag{6.4}$$

Fix $\gamma \in (0, \frac{1}{2})$ to be a small constant only depending on κ such that

$$\fint_{B(y,h)\cap\Omega^{\gamma h}} \nabla_\mu u(x,s)\mathrm{d}x \geqq \frac{\kappa}{2}.$$

Therefore there exists a point

$$z \in B(y,h) \cap \Omega^{\gamma h} \subset B(\hat{x}, C_2 h) \cap \Omega^{\gamma h}$$

such that

$$\nabla_\mu u(z,s) \geqq \frac{\kappa}{2}. \tag{6.5}$$

We will apply Harnack inequality to $\phi := \nabla_\mu u$, using the fact that it solves a locally uniform parabolic equation in the positive set of u.

Let us consider $\{u > 0\}$ and then u is C^1 inside the open region. Differentiating (1.5) in $\{u > 0\}$, we can check that ϕ satisfies the following parabolic equation

$$\phi_t = (m-1)\phi\Delta u + (m-1)u\Delta\phi + (2\nabla u + \vec{b})\cdot\nabla\phi + (m-1)\phi\nabla\cdot\vec{b} + f,$$

where

$$f := \nabla u \cdot \nabla_\mu \vec{b} + (m-1)u\nabla\cdot\nabla_\mu\vec{b}.$$

Since u is Lipschitz continuous and $\vec{b}$ is smooth, f is uniformly bounded. Then the new function

$$\tilde{\phi} := \phi e^{C_3(t-s)} + \|f\|_\infty(t-s) \quad \text{with } C_3 := (m-1)(C_0 + \|\nabla\cdot\vec{b}\|_\infty)$$

satisfies

$$\tilde{\phi}_t \geqq (m-1)u\Delta\tilde{\phi} + (2\nabla u + \vec{b})\cdot\nabla\tilde{\phi}. \tag{6.6}$$

Let us define

$$\Sigma_1^h := \Omega^{\gamma h} \cap \left(B(\hat{x}, C_2 h) \times (-h+\hat{t}, \hat{t})\right),$$
$$\Sigma_2^h := \Omega^{\gamma h/2} \cap \left(B(\hat{x}, 2C_2 h) \times (-2h+\hat{t}, \hat{t})\right),$$

where Ω^r is as given in (6.4). Then we have

$$(\hat{x}, \hat{t}),\ (z, s) \in \Sigma_1^h \subseteq \Sigma_2^h.$$

For any $(x,t) \in \Sigma_2^h$, it is $\frac{\gamma h}{2}$ away from Γ, and then by the cone monotonicity and (5.45) we have

$$u(x,t) \geqq \frac{\kappa_* \gamma h}{2}. \tag{6.7}$$

Consider $w(x,t) := \tilde{\phi}(xh + \hat{x}, th + s)$. Since $\hat{t} - s = h$, we have

$$w(0,1) = \tilde{\phi}(\hat{x}, \hat{t}), \quad \text{and } w(z', 0) = \tilde{\phi}(z, s) \quad \text{with } z' := \frac{z - \hat{x}}{h}.$$

Denote

$$\Sigma_1 := (\Sigma_1^h - (\hat{x}, s))/h, \quad \Sigma_2 := (\Sigma_2^h - (\hat{x}, s))/h.$$

Then Σ_1, Σ_2 are domains with Lipschitz boundary with Lipschitz constant depending only on C, σ, and

$$(0,1), (z', 0) \in \Sigma_1 \subseteq \Sigma_2,$$

Also writing $\Sigma_i(t) = \{x \mid (x,t) \in \Sigma_i\}$ for $i = 1, 2$, we get

$$\Sigma_2(t) + B_{\frac{\gamma}{2}} \subseteq \Sigma_1(t) \text{ for } t \in (-h+\hat{t}, \hat{t}).$$

From (6.6), w satisfies

$$w_t \geqq (m-1)\frac{u}{h}\Delta w + (2\nabla u + \vec{b}) \cdot \nabla w,$$

and the operator here is uniformly parabolic in Σ_2, due to (6.7). Applying Harnack's inequality to w in $\Sigma_1 \subseteq \Sigma_2$, we get

$$w(0,1) \geqq c\, w(z', 0)$$

for some constant $c = c(\theta, \kappa_*, C_2) > 0$. Write the above inequality in terms of ϕ to obtain

$$\phi(\hat{x}, \hat{t}) e^{C_3(\hat{t}-s)} + \|f\|_\infty (\hat{t} - s) \geqq c\,\phi(z, s),$$

which is larger than $\frac{c\kappa}{2}$, due to (6.5).

Since $\hat{t} - s = h \leqq \delta$, further assuming δ to be small enough, we get $\phi(\hat{x}, \hat{t}) \geqq \frac{c\kappa}{4} > 0$. Finally we conclude that $\nabla_\mu u \geqq \frac{c\kappa}{4} > 0$ in $\Omega \cap Q_\delta$.

□

Next we show the strict monotonicity of u along the streamlines.

Lemma 6.4. *Let u be given as in Proposition 6.3. Then there exist $\delta_2 \in (0, \delta_1)$ and $c_2 > 0$ such that, for $v(x,t) := u(x + X(t), t)$ with $X(t) = X(0,0;t)$, we have*

$$v_t \geqq c_2 \quad \text{in } Q_{\delta_2} \cap \{v > 0\}.$$

Proof. By definition, v solves $\mathcal{L}_2(v) = 0$, where $\mathcal{L}_2$ is as given in (5.4). By the equation, we have

$$\begin{aligned}
\partial_t v &\geqq -C_0(m-1)v + \frac{1}{2}|\nabla v|^2 - 4|\vec{b}(x + X(t)) - \vec{b}(X(t))|^2 - (m-1)v\|\nabla \vec{b}\|_\infty \\
&\geqq -\sigma C_0 \delta + \frac{c_1^2}{2} - 4|x|^2 \|\nabla \vec{b}\|_\infty^2 - C\delta \\
&\geqq -\sigma C_0 \delta + \frac{c_1^2}{2} - \sigma \delta^2 - C\delta \qquad\qquad \text{in } Q_\delta,
\end{aligned}$$

where the second inequality comes from the fact that $v \leqq C\delta$ due to (6.2), and the third inequality follows from Proposition 6.3.

Since c_1 is independent of δ, the last quantity is positive if $\delta = \delta_2$ is small enough compared to C_0, c_1, the Lipchitz constant of u and universal constants. We thus conclude. □

Now we are ready to follow the celebrated iteration procedure given in [10]. Their argument describes the enlargement of cone of monotonicity as we zoom in near a free boundary point. More precise discussions are below.

Our reference point is $(0,0) \in \Gamma$, and let v be from Lemma 6.4. For $\delta \in (0, \delta_2)$, define

$$v_\delta(x,t) := \frac{1}{\delta} v(\delta x, \delta t), \quad \vec{b}_\delta(x,t) := \vec{b}(\delta x, \delta t), \quad X_\delta := \frac{1}{\delta} X(\delta t). \tag{6.8}$$

Then X_δ is the streamline generated by $\vec{b}_\delta$ starting at $(0, 0)$. We have that v_δ is a solution to $\mathcal{L}_2(\cdot) = 0$ with $\vec{b}$, X replaced by $\vec{b}_\delta$, X_δ. From Lemmas 6.2–6.4, we have for some $L > 0$ independent of δ [depending on constants in (5.2)] such that

$$0 \leqq v_\delta \leqq L, \quad \frac{1}{L} \leqq |\nabla v_\delta|,\ \nabla_\mu v_\delta,\ \partial_t v_\delta \leqq L, \quad \Delta v_\delta \geqq -\delta L \quad \text{in } Q_1. \tag{6.9}$$

Denoting σ as the C^2 norm of $\vec{b}$, we have

$$\|\vec{b}_\delta\|_\infty \leqq \sigma, \quad \|\nabla \vec{b}_\delta\|_\infty + \|\partial_t \vec{b}_\delta\|_\infty \leqq \sigma\delta, \quad \|D^2 \vec{b}_\delta\|_\infty + \|\nabla\partial_t \vec{b}_\delta\|_\infty \leqq \sigma\delta^2. \tag{6.10}$$

Let $\widehat{W}_{\theta,\nu}$ be given as in (2.1). We say v has the *cone of monotonicity* $\widehat{W}_{\theta,\nu}$ in Q_1 if

$$\hat{\nabla}_p v \geqq 0 \quad \text{in } Q_1 \text{ for all } p \in \widehat{W}_{\theta,\nu}.$$

The next lemma, yielding the initial cone of monotonicity for v_δ, can be proven using (6.9)–(6.10) with a parallel proof to Proposition 2.1 of [10]. Let us denote the positive time direction as e_{d+1}.

Lemma 6.5. *Let v_δ be as given in (6.8). Then there exists $\theta_0 > 0$ such that*

$$\hat{\nabla}_p v_\delta \geqq \frac{1}{2L} \quad \textit{in } Q_1 \quad \textit{for all } p \in \widehat{W}_{\theta_0,\mu_0} \cap \mathcal{S}^{d+1},$$

where $\mu_0 := \frac{1}{\sqrt{2}}[(\mu, 0) + e_{d+1}]$ and L is as given in (6.9).

Now we begin our iteration procedure. Fix some $J(L) \in (0, 1)$ to be chosen later, define

$$v_k(x, t) := \frac{1}{J^k} v_\delta(J^k x, J^k t) \quad \text{for } k \in \mathbb{N}^+. \tag{6.11}$$

Then v_k satisfies

$$\begin{aligned}\partial_t v_k - (m-1)v_k\Delta v_k - |\nabla v_k|^2 - \nabla v_k \cdot (\vec{b}_k(x + X_k(t), t) - \vec{b}_k(X_k(t), t))\\ -(m-1)v_k \nabla \cdot \vec{b}_k(x + X_k(t), t) = 0,\end{aligned} \tag{6.12}$$

where $\vec{b}_k(x, t) := \vec{b}_\delta(J^k x, J^k t)$, $X_k(t) := \frac{1}{J^k} X_\delta(J^k t)$.

Due to (6.9)–(6.10) the following holds in Q_1:

(A_k) $0 \leqq v_k \leqq L, \quad \Delta v_k \geqq -L\delta, \quad |\nabla v_k| + |\partial_t v_k| \leqq L;$

(B_k) $\nabla_\mu v_k, \quad \partial_t v_k \geqq \frac{1}{L};$

(C_k) $\|\vec{b}_k\|_\infty \leqq \sigma, \quad \|\nabla \vec{b}_k\|_\infty + \|\partial_t \vec{b}_k\|_\infty \leqq \sigma\delta J^k, \quad \|D^2 \vec{b}_k\|_\infty + \|\nabla\partial_t \vec{b}_k\|_\infty \leqq \sigma\delta^2 J^{2k}$.

The main step in the proof of Theorem 6.1 is to show the following property inductively.

(D_k) there exist $s \in (0,1)$ and $\mu_k \in \mathbb{R}^{d+1}$ such that for $\theta_k := \frac{\pi}{2} - s^k(\frac{\pi}{2} - \theta_0)$,

$$\hat{\nabla}_p v_k \geqq \frac{1}{2L} J^k \quad \text{in } Q_1 \quad \text{for all } p \in \widehat{W}_{\theta_k,\mu_k} \cap \mathcal{S}^d. \tag{6.13}$$

Once establishing (D_k), it shows that the cone of monotonicity $\widehat{W}_{\theta_k,\mu_k}$ for v_k has strictly increasing θ_k, converging to $\pi/2$ as $k \to \infty$. The rate of its increasing angles leads to the $C^{1,\alpha}$ regularity of the free boundary.

In [10], (6.13) is stated with the weaker requirement $\hat{\nabla}_p v_k \geqq 0$. However for us the competition between diffusion and drift requires a stronger inductive property: see Remark 6.9. This extra observation follows from the enlargement of cones as well as the non-degeneracy of the solution.

We will proceed with several lemmas that lead to the enlargement of cones in Proposition 6.10. The proofs of the lemmas will be postponed until after the proof of the Proposition.

First we show that some improvements on monotonicity can be obtained on the set $\{v_k = \varepsilon\}$. Recall L from (6.9).

Lemma 6.6. (Enlargement of Cones) *For any $\varepsilon \in (0,1)$, there exist positive constants $r \leqq \frac{1}{10}$, $\delta_0 < \delta_2$, and C depending only on ε, L, σ such that the following holds. For any $\delta \in (0,\delta_0)$, $J \in (0,1)$ and $k \geqq 0$, let v_k be as given in (6.11), and suppose that v_k satisfies (D_k). Then for any $\gamma \in (0,\varepsilon)$, $p \in \widehat{W}_{\theta_k,\mu_k} \cap \mathcal{S}^d$ and $\tau := C\varepsilon^{-1}\cos\langle\, p, \hat{\nabla} v_k(\mu,-2r)\,\rangle$, we have*

$$\begin{aligned} &v_k \leqq \varepsilon \ \text{ in } Q_{2r}; \quad \text{and} \quad v_k((x,t)+\gamma p) \\ &\geqq (1+\tau\gamma)v_k \ \text{ on } (B_{\frac{3}{4}} \times (-2r,2r)) \cap \{v = \varepsilon\}. \end{aligned}$$

Next we show that this improvement can propagate to the zero level set of v.

Lemma 6.7. *Let $\varepsilon > 0$ be small enough depending only on L, σ. Take r, v_k, τ from Lemma 6.6. If w is a supersolution of (6.12) such that $w \geqq v_k$ in Q_1, and for some $\gamma \in (0,\varepsilon)$,*

$$w \geqq (1+\tau\gamma)v_k \text{ in } (B_{\frac{1}{2}} \times (-2r,2r)) \cap \{v_k = \varepsilon\},$$

then we have

$$w \geqq (1+\tau\gamma)v_k \text{ in } (B_{\frac{1}{4}} \times (-2r,2r)) \cap \{v_k \leqq \varepsilon\}.$$

Lastly we further improve the monotonicity in a smaller domain of size r.

Lemma 6.8. *Let $\varepsilon, r, v_k, \tau, w, \gamma$ be as in Lemma 6.7. There exists a small $\kappa > 0$ depending only on L, σ such that the following holds. Consider any smooth function $\phi : \mathbb{R}^d \to \mathbb{R}^+$ such that ϕ is supported in B_{2r} and $\phi, |\nabla\phi|, |D^2\phi| \leqq \kappa\tau\gamma$. If $v_k \leqq \varepsilon$ in Q_{2r} then we have*

$$w(x,t) \geqq v_k(x + (t+2r)\phi(x)\mu, t) \text{ in } Q_{2r}.$$

Remark 6.9. In [10] for the zero drift case, w in the above lemmas is chosen as a translation of v_k to derive monotonicity properties of v_k. Since our equation is not translation invariant, we instead choose w of the form $v_k((x,t)+p)+Et$ with $E>0$. To control the extra term Et we rely on the inductive property (D_k). The order between v_k, w is still enough to derive the Proposition below.

Now we state the main proposition.

Proposition 6.10. (Improvement of monotonicity) *There exist constants* $J, s \in (0,1)$ *such that the following holds for all* $k \geqq 0$. *Let* v_δ *be as given in* (6.8) *with* $\delta \in (0,\delta_0)$ *and* δ_0 *from Lemma* 6.6. *Suppose* $(0,0) \in \Gamma$ *and* (6.9)–(6.10). *Then there exist a monotone family of cones* $\widehat{W}_{\theta_k,\mu_k}$ *with* $\theta_k = \frac{\pi}{2} - s^k(\frac{\pi}{2} - \theta_0)$ *such that*

$$\hat{\nabla}_p v_\delta \geqq (2L)^{-1} J^k \quad \textit{in } Q_{J^k} \cap \{v_\delta > 0\} \quad \textit{for all } p \in \widehat{W}_{\theta_k,\mu_k} \cap \mathcal{S}^d.$$

The $C^{1,\alpha}$ regularity of Γ at $(0,0)$ is a result of the relation $\theta_k = \theta_{k-1} + S(\pi/2 - \theta_{k-1})$ which describes quantitatively the enlargement of cone of monotonicity of solutions near the free boundary. Then Theorem 6.1 follows. We omit detailed discussion of this part since it is parallel to Theorem 1 in [10].

Proof. Fix a small $\varepsilon > 0$ such that the conclusion of Lemma 6.7 holds, and let r, δ_0 be as given in Lemma 6.6. Then ε, δ_0, r only depend on L and universal constants. Define τ as in Lemma 6.6. Let v_k be as in (6.11), and set $\vec{b}_k$, X_k as before and we take $J \leqq r$ to be determined. It is straightforward that for all $k \geqq 0$, $(A_k) - (C_k)$ hold. When $k=0$, due to Lemma 6.5, (D_0) holds for $v = v_0$.

Let us suppose that (D_k) holds for some $k \geqq 0$ with $\mu_k, \theta_k \geqq \theta_0$ i.e. the hypothesis of Lemmas 6.6–6.8 are satisfied. We will show (D_{k+1}).

For any $\gamma \in (0,\varepsilon)$ and a unit vector $p \in \widehat{W}_{\theta_k,\mu_k}$, define

$$\widetilde{w}(x,t) := v_k((x,t)+\gamma p).$$

Note that $\widetilde{w} \geqq v_k$ in Q_1 due to D_k. Next, (6.12) implies that

$$\mathcal{L}_2(\widetilde{w}) \geqq -\gamma\left(|\nabla \widetilde{w}||\hat{\nabla}_p \vec{b}_k(x+X_k)| + (m-1)\widetilde{w}|\nabla \cdot \hat{\nabla}_p \vec{b}_k|\right).$$

By $(A_k) - (C_k)$ and the fact that $|\partial_t X_k| \leqq |\vec{b}_k| \leqq \sigma$, we have

$$\mathcal{L}_2(\widetilde{w}) \geqq -\gamma(\sigma\delta L J^k) =: -\gamma E_k.$$

Then for $w := \widetilde{w} + E_k(t+2r)$, we have $w \geqq v_k$ in Q_{2r}.

In view of Lemma 6.6, $v_k \leqq \varepsilon$ in Q_r and w satisfies the hypothesis of Lemma 6.7. Let τ be defined as in Lemma 6.6, and let $\kappa < \kappa_0$ be from Lemma 6.8. We select a smooth function $\phi\colon \mathbb{R}^d \to \mathbb{R}^+$ such that ϕ is supported in B_{2r}, and $\phi, |\nabla\phi|, |D^2\phi| \leqq \kappa\tau\gamma$, and

$$\phi \geqq \sigma r^2 \kappa\tau\gamma \quad \text{in } B_r \quad \text{for some universal } \sigma. \tag{6.14}$$

Clearly such ϕ exists.

It follows from Lemmas 6.6–6.8 that

$$w(x,t) \geqq v_k(x+(t+2r)\phi(x)\mu, t) \quad \text{in } Q_{2r}.$$

By (B_k) and (6.14), for $c' := \frac{\sigma r^3 \kappa}{L}$ we have

$$w(x,t) \geqq v_k(x,t) + \frac{t+2r}{L}\phi(x) \geqq v_k(x,t) + c'\tau\gamma \text{ in } Q_r.$$

This implies that

$$\begin{aligned}\hat{\nabla}_p v_k(x,t) &= \lim_{\gamma\to 0} \frac{v_k((x,t)+\gamma p) - v_k(x,t)}{\gamma} \\ &\geqq \lim_{\gamma\to 0} \frac{w(x,t)-v_k(x,t)}{\gamma} - 3E_k r \\ &\geqq c'\tau - \sigma\delta L J^k r \qquad\qquad \text{in } Q_r \cap \{v_k > 0\}.\end{aligned}$$

Using the definition of τ, we obtain

$$\hat{\nabla}_p v_k(x,t) = C_1 \cos\langle p, \hat{\nabla} v_k(\mu,-2r)\rangle - \sigma\delta L J^k r, \tag{6.15}$$

where $C_1 := c' C \varepsilon^{-1}$ only depending on L, σ (since ε is fixed).

It follows from (A_k) and (D_k) that

$$\cos\langle p, \hat{\nabla} v_k(\mu,-2r)\rangle = \frac{\hat{\nabla}_p v_k}{|\hat{\nabla} v_k|}(\mu,-2r) \geqq \frac{1}{L}\hat{\nabla}_p v_k(\mu,-2r) \geqq \frac{1}{2L^2} J^k. \tag{6.16}$$

Taking δ to be small enough only depending on L and σ, (6.15) yields

$$\hat{\nabla}_p v_k(x,t) \geqq \frac{C_1}{2}\cos\langle p, \hat{\nabla} v_k(\mu,-2r)\rangle \quad \text{in } Q_r \cap \{v_k > 0\}.$$

Thus in $Q_r \cap \{v_k > 0\}$,

$$\cos\langle p, \hat{\nabla} v_k(x,t)\rangle = \frac{\hat{\nabla}_p v_k}{|\hat{\nabla} v_k|}(x,t) \geqq \frac{C_1}{2L}\cos\langle p, \hat{\nabla} v_k(\mu,-2r)\rangle. \tag{6.17}$$

For $p \in \mathcal{S}^{d+1}$, set

$$\rho(p) := \frac{C_1}{8L}\cos\langle p, \hat{\nabla} v_k(\mu,-2r)\rangle.$$

For any $q \in B(p,\rho(p))$ we have $\sin\langle p, q\rangle \leqq \rho(p)$ and thus

$$\begin{aligned}\cos\langle q, \hat{\nabla} v_k(x,t)\rangle &\geqq \cos\langle p, \hat{\nabla} v_k(x,t)\rangle - 2\sin\langle p, q\rangle \\ &\geqq \frac{C_1}{2L}\cos\langle p, \hat{\nabla} v_k(\mu,-2r)\rangle - 2\rho(p) \quad (\text{ by } (6.17)) \\ &= \frac{C_1}{4L}\cos\langle p, \hat{\nabla} v_k(\mu,-2r)\rangle.\end{aligned}$$

In view of (A_k) and (6.16), we get

$$\hat{\nabla}_q v_k(x,t) \geqq \frac{C_1}{4L^2}\cos\langle p, \hat{\nabla} v_k(\mu,-2r)\rangle \geqq \frac{C_1}{8L^4}J^k.$$

Since the above holds for all $q \in B(p,\rho(p))$, there exists a larger cone $\widehat{W}_{\theta_{k+1},\mu_{k+1}}$ for some $\mu_{k+1} \in \mathbb{R}^{d+1}$, $S \in (0,1)$ and $\theta_{k+1} = \theta_k + S(\frac{1}{2}\pi - \theta_k)$ such that

$$\hat{\nabla}_p v_k(x,t) \geqq \frac{C_1}{8L^4}J^k \quad \text{for all unit vector } p \in \widehat{W}_{\theta_{k+1},\mu_{k+1}} \text{ and } (x,t) \in Q_r.$$

Here S is independent of k, because $\rho(p)$ only depends on the angle between p and $\hat{\nabla} v_k(\mu,-2r)$. From the iterative definition of θ_k, we obtain $\theta_k = \frac{\pi}{2} - s^k(\frac{\pi}{2} - \theta_0)$ with $s = 1 - S$. We refer readers to [8,10] for more details.

Let $J := \min\{C_1/(4L^3), r\}$. Recalling $v_{k+1}(x,t) = \frac{1}{J}v_k(Jx, Jt)$, we obtain for all unit $p \in \widehat{W}_{\theta_{k+1},\mu_{k+1}}$

$$\hat{\nabla}_p v_{k+1}(x,t) = \hat{\nabla}_p v_k \geqq \frac{C_1}{8L^4}J^k \geqq \frac{1}{2L}J^{k+1} \quad \text{for } (x,t) \in Q_1 \cap \{v_{k+1} > 0\}.$$

We checked (D_{k+1}) and therefore by induction we conclude the proof of the theorem. □

Now we give the proofs of Lemmas 6.6–6.8. To simplify notations, we write $v := v_k$, $\vec{b} := \vec{b}_k$ and $X := X_k$ in the that follow proofs.

Proof of Lemma 6.6. First note that if $r \leqq \frac{\varepsilon}{2L}$, then $v \leqq \varepsilon$ in Q_{2r} from (A_k) and the fact that $0 \in \Gamma_0$. Next observe that in Q_1, $g := \hat{\nabla}_p v$ solves

$$\begin{aligned} g_t = (m-1)g\Delta v + 2\nabla v \cdot \nabla g + (m-1)v\Delta g \\ +\nabla g \cdot (\vec{b}(x+X) - \vec{b}(X)) + (m-1)g\nabla\cdot\vec{b} \\ +\nabla v \cdot \hat{\nabla}_p \vec{b}(x+X) + (m-1)v\nabla\cdot\hat{\nabla}_p\vec{b}. \end{aligned} \tag{6.18}$$

By the condition $(A_k)(C_k)$,

$$|\nabla v \cdot \hat{\nabla}_p \vec{b}(x+X)| + |(m-1)v\nabla\cdot\hat{\nabla}_p\vec{b}| \leqq \sigma\delta L J^k.$$

Now we apply Harnack's inequality to g, using (6.18), in $(B_{\frac{7}{8}} \times [-3r, 3r]) \cap \{v \geqq \frac{1}{2}\varepsilon\}$. As done in Proposition 2.2 in [10], if we restrict to a smaller region $(B_{\frac{3}{4}} \times (-2r, 2r)) \cap \{v \geqq \varepsilon\}$ for r small enough (depending on ε), there exist C, C' (depending on L, r, ε) such that

$$\hat{\nabla}_p v(x,t) \geqq C\hat{\nabla}_p v(\mu,-2r) - C'\delta J^k.$$

By (D_k), we have $\hat{\nabla}_p v(\mu,-2r) \geqq J^k$. Thus we can select δ small enough such that for some $C > 0$

$$\hat{\nabla}_p v(x,t) \geqq C\hat{\nabla}_p v(\mu,-2r) \text{ in } (B_{\frac{3}{4}} \times (-2r,2r)) \cap \{u \geqq \varepsilon\}. \tag{6.19}$$

To show the assertion, we need to show

$$\frac{v((x,t)+\gamma p) - v(x,t)}{\gamma} \geqq \tau v(x,t) = \tau\varepsilon,$$

which holds by the definition of τ and (6.19). □

Proof of Lemma 6.7. Let $f \in C^1(B_{\frac{1}{2}})$ be a non-negative function such that

$$f = 0 \text{ in } B_{\frac{1}{4}}; \quad f = \varepsilon \text{ on } \partial B_{\frac{1}{2}}; \quad |\nabla f| \leqq 10\varepsilon; \quad |\Delta f| \leqq 10\varepsilon.$$

For $\alpha \in (-2r, 2r)$, define

$$\xi(x,t) := v(x,t) + \tau\gamma(v(x,t) + \varepsilon(t+\alpha) - f(x))_+.$$

We claim that ξ is a subsolution in $\Sigma := (B_{\frac{1}{2}} \times (-2r, -\alpha)) \cap \{v \leqq \varepsilon\}$ if ε is small enough, independent of r. Let us follow [10] and only point out the differences coming from the drift. We recall the operator $\mathcal{L}_2$ defined in (5.4) and denote the drift independent part as $\widetilde{\mathcal{L}}$ to get

$$\widetilde{\mathcal{L}}(\xi) := \xi_t - (m-1)\,\xi\,\Delta\xi - |\nabla\xi|^2. \tag{6.20}$$

Let $g(s) := \tau\gamma s_+$ and thus $g' = \tau\gamma\,\chi_{\{s>0\}}$, $g'' \geqq 0$ in the sense of distribution. Below, we write $g = g(v + \varepsilon(t+\alpha) - f)$. Direct computations yield

$$\begin{aligned} \xi_t &= (v+g)_t = (1+g')v_t + \varepsilon g', \\ \nabla\xi &= \nabla(v+g) = (1+g')\nabla v - g'\nabla f. \end{aligned}$$

Following the computations in Lemma 3.1 of [10] and using $|\nabla v| \geqq \frac{1}{L}$, we obtain

$$\widetilde{\mathcal{L}}(\xi) \leqq (1+g')\widetilde{\mathcal{L}}(v) - \left(L^{-2} - C\varepsilon\right) g' \quad \text{with } C \text{ only depending on } L \text{ and } \sigma.$$

Since $\mathcal{L}_2(\xi) = \widetilde{\mathcal{L}}(\xi) - \nabla\xi \cdot (\vec{b}(x+X) - \vec{b}(X)) - (m-1)\xi\nabla\cdot\vec{b}$, then

$$\begin{aligned} \mathcal{L}_2(\xi) &\leqq (1+g')\widetilde{\mathcal{L}}(v) - \left(L^{-2} - C\varepsilon\right) g' - \nabla\xi \cdot (\vec{b}(x+X) - \vec{b}(X)) - (m-1)\xi\,\nabla\cdot\vec{b} \\ &= (1+g')\mathcal{L}_2(v) + g'\nabla f \cdot (\vec{b}(x+X) - \vec{b}(X)) - (m-1)(g-g')\nabla\cdot\vec{b} - \left(L^{-2} - C\varepsilon\right) g' \\ &= g'\nabla f \cdot (\vec{b}(x+X) - \vec{b}(X)) - (m-1)g\nabla\cdot\vec{b} - \left(L^{-2} - C\varepsilon - (m-1)\nabla\cdot\vec{b}\right) g'. \end{aligned}$$

By (C_k), we have $\|\vec{b}\|_\infty \leqq \sigma$, $\|\nabla\vec{b}\|_\infty \leqq \sigma\delta J^k$. Since we assumed $\delta \leqq \varepsilon$ and $J < 1$, $\|\nabla\vec{b}\|_\infty \leqq \sigma\varepsilon$. Also for $(x,t) \in \Sigma$, $s := v + \varepsilon(t+\alpha) - f \leqq \varepsilon$ and hence $g(s) \leqq \varepsilon g'(s)$. We get

$$|g'\nabla f \cdot (\vec{b}(x+X) - \vec{b}(X)) + (m-1)g\nabla\cdot\vec{b}| \leqq \sigma\varepsilon^2 g' \text{ in } Q_{\frac{1}{2}}.$$

Thus $\mathcal{L}_2(\xi) \leqq 0$ if ε is small enough depending only on L, σ.

The rest of the proof follows from the proof of Proposition 2.3 [10], where we compare w and ξ in Σ to conclude that

$$w(x,-\alpha) \geqq (1+\tau\gamma)v(x,-\alpha) \quad \text{in } B_{\frac{1}{4}} \cap \{v \leqq \varepsilon\}$$

for all $\alpha \in (-2r, 2r)$. □

Proof of Lemma 6.8. Based on (A_k),(B_k) and the elliptic regularity estimate applied to v, one can argue as in Lemma 3.2 of [10] to conclude that

$$vD_{ij}v \geqq -C_4, \quad \text{for all } i, j = 1, \ldots, d \quad \text{in } Q_{2r}, \tag{6.21}$$

where C_4 depends only on L, universal constants and the Lipschitz constant of $\Gamma(v)$. We will use this fact in the computation below.

Define

$$h(x,t) := (1+\tau\gamma)v(x+(t+2r)\phi\mu, t), \quad y := x+(t+2r)\phi\mu.$$

Note that $|y-x| \leqq \kappa\tau\gamma$. Lemma 6.7 implies that $w \geqq h$ on the parabolic boundary of

$$\Sigma := (B_{\frac{1}{4}} \times (-2r, 2r)) \cap \{v \leqq \varepsilon\}.$$

We claim that $\mathcal{L}_2(h) \leqq 0$ in Σ. Write $\tau' := \tau\gamma$. We have

$$\begin{aligned}
h_t &= (1+\tau')(v_t + v_\mu \phi),\\
\nabla h &= (1+\tau')\left(\nabla v + v_\mu (t+2r)\nabla\phi\right),\\
\Delta h &= (1+\tau')\left(\Delta v + 2(t+2r)\nabla v_\mu \cdot \nabla\phi + v_{\mu\mu}(t+2r)^2|\nabla\phi|^2 + v_\mu (t+2r)\Delta\phi\right),
\end{aligned}$$

From (6.21) and the computations in Proposition 2.4 [10]

$$\widetilde{\mathcal{L}}(h) \leqq (1+\tau')\widetilde{\mathcal{L}}(v)(y,t) - \tau'\left(L^{-1} - C\kappa\right)$$

where $\widetilde{\mathcal{L}}$ is given by (6.20) and C depends only on m, L, C_4, σ. Thus

$$\begin{aligned}
\mathcal{L}_2(h) &\leqq (1+\tau')\widetilde{\mathcal{L}}v(y,t) - \tau'\left(L^{-1} - C\kappa\right) - \nabla h \cdot \left(\vec{b}(x+X) - \vec{b}(X)\right)\\
&\quad - (m-1)h\nabla\cdot\vec{b}(x+X)\\
&= (1+\tau')\mathcal{L}_2(v)(y,t) - \tau'\left(L^{-1} - C\kappa\right) - (1+\tau')\nabla v \cdot \left(\vec{b}(x+X) - \vec{b}(y+X)\right)\\
&\quad - (m-1)(1+\tau')v(y,t)\nabla\cdot\left(\vec{b}(x+X) - \vec{b}(y+X)\right)\\
&\quad - (1+\tau')v_\mu(t+2r)\nabla\phi)\cdot\left(\vec{b}(x+X) - \vec{b}(X)\right)\\
&\leqq -\tau'\left(L^{-1} - C\kappa\right) + (1+\tau')|\nabla v|\|D\vec{b}\|_\infty |x-y| + (m-1)(1+\tau')v\|D^2\vec{b}\|_\infty |x-y|\\
&\quad + (1+\tau')\left|v_\mu\right|(t+2r)\,|\nabla\phi|\;\|\nabla\vec{b}\|_\infty |x|.
\end{aligned}$$

Now apply (C_k) and since $\delta \leqq \varepsilon$, we have $\|D\vec{b}\|_\infty \leqq \sigma\varepsilon$, $\|D^2\vec{b}\|_\infty \leqq \sigma\varepsilon^2$. Since $|\nabla\phi| \leqq \kappa\tau'$, we obtain

$$\begin{aligned}
\mathcal{L}_2(h) &\leqq -\tau'\left(L^{-1} - C\kappa\right) - \sigma L\varepsilon\kappa\tau' - \sigma L\varepsilon r\kappa\tau' - \sigma L\varepsilon^2\kappa\tau'\\
&\leqq -\tau'\left(L^{-1} - C\kappa - \sigma L\kappa\right) \leqq 0 \quad \text{in } \Sigma,
\end{aligned}$$

if κ is small enough. By comparison principle applied to w and h in Q_{2r} we can conclude that

$$w(x,t) \geqq h(x,t) \geqq v(x+(t+2r)\phi(x)\mu, t) \text{ in } Q_{2r}.$$

□

7. Discussion of Traveling Waves and Potential Singularities

In this section we discuss evolution of solutions in two space dimensions, in several explicit scenarios.

7.1. A Discussion on Traveling Waves

For simplicity, we restrict to two space dimensions $d = 2$. The drift is chosen as

$$\vec{b}(x_1, x_2) := (\alpha(x_2), 0), \text{ where } \alpha \text{ is Lipschitz and bounded.} \tag{7.1}$$

When α is periodic and $\max\{\alpha\} < c$, it is shown in [21] that there exist traveling wave solutions of the form $U(x + cte_1)$ for the corresponding pressure equation (7.2), with the growth condition $\lim_{x_1\to\infty} \frac{U(x)}{x_1} = c$. While Lipschitz regularity of the solutions are established therein, the free boundary regularity and possibility of a corner remain open.

Our regularity analysis cannot address the traveling waves themselves, but we are able to say that such singularity, if at all, is of asymptotic nature. More precisely, we show that dynamic solutions, used in [22] to approximate the travelling waves, stay smooth in any finite time interval.

Theorem 7.1. *Let u solve (1.5) in $\mathbb{R}^2 \times (0, \infty)$, with $\vec{b}$ given in (7.1), with the initial data $u_0(x) = (x_1)_+$. Further impose that $\frac{u(x,t)}{x_1} \to 1$ as $x_1 \to \infty$. Then the following holds:*

(a) *u is uniformly Lipschitz continuous in $\mathbb{R}^2 \times [0, \infty)$.*
(b) *For any fixed $T > 0$, there exists $\tau_0(T) > 0$ such that for all $t \in [0, T]$ and $\tau \leqq \tau_0$*

$$\partial_{x_1} u \pm \tau \partial_{x_2} u \geqq 0.$$

(c) *u is non-degenerate, and $\Gamma(u)$ is $C^{1,\alpha}$ in $\mathbb{R}^2 \times [0, T]$.*

Proof. Let us rewrite (1.5) with our choice of $\vec{b}$:

$$\partial_t u - (m-1)u\,\Delta u - |\nabla u|^2 - \alpha(x_2)\,\partial_{x_1} u = 0. \tag{7.2}$$

Define $\varphi(x, t) := (x_1 + \sigma_1 t)_+$ with $\sigma_1 := \sup|\alpha| + 1$. Then φ is a supersolution of (7.2) with the same initial data as u, and thus $u \leqq \varphi$. In particular, for any $\varepsilon > 0$

$$u(x - \sigma_1\varepsilon e_1, \varepsilon) \leqq \varphi(x - \sigma_1\varepsilon e_1, \varepsilon) = (x_1)_+ = u(x, 0), \tag{7.3}$$

where we denote the positive x_1 direction as e_1.

For $\varepsilon > 0$ let $u^\varepsilon(x, t) := u(x - \sigma_1\varepsilon e_1, t + \varepsilon)$. From (7.3), it follows that $u^\varepsilon(\cdot, 0) \leqq u_0$. Since u^ε also solves (7.2), by comparison principle it follows that $u^\varepsilon \leqq u$, and thus

$$u_t - \sigma_1 u_{x_1} \leqq 0.$$

Above inequality with (6.1) yields that u is uniformly Lipschitz continuous in space and time.

Next to show (b), for $\varepsilon > 0$ and $\sigma_2 := \sup |\partial_{x_2}\alpha|$ we define

$$w(x,t) := \sup_{|y-x|\leqq \varepsilon e^{-\sigma_2 t}} u(y-\varepsilon e_1, t).$$

For each x, pick $y = y(x,t)$ that realizes the supremum. As in the proof of Lemma 5.4, for a.e. $(x,t) \in \mathbb{R}^2 \times (0,\infty)$ we have

$$w_t(x,t) = (u_t - \sigma_2 \varepsilon e^{-\sigma_2 t}|\nabla u|)(y,t).$$

Therefore for a.e. $(x,t) \in \mathbb{R}^2 \times (0,\infty)$,

$$\begin{aligned}
& w_t - (m-1)w\Delta w - |\nabla w|^2 - \nabla w \cdot \vec{b} - (m-1)w\nabla \cdot \vec{b} \\
&\leqq -\sigma_2 \varepsilon e^{-\sigma_2 t}|\nabla w| + |\nabla w| \sup_{\vec{y}\in B(x,\varepsilon e^{-\sigma_2 t})} |\vec{b}(y-\varepsilon e_1) - \vec{b}(x)| \\
&\leqq (-\sigma_2 \varepsilon e^{-\sigma_2 t} + \varepsilon e^{-\sigma_2 t}\|\alpha'\|_\infty)|\nabla w| \leqq 0,
\end{aligned}$$

where for the second equality above we used the fact that $\vec{b}$ only depends on x_2. Thus w is a subsolution. Since $w(\cdot, 0) \leqq u_0$, the comparison principle for (7.2) yields $w \leqq u$. In particular we have

$$u(x,t) \geqq \sup_{|y|\leqq \varepsilon e^{-\sigma_2 T}} u(x+y-\varepsilon e_1, t) \text{ for } 0 \leqq t \leqq T,$$

which yields (b) with $\tau \leqq \tan(\arcsin(e^{-\sigma_2 T}))$. Since (a)-(b) imply (1.10) and that u is cone monotone, Proposition 6.3 and Theorem 6.1 yield (c). □

Remark 7.2. Let us consider the travelling wave solution $u(x,t) = U(x + cte_1)$ of (7.2) with smooth and periodic α, studied in [21]. It was shown there that, assuming non-degeneracy, the free boundary $\Gamma_0 = \partial\{U(x) > 0\}$ can be represented by a Lipschitz graph $x_1 = f(x_2)$.

Our analysis shows that under the same assumption the graph function f is at least $C^{1,\alpha}$. Indeed $|\nabla U|$ is globally bounded due to Theorem 1 of [21] and thus (1.10) holds for u. Now Theorem 6.1 applies to yield the desired regularity of f. This improvement suggests that singularity of the free boundary such as corner formulation could happen only when non-degeneracy fails.

The rest of the section discusses examples of singular solutions that are not present in the zero drift problem. First we discuss global-time persistence and aggravation of corners.

Theorem 7.3. *There exist solutions u_1, u_2 to (1.5) in Q with bounded smooth spatial vector fields and non-negative, Lipschitz initial data such that*

1. *u_1 is stationary and $\Gamma(u_1)$ has a corner at the origin.*
2. *There is a corner of shrinking angles on $\Gamma(u_2)$.*

Proof. Write (x, y) as the space coordinate. Let

$$\vec{b} := -\nabla\Phi(x, y) \text{ for some smooth function } \Phi,$$

and then it can be checked directly that

$$u_1 := \max\{\Phi, 0\}$$

is a stationary solution to (1.5). Notice $\Gamma_0(u_1)$ is the 0-level set of Φ and we claim that if Φ is degenerate, the interface can be non-smooth.

For example, we can take

$$\Phi(x, y) = g(x)g(y)$$

where g is a function on $\mathbb{R}$ that it is only positive in $(0, 1)$. Then $\partial\{u_1 > 0\}$ is a square. In particular, $\partial\{u_1 > 0\}$ contains a Lipschitz corner at the origin.

Next we show (2). Take $\vec{b} := (ax, by)$ (for a moment) and

$$\varphi(x, y, t) := \begin{cases} \lambda(t)(x^2 - k(t)y^2)_+ & \text{if } x > 0, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$\lambda(t) = e^{\sigma_1 t}, k(t) = k_0 e^t \text{ for some } \sigma_1, k_0 > 0.$$

Then the $\Gamma_t(\varphi)$ contains a corner with vertex at the origin.

Let us show that φ is a supersolution to (1.1) for $t \in (0, 1/\sigma_1)$. Due to Lemma 2.6, we only need to check this for $x > k^{1/2}|y|$.

$$\begin{aligned}
\mathcal{L}(\varphi) &:= \varphi_t - (m-1)\varphi\Delta\varphi - |\nabla\varphi|^2 - \nabla\varphi\cdot\vec{b} - (m-1)\varphi\nabla\cdot\vec{b} \\
&= (x^2 - ky^2)\lambda' - \lambda k' y^2 - (m-1)\lambda^2(x^2 - ky^2)(2-2k) - 4\lambda^2x^2 - 4\lambda^2k^2y^2 \\
&\quad - 2a\lambda x^2 + 2bk\lambda y^2 - (m-1)\lambda(x^2 - ky^2)(a+b) \\
&= (x^2 - ky^2)(\lambda' - \lambda^2(m-1)(2-2k) - \lambda(m-1)(a+b) - 2a - 4\lambda^2) \\
&\quad + \lambda y^2(2bk - k' - 4\lambda k - 4\lambda k^2 - 2ak) \\
&\geqq (x^2 - ky^2)\lambda\,(\sigma_1 - \sigma(\lambda, m, k_0, a, b)) + \lambda y^2 k((2b-1) - (4\lambda + 4\lambda k + 2a)).
\end{aligned} \tag{7.4}$$

Now we fix a and take b such that

$$2b - 1 \geqq 4\lambda + 8\lambda k_0 + 2a \geqq 4\lambda + 4\lambda k(t) + 2a,$$

if $\sigma_1 \geqq 10$ and $t \leqq 1/\sigma_1$. Next we further take σ_1 to be large enough such that, the first part of (7.4) is also non-negative. We conclude that for $t \in (0, 1/\sigma_1)$, φ is indeed a supersolution and its support contains a corner with angles shrinking from $2\arctan(k_0^{-\frac{1}{2}})$ to $2\arctan(k(t)^{-\frac{1}{2}})$.

Now consider a solution u_2 with initial data u_0 such that $u_0 = \varphi(x, y, 0)$ in B_1 and $u_0 \leqq \varphi(x, y, 0)$. By comparison, $\varphi \geqq u_2$ for all times and so

$$\Omega_t(u_2) \subset \Omega_t(\varphi) \subset \{x > k^{1/2}(t)|y|\}.$$

Since $\vec{b} = 0$ at the origin, the origin is a one-point streamline. By Lemma 3.3, $0 \in \overline{\Omega_t(u_2)}$ for all $t \geqq 0$. Thus $\Gamma_t(u_2)$ has a shrinking corner for a short time. Lastly since u_2 is compactly supported, we can truncate $\vec{b}$ to be bounded which does not affect u_2 and its support. □

Next we consider formation of corners and cusps over time.

Theorem 7.4. *There is a solution u to (1.5) in Q with some bounded continuous vector field and non-negative, bounded and Lipshitz initial data u_0 such that:*

1. *$\Gamma_0(u)$ is smooth.*
2. *$\Gamma_t(u)$ contains a corner/a cusp for a range of time.*

Proof. First we consider $\vec{b} := -(x + |y|,\ y)$. We will construct a supersolution for this choice of $\vec{b}$. For some $\sigma_0, \sigma_1, \varepsilon > 0$, set $\lambda(t) = \sigma_0 e^{\sigma_1 t}$, $\alpha(t) = \varepsilon t$ and

$$\varphi(x, y, t) := \lambda(t)x(x - \alpha(t)|y|)_+.$$

When $t = 0$, the support of φ is a half-plane, while for any $t > 0$ there forms a corner on $\Gamma_t(\varphi)$.

In the positive set of φ $(x > \alpha|y|)$, we have

$$\begin{aligned}\mathcal{L}(\varphi) = \lambda' x(x - \alpha|y|) - \lambda\alpha' x|y| - (m-1)\lambda^2 x(x - \alpha|y|)(2 - \alpha x\delta_y) - \lambda^2 \\ \left|\left(2x - \alpha|y|, \alpha x\frac{y}{|y|}\right)\right|^2 + \lambda\left(2x - \alpha|y|, \alpha x\frac{y}{|y|}\right)\cdot(x + |y|, y) \\ + 2(m-1)\lambda x(x - \alpha|y|).\end{aligned}$$

Here δ_y is the Dirac mass of variable y. Since $\delta_y \geqq 0$, the above simplifies to

$$\begin{aligned}&\geqq (x - \alpha|y|)(\lambda' x - 2(m-1)\lambda^2 x + 2(m-1)\lambda x) - \lambda\alpha' x|y| \\ &\quad - \lambda^2|(x - \alpha|y|) + x|^2 - \lambda^2\alpha^2 x^2 \\ &\quad + \lambda((x - \alpha|y|) + x)(x + |y|) - \lambda\alpha x|y| \\ &\geqq (x - \alpha|y|)(\lambda' x - 2m\lambda^2 x + 2(m-1)\lambda x - \lambda^2(x - \alpha|y|)) - \lambda^2 x^2 - \lambda^2\alpha^2 x^2 \\ &\quad + \lambda x(x + |y|) - (\lambda\alpha + \lambda\alpha')x|y|.\end{aligned}$$

Select $\sigma_1 = 4m$, $\sigma_0 \leqq \frac{1}{2}e^{-4m}$, $\varepsilon \leqq 1/4$ and then $\lambda' \geqq 2(m-1)\lambda + 2m\lambda^2$. Therefore for $t \in [0, 1]$,

$$\begin{aligned}\mathcal{L}(\varphi) &\geqq -(\lambda^2 + \lambda^2\alpha^2)(x - \alpha|y|)^2 + \lambda x^2 + (\lambda - \lambda\alpha - \lambda\alpha')x|y| \\ &\geqq (\lambda - \lambda^2(1 + \varepsilon^2 t^2))|x|^2 + \lambda(1 - \varepsilon - \varepsilon t)x|y| \geqq 0.\end{aligned}$$

In the last inequality we used that $\lambda \leqq \frac{1}{2}$, $\varepsilon + \varepsilon t \leqq \frac{1}{2}$.

Thus φ is a supersolution in $\mathbb{R}^2 \times [0, 1]$. Now $u_0 = \varphi(x, y, 0)$ in B_1 and u be a solution with initial data u_0. Then by comparison we conclude that a corner forms on $\Gamma_t(u)$ for $t > 0$.

Next we show the possibility of the formation of cusps. Consider

$$\vec{b} := (x \log x - 10x^{1-\delta}, 0).$$

which is continuous but not Lipschitz continuous at $x = 0$. In particular in our barrier argument we will use approximations to avoid the origin. For some σ_2 is large enough, let

$$\alpha(t) := 1 + \tau(\tau - t),\ \lambda := e^{\sigma_2 t}$$

and let $\tau, \delta > 0$ be such that

$$1 > \delta \geqq \frac{2\tau^2}{1-\tau^2},\ \tau \leqq \frac{1}{2},\ e^{2\sigma_2\tau} \leqq 1. \tag{7.5}$$

For $\varepsilon \in (0, 1)$, set

$$\varphi_\varepsilon(x, y, t) := \begin{cases} \lambda(t)(x^2 - (|y| + \varepsilon)^{2\alpha(t)})_+ & \text{if } x \geqq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then as $\varepsilon \to 0$, for $x \geqq 0$,

$$\varphi_\varepsilon(x, y, t) \to \varphi(x, y, t) := \lambda(t)(x^2 - |y|^{2\alpha(t)})_+.$$

Directly from the definition, the support of φ is smooth when $\alpha > 1$, while a cusp appears when $\alpha = 1$ i.e. $t > \tau$. Set the domain

$$\Sigma_\varepsilon := \bigcup_{t\in[0,2\tau]} \left(\left(\frac{1}{2} \geqq x \geqq (|y| + \varepsilon)^{\alpha(t)} \right) \times \{t\} \right).$$

Let us check that φ_ε is a supersolution to (1.5) in Σ_ε. Notice

$$\begin{aligned} \partial_y(|y| + \varepsilon)^{2\alpha} &= 2\alpha(|y| + \varepsilon)^{2\alpha-1}\frac{y}{|y|}, \\ \partial_{yy}(|y| + \varepsilon)^{2\alpha} &= 2\alpha(2\alpha - 1)(|y| + \varepsilon)^{2(\alpha-1)} + 2\alpha(|y| + \varepsilon)^{2\alpha-1}\delta_y \\ &\geq 2\alpha(2\alpha - 1)(|y| + \varepsilon)^{2(\alpha-1)}. \end{aligned}$$

By direct computation, in Σ

$$\begin{aligned} \mathcal{L}(\varphi_\varepsilon) \geqq{} & (x^2 - (|y| + \varepsilon)^{2\alpha})(\lambda' - \lambda^2(m - 1)(2 - 2\alpha(2\alpha - 1)(|y| + \varepsilon)^{2(\alpha-1)}) \\ & - \lambda(m - 1)\nabla\cdot\vec{b}) \\ & - \lambda\alpha'(|y| + \varepsilon)^{2\alpha}\log(|y| + \varepsilon)^2 - \lambda^2(4|x|^2 + 4\alpha^2(|y| + \varepsilon)^{4\alpha-2}) \\ & - 2\lambda\left(\left(x, -2\alpha(|y| + \varepsilon)^{2\alpha-1}\frac{y}{|y|}\right)\cdot\vec{b}\right) \end{aligned}$$

Note we can assume $\alpha \geqq \frac{1}{2}$ and $\nabla\cdot\vec{b} \leqq \sigma$ for some universal σ in Σ, and therefore the above

$$\begin{aligned} &\geqq (|x|^2 - (|y| + \varepsilon)^{2\alpha})(\lambda' - 2\lambda^2(m - 1) - \sigma\lambda(m - 1)) \\ &\quad - \lambda\alpha'(|y| + \varepsilon)^{2\alpha}\log(|y| + \varepsilon)^2 \\ &\quad - \lambda^2(4(|y| + \varepsilon)^{2\alpha} + 4\alpha^2(|y| + \varepsilon)^{4\alpha-2}) + 2\lambda(-x^2\log x + 10x^{2-\delta}) \\ &=: A_1 + A_2 + A_3 + A_4. \end{aligned}$$

To have $A_1 \geqq 0$, we only need

$$\lambda = e^{\sigma_2 t} \leqq e^{2\sigma_2 \tau} \leqq 2 \text{ and } \sigma_2 \geqq (\sigma + 4)(m - 1).$$

Using that $r^2 \log r$ is negative and decreasing for $r \in [0, \frac{1}{2}]$ and (7.5), we have

$$\begin{aligned} A_2 &= \lambda\tau(|y| + \varepsilon)^{2\alpha} \log(|y| + \varepsilon)^2 && (\alpha' = -\tau) \\ &\geqq 4\lambda\tau(|y| + \varepsilon)^{2\alpha} \log(|y| + \varepsilon)^{\alpha} && (\alpha \geqq 2^{-1}) \\ &\geqq 4\lambda\tau x^2 \log x \geqq 2\lambda x^2 \log x && (x \geqq (|y| + \varepsilon)^{\alpha}, 2\tau \leqq 1). \end{aligned}$$

Also note by (7.5), we have $\lambda \leqq 1, \alpha \leqq 1 + \tau^2 \leqq 2, 4\alpha - 2 \geqq \alpha(2 - \delta)$. So

$$\begin{aligned} A_3 &= -4\lambda^2(|y| + \varepsilon)^{2\alpha} - 4\lambda\alpha^2(|y| + \varepsilon)^{4\alpha - 2} \\ &\geqq -4\lambda x^2 - 16\lambda^2 x^{(2\alpha - 1)/\alpha} \geqq -20\lambda x^{2-\delta}. \end{aligned}$$

In all $\Sigma_{i=1}^4 A_i \geqq 0$. We proved that φ_ε is a supersolution in Σ_ε, so by Lemma 2.6, it is a supersolution in $B_{\frac{1}{2}} \times [0, 2\tau]$.

Now for $h \in (0, 1)$, we select $u^h_{0,\varepsilon}$ to be smooth with initial data $u^h_{0,\varepsilon} = h\,\varphi_\varepsilon(\cdot, 0)$ in B_h and $u^h_{0,\varepsilon} = 0$ in B^c_{2h}. Let u^h_ε solve (1.1) with vector field $\vec{b}$ and initial data $u^h_{0,\varepsilon}$. By finite propagation property, we can take h to be small enough such that for all $\varepsilon \in (0, 1)$

$$u^h_\varepsilon(\cdot, t) = 0 \quad \text{on } (\partial B_{\frac{1}{2}}) \times [0, 2\tau].$$

By comparison (which is valid since $\vec{b}$ is smooth in Σ_ε), $u^h_\varepsilon \leqq \varphi_\varepsilon$ in $B_{\frac{1}{2}} \times [0, 2\tau]$. Now passing $\varepsilon \to 0$ gives a solution u^h with initial data $h\varphi(\cdot, 0)$ in B_h such that $u^h \leqq \varphi$ for $t \in [0, 2\tau]$. As before we conclude by the geometry of $\Omega(\varphi)$ and Lemma 3.3 that a cusp appears for $\tau < t < 2\tau$. □

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Appendix A: Proof of Lemma 2.6

Let us only consider the case when $U = \mathbb{R}^d$. The case of $U = B_1$ follows similarly. Fix one non-negative $\phi \in C_c^\infty(\mathbb{R}^d \times [0, T))$. Denote

$$U_0 := \{\phi > 0\} \cap \{\psi > 0\}.$$

For any $\varepsilon > 0$, take finitely many space time balls U_i, $i = 1, \ldots, n$ such that

1. for each $i \geqq 1$, $|U_i| \leqq \varepsilon^{d+1}$ and U_i is in the ε-neighbourhood of $\Gamma(\psi)$,

2. $\{U_i\}_{i=1,\dots,n}$ is an open cover of $\Gamma(\psi) \cap \{\phi > 0\}$.

Since $\Gamma(\psi)$ is of dimension d, we can assume

$$n \lesssim \varepsilon^{-d}. \tag{A.1}$$

Take a partition of unity $\{\rho_i, i = 0, \dots, n\}$ which is subordinate to the open cover $\{U_i\}_{i \geqq 0}$. Then for $i \geqq 1$,

$$|\nabla \rho_i| + |\partial_t \rho_i| \lesssim 1/\varepsilon. \tag{A.2}$$

By the assumption, ψ is a supersolution in the interior of its positive set. And since ε can be arbitrarily small, to show (2.4) we only need to show

$$I_\varepsilon := \sum_{i=1}^{n(\varepsilon)} \left(\int_0^T \int_{\mathbb{R}^d} \psi\,(\phi\rho_i)_t - (\nabla\psi^m + \psi\,\vec{b})\nabla(\phi\rho_i)\,\mathrm{d}x\mathrm{d}t - \int_{\mathbb{R}^d} \psi(0,x)\phi(0,x)\rho_i\,\mathrm{d}x \right) \to 0$$

as $\varepsilon \to 0$.

By property 1 of U_i and the regularity assumption on ψ, in all U_i, $i \geqq 1$ we have

$$\psi \leqq C\varepsilon^{\frac{1}{\alpha}}, \quad |\nabla\psi^m| \leqq C\psi^{m-\alpha}|\nabla\psi^\alpha| \leqq C\varepsilon^{\frac{m-\alpha}{\alpha}}.$$

Now from (A.1), (A.2) and $\alpha < m$, it follows that

$$\begin{aligned} |I_\varepsilon| &\leqq C\varepsilon^{-d} \left(\iint_{U_i} \frac{1}{\varepsilon}(\psi + |\nabla\psi^m|)\,\mathrm{d}x\mathrm{d}t + \int_{U_i \cap \{t=0\}} \psi(0,x)\mathrm{d}x \right) \\ &\leqq C(\varepsilon^{\frac{1}{\alpha}} + \varepsilon^{\frac{m-\alpha}{\alpha}} + \varepsilon) \end{aligned}$$

which indeed converges to 0 as $\varepsilon \to 0$.

Appendix B: Sketch of the proof of Lemma 5.3

We follow the idea of Lemma 9 [8] and compute

$$\Delta f(0) = \overline{\lim}_{r\to 0} \left(\fint_{B_r} f(x) - f(0)\mathrm{d}x \right).$$

Without loss of generality, suppose locally near the origin that

$$f(x) = \inf_{|\nu|=1} h\left(x + \psi(x)\nu\right),$$

because otherwise $\Delta f(0) = 0$. Choosing an appropriate system of coordinates, we can have

$$\begin{aligned} f(0) &= h(\psi(0)e_n); \\ \nabla\psi(0) &= \alpha e_1 + \beta e_n. \end{aligned}$$

We will evaluate $f(x)$ by above by choosing $\nu(x) = \frac{\nu_*(x)}{|\nu_*(x)|}$ where

$$\nu_*(x) := e_n + \frac{\beta x_1 - \alpha x_n}{\psi(0)} e_1 + \frac{\gamma}{\psi(0)} \left(\Sigma_{i=2}^{d-1} x_i \, e_i \right)$$

where γ satisfies

$$(1+\gamma)^2 = (1+\beta)^2 + \alpha^2.$$

With this choice of ν, we define $y := x + \psi(x)\nu(x)$ and so $y(0) = \psi(0)e_n$. After direct computations (also see [8]), we can write

$$y = Y_*(x) + \psi(0)e_n + o(|x|^2)$$

such that the first-order term, except the translation $\varphi(0)e_n$, satisfies

$$Y_*(x) := x + (\alpha x_1 + \beta x_n)e_n + (\beta x_1 - \alpha)e_1 + \gamma \, \Sigma_{i=1}^{d} x_i e_i.$$

Hence $Y_*(x)$ is a rigid rotation plus a dilation and we have

$$\left| \frac{D(Y_* - x)}{Dx} \right| \leqq \sigma \|\nabla \psi\|_\infty. \tag{B.1}$$

Then

$$\begin{aligned} \fint_{B_r} f(x) - f(0)\mathrm{d}x &\leqq \fint_{B_r} h(y(x)) - h(y(0))\mathrm{d}x \\ &\leqq \fint_{B_r} h(y(x)) \\ &\quad - h(Y_*(x) + y(0))\mathrm{d}x + \fint_{B_r} h(Y_*(x) + y(0)) - h(y(0))\mathrm{d}x. \end{aligned}$$

By the condition on ψ and the computations done in Lemma 9 [8], the first term is non-positive.

Since h is smooth, the second term converges to

$$\left(\left| \frac{DY_*}{Dx} \right|_{x=0} \right)^2 (\Delta h)(y(0)) \quad \text{as } r \to 0.$$

Now, using (B.1) and the assumption that $\Delta h \geqq -C$ and $\|\nabla\psi\|_\infty \leqq 1$, we get

$$\begin{aligned} \fint_{B_r} f(x) - f(0)\mathrm{d}x &\leqq \fint_{B_r} h(Y_*(x) + y(0)) - h(y(0))\mathrm{d}x \\ &\leqq (1 + \sigma\|\nabla\psi\|_\infty)(\Delta h)(y(0)) + \sigma\|\nabla\psi\|_\infty C. \end{aligned}$$

Thus we have finished the proof.

Appendix C: Proof of Lemma 5.4

Let us suppose $x = 0$ and $f(0) = h(y)$ for a unique y. We only compute $\partial_1 f(0) = \partial_{x_1} f(0)$. If $\nabla h(y) = 0$, it is not hard to see

$$\partial_1 f(0) = \partial_1 h(y) = 0.$$

Next suppose $\nabla h(y) \neq 0$. We know that h obtains its minimum over $B(0, \psi(0))$ at point $y \in \partial B(0, \psi(0))$. Let us assume

$$y = (y_1, y_2, 0, \dots, 0), \quad \text{and thus } |y_1|^2 + |y_2|^2 = (\psi(0))^2.$$

For smooth h, it is not hard to see that

$$\nabla h(y) = -ky \quad \text{with } k = \frac{|\nabla h|}{\psi(0)}.$$

Near point y

$$h(x) - h(y) = -ky_1(x_1 - y_1) - ky_2(x_2 - y_2) + o(|x - y|).$$

To estimate $w((\delta, 0, \dots, 0))$, consider the leading terms:

$$\begin{aligned} A(\delta) &:= -ky_1(x_1 - y_1) - ky_2(x_2 - y_2) \\ &= -ky_1(x_1 - \delta) - ky_2 x_2 + ky_1^2 + ky_2^2 - ky_1\delta. \end{aligned}$$

By a standard argument, under the constrain

$$|x_1 - \delta|^2 + |x_2|^2 + |x_3|^2 + \cdots + |x_n|^2 \leqq \psi(\delta, 0, \dots, 0)^2,$$

$A(\delta)$ achieves its minimum at

$$x_1 = y_1\psi(\delta, 0, \dots, 0)/(y_1^2 + y_2^2)^{\frac{1}{2}} + \delta, \; x_2 = y_2\psi(\delta, 0, \dots, 0)/(y_1^2 + y_2^2)^{\frac{1}{2}}$$

with value

$$\begin{aligned} &-k\psi(\delta, 0, \dots, 0)(y_1^2 + y_2^2)^{\frac{1}{2}} + ky_1^2 + ky_2^2 - ky_1\delta \\ &= -k\psi(\delta, 0, \dots, 0)\psi(0) + k\psi(0)^2 - ky_1\delta. \end{aligned}$$

Thus

$$\partial_1 f(0) = \lim_{\delta \to 0} A(\delta)/\delta = -k\psi(0)\,\partial_1\psi(0) - ky_1.$$

Notice that $\partial_1 h(y) = -ky_1$. So we find

$$\partial_1 f(0) - \partial_1 h(y) = -k\psi(0)\,\partial_1\psi(0) = -|\nabla h|\,\partial_1\psi(0).$$

This leads to the conclusion.

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INWON KIM
Department of Mathematics,
University of California,
Los Angeles,
USA.
e-mail: ikim@math.ucla.edu

and

YUMING PAUL ZHANG
Department of Mathematics,
University of California,
San Diego,
USA.
e-mail: yuz018@ucsd.edu

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